Fiscal Policy and Longevity

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Vor vorkunnarinnar

Gylltir geislar dekkja kroppa,
sumarkjólar þekja fátt.
Ungar meyjar með spékoppa,
drekka bjór, fá hópafslátt.

Bölvun mætir sólarilur,
sem áður kætti drengsins hjarta.
Lærdómur sem djúpur hylur,
drepur alla drauma bjarta.

Upprisinn á öðru ári,
með meirapróf í hagfræði.
Hans kjammar þaktir skegghári.
Nú frelsaður frá volæði.

-Þorsteinn Sigurður Sveinsson, 2014

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Abstract

The thesis uses the Blanchard-Yaari model to examine the macroeconomic implications of increased longevity. The model analysis demonstrates several economic implications of increased longevity and the capacity of fiscal policy to respond to these economic implications. Optimal retirement age is determined under conditions of age-dependent productivity.

Chapter 1

Introduction

A long and prosperous life is desired by most. This has been made possible by a paradigm shift in advancement of medical sciences and a great diversity of other science and technology disciplines. During the last decades it has been fueled by increased public and private funding of research and development of life sciences. The public sector has considered health care as one of the pillars of public policy and the private sector has seen opportunities for growth and prosperity in product developments.

This evolution will continue resulting in increased longevity of the global population. The consequences will, however, be more complicated and need to be approached in a more holistic manner, as longevity will create social and economic side-effects that may be difficult to foresee and resolve. This raises the question of the carrying capacity of nations and regions which in the past has repeatedly created political turmoil and caused serious conflicts and social disasters. This complex issue has caught my interest and motivated me to study the subject further.

Life expectancy has steadily been on the rise during the last decades. Figure 1.1 shows the average life expectancy at birth in Denmark for the past 160 years. An upwards slope follows the growth of economic and social prosperity. In 1840 the average life expectancy was 45 years for women and 43 years for men. In 2012 life expectancy at birth had nearly doubled and was 82 years for women and 78 years for men. This statistic is not completely descriptive since infant mortality was considerably higher in 1840 than now.

Life expectancy will continue to rise. In a 2004 UN report on the global demographic projections until the year 2300 life expectancy at birth was forecast. According to the report life expectancy in northern Europe will continue to rise at an even pace. By the year 2050 life expectancy will be

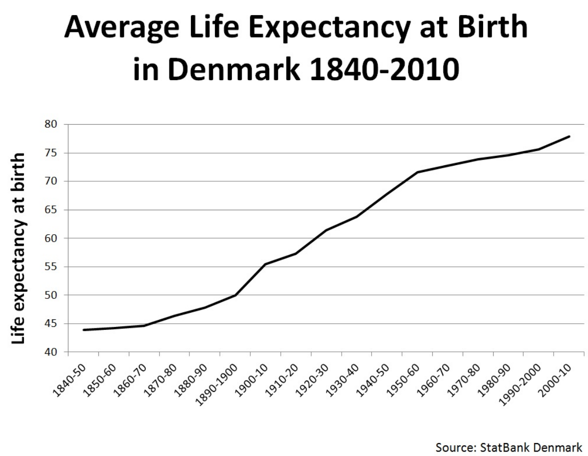

Figure 1.1: Life expectancy in Denmark 1840-2010

about 83 years, which is an improvement from the slightly less than 80 years observed presently in Denmark. Life expectancy will be at 90 years around the year 2125 and by 2300 the expected length of the lifetime of a newborn is projected to be above 100 years. The implications of this constant increase in life expectancy, both observed in the past and forecast into the future are the main motivations behind the thesis.

Increased longevity means that higher percentage of the population will be healthy, and active when they reach retirement age. This is especially relevant in developed countries, where the working population will not be able to sustain the rapidly increasing number of retired senior citizens and where the social security system is not able to maintain the present level of benefits to the needy. This problem is still further aggravated by high youth unemployment and the inability of the education system to attract qualified students to fields that are locomotives in cutting edge technologies and economic growth. Without a paramount change in the social security system and retirement policy this might lead to severe economic inefficiencies.

One fiscal response is to increase the retirement age considerably. With improved health and social support for the older citizens it should be possible to allow those that can and so wish to work until the age of 70. This has been a common practice in Iceland many years. Allowing for this flexibility would have a threefold effect. It would create higher incomes for individuals and families which results in higher turnover and economic growth. It would increase government income through taxes and mitigate social disadvantages resulting from unemployed seniors. It would as well be a tremendous relief on the pension funds that in most cases are totally incapable to sustain its present role and responsibilities.

A fiscal response could be based in coercive mandates or voluntary programs. For example, a forced retirement age might not suit some individuals and lead to economic inefficiencies. Ultimately the individual is most competent to make his own economic decisions. When designing a fiscal policy the pros and con have to be evaluated based on how can the government can positively influence the decision of individuals without coercion.

The present and foreseeable demographic changes are of significant importance both at the national and international level. Different cultures could respond differently. In the developed countries the implications of increased longevity would fall on the social security system and incentivizes the government to take appropriate action. However in impoverished countries with underdeveloped infrastructure the strain of increased longevity would fall mainly on family members.

An example of a response to demographic change comes from China. Facing unsustainable population growth China adopted the one-child policy as a response to the increased longevity of its population. This policy has had a socioeconomic impact on the country. Culturally male offspring are expected to take care of their parents. Therefore male babies were preferred leading to infanticide of female babies. This example demonstrates that the implications of the government's response to longevity can be substantial and unforeseen.

Increased longevity is also important at a political level. It calls for changes in government spending which could have different implications for different generations, causing intergenerational tensions. An example of this is a wave of protest in 2010 in France as a response to an increase in the retirement age. Furthermore the increased longevity will increase the age of the median voter which could affect the outcomes of the demographic process.

Fairy tales of the future often foster interesting questions without any real deliverance. It sparks the imagination about what the future will bring. Could there be a maximum life expectancy that humans can achieve, or are there no such upper bounds? Could the present generations be among the last to face their death with certainty within a number of decades? Will the individuals of the future plan their lives centuries into the future? Will the death rate become more predictable and uniform as diseases become increasingly preventable? These questions cannot be answered, but they have inspired, the author of this thesis, and created a keen academic interest about the abundant and complex effects of longevity.

The thesis sets up the Blanchard-Yaari macroeconomic model to analyze the economic effects of increased longevity, implications to the optimal retirement age and the fiscal response to these changes. A special focus will be set on the individual's decisions regarding retirement and the ability of fiscal policy to achieve desirable outcomes in face of increased longevity.

Chapter 2

The Yaari Model

In this section a model examining the economic decisions of individuals facing uncertainty about time of death will be set up. The model was developed by Yaari in his 1965 paper *Uncertain Lifetime, Life Insurance, and the Theory of the Consumer*. Yaari introduces the effects of uncertain lifetime length on the consumption choices of the agent along with presenting the effects of the availability of *actuarial notes* which act as an insurance policy against longevity risk.

In the next chapter we will examine Blanchard's additions to Yaari's model as proposed in his 1985 paper *Debt, Deficits and Finite Horizons*, accompanied with further extensions to the Blanchard-Yaari model to facilitate the research topic. Blanchard transforms the Yaari's findings into a model that can be manipulated in order to examine a specific fiscal policy. Importantly, Blanchard's extensions of the Yaari model allow for aggregation of values which allows for identification of macroeconomic effects.

In Yaari's model time of death, denoted by T, is stochastic and by implication lifetime consumption maximization is also treated as stochastic. Yaari also introduces *actuarial notes*, which can be treated as life insurance, and realizes their effect on the consumption choice of the consumer. These actuarial notes dampen the consumption loss associated with the lifetime uncertainty. Yaari specifies several cases in his article, firstly whether actuarial notes are available or not and secondly which utility function will be used, the Fisher utility function with wealth constraint or the Marshall utility function without wealth constraint. In this section the focus will be on the Fisher utility case. Yaari's model is a invaluable building block in researching the effects of longevity on a sustainable fiscal policy.

2.1 Consumption and Random Horizons

As specified above we begin with the reasonable assumption that T, the time of death, is a random variable that can take any value in the interval $[0, \bar{T}]$. The density function (DF) for T is denoted by $f(T)$ and satisfies:

$$f(T) \geq 0, \quad \forall\, T \geq 0, \qquad \int_0^{\bar{T}} f(T)\mathrm{d}T = 1$$

The time of death lies between zero and $\bar{T}$ with certainty. Yaari assumes further that $f(T)$ is actually positive for all T in the interval $[0, \bar{T}]$. This is resonates with reality since death can strike at any time. The agent's lifetime utility under certainty is defined by:

$$\Lambda(T) \equiv \int_0^T U[C(\tau)]e^{-\rho\tau}\mathrm{d}\tau \tag{2.1}$$

The instantaneous utility function is defined by $U[C(\tau)]$ where $C(\tau)$ is consumption at time τ and ρ is the agent's rate of time preference. The agent does not make consumption-leisure decisions. Since the time of death is assumed to be stochastic, and therefore lifetime length is also stochastic, it's an impossible task for the agent to maximize lifetime utility, denoted by $\Lambda(T)$. Therefore the agent selects a consumption path that makes the expected value of $\Lambda(T)$ the greatest. $E[\Lambda(T)]$ denotes expected lifetime utility.

$$\begin{aligned} E[\Lambda(T)] &\equiv \int_0^{\bar{T}} f(T)\Lambda(T)\mathrm{d}T \\ &= \int_0^{\bar{T}} \left[\int_\tau^{\bar{T}} f(T)\mathrm{d}T \right] U[C(\tau)]e^{-\rho\tau}\mathrm{d}\tau \\ &= \int_0^{\bar{T}} [1 - F(\tau)]U[C(\tau)]e^{-\rho\tau}\mathrm{d}\tau \end{aligned} \tag{2.2}$$

Where $F(\tau)$ is the cumulative distribution function (CDF) of T and $1-F(\tau)$ denotes the probability that the agent is alive at time τ. The optimal consumption path must satisfy a budget identity, which is defined as:

$$\dot{A}(\tau) = r(\tau)A(\tau) + W(\tau) - C(\tau) \tag{2.3}$$

Where $r(\tau)$ is the interest rate at time τ and $W(\tau)$ is the agent's non-interest income which can be interpreted as wages. $A(\tau)$ is the wealth of the agent at time τ and $\dot{A}(\tau) \equiv dA(\tau)/d\tau$. The change in assets over time is

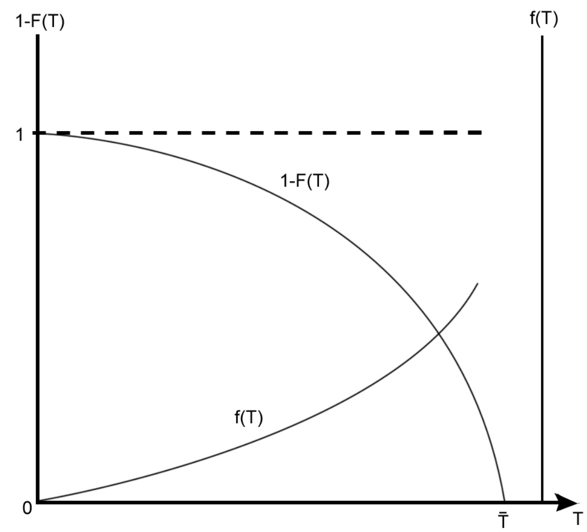

Figure 2.1: CDF and DF visualization

dependent on the interest income on existing assets, the non financial income and consumption on each time. To account for the stochasticity of the time of death the non-negative real asset constraint is introduced. Real assets at time of death must be non-negative, i.e., $Prob[A(T) \geq 0] = 1$. Which is the same as saying $A(\tau) \geq 0, \forall \tau \in [0, \bar{T}]$ because death can happen at any moment. This eliminates the possibility for agent to have substantial negative assets, which could've been used to facilitate increased consumption in previous periods, and die before they can balance their budget.

Since agents will maximize their consumption they choose to hold no assets at time $\bar{T}$, that is $A(\bar{T}) = 0$, because they have used all the assets to facilitate consumption and they know with certainty that they will not survive past time $\bar{T}$, even though they do not know their time of death with certainty. The non-negative assets constraint is summarized by stating:

$$A(\bar{T}) = 0 \quad and \quad \dot{A}(\tau) \geq 0 \Leftrightarrow C(\tau) \leq W(\tau) \quad if \quad A(\tau) = 0$$

The agent maximizes the expected lifetime utility function, subject to the budget identity and the non-negative real asset constraint. Initial wealth is defined as $A(0) = A_0$ and consumption is constrained by $C(\tau) \geq 0$. The maximization problem becomes:

$$
\begin{aligned}
\max_{\{C(\tau)\}} \quad & E[\Lambda(T)] = \int_0^{\bar{T}} [1 - F(\tau)] U[C(\tau)] e^{-\rho\tau} \mathrm{d}\tau \\
s.t. \quad & (i) \quad \dot{A}(\tau) = r(\tau)A(\tau) + W(\tau) - C(\tau) \\
& (ii) \;\; C(\tau) \geq 0, \forall \tau \in [0, \bar{T}] \\
& (iii) \; C(\tau) \leq W(\tau) \quad if \quad A(\tau) = 0 \\
& (iv) \;\; A(0) = A_0 \quad (predetermined) \\
& (v) \quad A(\bar{T}) = 0
\end{aligned}
\tag{2.4}
$$

Note that it is assumed that the agents have no bequest motive and only receive utility from their own consumption. Furthermore, the solution of this problem is composed of three segments. Consumption in the first segment is bound by constraint (*ii*) and set to $C(\tau) = 0$. Consumption in the second segment is bound by constraint (*iii*) and set to $C(\tau) = W(\tau)$. In the third segment the "interior solution" for consumption is found by solving the *current value Hamiltonian*:

$$
\begin{aligned}
\mathcal{H}_{\mathcal{C}} = [1 - F(\tau)]U[C(\tau)] + \lambda(\tau)[r(\tau)A(\tau) + W(\tau) - C(\tau) - \dot{A}(\tau)] \\
+ \mu(\tau)[A(\bar{T})]
\end{aligned}
\tag{2.5}
$$

Furthermore we get.

$$
\frac{\partial \mathcal{H}_{\mathcal{C}}}{\partial C(\tau)} = 0 \Leftrightarrow \lambda(\tau) = [1 - F(\tau)]U'[C(\tau)] \tag{2.6}
$$

$$
\dot{\lambda}(\tau) = -\frac{\partial \mathcal{H}_{\mathcal{C}}}{\partial A(\tau)} + \rho\lambda(\tau) = -\lambda(\tau)r(\tau) + \rho\lambda \Leftrightarrow \frac{\dot{\lambda}(\tau)}{\lambda(\tau)} = \rho - r(\tau) \tag{2.7}
$$

By combining equations (2.6) and (2.7) the agent's consumption *Euler equation* is obtained. The Euler equation denotes the proportional change in consumption over time.

$$
\frac{\dot{C}(\tau)}{C(\tau)} = \sigma[C(\tau)][r(\tau) - \rho - \beta(\tau)] \tag{2.8}
$$

Where $\sigma[C(\tau)] \equiv -U'[C(\tau)]/[C(\tau)U''[C(\tau)]] > 0$ is the *elasticity of intertemporal substitution*, which is defined as one over the relative risk aversion. This measures the responsiveness to a change in the interest rate on the agent's growth rate of consumption. $\beta(\tau) \equiv f(\tau)/[1 - F(\tau)] > 0$ is the *instantaneous probability of death* or *hazard rate* at time τ. The hazard rate will be manipulated in later chapters to simulate longevity, since a lower hazard rate implies that agents live longer. These results can be compared to results in a model with non-random time of death by substituting $F(\tau)$ with 0, because then it's certain that the agent is alive for every $\tau \in [0, \bar{T}[$ and every agent will die at time $\bar{T}$. The Euler equation under certainty about time of death is:

$$\frac{\dot{C}(\tau)}{C(\tau)} = \sigma[C(\tau)][r(\tau) - \rho] \tag{2.9}$$

We see that the Euler equations (2.8) and (2.9) differ only by the inclusion of $\beta(\tau)$ in the former. The uncertainty of survival leads agents to discount the future more heavily than when time of death is deterministic, this makes intuitive sense since there is a positive probability that the agent will not live long enough to enjoy the planned consumption path.

By assuming that an increase in longevity will homogeneously effect society its increase would have a direct effect of lowering the hazard rate for all $\tau \in [0, \bar{T}]$. This would in turn would lead agents to drift from equation (2.8) towards (2.9) raising the optimal consumption path as the future is discounted less.

2.2 Actuarial Notes

Up until this point no insurance possibilities have been introduced. The absence of insurance possibilities forces agents to hold assets in all periods to act as a *buffer* in the case of immediate death so that the non-negative wealth constraint holds. This forces agent to allocate less of their wealth towards consumption which in turn leads to lesser lifetime utility. The presence of some sort of insurance against lifetime uncertainty is a realistic assumption.

To address this Yaari suggests the existence of *actuarial notes* which agents can buy and sell. The purchaser of an actuarial note gets a constant stream of payment until his death. The notes are in a sense a annuity which at the time of the purchaser's death leave the seller (most likely an insurance company) free of any obligations. The rate of actuarial notes is denoted by $r^A(\tau)$ and we expect that $r^A(\tau) > r(\tau)$ where $r(\tau)$ is the rate on regular notes.

If an agent wishes to take out a life insurance policy he can simultaneously sell actuarial notes and purchase regular notes by the same amount.

The difference between the rate on the regular notes and the actuarial notes is the insurance premium. To simplify things the actuarial notes are assumed to be issued at a *fair* rate.

If a actuarial note is bought at time τ then two scenarios are possible, either the buyer will die before the time $\tau + d\tau$ and the note is canceled or the buyer survives until $\tau + d\tau$ and interest is received. Therefore actuarial fairness implies:

$$[1 + r^A(\tau)d\tau]\left(\frac{1 - F(\tau + d\tau)}{1 - F(\tau)}\right) = 1 + r(\tau)d\tau \tag{2.10}$$

The term in round brackets corrects for the possibility of the purchaser passing away in between τ and dτ. Rearranging (2.10) yields:

$$r^A(\tau) = \left(\frac{1 - F(\tau)}{1 - F(\tau + d\tau)}\right)r(\tau) + \frac{[F(\tau + d\tau) - F(\tau)]/d\tau}{1 - F(\tau + d\tau)} \tag{2.11}$$

By letting $d\tau \to 0$ the term $([1 - F(\tau)]/[1 - F(\tau + d\tau)])$ converges to 1 and the second term $[(F(\tau + d\tau) - F(\tau))/d\tau]/[1 - F(\tau + d\tau)]$ approaches the hazard rate $\beta(\tau)$. From this the following relation is obtained.

$$r^A(\tau) = r(\tau) + \beta(\tau) \tag{2.12}$$

The assumption of actuarial fairness is not necessarily a realistic one. Actuarial fairness can be broken by factors such as the operating costs of the issuer of the notes. It can also be broken because of *adverse selection* which is the result of the information gap between the purchaser and the issuer of the actuarial note. Adverse selection can result in only people that know they will live long, maybe because of good health, buying notes and only people with shorter life expectancy issuing the notes. In the next chapter the hazard rate is assumed to be constant for all individuals and therefore the adverse selection effect is neutralized.

The agent chooses to hold only actuarial notes, this is caused by three factors. Firstly, the agent has no bequest motive so holding excess wealth at death does not enter into consideration. Secondly, the agent is restrained to have non-negative wealth upon his death with certainty, if the agent were to hold any outstanding negative assets this constraint would be violated. Lastly, the actuarial notes provide greater return than regular notes, since the hazard rate is always positive. Therefore the budget identity from equation (2.3) can be modified:

$$\dot{A}(\tau) = r^A(\tau)A(\tau) + W(\tau) - C(\tau) \tag{2.13}$$

Under these assumptions an agent could "beat the system" and engage in unlimited consumption. He could sell as many actuarial notes as he pleases, because any debt to the buyer of the notes (the insurance company) will be canceled upon the agent's death. This would result in a "Ponzi scheme" situation as the agent pays of the current actuarial payment by taking out new actuarial notes.

To address this problem a "global constraint" on borrowing is introduced. The insurance company will refuse to buy actuarial notes from an agent after he reaches the age $\bar{T} - \Delta$ where $\bar{T} > \Delta > 0$. $A(\tau)$ is defined as the agent's stock of actuarial notes at time τ. This assumption is possible because the agent holds only actuarial notes. Here t is a previous time which the agent has made consumption saving decisions leading him to save or dissave by the amount $W(t) - C(t)$. Accumulated assets at time τ are therefore:

$$\int_0^\tau e^{\int_t^\tau r^A(s)\mathrm{ds}}[W(t) - C(t)]dt \tag{2.14}$$

This integral is defined for all $\tau \in [0, \bar{T}]$. Since $\bar{T} - \Delta$ is within these boundaries the constraint can be rewritten for $A(\bar{T} - \Delta)$:

$$\int_0^{\bar{T}-\Delta} e^{\int_\tau^{\bar{T}-\Delta} r^A(s)ds}[W(\tau) - C(\tau)]d\tau \geq 0 \tag{2.15}$$

By factoring out out the quantity $e^{\int_0^{\bar{T}-\Delta} r^A(s)ds}$ the following constraint is obtained:

$$\int_0^{\bar{T}-\Delta} e^{-\int_0^\tau r^A(s)\mathrm{ds}}[W(\tau) - C(\tau)]d\tau = 0 \tag{2.16}$$

Where the inequality has been replaced by an equality. By approximating for small Δ and introducing the agent's initial wealth the solvency condition becomes.

$$A(\tau) = A(0) + \int_0^{\bar{T}} e^{-\int_0^\tau r^A(s)ds}[W(\tau) - C(\tau)]d\tau = 0 \tag{2.17}$$

This constraint states that the present value of the consumption path must be equal to the present value of the initial wealth, $A(0)$, and present value of the current and future non-interest income stream, discounted by the rate of the actuarial notes. Using this condition and maximizing the consumption path for the agents the following Euler equation is obtained.

$$\frac{\dot{C}(\tau)}{C(\tau)} = \sigma[C(\tau)][r^A(\tau) - \rho - \beta(\tau)] = \sigma[C(\tau)][r(\tau) - \rho] \tag{2.18}$$

This result is remarkable since the introduction of the actuarial notes makes the Euler equation above identical to the Euler equation with deterministic lifetime length (2.9). Even though the Euler equations are identical the same lifetime consumption is not obtainable in both cases because the consumption possibility frontier will differ. There are considerable differences in the effect of longevity on the consumption path of agents depending on whether actuarial notes are present or not.

Chapter 3

The Blanchard-Yaari Model

This chapter introduces a macroeconomic model which will be used to understand the effects of longevity on the economy and specifically its fiscal policy implications.

Blanchard's extensions of Yaari's model facilitate macroeconomic analysis. In his 1985 paper *Debt, Deficits and Finite Horizons* he provided a model in which the horizon of the agents, dependent on the hazard rate, can be manipulated arbitrarily. In the model the steady state of the economy and the effects of fiscal policy can vary with the horizon of individuals. Furthermore Blanchard provides the effects of saving for retirement by assuming that labor productivity declines with the agent's age. Blanchard's model brings us one step closer to understand the thesis topic.

So far only the economics decisions of individuals have been examined and not of the aggregate values for the economy as a whole. To obtain these aggregate values representative consumer has to be obtained. There are two problems with finding a representative consumer in the model presented by Yaari. Agents in the model differ in two respects, they have different ages and different horizons. This leads the agents to have different wealth levels and consumption choices which makes aggregation a cumbersome task.

3.1 Individual Households

Blanchard addresses the problem of aggregation by assuming that all agents have identical *hazard rates*, β, which leads to all agents having expected lifetime of $1/\beta$ at all times. However the agents are not all identical because they might have differing levels of wealth depending on their age and therefore have a different propensity to consume. This assumption allows for aggregate values to be obtained even though agents are heterogeneous by age.

Here, contrary to the Yaari model in the previous subsection, the agent can hypothetically live until infinity even though the probability of that is minuscule, this is directly caused by the identical hazard rate. This seems far removed from reality. However by thinking of the agents as families and β as the probability of that family ending by leaving no descendants the assumption of a constant hazard rate seems more reasonable.

A large population is assumed, therefore generalization about each cohort can be made and eventually the aggregate values for the economy can be found. A cohort born at time 0 has the size $\beta e^{-\beta\tau}$ at time τ, since a proportion of the cohort has died in the time elapsed since birth. The size of the whole population at time τ is denoted as $P(\tau)$ and is normalized to 1 for all time periods. This results in $\int_{-\infty}^{\tau} \beta e^{-\beta(\tau-v)}\mathrm{d}v = 1$ where v represents the time of birth. This implies that the population is static and a cohort of size β is born at the same time as a equal number of agents die. The density function for agent's time of death, $f(T)$, is the exponential probability density function:

$$f(T) = \begin{cases} \beta e^{-\beta T} & for\ T \geq 0 \\ 0 & for\ T < 0 \end{cases} \tag{3.1}$$

The probability that an agent will be alive at time τ is given by $[1 - F(\tau)] = \int_{\tau}^{\infty} f(T)dT = e^{-\beta\tau} = f(\tau)/\beta$. The agent maximizes expected utility just as in the previous section and instantaneous utility is logarithmic. The agent maximizes:

$$E[\Lambda(v,t)] = \int_{t}^{\infty} [1 - F(\tau - t)]\log(C(v,\tau))e^{\rho(t-\tau)}d\tau \tag{3.2}$$

Where v is time of the agent's birth, t is the present time and τ is a future moment for which the agent is planing consumption. In the exponential distribution $[1 - F(\tau - t)] = e^{\beta(t-\tau)}$. Since the instantaneous probability of death, β, is constant across all agents and the only source of stochasticity is the time of death the expected utility function becomes:

$$E[\Lambda(v,t)] = \int_{t}^{\infty} \log(C(v,\tau))e^{(\rho+\beta)(t-\tau)}d\tau \tag{3.3}$$

The agent's budget identity assumes the existence of fair actuarial notes and lump-sum taxes levied by the government. Taxes at time τ are denoted as $T(\tau)$. The agents budget identity is:

$$\dot{A}(v,\tau) = [r(\tau)+\beta]A(v,\tau) + W(\tau) - T(\tau) - C(v,\tau) \tag{3.4}$$

The agent holds all of his assets in actuarial notes. Note that this is quite similar to the condition found in the previous chapter in equation (2.13), the only difference is that now the lump sum taxation levied by the government is included. To avoid Ponzi-scheme behavior as described in the previous section a *solvency condition* is introduced. To obtain the solvency condition the budget identity is premultiplied by $e^{-R^A(t,\tau)}$ where $R^A(t,\tau) \equiv \int_t^\tau r^A(s)ds = \int_t^\tau [r(s)+\beta]ds$ and rearranged to get the following.

$$[\dot{A}(v,\tau) - [r(\tau)+\beta]A(v,\tau)]e^{-R^A(t,\tau)} = [W(\tau) - T(\tau) - C(v,\tau)]e^{-R^A(t,\tau)} \tag{3.5}$$

R^A is differentiated by applying *Leibniz rule*:

$$\frac{d}{d\tau}\left(\int_t^\tau [r(s)+\beta]ds\right) = \int_t^\tau 0ds + (r(\tau)+\beta)*1 - (r(\tau)+\beta)*0 \tag{3.6}$$

By using the fact that $dR^A(t,\tau)/d\tau = r(\tau)+\beta$ the following can be obtained:

$$\frac{d}{d\tau}[A(v,\tau)e^{-R^A(t,\tau)}] = [W(\tau) - T(\tau) - C(v,\tau)]e^{-R^A(t,\tau)} \tag{3.7}$$

Integration on both sides yields:

$$\int_t^\infty dA(v,\tau)e^{-R^A(t,\tau)} = \int_t^\infty [W(\tau) - T(\tau) - C(v,\tau)]e^{-R^A(t,\tau)}d\tau \tag{3.8}$$

Solving the integrals, realizing that $e^{-R^A(t,t)} = 1$ we get the following relation.

$$\lim_{\tau\to\infty} e^{-R^A(t,\tau)}A(v,\tau) - A(v,t) = H(\tau) - \int_t^\infty C(v,\tau)e^{-R^A(t,\tau)}d\tau \tag{3.9}$$

From this we get the solvency condition below. It states that if the agent is alive at time τ then the present value, discounted by the annuity rate, of assets is equal to zero as τ approaches infinity. This is an extension of Yaari's *global constraint on borrowing* assuming that A(0)=0 and there is no upper bound on lifetime length.

$$\lim_{\tau\to\infty} e^{-R^A(t,\tau)}A(v,\tau) = 0 \tag{3.10}$$

The lifetime budget restriction is depicted in equation (3.11). On the left hand side is the accumulated wealth for an agent born at time v in addition to human wealth at time t. On the right hand side is the current value of the agent's consumption plan. Human wealth is defined in equation (3.12).

$$A(v,t) + H(t) = \int_t^\infty C(v,\tau)e^{-R^A(t,\tau)}d\tau \tag{3.11}$$

Human wealth, $H(t)$ is defined as the present value of future lifetime income. Note that even though agents have different ages and therefore different level of non-human wealth, $A(v,t)$, the human wealth will be identical for all agents because of the identical life expectancy.

$$H(t) \equiv \int_t^\infty [W(\tau) - T(\tau)]e^{-R^A(t,\tau)}d\tau \tag{3.12}$$

The utility maximization problem for the agent is:

$$\begin{aligned} \max_{\{C(v,\tau)\}} E[\Lambda(v,t)] &= \int_t^\infty \log(C(v,\tau))e^{(\rho+\beta)(t-\tau)}d\tau \\ s.t. \qquad A(v,t) + H(t) &= \int_t^\infty C(v,\tau)e^{-R^A(t,\tau)}d\tau \end{aligned} \tag{3.13}$$

To find a solution to this maximization problem the following Lagrangian is set up and solved.

$$\mathcal{L} = \log(C(v,\tau))e^{(\rho+\beta)(t-\tau)} + \lambda(t)\left[\int_t^\infty C(v,\tau)e^{-R^A(t,\tau)}d\tau - A(v,t) - H(t)\right] \tag{3.14}$$

The first order condition associated with the lifetime utility maximization is:

$$\frac{d\mathcal{L}}{dC(v,\tau)} = \left(\frac{1}{C(v,\tau)}\right)e^{(\rho+\beta)(t-\tau)} = \lambda(t)e^{-R^A(t,\tau)}, \quad \tau \in [t,\infty] \tag{3.15}$$

Where $\lambda(t)$ is the Lagrange multiplier associated with the lifetime budget restriction and represents the marginal expected lifetime utility of wealth. Realizing this, the first order condition implies that the agent plans his lifetime consumption such that the discounted marginal utility of consumption is equated with the discounted marginal utility of wealth. This makes intuitive sense. Here the agent's Euler equation is found by differentiating.

$$\frac{\dot{C}(v,\tau)}{C(v,\tau)} = r(\tau) - \rho \tag{3.16}$$

Assuming that $\tau = t$ then $C(t,v) = 1/\lambda(t)$. Having realizing this and by using the budget restriction in the utility maximizing problem along with the first order conditions a relation between consumption and total wealth can be obtained.

$$\begin{aligned}\int_t^\infty C(v,t)e^{(\rho+\beta)(t-\tau)}d\tau &= \int_t^\infty C(v,\tau)e^{-R^A(t,\tau)}d\tau \\ \left(\frac{C(v,t)}{\rho+\beta}\right)\left[-e^{(\rho+\beta)(t-\tau)}\right]_t^\infty &= A(v,t)+H(t) \Leftrightarrow \\ C(v,t) &= (\rho+\beta)[A(v,t)+H(t)]\end{aligned} \tag{3.17}$$

The agent's consumption in each period is dependent on the sum of accumulated wealth, $A(v,t)$ and human wealth, $H(t)$. Furthermore, the propensity of consumption depends on the effective rate of time preference, $\rho+\beta$.

3.2 Aggregate Values

As noted before a proportion of the population is assumed to die in each period and a equal size cohort of new agents is born in each period. This fraction of newly born agents is constant across all periods and therefore the population size is also constant, $P(t) = 1, \ \forall t \in]-\infty:\infty[$. It was also assumed that there is no bequest motive so that for all agents the financial wealth at birth is equal to zero. Since cohorts are big the size of each cohort can be traced across time even though the lifetime of an individual agent is stochastic.

As seen before a cohort born at time v will be of size $\beta e^{\beta(v-t)}$ at time t where $t > v$. Since the size of each cohort can be estimated and by assuming agents in each cohort maximize according to the maximization problem in equation (3.13) aggregate values can be formulated. The relation between a non-specific aggregate value, $X(t)$, and its individual counterpart is:

$$X(t) \equiv \beta \int_{-\infty}^{t} e^{\beta(v-t)} X(v,t)dv \tag{3.18}$$

This is intuitive since within each cohort the value $X(v,t)$ should be identical for all agents. $X(t)$ in equation (3.18) represents the value for $X(v,t)$ for all agents within all cohorts. $\dot{X}(t)$ can be found by applying the Leibniz rule:

$$\begin{aligned}\dot{X}(t) &= \frac{d}{dt}\left(\beta\int_{-\infty}^{t} e^{\beta(v-t)}X(v,t)dv\right)\\ &= \beta\int_{-\infty}^{t}\dot{X}(v,t)e^{\beta(v-t)} - \beta X(v,t)e^{\beta(v-t)}dv + \beta X(t,t)e^{\beta(t-t)} - 0\\ &= \beta X(t,t) - \beta X(t) + \beta\int_{-\infty}^{t}\dot{X}(v,t)e^{\beta(v-t)}dv\end{aligned} \tag{3.19}$$

The fact that the hazard rate is constant for all agents, indifferent of time of birth is a necessary condition for the relation above. The aggregate consumption in each period is:

$$C(t) \equiv \beta\int_{-\infty}^{t} e^{\beta(v-t)}C(v,t)dv \tag{3.20}$$

The time of birth, v, has been dropped from the aggregate consumption notation, because it doesn't refer to a specific cohort but rather all cohorts already born. Equation (3.20) is not of much use, however by applying equation (3.17) a meaningful relation can be found:

$$\begin{aligned}C(t) &\equiv \beta\int_{-\infty}^{t} e^{\beta(v-t)}(\rho+\beta)[A(v,t)+H(t)]dv\\ &= (\rho+\beta)\left[\beta\int_{-\infty}^{t} e^{\beta(v-t)}A(v,t)dv + \beta\int_{-\infty}^{t} e^{\beta(v-t)}H(t)dv\right]\\ &= (\rho+\beta)[A(t)+H(t)]\end{aligned} \tag{3.21}$$

The consumption function is a linear function of aggregate human and financial wealth. Now a closer look will be taken at the aggregate value for wealth accumulation, $\dot{A}(t)$. By plugging in values from equation (3.4) into equation (3.19) and applying the Leibniz rule the following can be obtained:

$$\begin{aligned}\dot{A}(t) &\equiv \beta A(t,t) - \beta A(t) + \beta\int_{-\infty}^{t}\dot{A}(v,t)e^{\beta(v-t)}dv\\ &= -\beta A(t) + \beta\int_{-\infty}^{t} e^{\beta(v-t)}[(r(t)+\beta)A(v,t) + W(v,t) - T(v,t,) - C(v,t)]dv\\ &= -\beta A(t) + (r(t)+\beta)A(t) + W(t) - T(t) - C(t)\\ &= r(t)A(t) + W(t) - T(t) - C(t)\end{aligned} \tag{3.22}$$

Where $A(t,t) = 0$ since newborns have no financial assets, they do however have a positive amount of human wealth at birth, $H(t,t) = H(t)$. The term $\beta A(t)$ represents the wealth of those who die, $\beta A(t)$ cancels out in the second line of equation (2.40) which represent the transfer to the insurance company. The time differentiation of aggregate consumption is:

$$\dot{C}(t) = \beta C(t,t) - \beta C(t) + \beta \int_{-\infty}^{t} \dot{C}(v,t) e^{\beta(t-v)} \tag{3.23}$$

Newborns consume a fraction of their human wealth at birth i.e., $C(t,t) = (\rho + \beta) H(t) \neq 0$. From equation (3.21) and equation (3.16) the Euler equation is obtained.

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho - \beta(\rho + \beta)\left(\frac{A(t)}{C(t)}\right) = \frac{\dot{C}(v,t)}{C(v,t)} - \beta\left(\frac{C(t) - C(t,t)}{C(t)}\right) \tag{3.24}$$

Which is the same as the Euler equation for individual agents except for the final term $\beta\left(\frac{C(t)-C(t,t)}{C(t)}\right)$ which represents a distribution effect of generational turnover. Every generation has identical optimal consumption *growth*, this is because all generations face the same interest rate. However older generations have accumulated more wealth and have a higher *level* of consumption compared to younger generations.

Now firms are introduced into the model. The aggregate production function that satisfies the *Inada conditions*.

$$Y(t) = F(K(t), L(t)) \tag{3.25}$$

Where $Y(t)$ is output, $F(\cdot)$ is the production function, the factor imputs are capital, $K(t)$, and labor, $L(t)$. Since this is a closed economy case the factor inputs are rented from households, in the open economy case the capital could be supplied by a foreign source. *Perfect markets* are assumed so wages are equal to marginal productivity of labor and the marginal productivity of capital is the sum of interest and depreciation. Depreciation is denoted as δ and is assumed to be constant over time. To summarize, the marginal productivity of these factors are equal to the producer's cost of applying them.

$$\begin{aligned} W(t) &= F_L(K(t), L(t)) \\ r(t) + \delta &= F_K(K(t), L(t)) \end{aligned} \tag{3.26}$$

The firm seeks to maximize it's value. The stockmarket's valuation of the firm is:

$$V(t) = \int_t^\infty [Y(\tau) - W(\tau)L(\tau) - I(\tau)]e^{-R(t,\tau)}d\tau$$
$$where \quad R(t,\tau) \equiv \int_t^\tau r(s)ds \tag{3.27}$$

Note that only the agents discount with the annuity rate, neither the government nor firms face the hazard rate that agents face and therefore they discount with the regular rate. The firm is subject to a capital accumulation constraint:

$$\dot{K}(t) = I(t) - \delta K(t) \tag{3.28}$$

Where I(t) denotes gross investment and $\delta K(t)$ is gross depreciated capital.

Government expenditure is denoted by $G(t)$ and is funded by the lump sum taxes, $T(t)$, and possibly government debt, $B(t)$. The government dept is subject to the interest rate $r(t)$, therefore interest on outstanding debt is $r(t)B(t)$. Just like private agent the government is subject to a solvency condition, specifically:

$$\lim_{\tau\to\infty} e^{-R(t,\tau)}B(t) = 0 \tag{3.29}$$

From these realizations the government's budget restriction is obtained.

$$B(t) = \int_t^\infty (T(\tau) - G(\tau))e^{-R(t,\tau)}d\tau \tag{3.30}$$

Which implies that any existing government debt on the left hand side, must be equal to a discounted future budget surplus on the right hand side. Since outside income streams are impossible it's concluded that $A(t) = K(t) + B(t)$. Furthermore flexible wages are assumed which ensures that labor supply equals labor demand by firms and there is always full employment in the economy, $L(t) = 1$, and goods markets always clear. The gross output of the closed economy is $Y(t) = C(t) + I(t) + G(t)$. Table 3.1 summarizes these results.

Table 3.1	
$\dot{C}(t) = (r(t) - \rho)C(t) - \beta(\rho + \beta)[K(t) + B(t)]$	(T1)
$\dot{K}(t) = F(K(t), L(t)) - C(t) - G(t) - \delta K(t)$	(T2)
$\dot{B}(t) = r(t)B(t) + B(t) - T(t)$	(T3)
$r(t) + \delta = F_K(K(t), L(t))$	(T4)
$W(t) = F_L(K(t), L(t))$	(T5)
$L(t) = 1$	(T6)

Where equation (T1) in the table above is found by isolating $\dot{C}(t)$ in equation (3.24) and applying $A(t) = K(t) + B(t)$. Equation (T2) is found by applying $Y(t) = C(t) + I(t) + G(t)$ and (3.25) to equation (3.28).

3.3 Phase Diagrams

The dynamic response of consumption and capital stock to changes in macroeconomic factors is analyzed using a *phase diagram*.

The phase diagram we analyze has consumption on the vertical axis and capital on the horizontal axis, it has two functions plotted, each of which represents a path in which either consumption or capital does not change, these functions are called loci. On the intersection of these functions there is a *steady state equilibrium*, at which point neither consumption or capital changes. For a shock either function could shift, depending on the shock, and a new equilibrium might form. To transition from the old equilibrium to the new one, following the shock, the economy moves on to a *saddle path*.

In this section insights from the Blanchard-Yaari model are applied to produce the phase diagram in the closed economy and the open economy. For simplicity it's assumed that there is no government intervention, i.e., $T(t) = G(t) = B(t) = 0$. By assuming that $\dot{K}(t) = 0$ in equation (T2) in table 3.1 and isolating C(t) the $\dot{K}(t) = 0$ locus is produced.

$$\dot{K}(t) = 0 \Leftrightarrow C(t) = F(K(t), 1) - \delta K(t) \tag{3.31}$$

Aggregate consumption is equal of the aggregate output less depreciation. The *golden rule* level of capital, which maximizes consumption growth, is achieved when the $\dot{K}(t)$ locus reaches its maximum.

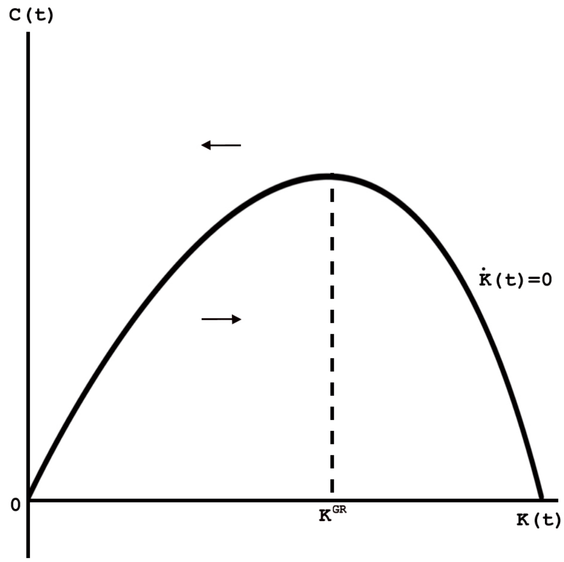

Figure 3.1: The $\dot{K}(t) = 0$ locus

$$\left.\frac{dC(t)}{dK(t)}\right|_{\dot{K}(t)=0} = 0 \Leftrightarrow F_K(K^G(t), 1) = \delta \tag{3.32}$$

Where K^{GR} represents the golden rule amount of capital. The production function $F(\cdot)$ fulfills the *Inada conditions* and therefore it's vertical at $K(t) = 0$. The maximum value $K(t)$ can take is where all income goes towards replacement investment to counter the depreciation of captial and by extension consumption is equal to zero, $C(t) = 0$:

$$\frac{F(K^{max}, 1)}{K^{max}} = \delta \tag{3.33}$$

From equation (T2) it can be seen that for points above the $\dot{K}(t) = 0$ line consumption is too high to maintain the level of capital and net investment becomes negative and leads to a fall in capital. Below the $\dot{K}(t) = 0$ line the opposite effect occurs. This is represented by the horizontal arrows in figure 3.1. By assuming that $\dot{C}(t) = 0$ in equation (T1) in table 2.1 we get the $\dot{C}(t) = 0$ locus:

$$\dot{C}(t) = 0 \Leftrightarrow C(t) = \frac{\beta(\rho + \beta)}{r(t) - \rho} K(t) \tag{3.34}$$

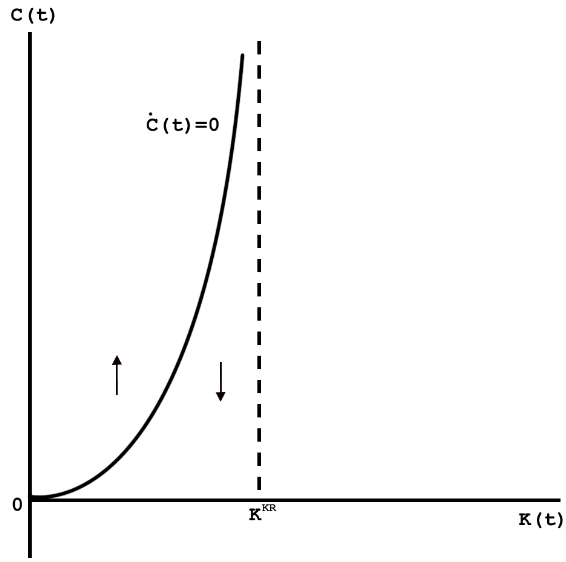

Figure 3.2: The $\dot{C}(t) = 0$ locus

The slope and shape of the $\dot{C}(t)$ line is dependent on the interplay between β, ρ and $r(t)$. The fraction in equation (3.34) approaches infinity as $r(t)$ approaches ρ. There is a negative relation between the amount of capital and the interest rate, that is, an increase in capital $K(t)$ leads to a fall in interest rate $r(t)$. The point for $K(t)$ where $r(t) = \rho$ is represented as K^{KR} and the interest rate associated with it is represented by r^{KR}. That is $r^{KR} = F_K(K^{KR}, 1) - \delta \equiv \rho$.

The $\dot{C}(t) = 0$ is upwards sloping from its origin and has a vertical asymptote at K^{KR}. The slope of the locus is steeper for higher amounts of capital stock. This due to the fact that $K(t)$ influences the denominator in equation (3.34) as described above. From equation (T1) it can be seen that an increase in $K(t)$ will cause $\dot{C}(t)$ to decrease, the opposite effect is true for a decrease in $K(t)$. This effect is represented by the vertical arrows in figure 3.2.

By combining figures 3.1 and 3.2 we get the *phase diagram* in figure 3.3. The equilibrium for the steady state where $\dot{C}(t) = \dot{K}(t) = 0$ is unique and saddle point stable. Equilibrium is represented at point E in the intersection of the two loci. Given arbitrary starting values for $C(t)$ and $K(t)$ we will arrive at a negative level for $C(t)$ or $K(t)$ if the saddle path, marked by S

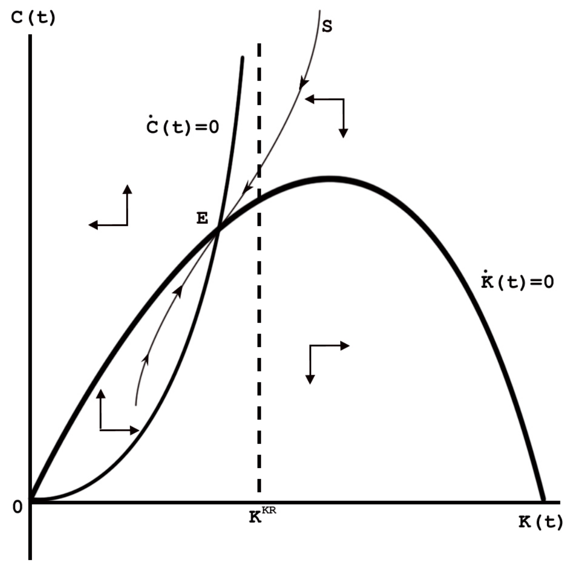

Figure 3.3: The Blanchard-Yaari Standard Phase Diagram

in figure 3.3, is not chosen. The saddle path is therefore the only acceptable choice for consumption-saving decision of the agents.

Open economy case

Let's now look at a simple *open economy* case, in a single-product world with perfectly mobile financial capital. In the open economy the interest rate is set at the exogenous world level, r^w, at which agent's in the economy can lend and borrow freely from the rest of the world. Parameter values distinguishes whether a nation is a creditor nation, inhabited by relatively patient agents, or a debtor nation, inhabited by relatively impatient agents. In this simple case all agent's wealth is held in foreign assets, denoted by $A_f(t)$. The production function is given by:

$$Y(t) = Z(t)L(t) \tag{3.35}$$

Z(t) is exogenous but potentially time varying and represents an index of technological change. Since perfect competition is assumed equation (3.35) implies that wages are exogenous and determined by $W(t) = Z(t)$. Like before $L(t) \equiv 1$ so $Z(t)$ determines the output of the economy. The equations of motion that characterize the economy become:

$$\dot{C}(t) = (r^w - \rho)C(t) - \beta(\rho + \beta)A_f(t) \tag{3.36}$$

$$\dot{A}_f(t) = r^w A_f(t) + Z(t) - C(t) \tag{3.37}$$

The loci for $\dot{C}(t) = 0$ and $\dot{A}_f(t) = 0$ are respectively:

$$\dot{C}(t) = 0 \Leftrightarrow C(t) = \frac{\beta(\rho + \beta)}{r^w - \rho} A_f(t) \tag{3.38}$$

$$\dot{A}_f(t) = 0 \Leftrightarrow C(t) = r^w A_f(t) + Z(t) \tag{3.39}$$

The $\dot{A}_f(t) = 0$ locus is a straight line that intersects the vertical $C(t)$ axis at point $Z(t)$. Its slope is the exogenous world interest rate r^w. For any level of consumption above the $\dot{A}_f(t) = 0$ locus foreign assets will deteriorate since the stock can not facilitate the consumption expenditure, the opposite will occur for consumption choice below the locus. This is represented by the horizontal arrows in figure 3.4. This effect can also be seen from equation (3.37) because of the negative relationship between $C(t)$ and $\dot{A}_f(t)$

The $\dot{C}(t) = 0$ locus has subtly changed, now it is linear because $A_f(t)$ does not effect r^w which causes the slope of the locus to be independent of the stock of foreign assets, this is different from the open economy case. The $\dot{C}(t) =$ locus intersects the vertical $C(t)$ axis at zero. From equation (3.36) we see that an increase in the stock of foreign assets, ceteris paribus, will lead to a drop in consumption. This effect is represented by the vertical arrows in figure 3.4. To determine the slope of the $\dot{C}(t) = 0$ we need to look at a few cases.

In the case of a relatively patient *creditor nation* the world interest rate exceeds the time preference of inhabitants, $r^w > \rho$. In this case a restriction must be imposed to ensure the stability of the model. In order for the loci to intersect the slope of the $\dot{C}(t) = 0$ locus needs to be greater than the slope of the $\dot{A}_f(t)$, that is $r^w < \beta(\rho + \beta)/(r^w - \rho)$. Assuming that the loci intersect we have a steady-state equilibrium at point E in panel A of Figure 3.4 and the country holds net positive foreign assets. The opposite is true if the nation is a relatively impatient *debtor nation* where $r^w < \rho$. In this case there will be a negative amount of foreign assets held and the economy, this case is represented in panel B of figure 3.4.

In the case where $r^w = \rho$ equation (3.36) becomes $\dot{C}(t) = -\beta(\rho+\beta)A_f(t)$ and the $\dot{C}(t) = 0$ locus coincides with the vertical axis and the stock of foreign assets is equal to zero. There is no saving or dissaving by agents in which case the model will still be saddle point stable.

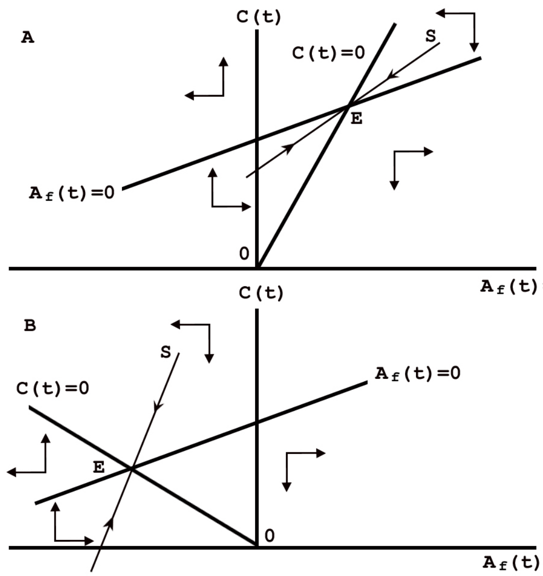

Figure 3.4: Open Economy Phase Diagram

In both cases there is a unique steady state equilibrium, at point E. The saddle path is depicted as the line S in figure 3.4. Similar to the closed economy case if we assume a arbitrary starting level of foreign assets we will arrive at infinite or negative values for $C(t)$ or $A_f(t)$ if the saddle path is not chosen.

Assuming that the only difference between two nations is their time preference where one nation has $r^w > \rho$ and the other one has $r^w < \rho$ it can be seen from figure 3.4 that the impatient nation will always have a lower level of steady state consumption than the patient nation.

An increase in r^w would result in a increase in holding of foreign assets through changes in the savings rate. To clarify this let's take a look at the case where $r^w > \rho$. In this case an increase in r^w would increase the slope of the $\dot{A}_f(t) = 0$ locus and decrease the slope of the $\dot{C}(t) = 0$ locus. This would push the equilibrium at the intersection of the two loci towards a greater value of foreign assets held. Because both loci have a positive slope this would increase consumption as well. Initially the economy would move towards the new saddle path and with time reach the new equilibrium.

An alternative way of looking at aggregate behavior is by giving aggregate consumption as a function of wealth which allows for a derivation of a savings function. Aggregate consumption becomes.

$$C(t) = (\beta + \rho)\left(\frac{Z(t)}{r^w + \beta + A_f(t)}\right) \tag{3.40}$$

The sum of financial and non-financial income is $Z(t) + r^w A_f(t)$. Savings, represented as $S(t)$, is total income less consumption.

$$S(t) \equiv \frac{r^w - \rho}{r^w + \beta} Z(t) + (r^w - \beta - \rho) A_f(t) \tag{3.41}$$

Assuming that $r^w < \beta + \rho$ savings becomes a decreasing function of wealth. The effect of an increase in $Z(t)$ depends on whether the nation is a debtor or a creditor.

In the following chapter a closer look at a fiscal policy in the open economy. In chapter 5 the effects of a decrease in the hazard rate on the open economy equilibrium will be examined.

Chapter 4

Fiscal Policy

Within the context of the Blanchard-Yaari model a fiscal policy is the sequence of current and anticipated taxes, government spending and government debt. In this chapter tax timing, increase in government spending and debt will be examined. This framework will be instrumental in examining the fiscal policy implications of increased longevity.

4.1 Fiscal Policy Under No Debt Constraint in the Closed Economy

The fiscal analysis begins with examining a policy constrained by a strictly balanced budget government. An increase in government spending $G(t)$ will be directly accommodated by an equal increase in lump sum taxes in the same period, $T(t)$, and government debt will be zero at all points in time that is, $G(t) = T(t), \dot{B}(t) = B(t) = 0$. Furthermore economy is initially in a steady state at point E_0. The equations of motion are as follows.

$$\dot{C}(t) = (r(t) - \rho)C(t) - \beta(\rho + \beta)K(t) \tag{4.1}$$

$$\dot{K}(t) = F(K(t), 1) - \delta K(t) - C(t) - G(t) \tag{4.2}$$

The beginning time is equalized at time $t = 0$ where $G(t) = 0$ and an increase in government spending will shift $\dot{K}(t) = 0$ locust downwards by the amount $dG(t)$ at time $t = 1$. This effect is represented in figure 4.1. This shift in the $\dot{K}(t) = 0$ locust will in turn cause the present consumption to jump downwards towards point A which is positioned on the new saddle path. This is due to the fact that the capital stock has not changed in the economy but there is only an increase in government spending that shifts

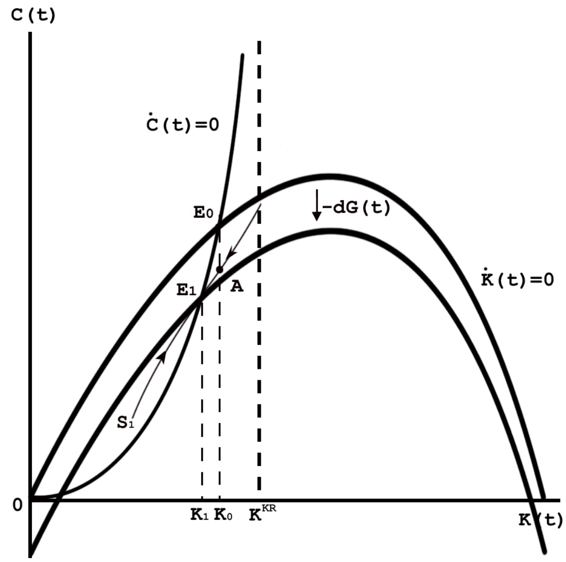

Figure 4.1: Increase in government expenditure

the $\dot{K} = 0$ locust downwards. Over time the economy will travel along the saddle path and reach the new equilibrium.

Initially the crowding out effect of private consumption caused by the increase in government spending is less than one to one, since $(E_0 - A) < dG(t)$. However in the long run, as the new equilibrium E_1 is reached the crowding out effect is greater than one to one, this can be seen from the slope of the $\dot{K} = 0$ and $\dot{C} = 0$ loci. The increase in lump sum taxes decreases the agent's human capital which leads to a instantaneous downwards shift in consumption, but this effect does not shift the $\dot{C}(t) = 0$ locus.

Agents discount their human capital by the annuity rate R^A which is higher than the rate on bonds. This leads agents to discount the future more heavily, because of the *hazard rate* and has the effect to dampen the response of the lump-sum tax increase.

The insufficient cut in consumption leads to a gradual drop in capital stock. Because of the structure of the production function this drop in capital leads to a less efficient workforce as the marginal productivity of labor, $F_L(K(t), 1)$, drops. This leads to a downwards pressure on wages and lower human capital for present and future generations. These effects are represented as the saddle path transition from A to E_1 in figure 4.1.

4.2 Time Reallocation of Taxes

To examine the effects of reallocating taxes through time a closer look is taken at two equations from before. Namely the equation for human wealth and the government's budget constraint.

$$H(t) = \int_t^{\infty} [W(\tau) - T(\tau)] e^{-\int_t^{\tau} [r(s)+\beta] ds} d\tau \tag{4.3}$$

$$B(t) = \int_t^{\infty} (T(\tau) - G(\tau)) e^{-\int_t^{\tau} r(s) ds} d\tau \tag{4.4}$$

The exponential discounting in equations (4.3) and (4.4) have been written with the integral notation for clarity. Let's assume a fiscal policy of lowering taxes at time t and then increasing them back again at the later time $t + \tau$, without changing the path of government expenditure. The changes in taxes at times t and $t + \tau$ are represented as $dT(t)$ and $dT(t + \tau)$ respectively. From the government's budget constraint (4.4) it can be seen that the change in the taxation given a level of debt will be dependent on the following relation to hold.

$$dT(t + \tau) = -e^{\int_t^{t+\tau} r(s) ds} dT(t) \tag{4.5}$$

From the human wealth equation above the impact of the taxation strategy on the consumer is determined.

$$-dT(t) - dT(t + \tau) e^{-\int_t^{t+\tau} [r(s)+\beta] ds} \tag{4.6}$$

By substituting (4.5) into (4.6) the effect of this particular taxation strategy on human capital is obtained.

$$-dT(t)(1 - e^{-\beta\tau}) \tag{4.7}$$

$dT(t)$ is a negative value, because there is a drop in taxation at time t. This implies that human capital increases with tax reallocation, assuming that $\beta > 0$. The increase in human capital is proportional to the length of the tax reallocation time. This is due to the probability that the agent will die before time $t + \tau$ and will therefore not be effected by the tax hike, and to a lesser extent due to the difference in the discount rate by the individuals and government. This leads to the taxation being partially shifted from the current generation towards a future generation. According to equation (4.8) the tax strategy will increase aggregate consumption through human wealth.

$$C(t) = (\rho + \beta)[K(t) + B(t) + H(t)] \tag{4.8}$$

4.3 Fiscal Policy and Debt in the Closed Economy

To understand the steady state effects of fiscal policy and government debt accumulation a specific policy experiment is produced. The government issues debt at time $t = 0$ and covers the increased debt payments by increasing future taxes. If interest rates change the government will adjust taxes to meet the interest payments. The tax increase satisfies $r(K(0))dB(0) = dT(0)$ and the new debt level is constant to eternity. The equations of motion are:

$$\dot{C}(t) = (r(K(t)) - \rho)C(t) - \beta(\beta + \rho)(B(t) + K(t)) \tag{4.9}$$

$$\dot{K}(t) = F(K(t), 1) - C(t) - G(t) - \delta K(t) \tag{4.10}$$

$$\dot{B} = r(K(t))B(t) + G(t) - T(t) \tag{4.11}$$

And the accompanying loci for the phase diagram are:

$$\dot{C}(t) = 0 \Leftrightarrow C(t) = \frac{\beta(\beta + \rho)}{r(K(t)) - \rho}[B(t) + K(t)] \tag{4.12}$$

$$\dot{K}(t) = 0 \Leftrightarrow C(t) = F(K(t), 1) - G(t) - \delta K(t) \tag{4.13}$$

Figure 4.2 summarizes the effects of the fiscal policy on the steady state of the economy. Since the interest rates will vary after $t = 0$ the taxes will also vary to cover the interest payments as a new steady state is reached. The $\dot{K}(t) = 0$ locus will not be effected by the fiscal policy since we have assumed that there is no change in government spending, in the phase diagram in figure 4.2 locus is drawn for $G(t) = 0$ for simplicity.

For a positive value of government spending there could be two or even zero equilibriums as the $\dot{K} = 0$ locus is shifted downwards providing two intersections of the loci and if shifted far enough the loci will not intersect at all. The increase in government debt will shift the $\dot{C}(t) = 0$ locus, this can be seen from equation (4.12). To determine shape of the $\dot{C}(t) = 0$ locus a few cases have to be distinguished.

Firstly if $B(t) > -K^{KR}$ the $\dot{C}(t) = 0$ locus goes through its origin, slopes upwards and approaches K^{KR} from the left. If $B(t) > 0$ then the locus will shift to the left while still going through its origin since $r(K(t))$ approaches infinity as $K(t)$ approaches zero, this results in a decrease in capital and consumption in the saddle point equilibrium. If $-K(t) < B(t) < 0$ the $\dot{C}(t) = 0$ locus shifts to the right and intersects the $\dot{K}(t) = 0$ locus at an equilibrium where the steady state level of both consumption and capital have increased. This case is interesting as the $\dot{C}(t) = 0$ locus has negative

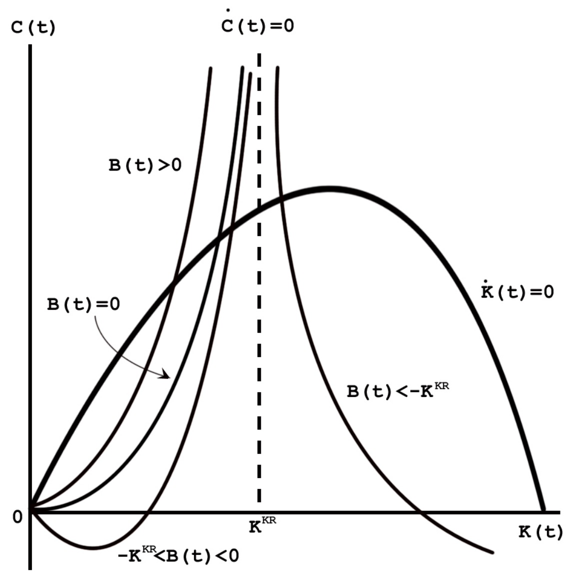

Figure 4.2: Debt Accumulation in the Closed Economy

values for consumption where $[B(t)+K(t)] < 0$ but as capital increases the curve become positive and reaches K^{KR} asymptotically form the left. If $B(t) < -K^{KR}$ then the $\dot{C}(t) = 0$ locus reaches the K^{KR} asymptote from the right and is downwards sloping. These effects can be seen in figure 3.2.

In any of these cases the change in government debt will lead agents to shift their consumption-saving choices to the appropriate saddle path. With time they will approach the new stable saddle point equilibrium at the intersection of the new $\dot{C}(t) = 0$ locus and the $\dot{K}(t) = 0$ locus.

4.4 Fiscal Policy and Debt in the Open Economy

The same fiscal policy as described above will be analyzed in the open economy case. The government issues debt at time $t = 0$ and covers the increased debt payments by increasing future taxes. Debt remains at a constant new level for eternity. We have:

$$C(t) = (\beta+\rho)\left(\frac{Z(t)-T(t)}{r^w+\beta} + B(t) + A_f(t)\right) \tag{4.14}$$

$$\dot{A}_f(t) = r^w A_f(t) + Z(t) - C(t) - G(t) \tag{4.15}$$

$$\dot{B} = r^w B(t) + G(t) - T(t) \tag{4.16}$$

The policy has constant levels of $G(t)$, $T(t)$ and $B(t)$ at all points in time except for time $t = 0$ where a permanent change occurs. Since the debt is issued at the exogenous world interest rate and future debt payments are covered by future taxes the change in policy satisfies $r^w dB(0) = dT(0)$. The steady state level of consumption and foreign assets can be determined by plugging equation (4.14) into equation (4.15) and allowing the level of taxation to be determined by the government's budget constraint in equation (4.16). In the steady state the stock of foreign assets and consumption is.

$$A_f^{SS}(t) = \frac{(r^w - \rho)(Z(t) - G(t)) - (\beta + \rho)\beta B(t)}{(\beta + \rho - r^w)(r^w + \beta)} \tag{4.17}$$

$$C^{SS}(t) = Z(t) - G(t) + r^w A_f^{SS} \tag{4.18}$$

The steady state level foreign assets is a decreasing function of government debt. Consumption is a positive function of $A_f(t)$ and therefore also a decreasing function of government debt.

$$\frac{dA_f^{SS}(t)}{dB(t)} = \frac{-(\beta + \rho)\beta}{(\beta + \rho - r^w)(r^w + \beta)} \tag{4.19}$$

More specifically the effect of government debt on the stock of foreign assets depend on the world interest rate.

$$\frac{dA_f^{SS}(t)}{dB(t)} \begin{cases} > -1 & for\ r^w > 0 \\ = -1 & for\ r^w = 0 \\ < -1 & for\ r^w < 0 \end{cases} \tag{4.20}$$

This is the result of government's replacement of foreign assets with debt in the agent's portfolio. If $r^w = \rho$ then this displacement is one-for-one and $A_f^{SS}(t) = -B(t)$. This effect is however more pronounced for cases where $r^w > \rho$. By manipulating the debt level the government can choose any level of steady state consumption deemed desirable. The effects of this debt policy can be seen in the agent's savings function below.

$$S(t) = \left(\frac{r^w - \rho}{r^w + \beta}\right)(Z(t) - G(t)) + (r^w - \beta - \rho)M(t) - \beta\left(\frac{\beta + \rho}{r^w + \beta}\right)B(t) \tag{4.21}$$

Where $S(t)$ is savings by the agent at time t. The effects of the fiscal policy implemented at time $t = 0$ is a decrease in savings by the amount $-\beta(\beta+\rho)/(r^w+\beta)$. The increase in debt does not affect the income of agents but rather makes them feel wealthier by the amount $[\beta/(r^w+\beta)]dB(0)$. This incentivizes agents to increase consumption and decrease saving until a new steady amount of foreign assets is reached. In the new steady state both consumption and the level of foreign assets will be lower.

Chapter 5

Increased Longevity

In this chapter the model developed in the previous section will be used to estimate the effect of increased longevity on the economy. Increased longevity can be analyzed in the Blanchard-Yaari model by manipulating the hazard rate, β. Change in the hazard rate will however both influence the death rate and the birth rate. This might cause a problem, since the economic results in the model is caused by a combination of increased longevity and lowered birth rate. The Buiter model, introduced in section 5.1, allows for lowering of the death rate while keeping the birth rate constant.

5.1 Manipulation of the Hazard Rate

Increased longevity can be simulated by lowering β resulting in expected lifetime length, $1/\beta$, to increase. The analysis begins by focusing on the closed economy without government then moving on to an open economy. In both cases an anticipated change in β is examined as well as an unanticipated change. In the real world the hazard rate would not change in shocks but rather continuously over time. However exogenous shock such as war or famine could result in an unanticipated shock to the hazard rate.

5.1.1 Closed Economy Case

To identify the impact of a change in the hazard rate, β, on the steady state values of consumption and capital stock the phase diagram is used. The equations for the $\dot{C}(t) = 0$ and $\dot{K}(t) = 0$ loci in the phase diagram were, respectively:

$$K(t) = \frac{r(t) - \rho}{\beta(\rho + \beta)} C(t) \tag{5.1}$$

$$C(t) = F(K(t), 1) - \delta K(t) \tag{5.2}$$

β enters into equation (5.1) and has no effect of (5.2). Equation (5.1) implies that a decrease in β would lead to a decrease in the level of consumption associated with a given value of capital. This leads to a shift in the $\dot{C}(t) = 0$ locus to the right with a more pronounced effect as capital increases. This increase in effect is caused by the fact that as $K(t)$ approaches K^{KR} the numerator in equation (5.1) approaches zero.

The $\dot{C}(t) = 0$ locus originates still at the intersection of the axis. This shift of the curve leads to a new steady state equilibrium at the intersection of the two loci, if the economy is dynamically efficient the increased life expectancy leads to both increased steady state consumption and greater capital stock. Increased capital stock implies a fall in the steady state interest rate. The $\dot{C}(t) = 0$ locus asymptotically approaches K^{KR} which was derived from the relationship:

$$r^{KR} = F_K(K^{KR}, 1) - \delta \equiv \rho \tag{5.3}$$

This level of capital, K^{KR} is not effected by a change in β. Therefore the new steady state equilibrium amount of capital stock does not exceed K^{KR}. This result is summarized in figure 5.1.

An unanticipated shock, at time t_1, would cause a drop in consumption at t_1 as the economy moves to the new saddle path. With time the new equilibrium would be reached as the economy travels along the saddle path.

An anticipated shock results in a slightly different transition to the new steady state equilibirum. At time t_0 the information that the hazard rate will drop at time t_1 is known to all agents in the economy. At time t_0 the economy slightly moves out of the equilibrium by decreasing consumption, at time t_1 the economy is at the new saddle path as the $\dot{C}(t) = 0$ locus shifts. This leads to an more efficient transition than if the shock is unanticipated. The levels of consumption and capital associated with these transitional paths are illustrated in figure 5.2. An anticipated and continuous decrease in the hazard rate most accurately describes the real world.

5.1.2 Open Economy Case

The open economy case is similar to the closed economy case to the extent that the β change does not effect the capital accumulation locus but rather only the $\dot{C}(t) = 0$ locus. The loci for $\dot{C}(t) = 0$ and $\dot{A}_f(t) = 0$ are respectively:

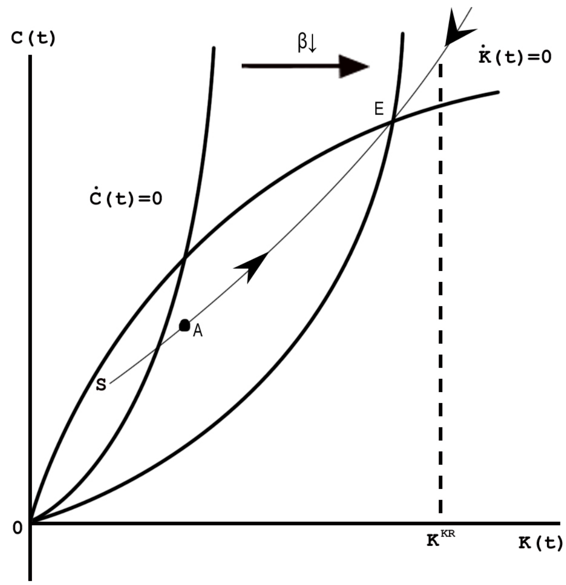

Figure 5.1: Shift in β in the closed economy

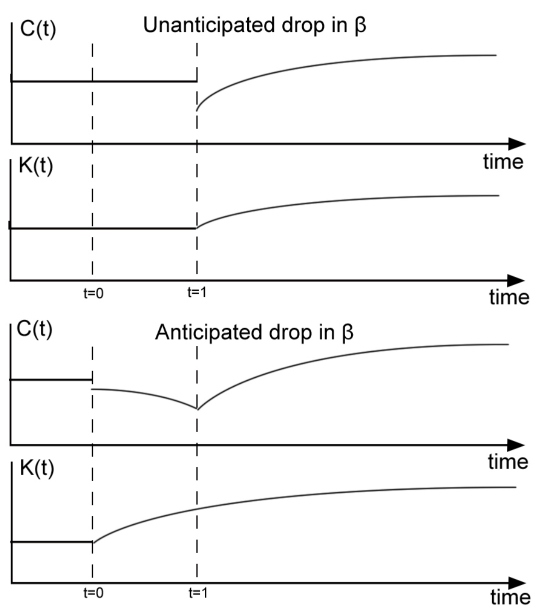

Figure 5.2: β drop, anticipated and unanticipated response

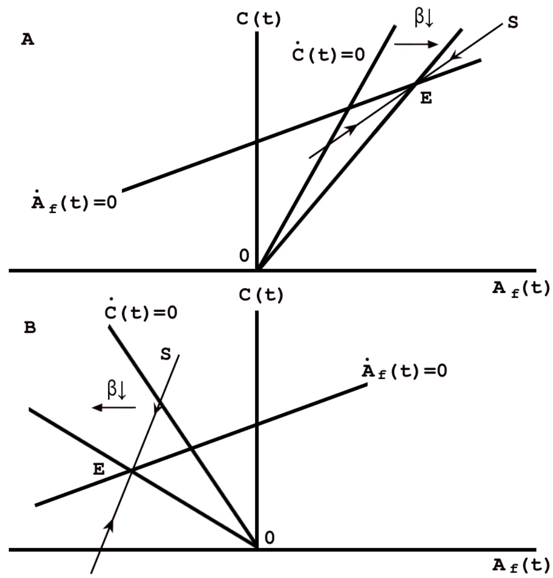

Figure 5.3: Shift in β in the open economy

$$C(t) = \frac{\beta(\rho + \beta)}{r^w - \rho} A_f(t) \tag{5.4}$$

$$C(t) = r^w M(t) + W(t) \tag{5.5}$$

Increased longevity has opposite steady-state effects depending on whether r^w is greater or less than ρ. A decrease in β increases the levels of foreign assets if $r^w > \rho$ and decreases them if $r^w < \rho$, this effect is caused by a change in the slope of the $\dot{C}(t) = 0$ locus which results in a new steady state equilibrium. These results are illustrated in figure 5.3. Like before, panel A depicts the case where $r^w > \rho$ and panel B depicts the case where $r^w < \rho$.

Like before, the transition to the new steady state equilibrium depends on whether the shock in β is anticipated or not. Like in the closed economy case, if the shock is anticipated the economy moves away from the equilibrium when the news of the change in the hazard rate become public. In the $r^w > \rho$ case the consumption drops by a marginal amount, disrupting the previous equilibrium causing the stock of foreign assets to accumulate. When the change in β occurs the economy is at the new saddle path associated with the new equilibrium.

If the change in β is unanticipated the economy moves straight to the new saddle path at the time of the change. The differing results from an unanticipated and anticipated drop in β in the open economy case where $r^w > \rho$ is similar to the open economy case, therefore figure 5.2. can be also used in this case to depict the results comparatively without much loss of precision.

In both the open economy (where $r^w > \rho$) and the closed economy case the increased longevity causes an increase in the capital stock through increased savings. People who expect to live longer are more inclined towards saving, this relation is true at an aggregate and individual level.

5.2 Differing Birth and Death Rates

In the model above the hazard rate, β, both describes the instantaneous death probability and the birth rate. This is leads to a constant population. When manipulating the hazard rate the both effects of increased longevity and a lower birth rate are represented in the model results. In 1988 Buiter published the article *Death, Birth, Productivity Growth and Debt Neutrality* which generalized the Blanchard-Yaari model by distinguishing the birth rate and the mortality rate. This allows for an isolation of the longevity effect from the birth effect on the steady state. The derivation of the Buiter model will not be traced in detail. The main building blocks of the model relative to the thesis topic will be discussed below.

The model defines the birth rate parameter as β and the death rate parameter as μ. The population grows an the rate $n_L \equiv \beta - \mu$ Consumption, $c(t)$, capital, $k(t)$, and output, $y(t)$, are measured in per capita terms, therefore the model allows for population growth if $\beta > \mu$. Like before ρ is the time preference of agents and δ is the depreciation rate of capital. Furthermore the model assumes a Cobb-Douglas production function. In the absence of government the model can be summarized to the following form:

Table 5.1	
$\frac{\dot{c}(t)}{c(t)} = r(t) - \rho - \beta(\rho+\mu)\frac{k(t)}{c(t)}$	(B1)
$\dot{k}(t) = y(t) - c(t) - (\delta+\beta-\mu)k(t)$	(B2)
$r(t) + \delta = \epsilon\frac{y(t)}{k(t)}$	(B3)
$w(t) = (1-\epsilon)y(t)$	(B4)
$y(t) = Z_0 k(t)^{\epsilon}, \quad 0 < \epsilon < 1.$	(B5)

From table 5.1 the $\dot{c}(t) = 0$ and the $\dot{k}(t) = 0$ loci can be determined. The $\dot{k}(t) = 0$ locus is:

$$c(t) = Z_0 k(t)^{\epsilon} - (\delta+\beta-\mu)k(t) \tag{5.6}$$

And the $\dot{c}(t) = 0$ locus is:

$$c(t) = \frac{\beta(\rho+\mu)k(t)}{\epsilon Z_0 k(t)^{\epsilon-1} - (\delta+\rho)} \tag{5.7}$$

The $c(t)$ locus is upward sloping and has vertical asymptote at $\bar{k}$, where:

$$\bar{k} \equiv \left(\frac{\epsilon Z_0}{\delta+\rho}\right)^{1/(1-\epsilon)} \tag{5.8}$$

The phase diagram is depicted in figure 5.4. In the Buiter model a change in either birth rate or death rate affects both loci while in the Blanchard-Yaari model a change in the hazard rate only affected the $\dot{C}(t) = 0$ locus.

To determine the effect of increased longevity the death rate, μ is decreased while keeping the birth rate β constant. We see that in the $\dot{c}(t) = 0$ loci a decrease in μ leads to a shift to the right which is more pronounced at higher levels of k(t). This effect of increased longevity on the $\dot{c}(t) = 0$ curve is similar to the shift associated with a decreased hazard rate in the standard Blanchard-Yaari model. The $\dot{c}(t) = 0$ still has the same vertical asymptote $\bar{k}$. The decreased death rate also has an effect on the $\dot{k}(t)$ locus. The locus shifts downwards and with a more pronounced downward shit at higher levels of $k(t)$. These results are illustrated in figure 5.5.

The economy moves from the initial equilibrium at point E1 and goes to the equilibrium in E2. The effect on long term steady state consumption is ambiguous since the shift in each curve has opposite effects. However the decreased death probability leads to a higher level of capital stock.

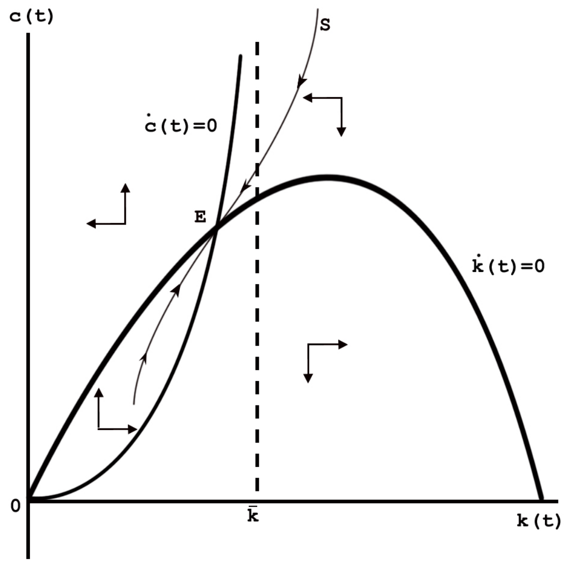

Figure 5.4: The Phase Diagram in the Buiter model

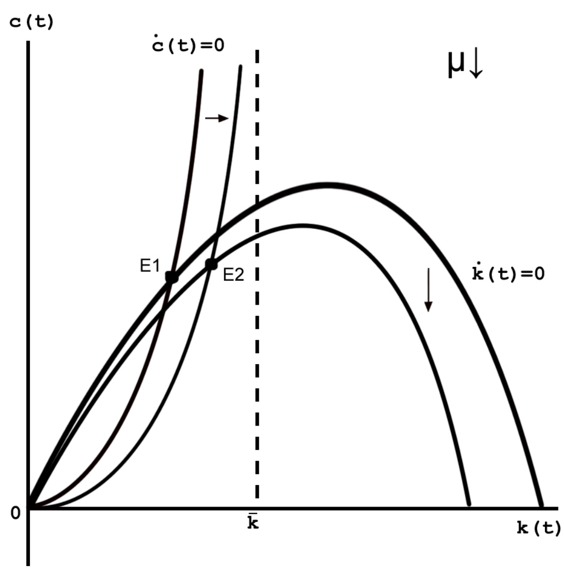

Figure 5.5: Decreased death rate

Chapter 6

Savings and Retirement

6.1 Age Dependent Productivity

To simulate the effects of saving for retirement Blanchard introduced declining productivity with age. Since there are perfect markets agents' wages will be directly dependent on the marginal productivity of their labor. This would lead to agents saving for those years where their wages will drop following a decrease in their productivity. Therefore the declining productivity will lead to a consumption smoothing behavior by agents who will save more in early parts of life to maintain a high level of consumption even though their non-interest income has dropped.

Increased longevity implies that agents are *active* and being able to participate in the labor market for a longer period of their lives. This implies that with increased longevity productivity declines at a slower rate with age. This extension of the Blanchard-Yaari model puts us one step closer to the thesis topic as agent's behavior portrays foresight regarding latter part of life.

To incorporate this age dependent productivity into the model all workers are assumed to supply one unit of *raw* labor over their lifetime, but the efficiency of this raw labor declines with age. $N(v,\tau)$ denotes agent's unit of effective labor, $L(v,\tau)$ is the units of raw labor supplied at time τ by a worker born at time v. Lastly $E(\tau - v)$ is defined as the efficiency of $L(v,\tau)$. Aggregate value of effective labor is as follows.

$$N(\tau) = \int_{-\infty}^{\tau} N(v,\tau)dv = \int_{-\infty}^{\tau} L(v,\tau)E(\tau - v)dv \tag{6.1}$$

The aggregate production function becomes.

$$Y(\tau) = F(K(\tau), N(\tau)) \tag{6.2}$$

Since the hazard rate is constant and all cohorts die at the rate β the following realization about raw labor units supplied by each cohort can be made. As before, $\beta e^{-\beta(v-t)}$ is the size at time t of the cohort born at time v.

$$L(v, \tau) = e^{-\beta(\tau - v)} L(v, v) = \beta e^{-\beta(\tau - v)} \tag{6.3}$$

That is, the labor supply at time τ of cohort born at time v is equal to its size since the raw labor units have been neutralized to one, because of full employment. Efficiency falls exponentially over lifetime and is defined as:

$$E(\tau - v) \equiv \left(\frac{\alpha + \beta}{\beta}\right) e^{-\alpha(\tau - v)} \tag{6.4}$$

Where the term in the brackets on the right hand side is a normalization made for convenience. Aggregate effective labor is:

$$\begin{aligned} N(\tau) &= \int_{-\infty}^{\tau} \left(\frac{\alpha + \beta}{\beta}\right) e^{-\alpha(\tau - v)} \beta e^{-\beta(\tau - v)} dv \\ &= (\alpha + \beta) \int_{-\infty}^{\tau} e^{-(\alpha+\beta)(\tau - v)} dv \\ &= (\alpha + \beta) \left[\frac{1}{\alpha + \beta} e^{-(\alpha+\beta)(\tau - v)}\right]_{-\infty}^{\tau} = 1 \end{aligned} \tag{6.5}$$

Aggregate effective labor is equal to one, this is due to the constant birth-death rate and large cohorts. The representative firm maximizes shareholder's profits and consequently also the stockmarket's valuation, which is:

$$V(t) = \int_{t}^{\infty} \left(F(K(\tau), N(\tau)) - \int_{-\infty}^{\tau} W(v, \tau) L(v, \tau) dv - I(\tau)\right) e^{-R(t,\tau)} d\tau \tag{6.6}$$

Note the difference between this notation and the one presented in equation (3.27) is only in the production function, which is depended now on effective labor, and the wages which are now also payed by effective labor rather than by raw labor. This is due to the fact that firms potentially hire agents from all cohorts but pay them wages according to their productivity. Furthermore we get the cost of production inputs:

$$r(\tau) + \delta = F_K(K(\tau), N(\tau)) \tag{6.7}$$

$$W(v,\tau) = E(\tau - v)F_N(K(\tau), N(\tau)) \tag{6.8}$$

The relation in equation (6.7) is approximately the same as (T.4) in the unchanged Blanchard-Yaari model in chapter 3. However the wage relation has changed from (T.5) and now includes the efficiency factor. In a steady state the wages of agents will decline over their lifetime and incentivize agents to increase savings in early stages of life to counteract this wage drop in the latter part of life, this simulates saving for retirement. We assume that there is no other change in the utility function of the agent and they are bound by the same budget identity from the constant lifetime productivity case. Assuming that there are no taxes and government spending we get the agent's consumption.

$$C(v,t) = (\rho + \beta)[A(v,t) + H(v,t)] \tag{6.9}$$

Before human wealth was identical for all agents, independent of age, because of identical hazard rates. Now however this it not the case.

$$\begin{aligned} H(v,t) &\equiv \int_t^\infty W(v,\tau)e^{-R^A(t,\tau)}d\tau \\ &= \int_t^\infty \left(\frac{\alpha+\beta}{\beta}\right)e^{\alpha(v-\tau)}W(\tau)e^{-R^A(t,\tau)}d\tau \\ &= e^{\alpha(v-\tau)}H(t,t) \end{aligned} \tag{6.10}$$

The first line is self explanatory as it is similar to the human capital relation seen before. The transition between the first two lines is due to the predictable productivity of the workforce when age is accounted for. That is, $W(v,\tau) = \left(\frac{\alpha+\beta}{\beta}\right)e^{\alpha(v-\tau)}W(\tau)$. Transition to the last line utilizes that human wealth of newborns, $H(t,t)$ is as follows.

$$H(t,t) = \frac{\alpha+\beta}{\beta}\int_t^\infty W(\tau)e^{-\int_t^\tau (r(s)+\alpha+\beta)ds}d\tau \tag{6.11}$$

Note that for this relation to hold β needs to be strictly positive. Aggregate human wealth in the economy is.

$$\begin{aligned} H(t) &\equiv \beta \int_{-\infty}^{t} e^{\beta(v-t)} H(v,t) dv \\ &= H(t,t) \int_{-\infty}^{t} e^{(\alpha+\beta)(v-t)} dv \\ &= \left(\frac{\beta}{\alpha+\beta}\right) H(t,t) \\ &= \int_{t}^{\infty} W(\tau) e^{-\int_t^\tau (r(s)+\alpha+\beta)ds} d\tau \end{aligned} \tag{6.12}$$

The first line in the equation above is a application of equation (3.18). Going form the first to second line in (6.12) we simply apply equation (6.10). By comparing the last line of (6.12) to (3.12) the change in effective labor leads agents to discount their future aggregate wage more heavily. The heavier discounting is caused by th fact that agents have a positive death probability, as before, and with age their non-interest income will dwindle. Now aggregate consumption can be determined.

$$C(t) = (\rho+\beta)[A(t) + H(t)] \tag{6.13}$$

$\dot{A}(t)$ is found similarly as in the previous section and $\dot{H}(t)$ can be found by differentiating (6.12) w.r.t. time.

$$\dot{A}(t) = r(t)A(t) + W(t) - C(t) \tag{6.14}$$

$$\dot{H}(t) = (r(t) + \alpha + \beta)H(t) - W(t) \tag{6.15}$$

The Euler equation becomes:

$$\frac{\dot{C}(t)}{C(t)} = (r(t) + \alpha - \rho) - (\alpha+\beta)(\rho+\beta)\frac{A(t)}{C(t)} \tag{6.16}$$

When assuming $\alpha = 0$ the Euler equation is identical to the one in the standard Blanchard-Yaari model. The equations of motion can be found by applying equation (6.7) to the Euler equation and noting that $A(t) = K(t)$ since there $B(t) = 0$.

$$\dot{C}(t) = [F_K(K(t), 1) + \alpha - (\rho+\delta)]C(t) - (\alpha+\beta)(\rho+\beta)K(t) \tag{6.17}$$

$$\dot{K}(t) = F(K(t), 1) - C(t) - \delta K(t) \tag{6.18}$$

Equation (6.18) is the equation of motion from (T2) while taking into account that $G(t) = 0$. We see that there is no change in the $\dot{K}(t) = 0$ locus

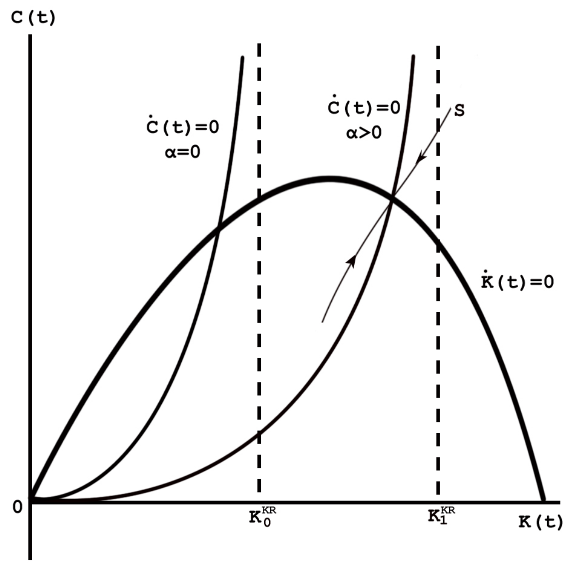

Figure 6.1: The Phase Diagram with Age Dependent Productivity

from the standard no-government case previously examined. However there is a change in the $\dot{C}(t) = 0$ locus, let's take a closer look at the locus.

$$C(t) = \frac{(\alpha + \beta)(\beta + \rho)}{r(t) + \alpha - \rho} K(t) \tag{6.19}$$

Now the C(t) function is upwards sloping and asymptotically reaching K^{KR} where $r^{KR} = F_K(K^{KR}, 1) - \delta = \rho - \alpha$. There is a negative relationship between $r(t)$ and $K(t)$. Since $\rho - \alpha < \rho$ if $\alpha > 0$ the position of the K^{KR} asymptote on the $K(t)$ axis is dependent on the size of α. in figure 6.1 two different K^{KR} asymptotes are illustrated. One associated with a drop in productivity with age, denoted by $\alpha > 0$, and the other associated with no drop. As α increases the K^{KR} moves along the $K(t)$ axis increasing the steady state capital. This is portrayed in figure 6.1.

The phase diagram is similar as in the constant productivity case except for the asymptote for the $\dot{C}(t) = 0$ locus. An increase in α leads agent's non-interest income to accrue at an early age, leading to more early age savings to maintain a smooth consumption over their expected lifespan. This in turn leads to more capital accumulation as a larger portion of non-interest income is shifted towards savings. This can however lead to sub optimal levels of over-savings if α is large enough to shift the $\dot{C}(t)$ locus across the K^{GR} point This is portrayed in figure 6.1 with the $\dot{C}(t) = 0$ locus labeled as $\alpha > 0$. The equilibrium associated with the $\alpha > 0, \dot{C}(t) = 0$ locus is saddle point stable but has over-accumulation of capital.

Increased longevity would lead to productivity to decline at a slower rate than before. This leads to the α in this model extension to follow β to a certain extent. This hypothesized correlation between α and β is not perfect since agents could expect a longer retirement with increased longevity. The fiscal policy described in chapter 4 where the permanent debt level is increased could shift the $\dot{C}(t) = 0$ locus towards the golden rule amount of capital.

6.2 Endogenous Labor Supply and Retirement Age

When determining the fiscal implication of increased longevity one must look at the labor decisions of households. In this section the endogenous labor decisions of households will be analyzed. Determinants of the optimal retirement age will be examined.

6.2.1 Life-cycle Labor Supply

By introducing leisure the expected lifetime utility function becomes:

$$E(\Lambda(v,t)) \equiv \int_t^{\infty} ln(C(v,\tau)^{\varepsilon_C}[1 - L(v,\tau)]^{1-\varepsilon_C})e^{(\rho+\beta)(t-\tau)}d\tau \qquad (6.20)$$

Where $0 < \varepsilon_C \leq 1$. This allows for consumption-leisure decisions by the individual. Utility is derived from both consumption, $C(v,\tau)$ and leisure $[1-L(v,\tau)]$. Leisure is, by definition, the time the individual spends outside of work. The agent's time endowment has be neutralized to 1 and $L(v,\tau)$ is time spent working. The original expected lifetime utility function can be obtained by setting $\varepsilon_C = 1$ and the agent chooses full employment since a positive amount of leisure results in less utility through a drop in income. Including leisure into the agent's utility function allows for the introduction various taxes, such as income tax. The agent's budget identity becomes:

$$\dot{A}(v,\tau) = [r(\tau) + \beta]A(v,\tau) + W(\tau)(1 - t_L) + TR(\tau) - X(v,\tau) \qquad (6.21)$$

Where $TR(\tau)$ are age independent transfers from government. Where $X(v,\tau)$ represents *full consumption* and is the sum of spending on consumption and leisure, that is:

$$X(v,\tau) \equiv (1+t_C)C(v,\tau) + W(\tau)(1-t_L)[1-L(v,\tau)] \tag{6.22}$$

Where t_L and t_C represent proportional taxes on labor income and consumption respectively. The solvency conditions are the same as before, namely:

$$\lim_{t\to\infty} e^{-R^A(t,\tau)}A(v,\tau) = 0, \quad R^A \equiv \int_t^\tau [r(s)+\beta]ds \tag{6.23}$$

The optimization problem is solved using two-stage budgeting. This method is valid if the utility function is intertemporally separable. In the first stage the optimal allocation of consumption and leisure is determined by the agent, conditional on a given level of full consumption, $X(v,\tau)$. The maximization problem for the first stage is:

$$\begin{aligned} &\max_{C(v,\tau),L(v,\tau)} \quad ln\big[C(v,\tau)^{\varepsilon_C}[1-L(v,\tau)]^{1-\varepsilon_C}\big] \\ &s.t.\;\; X(v,\tau) \equiv (1+t_C)C(v,\tau) + W(\tau)(1-t_L)[1-L(v,\tau)] \end{aligned} \tag{6.24}$$

The Lagrangian is:

$$\begin{aligned} \mathcal{L} =& ln\big[C(v,\tau)^{\varepsilon_C}[1-L(v,\tau)]^{1-\varepsilon_C}\big] \\ &+ \lambda((1+t_C)C(v,\tau) + W(\tau)(1-t_L)[1-L(v,\tau)] - X(v,\tau)) \end{aligned} \tag{6.25}$$

Solving the Lagrangian yields

$$\begin{aligned} \frac{d\mathcal{L}}{dC(v,\tau)} &= 0 \Leftrightarrow \frac{\varepsilon_C}{C(v,\tau)} + \lambda(1+t_C) = 0 \\ \frac{d\mathcal{L}}{dL(v,\tau)} &= 0 \Leftrightarrow \frac{\lambda(L(v,\tau)-1)(t_L-1)W(\tau) - \varepsilon_C + 1}{L(v,\tau)-1} = 0 \\ \frac{d\mathcal{L}}{d\lambda} &= 0 \Leftrightarrow X(v,\tau) = (1+t_C)C(v,\tau) + W(\tau)(1-t_L)[1-L(v,\tau)] \end{aligned} \tag{6.26}$$

By isolating λ in the the first two lines of equation (6.26) the following relation is obtained.

$$\frac{\frac{1-\varepsilon_C}{1-L(v,\tau)}}{\frac{\varepsilon_C}{C(v,\tau}} = W(\tau)\frac{1-t_L}{1+t_C} \tag{6.27}$$

Substituting equation (6.27) into equation (6.22) optimal consumption and leisure is found given a certain level of full consumption. The relationship is expressed by:

$$(1+t_C)C(v,\tau) = \varepsilon_C X(v,\tau) \tag{6.28}$$

$$W(\tau)(1-t_L)[1-L(v,\tau)] = (1-\varepsilon_C)X(v,\tau) \tag{6.29}$$

Equation (6.29) implies that it is optimal for agents in a steady state to spend a positive age-independent fraction of full consumption on leisure. This realization depends on $\varepsilon_C < 1$. To determine the leisure decisions of the agent the path of full consumption over the lifetime must be examined, this will be determined in the second stage of the agent's maximization. Before proceeding to the second stage of the agent's utility maximization the *true cost of living index*, $P_\Omega(\tau)$, is introduced:

$$P_\Omega(\tau) = \left(\frac{1+t_C}{\varepsilon_C}\right)^{\varepsilon_C}\left(\frac{W(\tau)(1-t_L)}{1-\varepsilon_C}\right)^{1-\varepsilon_C} \tag{6.30}$$

This index incorporates the true costs of consumption an leisure. Utility is the logarithm of the true cost of living index subtracted from the logarithm of full consumption. Now for the second stage the maximization problem becomes:

$$\begin{aligned}
&\max_{X(v,\tau)} E(\Lambda(v,t)) \equiv \int_t^\infty [ln(X(v,\tau)) - ln(P_\Omega(\tau))]e^{(\rho+\beta)(t-\tau)}d\tau \\
&s.t. \;\; \dot{A}(v,\tau) = [r(\tau)+\beta]A(v,\tau) + W(\tau)(1-t_L) + TR(\tau) - X(v,\tau), \\
&\qquad \lim_{t\to\infty} e^{-R^A(t,\tau)}A(v,\tau) = 0
\end{aligned} \tag{6.31}$$

Here the agent solves the lifetime utility function w.r.t. full consumption subject to equation (6.21) and (6.23). Solving this maximization problem yields similar results as presented previously:

$$X(v,t) = (\rho+\beta)[A(v,t)+H(t)] \tag{6.32}$$

$$\frac{\dot{X}(v,\tau)}{X(v,\tau)} = r(\tau) - \rho, \quad (for\ \tau \geq t) \tag{6.33}$$

$$H(t) \equiv \int_t^\infty (W(\tau)(1-t_L) + TR(\tau))e^{R^A(t,\tau)}d\tau \tag{6.34}$$

Now the path of full consumption over the lifetime has been identified and by extention how labor supply develops over the lifetime. Equation (6.32) tells us that steady state full consumption is proportional to total wealth. Equation (6.33) implies that steady state growth in full consumption is determined on the difference between the interest rate and the agent's time preference. The relation for human wealth differs from the relation obtained

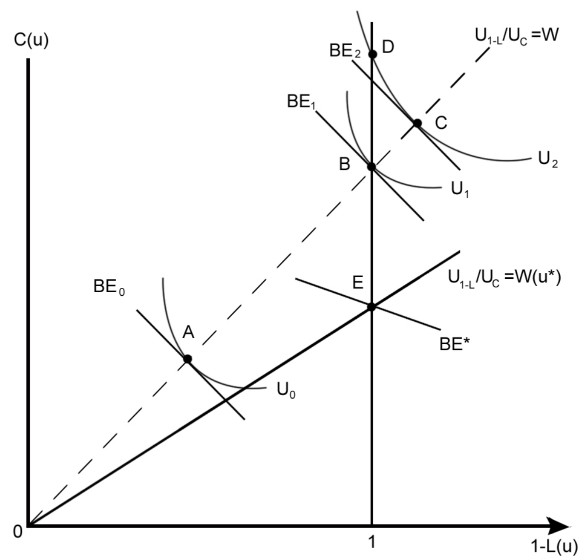

Figure 6.2: Life-cycle labor supply

in the previous chapter only by the inclusion of labor taxes and the lump sum transfers by the government.

Since $r(\tau) - \rho > 0$ equation (6.33) implies that full consumption is increasing exponentially over time. However in equation (6.28) we saw that leisure is a fixed proportion of full consumption. Since full consumption is increasing with age leisure must also increase with age, i.e., as agents age they work less.

In figure 6.2 the life-cycle labor supply of agents is demonstrated graphically. At point A the agent is in his working life, he equates the marginal rate of substitution between consumption and leisure to the wage rate, this is choice is represented as the tangent between the *budget line*, BE_0, and the indifference curve U_0. The slope of the budget line is given by $C(u) + W(u)[1 - L(u)] = X(u)$ and the slope of the indifference curve is $-U_{1-L}/U_C$.

As the agent gets older full consumption increases, given a constant wage the agent shifts upwards along the dashed line. When the agent reaches point B the whole time endowment goes towards leisure. Here a non-negative labor supply constraint has not been introduced. This allows for agents to move to point C in figure 6.2. and becoming demanding labor from others. This ignores retirement from the labor market altogether. In the next section a more concise representation of retirement decisions of agents will be developed.

6.2.2 Productivity and Retirement

Let's assume a small open economy where agents can lend and borrow at will at the exogenous interest rate r^w. Furthermore the analysis will be confined to the case where $r^w > \rho$. Agent's productivity is hump shaped over the agents life, it rises until a certain point and then decreases. There are perfect markets and productivity is dependent on age which implies age dependent wages $W(u)$. The agent's age is $u = t - v$, where t is the planning period and v is period of birth. Further assumption about the wage curve are as follows, $W'(u) > 0$ for $0 \leq u < \bar{u}$ and $W'(u) < 0$ for $u \geq \bar{u}$ and that $W(u) > 0 \; \forall \; u$. The agent chooses consumption and leisure to maximize expected lifetime utility, given by:

$$E(\Lambda(u)) \equiv \int_u^\infty [\varepsilon_C ln(C(s)) + (1-\varepsilon_C) ln(1-L(s))] e^{(\rho+\beta)(u-s)} ds \quad (6.35)$$

The choices of consumption and leisure to maximize the expected utility of the agent is subject to the budget identity:

$$\dot{A}(s) = (r^w + \beta)A(s) + W(s)L(s) - C(s) - T \quad (6.36)$$

The budget identity is similar to before. Age independent lump sum taxes T have been included. To exclude Ponzi behavior the following condition is included:

$$\lim_{s\to\infty} A(s) e^{-(r+\beta)s} = 0 \quad (6.37)$$

Initial assets in the planing period are $A(u)$ and time spent working is non negative $L(u) \geq 0$. The maximization problem for the planning period $s \geq u$ is:

$$\begin{aligned} \max_{C(s),L(s)} E(\Lambda(u)) = & \int_u^\infty [\varepsilon_C ln(C(s)) + (1-\varepsilon_C) ln(1-L(s))] e^{(\rho+\beta)(u-s)} ds \\ s.t. \quad & \dot{A}(s) = (r^w + \beta)A(s) + W(s)L(s) - C(s) - T, \\ & L(u) \geq 0 \end{aligned} \quad (6.38)$$

The current value Lagrangian becomes.

$$\begin{aligned}\mathcal{L}_{\mathcal{C}} \equiv \quad & \varepsilon_C ln(C(s)) + (1-\varepsilon_C) ln(1-L(s)) \\ & + \eta(s)[(r^w+\beta)A(s) + W(s)L(s) - C(s) - T] + \zeta(s)L(s)\end{aligned} \tag{6.39}$$

Where $\eta(s)$ and $\zeta(s)$ are continuous Lagrangian multipliers. Differentiation of the current value Lagrangian yields:

$$\begin{aligned} & \frac{d\mathcal{L}_{\mathcal{C}}}{dC(s)} = 0 \Leftrightarrow \frac{\varepsilon_C}{C(s)} = \eta(s) \\ & \frac{d\mathcal{L}_{\mathcal{C}}}{dL(s)} = 0 \Leftrightarrow \frac{1-\varepsilon_C}{1-L(s)} = \eta(s)W(s) + \zeta(s) \\ & \frac{\dot{\eta}(s)}{\eta(s)} = \rho - r^w \\ & L(s) \geq 0, \quad \zeta(s) \geq 0, \quad \zeta(s)L(s) = 0 \end{aligned} \tag{6.40}$$

From the first and the third line in equation (6.40) the Euler equation can be derived.

$$\frac{\dot{C}(s)}{C(s)} = r^w - \rho \tag{6.41}$$

The change in consumption is determined by the difference between the world interest rate and the agent's time preference. The lifetime budget constraint becomes.

$$A(u) + \int_u^\infty [W(s)L(s) - T] e^{(r+\beta)(u-s)} ds = \frac{C(u)}{\rho+\beta} \tag{6.42}$$

The right hand side consists of assets at the beginning of the planning period plus a present value of after tax wages. Now optimal labor planning can be determined. There are two distinguishing between two scenarios, whether the agent is retired in the planing period or not.

Let's first take a closer look at the scenario where the agent is working. If the agent is working $L(u) > 0$ which implies $\zeta(s) = 0$ because of (6.40). By plugging the first line of (6.40) into the second and isolating $1 - L(s)$ the following relation determining labor supply is obtained.

$$1 - L(u) = \frac{1-\varepsilon_C}{\varepsilon_C} \frac{C(u)}{W(u)} \tag{6.43}$$

The optimal amount of leisure is dependent on the age of the agent, because both consumption and wages are age dependent. If the agent has reached the downward slope in productivity the wages drop which will lead the individual to choose a higher level of leisure. To further understand this relationship between the agent's age and leisure decisions equation (6.43) is differentiated w.r.t. age:

$$\dot{L}(u) = \frac{1-\varepsilon_C}{\varepsilon_C} \frac{W(u)\dot{C}(u) - C(u)\dot{W}(u)}{W(u)^2} \tag{6.44}$$

From equations (6.44) and (6.43) and by applying the Euler equation in (6.41) the following relation is obtained.

$$\frac{\dot{L}(u)}{1-L(u)} = \frac{\dot{W}(u)}{W(u)} - \frac{\dot{C}(u)}{C(u)} = \pi(u) + \rho - r^w \tag{6.45}$$

Where $\pi(u) = \dot{W}(u)/W(u)$ is the proportional change in wages with age. The proportional change in leisure over the agent's lifetime is dependent on the time preference, world interest rate and the proportional growth in wages. During youth the wages grow with increased productivity but when the agent gets older the wages start to drop. The proportional change in leisure depends on whether $\pi(u)+\rho$ is greater or less than the world interest rate r^w, that is:

$$\pi(u) + \rho \quad \begin{cases} > r^w & \text{for } 0 \leq u < \bar{u} \\ < r^w & \text{for } u \geq \bar{u} \end{cases} \tag{6.46}$$

Time spent working is increasing while wages increase (in youth) and starts to drop as wages drop (later in life).

Now we can take a look at the retired agent. The assumptions about the productivity of agents imply that once an individual is retired he does not start working again, assuming that no significant unanticipated loss in financial wealth occurs after retirement. This is do to the fact that once the productivity starts declining it never inclines again. If $\zeta(u) > 0$ it follows from equation (6.40) that $L(u) = 0$ which implies that the agent is retired. From equation (6.40):

$$1 - \varepsilon_C = \eta(s)W(s) + \zeta(s), \quad (\text{for } u \geq u^*) \tag{6.47}$$

Here u^* is the retirement age. Before retirement labor supply is positive and at retirement it drops to zero. Therefore it follows, from the fact that the Lagrangian multiplier $\zeta(s)$ is continuous, that $\zeta(u^*) = 0$. Now we can differentiate w.r.t. u and plug in from equations (6.40) and (6.41) to get the following.

$$\dot{\zeta}(u) = \eta(u)W(u)[r^w - (\pi(u) + \rho)] > 0, \quad (\text{for } u \geq u^*) \tag{6.48}$$

The sign follows from equation (6.46), since the agent is retired $\pi(u) + \rho < r^w$. These results can be interpreted in figure 6.2. the non-negativity constraint on labor supply becomes binding at point E where the $U_{1-L}/U_C = W(u^*) < W$ and $L(u^*) = 0$. As the agent ages ($u > u^*$) the optimal consumption continues to grow and leisure remains at 1. This is represented in figure 6.2 as a movement along the vertical line from point E towards points B and D. Since labor supply cannot be negative the agent cannot reach point C.

6.2.3 Optimal Retirement Age

Now the consumption and leisure decisions of agents with non-constant productivity over lifetime have been examined. In this section the optimal retirement age of an agent facing a hump shaped productivity will be determined. This will shed light on the retirement decisions of agents and the possible effects a fiscal policy might have on those decisions.

For a retired agent $L(u^*) = 0$ and equation (6.43) becomes:

$$C(u^*) = \frac{\varepsilon_C}{1-\varepsilon_C} W(u^*) \tag{6.49}$$

This implies that retired agents will consume a fixed fraction of wealth at retirement. Furthermore we know that a agent will work until the retirement age u^* after which they will never work again. The lifetime budget constraint is:

$$\begin{aligned} \frac{C(u)}{\rho+\beta} &= A(u) - T\int_u^\infty e^{(r^w+\beta)(u-s)}ds + \int_u^{u^*} W(s)L(s)e^{(r^w+\beta)(u-s)}ds \\ &= A(u) - \frac{T}{r^w+\beta} + \int_u^{u^*}\left(W(s) - \frac{1-\varepsilon_C}{\varepsilon_C}C(s)\right)e^{(r^w+\beta)(u-s)}ds \end{aligned} \tag{6.50}$$

The first line of equation (6.50) splits the integral in equation (6.42). The second line is obtained by isolating work in equation (6.43) and plugging in for $L(s)$. Using the fact that $C(s) = C(u))e^{(r^w-\rho)(s-u)}$ the optimal retirement age and consumption to the age-dependent path of wages can be related.

$$A(u) - \frac{T}{r^w+\beta} + \int_u^{u^*} W(s)e^{(r^w+\beta)(u-s)}ds = \frac{C(u)}{\varepsilon_C(\rho+\beta)}(1-(1-\varepsilon_C)e^{(r^w+\beta)(u-u^*)}) \tag{6.51}$$

Using equation (6.49) and the fact that $C(u^*) = C(u))e^{(r^w-\rho)(u^*-u)}$ we get:

$$C(u) = \frac{\varepsilon_C}{1-\varepsilon_C} W(u^*)e^{(r^w-\rho)(u-u^*)} \tag{6.52}$$

By applying equation (6.52) into (6.51) a relation that determines the agent's optimal retirement age is obtained. The agent is time consistent and his choice of retirement age does not depend on his current age. Therefore it is possible to write the optimal retirement age from the perspective of a newborn without a loss of generality. However external economic factors could alter the agent's decisions for a optimal retirement age. The optimal retirement age at birth is determined by the following.

$$\int_0^{u^*} W(s)e^{-(r^w+\beta)s}ds = \frac{T}{r^w+\beta} + \frac{e^{-(r^w-\rho)u^*}}{1-\varepsilon_C}\frac{W(u^*)}{\rho+\beta}(1-(1-\varepsilon_C)e^{-(\rho+\beta)u^*}) \tag{6.53}$$

To further determine the agent's optimal retirement age, we summarize the realtion above into the two terms below. The left hand side of equation (6.53) is defined as $\Xi(u)$ and the right hand side as $\Psi(u)$. The optimal retirement age, u^* is where $\Xi(u^*) = \Psi(u^*)$

$$\Xi(u) \equiv \int_0^u W(s)e^{-(r^w+\beta)s}ds \tag{6.54}$$

$$\Psi(u) \equiv \frac{T}{\rho+\beta} + \frac{W(u)}{(1-\varepsilon_C)(\rho+\beta)}(e^{-(r^w-\rho)u} - (1-\varepsilon_C)e^{-(r^w+\beta)u}) \tag{6.55}$$

From equation (6.54) following is determined.

$$\begin{aligned}
&\Xi(0) = 0, \quad \Xi(u) > 0 \text{ (for } u > 0)\\
&\Xi'(u) \equiv W(u)e^{-(r^w+\beta)u} > 0\\
&\Xi''(u) \equiv -W(u)e^{-(r^w+\beta)u}[r+\beta-\pi(u)]
\end{aligned} \tag{6.56}$$

$\Xi(u)$ is a positive and increasing function of u and concave when $u > \bar{u}$. Similarly as for $\Xi(u)$ we get for $\Psi(u)$.

$$\begin{aligned}
&\Psi(0) = \frac{T}{r^w+\beta} + \frac{\varepsilon_C W(0)}{(1-\varepsilon_C)(\rho+\beta)} > 0. \quad \Psi(\infty) = \frac{T}{r^w+\beta} > 0\\
&\Psi'(u) \equiv \frac{W(u)\big[\pi(u)-(r^w-\rho))e^{-(r^w-\rho)u} - (1-\varepsilon_C)[\pi(u)-(r^w+\beta)]e^{-(r^w+\beta)u}\big]}{(1-\varepsilon_C)(\rho+\beta)}
\end{aligned} \tag{6.57}$$

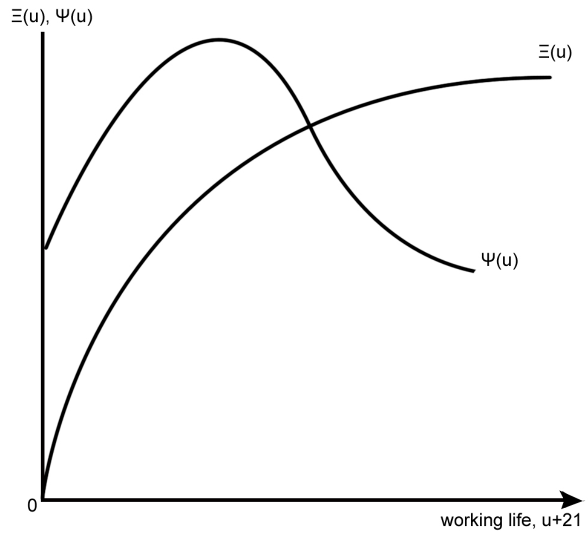

Figure 6.3: $\Psi(u)$ and $\Xi(u)$ over the working life

At birth $\Psi(0) > \Xi(0)$. $\Psi(u)$ and stays positive and approaches the value $T/(r^w + \beta)$ as age approaches infinity. Furthermore $\Psi'(u) > 0$ for any values of $0 \leq u < \hat{u}$ and $\Psi'(u) < 0$ for $u > \hat{u}$. Where $\hat{u}$ is the solution to:

$$\hat{u} = \frac{1}{\rho + \beta} ln \left[\frac{(1 - \varepsilon_C)[r^w + \beta - \pi(\hat{u})]}{r^w - (\rho + \pi(\hat{u}))} \right] \tag{6.58}$$

Equation (6.58) determines at which age the $\Psi'(u)$ is equal to zero. From equation (6.57) we see that $r^w > \rho + \pi(\hat{u})$ which implies that $\hat{u} > \bar{u}$. Therefore $\Psi(u)$ peaks later in the working life than labor supply does. $\Xi(u)$ starts out below the $\Psi(u)$ curve and is increasing. Therefore $\Xi(u)$ crosses $\Psi(u)$ from below which implies that the slope of $\Xi(u)$ is steeper than $\Psi(u)$ at their intersection, $\Xi'(u^*) > \Psi'(u^*)$.

Introduction, or an increase, of lump sum taxes, T, would shift the $\Psi(u)$ curve and increase the optimal retirement age, i.e. $du^*/dT > 0$. This is important to the thesis topic, as it demonstrates a way the fiscal policy can alter the optimal retirement age of agents without mandating the age itself. The increase in lump sum taxes was discussed within the context of increased government debt in chapter 4. In figure 6.3 possible $\Xi(u)$ and $\Psi(u)$ are illustrated graphically over the agent's working life.

Analysis of lowering the hazard rate to simulate increased longevity is relevant this extension. Equation (6.54) demonstrates that a drop in β would shift the $\Xi(u)$ curve upwards as wages are discounted less intensely. However a drop in β would shift the $\Psi(u)$ curve upwards as well, this can be seen in equation (6.55). The drop in the hazard rate would increase value of $\Psi(u)$ in the limit where age approaches infinity, this effect can be seen in the fist line of equation (6.57). This leads does not lead to a clear effect of increased longevity on the retirement age. A more precise relation between the hazard rate an optimal retirement age can be found by assuming a precise path for wages.

Considering a special case where wages are age-independent and zero lump sum taxes we get that $\pi(u) = 0$ and that wage itself does not effect the retirement decision. This is represented in figure 6.2 as an agent who ascends along the dashed line as he ages. In this special case the only thing that affects retirement decisions are the agent's initial position on the dashed line and the speed at which the agent moves along the dashed line towards point B.

In the book Foundations of Modern Macroeconomics, 2nd edition, by Ben J. Heijdra (2009) a calibration of this model was undertaken, based on data obtained about the Finnish labor market. $u = 0$ corresponded to an individual aged 21 as it was assumed that individuals enter the workforce at that age. Values for specific parameters were chosen as follows, $r^w = 0.06$, $\rho = 0.045$, $\beta = 1/62$, $\varepsilon_C = 0.25$. The value for β implies a life expectancy at birth of 83 years.

From the Finnish data it was observed that the optimal labor supply over the lifetime follows a hump shaped path, similar to the hump shaped wage path. Labor supply reaches its peak at $\hat{u} = 15.5$ or at the biological age of 36.5 years. As the wage grows sharply during youth the labor supply grows as well. The labor supply reaches its peak at an higher age than the wage rate does. The optimal retirement age, as found by the intersection of the $\Xi(u)$ and $\Psi(u)$ curves was found to be at the biological age of 66.6 years.

Over the lifetime the agent consumption grows at an exponential rate. At the time of retirement an agent meets his expenditure with existing wealth.The consumption is lower if the non-negativity labor supply constraint is relaxed because the agent then demands labor (buys leisure) in retirement. The young agent who anticipates higher wages in future borrows financial assets which he pays back at a later date when wages are higher.

Chapter 7

Concluding Remarks

The analysis presented in the thesis investigates the implications of increased longevity on an open and closed economy. It is based on the Blanchard-Yaari macroeconomic model with selected extensions. The model is not a perfect representation of reality as it does not incorporate all relevant factors of reality into the analysis. However it can be used as an important tool in obtaining a fairly clear view on the main elements of the subject

The analysis commenced by studying individuals. Individuals naturally face an uncertainty about their time of death and accordingly are forced to hold a portfolio of buffer assets to ensure positive wealth at death. The necessity of holding such buffer assets hinders optimal wealth allocation. By introducing life-insurance, in the form of actuarial notes, an individual can allocate assets optimally. The rate of the actuarial notes depends on the instantaneous death probability, or hazard rate, of the individual. Increased longevity implies that the death probability decreases, which in turn implies that the rate on the actuarial notes decreases. Furthermore, increased longevity would increase consumption growth because the higher probability of being able to enjoy the benefits from increased saving.

Blanchard's additions to the model assumed a homogeneous population where each individual faces the same instantaneous probability of death. This assumption allows for aggregation and analysis at the macroeconomic level. The analysis is performed by the help of a phase diagram which illustrates dynamic changes in consumption and capital stock. Two main cases where examined, an open economy case and a closed economy case.[1]

[1] The main difference between an open and closed economy is that in a closed economy the amount of capital stock affects the interest rate. The interest rate in turn affects the consumption path chosen by individuals. In an open economy the interest rate is

The conclusion of the fiscal policy analysis is that through an increase in government spending, financed by contemporaneous taxation, both consumption and capital stock would diminish. Furthermore, a permanent increase in government debt would further influence the capital stock and consumption. This is caused by a change in the savings rate of individuals. The effectiveness of such a policy is dependent on the hazard rate and time preference of the homogeneous population. A fiscal policy can affect the capital stock and consumption composition in a variety of ways. This implies that if inefficiency arises as a result of increased longevity a specific fiscal policy could influence the economy in beneficial ways.

Increased longevity is simulated in the model by lowering the hazard rate. This results in individuals being less likely to die at any point in time, which makes savings more attractive. In a closed economy increased longevity results in an increase in capital stock and consumption. In an open economy the effect of increased longevity is dependent on whether or not the population is relatively patient or not. In a relatively patient population the increased longevity leads to similar results as in the closed economy case. For a relatively impatient population the increased longevity, however, causes a drop in consumption and the stock of assets.

A change in life expectancy is generally gradual. In most cases it follows the country's development. This implies that the results from anticipated gradual changes in hazard rate are most representative of the real world. However, an unanticipated negative shock to life expectancy of agents can be realistic in specific isolated cases, such as war, famine or natural disasters. Even a positive shock to life expectancy can be representative of the real world. An example of this is an unanticipated scientific breakthrough in medicine.

Within the context of the Blanchard-Yaari model birth and death rates are both determined by the hazard rate. To isolate the effects of increased longevity from the effects of decreased birth rate an extension of the model was made. A decreased death rate resulted in an increase in the capital stock but the effects on consumption were ambiguous. A correlation between death- and birth rates can be argued on the grounds that if parents expect more children to reach adulthood they are less inclined to have many children. The model ignores any relationship between parents and offspring.

To simulate saving for retirement individuals were assumed to have age dependent productivity. As individuals get older their productivity and wages would drop. To smooth out lifetime consumption individuals in-

exogenous and is not affected by the amount of domestic capital stock.

creased savings while young. This resulted in increased consumption and capital stock at the macroeconomic level. However if the drop in productivity was rapid enough inefficient amounts of consumption and capital could be obtained. At any rate, an appropriate fiscal policy could correct inefficiencies obtained by diminishing productivity.

To understand the effects of leisure and life-cycle labor supply another extension was made to the Blanchard-Yaari model. Leisure was introduced into the lifetime utility function. Individuals were found to increase their recreational activities steadily with age until they reach retirement. To further the analysis individuals were characterized as having increasing productivity while young and decreasing productivity in latter part of their working life. Based on this an optimal retirement age was determined. Importantly an increase in lump sum taxes would result in increased retirement age. This provides the government with yet another tool to react to increased longevity.

Future research could futher examine the fiscal policy implications of increased longevity. Implications of a higher retirement age, provided that the working population is active, can be examined with special focus on the returns from education. Pension reforms could also be researched and the sustainability of pay-as-you-go pensions schemes in face of changing demographic structures. Future research could also analyze the effects increased longevity has on the median voter and democratic results in general.

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