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A modified wild bootstrap procedure for Laplace transforms of volatility ☆

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ABSTRACT

In this note, we propose a modified wild (MW) bootstrap-based procedure for the realized Laplace transform (RLT) of volatility. We establish its first-order asymptotic validity.

1. Introduction

The validity of wild bootstrap methods for the realized Laplace transform (RLT) of volatility has recently been discussed in our companion paper Hounyo et al. (2023). Specifically, we have shown that standard wild bootstrap procedures, including the variants provided by Wu (1986) and Liu (1988) as well as Gonçalves and Meddahi (2009) in different contexts, deliver inconsistent inference for the RLT of volatility. In this note, we introduce a modified wild (MW) bootstrap method and establish its first-order asymptotic validity under mild assumptions.

2. Main results

We suppose that the process X obeys an Itô semimartingale with stochastic differential equation of the form,

$$dX_t = \alpha_t dt + \sigma_t dW_t + \int_{\mathbb{R}} \delta(t-,x)\mu(dt,dx), \tag{1}$$

where the drift α_t and volatility σ_t are adapted processes with càdlàg paths, W_t is a standard Brownian motion, μ a homogeneous Poisson measure with compensator $dt \otimes \nu(dx)$, ν is the Lévy measure and $\delta(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is càdlà g in t. We follow Todorov and Tauchen (2012) by imposing mild restrictions:

Assumption A. The Lévy measure ν satisfies

$$\mathbb{E}\left(\int_0^t \int_{\mathbb{R}} \left(|\delta(s,x)|^p \vee |\delta(s,x)|\right) ds\, \nu(dx)\right) < \infty,$$

for every $t > 0$ and every $p \in (\beta, 1)$, where $0 \le \beta < 1$ is some constant.

Assumption B. The volatility, σ_t, is an Itô semimartingale, defined by

$$\sigma_t = \sigma_0 + \int_0^t \tilde{a}_s ds + \int_0^t \upsilon_s dW_s + \int_0^t \upsilon'_s dW'_s + \int_0^t \int_{\mathbb{R}} \delta'(s-,x)\tilde{\mu}'(ds,dx),$$

where W'_t is a Brownian motion, independent of W_t, $\tilde{\mu}'$ is a compensated homogeneous Poisson measure with Lévy measure $dt \otimes \nu'(dx)$, having arbitrary dependence with μ, and $\delta'(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is càdlàg in t. In addition, for every $t, s > 0$ and some $\iota > 0$, it is required that

$$\mathbb{E}\left(|a_s|^{3+\iota} + |\tilde{a}_s|^2 + |\sigma_t|^{3+\iota} + |\upsilon_t|^{3+\iota} + |\upsilon'_t|^{3+\iota} + \int_{\mathbb{R}} |\delta'(t,x)|^{3+\iota} \nu'(dx)\right) < C,$$

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$$\mathbb{E}\left(|a_t - a_s|^2 + |v_t - v_s|^2 + |v'_t - v'_s|^2 + \int_{\mathbb{R}} (\delta'(t,x) - \delta'(s,x))^2 \nu'(dx)\right) < C|t-s|,$$

where $C > 0$ is some constant that is free of t and s.

Moreover, we assume that T is fixed and, within the interval $[0,T]$, we observe the process X at the equidistant time points $\{0, \Delta_n, 2\Delta_n, \ldots, i\Delta_n, \ldots, n\Delta_n \equiv T\}$. The RLT is given by

$$RLT_n(u) = \Delta_n \sum_{i=1}^{n} \xi(u)_i^n, \quad \xi(u)_i^n \equiv \cos\left(\sqrt{2u}\Delta_n^{-1/2} \Delta_i^n X\right), \tag{2}$$

for some $u \geq 0$, where $\Delta_i^n X \equiv X_{i\Delta_n} - X_{(i-1)\Delta_n}$, see, e.g., Todorov and Tauchen (2012).

The main reason behind the failure of standard wild bootstrap procedures to replicate the distributional properties of the RLT is that their respective bootstrap observations are not centered appropriately. Hence, we propose a mean-correction to the standard wild bootstrap approach, which restores its asymptotic validity. Specifically, we generate the bootstrap observations to replicate the conditional independence and heterogeneity properties of the sequence $(\xi(u)_i^n)_{i=1}^n$ as follows,

$$\xi(u)_i^{n*} = \begin{cases} \xi(u)_{i+1}^n + \left(\xi(u)_i^n - \xi(u)_{i+1}^n\right)\eta_i^*, & \text{if} \quad 1 \leq i \leq n-1, \\ \xi(u)_n^n, & \text{if} \quad i = n, \end{cases} \tag{3}$$

where η_i^* is an external random variable. This MW resampling is related to the wild blocks-of-blocks bootstraps introduced by Hounyo et al. (2017) and Hounyo (2017) in different settings. In particular, we utilize their idea of centering the observations around $\xi(u)_{i+1}^n$ to accommodate the mean heterogeneity of the sequence. However, whereas the former two apply this centering to estimate the distribution of integrated covariance matrix estimators, it is important to note that the RLT does not belong to the general class of statistics covered by their respective frameworks. Specifically, in contrast with Hounyo (2017), who study inference for a random variable (or matrix), we are concerned with inference for a random process. This increases the level of technicality of the asymptotic analysis considerably as we have to provide both pointwise and uniform (in the argument u) bootstrap limit theory for the MW resampling procedure. Moreover, our bootstrap resampling scheme in (3) does not involve blocks since the observations, $(\xi(u)_i^n)_{i=1}^n$, are conditionally independent. Finally, the MW bootstrap is not a special case of the local Gaussian (LG) approach in Hounyo et al. (2023). While the LG analog of $\xi(u)_i^{n*}$ uses blocks of observations $\{\Delta_i^n X : i = 1, \ldots, n\}$, the MW uses only pairs $\Delta_i^n X$ and $\Delta_{i+1}^n X$. The LG bootstrap requires two additional tuning parameters not needed for the MW; the local block size k_n for a spot variance estimator and a threshold parameter to remove jumps.

Next, we define the bootstrap RLT as $RLT_n^*(u) \equiv \Delta_n \sum_{i=1}^n \xi(u)_i^{n*}$. Moreover, let

$$S_n(u) = \Delta_n^{-1/2}\left(RLT_n(u) - \int_0^T e^{-u\sigma_s^2} ds\right),$$
$$S_n^*(u) = \Delta_n^{-1/2}\left(RLT_n^*(u) - \mathbb{E}^*(RLT_n^*(u))\right), \tag{4}$$

and where $\mathbb{E}^*\left(RLT_n^*(u)\right) = \Delta_n \sum_{i=1}^{n-1}\left[\xi(u)_{i+1}^n + \left(\xi(u)_i^n - \xi(u)_{i+1}^n\right)\mathbb{E}^*(\eta_i^*)\right] + \Delta_n \xi(u)_n^n$.

Theorem 1. *Assume that $\xi(u)_i^{n*}$ are generated as in (3) and the external random variable is chosen as $\overset{i.i.d.}{\sim} (\mathbb{E}^*(\eta_i^*), \mathbb{V}^*(\eta_i^*))$, such that $\mathbb{V}^*(\eta_i^*) = 1/2$ with $\mathbb{E}^*(|\eta_i^*|)^{2+\delta} < \infty$ for some $\delta > 0$. Moreover, suppose that Assumptions A and B hold. Then, for every $u \in \mathbb{K}$, where $\mathbb{K}$ is a compact subset of $\mathbb{R}_+$, and as $\Delta_n \to 0$, it follows that*

(a) $S_n^*(u) \overset{d^*}{\Longrightarrow} \Psi_T(u)$ on $l^\infty(\mathbb{K})$ in probability-$\mathbb{P}$, for any compact subset $\mathbb{K}$ of $\mathbb{R}_+$.

(b) $\sup_{x\in\mathbb{R}} \left|\mathbb{P}^*(S_n^*(u) \leq x) - \mathbb{P}(S_n(u) \leq x)\right| \overset{\mathbb{P}}{\longrightarrow} 0.$

Theorem 1 demonstrates that the MW bootstrap may be utilized to estimate the entire distribution of $S_n(u)$ and, thus, to draw inference on the RLT. Moreover, we next show that the MW bootstrap method may also be applied to estimate the distribution of studentized statistics. To this end, define

$$T_n(u) = \frac{S_n(u)}{\sqrt{\widehat{C}_n(u,u)}}, \quad T_n^*(u) = \frac{S_n^*(u)}{\sqrt{\widehat{C}_n^*(u,u)}}, \tag{5}$$

where $\widehat{C}_n(u,v) = \frac{\Delta_n}{2} \sum_{i=1}^{n-1} \left(\xi(u)_i^n - \xi(u)_{i+1}^n\right)\left(\xi(v)_i^n - \xi(v)_{i+1}^n\right)$ and

$$\widehat{C}_n^*(u,v) = \Delta_n \frac{\mathbb{V}^*(\eta^*)}{\mathbb{E}^*[(\eta^*)^2]} \sum_{i=1}^{n-1} \left(\xi^*(u)_i^n - \xi(u)_{i+1}^n\right)\left(\xi^*(v)_i^n - \xi(v)_{i+1}^n\right), \quad u, v > 0, \tag{6}$$

are (consistent) estimators of the (conditional) asymptotic covariance for the unstudentized test statistic $S_n(u)$ and the corresponding MW bootstrap statistic $S_n^*(u)$, respectively.

Theorem 2. *Suppose the conditions for Theorem 1 hold. In addition, if $\mathbb{E}^*(|\eta_i^*|)^4 < \infty$, then for every $u \in \mathbb{K}$, where $\mathbb{K}$ is a compact subset of $\mathbb{R}_+$, and as $\Delta_n \to 0$, it follows that*

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}^*\left(T_n^*(u) \leq x\right) - \mathbb{P}\left(T_n(u) \leq x\right)\right| \overset{\mathbb{P}}{\longrightarrow} 0.$$

Theorems 1 and 2 demonstrate that the heterogeneous centering in (3) allows the MW bootstrap to overcome issues with bias, stochastic volatility and leverage effects, causing standard wild bootstraps to fail. Moreover, the functional limit theory is uniform (in u) with respect to the supremum norm, significantly generalizing the corresponding results in Hounyo et al. (2017) and Hounyo (2017).

Appendix

Before proceeding to the proofs, note that we adopt the same notation as in Hounyo et al. (2023).

A.1. Proof of Theorem 1

For part (a) of Theorem 1, we show that $S_n^*(u) \overset{\mathbb{P}^*}{\Longrightarrow} \Psi_T(u)$ in $l^\infty(\mathbb{K})$ in probability-$\mathbb{P}$, where $\mathbb{K}$ is a compact subset of $\mathbb{R}_+$. The process $\Psi_T(u)$ is defined by the limit (cf., Todorov and Tauchen, 2012, Theorem 1),

$$S_n(u) \overset{d_s}{\longrightarrow} \Psi_T(u) \quad \text{as} \quad \Delta_n \to 0, \tag{A.1}$$

is $\mathcal{F}$-conditionally Gaussian and has a zero mean function as well as a covariance function given by the integral $\int_0^T F\left(\sqrt{uc_s}, \sqrt{vc_s}\right) ds$ for every $u, v \in \mathbb{R}_+$, where $c_s = \sigma_s^2$ and

$$F(x,y) = \frac{e^{-(x+y)^2} - 2e^{-x^2-y^2} + e^{-(x-y)^2}}{2}, \quad \text{for} \quad x, y \in \mathbb{R}_+. \tag{A.2}$$

The proof proceeds in two steps.

Step 1. We show that

$$\left(S_n^*(u_1), \ldots, S_n^*(u_k)\right) \overset{d^*}{\to} \left(Z(u_1), \ldots, Z(u_k)\right),$$

in probability-$\mathbb{P}$, for any arbitrary $u_1, \ldots, u_k \in \mathbb{K}$, where, for every $u, v \in \mathbb{K}$, $Z(u) \sim N(0, C(u,u))$ with $C(u,u) = \int_0^T F\left(\sqrt{u}\sigma_s, \sqrt{u}\sigma_s\right) ds$ and $\text{Cov}(Z(u), Z(v)) = \int_0^T F\left(\sqrt{u}\sigma_s, \sqrt{v}\sigma_s\right) ds$.

Step 2. We show that for any $\epsilon > 0$, there exist $\delta > 0$ sufficiently small, such that

$$\limsup_{\Delta_n \to 0} \mathbb{P}\left(\mathbb{E}^*\left(\sup_{|u_1-u_2|<\delta,\ u_1,u_2\in\mathbb{K}} \left|S_n^*(u_1) - S_n^*(u_2)\right|\right) > \epsilon\right) = 0,$$

thereby establishing stochastic equicontinuity of the statistic.

Proof of Step 1. First, let us write $S_n^*(u) = \sum_{i=1}^n z_i^*(u)$, where $z_i^*(u) = \sqrt{\Delta_n}\left(\xi(u)_i^{n*} - \mathbb{E}^*\left(\xi(u)_i^{n*}\right)\right)$, and for which we have for any fixed $u, v > 0$, that $\mathbb{E}^*\left(z_i^*(u)\right) = 0$, and

$$\mathrm{Cov}^*\left(\sum_{i=1}^n z_i^*(u), \sum_{i=1}^n z_i^*(v)\right) = C_n^*(u,v) \xrightarrow{\mathbb{P}} \int_0^T F\left(\sqrt{u}\sigma_s, \sqrt{v}\sigma_s\right) ds,$$

as $\Delta_n \to 0$, such that $C_n^*(u,v) \equiv \frac{\Delta_n}{2}\sum_{i=1}^{n-1}\left(\xi(u)_i^n - \xi(u)_{i+1}^n\right)\left(\xi(v)_i^n - \xi(v)_{i+1}^n\right)$. The consistency follows using Hounyo et al. (2023, Proposition 1). Moreover, since $z_1^*(u), \ldots, z_n^*(u)$ are conditionally independent, it follows by the Berry–Esseen bound, for some small $\delta > 0$ and constant $K > 0$, that

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}^*\left(S_n^*(u) \le x\right) - \Phi\left(x/\sqrt{C(u,u)}\right)\right| \le K\sum_{i=1}^n \mathbb{E}^*\left|z_i^*(u)\right|^{2+\delta},$$

where $\Phi(\cdot)$ is the standard Gaussian cumulative distribution function. Next, we show that,

$$\sum_{i=1}^n \mathbb{E}^*\left|z_i^*(u)\right|^{2+\delta} \xrightarrow{\mathbb{P}} 0,$$

as $\Delta_n \to 0$. Indeed, using $|\xi(u)_i^n|^{2+\delta} \le 1$, we have

$$\begin{aligned}
&\sum_{i=1}^n \mathbb{E}^*\left|z_i^*(u)\right|^{2+\delta} \\
&= \Delta_n^{(2+\delta)/2}\sum_{i=1}^n \mathbb{E}^*\left|\left(\xi(u)_i^{n*} - \mathbb{E}^*\left(\xi(u)_i^{n*}\right)\right)\right|^{2+\delta} \\
&= \Delta_n^{(2+\delta)/2}\sum_{i=1}^{n-1}\left|\left(\xi(u)_i^n - \xi(u)_{i+1}^n\right)\right|^{2+\delta}\mathbb{E}^*\left|\eta_i^* - \mathbb{E}^*\left(\eta_i^*\right)\right|^{2+\delta} \\
&\le K\mathbb{E}^*|\eta^*|^{2+\delta}\Delta_n^{(2+\delta)/2}\sum_{i=1}^n|\xi(u)_i^n|^{2+\delta} \le K\mathbb{E}^*|\eta^*|^{2+\delta}\Delta_n^{(2+\delta)/2}\lfloor T/\Delta_n\rfloor = O_p(\Delta_n^{\delta/2}),
\end{aligned}$$

which is $o_p(1)$, where the first inequality follows from the C_r and Jensen inequalities, the second uses boundedness of the cosine function, and the last inequality follows as $\mathbb{E}^*|\eta^*|^{2+\delta} \le K < \infty$, by assumption of the external variables stated in the theorem, and $\Delta_n \to 0$ while T is fixed. Hence, it follows that $S_n^*(u) \xrightarrow{d^*} Z(u)$, in probability-$\mathbb{P}$, where $Z(u) \sim N(0, C(u,u))$.

Proof of Step 2. First, we apply the decomposition,

$$\begin{aligned}
&|S_n^*(u_1) - S_n^*(u_2)| \\
&= \Delta_n^{1/2}\left|\sum_{i=1}^n\left(\xi(u_1)_i^{n*} - \mathbb{E}^*\left(\xi(u_1)_i^{n*}\right)\right) - \left(\xi(u_2)_i^{n*} - \mathbb{E}^*\left(\xi(u_2)_i^{n*}\right)\right)\right| \\
&= \Delta_n^{1/2}\left|\sum_{i=1}^{n-1}\left[\left(\xi(u_1)_i^n - \xi(u_1)_{i+1}^n\right) - \left(\xi(u_2)_i^n - \xi(u_2)_{i+1}^n\right)\right]\left(\eta_i^* - \mathbb{E}^*\left(\eta_i^*\right)\right)\right| \\
&\le 2\Delta_n^{1/2}\sum_{i=1}^n\left|\left(\xi(u_1)_i^n - \xi(u_2)_i^n\right)\right|\left|\eta_i^* - \mathbb{E}^*\left(\eta_i^*\right)\right|.
\end{aligned}$$

Next, we use the trigonometric properties of cosine and sine functions stated in Hounyo et al. (2023, equation (S.7)) together with the definition of $\xi(u)_i^n$, and observe that

$$\left|\left(\xi(u_1)_i^n - \xi(u_2)_i^n\right)\right| \le \sqrt{2}\left|\Delta_n^{-1/2}\Delta_i^n X\right|\left|\sqrt{u_1} - \sqrt{u_2}\right| \vee |u_1 - u_2|.$$

Hence, we have

$$\begin{aligned}
&|S_n^*(u_1) - S_n^*(u_2)| \le 2\sqrt{2}\Delta_n^{-1/2}\left|\sqrt{u_1} - \sqrt{u_2}\right| \vee |u_1 - u_2| \\
&\times \Delta_n\sum_{i=1}^n\left|\Delta_n^{-1/2}\Delta_i^n X\right|\left|\eta_i^* - \mathbb{E}^*\left(\eta_i^*\right)\right|.
\end{aligned}$$

Since $\mathbb{K}$ is compact (thus, bounded), there exists $K > 0$ such that $\left|\sqrt{u_1} - \sqrt{u_2}\right| \le K$, and therefore,

$$|S_n^*(u_1) - S_n^*(u_2)| \le K\Delta_n^{-1/2}|u_1 - u_2|\,\Delta_n\sum_{i=1}^n\left|\Delta_n^{-1/2}\Delta_i^n X\right|\left|\eta_i^* - \mathbb{E}^*\left(\eta_i^*\right)\right|,$$

implying that, since $\mathbb{E}^*|\eta^*| \le \Delta < \infty$, we have

$$\begin{aligned}
&\mathbb{E}^*\Big(\sup_{|u_1-u_2|<\delta,\ u_1,u_2\in\mathbb{K}}\left|S_n^*(u_1) - S_n^*(u_2)\right|\Big) \\
&\le K\Delta_n^{-1/2}\delta\mathbb{E}^*|\eta^*|\left(\Delta_n\sum_{i=1}^n\left|\Delta_n^{-1/2}\Delta_i^n X\right|\right) \le K\Delta_n^{-1/2}\delta\Big(\Delta_n\sum_{i=1}^n\left|\Delta_n^{-1/2}\Delta_i^n X\right|\Big).
\end{aligned}$$

Then, for any $\epsilon > 0$, there exist $\delta > 0$, such that

$$\begin{aligned}
&\mathbb{P}\left(\mathbb{E}^*\left(\sup_{|u_1-u_2|<\delta,\ u_1,u_2\in\mathbb{K}}\left|S_n^*(u_1) - S_n^*(u_2)\right|\right) > \epsilon\right) \\
&\le \mathbb{P}\left(\left(\Delta_n\sum_{i=1}^n\left|\Delta_n^{-1/2}\Delta_i^n X\right|\right) > \frac{\epsilon}{K\Delta_n^{-1/2}\delta}\right),
\end{aligned}$$

which becomes arbitrarily small as $\Delta_n \to 0$. That is, the requisite limit result in Step 2 follows since the scaled sum $\Delta_n\sum_{i=1}^n|\Delta_n^{-1/2}\Delta_i^n X| = O_p(1)$, uniformly on the compact set $\mathbb{K}$ (since it does not depend on u), and we can always choose $\delta = \Delta_n^{1/2+\epsilon} \to 0$, such that

$$\mathbb{P}\left(\left(\Delta_n\sum_{i=1}^n\left|\Delta_n^{-1/2}\Delta_i^n X\right|\right) > \frac{\epsilon}{K\Delta_n^{-1/2}\delta}\right) \to 0,$$

uniformly on $\mathbb{K}$, thus establishing stochastic equicontinuity.

Finally, the proof of part (b) of Theorem 1 follows using the same arguments provided for Hounyo et al. (2023, Theorem 1(b)) and is, thus, omitted for brevity. □

A.2. Proof of Theorem 2

The proof follows by the same arguments as those provided for Hounyo et al. (2023, Theorem 2), that is, by verifying the condition $\widehat{C}_n^*(u,u) - C_n^*(u,u) \xrightarrow{\mathbb{P}^*} 0$ for the MW bootstrap procedure.

We readily have $\mathbb{E}^*(\widehat{C}_n^*(u,u) - C_n^*(u,u)) = 0$ by definition of $\widehat{C}_n^*(u,u)$ and $C_n^*(u,u)$. Hence, we need to show that,

$$\mathbb{V}^*\left(\widehat{C}_n^*(u,u) - C_n^*(u,u)\right) \xrightarrow{\mathbb{P}} 0, \qquad \text{as} \quad \Delta_n \to 0. \tag{A.3}$$

To this end, it holds that,

$$\begin{aligned}
&\mathbb{V}^*\left(\widehat{C}_n^*(u,u) - C_n^*(u,u)\right) \\
&= \left(\Delta_n\frac{\mathbb{V}^*(\eta^*)}{\mathbb{E}^*[(\eta^*)^2]}\right)^2\sum_{i=1}^{n-1}\left(\xi(u)_i^n - \xi(u)_{i+1}^n\right)^2\left(\xi(v)_i^n - \xi(v)_{i+1}^n\right)^2\mathbb{V}^*[(\eta^*)^2] \\
&\le 2\Delta_n^{3/2}\left(\frac{\mathbb{V}^*(\eta^*)}{\mathbb{E}^*[(\eta^*)^2]}\right)^2\mathbb{E}^*[(\eta^*)^4]\left(\sum_{i=1}^{n-1}\left(\left(\xi(u)_i^n\right)^4 + \left(\xi(u)_{i+1}^n\right)^4\right)^{\frac{1}{2}}\right) \\
&\quad + 2\Delta_n^{3/2}\left(\frac{\mathbb{V}^*(\eta^*)}{\mathbb{E}^*[(\eta^*)^2]}\right)^2\mathbb{E}^*[(\eta^*)^4]\left(\sum_{i=1}^{n-1}\left(\left(\xi(v)_i^n\right)^4 + \left(\xi(v)_{i+1}^n\right)^4\right)^{\frac{1}{2}}\right) \le O_p\left(\Delta_n^{3/2}\right),
\end{aligned}$$

since $\left(\xi(u)_i^n\right)^4 \le 1$, thereby concluding the proof. □

Data availability

No data was used for the research described in the article.

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