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How much can you claim?[☆]

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ABSTRACT

In a "claims problem" (O'Neill, 1982), a group of individuals have claims on a resource but there is not enough of it to honor all of the claims. A widely studied property of distribution rules, "claims truncation invariance", advances the existence of a maximal reasonable claim: the endowment of the resource. We examine the implications of imposing *any* maximal claim, no matter how large. Our conclusions establish the centrality of the "constrained equal awards rule" and its asymmetric generalizations in the class of asymmetric rationing rules (Moulin, 2000).

1. Introduction

When a firm goes bankrupt, how should its liquidation value be divided among its creditors? An extensive literature addresses this question, evaluating the possible division rules on the basis of normative properties.[1] This paper examines one such property, the existence of a maximal reasonable claim on what is available.

What is a reasonable claim? What is reasonable depends on how much is available. If a claim by far exceeds this amount and increases further still, should we, at some point, disregard the increase? If the answer is affirmative, then the excess of a claim over an amount many times greater than what is available should be disregarded. This property, which we call *maximal reasonable claims*, is the subject of this paper. A stronger and extensively studied property is *claims truncation invariance*, requiring that the excess of a claim over the endowment is considered irrelevant. As justified by Aumann and Maschler (1985), "you cannot get more than there is." Claims truncation invariance plays an important role in the literature on bankruptcy and claims problems.[2]

We use *maximum reasonable claims* as a starting point to reassess axiomatic analysis based on claims truncation invariance. We prove that the canonical *constrained equal awards (CEA) rule* is *the only* symmetric rule satisfying maximal reasonable claims and "composition up." Composition up (Moulin, 1987; Young, 1988) specifies that, upon an increase in the endowment, the rule can recommend the distribution in two equivalent ways: (i) Apply the rule directly to distribute the larger endowment. (ii) Apply the rule to distribute the initial endowment and, thereafter, apply it again to allocate the increment according to the outstanding claims. This shows that maximal reasonable claims can replace claims truncation invariance in one of the central characterizations of the CEA rule due to Dagan (1996). In fact, the only rules in the well known class of asymmetric rationing methods (Moulin, 2000) satisfying maximal reasonable claims are those generalizing the constrained equal awards rule. Maximal reasonable claims can also replace claims truncation invariance in the characterization of this asymmetric sub-class of rules (Flores-Szwagrzak, 2015).

2. Definitions and results

An endowment of a divisible resource among a group of claimants drawn from a finite set A. Let $\mathcal{N}$ denote the collection of claimant groups drawn from A. For each $N \in \mathcal{N}$, a **claims problem** is a pair $(c, e) \in \mathbb{R}_+^N \times \mathbb{R}_+$ consisting of a profile of claims c and an endowment e such that $\sum_N c_i \geq e$. For each $N \in \mathcal{N}$, let $\mathcal{P}^N$ denote the class of claims problems involving the claimants in N. An **allocation** for the problem

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[1] See Moulin (2002) and Thomson (2003, 2015, 2019) for surveys.

[2] The axiom enables the game theoretic approach pioneered by O'Neill (1982), as shown by Curiel et al. (1987). Thus, many correspondences between game theoretic solution concepts and rules (as surveyed by Theorem 2 in Thomson, 2003) depend on the rules satisfying the axiom. The axiom has also played a role in the study of the Talmud rule (Dagan, 1996) and its asymmetric generalizations (Hokari and Thomson, 2003). The axiom has also been used in the study of the structure of the space of rules (Thomson and Yeh, 2007; Hougaard et al., 2012).

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$(c,e) \in \mathcal{P}^N$ is a profile $z \in \mathbb{R}^N_+$ such that $\sum_N z_i = e$ and, for each $i \in N$, $z_i \leq c_i$. Let $Z(c,e)$ denote the collection of all allocations for claims problem (c,e). A **rule** is a function φ recommending an allocation for each possible claims problem: for each $N \in \mathcal{N}$ and each $(c,e) \in \mathcal{P}^N$, $\varphi(c,e) \in Z(c,e)$.

One of the central rules in the study of the claims problem is the **constrained equal awards (CEA) rule**, denoted by CEA, that recommends a uniform award across claimants under the constraint that no claimant receives more than her claim: For each $(c,e) \in \mathcal{P}^N$ and each $i \in N$, $CEA_i(c,e) = \min\{c_i, \lambda\}$ where $\lambda \in \mathbb{R}$ is such that $\sum_{i\in N} \min\{c_i, \lambda\} = e$.

We now introduce two classical properties of rules. A basic equity property is that claimants with equal claims receive equal awards:

Symmetry: For each $(c,e) \in \mathcal{P}^N$ and each pair $i,j \in N$ with $c_i = c_j$, $\varphi_i(c,e) = \varphi_j(c,e)$.

Another classical property, "composition up," specifies how a rule responds to endowment increments: Upon an increase, the endowment can be allocated either directly, or by first assigning the awards obtained by applying the rule to the initial endowment, and in a second step, after revising claims down by these awards, applying the rule to allocate the increment.[3]

Composition up: For each pair $(c,e),(c',e') \in \mathcal{P}^N$ such that $c = c'$ and $e \leq e'$, $\varphi(c,e') = \varphi(c,e) + \varphi(c - \varphi(c,e), e' - e)$.

We now propose the existence of a maximal reasonable claim: No individual ought to claim more than an amount arbitrarily many times larger than the endowment.

Maximal reasonable claims (MRC): There is a natural number k such that, for each $(c,e) \in \mathcal{P}^N$, $\varphi((\min\{c_i, ke\})_{i\in N}, e) = \varphi(c,e)$.

The standard **claims truncation invariance** property corresponds to the special case of the above axiom where $k = 1$.

Our first result strengthens a characterization of the constrained equal awards rule on the basis of symmetry, composition up, and claims truncation invariance (Dagan, 1996). We show that claims truncation invariance can be weakened to maximal reasonable claims:

Theorem 1. *A rule satisfies symmetry, composition up, and maximal reasonable claims if and only if it is the constrained equal awards rule.*

Proof. It is easy to check that the CEA rule satisfies symmetry, composition up, and MRC. Conversely, let φ denote a rule satisfying these properties and let $N \in \mathcal{N}$. By MRC, we can fix a positive integer denoted by k such that, for each $(c,e) \in \mathcal{P}^N$, $\varphi((\min\{c_i, ke\})_{i\in N}, e) = \varphi(c,e)$.

Let $(c,e) \in \mathcal{P}^N$. We will prove that $\varphi(c,e) = CEA(c,e)$. Without loss of generality, let $\{1,2,\ldots,m\} \subseteq N$ be such that claimant 1 is one of the claimants with the smallest claim, claimant 2 is one of the claimants with the second smallest claim, and so forth. Thus, $c_1 < c_2 < \cdots < c_m$. Define the sequences $\{c^n\}$, $\{e^n\}$, and $\{x^n\}$ recursively as follows:

$$c^1 = c, \qquad e^1 = \frac{1}{k}\min\{e, c_1^1\}, \qquad x^1 = \varphi(c^1, e^1),$$
$$c^2 = c^1 - x^1, \qquad e^2 = \frac{1}{k}\min\{e - e^1, c_1^2\}, \qquad x^2 = \varphi(c^2, e^2),$$
$$c^3 = c^2 - x^2, \qquad e^3 = \frac{1}{k}\min\{e - e^1 - e^2, c_1^3\}, \qquad x^3 = \varphi(c^3, e^3),$$
$$\vdots$$

Note that (c^1, e^1) is such that $ke^1 \leq c_1^1 = \min_{i\in N} c_j^1$. Thus, by MRC and symmetry, for each $i \in N$, $x_i^1 = \frac{e^1}{|N|}$. Similarly, for each $n \in \mathbb{N}$ and each $i \in N$, $x_i^n = \frac{e^h}{|N|}$. Thus, by composition up,

$$\text{for each } n \text{ and each } i \in N, \quad \varphi_i(c, e^1 + \cdots + e^n) = x_i^1 + \cdots + x_i^n = \frac{e^1 + \cdots + e^n}{|N|}. \tag{1}$$

The sequence $\{e^1 + \cdots + e^n\}_{n\in\mathbb{N}}$ is monotone increasing and bounded above by e. It thus has a limit e^*. Thus, by (1),

$$\text{for each } i \in N, \quad \varphi_i(c, e^*) = \frac{e^*}{|N|}. \tag{2}$$

Case 1: $e = e^*$. By (2), $\varphi(c,e) = CEA(c,e)$, as desired.

Case 2: $e > e^*$. The sequence $\{e^n\}_{n\in\mathbb{N}}$ is monotone decreasing and bounded below by 0. It is thus convergent. By the Cauchy convergence criterion, since $\{e^1 + \cdots + e^n\}_{n\in\mathbb{N}}$ is convergent, for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $e^n = \sum_{h=1}^n e^h - \sum_{h=1}^{n-1} e^h < \varepsilon$. Thus,

$$e^n \overset{n\to\infty}{\to} 0. \tag{3}$$

The sequence $\{c_1^n\}_{n\in\mathbb{N}}$ is monotone decreasing and bounded below and, thus, convergent. Let c_1^* denote its limit. We now prove that $c_1^* = 0$. Note that $\{e - \sum_{h=1}^n e^h\}_{n\in\mathbb{N}}$ converges to $e - e^* > 0$. Thus, if $c_1^* > 0$, $e^n \equiv \min\{e - \sum_{h=1}^{n-1} e^h, c_1^n\} \overset{n\to\infty}{\to} \min\{e - e^*, c_1^*\} > 0$, contradicting (3). Thus, $c_1^* = 0$. Thus, $c_1 - \frac{e^*}{|N|} = c_1 - \sum_{h=1}^\infty x_1^h = c_1^* = 0$. Thus, by (2),

$$\varphi(c, e^*) = (c_1, \ldots, c_1) = CEA(c, e^*). \tag{4}$$

Define the sequences $\{\tilde{c}^n\}$, $\{\tilde{e}^n\}$, and $\{\tilde{x}^n\}$ recursively as follows:

$$\tilde{c}^1 = c - \varphi(c, e^*), \qquad \tilde{e}^1 = \frac{1}{k}\min\{e - e^*, \tilde{c}_2^1\}, \qquad \tilde{x}^1 = \varphi(\tilde{c}^1, \tilde{e}^1),$$
$$\tilde{c}^2 = \tilde{c}^1 - \tilde{x}^1, \qquad \tilde{e}^2 = \frac{1}{k}\min\{e - e^* - \tilde{e}^1, \tilde{c}_2^2\}, \qquad \tilde{x}^2 = \varphi(\tilde{c}^2, \tilde{e}^2),$$
$$\tilde{c}^3 = \tilde{c}^2 - \tilde{x}^2, \qquad \tilde{e}^3 = \frac{1}{k}\min\{e - e^* - \tilde{e}^1 - \tilde{e}^2, \tilde{c}_2^3\}, \qquad \tilde{x}^3 = \varphi(\tilde{c}^3, \tilde{e}^3),$$
$$\vdots$$

Let $N_1 = \{i \in N : c_i = c_1\}$ and note that $(\tilde{c}^1, \tilde{e}^1)$ is such that $k\tilde{e}^1 \leq \tilde{c}_2^1 = \min_{i\in N\setminus N_1} \tilde{c}_i^1$. Thus, by MRC and symmetry, for each $i \in N \setminus N_1$, $\tilde{x}_i^1 = \frac{\tilde{e}^1}{|N\setminus N_1|}$. By (4), for each $i \in N_1$, $\tilde{c}_i^1 = 0$ and thus $\tilde{x}_i^1 = 0$. Similarly, for each $n \in \mathbb{N}$ and each $i \in N \setminus N_1$, $\tilde{x}_i^n = \frac{\tilde{e}^n}{|N\setminus N_1|}$ and, for each $n \in \mathbb{N}$ and each $i \in N_1$, $\tilde{x}_i^n = 0$. Thus, by composition up,

$$\text{for each } n \text{ and each } i \in N \setminus N_1, \quad \varphi_i(\tilde{c}^1, \tilde{e}^1 + \cdots + \tilde{e}^n) = \tilde{x}_i^1 + \cdots + \tilde{x}_i^n = \frac{\tilde{e}^1 + \cdots + \tilde{e}^n}{|N \setminus N_1|}. \tag{5}$$

The sequence $\{\tilde{e}^1 + \cdots + \tilde{e}^n\}_{n\in\mathbb{N}}$ is monotone increasing and bounded above by e. It thus has a limit $\tilde{e}^*$. Thus, by (5),

$$\text{for each } i \in N \setminus N_1, \quad \varphi_i(\tilde{c}^1, \tilde{e}^*) = \frac{\tilde{e}^*}{|N \setminus N_1|}. \tag{6}$$

By composition up,

$$\varphi(c, e^* + \tilde{e}^*) = \varphi(c, e^*) + \varphi(c - \varphi(c, e^*), \tilde{e}^*) = \varphi(c, e^*) + \varphi(\tilde{c}^1, \tilde{e}^*). \tag{7}$$

Recall that, for each $i \in N_1$, $\tilde{c}_i^1 = 0$. Thus, by (4), (6), and (7),

$$\text{for each } i \in N_1, \quad \varphi_i(c, e^* + \tilde{e}^*) = c_1 + 0 \quad \text{and}$$
$$\text{for each } i \in N \setminus N_1, \quad \varphi_i(c, e^* + \tilde{e}^*) = c_1 + \frac{\tilde{e}^*}{|N \setminus N_1|}. \tag{8}$$

Case 2.1: $e = e^* + \tilde{e}^*$. By (8), $\varphi(c,e) = CEA(c,e)$, as desired.

Case 2.2: $e > e^* + \tilde{e}^*$. Repeating the above arguments at most $m - 1$ more times we find that $\varphi(c,e) = CEA(c,e)$. ■

We now show that the only asymmetric rationing rules (Moulin, 2000) that satisfy MRC are the asymmetric generalizations of the CEA

[3] Composition up appears in the characterizations of the "equal sacrifice taxation rules" (Young, 1988), the "constrained equal awards rule" (Dagan, 1996), and in that of the asymmetric rules of Moulin (2000) and Chambers (2006).

rule.[4] The key property in Moulin's characterization is "consistency:" If an allocation is considered desirable for a group of claimants, then it should be considered desirable when restricted to each subgroup.[5]

Consistency: For each pair $N, N' \in \mathcal{N}$ such that $N' \subseteq N$, each $(c,e) \in \mathcal{P}^N$, and each $i \in N'$, $\varphi_i(c_{N'}, \sum_{N'} \varphi_j(c,e)) = \varphi_i(c,e)$.

We refer to the asymmetric generalizations of the CEA rule as priority-augmented weighted CEA rules, or **PWCEA** rules for brevity. Formally, a rule φ is a PWCEA rule if there is a partition of the set of potential claimants A into $n \leq |A|$ non-empty priority classes $A_1, \ldots, A_n$ and a weights profile $w \in \mathbb{R}^A_{++}$ such that, for each $N \in \mathcal{N}$ and each $(c,e) \in \mathcal{P}^N$, $\varphi_i(c,e) = c_i$ for each $i \in N_t$ and $\varphi_i(c,e) = \min\{c_i, \lambda w_i\}$ for each $i \in N \backslash N_t$ where

1. t is the smallest number in $\{0, 1, \ldots, n-1\}$ with $\sum_{N \cap [A_1 \cup \cdots \cup A_{t+1}]} c_i \geq e$,
2. N_t is the empty set if $t = 0$ and $N_t = N \cap [A_1 \cup \cdots \cup A_t]$ if $t > 0$,
3. $\lambda \in \mathbb{R}$ satisfies $\sum_{N \backslash N_t} \min\{c_i, \lambda w_i\} = e - \sum_{N_t} c_i$ under the convention that the summation over the empty set is zero.

Theorem 2. *A rule satisfies consistency, composition up, and maximal reasonable claims if and only if it is a PWCEA rule.*

See the Appendix for the proof.

Our results yield characterizations of other well known rules using the notion of "duality" between rules.[6] Given a rule φ, its dual φ^d is the rule that recommends, for each $(c,e) \in \mathcal{P}^N$, the allocation $c - \varphi(c, \sum_N c_i - e)$. That is, the dual rule φ^d allocates the total loss (the difference between the total amount claimed and what is available, $\sum_N c_i - e$) among the claimants using rule φ. The properties satisfied by a rule then have logical implications on the properties satisfied by its dual. We say that property P^d is the dual of property P when a rule satisfies P if and only if its dual rule satisfies P^d.

Theorem 1 then implies that a rule satisfies the dual properties of symmetry, composition up, and MRC if and only if it is the dual of the CEA rule.[7] The dual of the CEA rule is the also classical constrained equal losses rule. Similarly, Theorem 2 implies that a rule satisfies the dual properties of consistency, composition up, and MRC if and only if it is the dual of a PWCEA rule, an asymmetric generalization of the constrained equal losses rule.[8]

3. Final remarks

MRC weakens claims truncation invariance and can be used to provide axiomatic foundations for the CEA rule and its asymmetric generalizations. This suggests two avenues for future research: (i) exploring whether MRC can replace claims truncation invariance elsewhere, either in axiomatic characterizations (see Thomson, 2019) or to study the space of rules (Thomson and Yeh, 2007; Hougaard et al., 2012), and (ii) exploring the implications of similarly weakening other axioms in the literature.[9]

Appendix. Proof of Theorem 2

Given a natural number k, a rule φ satisfies **MRC*k*** if, for each $N \in \mathcal{N}$ and each $(c,e) \in \mathcal{P}^N$, $\varphi((\min\{c_i, ke\})_{i \in N}, e) = \varphi(c,e)$. If a rule satisfies maximal reasonable claims, then there is a k such that it satisfies MRCk and Theorem 2 follows from Theorem 3 below. We introduce a further property that will be useful in our proofs:

Converse consistency: For each $(c,e) \in \mathcal{P}^N$, $[x \in Z(c,e)$ and, for each $\{i,j\} \subseteq N$, $x_{\{i,j\}} = \varphi(c_{\{i,j\}}, x_i + x_j)]$ implies $x = \varphi(c,e)$.

Lemma 1 (*Flores-Szwagrzak, 2015*). *The PWCEA rules satisfy consistency, composition up, claims truncation invariance, and converse consistency.*

Lemma 2. *Let φ denote a rule satisfying composition up and MRCk. For each $N \in \mathcal{N}$, there is $r \in \mathbb{R}^N_+$ such that $\sum_N r_i = 1$ and such that, for each $(c,e) \in \mathcal{P}^N$ with $ke \leq \min_{i \in N} c_i$, $\varphi(c,e) = er$.*

Proof. Let φ denote a rule satisfying the properties in the Lemma. Let $N \in \mathcal{N}$, $(\bar{c}, \bar{e}) \in \mathcal{P}^N$ be such that $\bar{e} = 1$ and $k \leq \min_{i \in N} \bar{c}_i$. Let $r \equiv \varphi(\bar{c}, \bar{e})$. By feasibility, $\sum_N r_i = 1$. Let $(c,e) \in \mathcal{P}^N$ be such that $ke \leq \min_{i \in N} c_i$. We distinguish three cases:

Case 1: $e = 1$. By *MRCk*, $\varphi(c,e) = \varphi(\bar{c}, e) = r$.

Case 2: $e < 1$. By *MRCk*, $\varphi(c,e) = \varphi(\bar{c}, e)$. Thus, by *MRCk* we may assume without loss of generality that $k \leq \min_{i \in N} c_i$.

- Suppose that $e = \frac{1}{2}$ and let $x \equiv \varphi(c,e)$. Then, by Case 1 and composition up, $r = \varphi(c, 1) = x + \varphi(c - x, 1 - \frac{1}{2})$. Then, since $\frac{k}{2} \leq \min_{i \in N}(c_i - x_i)$, by *MRCk*, $\varphi(c - x, 1 - \frac{1}{2}) = x$. Thus, $r = 2x$. Thus, $\varphi(c, \frac{1}{2}) = x = \frac{1}{2} r$.
- Suppose that $e = \frac{1}{4}$ and let $x \equiv \varphi(c,e)$. Then, by composition up, $\frac{1}{2} r = \varphi(c, \frac{1}{2}) = x + \varphi(c - x, \frac{1}{2} - \frac{1}{4})$. Then, since $\frac{k}{4} \leq \min_{i \in N}(c_i - x_i)$, by *MRCk*, $\varphi(c - x, \frac{1}{2} - \frac{1}{4}) = x$. Thus, $\frac{1}{2} r = 2x$. Thus, $\varphi(c, \frac{1}{4}) = x = \frac{1}{4} r$.

Continuing in this way we can show that, for each pair of natural numbers m and n such that $m \leq 2^n$, $\varphi(c, \frac{m}{2^n}) = \frac{m}{2^n} r$. Let $e < 1$ and let $\{e_h\}$ denote an increasing sequence converging to e and such that, for each $h \in \mathbb{N}$, there are natural numbers m and n such that $e_h = \frac{m}{2^n}$. Thus, for each $h \in \mathbb{N}$, $\varphi(c, e_h) = e_h r$ and, by composition up, $\varphi(c,e) = e_h r + \varphi(c - e_h r, e - e_h) \geq e_h r$. Thus, $\varphi(c,e) \geq er$. Thus, $\varphi(c,e) = er$.

Case 3: $e > 1$. Let n be the largest integer such that $n \leq e$. By composition up, *MRCk*, and Case 1,

$$\begin{aligned}
\varphi(c,e) &= r + \varphi(c - r, e - 1) \\
&= 2r + \varphi(c - 2r, e - 2) \\
&\vdots \\
&= nr + \varphi(c - nr, e - n).
\end{aligned}$$

By Case 2, $\varphi(c - nr, e - n) = (e - n)r$. Thus, $\varphi(c,e) = nr + (e-n)r = er$. ■

Lemma 3. *Let φ denote a rule satisfying consistency, composition up, and MRCk. There is a PWCEA rule that coincides with φ on the subdomain of claims problems where k times the endowment is no larger than the smallest claim.*

[4] There has recently been much work on asymmetric rules (Hokari and Thomson, 2003; Chambers, 2006; Thomson, 2013; Stovall, 2014a,b; Harless, 2017; Flores-Szwagrzak et al., 2020; Stovall, 2020).

[5] Consistency has played a central role in the study of claims problems starting with the works of Aumann and Maschler (1985) and Young (1987, 1988). More recently, consistency has also been used to describe the structure of families of symmetric (Chambers and Moreno-Ternero, 2017) and asymmetric rules, i.e., rules that may recommend different awards for individuals with equal claims (Moulin, 2000; Hokari and Thomson, 2003; Chambers, 2006; Thomson, 2013; Stovall, 2014a,b; Harless, 2017; Stovall, 2020).

[6] See Chapter 7 in Thomson (2019) for a comprehensive overview of the duality relationship in the space of rules.

[7] The dual property of symmetry is symmetry itself and the dual property of composition up is the classical "composition down" (see Chapter 6 in Thomson, 2019). A rule φ satisfies the dual property of MRC if there exists a natural number k such that, for each $(c,e) \in \mathcal{P}^N$, $\varphi(c,e) = m^k(c,e) + \varphi(c - m^k(c,e), e - \sum_N m^k_j(c,e))$ where $m^k_i(c,e) = \max\{0, c_i - k(\sum_N c_j - e)\}$ and $m^k(c,e) = (m^k_i(c,e))_{i \in N}$. When $k = 1$, the dual of MRC coincides with "minimal rights first," the dual property of claims truncation invariance.

[8] The dual property of consistency is consistency itself.

[9] For instance, consider the following "securement" property (Moreno-Ternero and Villar, 2004): for each claims problem $(c,e) \in \mathcal{P}^N$, each claimant $i \in N$ receives at least $\frac{1}{|N|} \min\{c_i, e\}$. This can be weakened by requiring that there exists a very large natural number k such that each individual receives at least $\frac{1}{k|N|} \min\{c_i, e\}$.

Proof. Let φ denote a rule satisfying the properties in the Lemma.

Step 1: Constructing priority classes and a weights profile. By Lemma 2, for each $(c,e) \in \mathcal{P}^A$ such that $0 < ke \le \min_{i\in A} c_i$ there is $r^1 \in \mathbb{R}^A_+$ such that $\varphi(c,e) = r^1 e$. Let $A_1 \equiv \{i \in A : r^1_i > 0\}$. By Lemma 2, for each $(c,e) \in \mathcal{P}^{A\setminus A_1}$ such that $0 < ke \le \min_{i\in A\setminus A_1} c_i$ there is $r^2 \in \mathbb{R}^{A\setminus A_1}_+$ such that $\varphi(c,e) = r^2 e$. Let $A_2 \equiv \{i \in A \setminus A_1 : r^2_i > 0\}$. Continuing in this way, define $r^1, r^2, \ldots, r^n$ and $A_1, A_2, \ldots, A_n$ until the sets $A_1, A_2, \ldots, A_n$ form a partition of A. By construction, $n \le |A|$. Let $w \in \mathbb{R}^A_{++}$ be such that, for each $l \in \{1,\ldots,n\}$ and each $i \in A_l$, $w_i \equiv r^l_i$.

Step 2: Let ψ denote the PWCEA rule associated with the priority classes and weights profile constructed in Step 1. Then, φ coincides with ψ on the subdomain of claims problems where k times the endowment is no larger than the smallest claim. Let $N \in \mathcal{N}$. By Lemma 2, there is $r \in \mathbb{R}^N_+$ with $\sum_{i\in N} r_i = 1$ such that,

$$\text{for each } (c,e) \in \mathcal{P}^N \text{ with } ke \le \min_{i\in N} c_i, \quad \varphi(c,e) = re. \tag{9}$$

Let $(c,e) \in \mathcal{P}^N$ be such that $0 < ke \le \min_{i\in N} c_i$ and let $l \in \{1,\ldots,h\}$ denote the smallest number such that $N \cap A_l$ is non-empty. We first prove that

$$\text{for each } i \in N, r_i > 0 \text{ implies } i \in A_l \text{ and, for each pair } i,j \in A_l \cap N, \frac{r^l_i}{r^l_j} = \frac{r_i}{r_j}. \tag{10}$$

Otherwise there is an $h \in \{1,\ldots,n\}$ such that $h > l$ and a $j \in N \cap A_h$ such that $r_j > 0$. Let $M \equiv \cup^n_{g=l} A_g$ and let $(\tilde{c},\tilde{e}) \in \mathcal{P}^M$ be such that $k\tilde{e} = \min_{m\in M} \tilde{c}_m > 0$ and, for each $i \in N$, $\tilde{c}_i = c_i$. By the definition of r^l in Step 1, $\varphi(\tilde{c},\tilde{e}) = r^l\tilde{e}$ and, for each $i \in M \setminus A_l$, $r^l_i = 0$. Thus, $\sum_{i\in A_l} \varphi_i(\tilde{c},\tilde{e}) = \tilde{e}$. Let $\hat{e} \equiv \sum_{i\in A_l\cap N} \varphi_i(\tilde{c},\tilde{e})$ and note that, because $A_l \cap N$ is non-empty and $r^l_i > 0$ for $i \in A_l \cap N$, $\hat{e} > 0$. Since $N \subseteq M$, by consistency, for each $i \in A_l \cap N$, $\varphi_i(c,\hat{e}) = r^l_i\tilde{e} > 0$ and, for each $i \in N \setminus A_l$, $\varphi_i(c,\hat{e}) = 0$. However, by (9) and the assumption that there is $j \in N \cap A_h$ such that $r_j > 0$, $\varphi_j(c,\hat{e}) = r_j\hat{e} > 0$, a contradiction since $j \in N \setminus A_l$. Thus, for each $j \in N \setminus A_l$, $r_j = 0$, as desired. Moreover, for each pair $i,j \in A_l \cap N$, by (9), $\varphi_i(c,\hat{e}) = r_i\hat{e}$ and $\varphi_j(c,\hat{e}) = r_j\hat{e}$; by consistency, $\varphi_i(c,\hat{e}) = r^l_i\tilde{e}$ and $\varphi_j(c,\hat{e}) = r^l_j\tilde{e}$. Thus, for each pair $i,j \in A_l \cap N$, $r^l_i\tilde{e} = r_i\hat{e}$ and $r^l_j\tilde{e} = r_j\hat{e}$. Thus, for each pair $i,j \in A_l \cap N$, $\frac{r^l_i}{r^l_j} = \frac{r_i}{r_j}$, as desired.

By the definition of the PWCEA ψ, for each $i \in N \cap A_l$, $\psi_i(c,e) = w_i\lambda$ where λ is such that $\sum_{i\in N\cap A_l} w_i\lambda = e$ and, for each $i \in N \setminus A_l$, $\psi_i(c,e) = 0$. By (9) and (10), for each $i \in N \cap A_l$, $\varphi_i(c,e) = r_ie$ and, for each $i \in N \setminus A_l$, $\varphi_i(c,e) = r_ie = 0$. Thus, $\sum_{i\in N\cap A_l} \psi_i(c,e) = e = \sum_{i\in N\cap A_l} \varphi_i(c,e)$. Thus, by (10) and the definition of w in Step 1, for each pair $i,j \in N \cap A_l$,

$$\frac{\psi_i(c,e)}{\psi_j(c,e)} = \frac{w_i\lambda}{w_j\lambda} = \frac{r^l_i\lambda}{r^l_j\lambda} = \frac{r_ie}{r_je} = \frac{\varphi_i(c,e)}{\varphi_j(c,e)}.$$

Thus, $\varphi(c,e) = \psi(c,e)$. ■

Lemma 4. *Let φ denote a rule satisfying consistency, composition up, and MRCk. Let ψ denote the PWCEA rule coinciding with φ on the subdomain of claims problems where k times the endowment is no larger than the smallest claim, as identified in Lemma 3. For each $\{i,j\} \in \mathcal{N}$ and each $(c,e) \in \mathcal{P}^{\{i,j\}}$, such that $\min\{c_i,c_j\} < ke$, $\varphi(c,e) = \psi(c,e)$.*

Proof. Let φ and ψ be as specified in the Lemma. Let $\{i,j\} \in \mathcal{N}$ and $(c,e) \in \mathcal{P}^{\{i,j\}}$ be such that $\min\{c_i,c_j\} < ke$. Define the sequences $\{c^n\}$, $\{e^n\}$, and $\{x^k\}$ recursively as follows:

$$c^1 = c, \qquad e^1 = \frac{1}{k}\min\{c^1_i, c^1_j\}, \qquad x^1 = \varphi(c^1,e^1),$$
$$c^2 = c^1 - x^1, \qquad e^2 = \frac{1}{k}\min\{e - e^1, c^2_i, c^2_j\}, \qquad x^2 = \varphi(c^2,e^2),$$
$$c^3 = c^2 - x^2, \qquad e^3 = \frac{1}{k}\min\{e - e^1 - e^2, c^3_i, c^3_j\}, \qquad x^3 = \varphi(c^3,e^3),$$
$$\vdots$$

By construction, for each n, $(c^n,e^n) \in \mathcal{P}^{\{i,j\}}$ is such that $ke^n \le \min\{c^n_i,c^n_j\}$. Thus, by Lemma 2, there is $r \in \mathbb{R}^{\{i,j\}}_+$ with $r_i + r_j = 1$ such that

$$\text{for each } n, \quad x^n = re^n. \tag{11}$$

Thus, by composition up,

$$\text{for each } l, \quad \varphi(c, e^1 + \cdots + e^l) = x^1 + \cdots + x^l = re^1 + \cdots + re^l = r(e^1 + \cdots + e^l). \tag{12}$$

The sequence $\{e^1 + \cdots + e^l\}_{l\in\mathbb{N}}$ is monotone increasing and bounded above by e. It thus has a limit e^*. Thus, by (12),

$$\varphi(c,e^*) = re^*. \tag{13}$$

Let $\tilde{c}$ denote a profile of claims for i and j such that $\tilde{c} \ge c$ and $ke^* \le \min\{\tilde{c}_i,\tilde{c}_j\}$. By Lemma 2 and (13), respectively, $\varphi(\tilde{c},e^*) = re^* = \varphi(c,e^*)$. By feasibility, $\varphi(c,e^*) \le c$. By Lemma 3, $\varphi(\tilde{c},e^*) = \psi(\tilde{c},e^*)$. Thus, $\psi(\tilde{c},e^*) = \varphi(c,e^*) \le c$. Thus, from the definition of a PWCEA rule, $\psi(\tilde{c},e^*) = \psi(c,e^*)$. Thus,

$$\varphi(c,e^*) = \psi(c,e^*). \tag{14}$$

Case 1: $e = e^*$. By (14), $\varphi(c,e) = \psi(c,e)$, as desired.

Case 2: $e > e^*$. The sequence $\{e^n\}$ is monotone decreasing and bounded below by 0. It is thus convergent. By the Cauchy convergence criterion, since $\{\sum^n_{h=1} e^h\}_{n\in\mathbb{N}}$ is convergent, for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $e^n = \sum^n_{h=1} e^h - \sum^{n-1}_{h=1} e^h < \varepsilon$. Thus,

$$e^n \overset{n\to\infty}{\to} 0. \tag{15}$$

The sequences $\{c^n_i\}_{n\in\mathbb{N}}$ and $\{c^n_j\}_{n\in\mathbb{N}}$ are monotone decreasing and bounded below. They are thus convergent. Let c^*_i and c^*_j denote their respective limits.

We now prove that either $c^*_i = 0$ or $c^*_j = 0$. Note that $\{e - \sum^n_{h=1} e^h\}_{n\in\mathbb{N}}$ converges to $e - e^* > 0$. Thus, if $c^*_i > 0$ and $c^*_j > 0$,

$$e^n \equiv \min\{e - \textstyle\sum^{n-1}_{h=1} e^h, c^n_i, c^n_j\} \overset{n\to\infty}{\to} \min\{e - e^*, c^*_i, c^*_j\} > 0,$$

contradicting (15). Thus, without loss of generality, $c^*_i = 0$. Thus, by (12) and the definition of $\{c^n_i\}_{n\in\mathbb{N}}$,

$$c_i - \textstyle\sum^{n-1}_{h=1} r_ie^h = c_i - \sum^{n-1}_{h=1} x^h_i \equiv c^n_i \overset{n\to\infty}{\to} c^*_i = 0.$$

Thus, by (13),

$$\varphi_i(c,e^*) = r_ie^* = c_i. \tag{16}$$

By composition up,

$$\varphi(c,e) = \varphi(c,e^*) + \varphi(c - \varphi(c,e^*), e - e^*). \tag{17}$$

By (16), $\varphi_i(c,e^*) = c_i$. Thus, by feasibility, $\varphi_i(c - \varphi(c,e^*), e - e^*) = 0$. By (17), $\varphi_i(c,e) = c_i$ and, by feasibility, $\varphi_j(c,e) = e - c_i$.

Since ψ satisfies composition up (Lemma 1),

$$\psi(c,e) = \psi(c,e^*) + \psi(c - \psi(c,e^*), e - e^*). \tag{18}$$

By (14), $\psi(c,e^*) = \varphi(c,e^*)$. Thus, by (16), $\psi_i(c,e^*) = c_i$. Thus, by feasibility, $\psi_i(c - \psi(c,e^*), e - e^*) = 0$. Thus, by (18), $\psi_i(c,e) = c_i$. Thus, by feasibility, $\psi_j(c,e) = e - c_i$. Thus, again, $\varphi(c,e) = \psi(c,e)$, as desired. ■

Theorem 3. *A rule satisfies consistency, composition up, and MRCk if and only if it is a PWCEA rule.*

Proof. The proof of Theorem 3 relies on the above Lemmas and a standard "Elevator Lemma" argument (Thomson, 2011). By Lemma 1, each PWCEA rule satisfies consistency, composition up, and claims truncation invariance; the latter of these properties implies *MRCk*. Conversely, let φ denote a rule satisfying the properties in Theorem 3. Let $N \in \mathcal{N}$, $(c,e) \in \mathcal{P}^N$, and $x \equiv \varphi(c,e)$. By consistency, for each $\{i,j\} \subseteq N$, $x_{\{i,j\}} = \varphi(c_i,c_j,x_i + x_j)$. By either Lemma 3 or Lemma 4, there is a PWCEA rule ψ, such that, for each $\{i,j\} \subseteq N$, $\psi(c_i,c_j,x_i + x_j) = \varphi(c_i,c_j,x_i + x_j)$. Since ψ is conversely consistent, $\psi(c,e) = x = \varphi(c,e)$. ■

Data availability

No data was used for the research described in the article.

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