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# **Department of Economics**

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## **Rationalizing sharing rules**

Karol Flores-Szwagrzak  
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# Rationalizing sharing rules\*

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## Abstract

A partnership can yield a return—a loss or a profit relative to the partners’ investments. How should the partners share the return? We identify the sharing rules satisfying classical properties (symmetry, consistency, and continuity) and avoiding arbitrary bounds on a partner’s share. We show that any such rule can be rationalized in the sense that its recommendations are aligned with those maximizing a separable welfare function. Among these rules, we characterize those formalizing different notions of proportionality and, in particular, a convenient subclass specified by a single inequality aversion parameter. We also explore when a rule can be rationalized by a more general welfare function. Our central results extend to a wider class of resource allocation problems.

Keywords: Sharing, Consistency, Axioms, Welfare maximization

JEL classification: D70, D63, D71

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# 1 Introduction

A business partnership can yield a return—a loss or a profit relative to the partners’ investments. How should the partners share the return? A public good can end up costing more or less than the sum of the economic valuations of its beneficiaries, as is common for infrastructure projects. How should the beneficiaries share the cost? These are examples of the “sharing problems” we will study here, where an amount (e.g., a return or a cost) is to be split among a group of agents with different characteristics specifying their involvements in the problem (e.g., their investments in the partnership or their economic valuations for the public project).

A *rule* is a systematic method to recommend, for each sharing problem, a distribution of the available amount. Our goal is to identify rules standing out in terms of their normative properties. Our first result (Theorem 1) fully describes the rules satisfying three classical properties in the theory of distributive justice and a fourth more novel fairness condition:

1. The recommended distribution is such that agents with equal characteristics receive equal shares (*equal treatment of equals*).
2. The recommended distributions are consistent across sharing problems, i.e., if a distribution is recommended for a group of agents and it is possible to recommend the restriction of that distribution for a subset of agents, then the rule does so (*consistency*).
3. The recommended distribution varies continuously in the parameters of the sharing problem, i.e., in the agent characteristics and the amount to divide (*continuity*).
4. As the amount to divide becomes large enough, no agent’s share can be restricted to fall below a certain level (*non-stagnation*).

Remarkably, the rules satisfying these properties are precisely those that can be rationalized in the following sense: The recommendations of any such rule are aligned with those maximizing an additively separable welfare function. It is *as if* its allocations were chosen by a social planner maximizing a separable welfare function. In contrast, assuming a social welfare function from the start would be fraught with difficulties—the specification of utilities and their comparison across individuals.

Young (1987) proves a similar rationalization result for “claims problems” (O’Neill, 1982; Aumann and Maschler, 1985). Here, the relevant agent characteristic is a claim

on the resource and these claims add up to more than what is available.<sup>1</sup> For instance, this models a bankruptcy where a partnership’s liquidation falls short of the partners’ investments. Young shows that a full inventory of allocation rules, some of them ancient, all satisfy natural formulations of equal treatment of equals, consistency, and continuity. Moreover, Young establishes that these properties are necessary and sufficient for a rule’s recommendations to agree with those obtained by maximizing a separable welfare function *if no agent’s share can be greater than her claim*. Young also extends this characterization to “surplus sharing problems.” Here, the relevant agent characteristic is her investment in a partnership and the revenue from the partnership exceeds the investments of the partners. This extension holds *if no agent’s share is smaller than her contribution*. Our result applies beyond claims and surplus sharing problems, e.g., to partnerships that may either turn a loss or a profit and where restrictions on the shares relative to the investments can therefore not be imposed. It also applies to more general resource allocation problems where the agent characteristics can be more complex than (one-dimensional) claims or investments. The non-stagnation property provides the minimal structure that enables a rule to be rationalized by a separable welfare function in our sharing problems—a structure that is implicit in the claims and surplus sharing domains.

We also find that there are normatively compelling rules—that can be rationalized by the maximization of a separable welfare function—in the domain of sharing problems that do not induce rules for claims or surplus sharing problems when restricted to those domains. In particular, we characterize a family of weighted proportional rules (Theorem 3). Each rule in this family is specified by a function weighing the agent’s characteristic so that the allocation recommended by the rule is proportional to the weighted characteristics. Within this family, we derive a single-parameter weighted proportional rule where the weighing function is a power function (Theorem 4). We also identify the axioms necessary and sufficient for a sharing rule to be rationalized by a more general class of welfare functions.

There is substantial work on resource allocation problems generalizing the claims and surplus sharing problems; see, e.g., Chun (1988), de Frutos (1997), Herrero et al. (1999), Kaminski (2000, 2006), or Ju et al. (2007). These contributions have not aimed at extending Young’s fundamental rationalization result. However, Young also showed that a rule satisfying his three axioms can be described by means of

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<sup>1</sup>See Thomson (2003, 2015) and Moulin (2002) for surveys and Thomson (2019) for a full monograph.

“parametric” distribution schedule, whereby a single parameter moving in tandem with the endowment pins down the share recommended for each agent. [Kaminski \(2000, 2006\)](#) extends this result to more general environments.

The remainder of the paper is organized as follows: In [Section 2](#), we formally introduce sharing problems in the simplest setting, where an agent characteristic is a positive number, e.g., an investment in a partnership or a valuation for a public good. We also discuss a number of rules. In [Section 3](#), we present our main rationalization results. In [Section 4](#), we characterize the weighted proportional rules. In [Section 5](#), we show how our central results also hold for sharing problems where the agent characteristics can be more complex, enabling further applications. To keep the main text concise, the proofs are deferred to the Appendix.

## 2 Sharing problems and sharing rules

An amount of a divisible resource is to be shared among a group of agents drawn from a countably infinite set  $I$ . A (sharing) problem is a triple  $(N, c, e)$  where  $N$  is a finite subset of  $I$  specifying the agents involved in the problem,  $c \in \mathbb{R}_{++}^N$  specifies, for each agent  $i \in N$ , a normatively relevant characteristic  $c_i \in \mathbb{R}_{++}$  measuring the extent of her involvement in the problem, and  $e \in \mathbb{R}_+$  is the endowment of the resource. A feasible allocation of  $e$  among the agents in  $N$  is an  $x \in \mathbb{R}_+^N$  such that  $\sum x_i = e$ . Let  $X(N, e)$  denote the set of all such allocations.

**Example 1** (Investment partnership). *When  $(N, c, e)$  models an investment partnership,  $N$  is the set of partners,  $e$  is the monetary return of the partnership, and, for each  $i \in N$ ,  $c_i$  is the investment of  $i$  in the partnership.*

In the above example, the return and the investments are naturally assumed to be measured in the same monetary units. More generally, a partner’s investment could be the amount of time she devotes to the partnership. (See [Example 4](#) for a more thorough generalization.) Hence, an agent’s investment and her share of the revenue may not be comparable.

**Example 2** (Public project). *When  $(N, c, e)$  models public project finance,  $N$  is the set of project beneficiaries,  $e$  is the cost of the project, and, for each  $i \in N$ ,  $c_i$  is the monetary valuation of  $i$  for project.*

We study the possible *rules* recommending allocations for each problem. Formally, a rule  $\varphi$  specifies, for each problem  $(N, c, e)$ , a feasible allocation denoted by  $\varphi(N, c, e)$ . As usual,  $\varphi_i(N, c, e)$  denotes the amount awarded to individual  $i \in N$ .

Perhaps the simplest rule is that recommending an equal-split of the endowment. The *equal-split rule* denoted by  $E$  is the rule whereby, for each problem  $(N, c, e)$  and each  $i \in N$ , the amount awarded to  $i$  is

$$E_i(N, c, e) = \frac{e}{|N|}.$$

The underlying criterion is that an agent's share should only depend on her presence in the sharing problem, regardless of the extent of her involvement as measured by her characteristic. In contrast, in an investment partnership, the relevant criterion may require that each dollar invested should receive the same rate of return. The *proportional rule* denoted by  $P$  reflects this criterion. It is the rule whereby, for each problem  $(N, c, e)$  and each  $i \in N$ , the amount awarded to  $i$  is

$$P_i(N, c, e) = \frac{c_i}{\sum_{j \in N} c_j} e.$$

The equal-split and proportional rules belong to a broad family of rules that can be obtained by maximizing a separable welfare function. A rule in this class can be defined as follows:

**Definition 1.** Rule  $\varphi$  **maximizes a separable welfare function** if it can be specified by a continuous function  $u : \mathbb{R}_+ \times \mathbb{R}_{++} \rightarrow \mathbb{R}$  such that

- i.  $u$  is strictly concave and strictly increasing over its first variable, and
- ii. for each problem  $(N, c, e)$ ,

$$\varphi(N, c, e) = \arg \max \{ \sum_N u(z_i, c_i) : z \in X(N, e) \}.$$

To verify that the equal-split and proportional rules maximize a separable welfare function, observe that, for each problem  $(N, c, e)$ ,

$$\begin{aligned} E(N, c, e) &= \arg \max \{ \sum_N \sqrt{z_i} : z \in X(N, e) \}, \\ P(N, c, e) &= \arg \max \{ \sum_N c_i \ln(1 + \frac{z_i}{c_i}) : z \in X(N, e) \}. \end{aligned}$$

These equalities can readily be checked by examining the first-order conditions of the optimization problems.

### 3 Axiomatic analysis

The following three axioms are well known. The first axiom requires that two agents with identical characteristics receive the same shares.

**Equal treatment of equals (ETE):** For all  $(N, c, e)$  and all  $i, j \in N$  with  $c_i = c_j$ ,  $\varphi_i(N, c, e) = \varphi_j(N, c, e)$ .

The second axiom, consistency, reflects a fundamental principle applied widely in the axiomatic analysis of allocations rules in contexts ranging from taxation, apportionment, the adjudication of conflicting claims, game theoretic bargaining, exchange economies, and discrete resource allocation problems (Thomson, 1990, 2011, 2012b). The principle is that, if an allocation is recommended for a group of agents, then its restriction to each subgroup of agents ought be recommended as well. This is formally expressed as follows: Suppose that the rule selects an allocation  $x$  for a group of agents  $N$ . Then, the rule, confronted with a sharing problem where a subgroup of agents  $J$  has to divide the total amount it received under  $x$  should again recommend a share  $x_i$  for each member of  $J$ .

**Consistency:** For all  $(N, c, e)$  and  $J \subseteq N$ , if  $x = \varphi(N, c, e)$ , then

$$x_J = \varphi_i(J, c_J, \sum_{j \in J} x_j).$$

The third axiom requires that small variations in the sharing problem do not dramatically affect the amounts awarded:

**Continuity:** For each  $(N, c, e)$  and each sequence  $\{(c^k, e^k)\}$  converging to  $(c, e)$ , the sequence  $\{\varphi(N, c^k, e^k)\}$  converges to  $\varphi(N, c, e)$ .<sup>2</sup>

ETE, consistency, and continuity are standard axioms. They are satisfied by all the classical rules for the claims problem: the constrained equal awards, the constrained equal losses, the Talmud, the proportional rules as well as all the rules characterized by Young (1987, 1988) and Chambers and Moreno-Ternero (2017). Also for claims problems, consistency and continuity are central properties used to identify classes of rules that may not satisfy ETE: the asymmetric rationing rules (Moulin, 2000) and other closely related rules (Chambers, 2006), the asymmetric parametric rules (Stovall, 2014a), the monotone path rules (Stovall, 2014b), and the asymmetric equal sacrifice rules (Stovall, 2020).

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<sup>2</sup>The convergence is relative to the Euclidean product topology. The  $(c^k, e^k)$  and  $(c, e)$  belong to  $\mathbb{R}_{++}^N \times \mathbb{R}_+$ ; the  $\varphi(N, c^k, e^k)$  and  $\varphi(N, c, e)$  belong to  $\mathbb{R}_+^N$ .

A more novel axiom we will consider requires that, as the amount to divide becomes large enough, no agent's share can be restricted to fall below a certain level:

**Non-stagnation:** For each  $N$ , each  $c \in \mathbb{R}_{++}^N$ , each  $i \in N$ , and each  $x \in \mathbb{R}_+$ , there is  $e \in \mathbb{R}_+$  such that  $\varphi_i(N, c, e) \geq x$ .

Non-stagnation gives the minimal structure necessary to construct a separable welfare function. In a claims problem it is implied by the fact that no agent's share can exceed her claim. Thus, as the endowment increases, it will be impossible for any one agent to absorb all of the increases, *stagnating* the shares of other agents. Upon receiving her claim, further increases will land in the hands of other agents.

A property similar to non-stagnation has only been used by Naumova (2002) to extend the rules implementing the equal sacrifice principle in taxation (Young, 1988) to situations where agents with equal characteristics need not be treated equally.

The four axioms stated above are sufficient to ensure that the rule can be rationalized: *If a rule satisfies ETE, consistency, continuity, and non-stagnation, then it maximizes a separable welfare function.*

The four axioms above are sufficient for a rule to be rationalized by a separable welfare function but not necessary. To establish necessity, we refine Definition 1 as follows:

**Definition 2.** *A rule maximizing a separable welfare function has **diminishing individual marginal returns** if it can be specified by  $u : \mathbb{R}_+ \times \mathbb{R}_{++} \rightarrow \mathbb{R}$  satisfying the conditions in Definition 1 and, for all  $\gamma \in \mathbb{R}_{++}$ ,*

$$u(n+1, \gamma) - u(n, \gamma) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

We can now close the characterization:

**Theorem 1.** *A rule satisfies ETE, consistency, continuity, and non-stagnation if and only if it maximizes a separable welfare function with diminishing individual marginal returns.*

The idea underlying the proof of Theorem 1 is to infer a “standard of comparison” from how a rule allocates, as proposed by Young (1994) in the context of the claims problem and by Kaminski (2000, 2006) more generally. This standard captures how the rule—reflecting the distributional goals of the social planner—compares the relative priority of individuals  $i$  and  $j$  with characteristics  $c_i$  and  $c_j$  to awards  $x_i$  and  $x_j$ , respectively. Formally, this standard is a binary relation  $\succsim$  on  $\mathbb{R}_{++} \times \mathbb{R}_+$  where,

$$(c_i, x_i) \succsim (c_j, x_j)$$

reads as “awarding  $x_i$  to an individual with characteristic  $c_i$  is at least as justified as awarding  $x_j$  to an individual with characteristic  $c_j$ .” We construct  $\succsim$  from a rule  $\varphi$  satisfying the axioms in Theorem 1 as follows: for all  $(c_i, x_i), (c_j, x_j) \in \mathbb{R}_{++} \times \mathbb{R}_+$ , define  $(c_i, x_i) \succsim (c_j, x_j)$  if and only if

$$\inf\{e : \varphi_i(\{i, j\}, (c_i, c_j), e) \geq x_i\} \leq \inf\{e : \varphi_j(\{i, j\}, (c_i, c_j), e) \geq x_j\}. \quad (1)$$

That is, awarding  $x_i$  to an individual with characteristic  $c_i$  is at least as justified as awarding  $x_j$  to an individual with characteristic  $c_j$  if the allocation behavior of the rule reveals that the minimum endowment necessary for  $i$  to receive  $x_i$  is no greater than the minimum endowment necessary for  $j$  to receive  $x_j$ .

The key role of non-stagnation is to ensue that condition (1) yields a complete relation  $\succsim$  while consistency ensures the relation’s transitivity. We can then show, using continuity, that  $\succsim$  has a countable-order dense subset. Applying a classical representation result (e.g., see Chapter 3 in Fishburn, 1970),  $\succsim$  can then be represented by a function  $U : \mathbb{R}_{++} \times \mathbb{R}_+ \rightarrow (0, 1)$ . Now, from (1) we can deduce that  $(c_i, x'_i) \succ (c_i, x_i)$  whenever  $x'_i < x_i$ . In words, if an individual with characteristic  $c_i$  is justified to receive  $x_i$ , then she ought to be even more justified to receive a smaller amount. Thus,  $U$  is strictly decreasing over its second argument. The function  $u$  in Definition 1 can then be specified, for each pair  $(x_i, c_i) \in \mathbb{R}_+ \times \mathbb{R}_{++}$ , as follows:

$$u(x_i, c_i) = \int_0^{x_i} U(c_i, t) dt.$$

Since  $U(c_i, \cdot)$  is strictly decreasing,  $u(\cdot, c_i)$  is strictly concave. (See Appendix B for a complete proof.)

We now establish a more general rationalization result. Firstly, note that if a rule satisfies the axioms in Theorem 1 then, as the endowment increases, the recommendations of the rule are such that no agent’s share decreases. (See Lemma 2 in Appendix A). That is, the rule satisfies the following property:

**Resource-monotonicity:** For all  $(N, c, e)$  and  $e' \in [0, e]$ ,  $\varphi(N, c, e') \leq \varphi(N, c, e)$ .

Resource-monotonicity reflects the fundamental solidarity principle expressing that a change in the sharing problem (here, an increase in the available amount), affects all agents in the same direction. Nearly all rules for the claims problem satisfy resource-resource monotonicity, including all rules in within the major classes (e.g.,

those characterized by [Young, 1987, 1988](#); [Moulin, 2000](#); [Chambers, 2006](#); [Stovall, 2014a,b, 2020](#)).

We now describe the implications of resource-monotonicity and non-stagnation. The separable welfare functions in Definitions 1 and 2 are symmetric: They sum an identical function  $u$  for each agent where  $u$  depends solely on her own characteristic and share. In contrast, we now allow for individual-specific functions  $u_i$  depending on the share of the share of agent  $i$  and possibly on the characteristics of every agent. A rule is said to maximize a generalized welfare function if it aligns with the maximization of such a welfare function, which exhibits diminishing individual marginal returns.

**Definition 3.** Let  $\mathcal{C}$  denote the set of all possible characteristic profiles, i.e.,  $\mathcal{C} = \bigcup_{N \subseteq I, |N| < \infty} \mathbb{R}_{++}^N$ . Rule  $\varphi$  **maximizes a generalized welfare function** if, for each  $i \in I$ , there is a function  $u_i : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}$  such that

- i.  $u_i$  is continuous and strictly concave over its first variable,
- ii. for each problem  $(N, c, e)$ ,

$$\varphi(N, c, e) = \arg \max \{ \sum_{i \in N} u_i(z_i, c) : z \in X(N, e) \}.$$

Imposing only resource-monotonicity and non-stagnation, we obtain a rule aligned with the maximization of a generalized welfare function:

**Theorem 2.** *If a rule satisfies non-stagnation and resource-monotonicity, then it maximizes a generalized welfare function.*

The above result does not rely on the ETE axiom. Similarly, for the claims problem, [Moulin \(2000\)](#) provides a characterization of a class of rationing rules that do not necessarily recommend equal shares for agents with equal claims. Also for the claims problem, [Stovall \(2014a,b, 2020\)](#) axiomatizes rules that can be rationalized, in a sense similar to ours.

## 4 Proportionality and its variants

Consider a team of workers spending time on a project that will generate a return to be divided among them. Here, we could plausibly apply the proportional rule, whereby each worker's share is proportional to the time she contributed to the project—her characteristic. There are other forms of proportionality that may be

more compelling. For instance, when the marginal cost of contributing an extra hour to the project is increasing, we may require that each worker's share be proportional to her cost. On the other hand, it may be that what counts is having an agent on board the project, not how many hours the agent clocks. There is even a sense in which the equal-split rule is "proportional," when the rule imputes an equal contribution  $c_0 \in \mathbb{R}_{++}$  to each worker. Then, for each problem  $(N, c, e)$  and each  $i \in N$ ,

$$E_i(N, c, e) = \frac{c_0}{\sum_N c_0} e = \frac{e}{|N|}.$$

The point is that some notion of proportionality may apply. But, proportionality with respect to what?

To answer this question we introduce a family of rules that generalize the proportional rule and also maximize separable welfare functions (in the sense of Definitions 1 and 2). They are inspired by a class of "success functions" for contests (Skaperdas, 1996). In a contest involving a group of players, a success function determines the winning probability of each player on the basis of the efforts of every player. Given  $n$  players exerting efforts  $y_1, \dots, y_n \in \mathbb{R}_+$ , Skaperdas specifies necessary and sufficient axioms for the winning probability of player  $i$  to be determined by  $\frac{f(y_i)}{\sum_{j=1}^n f(y_j)}$  where  $f$  is a positive and increasing function of a player's effort. The winning probability is proportional to an  $f$ -transformation of the effort of a player. The choice of  $f$  determines the sensitivity of the winning probabilities to the players' efforts.

We can extend this idea to sharing problems. Here, the agents' characteristics will be *weighed* to reflect their relative desert. Each weighing scheme or function will specify a rule. The share recommended by this rule for each agent will then be proportional to her weighted characteristic. The weighing function then determines the relevant notion of proportionality.

**Definition 4.** A **weighted proportional rule** is a rule specified by a continuous (weighing) function  $w : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$  such that, for each problem  $(N, c, e)$  and each  $i \in N$ , the amount awarded to  $i$  is given by  $\frac{w(c_i)}{\sum_{j \in N} w(c_j)} e$ .

Note that every weighted proportional rule maximizes a separable welfare function. To see this, take any  $w : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$  continuous weighing function and let  $u : \mathbb{R}_+ \times \mathbb{R}_{++} \rightarrow \mathbb{R}$  be given by

$$\text{for all } (z_i, c_i) \in \mathbb{R}_+ \times \mathbb{R}_{++}, u(z_i, c_i) = w(c_i) \ln(1 + \frac{z_i}{w(c_i)})$$

Letting  $\varphi$  denote the separably welfare function specified by this  $u$ , it is straightforward to check that  $\varphi$  is the weighted proportional rule specified by  $w$ .

What properties characterize the weighted proportional rules? Additivity axioms have been studied widely in surplus and cost sharing, starting with (Moulin, 1987) and in cooperative game theory (Shapley, 1953; Roth, 1988).<sup>3</sup> The principle is that two allocation problems can be “added” or combined into a single problem and that the recommended share of each agent in the combined problem ought to equal the sum her shares in the two original problems. We formalize as follows: Consider two sharing problems identical in all respects except perhaps for the endowment of the resource. The requirement is that these problems can be solved independently or jointly, by adding up the endowments of the two problems, all else equal. The end result, the total share received by an agent, will be the same.

**Resource-additivity:** For all  $(N, c, e)$  and all  $e' \in \mathbb{R}_+$ ,

$$\varphi(N, c, e + e') = \varphi(N, c, e) + \varphi(N, c, e').$$

Resource-additivity properties have been consider by Moulin (1987) for surplus sharing problems and in a different setting by Chun (1988). For claims problems, resource-additivity clashes with feasibility (Bergantiños and Vidal-Puga, 2004), which has lead to the study of weaker additivity properties (Bergantiños and Méndez-Naya, 2001; Alcalde et al., 2014; Harless, 2017; Flores-Szwagrzak et al., 2020; García-Segarra and Ginés-Vilar, 2023).

In combination with ETE, consistency, continuity, and non-stagnation, additivity characterizes the weighted proportional rule:

**Theorem 3.** *A rule satisfies ETE, consistency, continuity, non-stagnation, and resource-additivity if and only if it is a weighted proportional rule.*

Theorem 3 (and Theorem 4 below) would hold also if we replaced resource-additivity by another additivity requirement: Consider a first and second sharing problem, both identical in all respects except for the endowment of the resource. Now, consider a third sharing problem where the amount to divide is the average of the amounts to divide in the two original problems, and all else is equal. Then, an agent’s average award over the first and second problems ought to be equal to her award in the third problem. One interpretation is that the amount to divide

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<sup>3</sup>See Moulin (2002) for a survey.

is uncertain, with two possible outcomes. Then, an agent's expected share of the uncertain amount coincides with her expected share across each of the two certain sharing problems corresponding to the two outcomes.

**Decomposability:** For all  $(N, c, e)$  and all  $e' \in \mathbb{R}_+$ ,

$$\varphi(N, c, \tfrac{1}{2}e + \tfrac{1}{2}e') = \tfrac{1}{2} \varphi(N, c, e) + \tfrac{1}{2} \varphi(N, c, e').$$

Chun (1988) considers a slightly stronger property, requiring the rule to be linear in the amount to divide.

The family of weighted proportional rules is broad. To narrow it down to a convenient one-parameter family, we focus on rules that satisfy homogeneity. A rule satisfies homogeneity if, for each problem, rescaling the characteristics and the endowment by a common factor rescales the recommended shares by the same factor. This is a natural requirement in many contexts and is assumed in Moulin (1987) as a defining property of a rule.

**Homogeneity:** For all  $(N, c, e)$  and all  $\lambda \in \mathbb{R}_{++}$ ,  $\varphi(N, \lambda c, \lambda e) = \lambda \varphi(N, c, e)$ .

By imposing homogeneity, we obtain a family of rules within the class of weighted proportional rules.

**Definition 5.** A weighted proportional rule **belongs to the power family** if its weighing function  $w : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$  is a power function, i.e., there exists an  $r \in \mathbb{R}_+$  such that, for all  $\gamma \in \mathbb{R}_{++}$ ,  $w(\gamma) = \gamma^r$ .

**Theorem 4.** A rule satisfies ETE, consistency, continuity, non-stagnation, resource-additivity, and homogeneity if and only if it is a weighted proportional rule belonging to the power family.

Applying insights from Thomson (2012a), it is easily shown that the lower the parameter  $r$ , the more evenly distributed the awards are, in the sense of Lorenz domination. Thus, the family is indexed by a parameter that determines the degree to which agents with large characteristics are awarded relative to agents with small characteristics.

The proportional rules we have described so far can be applied whether or not the agent claims or investments (their characteristics) are commensurable with the endowment to be divided. For instance, in a claims problem representing a bankruptcy situation, the amount claimed by each creditor, her eventual share, and the liquidation value are all measured in the same currency. Then, it is meaningful to require

that no agent’s share can exceed her claim. Similarly, in a surplus sharing problem, it is meaningful to require that no agent’s share of the surplus is less than her investment. These conditions can be extended to situations where the amount to divide can exceed or fall short of the total investments:

**Same-sidedness:** For all  $(N, c, e)$ , either  $\varphi(N, c, e) \leq c$  or  $\varphi(N, c, e) \geq c$ .

In words, for the investment partnership, if the amount to divide cannot cover the total investments, then nobody receives more than her investment. Otherwise, everyone receives at least her investment. However, the only weighted proportional rule satisfying same-sidedness is actually the proportional rule. Same-sidedness is a very strong requirement when combined with continuity and resource-additivity: It singles-out the proportional rule.

**Remark 1.** *A rule satisfies continuity, additivity, and same-sidedness if and only if it is the proportional rule (Theorem 5 in [Chun, 1988](#)). The proof is included in the Appendix for convenience.*

Thus, the weighted proportional rules may be especially relevant in situation where the agent characteristics are not necessarily directly comparable with the endowment—not measured in the same units. See Examples 3 and Example 4 in the following Section 5 for natural applications of these rules.

## 5 Further applications and generalizations

Our central results extend to situations where an agent’s characteristics can be more complex than a positive number.<sup>4</sup> A concrete example is revenue sharing among artists producing music for a streaming platform ([Bergantiños and Moreno-Ternero, 2023](#)), where the normatively relevant characteristic of an artist is how much each platform subscriber listens to her music.

**Example 3** (Revenue sharing in a streaming platform). *A streaming sharing problem  $(N, c, e)$  models a music streaming platform where  $N \subseteq I$  is a finite set of artists producing content for the platform,  $e \in \mathbb{R}_+$  is the part of the user subscription revenue that the platform will divide among the artists, and, for each artist  $i \in N$ , her characteristic specifies how many minutes each platform subscriber spends listening to*

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<sup>4</sup>Similarly, [Kaminski \(2000, 2006\)](#) examines a class of models generalizing the claims problem by replacing a claim with a “type” drawn from any separable topological space.

her music, i.e.,  $c_i = (c_{ij})_{j \in M} \in \mathbb{R}_+^M$  where  $c_{ij}$  measures how many minutes subscriber  $j \in M$  listened to the content generated by artist  $i$ .

Another example concerns a more general business partnership requiring a mix of production inputs from its partners. The inputs could include, capital, hours of product development expertise, hours of accounting expertise, etc.

**Example 4** (Multi-input partnership). *A multi-input sharing problem  $(N, c, e)$  models a partnership where  $N \subseteq I$  is a finite set of partners,  $e \in \mathbb{R}_+$  is the revenue of the partnership to be divided, and, for each partner  $i \in N$ , her characteristic specifies how many units of each production input she contributes with, i.e.,  $c_i = (c_{ij})_{j \in M} \in \mathbb{R}_+^M$  where  $M$  is set of production inputs and  $c_{ij}$  denotes the number of units of input  $j$  invested by partner  $i$ .*

With these Examples in mind, we can restate our definition of a sharing problem. Let  $C$  denote a set of normatively relevant characteristics describing the involvement of an agent in a sharing problem. In Sections 2, 3, and 4, we assumed  $C = \mathbb{R}_{++}$  for simplicity and easier reference to Young (1987). In the last two examples, we let  $C = \mathbb{R}_+^M$  where  $M$  is either the set of users of a streaming platform (Example 3) or the set of production inputs for the partnership (Example 4). A sharing problem is then a triple  $(N, c, e)$  where  $N$  is a finite subset of a countably infinite set  $I$ , and  $c \in C^N$  specifies, for each agent  $i \in N$ , a normatively relevant characteristic  $c_i \in C$  measuring her involvement in the problem, and  $e \in \mathbb{R}_+$  is the endowment of the resource to be shared.

The definition of a feasible allocation and a rule are then identical to those in Section 2 and the definitions of our axioms—ETE, consistency, and non-stagnation, and additivity—extend naturally. For the definition of continuity to be meaningful, we need to require that  $C$  is a separable topological space. For concreteness, we let  $C$  denote any convex subset of Euclidean space so the definitions follow through with no further issues.

**Remark 2.** *Let  $d$  denote a natural number and let  $C$  denote a convex subset of  $\mathbb{R}^d$  equipped with the usual topology. Theorems 1, 2, and 3 all hold for sharing problems where an agent's normatively relevant characteristic is drawn from  $C$ .*

The above can immediately be verified by examining our proofs: Simply replace instances of characteristics in  $\mathbb{R}_{++}$  with those in  $C$ . A key step in the proof of

Theorem 1 shows that  $\mathbb{Q}_{++} \times \mathbb{Q}_+$  is a countable  $\succsim$ -dense subset of  $\mathbb{R}_{++} \times \mathbb{R}_+$ . The same arguments show that  $[\mathbb{Q}_{++}^d \cap C] \times \mathbb{Q}_+$  is a countable dense subset of  $C \times \mathbb{R}_+$ .

Finally, we assumed that the set of potential agents  $I$  is infinitely countable. This is a technical assumption that is only used in the proof of Lemma 2 in the Appendix. It can be dispensed with by assuming resource-monotonicity.

**Remark 3.** *If  $I$  is finite, Theorem 1 would hold impose resource-monotonicity since it is no longer implied by the other axioms (Lemma 2). None of the other results require the assumption that  $I$  is countably infinite.*

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## A Preliminary results

**Lemma 1.** *If  $\varphi$  is resource-monotonic, then, for each  $N \subseteq I$  and each  $c \in \mathbb{R}_{++}^N$ ,  $\varphi(N, c, \cdot)$  is continuous.*

*Proof.* Let  $\{e_n\}_{n \in \mathbb{N}}$  denote a sequence in  $\mathbb{R}_+$  converging to  $e$ . For each  $n \in \mathbb{N}$ , let  $x^n \equiv \varphi(N, c, e_n)$  and  $y \equiv \varphi(N, c, e)$ . It suffices to prove that (i) the sequence  $\{x^n\}_{n \in \mathbb{N}}$  in  $\mathbb{R}_+^N$  has a limit denoted by  $x$  and that (ii)  $x = y$ .

By the Cauchy convergence criterion, if  $\{x^n\}_{n \in \mathbb{N}}$  does not have a limit, there is an  $\varepsilon > 0$  such that, for each natural number  $m$ , there are natural numbers  $h, l > m$  for which  $\|x^l - x^h\| \geq \varepsilon$ . This requires that  $e_l \neq e_h$  for otherwise, by definition,  $x^l = x^h$ . Without loss of generality,  $e_h > e_l$ . Then, by resource-monotonicity,  $x^h \geq x^l$ . However, since  $e_n \rightarrow e$ ,  $m$  can be chosen sufficiently large so that  $e_h - e_l < \varepsilon$ . This

is incompatible with  $x^h \geq x^l$  and  $\|x^l - x^h\| \geq \varepsilon$ , a contradiction. Thus,  $\{x^n\}_{n \in \mathbb{N}}$  has a limit that we will denote by  $x$ . Since  $e_n \rightarrow e$  and  $\sum_{i \in N} x_i^n = e_n$ , the sequence  $\{\sum_{i \in N} x_i^n\}_{n \in \mathbb{N}}$  converges to  $e$ . Thus,  $\sum_{i \in N} x_i = e$ . By way of contradiction, suppose that  $x \neq y$ . Then, since  $\sum_{i \in N} x_i = e = \sum_{i \in N} y_i$ , there is a pair  $i, j \in N$  such that  $x_i > y_i$  and  $x_j < y_j$ . Then, since  $x^n \rightarrow x$ , there is a sufficiently large  $n \in \mathbb{N}$  such that  $x_i^n > y_i$  and  $x_j^n < y_j$ . Then,  $e_n \neq e$  and without loss of generality, we may assume that  $e_n > e$ . Then, by resource-monotonicity,  $x^n = \varphi(N, c, e^n) \geq \varphi(N, c, e) = y$ . This contradiction establishes that, in fact,  $x = y$ .  $\square$

**Lemma 2.** *If  $\varphi$  is consistent, continuous, and satisfies non-stagnation and ETE, then it is resource-monotonic.*

*Proof.* The argument adapts that in [Young \(1987\)](#) for the domain of classical claims problems. Assume, by way of contradiction, that  $\varphi$  satisfies the axioms in [Lemma 2](#) except for resource-monotonicity. By consistency, there are  $i, j \in I$ ,  $c \in \mathbb{R}_{++}^{\{i,j\}}$ , and  $e, e' \in \mathbb{R}$  such that, letting  $x = \varphi(\{i, j\}, c, e)$  and  $x' = \varphi(\{i, j\}, c, e')$ ,

$$x_i < x'_i, x_j > x'_j, \text{ and } e < e'.$$

Thus, we can fix a sufficiently large  $n \in \mathbb{N}$  such that

$$x_i + nx_j > x'_i + nx'_j. \quad (2)$$

Now, let  $N \subseteq I$  be such that  $i, j \in N$  and  $|N| = n + 1$ . Let  $c' \in \mathbb{R}_{++}^N$  be such that  $c'_i = c_i$  and, for each  $k \in N$  such that  $k \neq i$ ,  $c'_k = c_j$ . For each  $E \in \mathbb{R}_+$ , define

$$f(E) = \varphi_i(N, c', E) + \varphi_j(N, c', E).$$

Since  $E = 0$  implies  $\varphi(N, c', E)$  is a vector of zeros,  $f(0) = 0$ . By non-stagnation,  $f(E) \rightarrow \infty$  as  $E \rightarrow \infty$ . Since  $\varphi$  is continuous,  $f$  is continuous. Thus, there is  $E'$  such that  $f(E') = e'$ . By consistency and ETE,  $\varphi(N, c', E') = y$  where  $y_i = x'_i$  and  $y_k = x'_j$  for all  $k \neq i$ . Thus,  $E' = x'_i + nx'_j$ . Since  $f$  is continuous, there is  $E \leq E'$  such that  $f(E) = e < e' = f(E')$ . By consistency and ETE,  $\varphi(N, c', E) = z$  where  $z_i = x_i$  and  $z_k = x_j$  for all  $k \neq i$ . Thus,  $E = x_i + nx_j$ . This however contradicts (2), establishing the Lemma.  $\square$

**Lemma 3.** *Suppose that  $\varphi$  satisfies resource-monotonicity and non-stagnation. For each finite  $N \subseteq I$ , each  $i \in N$ , and each  $(c, \alpha) \in \mathbb{R}_{++}^N \times \mathbb{R}_+$ , define*

$$G_i^N(c, \alpha) \equiv \inf\{e : \varphi_i(N, c, e) \geq \alpha\}.$$

Then, for each  $(c, \alpha) \in \mathbb{R}_{++}^N \times \mathbb{R}_+$ ,

- (i)  $G_i^N(c, \alpha) < \infty$ ,
- (ii)  $\varphi_i(N, c, G_i^N(c, \alpha)) = \alpha$  and  $[\varphi_i(N, c, e) < \alpha \Leftrightarrow e < G_i^N(c, \alpha)]$ ,
- (iii)  $G_i^N(c, \alpha) < G_i^N(c, \beta)$  for each  $\beta > \alpha$ .

*Proof.* Let  $(c, \alpha) \in \mathbb{R}_{++}^N \times \mathbb{R}_+$ .

- (i) By non-stagnation, there is  $e \in \mathbb{R}_+$  with  $\varphi_i(N, c, e) \geq \alpha$ . Thus,  $G_i^N(c, \alpha) < \infty$ .
- (ii) By way of contradiction, suppose that  $\varphi_i(N, c, g) \neq \alpha$  where  $g \equiv G_i^N(c, \alpha)$ . By Lemma 1,  $\varphi_i(N, c, \cdot)$  is continuous. By feasibility,  $\varphi_i(N, c, 0) = 0$ . Thus, by the Intermediate Value Theorem and non-stagnation,  $\varphi_i(N, c, \cdot)$  takes every value between 0 and  $\alpha$ . If  $\varphi_i(N, c, g) > \alpha$ , then, by the continuity of  $\varphi_i(N, c, \cdot)$ , there is  $\varepsilon > 0$  such that  $\varphi_i(N, c, g - \varepsilon) > \alpha$ . This would contradict the definition of  $g$ . If  $\varphi_i(N, c, g) < \alpha$ , by Lemma 1 and resource-monotonicity, there is  $\varepsilon > 0$  such that  $\varphi_i(N, c, g + \varepsilon) < \alpha$ . This would again contradict the definition of  $g$ . Thus, in fact,  $\varphi_i(N, c, g) = \alpha$ . By resource-monotonicity,  $e > g$  implies  $\varphi_i(N, c, e) \geq \varphi_i(N, c, g) = \alpha$ . By the definition of  $g = G_i^N(c, \alpha)$ , there is no  $e < g$  such that  $\varphi_i(N, c, g) \geq \alpha$ . Equivalently, for each  $e < g$ ,  $\varphi_i(N, c, g) < \alpha$ .
- (iii) Let  $\beta > \alpha$ . By (ii),  $\varphi_i(N, c, G_i^N(c, \alpha)) = \alpha$  and  $\varphi_i(N, c, G_i^N(c, \beta)) = \beta$ . Thus,  $\varphi_i(N, c, G_i^N(c, \alpha)) < \varphi_i(N, c, G_i^N(c, \beta))$ . By resource-monotonicity, this is incompatible with  $G_i^N(c, \alpha) \geq G_i^N(c, \beta)$ .  $\square$

**Lemma 4.** Suppose that  $\varphi$  satisfies ETE and consistency. For all  $i, j, k, l \in I$  such that  $i \neq j$  and  $k \neq l$ , all  $e \in \mathbb{R}_+$ , all  $c \in \mathbb{R}_{++}^{\{i, j\}}$  and all  $c' \in \mathbb{R}_{++}^{\{k, l\}}$  such that  $c_i = c'_k$  and  $c_j = c'_l$ ,

$$\varphi_i(\{i, j\}, c, e) = \varphi_k(\{k, l\}, c', e) \text{ and } \varphi_j(\{i, j\}, c, e) = \varphi_l(\{k, l\}, c', e).$$

*Proof.* Let  $i, j, k, l \in I$ , let  $c \in \mathbb{R}_{++}^{\{i, j, k, l\}}$  be such that  $c_i = c_k$  and  $c_j = c_l$ , let  $e \in \mathbb{R}_+$ , and let  $x = \varphi(\{i, j, k, l\}, c, 2e)$ . By ETE,  $x_i = x_k$  and  $x_j = x_l$ . Thus,  $x_i + x_j = e = x_k + x_l$ . By consistency,  $\varphi_i(\{i, j\}, c_{\{i, j\}}, e) = \varphi_k(\{k, l\}, c_{\{k, l\}}, e)$  and  $\varphi_j(\{i, j\}, c_{\{i, j\}}, e) = \varphi_l(\{k, l\}, c_{\{k, l\}}, e)$ .  $\square$

**Lemma 5.** Suppose that  $\varphi$  satisfies resource-monotonicity, non-stagnation, and consistency. For each finite subset  $N$  of  $I$  and each  $i \in N$ , let  $G_i^N$  be as defined in Lemma 3. Then, for each finite subset  $N$  of  $I$ , each pair  $i, j \in N$ , each  $c \in \mathbb{R}_{++}^N$ , and each pair  $\alpha, \beta \in \mathbb{R}_+$ ,

$$G_i^N(c, \alpha) \geq G_j^N(c, \beta) \text{ if and only if } G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) \geq G_j^{\{i,j\}}(c_{\{i,j\}}, \beta).$$

*Proof.* Let  $x \equiv \varphi(N, c, G_j^N(c, \beta))$  and  $y \equiv \varphi(N, c, G_j^N(c, \alpha))$ .

Step 1:  $G_i^N(c, \alpha) = G_j^N(c, \beta)$  implies  $G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) = G_j^{\{i,j\}}(c_{\{i,j\}}, \beta)$ .

Assume that  $G_i^N(c, \alpha) = G_j^N(c, \beta)$ . By Lemma 3,  $x_i = \alpha$  and  $x_j = \beta$ . By consistency,  $\varphi_i(\{i, j\}, c_{\{i,j\}}, \alpha + \beta) = \alpha$  and  $\varphi_j(\{i, j\}, c_{\{i,j\}}, \alpha + \beta) = \beta$ . Thus,  $G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) \leq \alpha + \beta$  and  $G_j^{\{i,j\}}(c_{\{i,j\}}, \beta) \leq \alpha + \beta$ . Since

$$\varphi_i(\{i, j\}, c_{\{i,j\}}, \alpha + \beta) + \varphi_j(\{i, j\}, c_{\{i,j\}}, \alpha + \beta) = \alpha + \beta,$$

$$G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) = \alpha + \beta = G_j^{\{i,j\}}(c_{\{i,j\}}, \beta).$$

Step 2:  $G_i^N(c, \alpha) > G_j^N(c, \beta)$  implies  $G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) > G_j^{\{i,j\}}(c_{\{i,j\}}, \beta)$ .

Assume that  $G_i^N(c, \alpha) > G_j^N(c, \beta)$ . By the definition of  $G_i^N$  and  $G_j^N$  and Lemma 3,  $x_i < \alpha$  and  $x_j = \beta$ . By consistency,

$$\varphi_i(\{i, j\}, c, x_i + x_j) = x_i < \alpha \text{ and } \varphi_j(\{i, j\}, c, x_i + x_j) = x_j = \beta.$$

Thus,  $G_j^{\{i,j\}}(c_{\{i,j\}}, \beta) \leq x_i + x_j$  and, by resource-monotonicity,  $G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) > x_i + x_j$ . Thus,  $G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) > G_j^{\{i,j\}}(c_{\{i,j\}}, \beta)$ .

Step 3:  $G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) > G_j^{\{i,j\}}(c_{\{i,j\}}, \beta)$  implies  $G_i^N(c, \alpha) > G_j^N(c, \beta)$ .

Assume that  $G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) > G_j^{\{i,j\}}(c_{\{i,j\}}, \beta)$ . By Step 1,  $G_i^N(c, \alpha) = G_j^N(c, \beta)$  would imply  $G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) = G_j^{\{i,j\}}(c_{\{i,j\}}, \beta)$  which is not the case. By Step 2,  $G_i^N(c, \alpha) < G_j^N(c, \beta)$  would imply  $G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) < G_j^{\{i,j\}}(c_{\{i,j\}}, \beta)$  which is not the case. The only remaining possibility is that  $G_i^N(c, \alpha) > G_j^N(c, \beta)$ .

Step 4:  $G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) = G_j^{\{i,j\}}(c_{\{i,j\}}, \beta)$  implies  $G_i^N(c, \alpha) = G_j^N(c, \beta)$ .

Assume that  $G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) = G_j^{\{i,j\}}(c_{\{i,j\}}, \beta)$ . By Step 2,  $G_i^N(c, \alpha) \neq G_j^N(c, \beta)$  would imply  $G_i^{\{i,j\}}(c_{\{i,j\}}, \alpha) \neq G_j^{\{i,j\}}(c_{\{i,j\}}, \beta)$ . Thus, in fact,  $G_i^N(c, \alpha) = G_j^N(c, \beta)$ .  $\square$

## B Proof of Theorem 1

Let  $\varphi$  satisfy the axioms in Theorem 1. By Lemma 2,  $\varphi$  is resource-monotonic. Thus, we also assume this axiom from here on. Let  $i, j \in I$  be distinct and let  $G_i^{\{i,j\}}$  and  $G_j^{\{i,j\}}$  be as defined in Lemma 3. For each  $(\gamma, \gamma', \alpha) \in \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_+$ , define

$$\phi(\gamma, \gamma', \alpha) = \varphi(\{i, j\}, c, \alpha) \text{ and}$$

$$G(\gamma, \gamma', \alpha) = G_i^{\{i,j\}}(c, \alpha) \quad \text{where } c \in \mathbb{R}_{++}^{\{i,j\}} \text{ is such that } c_i = \gamma \text{ and } c_j = \gamma'.$$

Note that  $\phi : \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^{\{i,j\}}$  is continuous. Define a binary relation  $\succsim$  on  $\mathbb{R}_{++} \times \mathbb{R}_+$  as follows: for all  $(\gamma, \alpha), (\gamma', \alpha') \in \mathbb{R}_{++} \times \mathbb{R}_+$ ,

$$(\gamma, \alpha) \succsim (\gamma', \alpha') \text{ if } G(\gamma, \gamma', \alpha) \leq G(\gamma', \gamma, \alpha').$$

Let  $\succ$  and  $\sim$  denote, respectively the asymmetric and symmetric parts of  $\succsim$ .

**Claim 1.** *Let  $\gamma \in \mathbb{R}_{++}$  and  $\alpha, \alpha' \in \mathbb{R}_+$  be such that  $\alpha' > \alpha$ . Then,  $(\gamma, \alpha) \succ (\gamma, \alpha')$ .*

Claim 1 follows from the definition of  $G$  and Lemma 3 (iii).

**Claim 2.** *The relation  $\succsim$  is complete and transitive.*

For any  $(\gamma, \alpha), (\gamma', \alpha') \in \mathbb{R}_{++} \times \mathbb{R}_+$ , either  $G(\gamma, \gamma', \alpha) \leq G(\gamma', \gamma, \alpha')$  or  $G(\gamma, \gamma', \alpha) > G(\gamma', \gamma, \alpha')$ . Thus,  $\succsim$  is complete.

We now prove that  $\succsim$  is transitive. Let  $(\gamma, \alpha), (\gamma', \alpha'), (\gamma'', \alpha'') \in \mathbb{R}_{++} \times \mathbb{R}_+$  satisfy

$$G(\gamma, \gamma', \alpha) \leq G(\gamma', \gamma, \alpha') \text{ and } G(\gamma', \gamma'', \alpha') \leq G(\gamma'', \gamma', \alpha'').$$

It suffices to show that the above imply  $G(\gamma, \gamma'', \alpha) \leq G(\gamma'', \gamma, \alpha'')$ . Let  $k \in I$  be different from  $i$  and  $j$  and let  $c' \in \mathbb{R}_{++}^{\{i,j,k\}}$  be such that  $c'_i = \gamma$ ,  $c'_j = \gamma'$ , and  $c'_k = \gamma''$ . By Lemmas 4 and 5,

$$\begin{aligned} G(\gamma, \gamma', \alpha) \leq G(\gamma', \gamma, \alpha') &\Rightarrow G_i^{\{i,j,k\}}(c', \alpha) \leq G_j^{\{i,j,k\}}(c', \alpha'), \\ G(\gamma', \gamma'', \alpha') \leq G(\gamma'', \gamma', \alpha'') &\Rightarrow G_j^{\{i,j,k\}}(c', \alpha') \leq G_k^{\{i,j,k\}}(c', \alpha''). \end{aligned}$$

Thus,  $G_i^{\{i,j,k\}}(c', \alpha) \leq G_k^{\{i,j,k\}}(c', \alpha'')$ . Hence, by Lemmas 4 and 5,  $G(\gamma, \gamma'', \alpha) \leq G(\gamma'', \gamma, \alpha'')$ . Thus,  $\succsim$  is transitive.

**Claim 3.** *Let  $(\gamma, \gamma', \alpha) \in \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_+$ . Then,*

- (i)  $\phi_i(\gamma, \gamma', G(\gamma, \gamma', \alpha)) = \alpha$  and  $[\phi_i(\gamma, \gamma', e) < \alpha \Leftrightarrow e < G(\gamma, \gamma', \alpha)]$ ,
- (ii)  $\phi_j(\gamma', \gamma, G(\gamma, \gamma', \alpha)) = \alpha$  and  $[\phi_j(\gamma', \gamma, e) < \alpha \Leftrightarrow e < G(\gamma, \gamma', \alpha)]$ .

Claim 3 (i) follows immediately from the definition  $G = G_i^{\{i,j\}}$  and Lemma 3 (ii). By Lemma 4,  $\phi_i(\gamma, \gamma', \cdot) = \phi_j(\gamma', \gamma, \cdot)$ . Claim 3 (ii) then follows from Claim 3 (i).

**Claim 4.** *Let  $(\gamma, \alpha), (\gamma', \alpha') \in \mathbb{R}_{++} \times \mathbb{R}_+$ . Then,*

$$(\gamma, \alpha) \succ (\gamma', \alpha') \Leftrightarrow \alpha + \alpha' > G(\gamma, \gamma', \alpha).$$

Let  $x = \phi(\gamma, \gamma', G(\gamma, \gamma', \alpha))$  and  $z = \phi(\gamma, \gamma', \alpha + \alpha')$ .

Suppose that  $(\gamma, \alpha) \succ (\gamma', \alpha')$ , i.e.,  $G(\gamma, \gamma', \alpha) < G(\gamma', \gamma, \alpha')$ . By Claim 3,  $x_i = \alpha$  and  $x_j < \alpha'$ . Then,  $\alpha + \alpha' > x_i + x_j = G(\gamma, \gamma', \alpha)$ , as desired.

Suppose that  $\alpha + \alpha' > G(\gamma, \gamma', \alpha)$ . By resource-monotonicity,  $z_i \geq x_i$ . By Claim 3,  $x_i = \alpha$ . Since  $z_i + z_j = \alpha + \alpha'$ ,  $z_j \leq \alpha'$ . Thus,  $G(\gamma', \gamma, \alpha') \geq \alpha + \alpha'$ . Hence,  $G(\gamma', \gamma, \alpha') > G(\gamma, \gamma', \alpha)$ , i.e.,  $(\gamma, \alpha) \succ (\gamma', \alpha')$ , as desired.

**Claim 5.** Let  $(\gamma, \alpha), (\gamma', \alpha') \in \mathbb{R}_{++} \times \mathbb{R}_+$ .

- (i)  $(\gamma, \alpha) \succeq (\gamma', \alpha')$  if and only if  $\alpha \leq \phi_i(\gamma, \gamma', \alpha + \alpha')$  and  $\alpha' \geq \phi_j(\gamma, \gamma', \alpha + \alpha')$ ,
- (ii)  $(\gamma, \alpha) \succ (\gamma', \alpha')$  if and only if  $\alpha < \phi_i(\gamma, \gamma', \alpha + \alpha')$  and  $\alpha' > \phi_j(\gamma, \gamma', \alpha + \alpha')$ ,
- (iii)  $(\gamma, \alpha) \succeq (\gamma', \alpha')$  if and only if  $\alpha' \geq \phi_i(\gamma', \gamma, \alpha + \alpha')$  and  $\alpha \leq \phi_j(\gamma', \gamma, \alpha + \alpha')$ ,
- (iv)  $(\gamma, \alpha) \succ (\gamma', \alpha')$  if and only if  $\alpha' > \phi_i(\gamma', \gamma, \alpha + \alpha')$  and  $\alpha < \phi_j(\gamma', \gamma, \alpha + \alpha')$ .

To prove (i), suppose that  $(\gamma, \alpha) \succeq (\gamma', \alpha')$ . By Claim 4,  $\alpha + \alpha' \leq G(\gamma', \gamma, \alpha')$ . By Claim 3,  $\alpha' \geq \phi_j(\gamma, \gamma', \alpha + \alpha')$ . Thus,

$$\phi_i(\gamma, \gamma', \alpha + \alpha') = \alpha + \alpha' - \phi_j(\gamma, \gamma', \alpha + \alpha') \geq \alpha.$$

Conversely, suppose that  $\alpha \leq \phi_i(\gamma, \gamma', \alpha + \alpha')$  and  $\alpha' \geq \phi_j(\gamma, \gamma', \alpha + \alpha')$ . By Claim 3, the first of these inequalities implies  $\alpha + \alpha' \geq G(\gamma, \gamma', \alpha)$  and the second one implies  $\alpha + \alpha' \leq G(\gamma', \gamma, \alpha')$ . Thus,  $G(\gamma, \gamma', \alpha) \leq G(\gamma', \gamma, \alpha')$ . Equivalently,  $(\gamma, \alpha) \succeq (\gamma', \alpha')$ . This establishes (i).

To prove (ii), suppose that  $(\gamma, \alpha) \succ (\gamma', \alpha')$ . By Claim 5 (i),

$$\alpha \leq \phi_i(\gamma, \gamma', \alpha + \alpha') \text{ and } \alpha' \geq \phi_j(\gamma, \gamma', \alpha + \alpha').$$

By Claim 5 (i), if both of these inequalities were equalities, then  $(\gamma, \alpha) \sim (\gamma', \alpha')$ . Thus, at least one of these inequality is strict. Since

$$\phi_j(\gamma, \gamma', \alpha + \alpha') + \phi_i(\gamma, \gamma', \alpha + \alpha') = \alpha + \alpha',$$

both inequalities are strict, as desired.

Conversely, suppose that  $\alpha < \phi_i(\gamma, \gamma', \alpha + \alpha')$  and  $\alpha' > \phi_j(\gamma, \gamma', \alpha + \alpha')$ . By Claim 5 (i),  $(\gamma, \alpha) \succeq (\gamma', \alpha')$ . By Claim 5 (i),  $(\gamma, \alpha) \sim (\gamma', \alpha')$  would imply  $\alpha = \phi_i(\gamma, \gamma', \alpha + \alpha')$  and  $\alpha' = \phi_j(\gamma, \gamma', \alpha + \alpha')$ . Thus,  $(\gamma, \alpha) \succ (\gamma', \alpha')$ .

By Lemma 4,  $\phi_i(\gamma, \gamma', \cdot) = \phi_j(\gamma', \gamma, \cdot)$ . Thus, statements (iii) and (iv) follow from statements (i) and (ii), respectively.

**Claim 6.** Let  $(\gamma, \alpha), (\gamma', \alpha') \in \mathbb{R}_{++} \times \mathbb{R}_+$  be such that  $(\gamma, \alpha) \succ (\gamma', \alpha')$ . Then, there is  $\alpha'' \in \mathbb{Q}_+$  such that  $\alpha'' > \alpha$  and  $(\gamma, \alpha) \succ (\gamma, \alpha'') \succ (\gamma', \alpha')$ .

Suppose that  $(\gamma, \alpha) \succ (\gamma', \alpha')$ . Define a sequence  $\{\alpha_n\}$  such that

$$\alpha_n \in (\alpha, \alpha + \frac{1}{n}) \cap \mathbb{Q}_+ \text{ for each } n.$$

By Claim 1,  $(\gamma, \alpha) \succ (\gamma, \alpha_n)$  for each  $n$ . By way of contradiction, suppose that  $(\gamma', \alpha') \succsim (\gamma, \alpha_n)$  for each  $n$ . By Claim 5 (iii),

$$\alpha_n \geq \phi_i(\gamma, \gamma', \alpha_n + \alpha') \text{ and } \alpha' \leq \phi_j(\gamma, \gamma', \alpha_n + \alpha') \text{ for each } n.$$

By continuity,  $\alpha_n \rightarrow \alpha$  thus implies  $\alpha \geq \phi_i(\gamma, \gamma', \alpha + \alpha')$  and  $\alpha' \leq \phi_j(\gamma, \gamma', \alpha + \alpha')$ . However, by Claim 5 (ii),  $(\gamma, \alpha) \succ (\gamma', \alpha')$  implies these inequalities cannot hold. This contradiction establishes Claim 6.

**Claim 7.** *Let  $(\gamma, \alpha), (\gamma', \alpha') \in \mathbb{R}_{++} \times \mathbb{R}_+$  be such that  $(\gamma, \alpha) \succ (\gamma', \alpha')$ . Then, there is  $(\gamma'', \alpha'') \in \mathbb{Q}_{++} \times \mathbb{Q}_+$  such that  $(\gamma, \alpha) \succsim (\gamma'', \alpha'') \succsim (\gamma', \alpha')$ .*

By Claim 6, we can fix an  $\alpha'' \in \mathbb{Q}_+$  with  $\alpha'' > \alpha$  and  $(\gamma, \alpha) \succ (\gamma, \alpha'') \succ (\gamma', \alpha')$ . Define

$$\Gamma_1 = \{\gamma'' \in \mathbb{Q}_{++} : (\gamma'', \alpha'') \succ (\gamma, \alpha)\} \text{ and } \Gamma_2 = \{\gamma'' \in \mathbb{Q}_{++} : (\gamma', \alpha') \succ (\gamma'', \alpha'')\}$$

If  $\gamma'' \in \Gamma_1 \cap \Gamma_2$ , then

$$(\gamma'', \alpha'') \succ (\gamma, \alpha) \succ (\gamma, \alpha'') \succ (\gamma', \alpha') \succ (\gamma'', \alpha'').$$

By transitivity, as established in Claim 2,  $(\gamma'', \alpha'') \succ (\gamma'', \alpha'')$  which is impossible. Thus,  $\Gamma_1$  and  $\Gamma_2$  are disjoint.

To prove the Claim 7, suppose, by way of contradiction, that  $\Gamma_1$  and  $\Gamma_2$  partition  $\mathbb{Q}_{++}$ . Let  $\{\tilde{\gamma}_n\}$  denote a sequence with elements in  $\mathbb{Q}_{++}$  that converges to  $\gamma$ . Then,  $\{\tilde{\gamma}_n\}$  contains a convergent sequence, denoted by  $\{\gamma_n\}$ , with elements fully contained in either  $\Gamma_1$  or  $\Gamma_2$  that converges to  $\gamma$ .

Case 1:  $\gamma_n \in \Gamma_1$  for each  $n$ . Then,  $(\gamma_n, \alpha'') \succ (\gamma, \alpha)$  for each  $n$ . By Claim 5 (ii), for each  $n$ ,

$$\phi_j(\gamma_n, \gamma, \alpha'' + \alpha) < \alpha < \alpha'' < \phi_i(\gamma_n, \gamma, \alpha'' + \alpha).$$

Thus, by continuity,  $\phi_j(\gamma, \gamma, \alpha'' + \alpha) \leq \alpha < \alpha'' \leq \phi_i(\gamma, \gamma, \alpha'' + \alpha)$ . This contradicts ETE and thus rules-out Case 1.

Case 2:  $\gamma_n \in \Gamma_2$  for each  $n$ . Then,  $(\gamma', \alpha') \succ (\gamma_n, \alpha'')$  for each  $n$ . By Claim 5 (ii), for each  $n$ ,

$$\alpha' < \phi_i(\gamma', \gamma_n, \alpha'' + \alpha') \text{ and } \alpha'' > \phi_j(\gamma', \gamma_n, \alpha'' + \alpha').$$

Thus, by continuity,  $\alpha' \leq \phi_i(\gamma', \gamma, \alpha'' + \alpha')$  and  $\alpha'' \geq \phi_j(\gamma', \gamma, \alpha'' + \alpha')$ . By Lemma 4,  $\phi_i(\gamma, \gamma', \cdot) = \phi_j(\gamma', \gamma, \cdot)$ . Thus, by Claim 5 (iv),  $(\gamma, \alpha'') \succ (\gamma', \alpha')$  implies

$$\alpha' > \phi_i(\gamma', \gamma, \alpha'' + \alpha') \text{ and } \alpha'' < \phi_j(\gamma', \gamma, \alpha'' + \alpha').$$

This contradiction rules-out Case 2.

Therefore,  $\Gamma_1$  and  $\Gamma_2$  do not partition  $\mathbb{Q}_{++}$ . This fact proves Claim 7.

**Claim 8.** *There is a function  $U : \mathbb{R}_{++} \times \mathbb{R}_+ \rightarrow (0, 1)$  that represents  $\succsim$  and is strictly decreasing in its second argument.*

By Claim 2,  $\succsim$  is complete and transitive. By Claim 7, has a countable  $\succsim$ -dense subset, namely  $\mathbb{Q}_{++} \times \mathbb{Q}_+$ . Thus, there exists a  $V : \mathbb{R}_{++} \times \mathbb{R}_+ \rightarrow \mathbb{R}$  that represents  $\succsim$  (e.g., see Chapter 3 in Fishburn, 1970). We can take a continuous resource-monotone transformation  $f$  to ensure that  $U = f \circ V$  has a range of values strictly between zero and one (e.g.,  $f(x) = \frac{e^x}{1+e^x}$ ) while also representing  $\succsim$ . By Claim 1,  $U$  is strictly decreasing in its second argument.

**Claim 9.** *Let  $U$  denote the function identified in Claim 8 and let  $u : \mathbb{R}_+ \times \mathbb{R}_{++} \rightarrow \mathbb{R}$  denote the function such that, for each  $(\alpha, \gamma) \in \mathbb{R}_+ \times \mathbb{R}_{++}$ ,*

$$u(\alpha, \gamma) = \int_0^\alpha U(\gamma, t) \, dt.$$

*Then,  $u$  is well-defined, continuous, and strictly increasing as well as strictly concave on its first variable.*

By Theorem 6.9 in Rudin (1976), for each  $(\alpha, \gamma) \in \mathbb{R}_+ \times \mathbb{R}_{++}$ ,  $U(\gamma, \cdot)$  is Riemann-integrable on  $[0, \alpha]$  since  $U(\gamma, \cdot)$  is bounded and decreasing. It is therefore also Lebesgue-integrable. Thus,  $u(\alpha, \gamma)$  is well defined.

We now prove that, for each  $\gamma \in \mathbb{R}_{++}$ ,  $u(\cdot, \gamma)$  is continuous: Let  $\varepsilon > 0$  and  $\delta \in (0, \varepsilon/U(\gamma, 0)]$ . Then, for all  $\alpha, \beta \in \mathbb{R}_+$  such that  $|\beta - \alpha| < \delta$ ,

$$|u(\beta, \gamma) - u(\alpha, \gamma)| = \left| \int_\alpha^\beta U(\gamma, t) \, dt \right| \leq U(\gamma, 0)|\beta - \alpha| \leq \varepsilon,$$

where the first inequality follows because  $U(\gamma, 0) \geq U(\gamma, \cdot) > 0$ . Thus,  $u(\cdot, \gamma)$  is continuous.

The strict concavity of  $u(\cdot, \gamma)$  follows from the fact that  $U(\gamma, \cdot)$  is strictly decreasing. The fact that  $u(\cdot, \gamma)$  is strictly increasing follows from the fact that  $U(\gamma, \cdot)$  is strictly decreasing and takes only positive values.<sup>5</sup>

We now show that  $u$  is continuous. We already established that  $u$  is continuous on its first variable. We will show that it is also continuous on its second variable. For this purpose, we will show that, given  $\gamma \in \mathbb{R}_+$ , a sequence  $\{\gamma_n\}$  converging to  $\gamma$ , the sequence  $\{U(\gamma_n, \alpha)\}$  converges to  $U(\gamma, \alpha)$  for all but possibly countably many  $\alpha \in \mathbb{R}_+$ . The argument will then be concluded by applying the Lebesgue Dominated Convergence Theorem.

Let  $\gamma \in \mathbb{R}_{++}$  and suppose that  $U(\gamma, \cdot)$  is continuous at  $\alpha \in \mathbb{R}_+$ . (Recall that since  $U(\gamma, \cdot)$  is decreasing, it is continuous almost everywhere on its domain.) Let  $\{\gamma_n\}$  denote a sequence in  $\mathbb{R}_{++}$  with limit  $\gamma$ .

Let  $n$  denote a positive integer. By Claim 3,  $\phi_i(\gamma_n, \gamma, G(\gamma, \gamma_n, \alpha)) = \alpha$ . Thus,  $\phi_j(\gamma_n, \gamma, G(\gamma, \gamma_n, \alpha)) = G(\gamma, \gamma_n, \alpha) - \alpha$ . By Claim 5 and since  $U$  represents  $\succsim$ ,

$$U(\gamma, G(\gamma, \gamma_n, \alpha) - \alpha) \geq U(\gamma_n, \alpha).$$

Similarly,  $\phi_j(\gamma, \gamma_n, G(\gamma_n, \gamma, \alpha)) = \alpha$  implies  $\phi_i(\gamma, \gamma_n, G(\gamma_n, \gamma, \alpha)) = G(\gamma_n, \gamma, \alpha) - \alpha$ . By Claim 5 and since  $U$  represents  $\succsim$ ,

$$U(\gamma_n, \alpha) \geq U(\gamma, G(\gamma_n, \gamma, \alpha) - \alpha).$$

Thus,

$$U(\gamma, G(\gamma, \gamma_n, \alpha) - \alpha) \geq U(\gamma_n, \alpha) \geq U(\gamma, G(\gamma_n, \gamma, \alpha) - \alpha). \quad (3)$$

As  $\gamma_n \rightarrow \gamma$ ,  $G(\gamma, \gamma_n, \alpha) \rightarrow 2\alpha$  and  $G(\gamma_n, \gamma, \alpha) \rightarrow 2\alpha$ . Since  $U(\gamma, \cdot)$  is continuous at  $\alpha$ , (3) implies that  $U(\gamma_n, \alpha) \rightarrow U(\gamma, \alpha)$ .

We have thus proven that  $U(\gamma_n, \alpha) \rightarrow U(\gamma, \alpha)$  whenever  $U(\gamma, \cdot)$  is continuous at  $\alpha$ . Since  $U(\gamma, \cdot)$  is resource-monotone, this holds for all but countably many  $\alpha \in \mathbb{R}$ . Moreover,  $U(\gamma, \beta) \in [0, 1]$  for all  $(\gamma, \beta) \in \mathbb{R}_{++} \times \mathbb{R}_+$ . Thus, by the Lebesgue Dominated Convergence Theorem, for each  $(\gamma, \beta) \in \mathbb{R}_{++} \times \mathbb{R}_+$  and each sequence  $\{\gamma_n\}$  converging to  $\gamma$ ,

$$u(\beta, \gamma_n) = \int_0^\beta U(\gamma_n, t) dt \rightarrow \int_0^\beta U(\gamma, t) dt = u(\beta, \gamma). \quad (4)$$

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<sup>5</sup>Let  $\alpha, \alpha' \in \mathbb{R}_+$  be such that  $\alpha < \alpha'$ . Then,  $U(\gamma, \alpha) > U(\gamma, \alpha') > 0$ . Then,  $u(\alpha', \gamma) - u(\alpha, \gamma) = \int_\alpha^{\alpha'} U(\gamma, t) dt \geq U(\gamma, \alpha')(\alpha' - \alpha) > 0$ . Thus,  $u(\cdot, \gamma)$  is strictly increasing.

To prove full continuity, let  $\{(\alpha_n, \gamma_n)\}$  denote a sequence in  $\mathbb{R}_+ \times \mathbb{R}_{++}$  that has a limit  $(\alpha, \gamma)$  in  $\mathbb{R}_+ \times \mathbb{R}_{++}$ . Then, for each  $n$ ,

$$|u(\alpha_n, \gamma_n) - u(\alpha, \gamma)| \leq |u(\alpha_n, \gamma_n) - u(\alpha_n, \gamma)| + |u(\alpha_n, \gamma) - u(\alpha, \gamma)|.$$

Since  $u$  is continuous on its first variable,  $u(\alpha_n, \gamma) \rightarrow u(\alpha, \gamma)$ . By (4),  $u(\alpha_n, \gamma_n) \rightarrow u(\alpha_n, \gamma)$  as  $\gamma_n \rightarrow \gamma$ . Thus,  $u(\alpha_n, \gamma_n) \rightarrow u(\alpha, \gamma)$ . Thus,  $u$  is continuous.

**Claim 10.** *Let  $u$  denote the function identified in Claim 9. Then, for all  $(\gamma, \gamma', \alpha)$  in  $\mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_+$  and all  $x \in X(\{i, j\}, \alpha)$  with  $x \neq \phi(\gamma, \gamma', \alpha)$ ,*

$$u(\phi_i(\gamma, \gamma', \alpha), \gamma) + u(\phi_j(\gamma, \gamma', \alpha), \gamma') > u(x_i, \gamma) + u(x_j, \gamma').$$

Let  $v = u(\cdot, \gamma)$  and  $w = u(\cdot, \gamma')$ . Then the mapping  $x \in \mathbb{R}_+^{\{i, j\}} \mapsto v(x_i) + w(x_j)$  is strictly concave and continuous by Claim 9. Thus, it has a unique maximizer in the compact and convex set  $X(\{i, j\}, \alpha)$ .

Before we continue, we introduce some notation: Given  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ , for each  $x \in \mathbb{R}_+$ ,  $\partial_+ f(x)$  denotes the right-hand derivative of  $f$  at  $x$  and  $\partial_- f(x)$  denotes the left-hand derivative of  $f$  at  $x$ .

Applying Theorem 8.1 in Fujishige (1991), the following conditions are equivalent:

$$z = \arg \max \{v(x_i) + w(x_j) : x \in X(\{i, j\}, \alpha)\} \quad (5)$$

$$[z_j > 0 \text{ implies } \partial_- w(z_j) \geq \partial_+ v(z_i)] \text{ and } [z_i > 0 \text{ implies } \partial_- v(z_i) \geq \partial_+ w(z_j)] \quad (6)$$

By Theorem 24.2 in Rockafellar (1970),

$$\begin{aligned} \partial_- w(z_j) &= \lim_{x_j \uparrow z_j} U(\gamma', x_j), \quad \partial_+ w(z_j) = \lim_{x_j \downarrow z_j} U(\gamma', x_j) \\ \partial_- v(z_i) &= \lim_{x_i \uparrow z_i} U(\gamma, x_i), \quad \partial_+ v(z_i) = \lim_{x_i \downarrow z_i} U(\gamma, x_i). \end{aligned}$$

Thus, condition (6) and the fact that  $U$  is strictly decreasing in its second variable imply the following conditions:

$$[z_j > 0 \text{ implies } U(\gamma', z_j - \frac{1}{n}) > U(\gamma, z_i) \text{ for each natural number } n > 1/z_j] \quad (7)$$

and

$$[z_i > 0 \text{ implies } U(\gamma, z_i - \frac{1}{n}) > U(\gamma', z_j) \text{ for each natural number } n > 1/z_i] \quad (8)$$

Then, since  $U$  represents  $\succsim$ , condition (7) and Claim 5 imply that

$z_j - \frac{1}{n} < \phi_j(\gamma, \gamma', z_i + z_j - \frac{1}{n})$  and  $z_i > \phi_j(\gamma, \gamma', z_i + z_j - \frac{1}{n})$  for each  $n > 1/z_j$ .

By continuity,

$$z_j \leq \phi_j(\gamma, \gamma', z_i + z_j) \text{ and } z_i \geq \phi_j(\gamma, \gamma', z_i + z_j). \quad (9)$$

Condition (8) similarly implies

$z_i - \frac{1}{n} < \phi_i(\gamma, \gamma', z_i - \frac{1}{n} + z_j)$  and  $z_j > \phi_i(\gamma, \gamma', z_i - \frac{1}{n} + z_j)$  for each  $n > 1/z_i$ .

By continuity,

$$z_i \leq \phi_i(\gamma, \gamma', z_i + z_j) \text{ and } z_j \geq \phi_i(\gamma, \gamma', z_i + z_j). \quad (10)$$

Combining (9) and (10) yields  $z = \phi(\gamma, \gamma', \alpha)$  and thus proves Claim 10.

**Claim 11.** *Let  $u$  denote the function identified in Claim 9 and let  $\psi$  denote a rule such that, for each problem  $(N, c, e)$ ,  $\psi(N, c, e) = \arg \max\{\sum_N u(z_i, c_i) : z \in X(N, e)\}$ . Then,  $\varphi = \psi$ .*

The argument that follows applies the logic of the ‘‘Elevator Lemma’’ (e.g., [Thomson, 2011](#)).

By Claim 10,  $\varphi(\{i, j\}, c, e) = \psi(\{i, j\}, c, e)$  for all  $c \in \mathbb{R}_{++}^{\{i, j\}}$  and  $e \in \mathbb{R}_+$ . Thus, by Lemma 4,

$$\text{for all } k, l \in I, \text{ all } c \in \mathbb{R}_{++}^{\{k, l\}}, \text{ and all } e \in \mathbb{R}_+, \varphi(\{k, l\}, c, e) = \psi(\{k, l\}, c, e). \quad (11)$$

Step 1:  $\psi$  is consistent.

Consider a problem  $(N, c, e)$  where  $|N| \geq 3$  and let  $x = \psi(N, c, e)$ . Suppose, by way of contradiction, that there are  $k, l \in N$  such that  $\psi(\{k, l\}, c_{\{k, l\}}, x_k + x_l) = y \neq x_{\{k, l\}}$ . Then,  $u(y_k, c_k) + u(y_l, c_l) > u(x_k, c_k) + u(x_l, c_l)$  and  $y_k + y_l = x_k + x_l$ . Let  $z \in X(N, e)$  be such that  $z_k = y_k$ ,  $z_l = y_l$ , and  $z_h = x_h$  for all  $h \in N$  with  $h \neq k, l$ . Then,  $\sum_N u(z_h, c_h) > \sum_N u(x_h, c_h)$ . This contradicts the definition of  $\psi(N, c, e)$ . Thus,  $\psi$  is consistent.

Step 2:  $\psi$  is conversely consistent, i.e., for each  $(N, c, e)$  and each  $x \in X(N, e)$ , if  $\psi(\{k, l\}, c_{\{k, l\}}, x_k + x_l) = x_{\{k, l\}}$  for all  $k, l \in N$ , then  $\psi(N, c, e) = x$ .

Let  $(N, c, e)$  and  $x \in X(N, e)$  be such that  $\psi(\{k, l\}, c_{\{k, l\}}, x_k + x_l) = x_{\{k, l\}}$  for all  $k, l \in N$ . Let  $y = \psi(N, c, e)$  and suppose, by way of contradiction, that  $x \neq y$ . Thus, since  $\sum_N x_h = \sum_N y_h$ , we can then fix  $k, l \in N$  such that  $x_k > y_k$  and  $x_l < y_l$ . However, by resource-monotonicity, if  $x_k + x_l \geq y_k + y_l$ , then

$$x_{\{k,l\}} = \psi(\{k,l\}, c_{\{k,l\}}, x_k + x_l) \geq \psi(\{k,l\}, c_{\{k,l\}}, y_k + y_l) = y_{\{k,l\}},$$

and, if  $x_k + x_l \leq y_k + y_l$ , then

$$x_{\{k,l\}} = \psi(\{k,l\}, c_{\{k,l\}}, x_k + x_l) \leq \psi(\{k,l\}, c_{\{k,l\}}, y_k + y_l) = y_{\{k,l\}}.$$

Thus, in fact,  $x = y$ . Since we proved this for an arbitrary  $(N, c, e)$ ,  $\psi$  is thus conversely consistent.

Concluding the proof of Claim 11: Consider a problem  $(N, c, e)$  and let  $x = \varphi(N, c, e)$ . By (11),  $\varphi(\{k, l\}, c_{\{k,l\}}, x_k + x_l) = \psi(\{k, l\}, c_{\{k,l\}}, x_k + x_l)$  for all  $k, l \in N$ . Thus, by Step 2 above,  $\psi(N, c, e) = x$ . Since  $(N, c, e)$  was chosen arbitrarily,  $\psi = \varphi$ .

**Claim 12.** *Let  $U$  denote the function identified in Claim 8. For all  $\gamma, \eta \in \mathbb{R}_{++}^N$ , the sequences  $\{U(\gamma, n)\}_n$  and  $\{U(\eta, n)\}_n$  are convergent and have a common limit.*

By Claim 8,  $\{U(\gamma, n)\}_n$  and  $\{U(\eta, n)\}_n$  are bounded and decreasing in  $n$ . Thus, they both converge. Let  $\ell_\gamma$  and  $\ell_\eta$  denote their respective limits. If these limits are equal, we are done. Therefore, without loss of generality, suppose that  $\ell_\gamma > \ell_\eta$ . Let  $\varepsilon \in (0, \ell_\gamma - \ell_\eta)$ . Then, we can fix a positive integer  $K$  such that,

$$\text{for each integer } n \geq K, U(\gamma, n) > \ell_\eta + \varepsilon > U(\eta, K).$$

Since  $U$  represents  $\succsim$ , Claim 5 (ii) implies that,

$$\text{for all } n \geq K, n < \phi_i(\gamma, \eta, n + K) \text{ and } K > \phi_j(\gamma, \eta, n + K).$$

Letting  $c \in \mathbb{R}_{++}^{\{i,j\}}$  be such that  $c_i = \gamma$  and  $c_j = \eta$  and applying the definition of  $\phi$ ,

$$\text{for each integer } n \geq K, \varphi_j(\{i, j\}, c, n + K) = \phi_j(\gamma, \eta, n + K) < K.$$

This implies that  $\varphi_j(\{i, j\}, c, \cdot)$  is bounded, contradicting non-stagnation. Thus, both sequences have the same limit, establishing Claim 12.

**Claim 13.** *Let  $u$  denote the function identified in Claim 9. For all  $\gamma \in \mathbb{R}_{++}$ ,*

$$u(n + 1, \gamma) - u(n, \gamma) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let  $U$  denote the function identified in Claim 8. By Claim 12, there is a common limit  $\ell \in \mathbb{R}$  such that, for each  $\gamma \in \mathbb{R}_{++}$ , the sequence  $\{U(\gamma, n)\}_n$  converges to  $\ell$ . Since  $U(\gamma, \cdot)$  is strictly decreasing and has a range in  $(0, 1)$ , for each positive integer  $n$ ,  $U(\gamma, n) > \ell \geq 0$ . If  $\ell > 0$ , we can replace  $U$  by  $U - \ell$  in all of our above proofs

since the resulting function will be a translation with the same domain and range. We then have,

$$\text{for each } \gamma \in \mathbb{R}_{++}, U(\gamma, n) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (12)$$

Note that, for each  $\gamma \in \mathbb{R}_{++}$  and each positive integer  $n$ ,

$$\begin{aligned} 0 < u(n+1, \gamma) - u(n, \gamma) &= \int_n^{n+1} U(\gamma, t) \, dt \\ &< U(\gamma, n)[(n+1) - n] = U(\gamma, n). \end{aligned}$$

where the first inequality follows because  $u$  is strictly increasing in its first argument, the first equality follows from the definition of  $u$ , and the second inequality follows from the fact that  $U(\gamma, \cdot)$  is strictly decreasing. By (12),  $u(n+1, \gamma) - u(n, \gamma) \rightarrow 0$  as  $n \rightarrow \infty$ .

**Claim 14.** *Let  $\rho$  denote a rule that maximizes separable welfare function with diminishing individual marginal returns. Then  $\rho$  satisfies ETE, consistency, continuity, and non-stagnation.*

By Definition 1, there is a continuous  $v : \mathbb{R}_+ \times \mathbb{R}_{++} \rightarrow \mathbb{R}$  such that

- (i)  $v$  is strictly concave and strictly increasing over its first variable,
- (ii) for each problem  $(N, c, e)$ ,

$$\rho(N, c, e) = \arg \max \{ \sum_N v(z_i, c_i) : z \in X(N, e) \},$$

- (iii) for all  $\gamma \in \mathbb{R}_{++}$ ,  $v(n+1, \gamma) - v(n, \gamma) \rightarrow 0$  as  $n \rightarrow \infty$ .

The facts that  $\rho$  satisfies ETE and is continuous are obvious. The proof that  $\rho$  is consistent is identical to that for rule  $\psi$  in the proof of Claim 11 (Step 1).

It remains to show that  $\rho$  satisfies non-stagnation. Suppose otherwise. Then, there are  $N \subseteq I$ ,  $c \in \mathbb{R}_{++}^N$ , and  $i \in N$  such that  $\rho_i(N, c, \cdot)$  is bounded. Thus,  $r_i \equiv \sup_{e \in \mathbb{R}_+} \rho_i(N, c, e) < \infty$ . Since  $v$  is strictly increasing over its first argument  $\Delta \equiv v(r_i + 1, c_i) - v(r_i, c_i) > 0$ . Since  $e \rightarrow \infty$ , feasibility implies that we can fix a  $j \in N$  such that  $\rho_j(N, c, e) \rightarrow \infty$  as  $e \rightarrow \infty$ . Since  $\rho_j(N, c, \cdot)$  is continuous (Lemma 1), resource-monotonic, and non-stagnant, for each positive integer  $n$ , there is  $e \in \mathbb{R}_+$  such that  $\rho_j(N, c, e) = n + 1$ . By property (iii) of  $v$  above, there is a large enough  $e^* \in \mathbb{R}_+$  such that  $\rho_j(N, c, e^*) = n + 1$  where the positive integer  $n$  satisfies  $v(n+1, c_j) - v(n, c_j) < \Delta$ . Let  $x = \rho(N, c, e^*)$  so that  $x_i \leq r_i$  and  $x_j = n + 1$ . Since  $v$  is concave in its first argument,  $v(x_i + 1, c_i) - v(x_i, c_i) \geq v(r_i + 1, c_i) - v(r_i, c_i) = \Delta$ .

Then,

$$v(x_i + 1, c_i) + v(x_j - 1, c_j) + \sum_{N \setminus \{i,j\}} v(x_k, c_k) > \sum_N v(x_k, c_k)$$

since increasing  $x_i$  by 1 increases the objective by at least  $\Delta$  and decreasing  $x_j$  decreases the objective by less than  $\Delta$ . Since this transfer is feasible, this contradicts property (ii) of  $v$  above. Thus,  $\rho$  satisfies non-stagnation.

## C Proof of Theorem 2

Theorem 2 is a consequence of the two lemmas proven here.

**Notation:** For each  $i \in I$ ,  $\mathbf{e}_i \in \mathbb{R}^I$  denotes the  $i$ th standard basis vector, i.e., the vector with a one in the  $i$ th coordinate and zeros elsewhere. Given a function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ , for each  $x \in \mathbb{R}$ ,  $\partial_+ f(x)$  and  $\partial_- f(x)$  denote the right hand and left hand derivatives of  $f$  at  $x$ , respectively.

**Lemma 6.** *For each  $i \in I$ , let  $u_i : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}$  denote a function that is continuous and strictly concave on its first argument. Define a rule  $\varphi$  to be such that, for each problem  $(N, c, e)$ ,*

$$\varphi(N, c, e) = \arg \max \{ \sum_{i \in N} u_i(z_i, c) : z \in X(N, e) \}.$$

*Then,  $\varphi$  is resource-monotonic.*

*Proof.* For each  $i \in I$  and each  $c \in \mathcal{C}$ , let  $F_i(\cdot) = -u_i(\cdot, c)$ . Then, each  $F_i$  is continuous and strictly convex. Thus,  $\varphi(N, c, e)$  can equivalently be obtained by minimizing  $\sum_{i \in N} F_i$  over  $X(N, e)$ . Let  $x = \varphi(N, c, e)$ , let  $e' \in \mathbb{R}_+$  be such that  $e < e'$ , and let  $y = \varphi(N, c, e')$ . We will prove that  $x_i \leq y_i$  for each  $i \in N$ .

Suppose, by way of contradiction, that there is  $k \in N$  with  $y_k < x_k$ . Then,

$$\sum_N x_i = e < e' = \sum_N y_i \text{ implies that there is } j \in N \text{ with } x_j < y_j.$$

Now,  $0 \leq x_j < y_j$  implies that  $y + \varepsilon(\mathbf{e}_k - \mathbf{e}_j) \in X(N, e')$  for each  $\varepsilon \in (0, y_j]$ . Thus,

$$y = \arg \min \{ \sum_N F_i(z_i) : z \in X(N, e') \} \text{ implies } \partial_- F_j(y_j) \leq \partial_+ F_k(y_k).^6$$

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<sup>6</sup>Otherwise, there is  $\varepsilon \in (0, y_j]$  with  $F_j(y_j - \varepsilon) + F_k(y_k + \varepsilon) + \sum_{N \setminus \{j,k\}} F_i(y_i) < \sum_N F_i(y_i)$ , contradicting  $y = \arg \min \{ \sum_N F_i(z_i) : z \in X(N, e') \}$ .

Since  $F_j$  and  $F_k$  are strictly convex,  $\partial_+ F_j(x_j) < \partial_- F_j(y_j)$  and  $\partial_+ F_k(y_k) < \partial_- F_k(x_k)$ . Thus,

$$\partial_+ F_j(x_j) < \partial_- F_j(y_j) \leq \partial_+ F_k(y_k) < \partial_- F_k(x_k). \quad (13)$$

However,  $0 \leq y_k < x_k$  implies that  $x + \varepsilon(\mathbf{e}_j - \mathbf{e}_k) \in X(N, e)$  for each  $\varepsilon \in (0, x_k]$ , and thus,

$$x = \arg \min \{ \sum_N F_i(z_i) : z \in X(N, e) \} \text{ implies } \partial_- F_k(x_k) \leq \partial_+ F_j(x_j).^7$$

This last inequality contradicts (13). Thus, in fact,  $x_i \leq y_i$  for each  $i \in N$ , as desired.  $\square$

**Lemma 7.** *Suppose that  $\varphi$  satisfies resource-monotonicity and non-stagnation. Then, for each  $i \in I$ , there is a function  $u_i : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}$  that is continuous and strictly concave over its first argument and such that, for each problem  $(N, c, e)$ ,*

$$\varphi(N, c, e) = \arg \max \{ \sum_{i \in N} u_i(z_i, c) : z \in X(N, e) \}.$$

*Proof.* Let  $N \subseteq I$ , let  $c \in \mathbb{R}_{++}^N$ , and let  $g_i(\cdot) = \varphi_i(N, c, \cdot)$  for each  $i \in N$ . By resource-monotonicity and Lemma 1 and non-stagnation, each  $g_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is non-decreasing, continuous, and onto. Thus, for each  $i \in N$ , we can define  $h_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  to be such that, for each  $\alpha \in \mathbb{R}_+$ ,  $h_i(\alpha) = \sup \{ e \in \mathbb{R}_+ : g_i(e) = \alpha \}$ , i.e.,  $h_i$  is a pseudo-inverse of  $g_i$ . Thus, each  $h_i$  is strictly increasing and satisfies

$$\begin{aligned} \text{for each } e \in \mathbb{R}_+, \quad & g_i(e) = 0 \Rightarrow e \in [0, h_i^+(g_i(e))] \text{ and} \\ & g_i(e) > 0 \Rightarrow e \in [h_i^-(g_i(e)), h_i^+(g_i(e))] \end{aligned} \quad (14)$$

where, for each  $\alpha \in \mathbb{R}_{++}$ ,

$$h_i^-(\alpha) = \lim_{\beta \uparrow \alpha} h_i(\beta),$$

and, for each  $\alpha \in \mathbb{R}_+$ ,

$$h_i^+(\alpha) = \lim_{\beta \downarrow \alpha} h_i(\beta).$$

By Theorem 6.9 in Rudin (1976), for each  $\alpha \in \mathbb{R}_+$ ,  $h_i$  is Riemann-integrable on  $[0, \alpha]$ . For each  $\alpha \in \mathbb{R}_+$ , let  $f_i(\alpha)$  denote the Riemann integral  $\int_0^\alpha h_i(t) dt$ .

We now prove that each  $f_i : \mathbb{R}_+ \rightarrow \mathbb{R}$  thus defined is continuous: Let  $\alpha \in \mathbb{R}_+$  and  $\varepsilon > 0$ . Since  $h_i$  is strictly increasing and injective,  $h_i(\alpha + 1) > 0$ . Let  $\delta \in (0, 1)$  be such that  $\delta < \frac{\varepsilon}{h_i(\alpha+1)}$ . Then, for each  $\beta \in (\alpha - \delta, \alpha + \delta)$ ,

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<sup>7</sup>Otherwise, there is  $\varepsilon \in (0, x_k]$  with  $F_j(x_j + \varepsilon) + F_k(x_k - \varepsilon) + \sum_{N \setminus \{j, k\}} F_i(x_i) < \sum_N F_i(x_i)$ , contradicting  $x = \arg \min \{ \sum_N F_i(z_i) : z \in X(N, e) \}$ .

$$|f_i(\beta) - f_i(\alpha)| = \left| \int_{\alpha}^{\beta} h_i(t) dt \right| < |\alpha - \beta| h_i(\alpha + 1) < \varepsilon.$$

where the first inequality follows from the fact that  $h_i(\gamma) < h_i(\alpha + 1)$  for each  $\gamma$  in the interval with endpoints  $\alpha$  and  $\beta$ . Thus,  $f_i$  is continuous at  $\alpha \in \mathbb{R}_+$ . Since  $\alpha$  was chosen arbitrarily,  $f_i$  is continuous on its domain. Additionally, because  $h_i$  is strictly increasing,  $f_i$  is strictly convex.

Since each  $f_i$  is continuous and strictly convex and, for each  $e \in \mathbb{R}_+$ ,  $X(N, e)$  is compact, the problem of minimizing  $\sum_N f_i$  over  $X(N, e)$  has a unique solution. Thus, fixing  $e \in \mathbb{R}_+$ , we can define

$$a = \arg \min \{ \sum_N f_i(z_i) : z \in X(N, e) \}.$$

We now prove that

$$a_i = g_i(e) \text{ for each } i \in N. \quad (15)$$

If  $e = 0$ , then  $X(N, e)$  is the singleton containing only the vector with zero coordinates. Thus,  $g_i(0) = 0 = a_i$  for each  $i \in N$  establishing (15).

It remains to consider the case where  $e > 0$ . By way of contradiction, suppose that (15) fails. Since  $\sum_N a_i = e = \sum_N g_i(e)$ , we can fix  $j, k \in N$  such that

$$0 \leq a_j < g_j(e) \text{ and } 0 \leq g_k(e) < a_k. \quad (16)$$

Then,

$$h_j^+(a_j) < h_j^-(g_j(e)) \leq e \leq h_k^+(g_k(e)) < h_k^-(a_k), \quad (17)$$

where the first and last inequalities follow because  $h_j$  and  $h_k$  are strictly increasing and the second and third inequalities follow from (14).

By (16), for each  $\varepsilon \in (0, a_k]$ ,  $a + \varepsilon(\mathbf{e}_j - \mathbf{e}_k) \in X(N, e)$ . Since  $a$  minimizes  $\sum_N f_i$  over  $X(N, e)$ , this requires that  $\partial_+ f_j(a_j) \geq \partial_- f_k(a_k)$ .<sup>8</sup> By the definitions of  $f_j$  and  $f_k$  and by Theorem 24.2 in Rockafellar (1970),  $\partial_+ f_j(a_j) = h_j^+(a_j)$  and  $\partial_- f_k(a_k) = h_k^-(a_k)$ . However, (17) implies  $\partial_+ f_j(a_j) < \partial_- f_k(a_k)$ . This contradiction again establishes (15).

Thus,

$$\text{for each } e \in \mathbb{R}_+, \varphi(N, c, e) = \arg \min \{ \sum_i f_i(z_i) : z \in X(N, e) \}.$$

Finally, for each  $i \in N$ , let  $u_i(\cdot, c) = -f_i$  and, for each  $i \in I \setminus N$ , let  $u_i(\cdot, c) : \mathbb{R}_+ \rightarrow \mathbb{R}$  be any strictly concave and continuous function. The  $u_i : \mathbb{R}_+ \times \mathcal{C} \rightarrow \mathbb{R}$  functions thus defined for each  $i \in I$  are then as claimed in Lemma 7.  $\square$

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<sup>8</sup>Otherwise, there is  $\varepsilon \in (0, a_k]$  with  $f_j(a_j + \varepsilon) + f_k(a_k - \varepsilon) + \sum_{N \setminus \{j, k\}} f_i(a_i) < \sum_N f_i(a_i)$ . This would contradict the definition of  $a$ .

## D Proof of Theorem 3

Every weighted proportional rule obviously satisfies ETE, consistency, continuity, non-stagnation, and additivity. Conversely, let  $\varphi$  denote a rule satisfying these axioms. We will show that  $\varphi$  is a weighted proportional rule.

**Claim 1.** For all  $(N, c, e)$  and all  $i \in N$ ,  $\varphi_i(N, c, e) = \varphi_i(N, c, 1)e$  and  $\varphi_i(N, c, 1) > 0$ .

Let  $N$  denote a finite subset of  $I$  and  $c \in \mathbb{R}_{++}^N$ . For each  $e \in \mathbb{R}_+$  and each  $i \in N$ , let  $f_i(e) = \varphi_i(N, c, e)$  and note that  $f_i$  is continuous by the continuity of  $\varphi$ . By feasibility,  $f_i \geq 0$ . By non-stagnation,  $f_i$  is non-constant. By additivity,  $f_i(e + e') = f_i(e) + f_i(e')$  for all  $e, e' \in \mathbb{R}_+$ . This is Cauchy's functional equation and since  $f_i$  is continuous, non-negative, and non-constant, there exist a constant  $a_i \in \mathbb{R}_{++}$  such that, for all  $e \in \mathbb{R}_+$ ,  $f_i(e) = a_i e$ . Thus,  $\varphi_i(N, c, 1) = f_i(1) = a_i > 0$ . Thus,

$$\varphi_i(N, c, e) = f_i(e) = a_i e = \varphi_i(N, c, 1)e.$$

**Claim 2.** For each finite  $N \subseteq I$ , each  $c \in \mathbb{R}_{++}^N$ , each  $M \subseteq N$ , and each  $i \in M$ ,

$$\varphi_i(M, c_M, 1) = \frac{\varphi_i(N, c, 1)}{\sum_{j \in M} \varphi_j(N, c, 1)}.$$

Consider a problem  $(N, c, e)$ , let  $x = \varphi(N, c, e)$ , and let  $M \subseteq N$ . By consistency,

$$\text{for each } i \in M, \varphi_i(M, c_M, e - \sum_{j \in N \setminus M} x_j) = x_i.$$

By Claim 1, for each  $j \in N$ ,  $x_j = \varphi_j(N, c, 1)e$ , and, for each  $i \in M$ ,

$$\varphi_i(M, c_M, e - \sum_{j \in N \setminus M} x_j) = \varphi_i(M, c_M, 1) \left[ e - \sum_{j \in N \setminus M} \varphi_j(N, c, 1)e \right].$$

Thus, for each  $i \in M$ ,

$$\begin{aligned} \varphi_i(N, c, 1)e &= \varphi_i(M, c_M, e - \sum_{j \in N \setminus M} x_j) \\ &= \varphi_i(M, c_M, 1) \left[ e - \sum_{j \in N \setminus M} \varphi_j(N, c, 1)e \right] \\ &= \varphi_i(M, c_M, 1) \sum_{j \in M} \varphi_j(N, c, 1)e. \end{aligned}$$

Solving for  $\varphi_i(M, c_M, 1)$  then concludes the argument.

**Claim 3.** There is a continuous function  $w : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$  such that, for each problem  $(N, c, e)$  and each  $i \in N$ ,  $\varphi_i(N, c, e) = \frac{w(c_i)}{\sum_{j \in N} w(c_j)} e$ .

Let  $k, l \in I$  be such that  $k \neq l$ . For each  $\gamma \in \mathbb{R}_{++}$ , let  $c^\gamma \in \mathbb{R}_{++}^{\{k, l\}}$  be such that  $c_k^\gamma = \gamma$  and  $c_l^\gamma = 1$ . Then,

$$\text{for each } \gamma \in \mathbb{R}_{++}, \text{ define } w(\gamma) \equiv \varphi_k(\{k, l\}, c^\gamma, 1).$$

By the continuity of  $\varphi$ , the function  $w$  thus defined is continuous. By Claim 1, for all  $\gamma \in \mathbb{R}_{++}$ ,  $w(\gamma) > 0$ .

By Lemma 4, for all  $i, j \in I$  such that  $i \neq j$  and all  $c \in \mathbb{R}_{++}^{\{i, j\}}$  such that  $c_j = 1$ ,  $\varphi_k(\{k, l\}, c^{c_i}, 1) = \varphi_i(\{i, j\}, c, 1)$ . Thus,

$$\text{for all } i, j \in I \text{ and all } c \in \mathbb{R}_{++}^{\{i, j\}} \text{ with } c_j = 1, \quad \varphi_i(\{i, j\}, c, 1) = w(c_i). \quad (18)$$

Let  $N$  denote a finite subset of  $I$  and  $c \in \mathbb{R}_{++}^N$ . Let  $j \in I \setminus N$  and let  $c' \in \mathbb{R}_{++}^{N \cup \{j\}}$  be such that  $c'_N = c$  and  $c'_j = 1$ . By Claim 2,

$$\begin{aligned} \text{for each } i \in N, \quad \varphi_i(N, c, 1) &= \frac{\varphi_i(N \cup \{j\}, c', 1)}{\sum_{k \in N} \varphi_k(N \cup \{j\}, c', 1)}, \\ \varphi_i(\{i, j\}, c_{\{i, j\}}, 1) &= \frac{\varphi_i(N \cup \{j\}, c', 1)}{\varphi_i(N \cup \{j\}, c', 1) + \varphi_j(N \cup \{j\}, c', 1)}. \end{aligned}$$

Thus,

$$\text{for all } i, h \in N, \quad \frac{\varphi_i(N, c, 1)}{\varphi_h(N, c, 1)} = \frac{\varphi_i(N \cup \{j\}, c', 1)}{\varphi_h(N \cup \{j\}, c', 1)} = \frac{\varphi_i(\{i, j\}, c_{\{i, j\}}, 1)}{\varphi_h(\{h, j\}, c_{\{h, j\}}, 1)}.$$

Thus, by (18),

$$\text{for all } i, h \in N, \quad \frac{\varphi_i(N, c, 1)}{\varphi_h(N, c, 1)} = \frac{w(c_i)}{w(c_h)}.$$

Let  $h \in N$  and add over  $i \in N$  above to obtain:

$$\frac{\sum_{i \in N} \varphi_i(N, c, 1)}{\varphi_h(N, c, 1)} = \frac{\sum_{i \in N} w(c_i)}{w(c_h)}.$$

By feasibility  $\sum_{i \in N} \varphi_i(N, c, 1) = 1$  and, upon rearranging,  $\varphi_h(N, c, 1) = \frac{w(c_h)}{\sum_{i \in N} w(c_i)}$ . Thus, we have proven that

$$\text{for all finite } N \subseteq I, \text{ all } c \in \mathbb{R}_{++}^N, \text{ and all } h \in N, \quad \varphi_h(N, c, 1) = \frac{w(c_h)}{\sum_{i \in N} w(c_i)}. \quad (19)$$

To conclude, take an arbitrary problem  $(N, c, e)$ . By Claim 1, for each  $h \in N$ ,  $\varphi_h(N, c, e) = \varphi_h(N, c, 1)e$ . By (19), for each  $h \in N$ ,  $\varphi_h(N, c, e) = \frac{w(c_h)}{\sum_{i \in N} w(c_i)} e$ .  $\square$

## E Proof of Theorem 4

Let  $\varphi$  denote a rule satisfying all of the above axioms. By Theorem 3,  $\varphi$  is a weighted proportional rule specified by a continuous weighting function that we will denote by  $w : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ . Then, for each  $(N, c, e)$ , each  $i \in N$ , and each  $\lambda \in \mathbb{R}_{++}$ ,

$$\lambda \frac{w(c_i)}{\sum_{j \in N} w(c_j)} e = \lambda \varphi_i(N, c, e) = \varphi_i(N, \lambda c, \lambda e) = \frac{w(\lambda c_i)}{\sum_{j \in N} w(\lambda c_j)} \lambda e$$

where the first and last equalities follow by the definition of a weighted proportional rule specified by  $w$  and the second equality follows from homogeneity. Thus, for all  $i \in N$ ,  $\frac{w(\lambda c_i)}{w(c_i)} = \frac{\sum_{j \in N} w(\lambda c_j)}{\sum_{j \in N} w(c_j)}$ . (The proof hereafter follows that of Theorem 2 in Skaperdas (1996) closely.) Therefore, for all  $i, k \in N$ ,  $\frac{w(\lambda c_i)}{w(c_i)} = \frac{\sum_{j \in N} w(\lambda c_j)}{\sum_{j \in N} w(c_j)} = \frac{w(\lambda c_k)}{w(c_k)}$ . Since this argument can be repeated for any profile of claims  $c$  and any  $\lambda > 0$ , we obtain:

$$\text{for all } \gamma, \gamma', \lambda \in \mathbb{R}_{++}, \quad \frac{w(\lambda \gamma)}{w(\gamma)} = \frac{w(\lambda \gamma')}{w(\gamma')}.$$

Letting  $\gamma' = 1$  above,  $\frac{w(\lambda \gamma)}{w(\gamma)} = \frac{w(\lambda)}{w(1)}$ . Equivalently,  $w(\lambda \gamma) = \frac{w(\lambda)}{w(1)} w(\gamma)$ . Dividing both sides of this equation by  $w(1)$ , we obtain:

$$\text{for all } \gamma, \lambda \in \mathbb{R}_{++}, \quad \frac{w(\lambda \gamma)}{w(1)} = \frac{w(\lambda)}{w(1)} \frac{w(\gamma)}{w(1)}. \quad (20)$$

Define  $F : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$  by  $F(\gamma) = \frac{w(\gamma)}{w(1)}$  for each  $\gamma \in \mathbb{R}_{++}$ . Then, (20) can be rewritten as

$$\text{for all } \gamma, \lambda \in \mathbb{R}_{++}, \quad F(\lambda \gamma) = F(\lambda) F(\gamma).$$

Moreover, since  $w$  is continuous,  $F$  is continuous. This is Cauchy's power functional equation and its only continuous and non-zero solution is given by  $F(t) = t^r$  for a constant  $r \in \mathbb{R}_+$  (e.g., see page 36 in Aczél, 2006). Thus, for each  $\gamma \in \mathbb{R}_{++}$ ,  $w(\gamma) = w(1) F(\gamma) = w(1) \gamma^r$ . Since  $\varphi$  is a weighted proportional rule, for each  $(N, c, e)$ ,

$$\varphi_i(N, c, e) = \frac{w(c_i)}{\sum_{j \in N} w(c_j)} e = \frac{w(1) c_i^r}{\sum_{j \in N} w(1) c_j^r} e = \frac{c_i^r}{\sum_{j \in N} c_j^r} e.$$

That is,  $\varphi$  belongs to the power family. □

## F Proof of Remark 1

It is easy to show that the proportional rule satisfies continuity, additivity, and same-sidedness. Conversely, let  $\varphi$  denote a rule satisfying these properties. Consider a

finite  $N \subseteq I$  and  $c \in \mathbb{R}_{++}^N$ . For each  $e \in \mathbb{R}_+$  and each  $i \in N$ , let  $f_i(e) = \varphi_i(N, c, e)$  and note that  $f_i$  is continuous by the continuity of  $\varphi$ . By feasibility,  $f_i \geq 0$ . By same-sidedness,  $f_i$  is non-constant. By additivity,  $f_i(e+e') = f_i(e) + f_i(e')$  for all  $e, e' \in \mathbb{R}_+$ . This is Cauchy's functional equation and since  $f_i$  is continuous, non-negative, and non-constant, we have that, for all  $e \in \mathbb{R}_+$ ,  $f_i(e) = a_i e$  for a constant  $a_i > 0$ . By feasibility,  $\sum_{j \in N} a_j e = \sum_{j \in N} f_j(e) = e$  and thus  $\sum_{j \in N} a_j = 1$ . By same-sidedness and feasibility,  $a_i e = c_i$  if  $e = \sum_{j \in N} c_j$ . Thus, for each  $i \in N$ ,  $a_i = \frac{c_i}{\sum_{j \in N} c_j}$ . Thus, for each  $i \in N$ ,  $\varphi_i(N, c, e) = f_i(e) = \frac{c_i}{\sum_{j \in N} c_j} e$ . That is,  $\varphi$  is the proportional rule.