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Platz, Trine Tornøe

Document Version

Accepted author manuscript

Published in:

Discrete Applied Mathematics

DOI:

[10.1016/j.dam.2018.11.029](https://doi.org/10.1016/j.dam.2018.11.029)

Publication date:

2019

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Citation for published version (APA):

Platz, T. T. (2019). On the Submodularity of Multi-depot Traveling Salesman Games. *Discrete Applied Mathematics*, 255, 75-85. <https://doi.org/10.1016/j.dam.2018.11.029>

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Journal article (Accepted manuscript*)

Please cite this article as:

Platz, T. T. (2019). On the Submodularity of Multi-depot Traveling Salesman Games. *Discrete Applied Mathematics*, 255, 75-85. DOI: 10.1016/j.dam.2018.11.029

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* This version of the article has been accepted for publication and undergone full peer review but has not been through the copyediting, typesetting, pagination and proofreading process, which may lead to differences between this version and the publisher's final version AKA Version of Record.

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On the submodularity of multi-depot traveling salesman games

Trine Tørnøe Platz^a

^a*Department of Economics, Copenhagen Business School, Porcelænshavn 16A, DK-2000 Frederiksberg, Denmark.*

Abstract

The Steiner traveling salesman problem is the problem of finding a minimum cost tour for a salesman that must visit a set of locations while traveling along costly streets before returning to his starting point at the depot. A solution to the problem is a minimum cost Steiner tour that both starts and ends at the depot and visits all the required locations. If different players are associated with the locations to be visited, the problem induces a cooperative traveling salesman (TS) game that poses the question of how to allocate the total cost of the tour between the different players involved. This cost allocation problem can be tackled using tools and solutions from cooperative game theory.

The purpose of this paper is to generalize the notion of a traveling salesman game to allow for multiple depots in the underlying traveling salesman problem (TSP) and to analyse the submodularity of such multi-depot TS games. A multi-depot TSP can be represented by a connected (di)graph in which a fixed set of vertices are denoted depots, and a non-negative weight function is defined on the edges of the graph. The submodularity of multi-depot TS games is analysed by characterizing graphs and digraphs that induce submodular multi-depot TS games for any choice of depots and for at least one choice of the depots, respectively.

Keywords: Steiner traveling salesman problem, cooperative game, submodularity
2000 MSC: 91A12

1. Introduction

The Steiner traveling salesman problem models the situation in which a salesman affiliated with a fixed depot (warehouse) has to visit a number of locations before returning to the depot. There is a cost associated with traveling along the roads that connect locations, and a solution to the problem is a minimum cost Steiner tour that visits the required locations while starting and ending at the depot. Since a Steiner tour allows multiple visits to the same vertices and also allows the set of vertices to be visited to be only a subset of the vertices of the graph, this variant of the traveling salesman problem can be considered more suitable for some applications, including applications on real world road networks, e.g. Letchford et al. [9]. The traveling salesman problem has been extended in a number of directions, for example by allowing for multiple salesmen, multiple warehouses, and by adding restrictions on visiting times and capacity.

A traveling salesman problem (TSP) can be represented by a (di)graph in which a fixed vertex is denoted the depot, and a weight function is defined on the edges (arcs). If the vertices to be visited are associated with different players, the TSP induces a cooperative cost allocation game denoted a traveling salesman game [12, 14] in which the cost of a coalition of players is the cost of a minimum weight tour that visits all players in the coalition. Once the cooperative game is defined, tools and solution concepts from cooperative game theory can be applied to analyze and solve the cost allocation problem.

The purpose of this paper is to extend the notion of a traveling salesman game to allow for multiple depots in the underlying TSP and to consider the submodularity of such multi-depot TS games. The

Email address: ttp.eco@cbs.dk (Trine Tørnøe Platz)

multi-depot setup considered here may be interpreted as a situation in which several depots exist, each with their own salesman/vehicle of unlimited capacity, such that an optimal tour may be a collection of subtours each originating from a different depot and returning to its point of origin.

The focus on submodularity is motivated by the desirable properties of submodular games. For one thing, submodular games are totally balanced, implying that the core of every subgame is non-empty. The Shapley value is in the core of a submodular game, and it is the barycenter of the core, Shapley [13]. Furthermore, several solution concepts coincide (the nucleolus and the kernel, the bargaining set and the core, Maschler et al. [10]), and others can be more easily computed for this class of games than is generally the case.

In the following, a multi-depot TSP with k depots will be denoted a k -depot TSP or simply, a k -depot TS problem. Furthermore, a graph G is said to be globally (locally) k -TS submodular if the game induced by a k -depot TS problem on G is submodular for every (some) location of the k depots and for every weight function. This paper characterizes the classes of globally and locally k -TS submodular undirected graphs and also provide some results on the characterization of globally and locally k -TS submodular digraphs. For the undirected case, it is shown that a graph G is globally k -TS submodular for $k \in \{2, \dots, |V(G)| - 3\}$ if and only if G does not contain a path of 5 vertices or more. Furthermore, a graph G is locally k -TS submodular if a set of k depots can be allocated such that a specific pattern is avoided, and if G at the same time fulfills a cut condition for all connected components in a specific subgraph of G .

Likewise, directed graphs are globally k -TS submodular only if they do not contain specific forbidden structures, and it is also shown that the class of locally k -TS submodular graphs is a strict superset of the globally k -TS submodular graphs.

For the standard case with just one depot in the underlying TSP, Herer and Penn [8] provided a characterization of the undirected graphs that induce submodular traveling salesman games. This result was extended to the directed case in Granot et al. [4]. The games analyzed in these papers, can be seen as special cases of the multi-depot TS game, for which $k = 1$. In a related string of research, Granot et al. [6] characterized for the case of Chinese postman (CP) games both CP-balanced and CP-submodular graphs. In Granot and Hamers [5], the authors distinguished between global and local requirements and characterized the class of locally CP-submodular graphs as well as locally TS-submodular graphs. The approach of analyzing both global and local requirements is followed in the present paper. Another related paper is Platz and Hamers [11] that introduce multi-depot Chinese postman games and characterize the classes of (globally and locally) k -CP balanced and k -CP submodular graphs. The modelling of the multi-depot TS game in the present paper follows a similar approach, but due to the different combinatorial nature of the two problems, the results and proofs differ.

The analysis of traveling salesman games (and Chinese postman games) is part of a larger literature that analyzes the properties of cooperative games that arise from underlying optimization problems. A few other examples of such classes of games are minimum cost spanning tree games [7], sequencing games [2], and minimum colouring games [3]. An overview can be found in Curiel [1].

The paper is structured as follows. Section 2 presents some useful terms and notation. In section 3, multi-depot traveling salesman games are introduced. Results are presented in sections 4 and 5 for undirected and directed graphs respectively.

2. Preliminaries

From cooperative game theory, we recall the following definitions: In a cooperative (cost) game (N, c) , $N = \{1, \dots, n\}$ denotes the finite playerset, and $c : 2^N \rightarrow \mathbb{R}$ is a function that assigns to every coalition $S \subseteq N$ a cost $c(S)$, with $c(\emptyset) = 0$. Let $x \in \mathbb{R}^N$ be an allocation of $c(N)$ between the players. A game (N, c) is monotonic if $c(S) \leq c(T)$ for all $S \subset T \subseteq N$, and it is subadditive if $c(S \cup T) \leq c(S) + c(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$. The core of a game (N, c) is given by

$$\text{core}(N, c) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = c(N), \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subseteq N\}.$$

The game (N, c) is said to be submodular if

$$c(T \cup \{i\}) - c(T) \leq c(S \cup \{i\}) - c(S) \quad (2.1)$$

for all $i \in N$ and all $S \subset T \subseteq N \setminus \{i\}$.

Next, we recall some notions from graph theory. An undirected (directed) *graph* $G = (V(G), E(G))$ consists of a non-empty, finite set of *vertices* $V(G)$ and a set of pairs of vertices $E(G)$ called *edges* (*arcs*). An edge $\{a, b\}$ in an undirected graph or an arc (a, b) in a directed graph joins the vertices a, b and is said to be *incident* to a and b . The vertices a and b are *adjacent*. The *degree* of a vertex a is equal to the number of edges incident to a . An arc (a, b) is directed from a to b and can only be traversed in this direction.

A *walk* (respectively, *directed walk*), w , in a graph G is a sequence of vertices and edges (respectively, arcs) of the form: $v_0, e_1, v_1, \dots, v_{m-1}, e_m, v_m$, where $v_0, \dots, v_m \in V(G)$, $e_1, \dots, e_m \in E(G)$, and $m \geq 0$ such that $e_j = \{v_{i-1}, v_j\}$ (respectively, $e_j = (v_{i-1}, v_j)$) for all $i \in \{1, \dots, m\}$. If $v_0 = v_m$, the walk is said to be *closed*. A closed walk may be empty, $w = \{v_0\}$. A (*directed*) *path* is a (*directed*) walk in which no edge (arc) or vertex is visited more than once, except v_0 in the case of $v_0 = v_m$. The *length* of a (*directed*) path is equal to the number, m , of edges (arcs) traversed. If there exists an undirected (directed) path between any two vertices in a (*directed*) graph G , then G is a (*strongly*) *connected* graph. A closed (*directed*) walk in which no edge is visited more than once will be denoted a (*directed*) circuit, whereas a closed (*directed*) path, i.e., a (*directed*) walk in which no edge (arc) or vertex is visited more than once, is denoted a (*directed*) cycle. A *subdivision* of a graph G is the graph G' that arises by (repeatedly) replacing an edge (arc) in G with a (*directed*) path of length two.

Let G be a graph, and let v_s and v_t be two vertices in G . Then an s - t vertex cutset is a set of vertices such that removing these vertices along with the edge $\{v_s, v_t\}$, if it exists, results in a disconnected graph in which v_s and v_t do not belong to the same component. A *minimal* s - t vertex cutset, K , is an s - t vertex cutset such that no proper subset of K is an s - t vertex cutset.

Finally, an *outerplanar graph* is a graph that can be embedded in the plane such that no edges cross, and such that all vertices of the graph lie on the boundary of the outer face of the embedding.

3. Multi-depot traveling salesman games

Let $G = (V(G), E(G))$ be a (strongly) connected (di)graph, and let $Q \subset V(G)$ be a fixed subset of the vertices of G . An element of Q is called a depot. A multi-depot traveling salesman problem is given by a tuple $\Gamma = (V^-(G), (G, Q), t)$ in which $V^-(G) = V(G) \setminus Q$ denotes the set of vertices in G that are not associated with depots, and $t : E(G) \rightarrow [0 : \infty)$ is a weight function defined on the edges (arcs) of G .

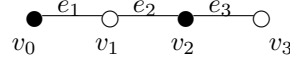
Let $w = (v_0, e_1, v_1, \dots, e_m, v_m)$ denote a walk in G . The cost of w equals the sum of the weights of the edges traversed, such that $cost(w) = \sum_{i=1}^m t(e_i)$. A closed walk that starts and ends at a vertex $v \in Q$ is denoted w_v . A closed walk $w_v = \{v\}$ is said to be empty, and the cost of an empty walk is 0.

Let G be a (strongly) connected (di)graph, and let $\Gamma = (V^-(G), (G, Q), t)$ be a multi-depot TSP defined on G . Next, let $S \subseteq V^-(G)$ be a subset of the non-depot vertices of G . Then for a given $Q = \{v_1, \dots, v_k\}$, an S -Steiner tour $d(S)$ is a collection of closed walks $d(S) = \{w_{v_1}, \dots, w_{v_k}\}$ (some of which may be empty) such that every vertex in S is visited. The set of all S -Steiner tours is denoted $D(S)$.

The following example illustrates the notion of an S -Steiner tour.

Example 3.1. Consider Figure 1 in which depots are located at vertices v_0 and v_2 , whereas v_1 and v_3 are associated with players. There are then at least two possible S -Steiner tours for $S = \{v_1, v_3\}$. One is the tour $v_2, e_2, v_1, e_2, v_2, e_3, v_3, e_3, v_2$, and the other is the tour consisting of the two subtours v_0, e_1, v_1, e_1, v_0 and v_2, e_3, v_3, e_3, v_2 . \triangle

Figure 1: S -Steiner tours in a multi-depot setting



When depots are located at the vertices of Q , the cost of an S -Steiner tour $d(S) = \{w_{v_1}, \dots, w_{v_k}\}$ is equal to:

$$C_Q(d(S)) = \sum_{i=1}^k \text{cost}(w_{v_i})$$

Now, the multi-depot TS game induced by a multi-depot TSP can be defined as follows. Let $G = (V(G), E(G))$ be a (strongly) connected (directed) graph, and let $\Gamma = (V^-(G), (G, Q), t)$ be a multi-depot TSP defined on G . Then (N, c_Q) is the induced multi-depot TS game in which $N = V^-(G)$ is the set of players, and $c_Q(S)$ is, for any $S \subseteq N$, the cost of a minimum weight S -Steiner tour, when the depots are located at the vertices of Q . That is:

$$c_Q(S) = \min_{d(S) \in D(S)} C_Q(d(S)).$$

Note that since $N = V^-(G) = V(G) \setminus Q$, different sets of depots imply different sets of players, and in particular, a greater number of depots implies fewer players. An illustration of two 2-depot TS problems and their induced games are given in the example below.

Example 3.2. The graph in Figure 2 illustrates two different 2-depot TS problems defined on the same graph. For the problem on the left, $Q = \{v_0, v_4\}$ while $Q = \{v_0, v_3\}$ for the problem on the right. For both TSPs assume that $t(\{v_1, v_4\}) = 10$ while $t(e) = 1$ for all other edges in the graph. The two induced 2-depot TS games $(\{1, 2, 3\}, c_{\{v_0, v_4\}})$ and $(\{1, 2, 4\}, c_{\{v_0, v_3\}})$ are shown in the table below. Note that as the location of the depots change, so does the player set. Furthermore, with the current choice of weight function, we see that $c_{\{v_0, v_4\}}$ is not submodular since $c_{\{v_0, v_4\}}(v_1, v_2, v_3) - c_{\{v_0, v_4\}}(v_2, v_3) = 2 > 0 = c_{\{v_0, v_4\}}(v_1, v_2) - c_{\{v_0, v_4\}}(v_2)$, whereas it is easy to verify that $c_{\{v_0, v_3\}}$ is (sub)modular. \triangle

S	v_1	v_2	v_3	v_1, v_2	v_1, v_3	v_2, v_3	v_1, v_2, v_3
$c_{\{v_0, v_4\}}(S)$	2	4	2	4	4	4	6
S	v_1	v_2	v_4	v_1, v_2	v_1, v_4	v_2, v_4	v_1, v_2, v_4
$c_{\{v_0, v_3\}}(S)$	2	2	2	4	4	4	6

Figure 2: Two 2-depot TS problems and their induced 2-depot TS games

Whereas not every k -depot TS game is submodular, it is straightforward to verify that for any multi-depot TS game (N, c_Q) induced by a multi-depot TSP, $\Gamma = (V^-(G), (G, Q), t)$, it holds that $c_Q(S) \leq c_Q(T)$ for all $S \subset T \subseteq N$ (the game is monotonic), and $c_Q(S \cup T) \leq c_Q(S) + c_Q(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$ (the game is subadditive).

We now turn to the analysis of multi-depot traveling salesman games. We will consider both undirected and directed graphs, and characterize classes of graphs that give rise to submodular multi-depot TS games. Since any subadditive two-player game is submodular, only games of at least three players are considered. Therefore, $|V(G)| \geq k + 3$ for all k -depot TS problems throughout the paper.

4. k -TS submodular undirected graphs

4.1. Globally k -TS submodular graphs

Before proceeding to the characterization of graphs, a definition and a few useful observations are stated.

Let P_m denote a path with m vertices. Let P_{m+1}^F denote the structure illustrated in Figure 3, in which depots are located only at the endpoints of a path with $m + 1$ vertices and $m \geq 4$.

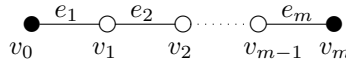


Figure 3: The P_{m+1}^F structure on a path of length $m \geq 4$

Definition 4.1. Let $\Gamma = (V^-(G), (G, Q), t)$ be a multi-depot TS problem defined on a connected graph G . Then G is P_{m+1}^F -free with respect to Q , if no path $v_0, e_1, v_1, \dots, e_m, v_m$ exists in G such that $v_0, v_m \in Q$, $v_1, \dots, v_{m-1} \in V^-(G)$ and $m \geq 4$.

It turns out that the presence of a P_{m+1}^F structure with $m \geq 4$ is incompatible with submodularity of the induced multi-depot TS game, as will be clear in what follows.

We are now ready to characterize globally k -TS submodular graphs for $1 < k < |V(G)| - 2$. Recall that for $k = 1$, a characterization of globally 1-depot TS-submodular graphs was given in Herer and Penn (1995), and for $k \in \{|V(G)| - 2, |V(G)| - 1\}$, all connected graphs are globally k -TS submodular, since there are at most two players in the induced game.

Theorem 4.1. Let $G = (V(G), E(G))$ be a connected, undirected graph, and let $k \in \{2, \dots, |V(G)| - 3\}$. Then G is globally k -TS submodular if and only if G contains no path of length four.

Proof. Let $\Gamma = (V^-(G), (G, Q), t)$ be a k -depot TS problem on G , and let (N, c_Q) be the induced k -depot TS game.

First, for the ‘only if’ part assume that G contains a path of length four. Then there exists a pair (v_i, v_{i+4}) of distinct vertices in G such that there is a path $P := v_i, e_{i+1}, v_{i+1}, \dots, e_{i+4}, v_{i+4}$, from v_i to v_{i+4} visiting four edges. However, since $|Q| = k \leq |V(G)| - 3$, we can then choose Q such that $v_i, v_{i+4} \in Q$, and $v_{i+1}, \dots, v_{i+3} \in V^-(G)$, implying that G is not P_{m+1}^F -free with respect to Q . Next, let $t(e_j) = 1$ for $j \in \{i + 1, i + 2, i + 3, i + 4\}$, and let $t(e) = 0$ for all other edges in G . Then $c_Q(v_{i+1}, v_{i+2}, v_{i+3}) - c_Q(v_{i+1}, v_{i+2}) = 6 - 4 = 2$ whereas $c_Q(v_{i+2}, v_{i+3}) - c_Q(v_{i+2}) = 4 - 4 = 0$, thus violating (2.1), and c_Q is not submodular. Therefore, since the induced game is not submodular for all weight functions, G is not globally k -TS submodular.

Now, consider the ‘if’ part and assume that G contains no path of length four. Since $|V(G)| \geq k + 3$, G has at least 5 vertices. Next, note that the only graphs with five vertices or more that do not contain a path of length four are: star graphs with at least four pendant vertices, graphs obtained by joining two star graphs by adding an edge between the two center vertices (a double star), and graphs obtained from a star graph by adding a single edge between two pendant vertices. The latter two types of graphs are illustrated in Figure 4. In all of these three cases, it holds that if $S = \emptyset$, then $c_Q(S \cup \{v\}) - c_Q(S) = c(v)$ for all $v \in N$, and it follows from subadditivity of c that (2.1) holds. In the remaining part of the proof, we may therefore assume that $S \neq \emptyset$. The three types of graphs will be considered separately.

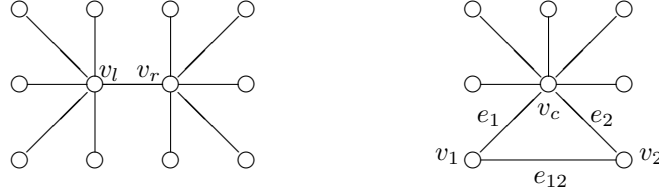


Figure 4: Globally k -TS submodular graph types

First, assume that G is a star graph, and let the single vertex with degree larger than one be denoted v_c . For any $v \in V(G) \setminus \{v_c\}$, let e_v denote the edge connecting v to the center vertex v_c . Now, if $S \neq \emptyset$, then $v_c \in d(S)$, irrespective of the location of depots in G . Then, for all $v \in N$ and all $S \subset T \subseteq N \setminus \{v\}$, we have

$$c_Q(T \cup \{v\}) - c_Q(T) = c_Q(S \cup \{v\}) - c_Q(S) = \begin{cases} 0 & \text{if } v = v_c \\ 2t(e_v) & \text{otherwise,} \end{cases}$$

and (2.1) holds.

Second, let G be a double star graph. Let the two vertices with degree greater than 1 be denoted v_l (left-center) and v_r (right-center) respectively, as in Figure 4. For $v_i \in V(G) \setminus \{v_l, v_r\}$ let e_i denote the edge incident to v_i . Assume that $S \neq \emptyset$ and note that when coalition S is non-empty, either v_l or v_r must belong to $d(S)$. Assume that $v_l \in d(S)$, and let e^* denote the cheapest edge connecting v_r to a vertex in $Q \cup \{v_l\}$ (symmetric arguments can be applied if $v_r \in d(S)$). Let $v_i \in V^-(G)$ and consider three separate cases:

Case 1. $v_i \in \{v_l, v_r\}$: If $v_i = v_l$ then $c_Q(T \cup \{v_i\}) - c_Q(T) = c_Q(S \cup \{v_i\}) - c_Q(S) = 0$ since $v_l \in d(S)$. If $v_i = v_r$ then $c_Q(T \cup \{v_i\}) - c_Q(T) = c_Q(S \cup \{v_i\}) - c_Q(S) = 0$, if $v_r \in d(S)$, and $c_Q(T \cup \{v_i\}) - c_Q(T) \leq c_Q(S \cup \{v_i\}) - c_Q(S) = 2t(e^*)$ otherwise.

Case 2. $v_i \in V^-(G) \setminus \{v_l, v_r\}$ and v_i is adjacent to v_l : then $c_Q(T \cup \{v_i\}) - c_Q(T) = c_Q(S \cup \{v_i\}) - c_Q(S) = 2t(e_i)$.

Case 3. $v_i \in V^-(G) \setminus \{v_l, v_r\}$ and v is adjacent to v_r : then $c_Q(T \cup \{v_i\}) - c_Q(T) = c_Q(S \cup \{v_i\}) - c_Q(S) = 2t(e_i)$ if $v_r \in \{Q \cup d(S)\}$, and $c_Q(T \cup \{v_i\}) - c_Q(T) \leq 2t(e_i) + 2t(e^*) = c_Q(S \cup \{v_i\}) - c_Q(S)$ otherwise.

It follows that (2.1) holds in all three cases, and the induced game is submodular for every $Q \subset V(G)$ and all weight functions.

Lastly, consider the graph in the right panel of Figure 4. Let the single vertex of degree larger than 2 be denoted v_c and refer to it as the center vertex. Let the two 2-degree vertices be denoted v_1 and v_2 respectively, and let e_{12} denote the edge incident to both v_1 and v_2 . Furthermore, for any $v_i \in V(G) \setminus \{v_c\}$, let e_i denote the edge connecting v_i to v_c , and let e^* denote the minimum weight edge connecting v_c to a vertex in $Q \cup \{v_1, v_2\}$. Assume that $S \neq \emptyset$, and consider two cases separately.

Case 1. $v_c \notin d(S)$: then since $S \neq \emptyset$, and the center vertex is not visited, it must be that one of the two vertices $\{v_1, v_2\}$ is in S while the other is in Q , and furthermore, that $\{v_1, v_2\}$ are the only vertices visited by $d(S)$. Thus, either $S = \{v_1\}$ and $v_2 \in Q$, or $S = \{v_2\}$ and $v_1 \in Q$. Either way, it must be that $v_c \in d(T)$ for all $S \subset T$. Therefore, if $v = v_c$, we get $c_Q(T \cup \{v\}) - c_Q(T) = 0 < 2t(e^*) = c_Q(S \cup \{v\}) - c_Q(S)$. For $v = v_i \in V^-(G) \setminus \{v_c, v_1, v_2\}$, we have $c_Q(T \cup \{v_i\}) - c_Q(T) = 2t(e_i)$, while $c_Q(S \cup \{v_i\}) - c_Q(S) = 2t(e_i) + 2t(e^*)$ when $v_c \notin Q$, and $c_Q(S \cup \{v_i\}) - c_Q(S) = 2t(e_i)$ when $v_c \in Q$.

Case 2. $v_c \in d(S)$: then it is trivial that $c_Q(T \cup \{v_c\}) - c_Q(T) = c_Q(S \cup \{v_c\}) - c_Q(S) = 0$, and that $c_Q(T \cup \{v_i\}) - c_Q(T) = c_Q(S \cup \{v_i\}) - c_Q(S) = 2t(e_i)$ for $v_i \in V^-(G) \setminus \{v_c, v_1, v_2\}$. Therefore, consider the case of $v = v_1$ (symmetric arguments can be applied to the case of $v = v_2$). Then either a) $v_1 \in d(S)$ which, again, trivially implies $c_Q(T \cup \{v_1\}) - c_Q(T) = c_Q(S \cup \{v_1\}) - c_Q(S) = 0$, or b) $v_1 \notin d(S)$, $v_2 \in d(S)$, in which case $c_Q(T \cup \{v_1\}) - c_Q(T) = c_Q(S \cup \{v_1\}) - c_Q(S) = \min\{2t(e_1), 2t(e_{12})\}$, or c)

$v_1, v_2 \notin d(S)$, implying that $c_Q(S \cup \{v_1\}) - c_Q(S) = \min\{2t(e_1), 2t(e_2) + 2t(e_{12})\} \geq c_Q(T \cup \{v_i\}) - c_Q(T)$, where the inequality follows, since $v_c \in d(S) \Rightarrow v_c \in d(T)$. Thus, (2.1) holds. \square

Now, consider the following definition:

Definition 4.2. Let G be a connected, undirected graph. Then G satisfies the *cut condition* if there is no vertex-pair $v_s, v_t \in V(G)$ such that there exists a minimal s-t vertex cutset of cardinality greater than 2.

From Herer and Penn (1995) it follows that an undirected graph is globally 1-TS submodular if and only if it satisfies the cut condition. Furthermore, from the proof of Theorem 4.1 it is easy to verify that all graphs that are globally k -TS submodular for $k > 1$ satisfy the cut condition. Hence, the class of globally k -TS submodular graphs is a subset of the globally 1-TS submodular graphs.

4.2. Locally k -TS submodular graphs

When modelling situations in which we can freely choose the location of depots, requiring the induced game to be submodular for all possible locations of depots seems too strong a restriction, and it may be more relevant to simply ask whether there exists at least one location of depots that induces a submodular game for every possible weight function. Therefore, it is also relevant to characterize the class of locally k -TS submodular graphs. Recall that a graph $G = (V(G), E(G))$ is said to be locally k -TS submodular if there exists $Q \subset V(G)$, with $|Q| = k$, such that the induced k -TS game (N, c_Q) is submodular for all weight functions.

In Granot and Hamers [5], it was shown that the class of locally 1-depot TS submodular graphs is equivalent to the class of globally 1-depot TS submodular graphs. A similar result does not hold for k -TS submodular graphs when $k > 1$. For $k > 1$, the class of locally k -TS submodular graphs is a strict superset of the class of globally k -TS submodular graphs. The class of locally k -TS submodular graphs can be characterized as follows.

Theorem 4.2. Let $G = (V(G), E(G))$ be a connected graph, and let $Q \subset V(G)$ be a subset of the vertices with cardinality $k \in \{2, \dots, |V(G)| - 3\}$. Let G_Q denote the subgraph induced by all paths between vertices of Q . Then G is locally k -TS submodular if and only if there exists $Q \subset V(G)$ such that G is P_{m+1}^F -free with respect to Q for $m \geq 4$, and such that all connected components of $G \setminus E(G_Q)$ satisfy the cut condition.

Proof. Let $\Gamma = (V^-(G), (G, Q), t)$, and let (N, c_Q) be the induced k -depot TS game. Consider first the ‘if part’ and assume that there exists $Q \subset V(G)$ such that G is P_{m+1}^F -free with respect to Q for $m \geq 4$. This implies that a path between vertices in Q can visit at most two player vertices in a row, and that every player vertex in $V(G_Q)$ is adjacent to at least one depot and at most one other player vertex within G_Q , as illustrated in Figure 5. A player vertex $v_j \in V(G_Q)$ can also be adjacent to players in $G \setminus V(G_Q)$. When this is the case, let G_j denote the connected component in $G \setminus E(G_Q)$ that contains v_j and note that v_j must be visited on any tour visiting players in G_j .¹ Therefore, it holds for all $S \subset T \subseteq N \setminus \{v_i\}$ that if $V(G_j) \cap S \neq \emptyset$, then v_j is visited by both $d(S)$ and $d(T)$. It remains to be shown that (2.1) holds for all players $v_i \in N$ and all $S \subset T \subseteq N \setminus \{v_i\}$. We distinguish between two cases.

Case 1. $v_i \in V(G_Q)$. For any vertex $v_l \in V(G_Q) \cap N$, let e_l denote the minimum weight edge connecting v_l to a depot. We start by showing that if $v_i \in d(S)$, then $v_i \in d(T)$ as well. To see this, note that if $v_i \in d(S)$, then it must be that either a) $V(G_i) \cap S \neq \emptyset$, or b) that v_i is adjacent to a player vertex v_j , $V(G_j) \cap S \neq \emptyset$, and $t(e_i) + t(e_{ij}) \leq t(e_j)$, where e_{ij} denotes the edge $\{v_i, v_j\}$. However, from a), it follows that $V(G_i) \cap T \neq \emptyset$, and hence that $v_i \in d(T)$, and likewise, if b) holds, it follows that $V(G_j) \cap T \neq \emptyset$

¹In case v_j is not adjacent to any players in $G \setminus V(G_Q)$, we simply have $G_j = \{v_j\}$.

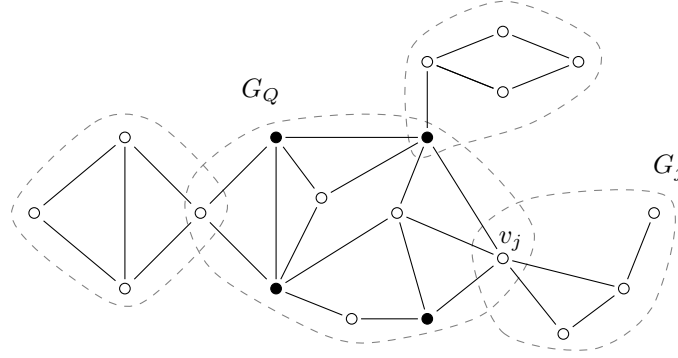


Figure 5: G locally k -TS submodular

(along with $t(e_i) + t(e_{ij}) \leq t(e_j)$), implying that $v_i \in d(T)$. Therefore, $v_i \in d(S) \Rightarrow v_i \in d(T)$. In general, we know that if $v_i \in d(T)$, then $c_Q(T \cup \{v_i\}) - c_Q(T) = 0 \leq c_Q(S \cup \{v_i\}) - c_Q(S)$, and condition (2.1) holds, so assume on the contrary that $v_i \notin d(S), v_i \notin d(T)$. First, if v_i is not adjacent to any player vertex, it follows immediately that $c_Q(T \cup \{v_i\}) - c_Q(T) = 2t(e_i) = c_Q(S \cup \{v_i\}) - c_Q(S)$. Next, assume that v_i is adjacent to player vertex v_j . Then if $v_j \in d(S)$, it follows that $v_j \in d(T)$ and therefore, $c_Q(S \cup \{v_i\}) - c_Q(S) = \min\{2t(e_i), 2t(e_{ij})\} = c_Q(T \cup \{v_i\}) - c_Q(T)$. On the other hand, if $v_j \notin d(S)$, then $c_Q(S \cup \{v_i\}) - c_Q(S) = \min\{2t(e_i), 2t(e_{ij}) + 2t(e_j)\} \geq c_Q(T \cup \{v_i\}) - c_Q(T)$. In all three cases, (2.1) holds.

Case 2. $v_i \notin V(G_Q)$. Then there exists a vertex $v_0 \in V(G_Q)$ that must be visited on any tour visiting v_i . Now, for all $S \subset T \subseteq N \setminus \{v_i\}$, let $P_S^*(v_0)$ denote a minimum cost path from v_0 to a vertex in $Q \cup d(S)$. If $v_0 \in Q \cup d(S)$, then the path is empty, and the cost is zero. Otherwise, it follows from the proof of case 1 that $\sum_{e \in P_T^*(v_0)} 2t(e) \leq \sum_{e \in P_S^*(v_0)} 2t(e)$. Next, consider $G \setminus E(G_Q)$, and let G_0 denote the connected component containing v_i and v_0 . Furthermore, let (N', c'_{v_0}) denote the 1-depot TS-game defined on G_0 in which v_0 is the single vertex associated with a depot, such that $N' = V(G_0) \setminus \{v_0\}$, and c' is the restriction of c to coalitions in N' . Then, since the cut condition holds for all $v_s, v_t \in V(G_0)$, it follows from Herer and Penn [8] and Granot and Hamers [5] that (N', c'_{v_0}) is submodular, which in turn implies that

$$\begin{aligned}
c_Q(T \cup \{v_i\}) - c_Q(T) &= c'_{v_0}(\{T \cap V(G_0)\} \cup \{v_i\}) - c'_{v_0}(\{T \cap V(G_0)\}) + \sum_{e \in P_T^*(v_0)} 2t(e) \\
&\leq c'_{v_0}(\{S \cap V(G_0)\} \cup \{v_i\}) - c'_{v_0}(\{S \cap V(G_0)\}) + \sum_{e \in P_S^*(v_0)} 2t(e) \\
&= c_Q(S \cup \{v_i\}) - c_Q(S),
\end{aligned} \tag{4.1}$$

for all $v_i \in N \setminus V(G_Q)$ and all $S \subset T \subseteq N \setminus \{v_i\}$, where the inequality follows from the submodularity of (N', c'_{v_0}) and since $\sum_{e \in P_T^*(v_0)} 2t(e) \leq \sum_{e \in P_S^*(v_0)} 2t(e)$.

For the ‘only if’ part: First, if G is not P_{m+1}^F -free with respect to Q for $m \geq 4$, it follows from the proof of Theorem 4.1 that the induced game is not submodular for every weight function. Next, let G_j be a connected component in $G \setminus E(G_Q)$, let $v_j \in V(G_j)$ be the unique vertex visited on any tour that visits G_j , and assume that there exists an $s-t$ vertex cutset of cardinality three or more for some $v_s, v_t \in G_j$. In Granot and Hamers [5], it was shown that a graph G is locally 1-TS submodular if and only if the cut condition is satisfied for all vertex pairs $v_s, v_t \in V(G)$. It therefore follows that there exists no location of a single depot v_0 in G_j , such that an induced 1-depot TS game (N', c'_{v_0}) on G_j is submodular for all weight functions. This in particular, is also the case for $v_0 = v_j$, implying that $c'_{v_j}(\{T \cap V(G_j)\} \cup \{v_i\}) - c'_{v_j}(\{T \cap V(G_j)\}) \leq c'_{v_j}(\{S \cap V(G_j)\} \cup \{v_i\}) - c'_{v_j}(\{S \cap V(G_j)\})$ does

not hold for all t . Now, if we choose $S \subset T \subseteq N \setminus \{v_i\}$ such that $S \cap V(G_j) \neq \emptyset$, then $v_j \in d(S)$ (and hence, $v_j \in d(T)$), implying that $\sum_{e \in P_S^*(v_j)} 2t(e) = \sum_{e \in P_T^*(v_j)} 2t(e) = 0$, since both paths are empty. It follows that (4.1) does not hold for all t , and (N, c_Q) is not submodular for all weight functions. Therefore, there does not exist Q with $|Q| = k$ such that every induced game on G is submodular, and it follows that G is not locally k -TS submodular. \square

5. k -TS submodular directed graphs

5.1. Globally k -TS submodular digraphs

Before we begin analyzing directed graphs, some notation is required. First, for any vertex pair $\{v_i, v_j\} \in V(G)$ such that a directed path from v_i to v_j exists, let P_{ij} denote such a path. Second, let a 1 -sum of two graphs G_1 and G_2 be the graph that is obtained by joining G_1 and G_2 by coalescing one vertex from G_1 with one vertex of G_2 . The vertex joining the two former graphs will be referred to as the *link vertex*. Figure 6 illustrates the case where G_1 and G_2 are directed cycles of length 3.

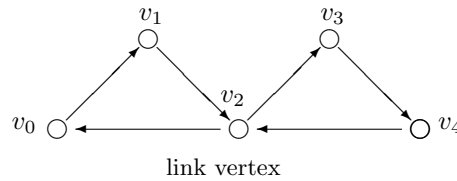


Figure 6: A 1-sum of two directed cycles

Next, note that a directed cycle C with arc set $E(C)$ is globally k -TS submodular for all $k \in \{2, \dots, |V(G)| - 3\}$, since $c(S) = \sum_{e \in E(C)} t(e)$ for all $S \subseteq N$. For digraphs in general, any induced game is submodular if $k \geq |V(G)| - 2$, since the game is then either a one or a two-player subadditive game. Granot et al. [4] consider the case of $k = 1$ and provide the equivalence theorem below. The graphs referred to as F_1 and F_2 are illustrated in Figure 7.

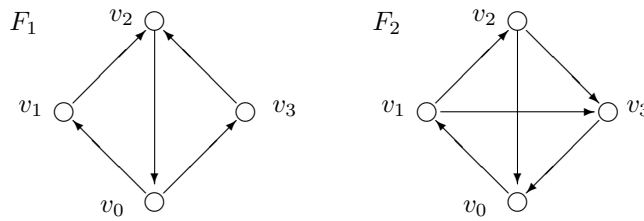


Figure 7: Forbidden subgraphs F_1 and F_2

Theorem 5.1. [Granot et al. [4]] *Let G be a strongly connected digraph. Then the following three statements are equivalent.*

- G is globally TS-submodular
- G does not contain a subdivision of F_1 or F_2

- G is a 1-sum of harmonic digraphs each of which is an outerplanar graph with a directed cycle on its outer boundary²

If a directed graph G contains a subdivision of F_1 or F_2 , then for every 1-depot TSP on G it is possible to choose a location of the single depot such that the induced game is not submodular for all weight functions t . Therefore, G is not globally 1-TS submodular. A similar result holds for the case of $k > 1$ as shown below.

Lemma 5.1. Let $G = (V(G), E(G))$ be a strongly connected, directed graph, and let $k \in \{2, \dots, |V(G)| - 3\}$. If G is globally k -TS submodular, then G does not contain (a subdivision of) F_1 or F_2 .

Proof. Let $\Gamma = (V^-(G), (G, Q), t)$ be a multi-depot TSP on G . To arrive at a contradiction, assume first that G contains (a subdivision of) F_1 . Refer to F_1 in Figure 7, and for all vertex pairs $v_i, v_j \in V(F_1)$, let $\text{cost}(P_{ij})$ denote the cost of traversing the path P_{ij} from v_i to v_j . Let t be a weight function on the arcs such that $\text{cost}(P_{ij}) = 1$ for all P_{ij} such that $(v_i, v_j) \in E(F_1)$, and let $t(e) = 100$ for all other arcs in G . Since we consider $k \leq |V(G)| - 3$, we can choose $Q \subset V(G)$ such that $v_0 \in Q$ and $v_1, v_2, v_3 \in V^-(G)$. Next, let $S = \{v_2\}$ and $T = \{v_1, v_2\}$. Then $c_Q(T \cup \{v_3\}) - c_Q(T) = 6 - 3 = 3 > 0 = 3 - 3 = c_Q(S \cup \{v_3\}) - c_Q(S)$, and the induced game is not submodular.

Assume instead that G contains (a subdivision of) F_2 . Refer to F_2 in Figure 7 and let $\text{cost}(P_{ij}) = 1$ for all $\{v_i, v_j\}$ such that $(v_i, v_j) \in E(F_2)$. Let $t(e) = 100$ for all other arcs in G . Then if $S = \{v_1\}$ and $T = \{v_1, v_2\}$, we have $c_Q(T \cup \{v_3\}) - c_Q(T) = 4 - 3 = 1 > 0 = 3 - 3 = c_Q(S \cup \{v_3\}) - c_Q(S)$. Again, the induced game is not submodular, and therefore, G is not globally k -TS submodular for any $k \in \{2, \dots, |V(G)| - 3\}$. \square

As noted above, any directed cycle C is globally k -TS submodular for all $k \in \{2, |V(C)| - 3\}$, and it follows from Theorem 5.1 that 1-sums of directed cycles are globally 1-TS submodular. However, 1-sums of directed cycles or more generally, 1-sums of directed circuits are not globally k -TS submodular for $k > 1$.³ In fact, if G contains a 1-sum of directed cycles (or circuits) of at least three vertices each, then G is not globally k -TS submodular for any $k \in \{2, \dots, |V(G)| - 3\}$.

Lemma 5.2. Let $G = (V(G), E(G))$ be a strongly connected, directed graph. Let $k \in \{2, \dots, |V(G)| - 3\}$. If G contains a 1-sum of two directed circuits C^1 and C^2 such that each circuit is a closed walk of length three or more, then G is not globally k -TS submodular.

Proof. Let $\Gamma = (V^-(G), (G, Q), t)$ be a multi-depot TSP on G , and let (N, c_Q) be the induced k -depot TS game. To arrive at a contradiction, assume that a 1-sum exists as described in the statement and note that $|V(C^1)|, |V(C^2)| \geq 3$. Figure 6 illustrates the case of G containing a 1-sum of two cycles with three vertices each, but the proof holds for cycles/circuits in general. Let v_2 denote the link vertex in this 1-sum, and let v_0, v_1 be vertices in C^1 such that paths P_{01}, P_{12}, P_{20} exist. Likewise, let v_3, v_4 be vertices in C^2 such that paths P_{23}, P_{34}, P_{42} exist. Now, since the number of depots is $2 \leq k \leq |V(G)| - 3$, we can choose $Q \subset V(G)$ such that $v_1, v_2, v_3 \in V^-(G)$ and $v_0, v_4 \in Q$, implying that there exists a depot in both C_1 and C_2 that is not the link vertex. Next, define the weight function t on G as follows: for any path $P_{ij} \in \{P_{01}, P_{12}, P_{20}, P_{23}, P_{34}, P_{42}\}$ let $\sum_{e \in E(P_{ij})} t(e) = 1$, and let $t(e) = 100$ for all other arcs in G . Then, if $S = \{v_2\}$ and $T = \{v_1, v_2\}$, we see that $c(S \cup \{v_3\}) - c(S) = 3 - 3 = 0$, while $c(T \cup \{v_3\}) - c(T) \geq 1$. Thus, the induced game is not submodular, and G is not globally k -TS submodular. \square

Let C_2 denote a directed cycle with only 2 vertices. Then from Theorem 5.1 and Lemmas 5.1 and 5.2, we can infer the following:

²In Granot et al. [4], a digraph G is said to be *harmonic* if every pair of cycles in G visit their common vertices in the same order, i.e., all pairs of cycles are *in harmony*.

³Recall that a directed circuit is a closed walk in which no edge is visited more than once, while a directed cycle denotes a closed path, implying that neither vertices nor edges are visited more than once.

Theorem 5.2. *Let $G = (V(G), E(G))$ be a strongly connected, directed graph, and let $k \in \{2, \dots, |V(G)| - 3\}$. If G is globally k -TS submodular, then G is a 1-sum of a harmonic, outerplanar graph, C , with a directed cycle on its outer boundary and (copies of) C_2 , such that no two C_2 graphs share a link vertex in $V(G) \setminus V(C)$.*

In the remainder of this section, we focus on the case where G is an oriented graph, implying that G contains no bi-directed edges, and hence no C_2 . It is shown that even for graphs fulfilling the requirements of Theorem 5.2, the answer to whether the graph is globally k -TS submodular depends on the specific structure of the graph and the number of depots.

It follows from Granot et al. [4] that if G is a harmonic and outerplanar graph with a directed cycle on its outer boundary, then this is sufficient for G to be globally 1-TS submodular. Theorem 5.3 shows that the same does not hold for the case of $k \geq 2$.

Theorem 5.3. *Let $G = (E(G), V(G))$ be a harmonic and outerplanar graph with a directed cycle on its outer boundary. Let G contain at least one directed cycle C_i such that $G \setminus V(C_i)$ is a collection of weakly connected components for which at least two distinct components contain 2 vertices or more. Let C_0 denote the C_i with the minimum number of arcs. Then G is globally k -TS submodular if and only if $k > |V(G)| - (|V(C_0)| + 2)$.*

Proof. Let $\Gamma = (V^-(G), (G, Q), t)$ be a multi depot TSP on G , and let (N, c_Q) be the induced k -depot TS game. For the if part, it needs to be shown that (2.1) holds for all $v_i \in N$ and all $S \subset T \subseteq N \setminus \{v_i\}$. Note that if $S = \emptyset$, the result follows from the subadditivity of c , so we consider $S \neq \emptyset$. Furthermore, if $v_i \in d(T)$, then $c(T \cup v_i) - c(T) = 0 \leq c(S \cup \{v_i\}) - c(S)$, and (2.1) holds. Therefore, it only remains to be shown that (2.1) holds when $v_i \notin d(T)$ and $S \neq \emptyset$.

Recall that G is an outerplanar graph with a directed cycle on its outer boundary and denote this cycle by C . Let $v_0 \in Q$ be a depot in $d(T)$ and number the vertices of C in the order they are visited in a tour starting from v_0 and ending at $v_{|V(G)|} = v_0$. Consider a player vertex $v_i \notin d(T)$, and let $h = \max\{l | l < i, v_l \in d(T)\}$. Then there exists a vertex pair $v_h, v_j \in d(T)$ such that $j > i$, the arc (v_h, v_j) - denoted e_{hj} - exists, and $e_{hj} \in d(T)$. Furthermore, since all the vertices of G lie on C , there also exists a path from v_h to v_j that visits v_i . Let P_{hij} denote the minimum cost path from v_h to v_j that visits v_i . Then since $d(T)$ is a minimum cost tour, we must have $t(e_{hj}) \leq \sum_{e \in P_{hij}} t(e)$. Next, consider $S \subset T$ such that $S \neq \emptyset$, and let $g = \max\{l | l < i, v_l \in d(S)\}$. We consider two cases separately, depending on whether v_i belongs to $d(S)$ or not.

Case 1. $v_i \notin d(S)$: there exists a vertex v_k , such that $k > i$ and such that the arc (v_g, v_k) - denoted e_{gk} - exists and belongs to $d(S)$, and such that for a minimum cost path P_{gik} from v_g to v_k that visits v_i , it holds that $t(e_{gk}) \leq \sum_{e \in P_{gik}} t(e)$. Furthermore, since G is harmonic and outerplanar with a directed cycle on its outer boundary, it follows from Theorem 5.1 that G does not contain the forbidden structure, F_2 . This in turn implies that for the vertices $\{v_h, v_j, v_g, v_k\}$, we cannot have $g < h < i < k < j$, or $h < g < i < j < k$, and therefore, it must be that either $g \leq h < i < j \leq k$, or $h \leq g < i < k \leq j$.⁴ We consider the two cases separately:

Case 1.1. $h \leq g < i < k \leq j$. Note that since only the endpoints of P_{hij} are in $d(T)$, no vertices on C that lie between v_h and v_j belong to T , implying that no vertices between v_h and v_j belong to S . Next, assume first that no depot exists on the part of $d(S)$ that lies between v_h and v_j (and goes via e_{gk}). Then we must have $t(e_{hj}) = \sum_{e \in P_{hij}} t(e) - \sum_{e \in P_{gik}} t(e) + t(e_{gk})$ for both $d(S)$ and $d(T)$ to be minimum cost tours. This implies $c(T \cup \{v_i\}) - c(T) = \sum_{e \in P_{hij}} t(e) - t(e_{hj}) = \sum_{e \in P_{gik}} t(e) - t(e_{gk}) = c(S \cup \{v_i\}) - c(S)$, and (2.1) holds.⁵ Now, let $C_{S \cup \{h, j\}}$ denote the smallest cycle that visits $S \cup \{v_h, v_j\}$. If $C_{S \cup \{h, j\}} \cap Q \neq \emptyset$, then the same reasoning as above applies. Therefore, consider the case of $C_{S \cup \{h, j\}} \cap Q = \emptyset$ and assume that there exists a depot on the part of $d(S)$ that lies between v_h and v_j . Then since G is outerplanar with a directed cycle on the outer boundary, we know that for $d(T)$ to be a minimum cost tour, there

⁴See Appendix A for illustrations of the two cases not considered.

⁵Note that in case $h = g, k = j$. We simply have $P_{hij} = P_{gik}$ and $e_{hj} = e_{gk}$.

must exist a vertex pair v_l, v_m on $C_{S \cup \{h,j\}}$ such that v_l, v_m are the endpoints of a path P_{lm} of which no other vertices belong to $S \cup \{v_h, v_j\}$, and such that at least one $v \in V(P_{lm}) \setminus \{v_l, v_m\}$ is a depot. First, in case $T \cap V(P_{lm}) \setminus \{v_l, v_m\} = \emptyset$, we must have $\sum_{e \in P_{lm}} t(e) = \sum_{e \in P_{hij}} t(e) - \sum_{e \in P_{gik}} t(e) + t(e_{gk})$, and again, the desired result follows. Secondly, if $T \cap V(P_{lm}) \setminus \{v_l, v_m\} \neq \emptyset$, this implies that $G \setminus V(C_{S \cup \{h,j\}})$ contains two weakly connected components with at least two vertices each, namely the paths P_{hij} and P_{lm} . This, however, implies that $k > |V(G)| - (|V(C_{h,j})| + 2)$, which contradicts that $(C_{S \cup \{h,j\}}) \cap Q = \emptyset$. *Case 1.2.* $g \leq h < i < j \leq k$. Now, since $g \leq h$ and $j \leq k$, it must be that $\sum_{e \in P_{hij}} t(e) \leq \sum_{e \in P_{gik}} t(e)$. Next, since v_h is the last vertex in $d(T)$ before v_i , and P_{hij} is the minimum weight path from v_h to v_j , we have $c(T \cup v_i) - c(T) = \sum_{e \in P_{hij}} t(e) - t(e_{hj}) \leq \sum_{e \in P_{gik}} t(e) - t(e_{gk}) = c(S \cup \{v_i\}) - c(S)$, where the inequality follows since $t(e_{gk}) \leq \sum_{e \in P_{gik}} t(e) - \sum_{e \in P_{hij}} t(e) + t(e_{hj})$ must hold for $d(S)$ to be a minimum cost tour. Thus, (2.1) holds.

Case 2. The case of $v_i \in d(S)$ remains to be considered. The reasoning is, however, similar to case 1 above. Like before, no vertex between v_h and v_j belongs to S , so if $V(P_{hij}) \setminus \{v_h, v_j\} \cap Q = \emptyset$ and/or $C_{S \cup \{v_h, v_j\}} \cap Q \neq \emptyset$, we must have $t(e_{hj}) = \sum_{e \in P_{hij}} t(e)$ for both $d(S)$ and $d(T)$ to be minimum cost tours. This implies $c(T \cup \{v_i\}) - c(T) = 0 = c(S \cup \{v_i\}) - c(S)$. Finally, if $C_{S \cup \{v_h, v_j\}} \cap Q = \emptyset$ and $V(P_{hij}) \setminus \{v_h, v_j\} \cap Q \neq \emptyset$, we know that for $d(T)$ to be a minimum cost tour, there must exist a vertex pair v_l, v_m on $C_{S \cup \{h,j\}}$ such that v_l, v_m are the endpoints of a path P_{lm} in $d(T)$ of which no other vertices belong to $S \cup \{v_h, v_j\}$, and such that at least one $v \in V(P_{lm}) \setminus \{v_l, v_m\}$ is a depot. Then either $P_{lm} \cap T = \emptyset$ implying that $\sum_{e \in P_{lm}} t(e) = \sum_{e \in P_{hij}} t(e)$ and in turn that $c(T \cup \{v_i\}) - c(T) = 0 = c(S \cup \{v_i\}) - c(S)$, or $P_{lm} \cap T \neq \emptyset$. However, as before, the latter implies that $G \setminus V(C_{S \cup \{h,j\}})$ contains two weakly connected components with at least 2 vertices each, contradicting that $C_{S \cup \{v_h, v_j\}} \cap Q = \emptyset$.

Next, consider the only if part and assume that $k \leq |V(G)| - (|V(C_0)| + 2)$. Recall that C_0 is the shortest directed cycle in G such that $G \setminus V(C_0)$ contains two weakly connected components with at least two vertices each, as illustrated in Figure 8. Let the two components be denoted W_1 and W_2 respectively. For each component W , let P_W denote the shortest path between two vertices of C_0 that visits at least two vertices of W . Next, let v_0 be a vertex in C_0 , let v_1, v_2 be vertices on P_{W_1} , and let v_3, v_4 be vertices on P_{W_2} .

Consider $\Gamma = (V^-(G), (G, Q), t)$, and note that for any $k \leq |V(G)| - (|V(C_0)| + 2)$, we can choose $Q \subset V(G)$ such that $V(C_0) \subset V^-(G)$, $v_0, v_2, v_3 \in V^-(G)$, and $v_1, v_4 \in Q$. That is, there are no depots located at any vertices of C_0 , and (at least) one depot is located on each of P_{W_1} and P_{W_2} . Next, define a weight function t on G as follows: for any $e \in E(C_0) \cup E(P_{W_1}) \cup E(P_{W_2})$, let $t(e) = 1$, and let $t(e)$ be arbitrarily high (e.g. $|E(G)|$) for all other arcs in G . Without loss of generality, assume that $|E(P_{W_2})| \geq |E(P_{W_1})|$.

Let $S = \{v_0\}$. Then since there are no depots in C_0 , a minimum cost tour of S visits $V(C_0) \cup V(W_1)$, and the cost of this tour is $c(S) = |E(C_0)| + |E(P_{W_1})| - 1$. It follows that $c(S \cup \{v_2\}) - c(S) = 0$. Furthermore, let $T = \{v_0, v_3\}$. Then we have $c(T) = |E(C_0)| + |E(P_{W_2})| - 1$. However, since any tour of $T \cup \{v_2\}$ must visit both $V(P_{W_1})$, $V(P_{W_2})$, and $V(C_0)$, it follows that $c(T \cup \{v_2\}) - c(T) = |E(P_{W_1})| - 1 > 0$. Thus, the induced game is not submodular, and G is therefore not globally k -TS submodular. \square

5.2. Locally k -TS submodular digraphs

Recall that for a graph G to be locally k -TS submodular, we only require that one location of k depots in G induces a submodular game for all weight functions. Whereas all globally k -TS submodular graphs are obviously also locally k -TS submodular, it is easy to show that the opposite is not true, and hence that the class of globally k -TS submodular graphs is a proper subset of the class of locally k -TS submodular graphs.

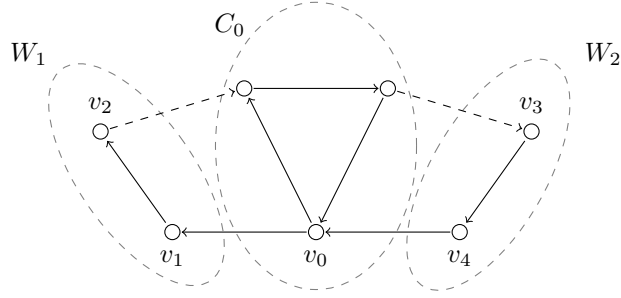


Figure 8: A globally 3-TS submodular graph that is not globally 2-TS submodular

Proposition 5.1. For $k \in \{2, |V(G)| - 3\}$, the set of globally k -TS submodular graphs is a proper subset of the set of locally k -TS submodular graphs .

Proof. We only need to show that there exist graphs that are locally k -TS submodular but not globally k -TS submodular for $k \in \{2, |V(G)| - 3\}$. To see this, consider the graph G in Figure 9. Since G is a subdivision of F_1 , it is not globally k -TS submodular, for $k \in \{1, 2, 3\}$. However, G is locally k -TS submodular for $k = \{2, 3\}$. For $k = 2$, choose $Q = \{v_0, v_1\}$. It can readily be verified that for this location of depots, the induced game is submodular for any weight function. Thus G is locally 2-TS submodular. Likewise, G is locally 3-TS submodular, since the induced game is submodular whenever Q is such that $\{v_0, v_1\} \in Q$. By further subdividing the graph in Figure 9 and increasing the number of depots correspondingly, the same argument can be applied for larger k .

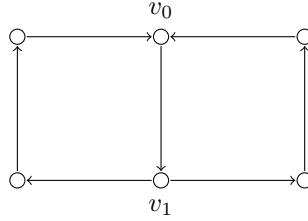


Figure 9: A locally (but not globally) 2-TS submodular graph

□

6. Concluding remarks

The current paper considers multi-depot traveling salesman games and analyzes the submodularity of such games by characterizing classes of graphs that induce submodular traveling salesman games. The main content of the results provided for undirected and directed graphs, respectively, is summarized in Table 1.

The traveling salesman problem has been extended and varied in many directions, and for many applications the standard model is not sufficient. Therefore, one aim for future research on cost allocations problems arising from underlying traveling salesman problems is to allow for different (sets of) variations and/or restrictions in the underlying optimization problem, thereby making the model more easily applicable to real world situations. Examples include imposing capacity constraints or time windows for delivery.

	Globally k -TS submodular	Locally k -TS submodular
Undirected graph	If and only if G contains no path of length four	If and only if a) there exists a location of k depots such that a forbidden structure is avoided, and b) the cut condition holds for specific connected components
Directed graph	Only if: a) G contains no forbidden subgraphs F_1 and F_2 , and b) G contains no 1-sum of directed circuits each of length three or more	The set of globally k -TS submodular digraphs is a proper subset of the locally k -TS submodular directed graphs

Table 1: Summary of results

Aknowledgements

Financial support from the Danish Council for Independent Research | Social Sciences (Grant IDs: DFF-1327-00097 and DFF – 6109-00132) is greatly acknowledged.

Appendix A. Appendix

To see that the remaining cases of $g < h < i < k < j$ and $h < g < i < j < k$ from the proof of Theorem 5.3 both imply that G contains F_2 , consider the illustrations in Figure A.1.

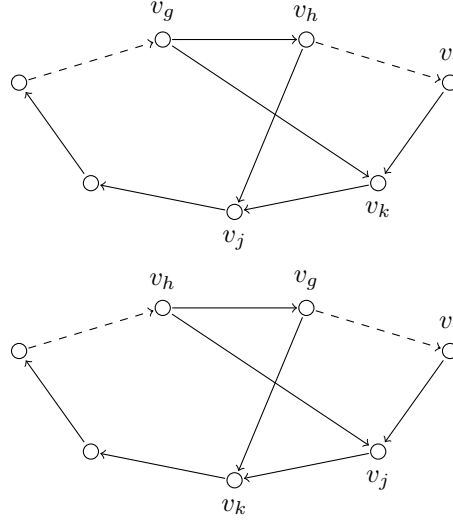


Figure A.1: G with $g < h < i < k < j$ and $h < g < i < j < k$, respectively

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