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Are Popular Recommendations Consistent with Rational Behavior?

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WP 2001-6

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ISBN 87-90705-54-8
ISSN 0903-0352

Portfolio Choice under Inflation: Are Popular Recommendations Consistent with Rational Behavior?*

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September 2001

* We thank participants at the Danske Bank Symposium at Stavrby Skov, the International Workshop on Stock Markets at Copenhagen Business School, the Mathematical Finance Network Meeting at Sørup Herregård, and at a seminar at Nordea Investment Management for comments and suggestions. Claus Munk and Carsten Sørensen gratefully acknowledge financial support from the Danish Research Council for Social Sciences.

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Abstract

We consider the optimal asset allocation choice of an investor who can invest in cash (a money market bank account), nominal bonds, and stocks (the stock index). The investor faces an incomplete market setting and is not able to perfectly hedge long run real interest rate risk using the available securities. The optimal investment strategy is consistent with the following features of popular investment advice which have been pointed out as puzzles: (i) a decreasing fraction of stocks in the portfolio as time passes towards the investment horizon, and (ii) a higher bond to stock ratio for more conservative (less risk tolerant) investors (Canner, Mankiw and Weil, 1997). The model for asset price dynamics is calibrated to US market data and, furthermore, risk aversion parameters and time horizons are calibrated so as to obtain a match between the optimal asset allocations and observed investment recommendations for "aggressive," "moderate," and "conservative" investor groups with different investment horizons.

1 Introduction

Investment advisors tend to recommend that younger investors – who have a long investment horizon – should invest a higher fraction of wealth in stocks than should older investors. As Samuelson (1963) originally demonstrated, this investment advice is not consistent with rational portfolio allocation in basic models of portfolio choice, and the phenomenon is thus often referred to as the Samuelson puzzle. A large literature has addressed this discrepancy and provided potential rational explanations; see e.g. the discussion in Samuelson (1989, 1994). More recently, Canner, Mankiw and Weil (1997) have pointed out another common feature of investment advice that is at odds with standard mean-variance two-fund separation results that prescribe all investors to invest in cash (short term risk-free money account) and a single tangency portfolio. The ratio of bonds to stocks should thus be the same for all investors and equal to the bonds/stocks ratio in the tangency portfolio, according to these two-fund separation results. However, the recommended asset allocations by professional investment advisors seem to systematically display a higher bonds/stocks ratio for less risk tolerant investors.

In this paper we use the general continuous-time modeling framework of Merton (1969, 1971, 1973) in analyzing rational asset allocation in a dynamic setting. We propose a specific model with inflation uncertainty that qualitatively and for some parameter values is able to resolve simultaneously both of the above mentioned puzzles. In order to analyze whether the optimal asset allocation behavior from the model is also quantitatively consistent with popular investment advice, we perform a calibration exercise. The parameters of stock, bonds, and inflation dynamics are estimated using US data from the period 1951-2001. Furthermore, we take a specific asset allocation advice as the "data" to be explained. The investment advice, which exhibits both of the above discussed features, is constructed based on the recommendations of Quinn (1991); one of the four sources used by Canner, Mankiw and Weil (1997). In particular, we calibrate the representative investors in the specific investment allocation advice by varying three relative risk aversion parameters and three investment horizons that are supposed to reflect the attitudes towards risk and investment horizons of "conservative," "moderate," and "aggressive" investors and investors with "short," "medium," and "long" investment horizons, respectively. When model parameters are allowed to vary in reasonable ranges, we demonstrate that the model is able to fit well the asset allocation advice used as " data" and resolve simultaneously the Samuelson puzzle as well as the asset allocation puzzle pointed out by Canner, Mankiw and Weil (1997).

In the formal modeling framework we consider the portfolio choice of a constant relative risk aversion (CRRA) investor who maximizes expected utility of wealth at a given investment horizon. The investor can continuously invest in cash, a nominal bond, and a stock (the stock index). The excess return on the stock is assumed constant while interest rates are stochastic. The model of the dynamics of the term structure of *nominal* interest rates is adopted from the no-arbitrage model of Vasicek (1977). Furthermore, there is inflation uncertainty in the economy but the investor does not have access to real bonds.

In a setting without inflation uncertainty, Brennan and Xia (2000), Omberg (1999), and Sørensen (1999) consider a similar problem and demonstrate how the optimal portfolio policy, in accordance with the general characteristics in Merton (1971,1973), can be decomposed into a "speculative term" and a term that prescribes how the investor should hedge changes in the opportunity set. They show that the relevant instrument for hedging changes in interest rates is the zero-coupon bond expiring at the investment horizon (or a portfolio replicating this zero-coupon bond). This result would also apply in our setting if markets were not incomplete and the appropriate real bond was available for investments. We demonstrate that instead the stock enters the hedge portfolio since the stock may be used as a non-perfect substitute for real bonds for hedging long term real interest rate risk in cases where the stock is negatively correlated with the real interest rate.

The model in this paper is close to the models applied in Brennan and Xia (2001) and Campbell and Viceira (2001). In both these papers the real interest rate is described by a Vasicek-model and the expected inflation dynamics is given by an Ornstein-Uhlenbeck process. The term structure of nominal interest rates is therefore described by a two-factor model. While Brennan and Xia (2001) assume CRRA utility preferences similar to those in this paper, Campbell and Viceira (2001) allow for intermediate consumption and a more general recursive utility specification in an infinite horizon setting. The infinite horizon assumption in Campbell and Viceira (2001), however, makes it difficult to address effects due to investors having different investment horizons and is thus not suitable for our calibration purposes. Also, while the model in this paper is conceptually very similar to the model in Brennan and Xia (2001), there are important differences. For example, in our setting the model of the term structure of nominal interest rates is a one-factor model while the implied term structure of real interest rates is described by a two-factor model; in Brennan and Xia (2001) it is the other way round. In Brennan and Xia (2001) it would thus be possible to implement the complete market solution by essentially mimicking the relevant real bond by investing in two different nominal bonds. On the other hand, in our model it is never possible to replicate a real bond by trading in any number of nominal bonds. Also, Brennan and Xia (2001) use a version of the martingale solution technique of Cox and Huang (1989,1991) which as a starting point requires specification of the relevant pricing kernel in the economy. We take a more direct dynamic programming approach to the incomplete market asset allocation problem.

The calibration of the model is done in two steps. In the first step we estimate parameters of stock returns, bond returns, and inflation parameters using historical US data from the period 1951–2001. The estimation is based on maximum likelihood and an application of the Kalman filter. In the second step, we use the obtained capital market parameters from the first step and calibrate the relevant preference parameters by minimizing the distance (sum of squared deviations) between the theoretical and observed asset allocation advice across investors with different risk attitudes and investment horizons. At the point estimates of capital market parameters obtained in the first step, we are not able to fit well the observed asset allocation advice since stocks do not correlate with the relevant real bond that should ideally be used as the instrument for hedging long term inflation risk and real interest rate risk. In this case the Samuelson puzzle applies. However, when we allow the capital market parameters to vary within intervals of plus-minus two standard deviations on the estimates (which could reflect reasonable uncertainty on the parameter estimates), we are able to obtain theoretical asset allocation results that closely mimic the observed asset allocation advice that is used as our data. In particular, in this case the model is able to provide simultaneously a resolution of the Samuelson puzzle as well as the asset allocation puzzle pointed out by Canner, Mankiw and Weil (1997).

The paper is organized as follows. In section 2 we present the formal model and the solution to the intertemporal portfolio problem. In section 3 we estimate capital market parameters and, subsequently, calibrate the model to observed asset allocation advice. Section 4 concludes.

2 The problem and optimal asset allocation

We consider the investment problem of an investor who has access to the capital market and wants to transfer current wealth W_0 into wealth at a specific investment horizon. We consider the basic asset allocation problem of how much to invest in cash (i.e. a money market bank account), nominal bonds, and stocks.

2.1 Preferences

The investor wants to choose a dynamic portfolio strategy in order to maximize the expected utility of wealth at a specific horizon T. The utility function displays constant relative risk aversion and the investor's portfolio policy is thus set so as to

$$\max \mathrm{E}\left[U(W_T)\right] \tag{1}$$

where

$$U(W) = \frac{W^{1-\gamma} - 1}{1-\gamma}$$

and $\gamma > 0$ is the parameter of constant relative risk aversion. For $\gamma = 1$, we have, as a limiting and special case, $U(W) = \log W$.

2.2 Investment asset dynamics

The stock index (with dividends reinvested) is assumed to evolve according to the stochastic differential equation

$$\frac{dS_t}{S_t} = (r_t + \lambda_S)dt + \sigma_S dW_{1t} \tag{2}$$

where r_t denotes the short nominal interest rate, the parameter λ_S is the expected excess return from investing in stocks, and σ_S is the stock index volatility. W_1 is a Wiener process. The (nominal) interest rate dynamics are described by an Ornstein-Uhlenbeck process,

$$dr_t = \kappa(\theta - r_t)dt - \sigma_r dW_{2t} \tag{3}$$

where the parameter θ describes the long-run mean of the interest rate, κ describes the degree of mean-reversion, and σ_r is the interest rate volatility. We will assume that the two basic Wiener processes W_1 and W_2 that generate the dynamics of the stock index and the interest rate are correlated with a constant correlation coefficient ρ_{12}. The instantaneous covariance rate between the stock return and the short interest rate is denoted σ_{Sr} $(= -\rho_{12}\sigma_S\sigma_r)$. Likewise, the instantaneous correlation rate between the stock return and the short interest rate is denoted ρ_{Sr} $(= -\rho_{12})$.[1]

The term structure of interest rates has the same form as in Vasicek (1977). In particular, the price of a zero-coupon bond with time to maturity τ is given by

$$P(r, t; \tau) = e^{-a(\tau)-b(\tau)r} \tag{4}$$

where

$$\begin{aligned} a(\tau) &= R(\infty)(\tau - b(\tau)) + \tfrac{\sigma_r^2}{4\kappa}(b(\tau))^2 \\ b(\tau) &= \tfrac{1}{\kappa}(1 - e^{-\kappa\tau}) \end{aligned}$$

and where $R(\infty) = \theta + \frac{\lambda_r}{\kappa} - \frac{1}{2}\frac{\sigma_r^2}{\kappa^2}$ describes the yield to maturity for a very long bond (as τ approaches infinity). The constant parameter λ_r describes the risk premium on interest rate risk.

[1] Throughout we will denote instantaneous covariance rates and correlation rates by σ_{ij} and ρ_{ij} where i and j indicate the relevant processes.

Any bond is essentially an interest rate contingent claim and, by Ito's lemma, the dynamics of the bond price B_t can be described by a stochastic differential equation on the form

$$\frac{dB_t}{B_t} = (r_t + \lambda_B)dt + \sigma_B dW_{2t} \tag{5}$$

with $\lambda_B = \lambda_r D(r,t)$, $\sigma_B = \sigma_r D(r,t)$, and where $D = -\frac{\partial B}{\partial r}\frac{1}{B}$ is the elasticity of the bond price with respect to the short interest rate; this elasticity is usually referred to as the *duration* of the interest rate contingent claim. We will assume that the bond available for the investor has a constant duration D. This can be thought of as reflecting the duration of the aggregate portfolio of bonds outstanding, or a bond index, where bonds that expire are always substituted with new longer term bonds. Also, note that the short interest rate and the return on the bond are perfectly negatively correlated and with covariance rate $\sigma_{Br} = -D\sigma_r^2 = -(1/D)\sigma_B^2$.

To sum up, the investor can invest in three securities: the bank account (cash), the stock index in (2), and the constant duration bond in (5). And, for future reference, the variance-covariance matrix and excess returns on the two risky securities can be summarized in matrix form by

$$\Sigma = \begin{pmatrix} \sigma_S^2 & \sigma_{SB} \\ \sigma_{SB} & \sigma_B^2 \end{pmatrix} \quad \text{and} \quad \lambda = \begin{pmatrix} \lambda_S \\ \lambda_B \end{pmatrix}$$

where $\sigma_{SB} = -D\sigma_{Sr}$.

2.3 Inflation uncertainty

The nominal price of the real consumption good in the economy at time t is denoted by Ψ_t. The real price of any asset in the economy is thus determined by deflating by the price index Ψ_t. The real price of the stock is, for example, given by S_t/Ψ_t.

The dynamics of the nominal price of the consumption good are given by the following system of differential equations:

$$\frac{d\Psi_t}{\Psi_t} = \pi_t\,dt + \sigma_\Psi\,dW_{3t} \tag{6}$$

and

$$d\pi_t = \beta(\alpha - \pi_t)\,dt + \sigma_\pi\,dW_{4t} \tag{7}$$

where π_t is the expected rate of inflation, α describes the long-run mean of the rate of inflation, β describes the degree of mean-reversion, σ_π is the volatility of the inflation rate, while σ_Ψ is the volatility of the price index and thus σ_Ψ determines the magnitude of the unexpected short-run inflation in the economy. Changes in the nominal price index and the inflation rate are correlated with the stock index return and interest rates. For example, the covariance rate between the return on the stock index and the price level is denoted $\sigma_{S\Psi}$ $(= \rho_{13}\sigma_S\sigma_\Psi)$, the covariance rate between the return on the stock index and the inflation rate is denoted $\sigma_{S\pi}$ $(= \rho_{14}\sigma_S\sigma_\pi)$, and so on.

2.4 Optimal asset allocation

We will assume that nominal stock returns are generated by the model of the stock index in (2), and that the nominal bond returns are generated by the nominal interest rate dynamics of the Vasicek type in (3) and the nominal bond index dynamics in (5). The real wealth of the investor at time t is denoted by W_t, and the Bellman equation associated with the problem of maximizing the expected utility in (1) has the form

$$\sup_{w=(w_S,w_B)\in\mathbb{R}^2} \Big\{\mu_W W J_W + \kappa(\theta - r)J_r + \beta(\alpha-\pi)J_\pi + \tfrac{1}{2}\sigma_W^2 W^2 J_{WW} + \tfrac{1}{2}\sigma_r^2 J_{rr} + \tfrac{1}{2}\sigma_\pi^2 J_{\pi\pi} + \sigma_{Wr} W J_{Wr} + \sigma_{W\pi} W J_{W\pi} + \sigma_{r\pi} J_{r\pi} + J_t\Big\} = 0 \tag{8}$$

where

$$\mu_W = r + w'\lambda - \pi_t + \sigma_\Psi^2 - w'\begin{pmatrix}\sigma_{S\Psi}\\ \sigma_{B\Psi}\end{pmatrix}, \quad \sigma_W^2 = w'\Sigma w + \sigma_\Psi^2 - 2w'\begin{pmatrix}\sigma_{S\Psi}\\ \sigma_{B\Psi}\end{pmatrix},$$

$$\sigma_{Wr} = w'\begin{pmatrix}\sigma_{Sr}\\ \sigma_{Br}\end{pmatrix} - \sigma_{\Psi r}, \quad \sigma_{W\pi} = w'\begin{pmatrix}\sigma_{S\pi}\\ \sigma_{B\pi}\end{pmatrix} - \sigma_{\Psi\pi}$$

and where $J(W, r, \pi, t)$ is the indirect utility function which must satisfy the boundary condition $J(W, r, \pi, T) = U(W)$. The first order condition of the problem in (8) provides the following characterization of the optimal risky asset proportions w:

$$w = \frac{-J_W}{WJ_{WW}}\Sigma^{-1}\lambda - \frac{J_{Wr}}{WJ_{WW}}\Sigma^{-1}\begin{pmatrix}\sigma_{Sr}\\ \sigma_{Br}\end{pmatrix} + \left(1 + \frac{J_W}{WJ_{WW}}\right)\Sigma^{-1}\begin{pmatrix}\sigma_{S\Psi}\\ \sigma_{B\Psi}\end{pmatrix} - \frac{J_{W\pi}}{WJ_{WW}}\Sigma^{-1}\begin{pmatrix}\sigma_{S\pi}\\ \sigma_{B\pi}\end{pmatrix}. \tag{9}$$

The expression in (9) provides a general characterization of the optimal portfolio weights in the specific market setting. The first term in (9) is the usual speculative portfolio which is optimal for an investor with short horizon or log utility. The other three terms in (9) describe how the investor optimally hedges changes in the opportunity set. The second term describes the hedge against the nominal interest rate and by using that

$$\begin{pmatrix}\sigma_{Sr}\\ \sigma_{Br}\end{pmatrix} = -\frac{1}{D}\begin{pmatrix}\sigma_{SB}\\ \sigma_B^2\end{pmatrix} = -\frac{1}{D}\Sigma\begin{pmatrix}0\\ 1\end{pmatrix},$$

it is seen that this term can be rewritten in the form

$$-\frac{J_{Wr}}{WJ_{WW}}\Sigma^{-1}\begin{pmatrix}\sigma_{Sr}\\ \sigma_{Br}\end{pmatrix} = \frac{J_{Wr}}{WJ_{WW}}\left(\frac{1}{D}\right)\begin{pmatrix}0\\ 1\end{pmatrix}.$$

Hence, the optimal hedge against changes in the interest rate is obtained by investing entirely in the bond; this is similar to the optimal strategy in the complete market Vasicek settings of Brennan and Xia (2000) and Sørensen (1999). The last two terms in (9) describe how the investor hedges against short-run unexpected inflation and changes in future inflation rates, respectively.

The optimal asset allocation of the CRRA investor is stated in the following proposition which can be verified by inspection and substitution in Equations (8) and (9).

Proposition 1 *(i) The indirect utility of wealth function which solves (8) is given by*[2]

$$J(W, r, \pi, t) = \frac{(We^{\alpha(T-t)+b(T-t)r-c(T-t)\pi})^{1-\gamma} - 1}{1-\gamma} \tag{10}$$

where $\alpha(\tau)$ is the solution to an ordinary differential equation with boundary $\alpha(0) = 0$, $b(\tau)$ is defined in (4), and $c(\tau) = \frac{1}{\beta}\left(1 - e^{-\beta\tau}\right)$.

(ii) The vector of optimal risky asset allocations at time t is given by:

$$w = \left(\frac{1}{\gamma}\right)\Sigma^{-1}\lambda + \left(1 - \frac{1}{\gamma}\right)\left(\frac{b(T-t)}{D}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \left(1 - \frac{1}{\gamma}\right)\Sigma^{-1}\left(c(T-t)\begin{pmatrix} \sigma_{S\pi} \\ \sigma_{B\pi} \end{pmatrix} + \begin{pmatrix} \sigma_{S\Psi} \\ \sigma_{B\Psi} \end{pmatrix}\right) \tag{11}$$

The residual $w_r = 1 - 1'w = 1 - w_S - w_B$ is invested in the bank account.

As formalized in Proposition 1, the optimal portfolio weights for CRRA investors are linear combinations of the speculative portfolio and the different hedge portfolios. In particular, for investors with the same investment horizon T the optimal portfolios are linear combinations of the speculative portfolio and a single hedge portfolio; the relative risk tolerance, $1/\gamma$, describes the weights on the two relevant portfolios.

As discussed above, the second term in (11) describes the hedge against changes in the nominal interest rate and consists entirely of a position in the bond. The occurrence of this hedge term can explain the bonds/stocks puzzle pointed out by Canner, Mankiw and Weil (1997); see e.g. Brennan and Xia (2000). On the other hand, the last hedge term in (11) describes the inflation hedge and involves the stock. The occurrence of this term may explain the Samuelson puzzle since there is a horizon effect reflected by the function $c(T-t)$. In particular, the parameter β determines the difference on the stock allocations for myopic and long term investors with the same relative risk aversion. If β is small, changes in the expected inflation rate are relatively permanent, and horizon effects may be significant. However, whether this horizon effect implies more or fewer stocks for the long-term investor depends on the precise correlation

[2]The expression is formally only defined for $\gamma \neq 1$. The case $\gamma = 1$ is described by the limiting result as γ approach 1.

structure and, intuitively, whether the stock serves as a relatively good substitute for the real bond that should ideally be used for hedging changes in real rates in a complete market setting. Moreover, while the last term in (11) can potentially explain horizon effects it may also in part explain a varying ratio of bonds to stocks. In a standard one-period mean-variance model, Elton and Gruber (2000) thus demonstrate that inflation uncertainty alone can explain the varying bonds/stocks ratio puzzle of Canner, Mankiw and Weil (1997) when investors focus on real returns and only nominal assets are available (and short sale constraints are imposed).

3 Calibration

In the following two subsections we will first calibrate the asset price and inflation parameters of the capital market model. Subsequently, we will use these parameters in a calibration exercise where the subjective risk aversion parameter and time horizon parameter are fitted to match observed asset allocation advice for different investor groups.

3.1 Calibration of asset price and inflation parameters

A Kalman filtering approach is adopted in order to estimate the parameters involved in the asset price and inflation dynamics in (2), (3), (4), (6), and (7). The specific approach is based on expressing the model in state space form and then using the Kalman filter to obtain the relevant log-likelihood function to be maximized; see e.g. Harvey (1989, chapter 3). The estimation approach is basically similar to the approach used by Pennacchi (1991) for term structure estimation and adopted by Campbell and Viceira (2001) and Brennan and Xia (2001) in a context similar to this paper.

The transition equation of the four state-variables $s_t = \log S_t$, r_t, $\psi_t = \log \Psi_t$, and π_t is described by the discrete time solution to the system of stochastic differential equations in (2), (3), (6), and (7). The relevant discrete-time moments of the Gaussian processes are obtained by a straightforward application of the results for linear stochastic differential equations in, e.g., Karatzas and Shreve (1991, pp. 354-358).

The measurement equation relates the state-variables to the empirical observations that consist in part of yields to maturity on n $(= 5)$ zero-coupon bonds with times to maturity τ_j, $j = 1, \ldots, n$ which are related to the state-variables by

$$y(t; \tau_j) \equiv -\frac{P(r_t, t; \tau_j)}{\tau_j} = \frac{a(\tau_j)}{\tau_j} + \frac{b(\tau_j)}{\tau_j} r_t + \tilde{\epsilon}_{t,j} \tag{12}$$

where $\tilde{\epsilon}_{t,j}$ is the measurement "noise" on the τ_j-maturity yield. The measurement noise terms on the zero-coupon yields are assumed uncorrelated and normally distributed with mean zero and standard deviation $\sigma_{\epsilon,j}$. In addition, the measurement equation consists of observing the (log) stock index value (i.e. s_t) and the (log) price index value (i.e. ψ_t) without measurement noise at the different sample dates.

In our implementation the shortest zero-coupon yield (0.25 years to maturity) is observed with limited measurement noise in order to obtain a close fit to the short end of the term structure which is closely determined by the short rate r_t which is one of the basic and important state-variables in our asset allocation model. In particular, the standard deviation on this observation is fixed at a value close to zero (0.0001).

The system is estimated using monthly US data which spans the half-century period from March 1951 until March 2001. Data on five constant maturity yields are used; the times to maturities are 0.25, 1, 2, 5, and 10 (years). The zero-coupon bond yields for the period March 1951 until February 1991 are adopted from McCulloch (1990) and McCulloch and Kwon (1993). The yields for the period March 1991 until March 2001 are based on constant maturity yields taken from the Federal reserve H.15 Statistical Releases; the specific constant maturity zero-coupon yields are constructed using bootstrapping (combined with linear interpolation when necessary to take account of missing maturities). The cum dividend stock returns and CPI-index data are identical to the S&P 500 data from Shiller (2000); the updated data were downloaded from Robert Shiller's homepage.

Maximum likelihood parameter estimates are given in Table 1.

[INSERT TABLE 1 ABOUT HERE]

Note that the zero-coupon yields are recorded at the end of the month whereas the Shiller (2000) stock returns are based on the average stock index value over each month. Besides the timing mismatch, the volatility on the stock index is potentially underestimated. Therefore, in the following exercise where the model is calibrated to asset allocation recommendations, the estimate (and standard error) on σ_S is increased by a factor 1.675 to correct for this feature.[3] The adjusted estimate on the stock index volatility is thus 0.1960 (= 0.1170 × 1.675) and with a standard error on the estimate of 0.0080 (= 0.0048 × 1.675).

The expected excess return on the stock index is estimated to be 5.12% while the expected interest rate risk premium implies that the expected excess return on, for example, a zero-

[3] If the stock index is described by a geometric Wiener process, as in the Black-Scholes framework, it can be shown that the monthly observed index value is higher than the (geometric) mean process with a factor

$$\sqrt{\frac{6n^2}{(n+1)(2n+1)}}$$

where n is the number of observation dates within the month. Substituting $n = 22$ trading days per month into this formula, one obtains the factor 1.675. Details can be obtained from the authors on request.

coupon bond with five years to maturity is 0.97%.[4] Furthermore, the long-run mean of the nominal interest rate is estimated to be 4.39% while the long-run expected inflation rate is estimated to be 3.57%.

In general, the estimates in Table 1 are compatible with similar estimates reported in e.g. Campbell and Viceira (2001) and Brennan and Xia (2001). For example, also the estimate of the inflation mean-reversion parameter β is in between the values considered in Brennan and Xia (2001) and which they induce from estimations with respectively monthly and annual data. In our case, the point estimate of β= 0.2936 implies that the half life of shocks to the inflation rate is 2.36 years. Also, the standard deviations on the noise terms imposed on the bond yields in the measurement equation are of the same magnitude as in Brennan and Xia (2001). However, the standard deviations on the noise term on the long zero-coupon yields are small, and the standard deviations on the noise term on short yields are relatively large in Brennan and Xia (2001) whereas the opposite is the case in our implementation (due to having fixed a small standard deviation on the 3 month zero-coupon yield).

The estimates on correlations imply that the stock index is slightly negatively correlated with the nominal interest rate ($\rho_{Sr} = -0.0029$) but slightly positively correlated with the real interest rate, $\rho_{S,(r-\pi)} = 0.0790$.[5] Since the stock is not negatively correlated with the real interest rate, it cannot serve as a natural substitute for a long term real bond in hedging long term real interest rate risk under inflation, as will be seen in the results to follow.

3.2 Calibration to asset allocation recommendations

The asset allocation recommendations that we take as our "data" and will try to match are given in Table 2.

[INSERT TABLE 2 ABOUT HERE]

These recommendations are generated from the advice in Quinn (1991). Quinn (1991, p. 489) provides three classic portfolios and rank them by their market risk: low risk, medium risk, and high risk. These portfolios are tabulated in Canner, Mankiw and Weil (1997) as Quinn's recommendation for "conservative," "moderate," and "aggressive" investors. Though, Quinn (1991, p. 489) is a bit more detailed and recommends to "Take these classics as baseline...Tip toward

[4] The interest elasticity of this bond is $D = b(5) = 4.585$ and its volatility is $\sigma_B = 4.585 \times \sigma_r = 0.0912$. The expected excess return is calculated as $4.585 \times \lambda_r$.

[5] Following e.g. Breeden (1986), it can be shown that in our continuous time setting the real interest rate is determined by: (r_t - π_t) plus a constant term. This fact is used in a straightforward way when calculating the specific correlations with the real interest rate as stated in the text.

higher risks...if you are young, or won't need the money for many years." Our construction in Table 2 is intended to quantitatively represent the qualitative guidance of Quinn (1991).

The portfolios in the diagonal are the so-called classic portfolios which we in our context have interpreted as being the portfolio recommendations for a short investment horizon "conservative" investor, a medium investment horizon "moderate" investor, and a long investment horizon "aggressive" investor. The off diagonal portfolio recommendations are constructed as weighted averages of the two diagonal (or classic) portfolio recommendations with the same attitude towards risk and the same investment horizon, respectively. The concrete recommendations are obtained by using a weight of 0.75 on the classic portfolio with the same attitude towards risk and a weight of 0.25 on the classic portfolio with the same investment horizon. For example, the stock allocation for a "conservative" long investment horizon investor is obtained as $0.75 \times 20\% + 0.25 \times 100\% = 40\%$.

The applied weighting scheme gives more weight to the risk attitude than the investment horizon, which seems appropriate in order to capture the qualitative statement of Quinn (1991) that investors with longer investment horizons should "tip" towards higher risks. However, we have tried other weighting schemes without affecting dramatically the calibration result reported in the following.

The recommendations in Table 2 are in accordance with the popular advice that investors with a long horizon should invest a higher fraction of wealth in stocks. Also, the investment recommendations are in accordance with the pattern pointed out by Canner, Mankiw and Weil (1997) and, in fact, for any investment horizon the bond to stock ratio is increasing with risk aversion.

We will calibrate parameters so as to minimize the sum of squared deviations between the asset allocation recommendations in Table 2 and the optimal asset allocations in the economic modeling framework in section 2. The summation of squared deviations that will be minimized is taken over all portfolio weights for the three horizons (short, medium, long), the three degrees of risk aversion (conservative, moderate, aggressive), as well as the allocations into stocks, bonds, and cash. This makes a total of 27 ($= 3 \times 3 \times 3$) squared deviations in the summation.

In calibrating the model, we vary three risk aversion parameters: $\gamma_{con} > \gamma_{mod} > \gamma_{agg} > 0$. Likewise, we vary three investment horizon parameters: $0 < H_{short} < H_{med} < H_{long} < 40$ (years). These parameters are meant to represent the relative risk aversion parameters of "conservative," "moderate," and "aggressive" investors as well as the investment horizon of investors with short, medium, and long horizons, respectively.

Furthermore, we allow investors with different investment horizons to use bonds that differ in duration. The individual investor can thus invest in cash, stocks and a bond with a duration that depends on the investment horizon. Without loss of generality the bond can be thought of as a zero-coupon bond and when we refer to the duration of the bond in the following, we are in fact referring to the time to maturity on the relevant zero-coupon bond. This duration concept is known as the *stochastic* duration; see e.g. Ingersoll, Skeldon and Weil (1978) and Cox, Ingersoll and Ross (1979). We calibrate the (stochastic) durations as part of the problem and we impose the intuitive restriction that investors with longer investment horizons should not use shorter duration bonds and the restriction that the duration on the bond is between 5 years and 15 years so that it could represent a realistic aggregate bond index; i.e. in the calibration we have the restriction: $5 \leq D_{short} \leq D_{med} \leq D_{long} \leq 15$.

The calibration is done in two versions. In the first version we only vary the risk attitude parameters, investment horizons, and relevant durations subject to the above restrictions. The point estimates of the asset price and inflation dynamics in Table 1 are applied in generating the optimal theoretical asset allocations as stated in Proposition 1. In the second version we also allow the estimated parameters in Table 1 to vary within a range that reflects the precision of the estimate. Formally, we allow each parameter to vary around the point estimates within intervals of plus-minus two standard deviations on the estimates which represent individual 95% confidence intervals for the parameters.

The calibration results in Table 3 are obtained using the point estimates of the asset price and inflation dynamics in Table 1.

[INSERT TABLE 3 ABOUT HERE]

It can be observed that the calibrated model asset allocations in Panel A of Table 1 do not conform to the advice that longer term investors should invest a higher fraction of wealth in stocks; i.e. the Samuelson puzzle applies in this case. In fact, the model prescribes a slightly lower fraction of wealth invested in stocks for longer term investors. This is similar to the results reported in Brennan and Xia (2001). On the other hand, the calibrated asset allocations are in accordance with the pattern observed by Canner, Mankiw and Weil (1997) that the bond to stock ratio is increasing with risk aversion.

The risk aversion parameters in Table 3 are so that "aggressive" investors have risk tolerance of 0.55 (= 1/1.815). Hence, for a given investment horizon these investors allocate 55% of wealth to the speculative portfolio and 45% of wealth to a hedge portfolio, according to Proposition 1. On the other hand, the "conservative" investors have a risk tolerance of 0.27 (= 1/3.771) and for a given investment horizon these investors thus allocate 27% of wealth to the speculative portfolio and 73% of wealth to a hedge portfolio. In our view, it seems reasonable that an "aggressive" investor uses at least 55% of wealth for speculative purposes while it also makes sense that a "conservative" investor is so cautious that 73% is invested with an eye on reducing intertemporal risks.

The relative short investment horizons in Table 3 are a consequence of the fact that the optimal asset allocations in the formal model dictates a smaller fraction invested in stocks for long horizon investors, contrary to the observed recommendations in Table 2. Moreover, note that the recommendations for medium horizon and long horizon investors are identical since their calibrated investment horizons are equal. This phenomenon is basically due to the above observation that the stock is a poor substitute for real bonds in hedging long term real interest rate risk under inflation since the stock is positively correlated with the real interest rate.

The calibration results in Table 4 are obtained by allowing the preference parameters to be varied, but also by allowing the capital market parameters in Table 1 to vary within intervals around the point estimates. The intervals are given as plus-minus two standard deviations around the point estimates.

[INSERT TABLE 4 ABOUT HERE]

Not all parameters estimated in Table 1 matter for the optimal asset allocation choice. This is the case for the long-run means of interest rates and inflation, θ and α, as well as the covariance parameter $\sigma_{\Psi\pi}$; these are thus not used in the calibration and, hence, their calibrated values are not reported in Table 4.

Comparing the calibrated asset allocations in Table 4 to the target asset allocation recommendations in Table 2, it is seen that the model is able to match the "data" very well. Only in a few cases do the model portfolios deviate by more than 10% from the targeted portfolio advice. And, in no case do the stock portfolio weights deviate by 10%. In fact, the calibrated asset allocations in Table 4 are well in accordance with the recommendation that long horizon investors should invest a higher fraction of wealth in stocks. Simultaneously, the model asset allocation in Table 4 also conforms to the pattern pointed out by Canner, Mankiw and Weil (1997). For any investment horizon the ratio of bonds to stocks is thus decreasing with risk tolerance.

The individual parameters calibrated in Table 4 are restricted, as described above. We have indicated whether the individual parameters take a value that hits the restriction boundary and whether the parameters hit a lower or upper boundary.

The representative investment horizons calibrated in Table 4 are reasonable in our view. Specifically, a short investment horizon investor has an investment horizon of 5.487 years while a long term investor acts so as to maximize utility of wealth at a forty-year horizon. On the other hand, the calibrated relative risk aversion parameters are unreasonably high in our view. For example, an "aggressive" investor has a relative risk tolerance of 0.28 (= 1/3.532). Hence, this investor will only allocate 28% of wealth to the speculative portfolio while the remaining 72% is allocated to the hedge portfolio. Also, the "conservative" investors are very cautious in the sense that they will only allocate 3.6% (= 1/27.99) of wealth to the speculative portfolio while 96.4% are invested in a hedge portfolio.

The reason that the relative risk aversion parameters are estimated rather high, at least in our view, is the heavy burden we place on the model in order to explain why long investment horizon investors should invest 20% extra in stocks. This horizon effect should be explained by intertemporal hedging behavior alone since the speculative portfolio remains the same for any horizon. Also, since we require the "aggressive" investor to hold 20% extra in stocks for a long horizon investor compared to a short investment horizon investor, even the induced "aggressive" representative investor must have a strong desire to buy the hedge portfolio.[6]

In order to resolve the Samuelson puzzle and make the stock a good substitute for a long-term real bond, the calibrated capital market parameters in Table 4 are so that the stock index and the real interest rate are negatively correlated, $\rho_{S,(r-\pi)} = -0.2934$ (compared to $\rho_{S,(r-\pi)} = 0.0790$ under the original estimates). Also, in order to obtain sufficient horizon effects the shocks to the inflation rate have to be quite permanent. This is reflected in the calibrated value of $\beta = 0.0954$ which is at its lower boundary value. The estimate implies that the half life of shocks to the inflation rate is 7.27 years (as compared to 2.36 years under the original estimates). Allowing these parameters to vary in larger intervals could reduce the deviation between the asset allocation advice in Table 2 and the asset allocations deduced from the formal model even more.

4 Conclusions

In this paper we have demonstrated that popular investment allocation advice is not necessarily at odds with rational portfolio choice. In particular, we have provided a model that is able to resolve simultaneously the Samuelson puzzle as well as the asset allocation puzzle pointed out by Canner, Mankiw and Weil (1997). However, it still seems that especially the recommendation that very "aggressive" investors should also have a significantly higher fraction of wealth invested in stocks is a heavy burden to put on the rational portfolio model – at least if this feature is to be explained by intertemporal hedging behavior.

[6]It may be noted that since the horizon effects in our theoretical asset allocation model are due to the allocation into the hedge portfolio, the model will never be able to explain why both "conservative" and "aggressive" investors should hold the same 20% extra in stocks for the long horizon investors.

Table 1: Estimates of Asset Price and Inflation Parameters

Parameter	Estimate	Standard error*
Stock Return Process: $\frac{dS}{S} = (r_t + \lambda_S)dt + \sigma_S dW_{St}$		
λ_S	0.0512	0.0164
σ_S	0.1170	0.0048
Nominal interest rate process: $dr_t = \kappa(\theta - r_t)dt + \sigma_r dW_{rt}$		
κ	0.0352	0.0549
θ	0.0439	0.0620
σ_r	0.0199	0.0006
λ_r	0.1067	0.0970
$\sigma_{\epsilon,1}$	0.0001	—
$\sigma_{\epsilon,2}$	0.0049	0.0002
$\sigma_{\epsilon,3}$	0.0071	0.0003
$\sigma_{\epsilon,4}$	0.0096	0.0003
$\sigma_{\epsilon,5}$	0.0119	0.0003
Price index Process: $\frac{d\Psi}{\Psi} = \pi_t dt + \sigma_\Psi dW_{\Psi t}$		
σ_Ψ	0.0081	0.0003
Inflation Process: $d\pi_t = \beta(\alpha - \pi_t)dt + \sigma_\pi dW_{\pi t}$		
β	0.2936	0.0991
α	0.0357	0.0089
σ_π	0.0213	0.0014
Correlations:		
ρ_{Sr}	-0.0029	0.0605
$\rho_{S\Psi}$	-0.0406	0.0457
$\rho_{S\pi}$	-0.0519	0.1054
$\rho_{r\Psi}$	0.0063	0.0535
$\rho_{r\pi}$	0.7948	0.1307
$\rho_{\Psi\pi}$	0.0168	1.0810

* Heteroscedasticity-consistent standard errors from White (1980).

Table 2: Asset Allocation Recommendations Used For Calibration

Horizon		Attitude towards risk		
		Conservative	Moderate	Aggressive
Short:	Stock	20.0%	42.5%	80.0%
	Bonds	30.0%	37.5%	7.5%
	Cash	50.0%	20.0%	12.5%
	Bonds/stock	1.50	0.88	0.09
Medium:	Stock	27.5%	50.0%	87.5%
	Bonds	32.5%	40.0%	10.0%
	Cash	40.0%	10.0%	2.5%
	Bonds/stock	1.18	0.80	0.11
Long:	Stock	40.0%	62.5%	100.0%
	Bonds	22.5%	30.0%	0.0%
	Cash	37.5%	7.5%	0.0%
	Bonds/stock	0.56	0.48	0.00

Table 3: Calibrated Asset Allocations and Investor Parameters

Panel A: Calibrated asset allocations

Horizon		Conservative	Moderate	Aggressive
		Attitude towards risk		
Short:	Stock	34.5%*	58.0%*	72.8%
	Bonds	17.4%*	24.2%*	28.4%**
	Cash	48.1%	17.9%	−1.3%*
	Bonds/stock	0.50	0.42	0.39
Medium:	Stock	34.3%	57.8%	72.7%*
	Bonds	22.2%*	27.8%*	31.3%**
	Cash	43.5%	14.4%	−4.0%
	Bonds/stock	0.65	0.48	0.43
Long:	Stock	34.3%	57.8%	72.7%**
	Bonds	22.2%	27.8%	31.3%***
	Cash	43.5%	14.4%	−4.0%
	Bonds/stock	0.65	0.48	0.43

Panel B: Calibrated parameters

	Parameter	Estimate	Boundary
Attitudes towards risk:	γ_{agg}	1.815	no
	γ_{mod}	2.272	no
	γ_{con}	3.771	no
Investment horizons:	T_{short}	2.424	no
	T_{med}	3.784	upper
	T_{long}	3.784	lower
Optimal durations of bonds:	D_{short}	15.00	upper
	D_{med}	15.00	lower/upper
	D_{long}	15.00	upper
Value of object function:		0.44686	

* Deviation from recommendation by more than 10%.
** Deviation from recommendation by more than 20%.
*** Deviation from recommendation by more than 30%.

Table 4: Calibrated Asset Allocations and Investor/Asset Price Parameters

Panel A: Calibrated asset allocations

Horizon		Conservative	Moderate	Aggressive
		Attitude towards risk		
Short:	Stock	21.2%	46.7%	83.1%
	Bonds	34.2%	24.2%*	10.0%
	Cash	44.6%	29.1%	6.9%
	Bonds/stock	1.61	0.52	0.12
Medium:	Stock	26.4%	51.3%	87.0%
	Bonds	39.1%	29.0%*	14.6%
	Cash	34.4%	19.6%	−1.6%
	Bonds/stock	1.48	0.57	0.17
Long:	Stock	37.5%	61.3%	95.2%
	Bonds	29.4%	20.4%	7.4%
	Cash	33.0%	18.4%*	−2.6%
	Bonds/stock	0.78	0.33	0.08

Panel B: Calibrated parameters

	Parameter	Estimate	Boundary
Attitudes towards risk:	γ_{agg}	3.532	no
	γ_{mod}	7.271	no
	γ_{con}	27.99	no
Investment horizons:	T_{short}	5.487	no
	T_{med}	9.337	no
	T_{long}	40.00	upper
Optimal durations of bonds:	D_{short}	5.000	lower
	D_{med}	5.506	upper
	D_{long}	5.506	lower

(Table 4, continued next page)

(Table 4, continued)

Stock Return Process: $\frac{dS}{S} = (r_t + \lambda_S)dt + \sigma_S dW_{St}$			
	λ_S	0.0840	upper
	σ_S	0.1800	lower
Nominal interest rate process: $dr_t = \kappa(\theta - r_t)dt + \sigma_r dW_{rt}$			
	κ	0.1362	no
	σ_r	0.0211	upper
	λ_r	0.0123	no
Price index Process: $\frac{d\Psi}{\Psi} = \pi_t dt + \sigma_\Psi dW_{\Psi t}$			
	σ_Ψ	0.0087	upper
Inflation Process: $d\pi_t = \beta(\alpha - \pi_t)dt + \sigma_\pi dW_{\pi t}$			
	β	0.0954	lower
	σ_π	0.0223	no
Correlations:			
	ρ_{Sr}	-0.1239	lower
	$\rho_{S\Psi}$	0.0508	upper
	$\rho_{S\pi}$	0.1589	upper
	$\rho_{r\Psi}$	-0.1007	lower
	$\rho_{r\pi}$	0.5334	lower
Value of object function:		0.10704	

* Deviation from recommendation by more than 10%.

References

Breeden, D. T. (1986). "Consumption, Production, Inflation and Interest Rates: A Synthesis". *Journal of Financial Economics*, 16, 3–39.

Brennan, M. J. & Xia, Y. (2000). "Stochastic Interest Rates and Bond-Stock Mix". *European Finance Review*, 4, 197–210.

Brennan, M. J. & Xia, Y. (2001). "Dynamic Asset Allocation under Inflation". Working paper, The Wharton School, University of Pennsylvania.

Campbell, J. Y. & Viceira, L. M. (2001). "Who Should Buy Long-Term Bonds?". *American Economic Review*, 91(1), 99–127.

Canner, N.; Mankiw, N. G., & Weil, D. N. (1997). "An Asset Allocation Puzzle". *American Economic Review*, 87(1), 181–191.

Cox, J. C. & Huang, C.-F. (1989). "Optimal Consumption and Portfolio Policies when Asset Prices follow a Diffusion Process". *Journal of Economic Theory*, 49, 33–83.

Cox, J. C. & Huang, C.-F. (1991). "A Variational Problem Arising in Financial Economics". *Journal of Mathematical Economics*, 20, 465–487.

Cox, J. C.; Ingersoll, Jr., J. E., & Ross, S. A. (1979). "Duration and the Measurement of Basis Risk". *Journal of Business*, 52, 51–61.

Elton, E. J. & Gruber, M. J. (2000). "The Rationality of Asset Allocation Recommendations". *Journal of Financial and Quantitative Analysis*, 35(1), 27–42.

Harvey, A. C. (1989). *Forecasting, structural time series models and the Kalman filter*. Cambridge University Press.

Ingersoll, Jr., J. E.; Skelton, J., & Weil, R. L. (1978). "Duration Forty Years Later". *Journal of Financial and Quantitative Analysis*, 13(4), 627–650.

Karatzas, I. & Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*. New York: Springer-Verlag, 2 edition.

McCulloch, J. H. (1990). "U.S. Term Structure Data, 1946-87". *Handbook of Monetary Economics*, Volume I, 672–715.

McCulloch, J. H. & Kwon, H.-C. (1993). "U.S. Term Structure Data, 1947-91". Working paper, Economics Department, Ohio State University.

Merton, R. C. (1969). "Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case". *Review of Economics and Statistics*, 51, 247–257.

Merton, R. C. (1971). "Optimum Consumption and Portfolio Rules in a Continuous-Time Model". *Journal of Economic Theory*, 3, 373–413.

Merton, R. C. (1973). "An Intertemporal Capital Asset Pricing Model". *Econometrica*, 41, 867–887.

Omberg, E. (1999). "Non-Myopic Asset-Allocation with Stochastic Interest Rates". Working paper, San Diego State University.

Pennacchi, G. G. (1991). "Identifying the Dynamics of Real Interest Rates and Inflation: Evidence Using Survey Data". *Review of Financial Studies*, 4(1), 53–86.

Quinn, J. B. (1991). *Making the most of your money.* New-York: Simon and Schuster.

Samuelson, P. A. (1963). "Risk and Uncertainty: A Fallacy of Large Numbers". *Scientia*, 98, 108–113.

Samuelson, P. A. (1989). "The Judgement of Economic Science on Rational Portfolio Management: Indexing, Timing, and Long-Horizon Effects". *Journal of Portfolio Management*, 16, 4–12.

Samuelson, P. A. (1994). "The Long-Term Case for Equities and How it can be Oversold". *Journal of Portfolio Management*, 21, 15–24.

Shiller, R. J. (2000). *Irrational Exuberance.* Princeton University Press.

Sørensen, C. (1999). "Dynamic Asset Allocation and Fixed Income Management". *Journal of Financial and Quantitative Analysis*, 34(4), 513–531.

Vasicek, O. A. (1977). "An Equilibrium Characterization of the Term Structure". *Journal of Financial Economics*, 5, 177–188.

White, H. (1980). "A heteroscedasticity-consistent covariance estimator and a direct test for heteroscedasticity". *Econometrica*, 48, 817–838.