Pricing Crude Oil Calendar Spread Options

Master Thesis

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Executive Summary

The aim of this Master Thesis is to estimate the prices of one-month calendar spread call options on crude oil, employing a stochastic pricing model for the underlying Futures prices of this energy commodity and a Monte Carlo simulation based model. The consistency of the results from our estimations is tested and a comparison with observed market prices is performed, in order to evaluate the accuracy of the sequential implementation of the two models.

We analyse the very particular commodity market, concentrating even more our efforts in the description of energy markets. Great attention is also given to the idiosyncratic characteristics of commodities prices: mean reversion, convenience yield, seasonality and jumps. We give particular emphasis to oil, the world’s most important commodity and the underlying commodity of the calendar spread options we are pricing.

In this study, we implement a three-factor model developed by Cortazar and Schwartz (2003). This model accounts for mean reversion and for the existence of convenience yield, while the other two characteristics of commodities prices are not included. In the same study, the authors proposed a technique to estimate the model parameters that is an alternative to the more demanding Kalman filter. We deeply discuss the former approach and individuate its advantages and disadvantages. We implement this technique in order to find the values for the parameters that, according to the three-factor stochastic model suggested by the authors, describe the price behaviour of the light sweet crude oil. These findings are later used as an input in the option pricing model.

Regarding the estimation of the parameters and Futures prices our findings suggest a mean reversion coefficient for the convenience yield that is higher than the one found in previous studies. We also obtained a level of the demeaned convenience yield volatility that is lower than the one we expected. The correlation between the spot price and the price appreciation of crude oil is also too low, which is usually not a characteristic of this market. The correlation between the convenience yield and the spot price is high and positive, which support the mean reversion of crude oil price. These results were then used in the Monte Carlo simulation model built to price one-month calendar spread options. Despite a correct implementation of the model, our estimates turned out to be excessively low compared to the prices observed in the market. To understand the reason of this mispricing we perform a sensitivity analysis and we compute, according to our model, the implied values of some pricing parameters. Low values for state variables volatilities and correlations seem to be the drivers of the underestimation of our option prices.

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Introduction

In the past decades, commodity markets have become central in the financial world. Within these markets, energy commodities have assumed a predominant role due to the recent worldwide energy liberalization process that is still taking place even in the most developed countries. Moreover, the significant increase in the demand for energy, in particular due to the spectacular economic growth of Asian economies such as India and China, has increased the prominence of energy trading and thus the number and complexity of transactions in energy markets. These circumstances have attracted market participants since there is empirical evidence that investing in commodities provides not only interesting diversification benefits to financial portfolio holders but also significant positive returns. Moreover, investors can take advantage of these benefits without even trading physical assets but using only financially settled products. Commodity traders can also use derivatives for hedging purposes. Liberalized and sophisticated markets provide investors with the chance of taking full profit from these benefits. Due to these advantages and to the abrupt increase of relevance of these markets, commodities, and energy commodities in particular, are usually seen as new or particular asset classes.

Inside the world of commodities, oil is undoubtedly one of the most important ones, as it is the number one energy source worldwide. It is portable, relatively safe, easy to handle and it is very powerful. It is particularly important for transportation but also intensively used for industrial purposes. Despite the recent technological development of cleaner energy sources, the prominence of oil as the main source of energy is expected to remain in the near future. Likewise, despite the general fears of oil scarcity in the short-term, the current proved reserves of oil are expected to cover worldwide production, given the current rate of consumption, for about 50 years (U.S. Energy Information Administration, 2012). Another reason that points out the importance of this commodity is the evidence that Gross Domestic Product (GDP) and dollar appreciation are closely related to the return in price of crude oil. All these reasons have contributed to our decision of focusing on this specific energy market.

Regarding the modelling of energy commodities price behaviour, this has been a very challenging task for academics and practitioners. The degree of complexity of energy price comportment is significantly higher than that of stock prices. Commodities in general, and energy commodities in particular, have very idiosyncratic characteristics that need to be taken into consideration when modelling prices. However, despite these difficulties, the pricing of energy commodities and their financial instruments is crucial for market participants: inaccuracies in the estimation of prices might lead to unexpected losses for speculators and hedgers, and to unfavorable investment decisions, where unprofitable project are accepted and profitable opportunities are denied. Moreover, misestimating may also lead to inaccurate interpretations of market information as, for example, implied volatilities.

In the last decades lecturers and financial practitioners have been coming up with different pricing models. These stochastic processes have been increasing in complexity in order to respond to the crescent market sophistication. Pricing models for different commodities need to account for different assumptions in order to perform on their aim of predicting prices. Academics and researches have reached a good level of approximation of the price of these particular assets, but there is still some work that needs to be done to improve even more their estimations. The fact that commodity markets are developing at a very fast pace and considerably increasing in complexity leads to the existence of some research gaps concerning pricing models and techniques. The financial industry and the actors active in the energy market ask for more reliable models. However, as models in finance are ultimately models of human behaviour, they are always likely to be, at best, approximations of the behaviour of market variables (Hull, 2009).

The aim of this Master Thesis is to estimate the prices of one-month calendar spread call options on crude oil traded on NYMEX, employing a stochastic pricing model for the underlying Futures prices of this energy commodity and a Monte Carlo simulation based model. The consistency of the results from our estimations is tested and a comparison with observed market prices is performed, in order to evaluate the accuracy of the sequential implementation of the two models.

Concerning the methodology used in this study, we initially estimate the parameters that describe the behaviour of Futures prices according to a particular stochastic model and, then, we perform the related Monte Carlo simulation. The model we employ for shaping the behaviour of Futures prices was proposed by Cortazar and Schwartz (2003) and it is based on previous commodity related stochastic pricing processes. In order to estimate the relevant parameters required by the model the authors suggest an alternative estimation technique. This procedure is relatively easy to implement and does not require high-level skills in advanced programming. We then use the results of this implementation in our Monte Carlo simulation model. We decided thus to price calendar spread options implementing an advanced pricing model that should permit to better represent the reality. To achieve a deep understanding of the functioning of this model and of the related estimation technique it is necessary, however, to provide solid background on the theory of commodity price modelling (Chapter 3).

About option pricing, after having provided a brief review on the existent literature, we support our decision of implementing Monte Carlo simulation. As we will have the chance to debate, there is no clear agreement on which closed-form approximation is the most reliable to price calendar spread options, even though most of these approximations appear to be relatively precise. Another reason for implementing such an intuitive pricing technique is related with our intention of presenting our model and its assumptions in a simple and intuitive manner. This is because it is also the purpose of this work to provide individuals that are not familiar with this topic with the opportunity of entering the world of commodity price modelling and option pricing.

We start this Master Thesis by presenting an overview on commodity markets in general and energy markets in particular. We further analyse the reasons for the crescent importance of these markets and we describe the main characteristics of the behaviour of energy prices. While the initial sections of this first chapter present a solid introduction to the core topics of this paper, the last point represents a first approach to the subject of price modelling, as it describes the characteristics that have to be taken into consideration when building an energy pricing model.

In the following chapter (Chapter 2) we dedicate our efforts to the oil market, providing an overview of its functioning and describing some of the financial products available in this market. In the last section of this chapter, we complete the analysis performed in Chapter 1, identifying the energy price characteristics that have to be included in crude oil pricing models.

Chapter 3 is entirely dedicated to the modelling of commodity prices: we start by describing the evolution of the models over time and by providing a mathematical background that is imperative in order to understand how these models are built and implemented. After introducing the two and three-factor commodity pricing models proposed by Eduardo S. Schwartz (1997), we analyse and concretely implement the Cortazar and Schwartz model and its related estimation technique, which we use to price light sweet crude oil. The results of this implementation are then presented and discussed with particular emphasis on the interpretation of the estimated coefficients of the model parameters. The output from this chapter is of great importance for Chapter 4, when different calendar spread call options are priced. This is because the estimated pricing model parameters will be used to compute the random paths for the three state variables needed to model crude oil prices.

We start the last chapter of our work introducing general option theory. Afterwards, we describe the characteristics of calendar spread options in particular and the theoretical foundations for closed-form solutions for pricing options. Lastly, the Monte Carlo simulation technique is presented, discussed and implemented in order to accomplish the final purpose of this study, which is to price calendar spread call options traded on the crude oil market. The results from this procedure are then presented and the reasons for pricing discrepancies are discussed.

1 Overview on Energy Markets

In this first chapter we start by presenting an overview on commodity markets, addressing specific attention to their characteristics and evolution. Additionally, we offer a comparison with securities markets. We also group commodities into three different categories, highlighting their most important particularities and identifying both participants and trading strategies. Later, we investigate the crescent relevance of commodities as an investment vehicle and we explain why they are seen as a new or different asset class. Following this general presentation of commodity markets, we focus on describing the specific segment that deals with energy commodities, known as energy markets. At this point, another comparison is offered, this time between energy and money markets, allowing for a better understanding of the specificities of the former market. Moreover, this comparison also provides suggestions for the specificities of the price behaviour of energy commodities, that we also have the chance to analyse with detail in the last section of this first chapter.

1.1 General Introduction to Commodity Markets

Commodity markets can be defined as physical or virtual marketplaces where raw or primary products can be bought, sell, or traded. These products include gold, sugar and oil, for example. Commodity markets constitute the only spot markets that have practically existed throughout the history or humankind. They have evolved not only in terms of complexity but also in terms of scope, changing from simple, physical trading based and agricultural markets to very complex physical and virtual markets where a broad range of commodities is traded. The nature of trading has advanced to include elaborated forward contracting, organized Futures markets, options and other diverse structured products. Most of the transactions that occur nowadays in commodity markets are related with Futures contracts. These products have become very liquid, have low transaction costs and are absent of credit risk (Geman, 2005).

Due to the specificities of its products, commodity markets have some important proprieties that contrast with those of bond and equity markets (Geman, 2005). First of all, commodity spot prices are not defined by the net present value of receivable cash flows, but by the intersections of supply and demand curves in a given location. Additionally, inventory is also an important issue to be taken into consideration. The second important feature is that the demand for commodities is generally inelastic to prices. This is due to the intrinsic importance of the goods being transacted. Most of the commodities are primary products that are essential for consumers and that cannot, in general, be easily substituted. The issue of demand inelasticity is addressed in one of the following sections (1.4.4 Jumps and Spikes), when jumps and spikes are analysed for the specific case of energy commodities.

Regarding commodity supply, it is defined by production, inventory and, in the case of energy commodities, underground reserves, due to their influence in long-term price levels. The balancing of supply and demand takes place at both regional and World level. This is in fact related with another distinctive characteristic of commodity markets: the specifications of the physical delivery of the good are attached to contracts, containing constraints in terms of product quality, shipping arrangements, delivery place, warehousing and others. The last important distinctive feature of commodities markets is the existence of quantity risk in transactions: while investors owning bonds or stocks are strictly concerned of price or interest rate fluctuations, commodity suppliers are affected by variabilities of demand that emerge from changes in consumer preferences, worldwide economic circumstances and/or weather conditions.

Besides these general distinctive characteristics, commodity markets have other particularities that depend on the specific commodity we are analysing. That is why it is worth to group commodities into major categories and describe each of them separately. Commodities can be than divided into three major categories: hard commodities, soft commodities and energy commodities. While the first group typically includes natural resources that must be mined or extracted (with exception of those that fit the last group), the second one encompasses agricultural products and livestock, that is, products that can be grown. The third group refers to energy generating commodities such as oil, gas, coal and electricity. However, it is also common to refer to hard and soft commodities according to the degree of spoiling ease. That is, commodities that are difficult to spoil are known as hard commodities and those that are easy to spoil are known as soft commodities. From this point of view lumber can be seen as a hard commodity, while it is understood as a soft commodity according to the previous definition. Moreover, this criterion implies that energy commodities are not seen as a separate commodity class anymore. For the purpose of this paper, we will stick to the three group criteria, as we believe that, besides being more unambiguous, it helps us addressing issues on energy markets more precisely.

Hard commodities include precious metals (such as gold and silver), base metals (as aluminium and copper) and ferrous metals (iron ore, for example), among others. Precious metals are very popular investments that are widely used for hedging against inflation and currency devaluation, while providing diversification benefits to investors. Gold, in particular, is generally highly demanded during periods of economic crisis. Precious metals were historically used as money and as relative standards since domestic currencies were, at some periods of the history, backed by gold (for example, during the gold standard era). The hard commodities group includes also other metals that are used as raw material in very dynamic industries like the automotive industry.

Soft commodities include grain, maize, coffee, cotton, sugar and cattle, for example. These products are usually produced in developing countries and exported to developed countries. The markets for these commodities are amongst the oldest markets in the world. Tropical commodities (or tropics) is a sub-group of soft commodities that encompasses goods that are mostly produced in tropical or subtropical regions like coffee, cocoa and tea. Soft commodities markets are mostly used by farmers that pretend to lock-in the future prices of their crops and by speculative investors. Being speculators or hedgers, soft commodity investors need to be aware of the seasonality price behaviour that occurs in many of these markets, due to both changes in demand and to the natural growing cycle of the commodities. Weather also plays a key role in soft markets as it turns supply predictions difficult, creating a considerable degree of uncertainty in the market.

Energy commodities are the focus of this study. They have become highly popular and dynamically traded in recent years. This is in part due to a profound deregulation process that is occurring throughout the world. In Europe, for example, while oil and natural gas markets have started to become less regulated by the 1980s and 1990s, respectively, electricity markets are still experiencing significant transformations that are changing the way the market operates. Even though the first stage for a region-wide deregulated electricity market has occurred in 1999, when the electricity market in Europe began to open up on an international basis (Whitwill, 2000), the liberalization process was not, by 2010, as advanced as anticipated by the European Union (Altmann, et al., 2010). According to the same report, while the Nordic countries are seen as the *fast movers* of the liberalization and integration process, other countries such as Poland and Greece are seen as *laggards*. The electricity deregulation process in Europe has been, thus, very heterogeneous and is far from being complete. However, it has been important enough to arise curiosity in market participants and to boost electricity markets and energy markets in general.

Another reason for the increasing dynamism of energy markets is the fact that the demand for energy has been increasing for a long time. Moreover, the need for energy is expected to continue to grow in the future, particularly in emerging markets. Some other reasons for the rising popularity of energy commodities are: their unique characteristics (such as storability limitations and seasonality); the increase in oil prices, which has captured the attention of the media and investors and the growing attention given to environmental problems such as climate changes. Energy markets are central commodity markets and will be analysed with detail later in this chapter.

Besides this rising popularity and dynamism, commodity markets have also become very sophisticated. This can be seen together as a consequence and a cause for the increasing number of participants in this market. On the one hand, investors are continuously requiring a bigger diversity of financial instruments that allows them to profit from the intrinsic characteristics of the behaviour of commodity prices. As we will have the chance to address in the last section of this first chapter, energy prices in particular have very distinctive characteristics that create high levels of volatility and difficulties in predicting future price behaviours. Moreover, commodity prices are easily influenced by political upheavals and worldwide economic changes (Geman, 2005). All these features explain why market participants have very significant hedging needs, and therefore require the existence of numerous market derivatives. On the other hand, the particularities of commodity markets have also attracted new participants, such as portfolio managers, pension funds and mutual funds. The existence of sophisticated commodity derivatives has provided investors that are not interested in the physical trading of the asset with the ability to capture the advantages of investing in the market through financially settled products. Thus, the increasing sophistication of the market has attracted diverse market participants.

In this section, we have provided a general overview on commodity markets. We have identified some of their most important features and described the reasons for the increasing complexity. We have also suggested why commodity markets have become so trendy. The next section provides a deeper look at the advantages of investing in commodities.

1.2 Commodities as a New Asset Class

As we have already pointed out when we introduced commodity markets, market participants frequently see commodities as a new, or particular, asset class. We believe it is worth to pay a bit more attention to this fact and to understand why commodity markets have attracted such a broad range of investors. From our point of view, this is mostly related with the increasing perception of the main direct benefits of investing in commodities: relative good performance in terms of returns and diversification benefits to financial portfolios.

Regarding the first benefit, one can observer that, during some recent periods of poor performance in stock markets, commodity related investments have performed considerably well (Geman, 2005). Considering long-term returns, Gorton and Rouwenhorst (2004) constructed an equally-weighted index of commodity Futures monthly returns over the period between July 1959 and December 2004, and found out that the average annualized return of a collateralized investment in commodity Futures has been comparable to the return of the S&P500 (Gorton & Rouwenhorst, 2004) and has, as predictable, outperformed the returns of corporate bonds. However, it is important to notice that, in this paper, the authors use an equally weighted index of commodity Futures. Therefore, these findings suggest that some of the commodities composing the above-mentioned equally weighted index have actually outperformed stock indexes and numerous individual stocks. In addition to this, both commodity spot and Futures prices have outpaced inflation over the last decades - Geman (2005) and Gorton and Rouwenhorst (2004).

The other main reason for the rising popularity of commodities is related with diversification benefits: commodities usually allow investors to reduce the overall risk of a financial portfolio. Considering, for example, an investor who holds a portfolio of stocks and bonds, an addition of commodities Futures to his position will shift his Markowitz frontier upward. Gorton and Rouwenhorst (2004) found out that over quarterly, 1-year and 5-year horizons, the total return of the equally weighted commodity index was negatively correlated with the return on the S&P500 and the return on long-term bonds. Moreover, this correlation was significant at 5% level for most of the cases, suggesting that commodity Futures are in fact effective in diversifying equity and bond portfolios. There is also suggestion that diversification benefits tend to be larger at longer horizons. Considering oil - the commodity we will focus later in this paper - Geman (2005) concluded that the correlation coefficients between oil and equity markets were negative between 1990 and 1999 (correlation of -0.16 between S&P 500 and NYMEX).

Gorton and Rouwenhorst (2004) also pointed out that diversification benefits of commodity Futures seem to work well when they are needed most, that is, when stocks earn below average returns, commodity Futures obtain above average returns. This is of course a very attractive feature that has caught the attention of investors. Needless to say, these perceptions and findings led to an increase in the demand for commodities. During the recent times of economic and financial crisis, investors have massively moved their funds to commodity markets. In fact, between 2008 and 2010, commodity assets under management more than doubled (Maslakovic, 2011). The amount of savings invested in commodities in a passive way has increased and thus prices have risen in general.

There are different ways of taking advantage of the above-mentioned benefits of holding positions in commodities. Besides the possibility to invest in the spot and derivatives markets (these are known as *direct investments*), investors can also purchase stocks of commodity-related companies and invest in commodity indexes. If purchasing shares of natural resources companies, an investor is anticipating the rise in the price of commodities. Considering a major oil firm for example, an increase of oil prices is expected to increase share prices due to an increase in revenues. That is, stock prices tend to move along with the price of the underlying commodity. However, one must be aware that there are obviously other factors influencing the stock price behaviour, meaning that buying shares of a natural resource company introduces a noise component in the desired exposure to the commodity (Geman, 2005). The other possibility is to invest in commodity indexes. It is an easy way of gaining exposure to commodity prices. Some examples of commodity indexes are: The Goldman Sachs Commodity Index (GSCI), The London Metal Exchange (LME) Index and The Commodity Research Bureau (CRB) Index. The possibility of investing directly in commodity markets will be analysed with detail in Chapter 2 for the particular case of oil.

Commodity markets have thus become very popular amongst financial practitioners. Their intrinsic characteristics, together with good historical performance and significant diversification properties, have attracted investor's attention. One of the most interesting and dynamic commodity markets is the energy market, which we now analyse with further detail, giving special importance to its characteristics and evolution.

1.3 Characteristics and Evolution of Energy Markets

Energy markets are those markets that deal with the trade of energy commodities, including oil, coal, natural gas and electricity. End users of energy include residential householders, commercial institutions, industrial companies and users of transportation means. Hence, energy utilization is crucial for the modern society.

Figure 1 in Appendix I shows the total primary energy consumption worldwide in quadrillion British Thermal Units (BTUs). The period considered goes from 1990 to 2009. The graph indicates a clear increase in worldwide energy consumption in the last 20 years. Total energy consumption has increased by 39.21%, with Asia and Oceania crucially contributing to this boost. In these regions, the primary electricity consumption has more than double in the past 20 years. The decrease of total primary energy consumption, which can be observed from 2008 to 2009, is certainly related with the recent global financial crisis.

This long-term positive trend in energy consumption is expected to continue in the future. The world's marketed energy consumption is expected to grow by 47 per cent from 2010 to 2035 (U.S. Energy Information Administration, 2012). Moreover, the same institute estimates that much of the growth in energy consumption will occur in non-OECD countries.

After having presented some basic facts and figures of energy markets, we now move to a first technical overview of these markets. In the previous section we have analysed the differences between commodity markets and equity/bond markets. We now establish a new comparison between energy markets and money markets, inspired by Pilipovic (2007). The following table (Table 1) illustrates some of the idiosyncratic characteristics of energy markets, suggesting why they are so particular and unique.

	Issue	In Money Markets	In Energy Markets
1	Maturity of the Market	Several Decades	Relatively New
2	Fundamental Price Drivers	Few, Simple	Many, Complex
3	Impact of Economic Cycles	High	Low
4	Frequency of Events	Low	High
5	Impact of Storage and Delivery	None	Significant
6	Correlation between Short and Long-Term Pricing	High	Lower, "Split Personality"
7	Seasonality	None	Key to Natural Gas and Electricity
8	Regulation	Little	Varies from Little to Very High
9	Market Activity ("Liquidity")	High	Lower
10	Market Centralization	Centralized	Decentralized
11	Complexity of Derivative Contracts	Majority of contracts are relatively simple	Majority of contracts are relatively complex

Table 1 - What Makes Energies Different? (Adapted from Pilipovic, 2007)

Energy markets are in general less mature than money markets. The oil market is the most developed energy market and only became really global in the 1980s. Nevertheless, even the relatively older markets of heating oil and crude oil continue to evolve in terms of theoretical sophistication and contract complexity (Pilipovic, 2007). Yet, despite their lower maturity, energy markets are replicating the evolution of money markets in a shorter period (Pilipovic, 2007).

The second key characteristic pointed by the author is that energies have a very significant number of fundamental price drivers. This is because their prices respond to the dynamic interaction between producing and using; transferring and storing; buying and selling – and ultimately "burning" actual physical products (Pilipovic, 2007). Each of the energy markets participants deals with a different set of fundamental drivers that cause extremely complex price behaviour. Money markets and equity markets are not subject to this degree of complexity regarding price behaviour and, therefore, are easier to understand and model.

Regarding the third issue, economic cycles have a lower impact on energy markets than on money markets, since they have a limited influence in both demand and supply. However, it is important to stress the fact that this impact is diverse between different energy commodities. While economic cycles have a big impact in oil consumption, electricity consumption is usually much more stable.

Issues 4 and 5 are inherently linked with each other. There are several news-making events that might generate imbalances between demand and supply. The frequency of events (the number of times an episode occurs in a certain period) is high on energy markets compared with money markets. Unexpected weather conditions, for example, can create significant changes in short-term market conditions, particularly due to abrupt changes in the demand for electricity. Moreover, the lack of flexibility on energy storability delays rapid responses to sudden changes in demand, which may lead to wide price fluctuations (Carmona & Durrleman, 2003). From the supply side, unforeseen changes in weather can also affect the production of energy, particularly in the case of renewable energies. The issue of storage limitation is therefore central, when discussing energy commodities, as it creates volatile behaviour of varying degrees for electricity, natural gas, heating oil and crude oil. Storage limitations cause energy markets to have much higher spot price volatility than is seen in money markets (Pilipovic, 1997). The extreme case of storage limitation occurs in electricity markets due to the non-storability of electricity. This causes demand and supply to be balanced *on a knife-edge* (Weron, et al., 2004). In fact, the occurrence of jumps and spikes is not uncommon in electricity markets. While the high volatility captures the frequency and magnitude of events, the mean reversion characteristic of commodities (that will be discussed later in this paper) illustrates how quickly the supply side of the market reacts to these events or how quickly they go away (Pilipovic, 2007). As we will have the chance to discuss later, lack of storability is also linked with the strong seasonality pattern that can be found in some energy markets.

Another characteristic that distinguishes energy markets from money markets is the lower correlation between short and long-term pricing, compared with money markets (Issue 6). This is referred to by Pilipovic (2007) as the "split personality" of energy markets: spot prices are primarily driven by the fundamentals of the short-term market factors but they are also influenced by long-term expectations. The opposite happens with Futures prices: they are primarily driven by the long-term market fundamentals but they might feel some effects of short-term fundamentals.

Seasonality was already mentioned as one of the characteristics of energy prices. It is particularly present in electricity and natural gas. In these markets, demand follows a strong seasonal pattern. Due to storage limitations, seasonality is also seen in the prices. As expected, when storage capacity is available, seasonality is less pronounced (Burger, et al., 2008). Seasonality can be observed both in historical spot prices and in the shape of the Futures curve.

Energy markets regulation varies from little to very high depending on the location and on the energy commodity (Issue 8). As we have already mentioned before, a process of deregulation in energy markets is occurring. The natural gas market deregulation started in the early 1990s in the United States and it is still occurring in continental Europe. Regarding European electricity markets, as we have already mentioned, the deregulation process started in the late 1990s and it is also far from being concluded. At this point, energy markets are still largely national in Europe. This process is expected to increase competition and thus to provide strong incentives for the improvement of operational efficiency. Nevertheless, we believe that market integration will occur in the next years.

Regarding liquidity (Issue 9), energy markets have become more liquid in recent years but they are still not as liquid as money markets. This fact is unsurprising, since energy markets are still relatively new and less mature than other markets. Another possible explanation for the existence of low liquidity is the fact that derivative contracts on energy markets are very complex and little standardized compared to other markets (Issue 11). This is mainly largely due to needs of end users that often request energy contracts to exhibit a complexity of price averaging and customized characteristics of commodity delivery (Pilipovic, 2007). In the next chapter we will return to this issue, as we analyse some of the existent financial products in the oil market.

In opposition to money markets, energy markets are very decentralized in terms of location, capital and expertise (Issue 10). Location is a fundamental price driver as the energy commodity is priced according to the delivery point (this is a critical issue for electricity in particular). Transportation of energy commodities can be costly or dependent on access to a network. This is of course not the case with the majority of other markets such as the money market.

For all these reasons, it is clear that energy markets are very particular markets. The complex characteristics we have just mentioned explain why these markets are so attractive and, at the same time, so challenging. After having acquired a general knowledge on energy markets key features, we believe that a deeper understanding of the behaviour of energy prices is essential for the purpose of this study. We therefore analyse with detail the mean reversion characteristic of energy prices, introducing afterwards the concept of convenience yield. We also develop the issues of seasonality and the occurrence of jumps and spikes in energy prices.

1.4 Characteristics of Energy Prices

The complex characteristics of energy commodities that we have described in the former section are in the origin of very particular price behaviours. In this section, we will address: the mean reversion characteristic of commodity prices; the convenience yield of commodities; the seasonal pattern of commodity prices and the existence of jumps and spikes in price paths.

1.4.1 Mean Reversion

A mean reverting process refers to a situation where prices do not grow indefinitely. In the short run fluctuations might occur, but in the long run prices should revert towards their marginal cost of production (Dixit & Pindyck, 1994). Bessembinder et al. (1995) conducted an analysis showing that investors expect mean reversion to exist for agricultural commodities, crude oil and metals, being the magnitude of this behaviour larger for the first two commodities. In the same paper, the authors found very weak evidence for mean reversion in financial markets, confirming the idea that mean reversion is mainly a feature of commodity prices (Lutz, 2010). Therefore, the mean reversion property has been recognized as the "*most noticeable price behaviour of energy commodities*" (Deng, 2000) and it undoubtedly defines a critical difference between energy and financial markets (Pilipovic, 1997). A graphical example of mean reversion processes can be seen in the following figure (Figure 1):

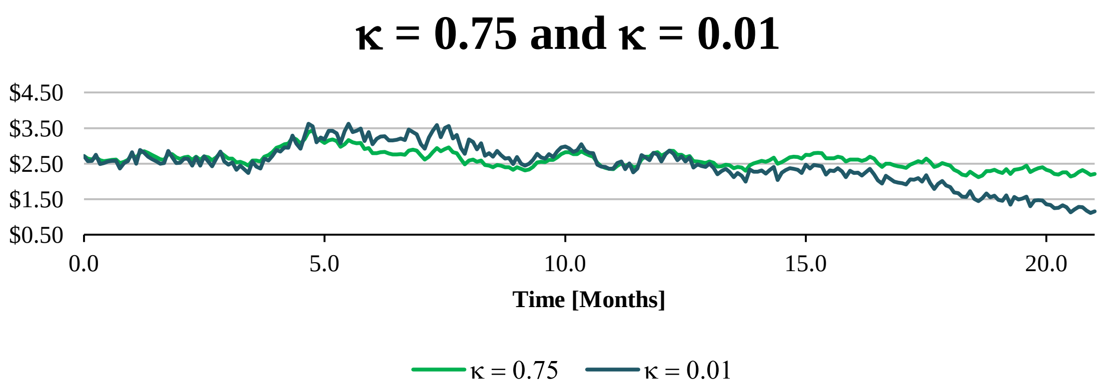

Figure 1 – Example of two mean reverting processes: $dx_t = \kappa(\bar{x} - x_t)dt + \sigma dW_t$ *where:* $x_t = \ln S_t$ *with different mean reversion speeds (*κ*)*

A theoretical explanation for the existence of mean reversion in energy prices arises from the supply-demand effect: when the price of a commodity is high (low), the supply tends to increase (decrease), pressuring the prices down (up). As it is so, energy prices tend to return to a local or asymptotic long-term level (Carmona & Durrleman, 2003). However, this does not occur instantaneously due to the significant inelasticity of energy markets. The speed of this adjustment is known as the mean-reversion speed and depends on how quickly the supply can react to events or how quickly the events disappear.

1.4.2 Convenience Yield

The convenience yield is a demand driven characteristic of commodity markets that, according to Brennan and Schwartz (1985), can be defined as "*the flow of services that accrues to the owner of the physical commodity but not to the owner of a contract for future delivery of the commodity*". It is thus the measure of the net benefits of physically holding a commodity. One of the reasons why agents hold inventories is to be able to profit from temporary local shortages and unexpected demand, as earnings may arise from local price variations. Additionally, by holding inventories, the holder is able to avoid the cost of frequent revisions in the production and to eliminate manufacturing disruptions, if the commodities owned are raw materials (Geman, 2005)[1]. Unsurprisingly, the greater the chance of occurring shortages, the higher the convenience yield (Hull, 2008). Moreover, it is important to

[1] Additional information regarding the Theory of Storage can be found in the pioneer works of Nicholas Kaldor (1939) and Holbrook Working (1948, 1949).

underline the fact that the magnitude of the convenience yield is user-specific, that is, dependent on the identity of the commodity owner.

The costs of holding the commodity, such as time lost, transportation costs and storage costs, are also considered for the measurement of the convenience yield, which means that its final value can be either positive or negative, depending on the magnitudes of both effects. According to the economic theory, one should expect the individuals with highest marginal convenience yield net of physical storage costs to hold the inventories, in an equilibrium situation (Brennan & Schwartz, 1985).

An analogy can be made between the concepts of convenience yield and stock dividend. When a shareholder buys a stock price prior to the ex-dividend date, he will pay a higher price relative to the price paid post ex-dividend date. He will, however, capture the value of the dividend in the payment date. This is very similar to what happens with commodities: in the particular case of energy markets, the holder of the commodity will have to pay a higher price for the immediate energy supply but he will be rewarded by the benefits that he can obtain from that commodity – that is, by holding it, he will be able to capture his very specific in-house dividend (Pilipovic, 2007).

The convenience yield helps understanding the relationship between short and long-term price behaviour in energy markets, as it reflects the benefits of holding commodities. As we have already mentioned before, in the short-term markets reflect the fundamentals of readily available and stored energy. In opposition, in the long-term, markets mirror the fundamentals of energy to be produced and put into storage. The convenience yield reflects the short-term supply-demand imbalances, since users are willing to pay a premium for near-term delivery in response to a supply shortage (Pilipovic, 2007).

1.4.3 Seasonal Pattern

We have already introduced the issue of energy prices seasonality in the sub-section 1.3 Characteristics and Evolution of Energy Markets. We have then said that it is one of the idiosyncratic features of energy markets, but we can additionally say that it is one characteristic that has no parallel in money markets (Burger, et al., 2008). Some energy commodities exhibit a stronger price seasonality pattern than others: while electricity and natural gas display a strong and easily identifiable seasonal component, oil prices do not usually exhibit a clear seasonal pattern. We will address further details on seasonality in oil prices in the section 2.3 Characteristics of Oil Prices and Consequences for Price Modelling. The following chart (Figure 2) provides an example of a seasonal pattern:

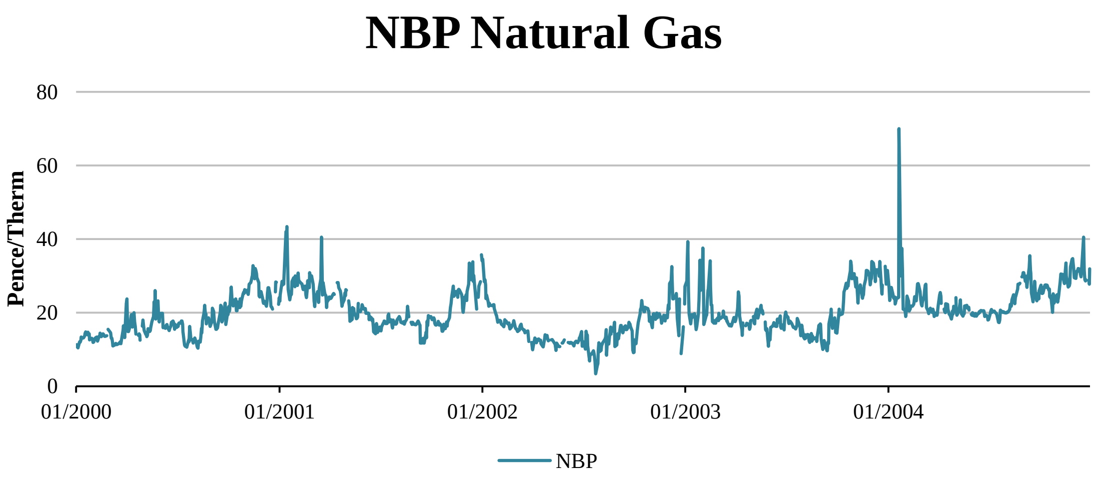

Figure 2 – Day ahead price of natural gas (NBP) in pence per therm

As we can see in the above graphical representation the price of natural gas follows the similar path of a sinusoidal function. In fact, sine waves are frequently used to model seasonal patterns in the prices of energy commodities.

In general, seasonality patterns in energy markets result from the actions of residential users. The householders' demand for energy is not constant throughout the year as there are different consumption patterns in different seasons: consumers demand electricity to power air conditioners in summer months and sources of heating during the winter period (heating oil, for example). Moreover, the demand for energy varies during different times of the day, creating intra-day seasonality. The reason why the seasonal pattern of demand can be seen on the short-term prices of energy is related with storage limitation: the lower the storage capacity, the higher the seasonality, due to the inflexibility of the energy supply. As a result, aggregate residential demand has a powerful effect on the short-term prices of energy.

1.4.4 Jumps and Spikes

Another particular characteristic of the prices of energy commodities is the presence of price jumps and spikes. The higher the storability limitation of a commodity, the higher the occurrence of jumps. A perfect example would be electricity, which is in fact almost non-storable, and where the demand is quite inelastic. Spot electricity prices exhibit, in fact, infrequent but large jumps and spikes. Spikes are usually short lived and prices use to return back to the normal level quickly (Weron, et al., 2004). The following figure (Figure 3) represents an example of a process with spikes for natural gas:

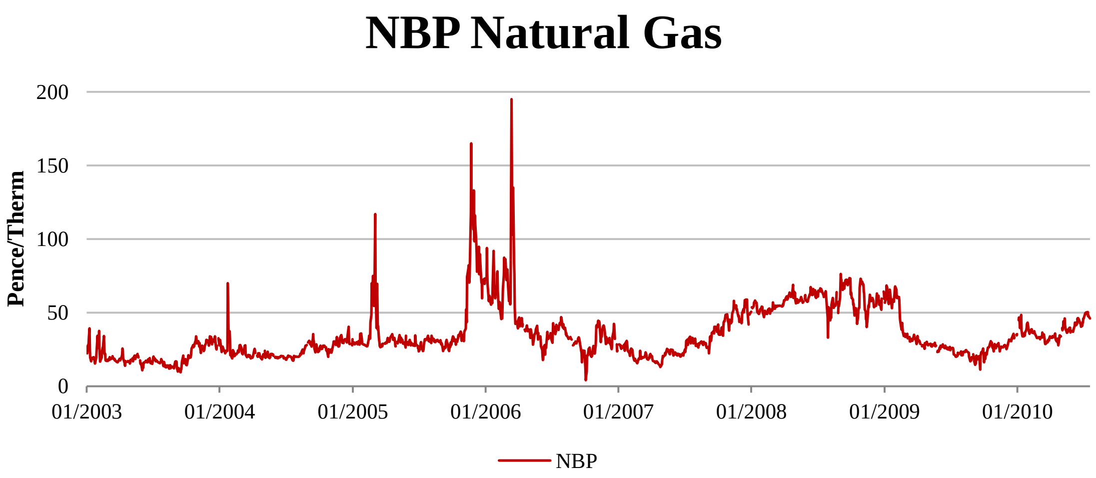

Figure 3 – Day ahead price of natural gas (NBP) in pence per therm

Electricity prices are, in a competitive market, determined by the intersection of aggregate demand and supply. A price jump can be caused by: a supply contraction (for example, during a forced outage of a major power plant), sudden surging demand or both at the same time. There would be a shift of the supply curve to the left, in the first case, or a lift up in the demand curve, in the second (Deng, 2000).

After having provided a general overview on energy markets, we now focus on the specific characteristics and features of the oil market. The next chapter (Chapter 2) provides a very good understanding of the core commodity of this study. A good knowledge of the characteristics of oil is essential in order to understand the pricing models that are described in Chapter 3.

2 The Oil Market

We devote this chapter to the crude oil market. We start with a general overview and then we focus on the identification of the main crude oil financial products traded on NYMEX. In this chapter we also identify and discuss the main characteristics of the crude oil price that we are going to model in Chapter 3.

2.1 Overview

The oil market is probably the most important commodity market in the world. As already mentioned in Chapter 1, it is also the most developed energy market. Physical crude oil markets are very fluid and universal. These markets deal with the main primary energy source worldwide. In the United States, e.g., oil covers 37% of the total energy consumption (U.S. Energy Information Administration, 2012). For transportation purposes, for example, oil has quite unique features and has almost no substitute - electric vehicle technology is emerging but still in the primary stages of development. It is estimated that, in 2004, annual oil sales corresponded to 2% of the world's GDP (Geman, 2005).

As we have already mentioned in Chapter 1, the use of energy is expected to increase in the future. Moreover, the Annual Energy Outlook (2012) from the Energy Information Administration (a well-known statistical and analytical agency within the U.S. Department of Energy) predicts that the worldwide use of energy from all sources will increase in the next 20 years. However, the growth rates for petroleum and other liquids are amongst the slowest growing energy sources. The main reasons for these low growth rates are the relatively high prices of oil and the rising concerns about environmental issues. Recently, some national governments have provided strong incentives that support the development of alternative (and cleaner) energy sources. All these reasons have contributed to the fact that renewables are the world's fastest-growing source of energy according to the same Annual Energy Outlook (U.S. Energy Information Administration, 2012). By 2030, the same outlook predicts that oil will have a reduced share of 32%, in the U.S. energy market (Figure 4). Nevertheless, due to its maturity and importance in energy consumption, the oil market is exepted to prevail as one of the most important markets in the world.

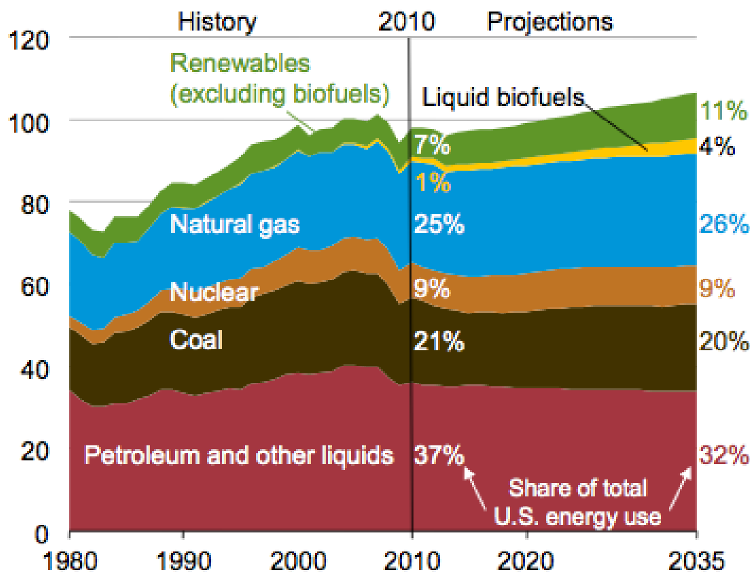

Figure 4 – Primary Energy use by Fuel in the United States (1980-2035) in quadrillion BTU.
Source: U.S. Energy Information Administration – Annual Energy Outlook 2012

A very important characteristic of the oil market is the significant geographical imbalance between producers and consumers of oil. While the main producers include North America, Saudi Arabia, Russia and Iran, the major consumers are the United States, Western Europe, China and Japan (Geman, 2005). This situation creates a need for massive flows between world regions that also creates opportunities to location investors due to market imbalances.

Besides the importance for energy consumers and international trade, oil markets are also very influential due to the fact that oil prices are usually taken as a benchmark for the price of energy. Oil prices have a big effect on the prices of other primary fuels (Geman, 2005), as its evolution tends to be replicated by other energy commodities: when oil becomes too expensive, the prices of other energy commodities follow the price of oil due to a substitution effect.

Oil can also be seen as a major financial indicator of the world's economy: its prices are very volatile and sensible to political and economic factors. In fact, as we will have the chance to discuss in Chapter 3, the most important political and economic events of the last years are easily identifiable in a historical oil price chart.

It is important to understand that oil is traded twice: first as a refinery feedstock and later as a refined product, since crude oil has to be transformed in order to become marketable. Moreover, oil is a non-standard commodity: there is a large variety of crude oil grades (more than 400 traded around the world) and thus different market values. These values depend essentially on two factors (Geman, 2005): the relative yield of products that can be extracted in the refining process (that is related to the density of the grade of crude oil) and the energy that must be spent in refinery treating units in order to meet the quality specifications for refined products, regarding the sulfur content. Thus, sulfur content and density define the attractiveness of a crude grade and, ultimately, the price refineries are willing to pay for it. Regarding sulfur content, sweet crudes are the most desirable (they have less than 1% sulfur by weight) and sour crudes the less desirable (more than 1% sulfur by weight). Concerning density, light oils are most valuable due to their low viscosity.

Despite the diversity of crude oil grades, the oil market is quite liquid because it is globally integrated and because it uses few reference oil qualities as benchmarks for pricing an individual oil quality (Burger, et al., 2008). Therefore, the prices of different types of crude oil tend to move together. Those benchmarks are standard crude oil prices against which other grades are compared and prices are set. The most important benchmark oils are the West Texas Intermediate (WTI) crude, North sea Brent Crude and UAE Dubai Crude and they are used as references in their respective locations (Geman, 2005). In this study, we will focus on the WTI, also known as Light Sweet Crude Oil. It is a low-density crude oil that is usually priced higher than Brent[2], due to its high quality and ease to refine. It is delivered in Cushing, Oklahoma and it is mostly refined in the Midwest and Gulf Coast regions in the U.S.

Refineries transform crude oil into various products, such as gasoline, liquefied petroleum gases (LPG), naphtha, middle distillates and fuel oil. Each of these products is valued differently and has distinctive pricing agreements. In general, lighter products are priced higher than heavier products. Some of the issues influencing the price differentials between refined products are: the availability of substitutes (for example, LPGs face competition from natural gas) and transportation and storage issues (jet fuel, e.g., requires special care and thus more expenses).

Despite their different characteristics, prices of crude and refined products are intrinsically linked by the technologies and economics of refining (Geman, 2005). Prices do fluctuate widely relative to each other but markets impose a limit on how much they differ: refineries will not operate in the long run with negative margins and high margins will disappear through competition. Nevertheless, it is possible and frequent to find temporary price discrepancies, just like in any other market.

Due to the characteristics of the refining industry, the supply of refined products is quite inflexible: refining is a complex and large-scale business and, therefore, suppliers are not able to completely respond to sudden demand increases. Moreover, refineries also have limited flexibility to change the production ratios among different products. Oil markets are, thus, quite volatile, just like the majority

[2] However, since mid-2012, there has been a significant change in the relationship between the prices of these two grades as the prices of WTI have become considerably lower than those from Brent.

of energy markets. Additionally, refined markets are much more regional than crude oil markets as refineries have historically been built close to consuming centers.

2.2 Financial Products

The oil market has evolved into a very sophisticated market with a huge variety of derivative contracts that have changed the way oil is priced. Nowadays, a financial sphere of derivative contracts - that includes Futures, forwards, swaps and options - dominates the process of worldwide oil price formation. Oil bonds and loans are other examples of derivatives existent in the oil market.

The New York Mercantile Exchange (NYMEX) is the world's largest physical commodity Futures exchange. Some of the most important oil derivatives are traded in this market, such as Futures and options contracts on light sweet crude oil. The Intercontinental Exchange (ICE) is another important energy commodity exchange market, where it is possible to find Futures and options for Brent contracts.

Trading has become primarily a financial activity. Due to the above-referred non-standardization of oil, a considerable part of oil trading is concerned with price differentials between grades, locations, markets and delivery periods (Geman, 2005). These differentials are constantly changing, opening profit opportunities for a broad range of traders.

2.2.1 Forwards and Futures in the Oil Market

A commodity forward contract is an agreement between two parties to sell or purchase a certain amount of a commodity on a fixed future date (delivery date) at a predetermined contract price (Burger, et al., 2008). Usually, the payment date is also the same (or near) the delivery date. In this case, there are no cash transfers until delivery. While forward contracts are Over-the-Counter (OTC) transactions, Future contracts are standardized forward contracts that are traded at commodity exchanges, where a clearing house serves as a central counterparty for all transactions (Burger, et al., 2008). This clearing house requires both parties to put an initial amount of cash (the margin) as a guarantee and, at each trading day, a settlement price for the contract is determined and gains or losses are immediately realised on the margin account. Consequently, there is no substantial credit risk in Future contracts, in opposition with forward contracts, where a central counterparty for transaction does not exist.

Forwards and Futures are usually very liquid instruments compared with other types of commodity derivatives. Most of the Future contracts are financially settled, which means that they often do not lead to physical delivery. Moreover, the majority of the physically settled Futures contracts are not held until maturity but closed out in advance, as most of the investors in this market are not interested on the physical delivery of the commodity. To close a Futures position in a commodity, the investor just needs to execute a trade that is opposite to the first one.

According to Burger, Graeber and Schindlmayr (Burger, et al., 2008) forward contracts are mainly used in commodity markets for the following purposes: to hedge the obligation to deliver or purchase a commodity at a future date; to secure a sales profit from a commodity production and to speculate on rising or falling commodity prices in case there is no liquid Futures market. Futures contracts have similar uses, despite the differences above mentioned.

In the oil market, these instruments are very important and widely used. The Futures on Light Sweet Crude Oil, traded in NYMEX, are amongst the most traded derivatives in the world. These Futures are listed nine years forward[3], with physical delivery in Cushing, Oklahoma (United States).

2.2.2 Commodity Swaps

Just like Futures and forwards, swap contracts have no optionality. They are used to lock in a fixed price for a commodity over a specific time period. A swap agreement defines a number of fixing dates and, on each of the fixing dates, one counterparty (payer) pays the fixed price whereas the other counterparty (receiver) pays the variable price given by the commodity index. In practice, only net amounts are paid. The fixed payment is also known as the fixed leg of the swap and the floating payments as the floating leg of the swap.

2.2.3 Options in the Oil Market

In the oil market and in energy markets in general, there are several different types of options. Options together with Futures contracts and spread options are among the most important ones.

An option is a contract between two parties for a future possible transaction on a defined underlying asset at a specified predetermined price. The holder (buyer) of the option has the right, but not the obligation, to exercise the option and receive the underlying assets at the predetermined price. The seller (writer) of the same option has the obligation to fulfil the transaction and provide the buyer with the underlying security at the prearranged price, in case the option is exercised. This type of option is defined as a plain vanilla[4] call option. On the other hand, an option that provides the buyer of the contract with the right to sell the underlying asset at a fixed price is called a put option. In this situation, when the option is exercised, the seller has the obligation to buy the underlying security at the appointed price.

[3] Consecutive months are listed for the current year and for the next five years Additionally, the June and December contract months are listed beyond the sixth year.

[4] *Plain vanilla* options are the most basic and standard options. They are the opposite of *Exotic* options, which are more complex and customized, according to the needs of financial actors.

Options together with Futures and forward contracts are generally used for hedging purposes: the holder of a long forward or Futures position can hedge his exposition by buying a put option. This financial product permits him to protect his portfolio against an unfavourable price evolution (price drop). In this situation, the put option is known as a protective put and allows the investor to take advantage of potential rising commodity prices while ensuring a minimum price for selling the commodity. On the other side, a covered call strategy is employed to increase profits if the market is expected to stay near the current price level. An investor holding a long Futures or forward position sells a call option, receiving a call premium and losing the upside potential above the strike level.

Multi-underlying options are increasingly common in energy markets. The price of these products depends not only on the volatilities of the underlying assets, but also on the correlation structure between them (Burger, et al., 2008). Basket options, spread options, quanto options and composite options are all multi-underlying options. For the sake of simplicity, we will only focus on spread options in this work. As our study focus on the pricing of one-month calendar spread options, we will further develop the theory and characteristics of this instrument in Chapter 4.

Before moving to the next chapter, we now present a discussion on the characteristics of oil prices and their consequences for modelling prices.

2.3 Characteristics of Oil Prices and Consequences for Price Modelling

After providing a general overview of the oil market, we now focus on the analysis of the price behavior of oil, with particular emphasys on the four most important characteristics of commodity prices: mean reversion, convenience yield, seasonality and jumps and spikes.

Concerning the mean reversion property of commodities, empirical evidence generally supports this behavior in oil markets. As we have already pointed out in Chapter 1, Bessembinder et al. (1995) conducted an analysis that identified the existence of mean reversion in several commodities, including crude oil. Moreover, Pindyck (2001) analyzed 127 years of data on crude oil concluding that the prices of oil mean revert to the long-run total marginal cost of production. As it is so, when modeling the price behavior of oil, mean-reversion has to be taken into account, so that the model is consistent with the fact that commodity prices do not grow up indefinitely.

Another important characteristic of the price behavior of oil is the existence of a convenience yield. That is, there are marginal benefits (or costs, if the convenience yield is negative) of holding oil in reserve. For example, the owner of an oil field can retain the option of increasing production at a later date by not pumping the oil in the first place (Pakko, 2005). This will allow him to meet unanticipated needs for oil more easily, and to profit from potential increases in prices.

The convenience yield is in the origin of a significant part of the difficulties in modeling commodity prices. One of the reasons for this is the fact that the value that is generated by holding the commodity instead of holding a Futures contract is user-specific, as we have already analyzed in the former chapter. This has important implications for modeling purposes, since there is no standardized formulation for the convenience yield (Pilipovic, 1997). Despite that, modeling this property is essential in order to capture this characteristic of oil prices. As it is so, our model will account for the existence of a convenience yield. We return to this issue in the next chapter (Chapter 3).

Regarding seasonal patterns, some authors have found weak evidence on the existence of price seasonality in oil. Therefore, this feature is far from being as important for this commodity as it is for other energy commodities such as electricity or natural gas. There are several reasons that help explaining why seasonality is weak in crude oil. From the demand side of the market, while electricity and natural gas are often used for heating and cooling, most of the products that can be refined from oil are used for industrial and transportation purposes. A clear exception is the heating oil that, as the name suggests, is commonly used for heating, especially in the United States. Nevertheless, the demand for crude oil is in theory less seasonal than the demand for electricity or natural gas. Moreover, being crude oil a much more international market than the markets for refined products, seasonal patterns tend to disappear at a global level, due to the established networks and flows. From the supply side of the market, there is usually storage capacity available for crude oil, which is not the case, for example, for electricity that is almost impossible to store. As it is so, crude oil suppliers are able to respond to unexpected demands with more effectiveness. For the purpose of this paper, we decided not to model this element, since we believe it is not very relevant in the price behavior of oil.

Another important issue to analyze here is the existence of price jumps and spikes. According to Pilipovic (2007), event corrections and supply/demand imbalances tend not to vary beyond three to six months, in the oil market. However, these imbalances are shorter-living in the majority of the other energy markets: in electricity markets, the short term is truly *short-term* in the sense that the imbalances last for very short periods of time (a couple of weeks or even less). As a result, in electricity markets, price imbalances create big jumps and spikes. When modeling the behavior of electricity prices, this is an issue that is generally incorporated in the models. For oil price modeling, this is not the case. This can be supported by a graphical analysis of the crude oil historical prices. A chart of the WTI price evolution between 2006 and 2011 can be found in page 52 (Figure 5, Chapter 3). In this figure, we can clearly see that, despite significant changes in prices, there are no relevant jumps and spikes in the price path. This supports our initial suggestion. Therefore, for the purpose of our paper, this issue will not be further addressed or modeled.

Pricing commodities is a very complex and demanding task that has been intensively challenging lecturers and financial actors in the last 40 years. In the next chapter we will present some of the most important price models employed by academic and risk managers to understand the behaviour of commodity prices. The price characteristics we have just analysed allow us to understand which factors should be taken into consideration when modelling crude oil prices.

3 Modelling Energy Commodity Prices

"All models are wrong. Some are useful." George P.E. Box

3.1 Introduction

After having intensely described the characteristics of the oil market, the aim of this chapter is to provide the reader with a deep understanding of pricing models for energy commodities, which are employed for the purpose of this study. The models we present are the ones commonly used by financial analysts and lecturers to price commodities and the financial derivatives related to them.

At the moment, due to the important deregulation phase that is occurring in energy markets and to the increase of trades in these markets, both the financial and energy industries claim for more effective and efficient pricing models. These tools permit to understand and forecast the price behaviour of commodities and their related financial products. Since the purpose of this study is to evaluate a calendar spread option related to the crude oil market, we mainly focus on the pricing models dedicated to this widely used fuel. It is clear that these models, with the due adjustments (assumptions), can also be employed to other commodities as precious metals or other energy commodities.

We start with an overview of the evolution of pricing models in the commodity industry. Afterwards, we set the mathematical background needed to understand the stochastic models used to price commodities and then we present the fundamentals of their structure. Later, we define the relationship between spot prices and Futures prices, which is crucial in energy markets and even more for the purpose of this study. Once we have a broader and general overview on commodity modelling, we describe in detail the pricing model employed in our study and the technique necessary to derive the parameters used for modelling crude oil prices.

After having set the theoretical background and describing the method employed to extrapolate the model factors, we apply our knowledge in the real world through an empirical study. We seek to estimate reasonable and performing oil pricing parameters that will be important for the continuation of this study. To conclude this chapter we largely discuss our findings and their pertinence; we also comment on the applicability of the employed technique and the possible challenges in using it.

3.1.1 The Evolution of Commodity Prices Models

In the last 30 years the models that have been employed by academics and practitioners to understand the behaviours of commodity prices have been notably improving. These models have been mainly focusing on the oil market, due to its high liquidity and ease in accessing market data.

The first commodity related model was the one-factor process proposed by Brennan and Schwartz (1985). This model was developed with regards to the oil market and it was created assuming the price of oil following a simple Geometric Brownian motion and implying a constant convenience yield. This was the kick off for the commodity pricing evolution.

Although this model slightly helped financial practitioners and researchers to better forecast prices in the oil market, it shortly showed its limitations and it was replaced by more sophisticated ones, which better accounted for the idiosyncratic characteristics of commodities.

In fact, due to their particular mean reversion behaviour, the simple Geometric Brownian motion was replaced by more elaborated models. A well-know and easy-to-implement model, which accounts for mean reversion, is the one proposed by Ornstein and Uhlenbeck (1930). Lecturers started to use this model, because it permits to capture the nature of commodities prices to revert to a long-term value, usually considered to be their long-term marginal cost of production, as we explained in the first two chapters.

Afterwards, in their brilliant and inspiring paper, Gibson and Schwartz (1990) suggested an even more advanced and realistic model that replaced the simple one-factor mean-reverting model. In effect, the one proposed by Gibson and Schwartz was a more complex two-factor model, which included a mean reverting stochastic expression of the convenience yield. This was a substantial improvement, because this new two-factor process permitted to stochastically model the intrinsic value generated by physically owning the commodity. This model has the particularity of having two levels of mean-reversion: the explicit mean-reversion of the convenience yield and the intrinsic reversion of the spot price to its *normal* level, through the positive correlation between the two Wiener processes.

In the following years, researchers and financial analysts revised this two-factor model. Other examples of two-factor models are Schwartz (1997) and Schwartz and Smith (2000). Schwartz, in particular, came up with a new model composed of three stochastic equations (1997). In this model not only the commodity price and its convenience yield were modelled, but also the stochastic risk free interest rate. This last factor also followed a stochastic process and it was modelled according to the bond price interest rate term-structure instead of using the commodity Futures prices curve (Cortazar & Schwartz, 2003).

Subsequently, Cortazar and Schwartz (2003) revised the models suggested by the latter in his article of 1997. They developed a new one reformulating the well-known two-factor process (Schwartz, 1997), and deriving a *parsimonious* two-factor model. Then, starting from this new simplified one, they developed the new *parsimonious* three-factor stochastic process. Its particularity is that the third factor is the long-term price return (i.e. price appreciation, whereas before it was the risk-free interest rate from the term structure of bonds). This model permits to account for short-term and long-term price deviations. In fact, while the convenience yield captures the change in price due to variations in inventories (short-run), the long-term deviations, due to technology improvements and other macroeconomic factors, are modelled by the long-term price return (Cortazar & Schwartz, 2003). It is relevant to underline that all three factors are calibrated exclusively employing commodity Futures prices, instead of using the term structure of bonds for modelling the risk free rate as it was in Schwartz (1997). Using only Futures prices to estimate parameters reduces the magnitude of the estimation risk and the time necessary to collect data. Moreover we are able to capture all the relevant market information for commodities from a single source.

Now that we have a better understanding of the evolution of the models used by practitioners to price commodities, we dedicate an entire section to the theoretical and mathematical background necessary to understand their functioning. Afterwards, we present, in a more technical and formal form, most of the models introduced in this subsection. We deeply focus on the stochastic process we use for the purpose of this study, highlighting its characteristics.

3.2 The Mathematical Background

Before starting explaining the pricing models used for the purpose of this study, it is necessary to introduce some mathematical techniques and financial terms, which will be also relevant for the option pricing theory in Chapter 4.

3.2.1 The Stochastic Process

"A stochastic process is defined as a variable that evolves over time in a way that is at least in part random" (Dixit & Pindyck, 1994).

An example of a stochastic process in our everyday life is the patient's EEG[5]; the price of a stock traded on a stock exchange is another example to describe a stochastic process (Figure 5). The idea of a stochastic process is that, in the short run, the price fluctuates in an unpredictable way, but over the long run it displays a general trend.

5 Electroencephalogram: is the resulting chart of a recording technique (electroencephalography) of the brain electrical activity.

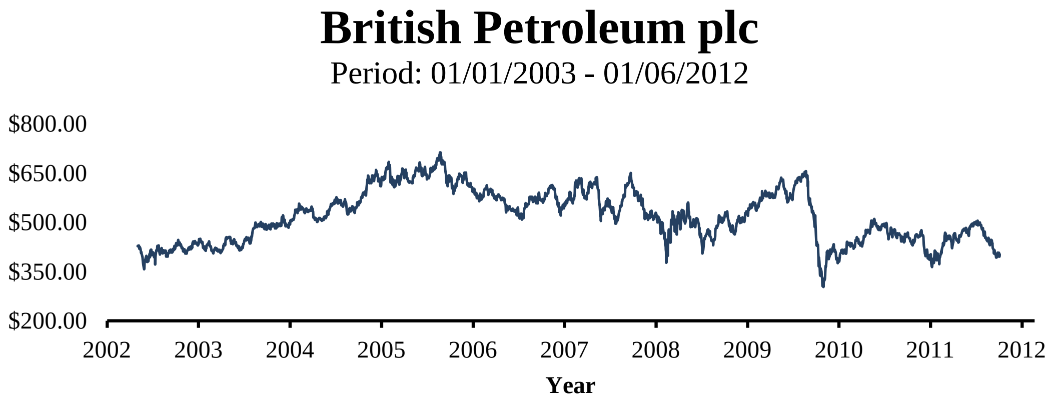

Figure 5 - Evolution of the stock price in USD from 2003 to 2012 of British Petroleum (BP.L)
Source: www.finance.yahoo.com

In a more formal and rigorous way we can define a stochastic process as an ordered collection of random values in ℝ, indexed by a totally ordered set T (time) ⊂ ℝ in a given probability space (Ω, $\mathcal{F}$, ℙ), (Trivedi, 2002) (Knill, 2011). Therefore, a stochastic process X can be defined as a collection of elements such as:

$$X = \{X_t : t \in T\}$$

For given time $t_1 < t_2 < t_3 < \cdots etc.$ we are given, or it can be computed, the probability that the corresponding value X_t lies in some specific range (Dixit & Pindyck, 1994).

It is possible to classify the stochastic processes in two main categories: *stationary* and *non-stationary*. The former are such processes whose stochastic variable's statistical proprieties are constant over long periods of time. A classic example of a stationary process is the yearly temperature of a city. On the other hand, the non-stationary processes are those whose expected values are not subject to boundary conditions, as for instance a security price.

We can also recognise two other types of stochastic processes: *discrete-time* and *continuous-time* processes. In a more formal way we can define a discrete-time stochastic process as:

$$X_t = X_{t-1} + \varepsilon_t$$

where ε_t is a random variable with a determined distribution of probabilities (Dixit & Pindyck, 1994). For the purpose of this study, in order to represent reality in a more efficient way and thus to achieve more efficient results, we decided to focus on continuous-time stochastic processes. Consequently, we now introduce and define the well-known *Wiener* process, considered as the continuous limit of the discrete-time process (Dixit & Pindyck, 1994).

3.2.2 The Standard Wiener Process

A standard way to model a stochastic path of a certain variable in continuous time consists of employing the well-known and widely used Wiener process (also known as Brownian motion). The model takes its name from Norbert Wiener, who based his studies on stochastic and noise processes and made a rigorous formulation of the motion (1923).

The Wiener process has three important proprieties (Dixit & Pindyck, 1994):

- Firstly, the Wiener process is defined as being a *Markov* process. This is an important feature for a process, because it helps to simplify the analysis. A stochastic process is defined as *Markovian* if the distribution of the probabilities of x_{t+1} depends only on the element x_t and not on its historical path before time t. In more modest terms we can define a Markovian process in the following way: the future value of the process depends only on the present value of it and it is unaffected by the past values of the process or any other available information before time t. Consequently, to predict the future value of the process it is just needed to know the current value of the process (Dixit & Pindyck, 1994);
- Secondly, the Wiener process is characterised by having independent increments, meaning that the probability distributions for the changes at each point in time are independent form each other. This property permits to consider the Wiener process as a continuous-time version of a random walk;
- Thirdly, the changes of the process over time are normally distributed, with a variance that linearly increases with respect to time. This simply means that the uncertainty about the value of the stochastic variable in the future, expressed as its standard deviation, increases by the square root of the interval of time between t and $t + dt$, which is equal to: $\sigma_S\sqrt{dt}$ (Hull, 2008).

It is time now to define these properties in a more formal manner. We denote the standard stochastic Wiener process as W and W_t as its value at time t. Therefore, W_t is a Wiener process if it satisfies the following features (Miltersen, 2011):

- $W_0 = 0$
- The increments are Gaussian: $W_t - W_s \sim \Phi\left(0, \sqrt{t-s}\right)$, for $s \leq t$, in discrete time; whereas in continuous time we have: $dW \sim \Phi\left(0, \sqrt{dt}\right)$
- $W_{t_0}, W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}}$ are independent, for $0 \leq t_0 < t_1 \dots < t_n$
- W has a continuous trajectory (path), i.e., $t \to W_t(\omega)$ is continuous for all $\omega \in \Omega$

According to the four characteristics just mentioned, we have:

$$W_t = W_{t-dt} + \epsilon_t$$

where $\epsilon_t \sim \Phi(0, \sqrt{dt})$, therefore[6]:

$$W_t = W_{t-dt} + \sqrt{dt} \cdot \eta_t$$

where $\eta_t \sim \Phi(0,1)$. We can easily see from the formula above that the Markov property is respected, since W_t depends only on W_{t-dt}.

We now have to underline several important elements of the standard Wiener process. Firstly, it is important to notice that the process randomly changes direction every instant, but in a continuous manner, meaning that it never jumps. As we explained in the section 1.4.4 Jumps and Spikes, commodities might include jumps in their price paths, especially in the electricity and natural gas markets. Consequently, the simple Wiener process does not capture this aspect and it should not be employed by itself when modelling such kind of commodities.

Secondly, we have to point out that the standard Wiener process cannot be considered as a valuable and reliable stochastic model for traded securities in general (Miltersen, 2011). Although it seems reasonable to consider a stock price satisfying the first two conditions – Markov property and independent increments – it is not possible to assume that the values of the process are normally distributed. This is derived from the limited liability characteristics of stocks, which does not permit stock prices to have a negative value, but it must be always greater than zero. If we assume the process to be normally distributed the stochastic variable can take any real value (Miltersen, 2011). Therefore, it would be more realistic to consider changes in the process to be log-normally distributed, which means to model the logarithm of price, instead of the price itself (Dixit & Pindyck, 1994).

Lastly, the Wiener process is a non-stationary process and, according to our definition provided in the previous section, it does not have any boundary. Moreover in the long run its volatility goes to infinity.

3.2.3 The Generalized Wiener Process (Brownian Motion With Drift)

In the previous section we explained the standard Wiener process, but we also enlightened its limitations. In order to reach our aim and be able to have an efficient and performing stochastic model for pricing commodities, we definitely need to extend the standard Wiener process. The first extension is to include mean and variance in the Wiener process, where both can be time-dependent functions (Miltersen, 2011).

This simple generalization of the standard Wiener process is also known as the Brownian motion with drift (Dixit & Pindyck, 1994), here denoted X, and it has the following form:

[6] $\Phi(\mu, \sigma)$ is defined as the comulative normal distribution with mean μ and standard deviation σ.

$$dX_t = \mu_X \cdot dt + \sigma_X \cdot dW_t$$

where dW_t is the increment of the standard Wiener process at time t. Moreover, μ_X is defined as the drift parameter of the process and σ_X is the variance parameter.

If both parameters are time dependent, the Brownian motion with drift takes the following form:

$$dX_t = \mu_X(t) \cdot dt + \sigma_X(t) \cdot dW_t$$

where $\mu_X(t)$ and $\sigma_X(t)$ are deterministic functions with respect to time.

In this last case, over any interval of time dt, the change in X_t is normally distributed with an expected value $\mu_X(t)dt$ and a variance of $\sigma_X^2(t)dt$ (Dixit & Pindyck, 1994).

This generalised Wiener process in continuous time has the following features, which are similar to the standard Wiener process (Miltersen, 2011):

- $X_0 = x$
- $X_t - X_s \sim \Phi(\int_s^t \mu_X\,(u)du, \int_s^t \sigma_X(u)du)$, for $s \leq t$
- $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent, for $0 \leq t_0 < t_1 \dots < t_n$
- X_t has a continuous trajectory (path), i.e., $t \to X_t(\omega)$ is continuous for all $\omega \in \Omega$

We can rewrite the previous equation as:

$$X_{t+dt} = X_t + \mu_X(t) \cdot dt + \sigma_X(t) \cdot dW_t$$

this means that the future price, X_{t+dt}, depends only on the value X_t and not on the historical price path. Therefore this is once more consistent with the Markov property we just explained before.

3.2.4 Itô's Process

As we have been appreciating so far, the Wiener process is a valuable building block for creating a broad range of stochastic variables. Up to this point, we have implied that the drift parameter and the variance parameter of the Brownian motion with drift were both deterministic functions of time. The next step is to extend the generalized Wiener process in order to have a more realistic and performing model. This can be done by making $\mu(\cdot)$ and $\sigma(\cdot)$ to be two stochastic functions. This model can be expressed by the following stochastic differential equation:

$$dX_t = \mu_X(X_t, t) \cdot dt + \sigma_X(X_t, t) \cdot dW_t$$

which is known as the *Itô's process*, or diffusion. In this model the functions $\mu_X(X_t, t)$ and $\sigma_X(X_t, t)$ are deterministic functions of the time and of the stochastic variable X_t (Miltersen, 2011). As we can easily reckon the process is still Markovian, since it depends only on the information available at time t.

At this point is interesting to notice a particularity of this stochastic process: while in the short run the volatility of the process is the dominant determinant, in the long run the drift parameter dominates the process (Hull, 2008). This is exactly the characteristic we would like to have when modelling prices. We allow for random movements in the short term, but on the long term it experiences a certain trend: $\mu_X(X_t, t)$.

3.2.5 The Geometric Brownian Motion

A crucial and special case of the Itô's process is the Geometric Brownian Motion, which can be expressed according to the following stochastic differential equation:

$$dX_t = \mu_X \cdot X_t \cdot dt + \sigma_X \cdot X_t \cdot dW_t$$

In this particular case the drift function $\mu_X(X_t, t)$ is equal to $\mu_X \cdot X_t$, while the diffusion function $\sigma_X(X_t, t)$ corresponds to $\sigma_X \cdot X_t$, and where μ_X and σ_X are constant parameters.

In the Geometric Brownian motion the absolute changes in X_t (dX_t) are absolute changes in the logarithm of X_t, meaning that they are log-normally distributed (Dixit & Pindyck, 1994). This means that the non-negativity property of stock and prices is respected. This log-normally distribution of the change is the main property of this process.

The Geometric Brownian Motion has some suitable proprieties (Miltersen, 2011):

- The process X_t has always a positive value (with probability one) at any point in time if the starting point x is positive. This is the main feature of the stochastic process that, in fact, permits to model stock prices observing their limited liability characteristic;
- Stock returns over non-overlapping time intervals are independent;
- This process is Markovian. The current value of the process reflects all the pertinent information;
- The distribution of the future value returns of the process is independent of its history and its current value.

3.2.6 Itô's Lemma

The Itô's process and the Geometric Brownian motion we have just developed in the previous sections are great starting stochastic expressions for modelling prices in continuous time. Unfortunately, they are not differentiable (Hull, 2008). This is a necessary step to understand pricing models, but it is also relevant for other financial problems, e.g. performing portfolio analysis (Pilipovic, 2007).

In the subsequent sections we work with some Itô's process style functions. For example, in Chapter 4 we derive the Black & Scholes model to price an option. The underlying asset of this option is modelled according to a Geometric Brownian motion and we would like to be able to understand the behaviour of this stochastic variable. In order to be able to differentiate such function, and comprehend its evolution, we have to employ the *Itô's Lemma* (Itô, 1951). This technique permits us to differentiate or integrate functions of the Itô's style (i.e. stochastic processes).

The Itô's lemma can be seen as a Taylor series expansion (Dixit & Pindyck, 1994). In fact, the Taylor series permits to express the change in value of a function as an infinite sum of the changes in the variables, on which the function depends. We can express this definition in a more formal way as it follows:

Consider x_t following an Itô process as we previously defined it, and suppose a function $f(x,t)$, which is a function that is at least twice differentiable in x and in t. Then the Taylor expansion has the following form:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2 + \frac{1}{6}\frac{\partial^3 f}{\partial x^3}(dx)^3 + \cdots$$

The higher-order terms, starting from $(dx)^3$ will disappear when we take the limit of the expression. If we substitute dx in the expression $(dx)^3$, we would note that all the terms dt have a power greater than 1 and thus they tend to zero faster than dt – since the differential of time is infinitesimal small. This is why these terms can be omitted from the expression.

Hence Itô's lemma gives the differential df as:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2$$

We can rewrite this expression substituting dx for $dx = \mu_x(x,t) \cdot dt + \sigma_x(x,t) \cdot dW$:

$$df = \left[\frac{\partial f}{\partial t} + \mu_x(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma_x^2(x,t)\frac{\partial^2 f}{\partial x^2}\right]dt + \sigma_x(x,t)\frac{\partial f}{\partial x}dW$$

To better understand this technique we provided an example of it in Appendix II.

3.2.7 Risk Neutral Probabilities

Before starting the discussion on the different pricing models that can be applied to crude oil, we need to introduce another crucial tool for pricing Futures contracts on commodities and generally speaking for valuing financial derivatives.

If we assume the market to be complete, meaning that it is possible to replicate any security combing any other traded products, we can employ risk-neutral probabilities instead of real (historical) probabilities for valuing such financial products. The idea consists in: instead of valuing the derivative taking the expected payoff and discount it at the adjusted discount rate for the investor´s risk preferences, we price the financial product computing its expected outcome using the risk-adjusted probabilities of the future payoff and discounting it at the risk-free interest rate. In a risk neutral world investors are indifferent to risk (Hull, 2008).

This technique is heavily employed for pricing financial derivatives, because of the assumption regarding the stochastic behaviour of the underlying assets. It is important to emphasise the fact that these risk-neutral probabilities are strictly bond to the notion of non-arbitrage. In fact, a complete market, without any arbitrage opportunities, implies the existence of such unique set of risk-neutral measures[7]. This new probabilities are *equivalent* to the real probabilities, meaning that they satisfy the mathematical condition of the probability definition and they also mimic the historical probabilities[8]. Of course the two concepts are closely related and the real probabilities exercise an influence on the risk-neutral measure (Gisiger, 2010). Lastly, these risk adjusted probabilities are also known as *martingale measures*.

The concepts of risk-neutral probability and non-arbitrage environment are two of the building-blocks of the famous Black-Scholes and Merton model that we present in detail in the last chapter of this study (Chapter 4).

In one previous subsection dedicated to stochastic models, we introduced the Geometric Brownian motion, which is the foundation for our pricing model. This process is continuous, therefore it is not evident how we can specify the value of the underlying asset at a certain time during its life (Gisiger, 2010).

Assume a financial product, which has an underlying asset following a Geometric Brownian motion, we know that the generalized expected payoff can be expressed in the following from:

$$E_{t=0}^{P}[X(S_T)] = \int_{-\infty}^{+\infty} X\left(S_0 e^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma W_t^P}\right) p(\omega)d\omega$$

where $p(\omega)$ is defined as the *normal* (real) probability density of $e^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma W_t^P}$. However, we would like to know the price of this product at time $t = 0$. A well-known method to do this is to transform the real probabilities into risk-neutral probabilities and discount the payoff at the risk-free rate. To do this we need to apply the brilliant Girsanov Theorem (Girsanov, 1960) in combination with the Randon-Nikodym derivative. The Girsanov Theorem permits to shift probabilities measures for a stochastic process modelled according to a Brownian motion (Gisiger, 2010). Such probability transformation is unique and if we are able to evaluate the payoff at each point in time – we know its

7 If we do not assume a complete market, then there does not exist a unique set of risk neutral probabilities but several sets.

8 If the historical probability is positive then the risk neutral probability is also positive. Moreover, if the historical probability allocates zero probability to an event then the risk neutral probability is also zero.

evolution – we can discount it at the risk-free rate (Gisiger, 2010). This transformation is done thanks to the Randon-Nikodym derivative:

$$RN(\omega) = \frac{q(\omega)}{p(\omega)}, \omega \subset \Omega$$

where $RN(\omega)$ is the Randon-Nikodym derivative and ω is a particular event in the universe of scenarios Ω. Moreover, $q(\omega)$ is defined as the risk neutral probability. Therefore we can rewrite the previous definition of the expected payoff in the following way:

$$E_{t=0}^{Q}[X(S_T)] = \int_{-\infty}^{+\infty} X\left(S_0 e^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma W_t^P}\right) p(\omega) \frac{q(\omega)}{p(\omega)} d\omega$$

In order to find the price of this financial product, we need to use another important result. In fact, since we are in a complete market, meaning that there only exists a risk-neutral probabilities density, it is implied the existence of a unique state price density. Thus shifting from risk-neutral probabilities to state price measures can be simply done by discounting the former at the risk-free interest rate (Gisiger, 2010):

$$\vartheta(\omega) = e^{-rT} q(\omega)$$

Therefore we can obtain the price of our financial product in the following way:

$$\pi_{t=0} = e^{-rT} E_{t=0}^{Q}[X(S_T)] = \int_{-\infty}^{+\infty} X\left(S_0 e^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma W_t^P}\right) p(\omega) \frac{e^{-rT} q(\omega)}{p(\omega)} d\omega$$

Girsanov derived another important result, which says that a Brownian motion under normal probabilities converts to a Brownian motion under equivalent risk-neutral measures minus a drift component, risk premium (Girsanov, 1960):

$$dW_t^P = dW_t^Q - \lambda_t dt$$

Furthermore we already know that the drift in a risk-neutral space is r (Gisiger, 2010), thus:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q$$

This is an interesting result, since we do not have to estimate μ because the risk premium in it is offset by λ_t. Thanks to this element we are allowed to discount at the risk-free rate. This definition of risk-neutral probabilities and the Girsanov transformation will be used along our study, in order to avoid accounting for the different risk premium associated to the diverse parameters.

3.3 Introduction to Commodity Price Models

Now that we have set most of the mathematical background necessary for the purpose of this study, it is time to focus on some available models that can be used to price commodities.

As we already explained in the section 2.3 Characteristics of Oil Prices and Consequences for Price Modelling, commodities have some idiosyncratic characteristics, as for instance mean reversion, which must be taken into account when modelling their prices.

The most known and simple model, which accounts for the mean reversion feature is the Ornstein-Uhlenbeck process (1930), which as the following form:

$$dx_t = \kappa(\bar{x} - x_t)dt + \sigma dW_t$$

where the term $\bar{x}$ is the long-term value of the process and x_t the value of the stochastic model at time t. In order to model prices we can set x_t as the natural logarithm of the spot price. The parameter κ is called the speed of the mean reversion – the higher is the value of this coefficient, the faster the process reverts to it is long-term value.

The equation of the expected change in x_t depends on the difference between the *normal* level of the process and its current value: $(\bar{x} - x_t)$. Therefore, if x_t is higher (lower) than the value of $\bar{x}$ the difference is negative (positive) and consequently it is more likely that the process will fall (rise) over the next infinitesimal interval of time. It is significant to extend this aspect, because this implies that the process does not have independent increments anymore, even though it is still a *Markovian* process (Dixit & Pindyck, 1994).

Although this model helps us capturing the singular characteristic of commodity prices, it is not enough effective and performing for the purpose of this study, since it does not account for the other characteristics, in particular the effects of the convenience yield on the spot price. The convenience yield, as defined in Chapter 2, it is a crucial element in the behaviour of crude oil prices. This is because of the relatively ease of accessing oil storage facilities and the fairly stable worldwide demand for oil compared to other energy commodities. Hence we need to improve our pricing model to account for this feature. It is in this direction that in the next subsection we present Schwartz's models (Schwartz, 1997) and discuss their proprieties, as both account for the convenience yield.

3.3.1 Two-Factor Model

This section is dedicated to the explanation of the two-factor model developed by Gibson and Schwartz (1990) and revised by Schwartz (1997). These papers focused on the pricing of commodities (crude oil and precious metals) and their related Futures contracts.

The two factors that are shaped in this model are: the spot price of a commodity (S_t) and its instantaneous convenience yield (δ_t). The model has the following stochastic differential structure[9]:

$$dS_t = (\mu_S - \delta_t) \cdot S_t \cdot dt + \sigma_S \cdot S_t \cdot dW_t^S$$

$$d\delta_t = \kappa \cdot (\theta_\delta - \delta_t) \cdot dt + \sigma_\delta \cdot dW_t^\delta$$

with dW_t^S and dW_t^δ being two standard Wiener processes following $\Phi(0, \sqrt{dt})$ and the correlation between the two stochastic processes is defined as: $dW_t^S dW_t^\delta = \rho dt$, where $\rho dt > 0$. Moreover, in this model μ_S represent the long-term return on oil (convenience yield plus price appreciation (Cortazar & Schwartz, 2003)), while κ is the speed of the mean reversion of the process modelling the convenience yield. The volatility of the spot price and of the convenience yield are expressed by: σ_S and σ_δ, respectively; whereas the parameter θ_δ is the long term convenience yield.

It is interesting now to analyse both equations and understand their relation. Firstly, we can see that the second equation, which defines the convenience yield, has the form of an Ornstein-Uhlenbeck model; the mean reversion parameter is explicitly expressed in the equation – i.e. κ. If we focus now on the first equation, which describes the spot price of the commodity, it seems that the spot price does not revert to a long-term value. However, when we look a bit further we can easily see that the mean reversion of the spot price is implicit in the model and it has the following form:

I. We know that the two Wiener processes are positively correlated. Therefore, if dW_t^S is positive, we have a higher chance that dW_t^δ is positive too.
II. Having a positive dW_t^δ means that the instantaneous convenience yield δ_t is more likely to increase as well.
III. Since the continuous convenience yield is part of the drift of the stock price - $(\mu_S - \delta_t)$ - its value will affect the change of the commodity price. Following the same logic as in the explanation of the Ornstein-Uhlenbeck model, a convenience yield δ_t higher (lower) than the long-term total return μ_S (or when working in a risk-neutral environment, the risk free rate) results in a more likely negative (positive) drift, affecting the commodity price.

[9] Note that the model is expressed in $\mathbb{P}$ measure – observed probabilities. We can also define the model in a risk neutral world $\mathbb{Q}$, where we avoid the risk premium of the convenience yield, λ_δ. In this last case the risk-adjusted stochastic model would have the following form:

$$dS_t = (r - \delta_t) \cdot S_t \cdot dt + \sigma_S \cdot S_t \cdot d\widetilde{W}_t^S$$

$$d\delta_t = \kappa \cdot \left(\tilde{\theta}_\delta - \delta_t\right) \cdot dt + \sigma_\delta \cdot d\widetilde{W}_t^\delta$$

where $\tilde{\theta}_\delta = \theta - \frac{\lambda}{\kappa}$.

We now have a model that accounts for both crucial oil market characteristics: mean-reversion and convenience yield. Later in the section 3.4 Futures Price and Spot Prices we explain the relation between spot prices and Futures prices and we analyse the closed-form solution for Futures prices derived from this model, as defined in Schwartz (1997).

Another important element that needs to be underlined is that this two-factor model still satisfies the Markov property. Both stochastic equations, in fact, still depend only on their current values and not on their historical paths. However, since the convenience yield follows a path that it is described according to an Ornstein-Uhlenbeck process, its increments are not independent anymore.

3.3.2 Three-Factor Model

In Schwartz (1997) the two-factor model, we have just presented, is extended to include a third stochastic factor: the risk free interest rate. The author of the paper choses to model this third factor according to a simple mean reversion – Ornstein-Uhlenbeck process – in the same way as the instantaneous convenience yield is modelled. Under the equivalent martingale measure the model can be expressed as:

$$dS_t = (r_t - \delta_t) \cdot S_t \cdot dt + \sigma_S \cdot S_t \cdot d\widetilde{W}_t^S$$

$$d\delta_t = \kappa\left(\tilde{\theta}_\delta - \delta_t\right) \cdot dt + \sigma_\delta \cdot d\widetilde{W}_t^\delta$$

$$dr_t = \beta(\tilde{\pi} - r_t) \cdot dt + \sigma_r \cdot d\widetilde{W}_t^r$$

The first two equations are identical as in the two-factor model, except for the risk free interest rate in the drift of the spot price, which is now stochastic. The terms β and $d\widetilde{W}_t^r$ are respectively: the speed of mean-reversion of the risk free interest rate and the Wiener process for the stochastic interest rate. Furthermore the parameter $\tilde{\pi}$ represent the risk adjusted short rate mean of the interest rate process (Schwartz, 1997).

We have to notice that the three equations are interlinked among them, because of the correlations among the three different standard Wiener processes, as it was in the two-factor model:

$$d\widetilde{W}_t^S d\widetilde{W}_t^\delta = \rho_{S,\delta} dt \ ; \ d\widetilde{W}_t^S d\widetilde{W}_t^r = \rho_{S,r} dt \ ; \ d\widetilde{W}_t^\delta d\widetilde{W}_t^r = \rho_{\delta,r} dt$$

The mean reversion of commodity prices is mainly guaranteed by the positive correlation of the two stochastic processes (commodity price and convenience yield), as it was in the case of two-factor model. Furthermore, the correlation between the interest rate and the spot price has also an influence on the mean reversion of the process (Lutz, 2010).

This more sophisticated model provides us with more evidence about price information and macroeconomic situation. For example we can observe the level of interest rate that is implied by the market. However, we have to be aware of two important elements: firstly, if we use this model the value of the Forward contracts will not be equal to the one of Futures contracts, because of the stochastic interest rate. Schwartz (1997) found a closed-form solution for both contracts, but the difference between the findings of the two-factor model and the three-factor model is not significant. Consequently a non-stochastic risk free interest rate is still a good assumption for modelling. Secondly, it is important to note that, to estimate the parameters of the third stochastic equation, Schwartz (1997) employs the term structure of governmental bonds. Therefore, the commodity Futures market prices are used only to estimate the parameters of the change in the commodity spot price and in the change in the instantaneous convenience yield.

This section was an introduction on some of the commodities models that have been used by researchers and financial analysts in the last three decades. We largely explained how the different pricing characteristics are included in these models in order to have a more performing estimation of commodity spot prices. This permits the user to set more efficient investment strategies. In the next section we explain the relation between spot prices and Futures prices.

3.4 Futures Price and Spot Prices

Before starting with the analysis and empirical implementation of the model used in our study, we dedicate few paragraphs to the essential relation between Futures contracts and the spot price of commodities.

Futures contracts are essential in the everyday trading activities. In fact, they allow financial participants to have an indication of where the price – e.g. of a barrel of oil or a megawatt-hour of electricity – is heading to. Futures prices are the result of open and competitive trades on the floors of exchange, they are treated as securities and thus they are able to reflect the expected underlying supply or demand at different dates in the future (Hull, 2008). Consequently, when more traders decide to go short than long, the price of the commodity Futures goes down and *vice versa*.

There are also other reasons that make Futures prices important for market participants: firstly, Futures contracts are highly related to the storage decisions of firms (Geman, 2005). In fact, a high Futures price level suggests the need for increasing the level of storage, whereas low Futures prices point to diminishing the level of the company's stock. This may be less likely in the oil industry (limited seasonal effect), but if we think about the natural gas industry, storage has a greater impact on business decisions. Secondly, they do not only trigger the decision regarding storage, but they also contribute to the choices regarding production and consumption, since they replicate the expectations about future demand and supply (Geman, 2005). Lastly, we have to underline the importance of such contracts in risk management strategies, since they can be used to hedge risk in investments. Actually, once we have set our optimal investment decision, the hedging strategy is aimed for being flexible in order to account not only for the present price level but also for future changes (Cortazar, et al., 2000). This consideration definitely stresses the importance of a reliable and efficient model for estimating Futures prices for an energy business company.

The market's general opinion about what the future spot value of a security will be at a certain future date is expressed as today's expected spot price of asset at that future certain date. The current Futures price is the best estimator of the future spot price. In mathematical terms we can write (Hull, 2011):

$$F_T(t) = E[S_T|\mathcal{F}_t]$$

where $\mathcal{F}_t$ is all the available information at time t.

In order to avoid arbitrage opportunities this relations must hold. The Future price is an unbiased estimate of the expected future spot price when the market is not correlated with the returns of the underlying assets, meaning that there is no systematic risk (Hull, 2011). When the equality does not hold, the Futures prices $F_T(t)$ is said to be a poor estimate of S_T. If $F_T(t)$ is an upward-biased estimate, this mirrors the risk aversion of financial participants such that the ones who want a long position are willing to pay more than the expected spot price in order to secure their access to the commodity at a future date T - e.g.: in the oil market during worldwide political tensions (Geman, 2005), (Borovkova & Permana, 2010). As a result the Futures values other than reflecting the information regarding supply, demand, level of stock and the expectation of the future spot prices, provide also information about the risk aversion of investors.

This possible inequality could be a drawback when modelling Futures prices. However, we can avoid this problem if we work in an environment with risk-neutral probabilities. In such a world we are able to discount the payoff of the Futures contract without accounting for any risk premium, as we explained before.

According to the non-arbitrage assumption (*buy-and-hold strategy*), we know that the relation between the Futures price[10] and the spot price, in a risk neutral world is[11]:

$$F_t(T) = S_t \cdot e^{(r-\delta)(T-t)} = E_t^Q[S_T]$$

By definition we can describe the Futures price as the unbiased expected spot price at time T, r is the risk free interest rate and δ represents the instantaneous convenience yield (according to the storage theory). This equality has to be valid, because we can note that the Futures price does not depend on

[10] When there is no uncertainty about the term structure of interest rates, a forward contract with the same sure delivery date of a Future contract and written on the same commodity have in theory the same value. The difference is usually small and it can be ignored. Therefore the following demonstration can also be used for forward contracts.

[11] In Appendix III we offer a more detailed demonstration of the buy-&-hold strategy.

the volatility of the spot price, which is not the case when dealing with option prices (Geman, 2005). This is the common relationship between Futures and spot price.

However, the expression only shows the relationship between Futures and spot prices in a case where the underlying assets are modelled according to a single Geometric Brownian Motion with a determinist drift. Assuming now we decided to employ in our study the following two-factor model proposed by Gibson and Schwartz (1990) expressed in risk-neutral measures:

$$dS = (r - \delta) \cdot S \cdot dt + \sigma_S \cdot S \cdot d\widetilde{W}^S$$

$$d\delta = [\kappa(\alpha - \delta) - \lambda_\delta] \cdot dt + \sigma_v \cdot d\widetilde{W}^v$$

Using contingencies claim analysis we would expect the Futures price to respect the following partial differential equation (PDE), derived employing Itô's lemma:

$$\frac{1}{2}\sigma_S^2 S^2 F_{SS} + \frac{1}{2}\sigma_\delta^2 F_{\delta\delta} + \sigma_S \sigma_\delta \rho_{S\delta} S F_{S\delta} + (r - \delta) S F_S + [\kappa(\alpha - \delta) - \lambda_\delta] F_\delta - F_T = 0$$

with the boundary condition: $F(S, \delta, T) = S$ when $T = 0$.

Lecturers and financial practitioners (Schwartz, 1997) solved the previous PDE and came up with the following formula for the Futures prices in the two-factor model:

$$F(S, \delta, T) = S \cdot \exp\left[-\delta \frac{1 - e^{-\kappa T}}{\kappa} + A(T)\right]$$

where:

$$A(T) = \left(r - \alpha - \frac{\lambda}{\kappa} + \frac{1}{2}\frac{\sigma_\delta^2}{\kappa^2} - \frac{\sigma_S \sigma_\delta \rho}{\kappa}\right) T + \frac{1}{4}\sigma_\delta^2 \frac{1 - e^{-2\kappa T}}{\kappa^3} + \left(\alpha - \frac{\lambda}{\kappa} + \sigma_S \sigma_\delta \rho - \frac{\sigma_\delta^2}{\kappa}\right) \frac{1 - e^{-2\kappa T}}{\kappa^2}$$

We need to remember this procedure, because it will be helpful to understand the Futures price formula of the model we apply in this study.

3.5 The Cortazar and Schwartz Model and Estimation Technique

We now present the model and its related estimation technique we have employed in this work in order to estimate the behaviour of the spot price of crude oil and the value of the parameters that compose the model. The estimation method we use was developed by Cortazar and Schwartz (2003) and it is based on some studies of Cortazar *et al*. (2000), where the authors developed and described a *parsimonious* two-factor and three-factor model derived from the previous studies of Schwartz (1997). We extensively present the estimation technique, pointing out its advantages and disadvantages.

Afterwards we apply the technique to extract the model parameters in the light sweet crude oil market according to the *parsimonious* three-factor model. We decided to implement the technique in

Microsoft Excel®, as it is suggested in the Cortazar and Schwartz study, and also to test the effectiveness of this method in providing the possibility of estimating pricing parameters to a broader group of users.

3.5.1 Background

The technique we decided to employ for estimating the parameters of our pricing model is the one presented by Cortazar and Schwartz (2003), which is mainly based on the previous paper written for the Commodity Conference in 2000 by the same authors (Cortazar, et al., 2000). This technique has been used in practice, in order to estimate Futures prices of crude oil and copper by oil companies and by the Risk America Plus, a company that analyses the commodity market in Chile.

In order to implement this technique we need to slightly modify the stochastic equations composing the three-factor model we presented in one of the previous section of this chapter. The advantage of using this process and technique is that we can derive a three-factor model only utilising the Futures curves, which is a significant improvement, as we explain below.

The starting point for using this estimation method is the well-known two-factor model of Schwartz (1997), where the first stochastic equation represents the commodity spot price and the second one denotes the convenience yield related to the analysed commodity. This is exactly the model we explained in the previous section dedicated to the two-factor model.

In their paper, Cortazar and Schwartz modified this bi-factor model to derive a more *parsimonious* one. In fact, they derived and rewrote the risk-adjusted model in the following way[12]:

$$dS = (v - y - \lambda_S) \cdot S \cdot dt + \sigma_S d\widetilde{W}^S$$

$$dy = (-\kappa y - \lambda_y) \cdot dt + \sigma_y d\widetilde{W}^y$$

where:

- dS represents the change in the spot price, as it was in the old version;
- y is defined as the *demeaned convenience yield*, and it is the difference between the instantaneous convenience yield (δ) and the long-term convenience yield (α), meaning: $y = \delta - \alpha$;
- v is the *long-term price return on oil* (i.e. oil price appreciation) and it is derived by subtracting the long-term convenience yield (α) to the long-term total return (μ), that is: $v = \mu - \alpha$;
- λ_S and λ_y are the risk premium associated to their respective factors;

[12] The model is expressed under risk neutral probabilities.

- The two Wiener processes are positively correlated: $d\widetilde{W}^S d\widetilde{W}^y = \rho dt$ and $\rho dt > 0$.

It must be stressed that this new model formulation is more natural for financial analysts and researches. In fact, the returns are expressed in terms of long-term price appreciation, which is more intuitive to financial practitioners than using the abstract method of the long-term convenience yield (Cortazar & Schwartz, 2003). In fact, the convenience yield is not directly observable in the market and it has to be computed by analysts using indirect techniques. This new pricing model formulation permits to avoid this weakness of the Schwartz 1997 two-factor model.

The new model is defined *parsimonious* because it reduces the number of parameters to be estimated. This simplification is made in the second stochastic equation, which explains the convenience yield, and, as we can see, we now have to estimate only the demeaned convenience yield. The long-term convenience yield and its instantaneous value are substituted by this unique term; hence we reduce the number of parameters to be estimated, with the idea of making the model clearer to financial practitioners, as we have just explained. Furthermore, we also have to underline that reshaping the *normal* two-factor model with the version we just presented does not influence the explanatory power of the model (Cortazar & Schwartz, 2003).

It has to be noticed that the two-factor model built by Schwartz (1997), although it still had some limits, had more than satisfying results. However, in certain specific dates the model was not able to fully capture the market behaviour of oil. Undoubtedly, this is not desirable for any model, as we would expect it to mirror as much as possible the commodity price path. It was shown by Cortazar and Schwartz (2003) an example of the limited fit between market prices and the results from a two-factor model for Futures contracts traded at NYMEX on January 1999.

Therefore, Cortazar and Schwartz did not stop enhancing the quality of their process. Despite the reasonable explanatory power of the *parsimonious* two-factor model, they decided to elaborate a more sophisticated one composed of three factors. This should permit to better fit the crude oil price behaviour, also in those days where the two-factor model was not able to reproduce the correct value.

Accordingly, the *parsimonious* three-factor model has the following configuration in $\mathbb{Q}$ measures[13]:

$$dS = (v - y - \lambda_S) \cdot S \cdot dt + \sigma_S \cdot S \cdot d\widetilde{W}^S$$

$$dy = \left(-\kappa y - \lambda_y\right) \cdot dt + \sigma_y \cdot d\widetilde{W}^y$$

$$dv = [a(\bar{v} - v) - \lambda_v] \cdot dt + \sigma_v \cdot d\widetilde{W}^v$$

where the differences from the *parsimonious* bi-factor model are:

[13] In the paper of Cortazar and Schwartz (2003) we think there is a typo in the third expression. In fact a multiplies the entire drift $[(\bar{v} - v) - \lambda_v]$, but it should only multiply the difference between the long-term value of the oil price appreciation and the current value of v, as it is presented in our model.

- dv is the third stochastic factor and it is defined as the *long-term spot price return*;
- a is defined as the mean reverting coefficient of v;
- σ_v is the volatility of the *long-term price return on oil;*
- λ_v is the risk premium associated to the factor v;
- The relation between the three stochastic equation is expressed by the correlations' structure:
    - $d\widetilde{W}^S d\widetilde{W}^y = \rho_{S,y} dt$
    - $d\widetilde{W}^S d\widetilde{W}^v = \rho_{S,v} dt$
    - $d\widetilde{W}^y d\widetilde{W}^v = \rho_{y,v} dt$

We now want to enlighten the advantages of using such a more sophisticated process. Firstly, it is evident that modelling three different parameters provides the user with more information about the behaviour of the oil price. Secondly, Cortazar & Schwartz underlined the importance of the third factor, especially in those days where the two-factor model was not performing so well. The third stochastic state variable has a statistical explanatory power in estimating crude oil price behaviour. This third factor enhances the precision of the model and helps to adapt it to different shapes of the Futures curve. Lastly, the only source of information that is used to estimate the parameters is the Futures term structure, which should reduce the effort of collecting input. These advantages permit to have a greater adaptability in the shape of the Futures price term structure and they enhance the interpretation of the model.

Furthermore, from a pure technical perspective it can be remarked that the drift of the oil spot price is the difference of two stochastic processes: the demeaned convenience yield and the long-term oil price appreciation. This allows for greater flexibility in the price behaviour, since both parameters are stochastic and change overtime. This is definitely an important improvement of this three-factor model, because the process, and in this case the spot price, will react quickly to sudden changes in the drift, since an unexpected effect would be immediately captured by the other two stochastic equations and reproduced in the drift of the spot price.

The Kalman filter[14] technique is often used to accurately estimate model parameters. The Cortazar and Schwartz technique is a valid alternative that permits to avoid the implementation of the more demanding Kalmar filter and that, according to its authors, also provides good estimations of the model factors.

[14] The Kalman filter approach is an estimation technique that optimizes time series and cross sections data at the same time (Cortazar & Schwartz, 2003).

Lastly, Cortazar and Schwartz (2003) remarked the fact that, despite the significant new features of their *parsimonious* model, the third factor is a bit more abstract than the one modelled in Schwartz (1997). In the first three-factor model the risk-free interest rate was expressly modelled since the term structure of governmental bonds was used to estimate it. On the contrary, in the *parsimonious* three-factor model the sole Futures curve is used to estimate all parameters, therefore the third factor is more general (Cortazar & Schwartz, 2003). Perturbations in the risk free interest rate might be included in this factor, but its explanatory power is not limited to that.

3.5.2 The Estimation Technique: Implementation and Functioning

We now want to present the relatively simple procedure that permits to estimate the parameters that are necessary for the purpose of this study. The method designed by Cortazar and Schwartz can be implemented in a normal Microsoft Excel® spreadsheet and running the Solver® add-in of Excel® (Cortazar & Schwartz, 2003). This is a great advantage; in fact, it permits people, who do have few or no skills in advanced programming codes or mathematical software, to still have access to a considerable level of information regarding commodity prices only utilizing the worldwide used Excel®.

The idea behind this technique is very intuitive and it consists in a double optimization problem. The calibration of the model is performed using all Futures prices [15] available at time t_i with $i = 1, 2, 3, \dots, N$. At each time t_i there are M_i contracts with different maturities.

The first step is to estimate the three state variables, which are the variables that define the current state of the process and are relevant for predicting the future price behaviour. In our case the state variables correspond to the three stochastic variables of the model: $\{S(t_i),\ y(t_i),\ v(t_i)\}$. At each point in time t_i all Futures contracts have in common these three variables. Therefore, for a given set of initial parameters $\{\Omega\}$ we estimate the state variables of that single day employing the cross section of the Futures contracts' prices.

We repeat this procedure for all t_i in our analysis. Once we estimated all state variables for each day we use the entire time series to estimate a new set of parameters $\{\Omega\}$. We repeat this procedure until convergence.

[15] Not all Futures contracts are necessary but it can be used a representative sample of them. In this study we decided to limit the number of Future contracts to 11, with different maturities from 1 month to 35 months.

We can write this procedure in a more formal and technical way, hence the first part of the procedure has the following form:

$$\{S(t_i), y(t_i), v(t_i)\} \in \underset{S(t_i),y(t_i),v(t_i)}{\operatorname{argmin}} \sum_{j=1}^{M} \left(\ln \hat{F}_{ij}\big(S(t_i), y(t_i), v(t_i), T_j - t_i; \{\Omega\}\big) - \ln F_{ij}\right)^2$$

$$\text{where: } \{\Omega\} = \left\{\kappa, a, \bar{v}, \lambda_S, \lambda_y, \lambda_v, \sigma_S, \sigma_y, \sigma_v, \rho_{Sy}, \rho_{Sv}, \rho_{yv}\right\}$$

For each date t_i we minimise the sum of the differences between the logarithm of the estimated Futures price and the logarithm of observed Futures price for a set of parameters: $\{\Omega\}$.

Taking the natural logarithm of the Futures values, we can express them in a linear form of the state variables, meaning:

$$\hat{F}_{ij}(S(t_i), y(t_i), v(t_i), \tau_i; \{\Omega\}) = S(t_i) \cdot \exp(a_0(\Omega, \tau_i) + a_1(\Omega, \tau_i) \cdot y(t_i) + a_2(\Omega, \tau_i) \cdot v(t_i))$$

where $\tau_i = T_j - t_i$, so when taking the natural logarithm we have:

$$\ln \hat{F}_{ij}(S(t_i), y(t_i), v(t_i), \tau_i; \{\Omega\}) = \ln S(t_i) + a_0(\Omega, \tau_i) + a_1(\Omega, \tau_i) \cdot y(t_i) + a_2(\Omega, \tau_i) \cdot v(t_i))$$

Rewriting the Futures price formula in its logarithm form reduces the optimisation program to a simple linear regression problem, where we minimise the sum of the squared difference between estimated and observed values, exactly as the optimisation problem above suggests. Therefore, we can now rewrite the minimisation program in the following way:

$$\ln \hat{F}_{ij}(S(t_i), y(t_i), v(t_i), \tau_i; \{\Omega\}) - a_0(\Omega, \tau_i) = \ln S_t + a_1(\Omega, \tau_i) \cdot y + a_2(\Omega, \tau_i) \cdot v + \varepsilon_t$$

This formula refers to a specific day for all the Futures contracts traded in that day. Given that we are working in a three dimensional space it is easier to rewrite the problem in its matrix form, consequently we have:

$$\boldsymbol{Y_t} = \boldsymbol{X_t} \cdot \boldsymbol{\hat{\beta}_t} + \boldsymbol{\varepsilon}$$

where:

$$\boldsymbol{Y_i} = \begin{bmatrix} \ln \hat{F}_{i1}(S(t_i), y(t_i), v(t_i), \tau_1; \{\Omega\}) - a_0(\Omega, \tau_1) \\ \vdots \\ \ln \hat{F}_{ij}(S(t_i), y(t_i), v(t_i), \tau_i; \{\Omega\}) - a_0(\Omega, \tau_i) \end{bmatrix}_{j \times 1}$$

and

$$\boldsymbol{X_i} = \begin{bmatrix} 1 & a_1(\Omega, \tau_1) & a_2(\Omega, \tau_1) \\ \vdots & \vdots & \vdots \\ 1 & a_1(\Omega, \tau_i) & a_2(\Omega, \tau_i) \end{bmatrix}_{j \times 3}$$

We need to highlight that in the case a contract is not traded for a day t_i, the row j is composed only of zeros in both matrix, $\boldsymbol{Y_i}$ and $\boldsymbol{X_i}$ and it will not influence the estimation error. Lastly, we have to define the state variables vector, which is:

$$\widehat{\boldsymbol{\beta}}_{\boldsymbol{i}} = \begin{bmatrix} \ln S(t_i) \\ y(t_i) \\ v(t_i) \end{bmatrix}_{3\times 1}$$

And from the least squared linear regression we know that we can find $\widehat{\boldsymbol{\beta}}_{\boldsymbol{i}}$ according to the following formula:

$$\widehat{\boldsymbol{\beta}}_{\boldsymbol{t}} = (\boldsymbol{X_i^T} \cdot \boldsymbol{X_i})^{-1} \cdot \boldsymbol{X_i} \cdot \boldsymbol{Y_i}$$

Consequently, implementing the least squared linear regression we can obtain the value of the unobservable state variables at each point in time, only using the cross section of the Futures prices and a given set of parameters $\{\Omega\}$. We do not need any other information besides the price information available at each date for each contract. This element makes easier to update the parameters, because of the limited work financial practitioners have to do in formatting data.

Then, once we have estimated the unobserved state variables for each date t_i for our arbitrary set of parameters, we find the optimal set of parameters running the following minimization problem:

$$\min_{\{\Omega\}} \sum_{i=1}^{N} \sum_{j=1}^{M} \left(\ln \hat{F}_{ij}\left(S(t_i), y(t_i), v(t_i), T_j - t_i; \Omega\right) - \ln F_{ij}\right)^2$$

$$s.t.$$

$$\{S(t_i), y(t_i), v(t_i)\} \in \underset{S(t_i), y(t_i), v(t_i)}{\operatorname{argmin}} \sum_{j=1}^{M} \left(\ln \hat{F}_{ij}\left(S(t_i), y(t_i), v(t_i), T_j - t_i; \Omega\right) - \ln F_{ij}\right)^2$$

$$\text{where: } \{\Omega\} = \left\{\kappa, a, \bar{v}, \lambda_S, \lambda_y, \lambda_v, \sigma_S, \sigma_y, \sigma_v, \rho_{Sy}, \rho_{Sv}, \rho_{yv}\right\}$$

In reality, not all the parameter in $\{\Omega\}$ are estimated form the minimization procedure , but volatilities, correlations and risk premium associated to each factor are consistent with the time series of the state variables. In particular, the volatilities and correlations among the three factors are computed from the estimated time series of each state variable. This assures that the covariance matrix is positive and definite (Cortazar & Schwartz, 2003), which permits to compute the inverse of this matrix. This is essential in order to compute the state variable vector $\widehat{\boldsymbol{\beta}}_{\boldsymbol{t}}$ of the linear regression.

Up to this point we have presented the three-factor model for the spot price and also the optimisation program we are using for estimating the different parameters that compose the universe of value $\{\Omega\}$, however, we haven't discussed yet the formula that permits to obtain the value of the Future contracts at each point in time implied in our model. In the section 3.4 Futures Price and Spot Prices we showed how these prices are closely related. According to the same procedure we adopted for deriving the formula of the Gibson and Schwartz (1990) two-factor model, we can obtain the formula for pricing Futures contracts for the Cortazar and Schwartz (2003) model.

In fact, applying Itô's lemma, it can be shown that the Futures price must satisfy the following partial differential equation:

$$\frac{1}{2}\sigma_S^2 S^2 F_{SS} + \frac{1}{2}\sigma_y^2 F_{yy} + \frac{1}{2}\sigma_v^2 F_{vv} + \sigma_S\sigma_y\rho_{Sy} S F_{Sy} + \sigma_S\sigma_v\rho_{Sv} F_{Sv} + \sigma_y\sigma_v\rho_{yv} F_{yv} + (v - y - \lambda_S) S F_S + (-\kappa y - \lambda_y) F_y + [a(\bar{v} - v) - \lambda_v] F_v - F_T = 0$$

with the boundary condition: $F(S, y, v, T) = S$ when $T = 0$.

Schwartz and Cortazar solved this PDE and came up with the following formula for the value of Futures contracts in the three-factor model[16]:

$$F(S, y, v, T) = S \cdot \exp\left[-y\frac{1 - e^{-\kappa T}}{\kappa} + v\frac{1 - e^{-aT}}{a} + B(\Omega, \mathrm{T})\right]$$

where:

$$\begin{aligned} B(\Omega, \mathrm{T}) = {} & -\lambda_S T + \frac{\lambda_y - \sigma_S\sigma_y\rho_{Sy}}{\kappa^2}(\kappa T + e^{-\kappa T} - 1) + \frac{\sigma_y^2}{4\kappa^3}(-e^{-2\kappa T} + 4e^{-\kappa T} + 2\kappa T - 3) \\ & + \frac{a\bar{v} - \lambda_v + \sigma_S\sigma_v\rho_{Sv}}{a^2}(aT + e^{-aT} - 1) - \frac{\sigma_v^2}{4a^3}(e^{-2aT} - 4e^{-aT} - 2aT + 3) \\ & - \frac{\sigma_y\sigma_v\rho_{yv}}{\kappa^2 a^2(\kappa + a)}\begin{pmatrix} \kappa^2 e^{-aT} + \kappa a e^{-aT} + \kappa a^2 T \\ + \kappa a e^{-\kappa T} + a^2 e^{-\kappa T} \\ -\kappa a e^{-(\kappa + a)T} - \kappa^2 \\ -\kappa a - a^2 + \kappa^2 a T \end{pmatrix} \end{aligned}$$

Given the fact that the volatilities and the correlation between the three different state variables are computed from their estimated time series, we can further simplify the optimization problem and reduce the number of parameter to be estimated.

In fact, we can rewrite the above expression of the Futures prices in a slightly difference way, without reducing its estimation power. Therefore, the new formula will be:

$$F(S, \hat{y}, \hat{v}, \tau) = S \cdot \exp\left[-\hat{y}\frac{1 - e^{-\kappa\tau}}{\kappa} + \hat{v}\frac{1 - e^{-a\tau}}{a} + B(\Omega', \tau)\right]$$

where,

[16] In our opinion there is a typo in the original paper by Cortazar and Schwartz (2003). In fact, in the formula they provide for pricing Futures contracts, the coefficient that multiplies v is not correct. We adjust the formula and we presented the correct expression in this paper.

$$B(\Omega', \tau) = -\frac{\sigma_{S,\hat{y}}}{\kappa^2}(\kappa\tau + e^{-\kappa\tau} - 1) + \frac{\sigma_{\hat{y}}^2}{4\kappa^3}(-e^{-2\kappa\tau} + 4e^{-\kappa\tau} + 2\kappa\tau - 3) + \beta(a\tau + e^{-a\tau} - 1)$$
$$- \frac{\sigma_{\hat{v}}^2}{4a^3}(e^{-2a\tau} - 4e^{-a\tau} - 2a\tau + 3) - \frac{\sigma_{\hat{y},\hat{v}}}{\kappa^2 a^2(\kappa + a)}\begin{pmatrix} \kappa^2 e^{-a\tau} + \kappa a e^{-a\tau} + \kappa a^2\tau \\ + \kappa a e^{-\kappa\tau} + a^2 e^{-\kappa\tau} \\ -\kappa a e^{-(\kappa+a)\tau} - \kappa^2 \\ -\kappa a - a^2 + \kappa^2 a\tau \end{pmatrix}$$

As we can see the formula for the Futures prices – especially in $B(\Omega', \tau)$ – does not depend anymore on the different risk premiums associated to the three different state variables. In this new formula we maintained the number of state variables unchanged, but we slightly have to modify them in order to accommodate for this simplification. In fact, the demeaned convenience yield and the price appreciation of light sweet crude oil are now defined according to the following expressions:

$$\hat{y} = y + \frac{\lambda_y}{\kappa} \Leftrightarrow y = \hat{y} - \frac{\lambda_y}{\kappa} \quad \text{and} \quad \hat{v} = v - \lambda_S + \frac{\lambda_y}{\kappa} \Leftrightarrow v = \hat{v} + \lambda_S - \frac{\lambda_y}{\kappa}$$

Reshaping these two state variables in the way we just presented permits to reduce the number of parameters to be estimated in the optimisation program, from twelve down to seven. Therefore the new set $\{\Omega'\}$ includes: $\{\kappa, a, \beta, \sigma_{\hat{y}}^2, \sigma_{\hat{v}}^2, \sigma_{S,\hat{y}}, \sigma_{\hat{y},\hat{v}}\}$. As we can see we do not estimate the volatility related to the spot price. Moreover, for the other two state variables, we only estimate their variances and not their volatilities, thus when interpreting the results we should transform them in order to have a better understanding. The estimated relationship between the three factors is expressed by the co-variances among them and not through correlations. Variance and covariances of $\{\Omega'\}$ are still computed using the estimated times series of the three state variables.

In order to be consistent with this new reformulation we also have to define three other expressions that permit us to obtain the other parameters of $\{\Omega\}$ and enhance our understanding regarding the results. These equations are[17]:

$$\beta = \frac{a\tilde{v} - \lambda_v + \sigma_{S,\hat{v}}}{a^2}$$

$$\tilde{v} = \bar{v} - \lambda_S + \frac{\lambda_y}{\kappa} \Leftrightarrow \bar{v} = \tilde{v} + \lambda_S - \frac{\lambda_y}{\kappa}$$

$$\sigma_{S,\hat{v}} = \sigma_S \sigma_{\hat{v}} \rho_{S,\hat{v}}$$

[17] The standard deviation of the first state variables (σ_S) and the correlation between the spot price and the return of the oil price ($\rho_{S,\hat{v}}$) are still computed using their estimated time series.

This simplification and reduction of the number of the parameter to be estimated, obviously influences the structure of the model. Taking into account the modifications of the two state variables the three stochastic differential equations compose the following model[18]:

$$dS = (\hat{v} - \hat{y}) \cdot S \cdot dt + \sigma_S \cdot S \cdot d\widetilde{W}^S$$

$$d\hat{y} = (-\kappa\hat{y}) \cdot dt + \sigma_y \cdot d\widetilde{W}^{\hat{y}}$$

$$d\hat{v} = \left[\beta a^2 - \sigma_{S,\hat{v}} - a\hat{v}\right] \cdot dt + \sigma_v \cdot d\widetilde{W}^{\hat{v}}$$

As we expected, the risk premiums are not part of the model. If we want to compute them we need to use the different time series of the estimated state variables. In fact, if we only use the estimated parameters and we reverse the five expressions we used to simplify the model, it is not possible to compute the three different risk premium. The resulting system of equation will be indefinite. For the purpose of this study is not necessary to compute the different risk factors, but it might be relevant for others analysis. This could be a possible drawback in using this model.

Since the different variances and co-variances are compute from the time series of the model state variables, we ask Solver® to find three parameters: κ, a, β. If we look at the problem from a more technical perspective, this is also an improvement. In fact, reducing the number of objective cells should permit Solver® to work in a more efficient and faster way.

Once we have understood and simplified the minimisation program, we can then implement it in a simple Excel® spreadsheet. In Appendix V we show how we structured the optimisation problem in Microsoft Excel.

3.6 Empirical Implementation

Having defined the model and the calibration technique, in this section we want to apply them to the recent light sweet crude oil Futures market data. We first describe the different data we used, subsequently we apply the Cortazar & Schwartz estimation technique and deeply analyse its results. Lastly we comment on this spreadsheet technique and the difficulties we encountered along the way.

3.6.1 Data

Our set of data is composed of Futures contracts on *Light Sweet Crude Oil* (CL). The Futures prices of the commodity used for the purpose of this study were collected through Bloomberg®. The period of the analysis goes from the 3rd January 2006 to the 30th of December 2011. We used 16,580 daily Futures prices for crude oil corresponding to the daily values of all light sweet crude oil Futures considered in this study.

[18] For the more detailed transformation of the model please refer to Appendix IV.

We decided to use a set of 11 Futures contracts: CL1, CL2, CL3, CL4, CL5, CL6, CL12, CL18, CL24, CL30 and CL35[19]. The delivery maturities of the contracts for crude oil vary from 1 month up to 3 years. These contracts were selected, and the maximum delivery was limited to 36 months, because of their higher liquidity in the market. Therefore we consider them to incorporate more information with respect to the ones rarely traded. This should allow our model to capture both the short-term behaviour of oil prices and the long-term movements.

The two contracts with the highest maturities (CL30 and CL35) were not daily traded at the beginning of the analysis. However, during the six-year period analysed, they also started to be daily traded. The impact of this lack of information should not be highly significant, since we do not have data for these two contracts only for the first couple of months.

Following the idea of Cortazar and Schwartz we did not aggregate the single data, but we used all the observations as independent inputs for our model, in order not to lose some explanatory power.

We did not only gather the prices for the different contract but we also extrapolate and computed the exact time-to-maturity for each observation, instead of using an average time-to-expiration. The value of the time-to-maturity of each single Futures contract varies overtime, since they expire at a specific day each month. This more precise use of time to maturity permits to enhance the reliability of our results. We have to point out that collecting these time series of time-to-expiration for each contract it is not straightforward using Bloomberg®. It requires a certain amount of time, since this financial platform does not have a specific function to compute the time-to-maturity for Futures contracts.

[19] CL contract are rolling maturity indexes: in CL1 for example we have always the Future with delivery in one month. Therefore when the contract that is currently in CL2 has a time to maturity of one month, it leaves the CL2 index and it is incorporated in the CL1 one.

Below Table 2 helps us to summarize and describes the Futures data we used in this study:

Crude Oil Data on NYMEX: from 03/01/2006 to 30/12/2011					
Future Contract	**Price [in $]**		**Maturity [Years]**		**OBS**
	μ	**St. Dev.**	μ	**St. Dev.**	#
CL1	$79,23	20,23	0,12	0,02	1511
CL2	$80,04	19,64	0,21	0,02	1511
CL3	$80,66	19,23	0,29	0,02	1511
CL4	$81,13	18,93	0,38	0,02	1511
CL5	$81,51	18,68	0,46	0,02	1511
CL6	$81,83	18,45	0,55	0,02	1511
CL12	$82,96	17,35	1,06	0,02	1511
CL18	$83,37	16,56	1,56	0,02	1511
CL24	$83,53	16,09	2,07	0,02	1511
CL30	$83,94	15,82	2,58	0,02	1484
CL35	$83,84	15,74	3,00	0,02	1497

Table 2 – Summary of the Light Sweet Crude Oil Data: daily observation from 03/01/2006 to 30/12/2011.

The chart below (Figure 6) represents the evolution of the price of crude oil during the period analysed in this study.

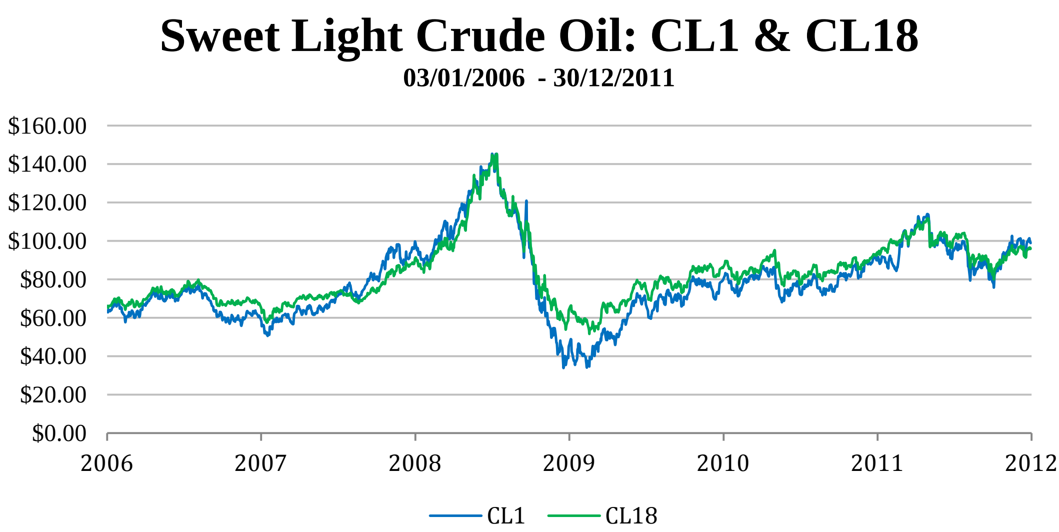

Figure 6 - Light Sweet Crude Oil Futures prices CL1 and CL18 from 03/01/2006 to 30/12/2011. Source: Bloomberg

At first sight we can easily observe that the price behaviour of oil does not have a flat evolution. After a steady period from winter 2006 to the end of 2007, the prices radically changed growth pace and it sharply started to increase until summer 2008, when it suddenly dropped to the levels of 2004 (~$40.00) in only 6 months. From the first quarter of 2009 crude oil started again a phase of consistent appreciation, which lasted until spring 2011. During the summer and autumn 2011 the price of the light sweet crude oil felt by 34% touching $75.67 in October 2011.

The minimum during the analysed period was touched in December 2008, when the crude oil was traded on NYMEX at $33.87 per barrel. On the contrary, after a quasi-exponential growth, the highest price for crude oil was reached on July 3rd 2008, when a barrel of crude oil was exchanged for $145.29.

In Chapter 1 we explained the different international interconnections between consumers and producers of oil. This element can allow us completing our understanding of the petroleum's price path and clarifies the reasons why it evolved in this way. Moreover, the price of crude oil is a good indicator of the world's macroeconomic situation. This supports the need of updating our parameters with a certain recurrence, in order to guaranty a reliable and representative pricing model. The updated parameters will better mirror the current macro and energy specific situation.

There are different elements on which the oil price depends. They can be classified in two main categories: *quantifiable* and *non-quantifiable*. Under the former categories we can include: GDP of countries and USD evolution, whereas in the latter group we can find speculation and geopolitical instability, which are harder to quantify.

The GDP level of the largest economies is without any doubt a driving factor of the crude oil price. In fact, Riha *et al.* (2011) discovered, in their study on oil analysis, a statistically significant and positive correlation between the crude oil price and the GDP of the seven most developed countries[20] (+0.79). Actually, the GDP indirectly reflects the demand of these countries for oil. In their study China was not taken into consideration, but since its high GDP growth level it is reasonable to assume that it has an effect on the worldwide consumption of oil and consequently on its price. The GDP growth can aid us to explain the stability in petroleum price from 2006 and 2007. During this period most of the G7 countries and China dealt with positive GDP growth rates.

[20] United States of America, United Kingdom, Japan, Italy, Germany, France and Canada.

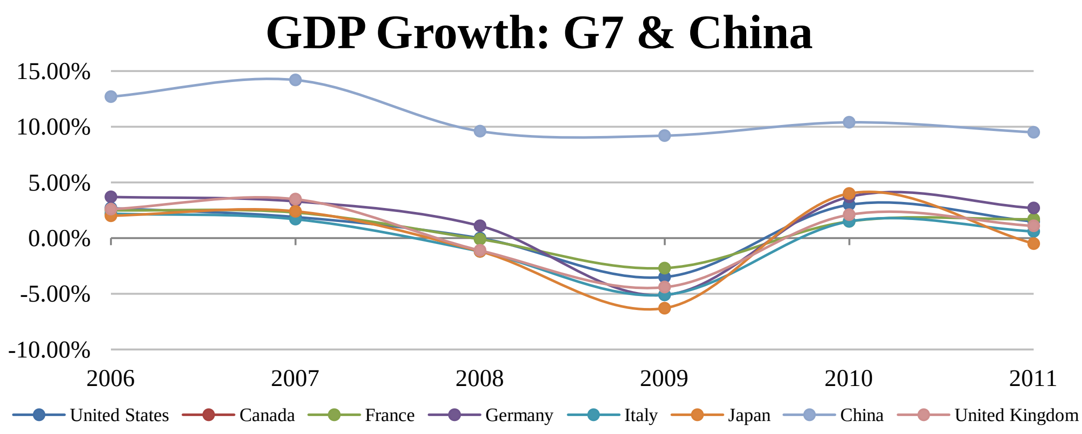

Figure 7 - GDP growth of G7 and China. Source: World Bank

Afterwards when the subprime and the consequent financial crisis impacted the worldwide economy, the GDP growth rates squeezed and the majority of them turned below zero. The consumption of crude oil drastically decreased due to the unfavourable global financial situation (U.S. Energy Information Administration, 2012). This could be claimed as one of the major reasons that rapidly made the light sweet crude oil to sink to 2004 levels. The consequent raise that started in the second part of 2009 can also be explained with the partial revival of the economy and the return to positive GDP growths.

Although the growth of the economy aids to explain the evolution of the price of oil, it is not the only driver. The value of the US dollar can also help to clarify the movement of the price. In fact, this is supported by the finding of Riha *et al.* (2011), which computed the correlation between the crude oil price and the value of USD. They discovered a strong inverse correlation (-0.75), which suggest that to a depreciation of the US dollar corresponds a petroleum appreciation. This is what happened after the World Trade Centre terrorist attacks in 2001 and the consequent financial policy of the FED, which depreciate the dollar (Riha, et al., 2011).

These two quantifiable factors are not sufficient to support the evolution of the crude oil price, especially for the sharp raise leading to the maximum level in 2008. There are many other small causes, which summed up, could also have influenced the price of sweet light crude oil. For example the countries, which adhere to OPEC, slowly reduced the extraction of raw oil by introducing production quotas, meaning a consequent raise in price. Moreover the fear of new terrorist attacks influenced the feeling that non-OPEC countries were not able to fulfil their obligations (Riha, et al., 2011). Another factor that boosted the price of oil was the political tension between USA and Iran on the nuclear power project and long distance rocket programs of the latter country. These are some example of the geopolitical instability that can affect the price of oil in the short term.

We listed some elements that have been contributing in shaping the petroleum price evolution. All of those reinforce the assumption that the oil price is a reliable macro indicator.

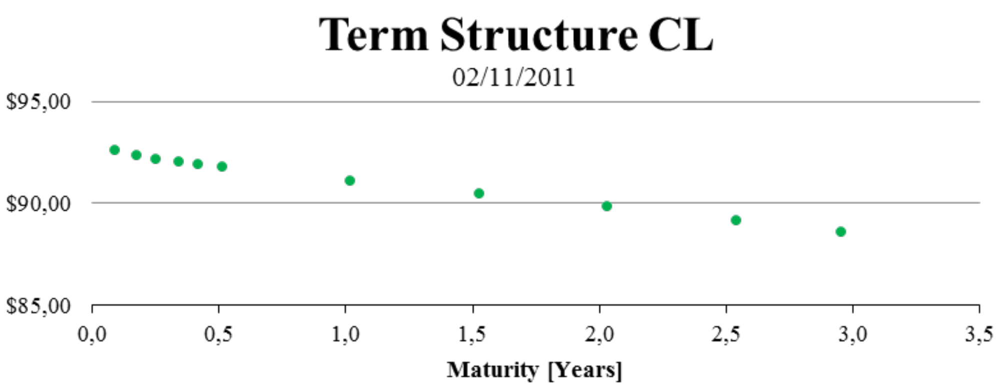

Figure 8 - Term structure of the CL for the 30th December 2011. Source Bloomberg

Additionally, looking at Figure 8 we can notice that the price does not follow a seasonal path. Even if the chart considered only a particular day we can infer that there are no observable sinusoidal movements in the price's path, which it is consistent with our assumption in modelling crude oil price, without accounting for seasonality. As we explained before seasonality is not a major issue when dealing with crude oil, but for instance it is determinant when pricing gasoline.

We can see, from this chart, that there is mean reversion in oil prices. In this particular day the prices of the Futures contracts are decreasing over time. The chart shows a backwardation situation, where the market participants believe the spot price to be above its *natural* level. This means that the market expects the price to decrease in the long run. This is consistent with the assumption of mean reversion and with the theory suggested by the study conducted by Baker *et al.* (Baker, et al., 1998). They pointed out that crude oil prices do not follow a random walk, but they revert to a long-term value, instead. When the price is considered too high compared to its normal level the market anticipates a decrease in its values, whereas if the market consider the spot price to be too low, the prices for Future contracts with a long maturity tend to be higher than the spot price. We already discussed this element when presenting commodities and also we account for this characteristic in modelling the light sweet crude oil price.

3.6.2 Implementation

For the purpose of this study we decided to focus and use only a single set of data. In fact, we only use available Futures prices for the period from January 2006 to December 2011 in order to extract the necessary parameters for pricing light sweet crude oil according to our three-factor model. We did not divide our sample in two data sets (in-sample and out-of-sample), since the purpose of our study is not to confirm the validity of the Cortazar & Schwartz technique for the estimation of Futures prices. In their paper they provide satisfactory results and they also support their findings testing and comparing them to the results of the implementation of the more complex Kalman filter.

As it was suggested in the paper of Cortazar & Schwartz (2003) we decided to construct our model in a spreadsheet in Excel®. Before starting to use the spreadsheet itself we choose to program some basic and recurrent functions in Visual Basic for Applications (VBA). Programming functions can help in speeding up calculations. The result is a model that is more time efficient in providing the user with an optimal solution for the problem. For the sake of clarity in the Appendix VI we included the VBA codes we used in our model.

After having programmed all the essential VBA functions and having entered all the prices and maturities of the different Futures contracts we started the iteration procedure to estimate our parameters. For getting the model started we need to provide it with some initial parameters[21]. Looking at the brilliant findings of Cortazar and Schwartz we decided to use them as our starting values. We used the results they had for the period January 1991 – December 2001, since those are the most recent values. We also confronted these parameters with another study on oil prices (Casassus & Collin-Dufresne, 2005). In this last paper the parameters for oil were estimated from a more complex model, but the period analysed included more recent data (i.e. 1990 – 2008). Although their findings refer to another model, the results can still be interpreted and compared. They are reasonably consistent with the ones found by Cortazar & Schwartz, which reinforce the use of the latter as initial parameters.

After setting the initial values we started to optimise the model using Solver®. We set a maximum number of iterations in order to control the optimisation problem. In fact using reasonable number of iterations the optimisation procedure should not become ungovernable and we would be more precise in find a global minimum of the error, instead of finding only a local minimum.

It is important to underline the fact that the variance and covariance of the three state variables $(S, \hat{y}, \hat{v})$ are computed using their time series. This has an implication on the functioning of the model since we have to allow for circular references. Consequently we have to be really careful when constructing the model and make it as simple as possible in order to ease the calculations and reduce the impact of circular procedures.

[21]The initial parameters needed for the model are: k, a, β. The variances and the covariance between the state variables are estimated from their estimated time series.

3.6.3 Results & Findings

This section of Chapter 3 is entirely dedicated to the presentation and explanation of the results and findings we obtained implementing the estimation technique developed by Cortazar and Schwartz (2003) We decided to comment our results using different approaches in order to have a broader overview on our estimations and their performance.

3.6.3.1 Estimated Price Parameters

Table 3 below summarizes the value for the parameters for crude oil we obtained implementing our estimation technique:

Estimated Parameters CL 2006 - 2011	
κ	4.2556
a	0.4308
β	-0.0422
$\sigma^2_{\hat{y}}$	0.1235
$\sigma^2_{\hat{v}}$	0.0045
$\sigma_{S,\hat{y}}$	0.0648
$\sigma_{\hat{y},\hat{v}}$	-0.0076

Table 3- Estimated Light Sweet Crude Oil Pricing Parameters of the three-factor model

In order to be able to better interpret and have a larger impression of our findings we can use the time series of the estimates state variables. In fact, we need these time series and the formulas explained in the previous section to compute the missing parameters of our pricing model necessary for this study.

Therefore we have:

Estimated Parameters CL 2006 - 2011	
κ	4.2556
a	0.4308
σ_S	0.2841
$\sigma_{\hat{y}}$	0.3514
$\sigma_{\hat{v}}$	0.0667
$\rho_{S,\hat{y}}$	0.6847
$\rho_{S,\hat{v}}$	-0.6707
$\rho_{\hat{y},\hat{v}}$	-0.3240
$\alpha\tilde{v} - l_{\hat{v}}$	0.0049

Table 4 – Complete Estimated Light Sweet Crude Oil Pricing Parameters

Firstly, we want to analyse the coefficients that reflect the level of mean reversion inside our model (κ and a). As a starting point we have to notice that they are strictly different from zero, meaning that the two stochastic expressions modelling the demeaned convenience yield and the crude oil price appreciation imply mean reversion. This indirectly supports the mean reversion behaviour of the spot price as we have already explained when we presented the model.

The coefficient of mean reversion for the price appreciation (a) is in line with the one found by Cortazar and Schwartz (2003): 0.4950. Even though Casassus *et al.* (2005) used a slightly different model for estimating the behaviour of crude oil price, we can still notice that our results for the mean reversion of the price appreciation are reasonable compared to their findings too.

We can now concentrate on the parameter that defines the reversion of the demeaned convenience yield to its long-term value. We obtained a κ that is much higher than the one found in previous studies. Cortazar and Schwartz found a coefficient of mean reversion equal to 1.6480 and, before them, Schwartz (1997) obtained a level of 1.3140. Previous studies were based on different models and they also analysed different periods, which did not experienced such a high appreciation of oil as in the year 2008 did. Consequently we have to stress the fact that, for both parameters, a deep comparison with previous models is difficult to perform and might be misleading.

Our higher value for the speed of adjustment coefficient (Schwartz, 1997) means that the convenience yield tends to revert faster to its *natural* level. An important feature of the mean reversion process is the fact that we can compute the *half-life* statistic This is a measure that tells us what is the necessary period for the variable to revert half way to its *normal* level, starting from its current value (Clewlow, et al., 2000). The formula to compute the half-life statistic is:

$$hl = \frac{\ln(2)}{\gamma}$$

where γ is the coefficient of mean reversion. Therefore, in our case, we have that the demeaned convenience yield needs on average 8 weeks[22] to be half way to its long-term level, if no random shocks occur during this period. The same statistics for the price appreciation of oil says that it takes more than a year to revert to its long-term value.

We consider the mean reversion of the price appreciation to be a good value, whereas the one of the convenience yield to be a bit too high. We would expect the cycle for the demeaned convenience yield to be longer, which will be reflected in a lower value for the speed adjustment coefficient. In our study the mean reversion of the convenience yield also influences spot prices. A previous study found that the crude oil spot price should revert in 8 months (Bessembinder, et al., 1995). This high mean reversion coefficient makes the convenience yield to return to a zero level in much less than 8 months. A value of zero for the demeaned convenience yield does not affect the drift of the spot price, which will be only influenced by the long-term appreciation. Consequently, this fact might soften the intrinsic mechanism of mean reversion of the pricing model. Since there is a strong relation between spot prices and convenience yield for consumption goods, according to theory of storage (Working, 1948), (Working, 1949) and (Brennan, 1958), this might have an impact on the storage policies of certain companies active in the industry, and on their related production decisions. The mean reversion parameters are also interesting for discussing the volatility structure of the Futures prices. This volatility analysis ends this section on the interpretation of results.

We now continue our analysis of the parameters discussing an important element in our model and that is still related to mean reversion. As it can be seen from Table 4, in the crude oil market there is a positive correlation between the first two state variables (+ 0.6847). This observation is consistent with the assumption we made when defining our three-factor model. Earlier in the study we discussed the fact that the model does not have an explicit mean reversion coefficient in the stochastic differential expression of the spot price. However, the mean reversion of the price is due to the positive correlation between the spot price and the convenience yield (Schwartz, 1997). A positive shock in the convenience yield means that it is more likely also for the spot price to have a positive shock, but since the convenience yield decreases the drift of the stock price, it makes the price to adjust to its long-term value. Hence he positive level of correlation between the two state variables supports the assumption of indirect mean reversion of the crude oil price. This is not a surprising result (we expected this), because of the well proved evidence of the direct correlation between spot price and convenience and the inverse relation of both with respect to inventory levels (Brennan , 1991),

[22] $\text{half} - \text{life} = \frac{\ln(2)}{4.2556} = 0.1629 \;\rightarrow 360 \text{ days } \cdot 0.1629 = 58.64 \text{ days} = 8.38 \text{ weeks}$

(Richter & Sørensen, 2002). It is interesting to note how this result is reasonably in line with the findings of other studies: (Schwartz, 1997), (Casassus & Collin-Dufresne, 2005) and (Trolle & Schwartz, 2009).

We can now focus our effort on the other two parameters that model the correlations among the three state variables. We can notice that both of them are negative and strictly different from zero. The correlation between the spot price and the appreciation of the light sweet crude oil is -0.6707, whereas the one between the convenience yield and the price return is -0.3240. Cortazar and Schwartz also found, for some periods, negative values for the correlation between the first and the third state variable. However, this was not the case for the correlation between the convenience yield and the price return, which was always significantly positive. Having also a look to the value found by Casassus *et al.* (2005), we would have expected a positive value for both of them, or at least smaller for $\rho_{S,\hat{v}}$. We tried diverse initial parameters in order to see if these values could be different, but Excel Solver® always provided us with negative values for these two measures. This aspect can influence the mean reversion process inside our model and making it less powerful. In fact, if we centre our attention to the correlation between the second and third state variables we can see that a positive shock in the return is mirrored by a more likely negative value in the convenience yield, which will be more probable to make the differential negative. These behaviours might impact the drift of the stock price and weaken the mean reversion feature of our model. The mean reversion deriving from negative correlation between the spot price and the interest rate is usually experienced in precious metals that are used as a store of value, which is not the case of crude oil (Lutz, 2010). If our correlation measure is correct we might infer that there is speculation in the market (also suggested by Riha, Jirova and Honcu (2011)), and now petroleum is also used as a kind of store value commodity and not only as a consumption good.

It is time to analyse the volatilities of the three state variables. The volatility of the spot price that we obtained in our model is 28.41%. The volatility of the demeaned convenience yield is around 35% and the one of the price appreciation is the lowest with 6.67%. These values are different from the previous studies we used as benchmark to compare our results. We discuss the volatilities figures more in detail in the subsection dedicated to the analysis of the volatility and its structure.

The last parameter we want to discuss is the element that is part of the drift of the price appreciation: $(\alpha\tilde{v} - \lambda_{\hat{v}})$. Analysing this parameter we should be able to understand the dynamics of the long-term oil price appreciation, since we can notice that one of the elements composing it is $\tilde{v}$. We have to say that it is more difficult to interpret and understand this parameter. In fact, it is composed of more than just one variable and accounts for multiple dynamics. However, the value of $a\hat{v} - \lambda_{\hat{v}}$ is positive, which means that the long term appreciation is positive and higher than the risk associated to this element. Since it is higher than zero we should expect the price of oil to increase with time. This seems to be the case for the period we decided to analyse. In order to deliver a more detailed interpretation of this parameter, we would have to compute the risk-factor $\lambda_{\hat{v}}$. Since this element is not relevant for the final purpose of this study (option pricing) we are not dedicating time to compute the value of the risk factors.

3.6.3.2 Analysis of the price evolution

After having individually discussed the different estimations and understood their values, it is now time to show how well our estimations fit the observed prices. This analysis has a significant importance because it permits to judge the reliability of our model. In their study of 2003, Cortazar & Schwartz support the use of the proposed model with satisfying results.

A first representative analysis we can do to prove how well our results fit the reality, is to compare the estimated time series for some contracts with the times series of the prices observed on the market. Figures from 1 to 4, in Appendix VII, show the evolution from 2006 to 2011 of four particular Futures contracts (CL1, CL12, Cl24 and CL30).

It is easy to see that, for all the four contracts, our estimations match the observed prices. In fact, it is nearly impossible to distinguish the two lines. Our model and the parameters we estimated imply estimations of Futures prices that almost perfectly match the ones observed on NYMEX. This is not surprising, since the purpose of the model is to replicate the path of prices. In effect, the optimization algorithm tries to find the set of parameter that best fits the observed data. However, what it is interestingly to see is the fact that all four contracts have a quasi-perfect match.

3.6.3.3 Analysis of Futures Price Term Structure

As we just discussed the analysis of each individual contract shows satisfying good results. It is interesting now to analyse the contracts in a cross section perspective. A simple way to do a cross sectional study is to look at the term structure of the contracts for a specific day and see how well the estimated term structure fits the observed one.

Before starting to analyse the Futures prices structure it is worth to spend few words on the total dollar error of the model. Using the parameter of Table 3 we obtained a value for the total estimation error of \$0.2280. This is clearly close to zero, which is what we would expect from a good estimation model. Of course this is a total model error, which means that there are some dates for which the term structure on average overestimates the prices of the Futures contracts and days in which it underestimates their values. We consider this level of estimation error a good value that supports the reliability of our findings. We have more than 16,500 observed prices in our model, which correspond to 1,500 observed days, thus the total estimation error in absolute terms is definitely tiny.

It is now more relevant to look at some specific dates and see the difference between the estimated and observed prices. The Figure 9 below shows three specific dates of the period analysed. The first two charts correspond to the days where the term structure has the minimum and maximum estimation error. The third one is a random observation.

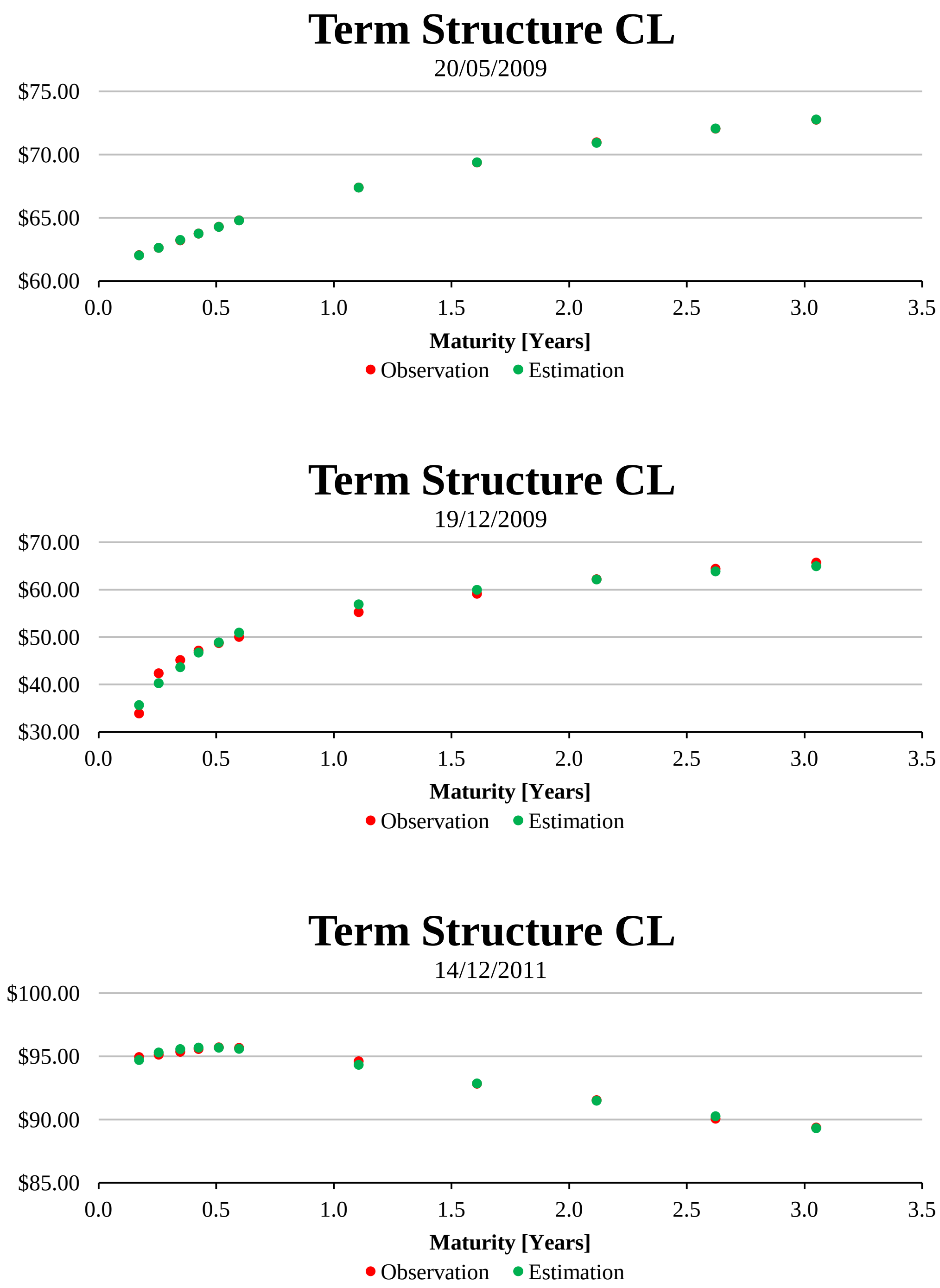

Figure 9 - Term Structure of CL for three particular dates

All the three charts demonstrate the performance of our model. Obviously there are some differences in prices, but they are very limited. What it is remarkable from the three graphs above is the good fit of the estimations even when the slope of the observed term structure is not monotone. This aspect is illustrated in the third chart, where after the third contract (CL4) there is a smooth change in the Futures curve. However, the model is still able to well capture the downward shift of the contracts with higher maturity. In the second chart, for example, the model does not have a fit as good as in the first or in the third; however, it is still possible to see how well the model follows the path of Futures prices on the market.

When we presented the pricing model of this study we underlined the importance of using three factors, instead of modelling only two state variables. Cortazar & Schwartz (2003) had the possibility to demonstrate this point by computing the term structure for a two-factor and a three-factor model. They highlighted how the third factor helps enhancing the precision of the model. The use of a three-factor model of course improves the performance and the flexibility of the model, since the introduction of an additional factor allows for the incorporation of more information in the model. Based on their conclusions, we can infer that the good performance of our estimated term structure, and thus the ability of our model to adapt to different Futures curves, is also due to the price appreciation of oil (v). Most likely we would not have obtained such a good fit if we were modelling prices using only two state variables.

This element sustains the thesis that a good pricing model for oil should be based on three different state variables in order to capture more information and therefore have more consistent price estimations.

3.6.3.4 Analysis of the Root Mean Square Error

In order to support even more our results and have a more reliable measure that permits to be more objective in the analysis of the model and our findings, we decided also to compute the Root Mean Square Error (RMSE). This is a good measure for evaluating the accuracy of a model. It considers the difference between the value predicted by our model and the value observed on the market. The RMSE is representative of the size of the typical error. The formula for computing this model performance measure for a specific Futures contract F_j is:

$$RMSE\left(\hat{F}_{i,j}, F_{i,j}\right) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\hat{F}_{i,j} - F_{i,j}\right)^2}$$

This is an absolute measure. The price between 2006 and 2011 reached sensible high prices, which would make the interpretation of the measure a bit misleading. In order to have a better understanding and also to be able to compare our findings with the ones of Cortazar and Schwartz, we decided to compute the relative RMSE. This means looking at the error in percentage terms, instead of in absolute values. Therefore the formula for the relative RMSE is:

$$RMSE_{\%}\left(\hat{F}_{i,j}, F_{i,j}\right) = \sqrt{\frac{1}{n}\sum_{1}^{n}\left(\frac{\hat{F}_j - F_j}{F_j}\right)^2}$$

We compute the RMSE in percentage terms for each contract used in this study. Figure 10 below summarizes the result of the RMSE analysis.

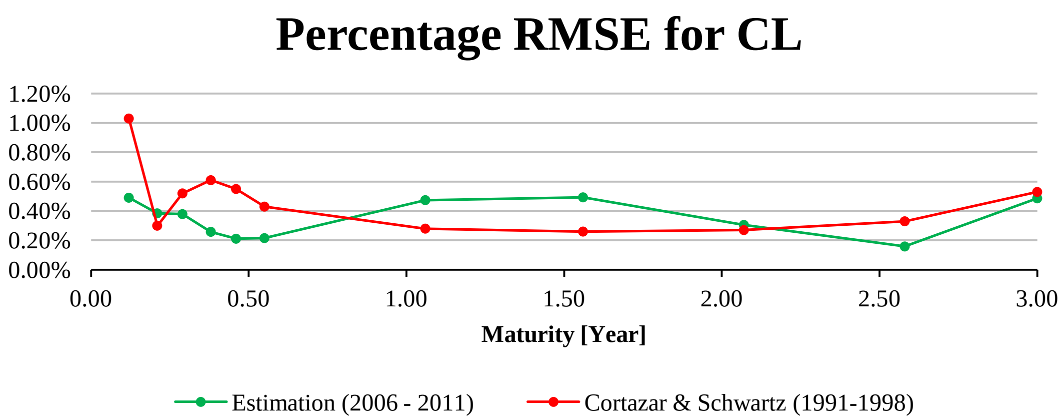

Figure 10 – Comparison of percentage RMSE of our model (2006 – 2011) and Cortazar and Schwartz (1991 – 1998)

The average percentage RMSE in our study is 0.35%, which is lower than the one of the previous study by Cortazar and Schwartz (0.465%)[23]. We recognize the fact that both errors are compared from the in sample estimations. Cortazar and Schwartz performed an out-of-sample analysis that supported the accuracy of their three-factor model. As it is so, we believe that our results also have a good out-of-sample performance. Apart from the average of the RMSE in percentage term is interesting to have a closer look at the errors across maturities. The worst fit in our model is for the first contract and for the one with delivery in one year and a half (0.49%). This is reasonable for CL1, since it has the highest volatility (as we show in the next section) and so it might be more difficult to replicate the price with a static set of parameters. The rest of the contracts have a percentage RMSE included in a range from 0.21% to 0.38%. We have the best fit in our study for the CL5 Futures contract (0.21%), but also the six-month delivery has a really low percentage RMSE value.

Given the fact that we decided not to include all the contracts traded each day we cannot have a deep discussion on the difference between long-term and short-term estimation. However, what we can still

[23] This average RMSE is computed only employing the contracts used in our study. The original total average RMSE was 0.42%.

infer from our results is that the model seems to work quite well for both short and long maturities, since the error level is low for both terms. Compared to Cortazar and Schwartz findings we have a slightly worse fit for medium term maturities (CL12, CL18), possibly due to the fact that we do not use all the contracts as they do in their study. In fact some missing contracts might help to include certain information that is not available in the ones we decided to use. Including more contracts could improve the reliability of our findings. However, the increase of the dimension of the model input would impact the smoothness of the Excel spreadsheet.

3.6.3.5 Analysis of Volatility

Regarding volatility, we have computed and analysed historical figures for oil price returns before implementing our model. This has allowed us to better understand the magnitude of oil Futures prices fluctuations and to form expectations for the results of our iterations. We have chosen to start our analysis by calculating the historical volatility of the returns on the prices of the first contract (CL1) since it is the one we should expect to find higher volatility. This is because it is the contract with shorter time-to-maturity and higher trading volumes. Nevertheless, we should expect the other contracts to show similar volatility patterns in terms of trend but lower values and smoother oscillations. In fact, the longer the maturity of the contract, the less pronounced the changes in volatility should be. This is related with the mean reversion characteristic of oil prices.

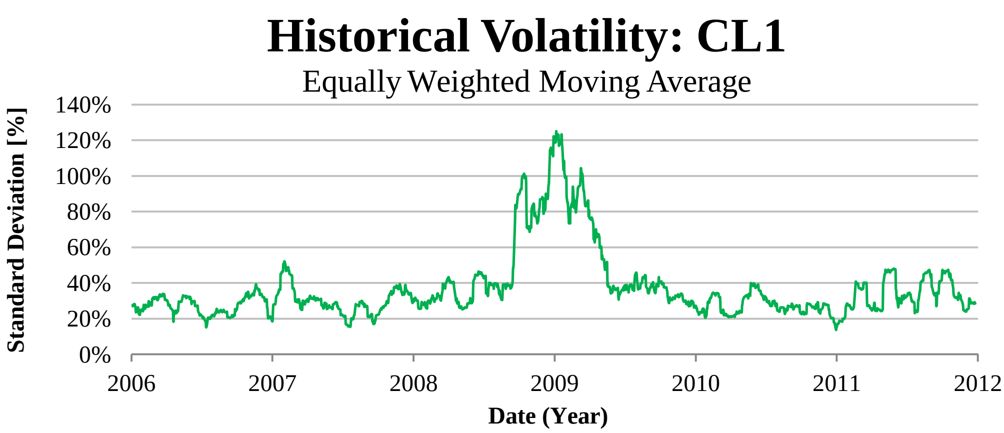

Figure 11 - Standard Deviation of Oil Price returns, using an Equally Weighted Moving Average (based on the returns of the previous 22 days). Results are presented in a yearly basis.

Figure 11 suggests significant changes in volatility during the period 2006-2011. This is not by any means surprising taking into consideration the abrupt changes in oil prices that were already analysed when we presented Figure 6. In fact, while 2006 returns have exhibited an unusually low level of volatility (yearly standard deviation of approximately 28%), the returns on the first contract prices during 2008 have exhibited roughly 62% of yearly standard deviation. In 2009, the volatility remained high followed by a sharp decline until 27.57% in 2010. In the last year of our analysis, the volatility of the returns on the CL1 contract was approximately 35%. For the total length of the considered period (2006 - 2011) the yearly volatility of the returns was 41.7%. We can then say that oil markets were very unstable in this period.

The next figure (Figure 12) presents a comparison between the observed volatilities for the considered period and the volatilities implied by the model we have built, according to the average time-to-maturity. The latter are computed using the formula provided by Cortazar and Schwartz (2003), applying Ito's lemma:

$$dF(S,y,v,T) = F_S dS + F_y dy + F_v dv + F_T dT + \frac{1}{2} F_{SS} dS^2$$

from where $\sigma_F^2(T)$ can be obtained using the following formula:

$$\sigma_F^2(T) = E\left[\left(\frac{dF}{F}\right)\right]^2$$

$$= \sigma_1^2 + \sigma_2^2 \frac{(1-e^{-kT})^2}{k^2} + \sigma_3^2 \frac{(1-e^{-aT})^2}{a^2} - 2\sigma_1\sigma_2\rho_{12}\frac{(1-e^{-kT})}{k} + 2\sigma_1\sigma_3\rho_{13}\frac{(1-e^{-aT})}{a} - 2\sigma_2\sigma_3\rho_{23}\frac{(1-e^{-aT})(1-e^{-kT})}{ak}$$

Representing the results under the form of a chart, we have:

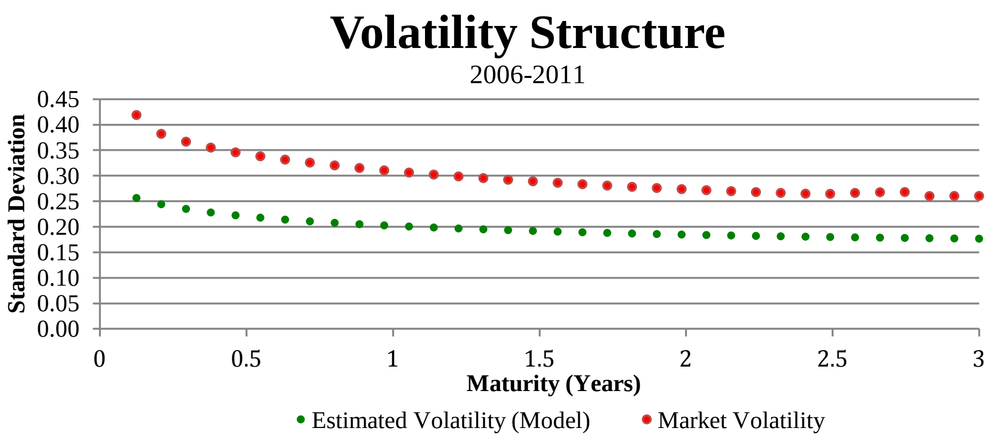

Figure 12 - Comparison of the volatility of Future returns implied by the implemented model with observed volatilities. The comparison is performed for different average maturities of the contracts.

We can observe a decline of Futures return's volatility when maturity is increasing, considering both historical and model figures. This is an expected result since it is an indicator of the existence of mean reversion in oil prices: the expectation that the prices will revert to a long-term level implies a decreasing volatility for increasing maturities (Samuelson, 1965) (Cortazar & Schwartz, 2003). In the case of random-walk price behaviour, volatility of Futures prices should be equal to the volatility of spot prices. Therefore, while historical figures suggest that the mean-reversion specificities of our model are appropriated; our findings are also theoretically consistent with this key characteristic of commodity price behaviour.

Despite this very positive aspect of our estimations, we can also see that the historical and modelled volatilities are not very similar. Model implied volatilities are lower than the observed ones and the magnitude of these differences is higher for the first contracts. This can be the consequence of the fact that our model implementation has predicted quite low volatility for the first two state variables (S and $\hat{y}$) compared to our expectations. Also, the estimated correlation between the first and third state variables, $\rho_{S\hat{v}}$, was unexpectedly high and negative. Concerning these three parameters ($\sigma_S, \sigma_{\hat{y}}, \rho_{S\hat{v}}$), our predictions also are quite different from the results of Cortazar & Schwartz (2003), even though the time periods considered for analysis are different. As these results are highly relevant for the option pricing we will perform in the next chapter, we will return to the analysis of these parameters later on. However, at this point, one should anticipate that these results might have an undesirable effect in the approximation of the value of the calendar option.

Originally, we would expect the volatile behaviour of oil price returns to be replicated by our model in a more exact way. However, considering the fact that this model is designed to capture price movements and not volatilities, and that volatility of Futures returns is not an input in the estimation, we believe this is not a very significant shortcoming of our findings. Regarding this issue, it is important to recall at this point the fact that, in the iterative procedure, the correlations and volatilities of state variables are computed from the time series of the estimated values of these state variables. We believe this might probably have had an important influence on the estimation of parameters due to the fact that circular references are difficult to handle by the optimization procedure.

3.7 Conclusion

Before moving to the last chapter of our study, which is dedicated to option pricing, we would like to conclude this long section on crude oil pricing models highlighting some important elements. We mainly focus on the aspects that are related to the technique and its implementation, that were employed to estimate the parameters necessary for the purposes of this study.

Firstly, it is important to stress the fact that the optimization program described in the Cortazar & Schwartz (2003) is extremely similar to the more complex and sophisticated Kalman filtering technique. This is a particular method used in digital computer of control system and avionics to extract a signal from a long sequence of noise. However, recently it has been applied even more in the field of finance and economics (Arnold, et al., 2008). E. Schwartz in his paper of 1997 used this technique and also Cortazar and Schwartz (2003) employed the Kalman technique to compare the results of their model with the ones that arise from this more demanding procedure. The technique we employed in this paper is just an easier alternative to the Kalman algorithm for estimating parameters. The main difference between the two the procedures, which according to Cortazar and Schwartz has a limited effect on the efficiency of the results, is related to the estimation of the volatilities and correlation among the three state variables. In the more advanced Kalman filter estimation procedure, the volatilities and correlation are also jointly estimated in the optimization program, whereas in our technique they are estimated using the time series of the three state variables. This can be a reason for the differences between the results of the two techniques.

Despite the fact that Cortazar & Schwartz designate the spreadsheet implementation of their three-factor model as an easy process, we believe that some steps of the employment of this technique are not completely well defined. In this sense, we think there is some lack of information on how to concretely implement the spreadsheet procedure. It is especially not clear how to deal with the circular reference in the algorithm. As we have just pointed out, the volatilities and correlations (which are part of the set of parameters Ω) are extracted from the time series, creating recursive iterations. We have tried different ways of avoiding this problem without success. The existence of circular references creates difficulties in the minimization process particularly regarding the speed of calculations. An alternative could be to program some codes, for example in VBA, in order to limit this iteration problem, however, by doing this we believe that this technique loses one of its advantages. It is also important to emphasize that, for users that are not familiar with estimation techniques, the implementation of this model in a spreadsheet might still require a certain effort.

There is an additional element that influences the results of our study: in order to start the iteration procedure to estimate parameters we need to provide the model with some reasonable initial parameters. These initial parameters have a significant impact on the output of the optimization procedure. In effect, using different sets of variables $\{\Omega'\}$ as initial parameters it is possible to reach significantly different results. At the beginning of the procedure we had difficulties in finding a reasonable output. By contrast we also found other sets of parameters that provided very similar results regarding the level of estimation errors, when compared to the one presented in this section. Therefore it is also relevant to note that, for very complex optimisation processes, the optimizer provides solution for the problem that might not be global minimum, but only local optimum. This is the reason why we recommend using different initial starting parameters and then comparing among them the diverse results obtained. Using different starting parameters permits to start from different points of the estimation error function, increasing the chances to find the global minimum.

We already pointed out that the usage of more contracts could permit to incorporate more information in the model. However, we consider that, as some of these contracts have low liquidity, when used to estimate parameters they might negatively impact the accuracy of the results. Of course it is easy to understand that depending on which contracts we use, our results will most likely fit them. The larger the number of contracts used, the higher is the number observations that need to be used in the spreadsheet. This would definitely slower the model and lead to an increase of the likelihood of sudden stop in the procedure.

Lastly, when using the Cortazar & Schwartz estimation procedure we will not be able to analyse the statistical significance of the different parameters, which is possible when employing the Kalman filtering procedure. This would permit to have a deeper discussion on the significance of the parameters and would allow setting stronger conclusions on the behaviour of crude oil price. We would be able to test some hypothesis on the different parameters and understand their importance.

Beside these points where we believe the model might be improved, we consider that this is a helpful tool particularly for practitioners, who are not familiar with advanced estimation techniques. This could be a good starting point for understanding how more sophisticated estimation procedures work.

4 Pricing Calendar Spread Options

This last chapter is entirely dedicated to the core topic of this study. After having introduced and discussed the diverse commodity markets, with a particular regard to the crude oil market in the US, we focused on pricing models and on a particular technique for estimating parameters. What we have been analysing so far was definitely necessary in order to have a solid and reliable background of the petroleum market. This allows us to move to the more complex world of options; we now dedicate our efforts to the theories and techniques that can be used to price calendar spread options in the oil industry, applying one of them to a concrete option pricing problem.

We start this chapter with a brief introduction and recapitulation of the basic concepts regarding option theory and the importance of this financial product for risk managers and other market participants. Afterwards we focus on calendar spread options traded in the oil market, defining them and highlighting their characteristics and importance in the energy industry.

Then, we dedicate a subsection to the more technical part of pricing options according to the Black-Scholes and Merton model (Hull, 2008), which is considered to be the starting point for closed-form solutions. Later, we summarize part of the available literature regarding analytical methods to price calendar spread options, which was inspired by the Black-Scholes and Merton model. We focus on the closed-form solution proposed by Kirk (1995), Carmona & Durrleman (2003) and Bjerksund & Stenslund (2011). These are considered to be good approximations for valuing spread options. As we will explain in detail in the subsequent sections, the literature in spread option pricing is broad. However, this is still an open topic for lecturers and financial practitioners, which keep working to reach the best closed-form solution possible. This is the reason why we decided to implement the Monte Carlo simulation to find the prices of such derivative products.

After having introduced the Monte Carlo technique and explained the necessary steps to price an option (including how to generate the paths for the two different Futures contracts), we apply our knowledge to an empirical study, as we did for crude oil, pricing a calendar spread option. Lastly, we interpret and comment on our results. The findings of Chapter 3 are very important, since they are used as input for the crude oil one-month calendar spread options pricing model.

4.1 Introduction to Option Theory

4.1.1 Fundamentals

The aim of this subchapter is to provide an introduction to the complex world of options and to the use that financial practitioners can make of them. As we defined in Chapter 2, a *call* option gives the right to buy the underlying asset at the reference price; obviously the holder of the option will exercise his right only when the option is *in-the-money*, meaning that the current value of the underlying asset (S_T) is higher than the *strike* price (K)[24].

It is useful to describe a European option in terms of its payoff; a long position in a European *call* option has the following payoff at maturity:

$$\text{Payoff}_c = \max(S_T - K; 0)$$

This expresses the fact that the *call* option is exercised only when $S_T > K$. Additionally the payoff of a European *put* option for the buyer has the subsequent form, at expiration date:

$$\text{Payoff}_p = \max(K - S_T; 0)$$

Hence a put option it is only exercised when $S_T < K$. The buyer of a *call* option expects the value of the underlying asset to rise, whereas the buyer of the put option anticipates a fall in value of the underlying security. We can easily see, by looking at charts below (Figure 13), that the payoffs at maturity of a call and a put option are not linear, which increases the degree of complexity in computing the option values.

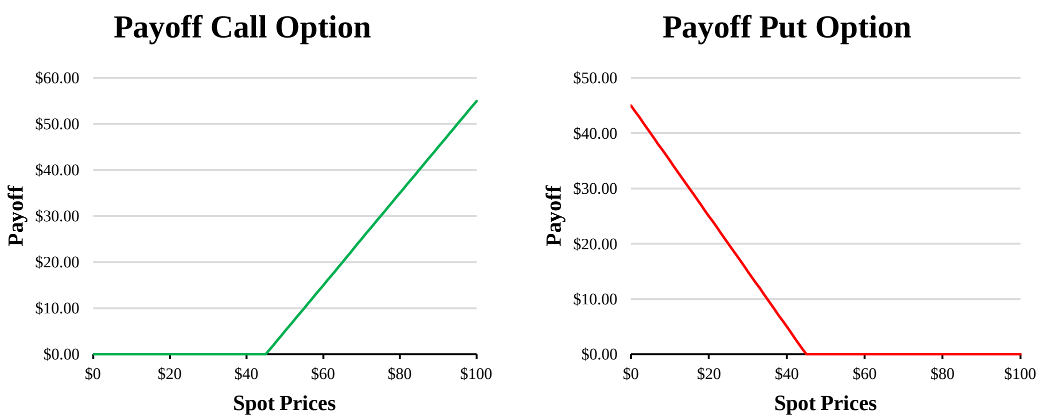

Figure 13 - Call option payoffs at maturity with strike price K = $45 and Put option payoff at expiration with strike price K = $45

4.1.2 Who is interested in the value of options?

The option market, as well as the Futures and forwards markets, are extremely important for financial actors, since they permit to hedge against financial sources of risk. Options can offer this protection, since they permit traders and companies to secure their payoffs in the eventuality of an unfavourable and unexpected price evolution. The protective effect of options helps risk managers in balancing the risks of the company, achieving more stable and solid cash flows. Therefore, in energy markets,

24 An option is said to be *out-of-the-money*, when the strike price is higher than the spot price at expiration. *At-the-money* options have the characteristic that the reference price and the spot price at maturity are identical, therefore the payoff is neutral.

options allow companies that are active in the industry to hedge their exposure to raw materials that are necessary, for example, to produce power or heating, or to secure against drops in price. The same reasoning can be expanded to the others commodity markets, considering options as an insurance policy. This is one of the most important reasons for buying options in energy markets.

A second reason for using options in financial markets is for speculation purposes. In fact, when taking a long position in a *call* option on a single underlying the buyer has the advantage of making an unlimited profit when the market raises and it is unaffected when the market turn downwards. Therefore, it is crucial for the buyer to understand and predict the movement of the underlying asset in order to subscribe the right contract (level and time) and profit at maximum of the characteristic of the option. This is one of the reasons why we decided to include and deeply discuss pricing models in our study and not focusing only on pure option price models, which assume a simplified price evolution of the underlying. In theory, an accurate approximation of the future evolution of prices, allows for more accurate estimations of the real values of options. This also permits us to clarify the difficulties that practitioners face in their activities, when they need to take crucial decisions.

There is another motive for following the price behaviour of options. In fact, options are a great source of information for market players. Different measures and parameters can be inferred from the price at which options are traded. One of the most useful measures that can be extracted from the option prices is the implied volatility. This parameter is simply the volatility of the underlying security that permits to match the current market price of the derivative financial instrument according to an option pricing model. In simpler terms, when using the Black-Scholes model, we present later in this chapter, the implied volatility is the parameter, which yields the theoretical value of the option to the actual price of the option observed on the market. This is the only parameter in the option pricing formula, which cannot be directly observed. Unfortunately, it is not possible to inverse the function of the Black-Scholes model and to express the volatility as a function of the spot price and other parameters. We have to use an iterative search procedure in order to find it (Hull, 2008).

This is an important source of information for financial practitioners, because the implied volatility is defined as a *forward-looking* measure. In effect to compute the implied volatility we do not need the complete time series of the underlying asset, but we just need the current value of the other elements for pricing (i.e. spot price, strike price, time to maturity and risk free rate in case we are using the Black-Scholes model). In the previous section, when we estimate the parameters for the oil market, we computed the different volatilities according to the estimated time series of the state variables or using the volatility formula derived from the Futures expression.

We also have to stress the fact that sometimes there is no option market for a specific asset meaning that traders and risk managers might have to use the historical time series to compute volatilities and correlations.

Moreover it is well know that implied volatility does not have a flat structure, meaning that it is not constant for different levels of the strike price. Actually when computing the implied volatility from options using the Black-Scholes model on a particular day for a specific maturity, with the same underlying but different reference prices, the implied volatility structure assumes an upward looking parabola shape, which is known as the *smile* of the implied volatility. In fact at-the-money options have relatively low volatility, but it increases when the spot price moves far from the strike price (Hull, 2009). There are a couple of reasons for this behaviour, in effect when using the Black & Sholes option pricing model we assume that the volatility is constant and the price of the underlying asset does not experience jumps. Of course in the real market these assumptions are far from being realistic, and that is why traders and risk managers prefer to talk about volatility *skew*. In effect they claim that volatility decreases as the strike price increases. This is also true for options based on Futures contracts written on crude oil. In the case of a stock option a possible reason for such behaviour of volatility is linked to leverage and the capital structure of the company (Hull, 2009). In effect, if the equity decreases its value the ratio debt-to-equity increases, which makes the equity riskier and so its volatility increases. On the contrary if the equity increases in value, the ratio debt-to-equity decreases and the volatility of the stock it is reduced. Consequently the implied volatility has a downward curve shape. If we extend the analysis of volatility also for different maturities we are able to construct the volatility surface.

From the Black-Scholes option-pricing model the only parameter we can extract is the implied volatility. In the case we employ closed-form solution to approximate the price of spread option we are able to compute the implied correlation between the two underlying assets that compose the spread. As for the implied volatility, the implied correlation is a forward-looking measure. Therefore we can see the importance of spread options in energy markets. For instance this derivative financial instrument can provide us with a measure of the level of correlation between the crude oil and the heating oil, which has significant importance for production for refinery companies. Borovkova and Permana (2010) studied the volatilities and correlation for certain Asian and spread option on the NYMEX market. Their findings suggest that volatilities have a skew form, but more surprisingly correlation similarly have the same skew form. We will use this concept of implied measure when analysing the pertinence of our results.

4.1.3 Zoology of Options

From the previous general definition of options it is possible to infer that the underlying asset which options are written on, can be any financial instrument. For example, an option can be written on an interest rate, currency exchange or a Futures contract. In this study, we analyse options that are based on the difference (*spread*) between the prices of two Futures contracts of the same commodity but with different expiration dates. These are known as a calendar spread options.

Options are not only classified according to their underlying assets, which permit only to identify the market where they are traded. They are also classified according to their exercise characteristics. When defining options we omitted the discussion on the different possibilities to exercise them. The simpler ones are *European* options, which can be exercised only at their maturity date. Instead, an *American* option can be exercised at any time before the expiration date. Due to this specific characteristic it is more difficult to analyse and model American options compared to European ones, however, some proprieties of American options can be still deducted from the simpler European-style. A considerable amount of the options that are traded on exchanges are American-style (Hull, 2008), more than European-style. In theory an American call option, which does not pay any dividend will not be exercised earlier than expiration (since the underlying asset price is not cap), so it has the same value of the European options. However, when paying dividend, it might be beneficial to exercise the American option earlier (before final *ex-dividend* date, (Hull, 2008)). An American put option has always a greater value than a European put option, due to the possibility to use the intrinsic right to exercise it at any time before maturity is valuable.

Lastly, there is a great amount of options, which do not exactly fit the definition of *plain vanilla* options. In the oil industry a significant amount of exotic options[25] are traded as in other financial markets. In the next section we explain in detail calendar spread options, which are the focus of this study, but those are only the pick of the iceberg.

4.2 Calendar Spread Options

In this section, we provide a general overview of the commodity markets spread options, with particular emphasis on the type of option we are going to price: calendar spread options. After providing the reader with basic knowledge on options between two different commodities, we focus on describing calendar spread options in terms of definition, payoff, settlement types, strike prices, importance and purposes of using this derivative product.

Spread options are multi-underlying options that are very popular in commodity markets. There are two broad types of spread options (Geman, 2005): spreads between two commodities and calendar

[25] Examples of exotic options are: basket/barrier option, Asian options or swing options.

spreads. Regarding spreads between two commodities, a spread option is an option whose payoff depends on the price spread between two correlated underlying assets. It can, thus, be regarded as a quality spread, usually rewarded by a premium (rebate) paid (received) by the buyer of a Futures contract when the physical commodity that is delivered is of a higher (lower) grade (Geman, 2005). The payoff at maturity of a spread call option between two commodities is typically:

$$c_T = \max\left(q_1 S_{1,T} - q_2 S_{2,T} - K, 0\right)$$

- Where q_1 and q_2 represent quantities and are positive constants;
- K is the strike price.

The following figure illustrates the payoff of a spread call option and a spread put option, considering two underlying securities S_1 and S_2, quantities q_1 and q_2 equal to 1 and strike price K equal to zero:

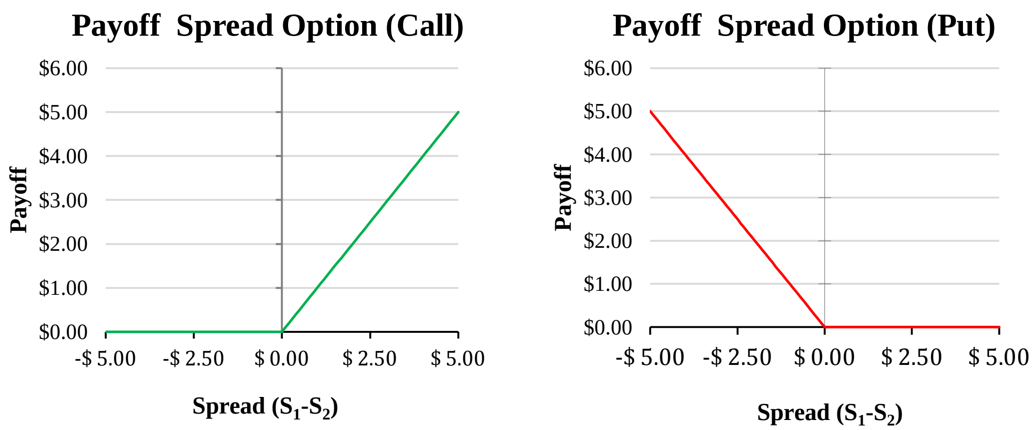

Figure 14 – Payoff at maturity of a Spread Call Option and a Spread Put Option, considering a strike price equal to $0.00

As we can see, and in opposition with plain vanilla call options, the value of the underlying of a spread option, at maturity, can be negative. In case of strike prices equal to zero, negative values of the spread provide put options with positive payoffs.

Spread options between two commodities are very important in energy markets and are amongst the most traded instruments in the commodity world. In energy markets, spreads between two commodities can be crack spreads, spark spreads or dark spreads. A crack spread is the differential between the prices of refined products and the price of crude oil input and it is heavily traded on the NYMEX and over-the-counter (OTC). A spark spread represents the difference between the price of electricity and the price of the corresponding quantity of primary fuel, taking into account the conversion rate or "heat rate". It is very popular in Europe and traded OTC. A dark spread is a spread between coal and electricity.

Calendar spread options were introduced in the NYMEX in June 2002. They are our focus in this study and differ from the above mentioned spread option between commodities because they involve only one underlying commodity: they are based on the differences between the prices of two Futures contracts of the same commodity, but with different maturities. A standard calendar spread call option has generally the following payoff at maturity (Geman, 2005):

$$c_T = \max\left(F_{T,T_1} - F_{T,T_2} - K, 0\right)$$

- Where $T < T_1 < T_2$, which means that the option is a Futures calendar spread.
- K is the strike price that is calculated according to the rules defined in the NYMEX rulebook.

Regarding Light Sweet Crude Oil, the NYMEX offers the possibility of investing in *WTI Crude Oil Calendar Spread Options* and also in *Crude Oil Financial Calendar Spread Options*. While the first contract is settled into Futures contracts, the second one has a financial settlement instead. At exercise, and regarding the *WTI Crude Oil Calendar Spread Options*, the buyer of a call option will obtain a long position in the Futures market for the closest month and a short position for the other month. One the other side, the buyer of a put option will obtain a short position in the Futures market for the closest month and a long position in the same market for the farther-dated month (James, 2003). The financially settled spread option does not require the actual delivery of the underlying assets. Instead, the investor will receive a cash amount that corresponds to the associated position at expiration.

Both options are European-style and expire one business day before the expiration of the first expiring Futures contract in the spread, at the close of trading. Each of the mentioned calendar spread options are options on the price differential between two different delivery dates for the Light Sweet Crude Oil. The corresponding underlying Futures are, thus, the Light Sweet Crude Oil Futures (CL), that is, the same contracts we have used as an input for our model in Chapter 3. Both WTI Crude Oil Calendar Spread Options and Crude Oil Financial Calendar Spread Options have several listed contracts according to the available lengths of the spread: 1-month spreads, 2 month-spreads, 3-month spreads, 6-month spreads and 12-month spreads.

One interesting thing to note about calendar spread options is that strike prices can be negative. This arises from the specifications of strike prices calculations and happens whenever the price curve of oil is in contango, that is, the price of the short-term delivery date is lower than the price of the relatively longer-term delivery date (James, 2003)[26].

[26] In case the price of the contracts delivering short-term is higher than the price of the relatively longer-term delivery dates, the market is said to be in *backwardation*.

Calendar spread options are essentially a correlation product: all the sensitivities of the option (that is, the sensitivity to the underlying Futures prices and Futures volatilities), with the exception of the correlation between the two underlying Futures contracts, can be hedged using other Futures contracts and options on Futures.

Recall from Chapter 1 and Chapter 2 that prices in energy markets can be very volatile due to the occurrence of news-making and other unexpected events. This will also have a reflection on the price spread between Futures contracts with different maturities. The existence of oil calendar spread options offers market participants the possibility to mitigate the considerable price risk that often exists between contract with different expiration dates (James, 2003). Moreover, they also create opportunities for investors to take advantage from the evolution of spread prices in the oil markets.

This financial instrument can be used for several purposes. One of the most interesting is the possibility of being used to maximize the benefits of storage facilities. Therefore, they are very appealing for those who have storage capacity. Considering the case of a contango period, a company with excess storage capacity can profit from the differential in prices, purchasing calendar spread call options, which means creating a long position in the short-term contract and a short position in the deferred contract. In this way, cheaper oil will be stored and delivered for a higher price in a further date[27]. In opposition, in backwardation periods, storage facilities should sell calendar spread puts. This is an effective way of profiting from the more expensive farther-dated prices. However, it requires that the company has the asset as a backstop, otherwise it will not be able to sell the put.

Another way of using calendar spread options to optimize the utilization of storage facilities is to build inventories (buying calendar spread calls).

Calendar spread options also provide benefits for hedgers. For downside protection, sellers of oil seek to buy puts while oil buyers seek to purchase calls, when the market is in contango (James, 2003). Another advantage of using calendar spread options is the fact that, when a steeply backwardated market exists, buying calendar spread call options can reduce the costs of buying back a hedge (that can become considerably expensive due to appreciation in value that occurs when the expiration date is approaching).

Besides these usages, calendar spread options can, as expected, also be used to speculate in the market and to capture volatility in the oil market.

[27] It is important to stress the fact that the size storage costs is a crucial element here, since costs that are higher than the differential of prices will make this operation unprofitable.

Calendar spread options are also very popular in other energy markets rather than oil, mostly due to seasonality (Geman, 2005). They are also widely used in other commodity markets. For the purpose of this paper, we will focus on pricing crude oil one-month calendar spread options

In the following sections we introduce the Black-Scholes and Merton model, the Margrabe Formula and other closed-form solutions for spread options. Before the empirical study, we present the Monte Carlo Simulation technique that is of crucial importance for the final purpose of our study.

4.3 The Black-Scholes and Merton model: A Building Block Model in Option Pricing

In this study we restrict our observation to one-month calendar spread options, whose underlying is the difference between two Futures contracts written on the same commodity – in our case light sweet crude oil. However, some of theories and results, we present along the way, can be modified and they might be applied also to more complex basket of underlying assets (Borovkova & Permana, 2010). For example, they might be used for pricing a crack spread option or a spark spread option. For the sake of simplicity we decided to focus on European-style spread options, as we already suggested they are easier to evaluate compared to American-style option.

Before starting our review of the available literature for pricing spread options, we want to spend some words on the first tool investors can use to price options, meaning the famous Black-Scholes formula. This is the building block of option pricing and its one of the most important model in the financial theory. We now give a brief overview of this, which is the starting point of our literature review on spread options.

As we explained earlier, when the investor exercises a European call option he receives the following payoff at maturity (T):

$$\text{Payoff}_T = \max(S_T - K; 0)$$

However, what we are interested in; it is the price of the call option (c_t) with $t < T$. Black, Scholes and Merton (Black & Scholes, 1973) and (Merton, 1973) proposed a technique to value derivative contracts. Today their studies are still highly appreciated, as much as that in 1997 Merton and Scholes won the Nobel Prize for their researches on pricing derivatives. The basic idea behind their model is that uncertain cash flows, like the one of options, can be replicated by a self-financial strategy. This means that the value of the option today must be equal to the cost today of the replicating portfolio strategy. If this would not be true, rationale investor would take advantage of the mispricing situation and immediately set up an arbitrage strategy. Therefore one of the assumptions of the Black-Scholes model is that there are no riskless arbitrage opportunities, the market is complete instead. Black and

Scholes heavily relied on the solution of their parabolic PDE (Black & Scholes, 1973) and (Carmona & Durrleman, 2003).

In order to explain the PDE formula we assume that the underlying asset – e.g. a stock – follows a one-factor Geometric Brownian motion as:

$$dS_t = \mu_S S_t dt + \sigma_S S_t dW_t$$

The price of the call option c_t must depend on the value of the underlying asset S_t and on the current date t. Therefore, assuming c_t to be the value of the call option and applying Itô's lemma, developed in 3.2 The Mathematical Background section, we can expressed the value of the call in function of its set of inputs:

$$dc_t = \left(\frac{\partial c_t}{\partial S_t}\mu_S S_t + \frac{\partial c_t}{\partial t} + \frac{1}{2}\frac{\partial^2 c_t}{\partial S_t^2}\sigma_S^2 S_t^2\right)dt + \frac{\partial c_t}{\partial S_t}\sigma_S S_t dW_t$$

Then the riskless arbitrage portfolio (π) replicating the option will have the following structure:

$$\pi = -c_t + \frac{\partial c_t}{\partial S_t} S_t$$

This mean that the investor goes short one unit of the derivative and at the same times he buys $\frac{\partial c_t}{\partial S_t}$ units of the underlying asset. Consequently a change in value of the portfolio is given by:

$$d\pi = -dc_t + \frac{\partial c_t}{\partial S_t} dS_t = \left(-\frac{\partial c_t}{\partial t} - \frac{1}{2}\frac{\partial^2 c_t}{\partial S_t^2}\sigma_S^2 S_t^2\right)dt$$

It is easy to note now that portfolio does not depend anymore on the Wiener processes of the stochastic differential equation (dW_t). This simply means that the portfolio is now riskless and consequently its return has to be equal to the short term risk free rate: r (Hull, 2008). In more formal terms we can rewrite the differential equation of the return of the replicating portfolio as:

$$d\pi = r\pi dt$$

If we now substitute the different variables with the results above, we obtain:

$$\left(\frac{\partial c_t}{\partial t} + \frac{1}{2}\frac{\partial^2 c_t}{\partial S_t^2}\sigma_S^2 S_t^2\right)dt = r\left(c_t - \frac{\partial c_t}{\partial S_t} S_t\right)dt$$

which it can be rewritten in the following way[28]:

[28] This procedure is basically the same we used in 3 for defining the closed-form expression of the price of Futures contracts. This PDE is particular for option pricing, the one of the Black-Scholes and Merton model has the same form but for a general function $f(\cdot)$.

$$\frac{\partial c_t}{\partial t} + \frac{\partial c_t}{\partial S_t} r S_t + \frac{1}{2}\frac{\partial^2 c_t}{\partial S_t^2}\sigma_S^2 S_t^2 - r c_t = 0$$

This is the Black-Sholes PDE with the boundary condition $c_t = \max(S_T - K; 0)$.

An alternative method to compute the value of derivative products, which consistently leads to the same result, consists in discounting the uncertain future cash flow for a probability structure, which has the particularity to be risk-neutral (Carmona & Durrleman, 2003). As defined earlier in this study, the structure of risk neutral probabilities consists of the set of probabilities that allows excluding possible source of risk from future cash flows. One example of such source of risk is the market risk. This technique it is less complex and more straightforward than solving the previous PDE (Carmona & Durrleman, 2003), which sometimes require lot of effort or even does not lead to a nice closed-form solution. We do not want to go into details, as we did in the section dedicated to the risk adjusted probabilities, but we just limit ourselves in repeating the results of the previous subchapter, reminding that the drift of the underlying asset under risk-neutral probabilities is equal to the short term risk free interest rate. However, we obviously need to know the evolution of the underlying asset under these risk neutral probabilities, too.

We already explained that the theory, which explains the following passage (or transformation) from historical probabilities to risk-neutral measures, is known as the Girsanov theorem. According to what we have just suggested the value of the call option c_t can be expressed in the following way:

$$c_t = E_t^Q\left[e^{-r(T-t)}\max(S_t - K; 0)\right]$$

Both techniques lead to the well-known formula of the Black-Scholes option pricing model, since the stock price is assumed to be log-normal distributed. Therefore we have:

$$c_t = S_t \Phi(d_1) - K e^{-r(T-t)}\Phi(d_2)$$

where:

$$d_1 = \frac{\ln\left(\frac{S_t e^{r(T-t)}}{K}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

$\Phi(\cdot)$ is the cumulative distribution of the standard normal distribution, $\mathcal{N}(0,1)$.

We have to point out that the discount rate (r) is assumed to be the risk free interest rate, if we decided to include dividends or even more relevant in energy markets the convenience yield, we might need to adjust this riskless interest rate.

4.4 The Margrabe Formula

The Black-Scholes and Merton model we have just describe, is a brilliant finding that has delivered a huge contribution to financial markets. However, this model is not suitable for the purpose of this study, since it only accounts for options with one underlying asset. As we already pointed out, the underlying of a calendar spread option is the price differential between two Futures contracts written on the same commodity but with different delivery dates. Therefore, we need to extend the Black-Scholes model in order to account for this specific characteristic in our option pricing model.

William Margrabe (1978) inspired by the revolutionary work of Black, Scholes and Merton (1973) derived a formula to price an *exchange* option[29], which it is a special type of a spread option with $K = 0$ (Poitras, 1998). Formally, the payoff at maturity of this type of option is the following:

$$\text{Payoff}_T = \max(S_2 - S_1; 0)$$

Assuming that both underlying assets follow a risk neutral Geometric Brownian motion and the two Wiener process are correlated between them, replicating the procedure used by Black-Scholes in their study, Margrabe was able to attain the following formula[30]:

$$\pi_t^{\text{Margrabe}} = S_{2,t}\Phi(d_1) - S_{1,t}\Phi(d_2)$$

This Margrabe solution for exchange options is the unique pricing formula in closed-form in the *Black-Scholes* style. This paper had a great impact on the financial community; in particular Carmona and Durrleman (2003) raised some important remarks regarding this formula. Firstly, it permits a good implementation and use of the Girsanov's transformation. Secondly, it is relatively straightforward to account for convenience yield or dividends; we just need to adjust the drift of our Geometric Brownian motions, which model the two underlying assets. Thirdly and most interesting the Margrabe formula does not depend on the risk free interest rate. Since we are in a risk neutral world, both underlying assets have the same drift. Lastly, this formula is definitely an advantage for the financial industry, because of its solid background and its user friendly form. Unfortunately, the Margrabe formula works only if the strike price (K) is equal to zero, which is rarely the case. In fact, since the reference price is equal to zero, it permits to compute the double integration implied in the option and thus the value of the option is unique (Deng, et al., 2008). Therefore this closed form solution cannot be applied to more sophisticated cases ($K \neq 0$), because the exercise boundary becomes nonlinear, which does not give the possibility to find a close-form solution to the double integration (Deng, et al., 2008).

[29] This is a contract, which permits to change $S_{2,t}$ for $S_{1,t}$, if the former outperforms the latter. The option is simultaneously a call option on $S_{2,t}$ with strike price $S_{1,t}$ and a put option on $S_{1,t}$ with reference price $S_{2,t}$.

[30] For more details on the method to derive the Margrabe formula please refer to the original paper (Margrabe, 1978).

4.5 Most Recent Closed-Form Solution for Spread Options

Margrabe formula is an excellent building block for pricing spread options in closed-form. However, we highlighted some of the possible drawbacks of using this formula. In this section we try to shine a light on the contemporary literature dedicated to analytical methods, which seeks to obtain good approximation in closed-form solution for estimating a fair price of spread options.

4.5.1 Kirk's Approximation

In his studies during the '90, E. Kirk was one of the firsts to suggest a closed-form solution for pricing spread options – on Futures or forwards contracts – when the strike prices was assumed to be different from zero. The formula, which uses as starting point the earlier findings of Margrabe (1978), improves them, accounting for small strike values (Alexander & Venkatramanan, 2009).

The idea behind this financial product is to exercise the spread option only if the value of the long assets $\left(S_{1,t}\right)$ exceeded a predetermined power function of the short asset $\left(S_{2,t}\right)$ (Bjerksund & Stenslund, 2011). The second asset is combined with the fixed strike price into a single asset, which is then log-normally distributed (Deng, et al., 2008). The formula suggested by Kirk for an European call option on the spread $S_{1,t} - S_{2,t}$ (Bjerksund & Stenslund, 2011) is:

$$\hat{c}_t^{Kirk} = S_{1,t}\Phi\left(\frac{\ln\left(\frac{S_{1,t}}{S_{2,t}+Ke^{-r\tau}}\right)}{\sigma_{Kirk}} + \frac{\sigma_{Kirk}}{2}\right) - \left(S_{2,t}+Ke^{-r\tau}\right)\Phi\left(\frac{\ln\left(\frac{S_{1,t}}{S_{2,t}+Ke^{-r\tau}}\right)}{\sigma_{Kirk}} - \frac{\sigma_{Kirk}}{2}\right)$$

where:

$$\sigma_{Kirk} = \left(\sigma_1^2 - 2\rho\sigma_1\sigma_2\frac{S_{2,t}}{S_{2,t}+Ke^{-r\tau}} + \sigma_1^2\left(\frac{S_{2,t}}{S_{2,t}+Ke^{-r\tau}}\right)^2\right)^{\frac{1}{2}}$$

Even if it's unknown its exact origin (Alexander & Venkatramanan, 2009), the formula gives a relatively good approximation of the price of spread options. This is the reason why this formula is well known and used by financial practitioners (Deng, et al., 2008) as standard practice (Bjerksund & Stenslund, 2011). The structure of the formula reminds us the Black-Scholes form, which eases its understanding.

4.5.2 Carmona and Durrleman Approximation

Lately, in their prodigious and detailed paper of 2003, Carmona and Durrleman derived a good formula to approximate the value of a spread option with a reference price, which differs from zero. They develop even further Margrabe's studies and provide a more sophisticated way to estimate value of a spread option with considerable precision (Carmona & Durrleman, 2003). This solution is still interesting for the purpose of this paper since both underlying asset dynamics follow a Geometric

Brownian motions, as we assumed in the previous sections and also because it is strictly related to the assumptions of Black-Scholes formula.

In fact, if we assume that the underlying is the spread between two indexes (or stocks) and the reference price is different from zero, according to the Black-Scholes model, the price of a spread option would be (Carmona & Durrleman, 2003):

$$c_t = E_t^Q\left[e^{-r(T-t)}\max\left(\left(S_{1,t} - S_{2,t}\right) - K; 0\right)\right]$$

this can be reformulated in the following way:

$$c_t = e^{-r(T-t)}E_t^Q\left[\left(S_{1,t}e^{\left(r-\delta_1-\frac{\sigma_1^2}{2}\right)(T-t)+\sigma_1 W_1(t)} - S_{2,t}e^{\left(r-\delta_2-\frac{\sigma_2^2}{2}\right)(T-t)+\sigma_2 W_2(t)} - K\right)^+\right]$$

This formula shows that c_t is derived from the integration of a function of two variables $(S_{1,t}, S_{2,t})$ with respect to the joint distribution of the two Wiener processes (Carmona & Durrleman, 2003).

In the case the two underlying assets are not stocks but instead they are two Futures contracts we have to set δ_1 and $\delta_2 = r$ (Carmona & Durrleman, 2003).

Thanks to their earlier studies (Carmona & Durrleman, 2003b) the authors were able to derive a formula for approximating the value of a spread option (c_t) of the previous formula. The approximation has the following form[31]:

$$\hat{c}_t^{CD} = S_{1,t}e^{-\delta_1(T-t)}\Phi(d^* + \sigma_1\cos(\theta^* + \varphi)\sqrt{T-t}) - S_{2,t}e^{-\delta_2(T-t)}\Phi\left(d^* + \sigma_2\sin\theta^*\sqrt{T-t}\right) - Ke^{-r(T-t)}\Phi(d^*)$$

It can be easily seen that if the strike price is equal to zero the formula reduces to Margrabe solution and if $S_{1,t}$ or $S_{2,t}$ equal to zero we have the Black-Scholes formula. In these cases and also when the correlation between the two underlying assets is perfectly negative or positive the Carmona and Durrleman approximation $\hat{c}_t^{CD}$ is equal to the true price c_t (Carmona & Durrleman, 2003).

This approximation permits to accurately estimate the price of a spread option even with a strike price, which is different from zero. This element definitely enhances and reinforces the use of the Carmona and Durrleman approximation when pricing such particular options.

Another reason that strengthens the usage of this approximation is the possibility of computing different risk management measures. In effect the closed-form expression of $\hat{c}_t^{CD}$ allows computing the diverse *Greek* letters, which are extremely useful for hedging purposes, since they permit to measure

[31] Appendix VIII shows how to compute the variables: d^* and θ^*.

different level of risk associated to the derivative product[32]. Traders and investors are not only interested in the price of the option, but they would like to know how the price behaves when one of the parameter changes as well. This approximation enables financial practitioners to compute these different partial derivatives necessary for having a rigorous hedging strategy. Some of the parameters of the formula are not directly observable (i.e. volatilities and correlations), thus the use of the closed-form solution permits to understand how price and hedging strategy are affected by estimation errors (Carmona & Durrleman, 2003).

4.5.3 Bjerksund and Stensland Approximation

More recently Bjerksund and Stensland (2011) derived a new approximation formula for pricing spread option, having as a starting point the Kirk's approximation we just presented. In particular they used the idea of exercising the option only when the value of the long underlying asset exceed the value of a power function based on the short asset with exponent b and scalar $\frac{a}{E_0\left[(S_{1,t})^b\right]}$. The resulting formula has the following structure[33]:

$$\hat{p}_t^B(a,b) = e^{-r\tau}E_0\left[\left(S_{2,t} - S_{1,t} - K\right)\cdot I\left(S_{2,t} \geq \frac{a\left(S_{1,t}\right)^b}{E_0\left[\left(S_{1,t}\right)^b\right]}\right)\right] = e^{-r\tau}\{S_{2,t}\Phi(d_1) - S_{1,t}\Phi(d_2) - K\Phi(d_3)\}$$

This closed form approximation of the value of a spread option is similar also to the Carmona and Durrleman one. However, Bjerksund and Stensland claim that their solution is more in line with the Black-Sholes and Margrabe models, because of the arguments d_1, d_2 and d_3 do not incorporate trigonometric functions as it is in the plain vanilla models.

4.5.4 Literature Resume

We have shown four different methods to price, or better say, to accurately approximate the price of spread options. All of the methods are in closed-form expressions, which are highly appreciated by practitioners, since their desire to achieve real solutions and simple calculations (Bjerksund & Stenslund, 2011). In fact, approximation formulae permit to quickly compute the value of the financial derivatives and simplify the analytical tractability. Option traders also prefer to employ such analytic approximations because of the closed-form formulae for hedge ratios (Alexander & Venkatramanan, 2009). This is the reason why we have dedicated greater space to this spread option theory on closed-form solutions.

It is important to highlight the fact that we decided to only include *Black-Scholes* & *Merton* related theories and formulae in our theoretical framework on spread options. Since in Chapter 3 we chose to

[32] The *Greek* letters are: $\Delta, \Gamma, \Upsilon, \Theta$ and P. They respectively measure the sensitivity of the portfolio to: price, delta, volatility, time and interest rate (Hull, 2009)

[33] For the complete derivation of the closed-form solution, please refer to the original paper (Bjerksund & Stenslund, 2011).

present prices that behave according to a Geometric Brownian motion, we decided to be consistent with this assumption and hence analyse only analytical option pricing methods, which are based on underlying assets that move according to an Itô process as in the first Black-Sholes and Merton model (Hull, 2008).

In the available literature for pricing spread options there are also different techniques that are based on different assumption regarding the evolution of the underlying assets. A model, which is also widely used to mimic the value of the underlying, is the arithmetic Brownian motion. The use of this type of model comes from the fact that spreads can also assume negative values, being the difference of two positive values. Assuming an arithmetic behaviour permits to compute the price of the option by simply solving Gaussian integrals (Carmona & Durrleman, 2003). This leads to simple closed-form solutions. However, we consider reasonable to think that the underlying assets move according to a Geometric Brownian motion as most of the literature – also no-option related – suggests. Moreover we decided not to model the spread itself, but the two securities that compose the spread, which are not normally distributed but follow a log-normal distribution.

After having enlightened the reason why we decided to use *Black-Scholes* related closed-form solutions, we discuss the different performance of the formulas, in order to individuate the one that should perform the best.

In their paper Carmona and Durrleman (2003) analyse the performance of three different methods for pricing spread options, meaning: Bachelier model[34], Kirk's formula and the Carmona and Durrleman approximation. According to their findings the Bachelier model should not be applied when compared to their approximation. In fact, this method provides the user with a value, which is always smaller than the approximation of the Carmona and Durrleman and far from the market price. This strongly sustains the use of the latter method, even if it is more complex than the Bachelier model. The main pitfall of the Bachelier model is the fact that is a single-factor model, which tries to mirror the dynamic of the difference between two assets. Therefore, it is not able to capture the real nature of the two log-normally distributed underlying assets.

Moreover they compared their technique to the Kirk's one. In this case they did not base their analysis on the value of the price but, since both techniques permit to compute with relative ease the different Greeks, they decided to focus on them. Therefore Carmona and Durrleman studied, for a given scenario, the performance of two hedging strategies based on the two different techniques. They repeat the analysis for a large number of scenarios in order to have a better understanding of the two strategies. Their findings show that the Kirk's approximation, even though providing satisfactory results, lacks of some precision compared to the Carmona and Durrleman method. Consequently the

[34] This model assumes the price of the underlying evolving according to an arithmetic Brownian motion.

Carmona and Durrleman proposition seems to be a valid and outperforming method for pricing options respect to the poorly performing Bachelier model and the quasi-satisfying Kirk's formula.

In 2011 Bjerksund and Stensland proposed their method for approximating the value of spread options. They also analysed the performance of some other different formulas. Looking at the error estimations of the Kirk's formula and the Bjerksund and Stensland approximation with respect to the Monte Carlo simulation[35], it seems that the latter method is more effective in estimating the correct value of the option. In fact the Kirk formula misprices the financial product when the strike price is close to zero or further away from it. This suggests that practitioners are better using the most recent methods instead of the Kirk formula. Bjerksund and Stensland do not limit their performance analysis of their model only to the comparison with the Kirk's method. Actually they also assessed the performance of it with respect to the Carmona and Durrleman method. The two methods are closely related; in fact, when optimising the Bjerksund and Stensland formula for the parameter a and b, the results are comparable to the Carmona and Durrleman approximation (Bjerksund & Stenslund, 2011). The findings suggest that the two models are extremely precise compared to the Monte Carlo simulation value and their difference is minim. Therefore Bjerksund and Stensland incentive the use of their closed form solution for pricing, which is simpler to implement respect the two-dimensional optimisation scheme of Carmona and Durrleman (Bjerksund & Stenslund, 2011).

Researches are still working on this challenging topic to find a perfect closed-form solution for spread options, the literature is huge regarding this topic and it quickly turns to be very complex. It is extremely challenging to compute the exact price of such derivative products and there is no clear consensus on the most reliable and performing method for approximating the value of spread options. Examples of other closed-form expressions are suggested by Deng *et al.* (2008), Trolle & Schwartz (2009) and Alexander & Venkatramanan (2009). As the different studies shown some analytic techniques achieve better results than others under some specific assumptions, while they poorly perform under other conditions.

Even though the nice proprieties of the close form solution, practitioners often employ numerical techniques to price options, which include numerical integration, Monte Carlo simulation, and fast Fourier transform (Deng, et al., 2008), (Carmona & Durrleman, 2003). Accurate results can also be achieved implementing numerical methods, for instance Monte Carlo simulation (Bjerksund & Stenslund, 2011). Even though numerical methods require more time than desirable, they often offer more precise results (Deng, et al., 2008). Bjerksund and Stensland computed the value of the spread option employing Monte Carlo simulation and used it as a benchmark to compare the different analytical solutions. However, when using this numerical technique we lose the possibility to compute

[35] The Monte Carlo simulation value was the reference point of the analysis of the two approximation methods.

hedge ratios, but on the other hand it permits for more flexibility. In fact, as Trolle and Schwartz suggested in their paper, when calendar spread options have a strike price different from zero, options should be priced performing a Monte Carlo simulation (Trolle & Schwartz, 2009). According to this aspect and since the purpose of this paper is not to have an understanding of the different Greeks, we chose to price calendar spread options using Monte Carlo simulation, which we present in the next section.

Using the Monte Carlo simulation we are able to generate the full path of the underlying assets, employing the three-factor model and the estimated parameters we derived in the previous chapter. This permits to have a better estimate of the final value of the Futures contracts for computing the expected payoff. The closed-form solution we presented before had different assumptions on the model of the underlying assets. This estimation technique allows for more assumptions. In fact, since it is a numerical method, we do not have a single formula that provide us with the final value, but for example we could add elements to the payoff – e.g. another contract – or we can cap/floor the price of oil based on previous observations. All this elements support the use of this method to price options, because it allows for more flexibility and thus users can easily modify their assumptions and immediately interpret the results of such different hypothesis. Using Monte Carlo simulation we can still compute the implied values of volatilities and correlations. In effect, we just need to slightly modify the Monte Carlo simulation model in order to transform it into an optimization model. If we implement the model in Excel we can use the add-in Solver to compute the value of the implied parameter. We need to create an objective formula, for example the sum of the squared estimation errors, and ask Solver to find the right value of the parameter that minimises this sum. The result will be the implied value of the parameter implicit in the market.

4.6 Monte Carlo Simulation for Option Pricing

The Monte Carlo Simulation is an easy to implement and very popular tool amongst financial practitioners that is widely used for financial purposes such as project valuation, portfolio analysis and option pricing. It is a procedure for randomly sampling changes in market variables and is therefore used to calculate numerical approximations of an expectation value $E[X]$. It has an immediate application to option pricing because the fair value of an option is given as the conditional expectation of a discounted payoff π:

$$U = e^{-r(T-t)}E[\pi]$$

The Monte Carlo method relies on risk-neutral valuation[36] and has theoretical foundation on the law of large numbers. According to this statistical theorem, the average of the results obtained from a large number of trials should be close to the expected value of the process. Moreover, these results will tend to become closer to the expected value as the number of trials performed increases. The idea of the Monte Carlo technique is therefore to generate a large number of sample price paths of the underlying security, compute the sample option payoffs, discount them to time t and then compute the average. The average of discounted payoffs will then be the Monte Carlo estimate of the option price at time t.

In a more formal way, let us consider the simple case of an European call option[37], with strike price K, maturity T and payoff at maturity π which depends only the underlying price at moment T, that is, $\pi = f\left(S(T)\right) = \max[S(T) - K, 0]$. The following steps are the necessary ones to implement Monte Carlo estimation technique:

I. Compute N sample paths $S_i(t)$ evaluated at times from $t = 1$ to $t = T$, where $(i = 1, \dots, N)$;

II. Compute each of the sample option payoffs π_i, that are, in this case, a function of the price of the underlying at the expiration date at each sample path, $S_i(T)$;

III. Discount each of the N sample option payoffs to time t at the risk-free interest rate that is:

$$U_i = e^{-r(T-t)}\pi_i = e^{-r(T-t)}f\left(S_i(T)\right) = e^{-r(T-t)}\max[S(T) - K, 0]$$

IV. The Monte Carlo estimate of the option price at time t is the average of the discounted sample payoffs at time t:

$$\widehat{U} = \frac{1}{N}\sum_{i=1}^{N} U_i$$

The standard estimation error can be calculated as follows. Let

$$s_{\widehat{U}}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(U_i - \widehat{U})^2$$

be the variance estimator of the sample option values. Then the standard pricing error of the Monte Carlo approximation is

[36] Risk-neutral probabilities were discussed in the Risk Neutral Probabilities section in Modelling Energy Commodity Prices, Chapter 3.

[37] The Black-Scholes model is appropriate for this situation, such that the usage of the Monte Carlo method is not really needed for pricing a European Option. However, this is just an illustrative example that aims to explain this technique in the simplest way possible.

$$\epsilon_U = \frac{s_U}{\sqrt{N}}$$

This means that the standard error reduces at the rate of the square root of the sample size. To reduce the error by a factor of ½, the number of sample paths has to increase by a factor 4. Compared to other numerical methods, as PDE solvers, this convergence is slow and demands a large number of sample paths to be generated, in order to achieve good accuracy (Burger, et al., 2008). This is the reason why practitioners usually employ approaches that aim to reduce the number of samples needed in Monte Carlo simulation. These procedures are known as variance reduction methods, as they decrease the variance of the sample option values s_U^2. The most commonly used methods are the antithetic variate method and the control variate method. Even though we are using only the first method for the purpose of this paper, we provide the reader with a compact description of both methods, as we believe this issue is very important for Monte Carlo practitioners.

The antithetic variate method is a very simple procedure that automatically doubles the sample size with a very short increase in computational time. Using the standard Monte Carlo simulation, the generated observations come from a standard normal random variable, distributed with mean 0 and variance 1 and, of course, symmetric. Thus, for each value we draw, there is a similar chance of having drawn the exact symmetric value, that is, the observed value times (-1). This means that we can legitimately give rise to a different sample path using the symmetric value $(-\varepsilon)$ for each ε number we generate. We can thus use these new figures in the exact same way we used the previous ones (ε), doubling the number of sample paths we are employing, without increasing the number of random drawings. The advantages of this method are likely to be greater the smaller the sample size (Chance, 2011).

The control variate method consists of using the Monte Carlo method not for the option itself but for the difference between the option and a similar option, called the control variate, for which an exact analytical solution is known (Burger, et al., 2008). Being V the control variate and $Z = U - V$ the difference for which the Monte Carlo method will produce an estimate, the option price approximation will be given by $\widehat{U} = V + \hat{Z}$. This technique is particularly suited for pricing arithmetically averaged Asian options, using geometrically averaged Asian options as the control variate.

For the purpose of this project, we will rely on the antithetic variate method since it is a theoretically efficient method to improve the accuracy of results. At the same time, it is not time-consuming and it is easy to implement.

We have already mentioned that the Monte Carlo simulation technique is widely used for pricing options. However, as it can be a relative time-consuming technique, it is a particularly popular solution for more complex option pricing problems such as the followings (Burger, et al., 2008):

- When the stochastic model for the underlying is complex, analytical or other numerical methods (tree, PDE) may not be available or tend to be inefficient. In this case the Monte Carlo method is usually a good alternative;
- The payoff at maturity π depends on underlying prices at different times and not only on the value of the underlying asset at time T. These are exotic options termed path-dependent. Monte Carlo methods are an excellent tool for handling these kind of options, whether they are written on a single or several underlying;
- The option is multi-asset, that is, it has multiple underlying $S_1(t), \dots, S_n(t)$. In the particular case when n >= 2, other numerical methods such as tree or PDE methods become inefficient and thus Monte Carlo methods turn to be a good pricing solution.

In this paper, we are pricing an option whose payoff depends on the spread between two different Futures prices of the same underlying commodity. We understand the Monte Carlo method as an applicable and very intuitive solution for our problem.

4.6.1 The Monte Carlo Simulation for Spread Option Pricing

The Monte Carlo Simulation method we have just described is a good solution for option pricing problems. As we have already mentioned, closed form solutions, despite being attractive for their quick implementation and for giving the possibility of computing important risk measures such as the Greeks, are not consensual between academics. Numerical techniques are more time consuming but are able to provide very satisfactory results. The Monte Carlo Simulation technique is widely used to find the prices of relatively complex options and it is also the most intuitive solution for spread option pricing problems.

Regarding spreads between two commodities, a Monte Carlo simulation of the bi-dimensional process (S_1 and S_2) is an easy-to-implement solution, but the consistency in the construction of the trajectories for both underlying assets needs to be ensured since there is a dependence structure between the two price processes $S_1(t)$ and $S_2(t)$ (Geman, 2005). That is, the price paths for both assets can no longer be independently drawn from each other. Geman (2005) provides the following simple example for two processes that follow mean-reverting diffusions and that are driven under the $\mathbb{Q}$ pricing measures by the following stochastic equations:

$$\begin{cases} dS_1(t) = a\big(b - lnS_1(t)\big)S_1(t)dt + \sigma_1 S_1(t)d\widetilde{W}_t^1 \\ dS_2(t) = c\big(e - lnS_2(t)\big)S_2(t)dt + \sigma_2 S_2(t)d\widetilde{W}_t^2 \\ \qquad d\widetilde{W}_t^1 \cdot d\widetilde{W}_t^2 = \rho dt \end{cases}$$

To make sure that the dependence structure between the price processes is correctly addressed, we need to guarantee that the correlation property is properly represented. This can be done by adjusting the normal distributed random variables:

$$\widetilde{W}^2(t) = \rho\widetilde{W}^1(t) + (1-\rho)\widetilde{W}^3(t)$$

being $W^3(t)$ a Brownian motion independent of $W^1(t)$.

The above represented geometric decomposition of $W^2(t)$ refers to the Cholesky-decomposition technique: the idea consists in finding a matrix $\boldsymbol{A}$, that satisfies the relation $\boldsymbol{AA^T} = \boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma}$ is the matrix of the correlations between the two factors. Then the matrix $\boldsymbol{A}$ is used to shape, in line with their correlations, the random numbers needed for modelling the stochastic path of the spot price, in the following way:

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \begin{bmatrix} A_{11}\varepsilon_1 \\ A_{21}\varepsilon_1 + A_{22}\varepsilon_2 \end{bmatrix}$$

ε_1 and ε_2 are normally distributed with mean zero and standard deviation one. Employing this method, we obtain two random numbers that are related among them according to precise conditions and ready to be used in the stochastic equations.

The above-mentioned Cholesky-decomposition is a technique that allows us to find the triangular matrix $\boldsymbol{A}$ using some elementary linear algebra. This lower triangular matrix is unique and defined, whose entries on the main diagonal are non-negative. This is only valid for positive definite symmetric matrices, as the correlation matrix. Appendix IX describes a trivial explanatory example of the Cholesky algorithm.

The former step is of crucial importance for accurate estimations of spread options prices. After ensuring the existence of a dependence structure between the two price processes, one can simulate the evolution of the underlying products of the option, in the case S_1 and S_2. The computation of the option price is now relatively straightforward, according to the general application of the Monte Carlo simulation technique: recalling the payoff at maturity of a spread option between commodities, we have,

$$c_T = \max(0, q_1S_1(T) - \; q_2S_2(T) - k)$$

Accounting for the specificities of the spread option (regarding q, q_2 and k) and plugging in the n generated values for both underlyings, it is possible to generate n payoffs (π_n). The Monte Carlo price of the spread option, at time 0, will then be:

$$c_0 = e^{-rT}\frac{\pi_1 + \pi_2 + \cdots + \pi_n}{n}$$

Regarding calendar spread options there are two fundamental ways of pricing these financial products (Geman, 2005) using the Monte Carlo procedure: the first one refers to situations where, due to the existence of seasonality, the underlying Futures prices can be seen as the prices of two different instruments. For example, the prices of RBOB gasoline in the United States exhibit some seasonality in prices, which tend to be higher during summer periods and lower in the winter. In this case, it can be reasonable to regard gasoline in summer as a different commodity from gasoline in winter. The implementation of the Monte Carlo technique is, in this case, similar to the one described before for spread options between commodities. The only considerable difference is the fact that, instead of generating paths for spot prices, one needs to generate paths for Futures prices, since in a calendar option the payoff depends on the difference between two different Futures prices instead of the difference between spot prices.

The second way of pricing a calendar spread option, according to the same author - Geman (2005) -, is to incorporate all the characteristics of the underlying commodity in the same stochastic process, containing a term structure of volatilities $\sigma(t,T)$. The calendar spread option can then be priced, at time 0, as the expectation of the discounted payoff at maturity: this step is implemented in the same way as in the former cases.

Our implementation of the Monte Carlo Simulation for calendar spread option pricing refers to the second method described by (Geman, 2005). This is because the three-factor model developed in Chapter 3 incorporates all the relevant characteristics of the underlying commodity. The parsimonious stochastic triple system of equations presented in Chapter 3 allows us to predict the evolution of the three state variables of our model and to compute expected Futures prices for every single maturity T, using the formula when presenting the estimation technique. Moreover, this model implicitly assumes a term structure of volatility for Futures prices, which we have also analysed in Chapter 3.

We have now reinforced the advantages of the Monte Carlo simulation technique for option pricing: in particular, the description of its implementation for pricing calendar spread options has demonstrated that it is easily adaptable to different types of options. A full description of the implementation is provided in the next section, after we introduce the data used in the analysis. Afterwards, we analyse the results and findings, comparing our price estimations with the observed market prices and identifying the possible reasons for the differences between the values.

4.7 Empirical Study

After having provided a broad analysis on the Monte Carlo simulation technique, we will now describe its implementation for the specific purpose of this project. We will start by analysing the data, including not only the inputs for our model but also an example of the observed price evolution of one of the calendar spread options we are going to price. Later, we will move to a step-by-step explanation of the model implementation and finally to an extended analysis of the results of our estimations.

4.7.1 Data

Regarding the input data for the employment of the Monte Carlo technique it is crucial to recall the importance of the oil pricing model built in Chapter 3. To achieve adequate results in the Monte Carlo simulation technique for option pricing, it is indispensable to be in possession of a pricing model that accounts for the characteristics of the underlying asset. Opportunely, we have seen that our iterations provided us with a model that is able to capture with accuracy, in an in-sample basis, the Futures prices movements of the WTI Light Sweet Crude Oil. Moreover, the Cortazar and Schwartz model has also proved to have a good out-of-sample fit, according to the author's findings (Cortazar & Schwartz, 2003). As it is so, we should expect this model to be a correct provider of inputs for the implementation of the Monte Carlo technique. We here recall the configuration of the *parsimonious* Cortazar and Schwartz model developed in section 3.5 The Cortazar and Schwartz Model and Estimation Technique:

$$dS = (\hat{v} - \hat{y}) \cdot S \cdot dt + \sigma_S \cdot S \cdot d\widetilde{W}^S$$

$$d\hat{y} = (-\kappa\hat{y}) \cdot dt + \sigma_y \cdot d\widetilde{W}^{\hat{y}}$$

$$d\hat{v} = \left[\beta a^2 - \sigma_{S,\hat{v}} - a\hat{v}\,\right] \cdot dt + \sigma_v \cdot d\widetilde{W}^{\hat{v}}$$

Therefore, in order to perform the Monte Carlo Simulation, we need to use the parameters we have estimated in the previous chapter: κ, a, β, σ_S, $\sigma_{\hat{y}}$, $\sigma_{\hat{v}}$, $\sigma_{S,\hat{v}}$, $\sigma_{S,\hat{y}}$ and $\sigma_{\hat{v},\hat{y}}$. While the first six parameters are explicitly represented in the above equations, the last two ($\sigma_{S,\hat{y}}$, $\sigma_{\hat{v},\hat{y}}$) are particularly important for the construction of the random paths. Furthermore, we also need to use the initial state variables S, $\hat{y}$ and $\hat{v}$ for each day of the estimation. We will return to issues regarding the input data when describing the implementation of the Monte Carlo simulation (in the following subsection Implementation).

Regarding the choice of which options to price, we have decided to focus on 15 different 1-month calendar spread call options on the WTI Light Sweet Crude Oil. The next table (Table 5) summarizes the most relevant information about the financial products we selected:

WTI Calendar Spread Options - Summary				
Bloomberg® Ticker	**Spread**	**Expiration**	**Period of Analysis**	**# Obs.**
WAX1C Z1 .00 Comdty WAX1C Z1 .05 Comdty WAX1C Z1 .10 Comdty	Nov. 2011 / Dec. 2011	10/19/2011	01/03/2011 to 10/19/2011	190
WAZ1C F2 .00 Comdty WAZ1C F2 .05 Comdty WAZ1C F2 .10 Comdty	Dec. 2011 / Jan. 2012	11/17/2011	01/03/2011 to 11/17/2011	211
WAF2C G2 .00 Comdty WAF2C G2 .05 Comdty WAF2C G2 .10 Comdty	Jan. 2012 /Feb. 2012	12/19/2011	01/03/2011 to 12/19/2011	232
WAG2C H2 .00 Comdty WAG2C H2 .05 Comdty WAG2C H2 .10 Comdty	Feb. 2012 / Mar. 2012	01/19/2012	01/03/2011 to 12/30/2011	240
WAH2C J2 .00 Comdty WAH2C J2 .05 Comdty WAH2C J2 .10 Comdty	Mar. 2012 / Apr. 2012	02/17/2012	01/03/2011 to 12/30/2011	240

Table 5 – Summary of WTI Calendar Spread Options data

As we can see, we have chosen contracts with three different strike prices ($0.00, $0.05 and $0.10) for each of the five one-month spreads. In practice, this requires the estimation of the prices for 15 different options. Regarding the period of analysis, it is worth to note that it is restricted to 2011, even though some of the options were traded also in 2012. This is because, in order to implement the Monte Carlo technique to our option pricing problem, we need to have access to the computed state variables for the day we are pricing the option. As the period of analysis of the model built in Chapter 3 goes from the first trading day of 2006 to the last trading day of 2011, we decided to refer to this period for the option pricing analysis too. However, from our point of view, this restriction does not establish a significant limitation to our findings. Regarding options with expiration in 2012, we can still compute a long path of option prices for days in the year 2011, which provides us with very relevant information. Concerning the time series of spread options, we decided to adjust it in order to account only for those days on which there was available observed data for the three contracts with the same underlying (but different strike prices). This allows for an easier and more comprehensive comparison of the estimated prices of these contracts. Nevertheless, we comprehend that this might have a small impact on the estimated prices due to missing days in the path generation. We believe, however, that this effect should be minimal. It is also important to clarify that the above-mentioned options are not financially settled.

The gathering of market prices of the five above-mentioned contracts is imperative since it allows for a comparison of estimated and historical figures. This, in turn, permits us to evaluate the capacity that our model has to guess option prices. For this purpose, we have downloaded historical data using Bloomberg®, in particular the price of the last trade at each trading day ("PX_LAST"). The following figure is an example of the price evolution of a calendar spread call option, in particular, the WAF2C G2 .00. This is a call option on a spread between the Futures contract with delivery in January 2012 (F2) and the Futures contract for delivery in February 2012 (G2), with strike price zero:

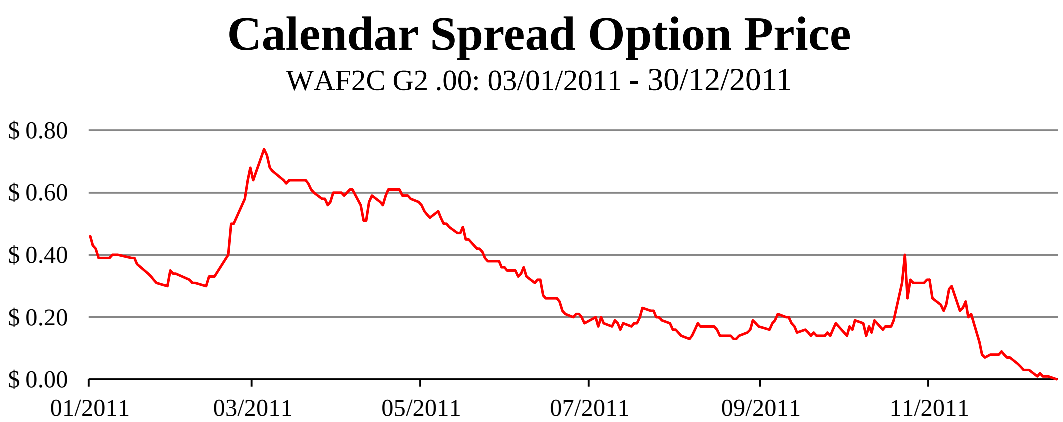

Figure 15 - Historical Prices for the WAF2C G2 .00 calendar spread option (call).

Figure 15 provides us with some intuition regarding the behaviour of the price of calendar spread call options. We can observe a general decline in the price of the option. At the last day of trading, the price of this calendar spread option is $0.00. This is because there was no payoff for this option, since the spread $F2(T) - G2(T)$, being T the maturity, was negative (remember the fact that, because the strike price of this specific option is zero, it has no influence on the final payoff). This is not surprising since we have already seen that the market is generally in contango: $F_2(T) > F_1(T)$. The only one-month calendar spread call options expiring at the same day as the above one that have paid off had negative strike prices: since the observed spread on the 19th of December 2011 was -0.17, options with strike prices -0.20 or lower have delivered a positive payoff[38].

Another interesting fact that can be observed in this graph is the increase of the spread option price by the end of October 2011. This happened because the market has temporarily priced short-term contracts higher than long-term contracts. If this had continued until the expiration date of the spread

[38] The last prices for December 19 2011 respectively were: CL1 = 93.88 and CL2 = 94.05.

option, it would have paid off. However, the contango state of the market was quickly restored: from the 15th November, the spread between the first and second contract has become negative again.

4.7.2 Implementation

After having analysed the data used in our empirical study, we will now describe in detail the implementation of the Monte Carlo simulation for the purpose of pricing calendar spread options.

The first step consisted in building a set of random numbers to be used in the trajectories of the state variables. We started by generating independent identically standard normal distributed random numbers: for each of the state variables, we decided to build 500 random numbers per day and to use the antithetic variate method we have previously described, which automatically doubles the number of generated paths to 1000. At this point, the random figures are not yet accurate for being used in the computation of the state variables' random paths since they are still not accounting for the dependence structure existent between S, $\hat{y}$ and $\hat{v}$. Therefore, in order to easily replicate the interdependences in the construction of the random paths, we had to perform the Cholesky decomposition technique to the matrix of variances and co-variances of the state variables. After being transformed into a lower triangular matrix, it was used to shape the random numbers needed for modelling, following the same procedure described in the previous section.

At this stage, it is important to recall the *parsimonious* Cortazar and Schwartz model implemented in Chapter 3. It is originally represented in a continuous form, but it can also be written in a discrete form:

$$S_{t+1} = S_t + (\hat{v}_t - \hat{y}_t) \cdot S_t \cdot \Delta t + \sigma_S \cdot S_t \cdot \sqrt{\Delta t} \cdot \bar{\varepsilon}_S$$

$$\hat{y}_{t+1} = \hat{y}_t + (-\kappa \hat{y}_t) \cdot \Delta t + \sigma_y \cdot \sqrt{\Delta t} \cdot \bar{\varepsilon}_{\hat{y}}$$

$$\hat{v}_{t+1} = \hat{v}_t + \left[\beta a^2 - \sigma_{S,\hat{v}} - a\hat{v}_t\right] \cdot \Delta t + \sigma_v \cdot d\sqrt{\Delta t} \cdot \bar{\varepsilon}_{\hat{v}}$$

Having access to the initial state variables and to the estimated parameters, and as we had already generated random numbers that account for the existent dependence structure, the construction of the 1000 paths for the state variables was performed. We have chosen to value them in a daily basis. Being 252 the average number of trading days a year, Δt is then equal to 1/252.

At the expiration day of the option, we can use the last set of estimated state variables to compute the value of the underlying assets of the options at maturity for each path, using the formula of Futures prices provided by (Cortazar & Schwartz, 2003) and that was presented in Chapter 3. Therefore, we calculated the 1000 estimations for F_1 and F_2 at the expiration of the option. Afterwards, for each of the paths, we computed the expected payoff of the option given the strike price and the estimates for F_1 and F_2 and discounted each of them to the day of analysis using the 1-Year Libor (100.66 basis points). By computing the average of the discounted values of the expected payoffs, we obtain the Monte Carlo estimation of the calendar spread option price.

4.7.3 Results

Up to this point we have explained the theory behind our model and its implementation. In this section we present the results generated using our model. In this subchapter we limit ourselves in presenting the results and give a general interpretation of them. A deeper discussion on our findings is left for the next section.

As we did in Chapter 3 we start our technical analysis of the results with a graphical comparison of the observed prices with the estimated ones. The chart below shows the evolution of the price for the option WAZ1C F2 .00[39], which expired in November 2011:

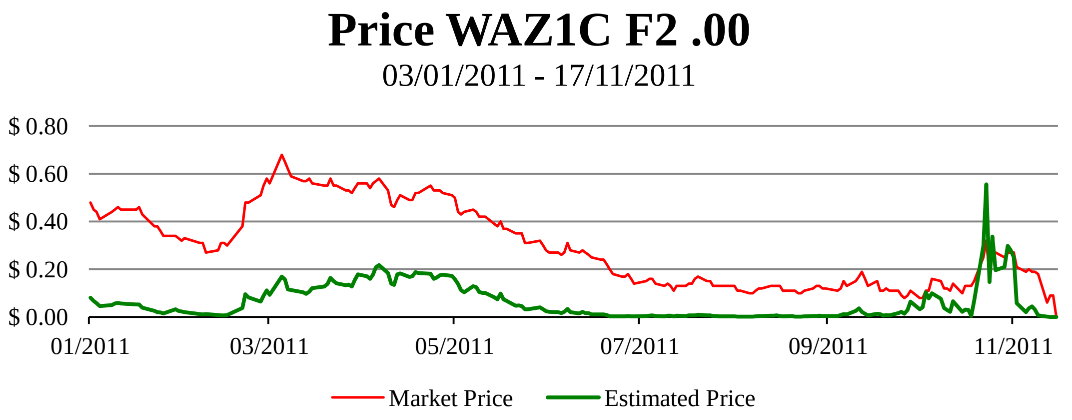

Figure 16 - Evolution of the one-month calendar spread option price WAZ1C F2 .00 from January 2011 to December 2011.

We decided to use this option for our discussion because it is the one, whose estimations fit the observed market prices better. We can easily notice from the chart that our estimations are significantly far from the observed market values. The green line, which represents our estimation, is always lower than the price observed on the trading floor, except for very few observations at the end of the period.

We can see that at the beginning of the period, our estimations are well below the price of the market, but in a certain way we can notice that both lines follow the same negative trend. From March to July our estimation of the price is lower than the observed one. Between the end of February and the beginning of March the price observed on the market starts to rise; our estimation rises as well, but with a certain delay. Then our price has a slightly positive trend that lasts until the beginning of May,

[39] This is the option, whose underlying assets are the Future contract with delivery in December 2011 (CLZ1) and the Future contract with delivery in January 2012 (CLF2).

whereas the traded price has a negative trend from March to July. During this period the market has different expectations for the prices of the Futures contracts at maturity. We can infer that the market implies the spread will decrease, while we still believe in a small increase in the spread until the end of April.

After the month of May our estimations decrease and from the beginning of July until the end of August the option is valueless, while the price observed on the market is on average $0.15. We are still far from maturity, but our model already considers the option to be worth zero, meaning that it predicts that the spread between the two Futures contracts (CLZ1 and CLF2) will always be negative. In this situation, a rational investor will not exercise the option.

At the end of the observed period the option starts to gain value again. In effect, the spread returns to be positive, which makes the option to be valuable. Another reason that could also explain this increase in price is the increase in trading volume of the two contracts, since we are closer to the expiration date of the option and of the first Futures contract. Traders start to settle and close their exposures in this period and they might be able to influence the price of financial products. During this last period our option provides a fair estimation of the price and our model understand the expected increase in the spread.

In order to have a clearer and complete understanding of our results we decided also to compute the percentage RMSE of our estimations, as we did when analysing the results for the crude oil price parameters. Table 6 and its respective chart below (Figure 17) summarize the different measures for the 15 calendar spread options we analysed:

Percentage RMSE for WA				
		***K* = $0.00**	***K* = $0.05**	***K* = $0.10**
1	WAX1C Z1	0,9409	1,0091	0,9136
2	WAZ1C F2	0,7052	0,7441	0,7809
3	WAF2C G2	0,7473	0,7818	0,8143
4	WAG2C H2	0,7710	0,8055	0,8368
5	WAH2C J2	0,7797	0,8116	0,8408

Table 6 - RMSE in percentage terms for the 15 options analysed in this study

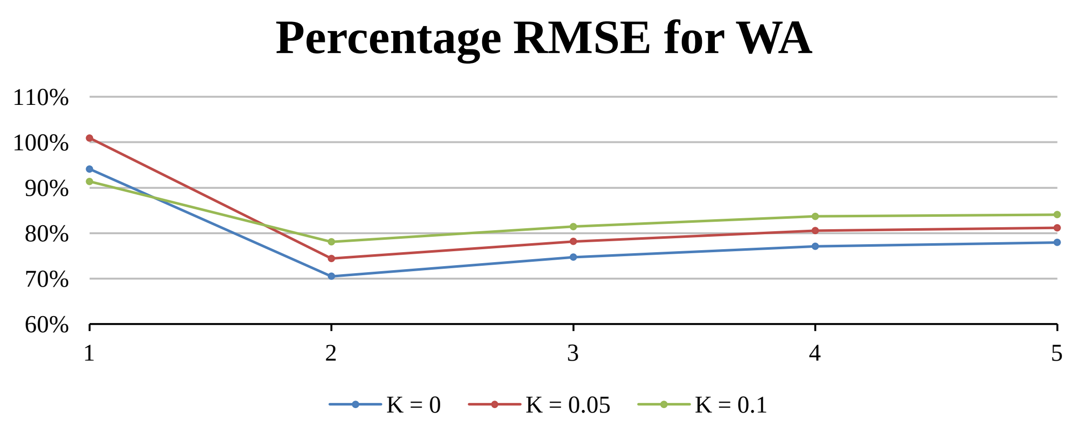

Figure 17 - Root Mean Squared Error in percentage terms for the call options for the three different strike prices

We can see that the RMSE confirms our previous observations. Our Monte Carlo simulation poorly estimates the price of the option. We obtained the minimum RMSE for the option written on the contracts with delivery in December 2011 and January 2012, however, the level of the error, in percentage terms, is very high (70%). This does not to support the use of our model for pricing these specific options. The RMSE in percentage is higher for the other options, with the one with strike price equal to $0.1 that reaches a mispricing of 100%. We mainly focus on the results of one particular option. The other results can be found in the Appendix X.

Unfortunately, it seems that we are not able to reach satisfactory option prices using our model. Generally speaking we always estimate a price that is below the one traded on the market. Our estimation of the spread does not replicate the one expected in the market. The spread we compute seems to be always lower than the one that the option implies. The analysis of the evolution of the price and the RMSE measure do not provide us with satisfying results compared to the ones we found when pricing Futures contracts in Chapter 3. Therefore, all these elements do not seem to support the use of our model for estimating the price of calendar spread option in the crude oil market.

4.7.4 Discussion On Findings

After having presented our results of the Monte Carlo simulation used to price the one-month calendar spread option traded on the NYMEX, it is now time for us to discuss our findings and list some possible reasons that could have led to such unsatisfactory performance of this numerical approach.

The technique we used for the purpose of this valuation of this particular financial product usually provides highly satisfactory results (Trolle & Schwartz, 2009). The theory regarding Monte Carlo simulation and its accuracy in option pricing is well supported by academic literature (Carmona & Durrleman, 2003). We consider our model and its structure to be correct. The sequence of steps used in order to obtain the final price of the calendar financial product is consistent with the standard procedure for Monte Carlo simulation. We account for the dependence between the three state variables, which has a strong relevance for our model, due to the particular mean reversion process of the pricing model we employed.

Given the fact that we decided to price the options only when all of them were traded, for all the different strike prices, should not have a great relevance for the meagre performance of our model. In effect, this decision led us to not consider only few days in our analysis. Obviously, this assumption has an influence on the estimated paths of the three different state variables, which are very important for the pricing of Futures contracts. However, we do not believe that this assumption has a significant impact on our results. The omitted observations were all from the first quarter of the year, which cannot explain the poor performance of our estimation for the rest of 2011, where there are no missing dates. Therefore the reasons for the underperformance of our model are linked to other elements, which are not related to the Monte Carlo simulation structure.

We can now analyse how our estimations mirror the evolution of the spread. In the chart below (Figure 18) we can observe the movement of the option prices compared to the differential between the two Futures contracts.

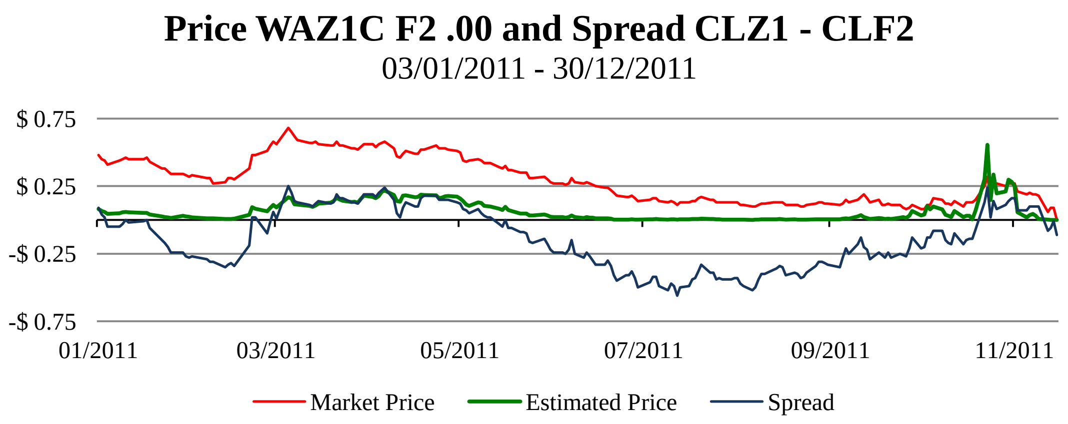

Figure 18 - Comparison of the spread between Z1 and F2 with the observed price and the estimated price

It is easy to notice that our estimation for the option price mimic the evolution of the spread between the two contracts. The underlying spread is positive from March to May, exactly when our option also has a value different from zero. The same can be said for the end of the period, when the spread returns to be above zero for some days. On the other hand, when the spread is negative the option as a value of zero even if it is far from maturity. The value of the option traded on the market is definitely higher. Even though the spread is low or even negative the option value is always different from zero before maturity. This is a relevant difference between our model and the reality. With our model we are able to replicate the evolution of the spread, but we are not able to anticipate the expected payoff that the market predicts. It seems that the market expects the payoff of the option at expiration to be more valuable and/or it forecasts a higher chance of a positive payoff. These two aspects seem not to be reflected in our calculations.

Consequently, this analysis of the spread suggests us that we have to search the poor fit of our model in the level of parameters that we estimated in Chapter 3. They are in fact the drivers of the Futures prices and they are also indirectly the determinants of the evolution of the spread.

Therefore, according to our reasoning, the most relevant motive for the underestimation of the estimated option prices is related with the quality of the inputs used in the Monte Carlo model. In particular, we believe that some of the variance and covariance figures of the state variables are not precise. Consequently, we understand this as the main reason for the poor results we achieved.

The variance and covariance figures of the state variables have an impact on the path generation of the three state variables and thus also in the estimated prices of the underlying contracts at the expiration date of the option. Moreover, there might also be a significant effect in the volatility of the underlying spreads $(CLF2 - CLG2)$. Even though this last volatility figure is not directly observable in our model, it is of crucial importance to the price of the option and it is implicitly present in the implementation of the model through the relationship between Futures prices and state variables. Therefore, the variance and covariance figures of the state variables are likely to have a significant influence on the price estimation of the options.

When we presented the results of the pricing model in Chapter 3, we extensively analysed the volatility structure implied by the parameters found through the implementation of the Cortazar and Schwartz's technique. We saw that this structure indicates lower values for the volatility of Futures contracts than the ones observed on the market. This is highly relevant for the purpose of this chapter since volatility is one of the key drivers of options prices. In the case of calendar spread options, the volatility pattern of the underlying Futures prices is highly relevant as it strongly influences option prices. As our model parameters underestimate the volatilities of Futures contracts, one should expect the estimated prices of the options to be lower than the ones observed in the market. This is why, from our point of view, the results from the implemented Monte Carlo simulation are not particularly surprising. Therefore it is time for use to take a more meticulous look at the importance that each volatility and covariance input has on option pricing.

In order to understand the influence that each of the inputs has on the estimated price of the calendar spread options we are studying, we have decided to utilize a sensitivity analysis approach. In fact, we arbitrarily changed the model's variances and covariances inputs in order to understand the consequences that these modifications would have in terms of option pricing. For this purpose, we have analysed the price of the option WAF2C G2 .00. For the first trading day of January 2011

(03/01/2011), our model has predicted an option price of \$0.08, when the observed one on the market was \$0.46.

The results of the implementation of this sensitivity analysis provided us with interesting results. The most important finding was that the estimated price of the option being analysed is very sensitive to changes in the volatility of the convenience yield ($\sigma_{\hat{y}}$). An increase of the variance of the convenience of 10% (that implies a change of the standard deviation from $\sigma_y = 0.3514$ to $\sigma_y = 0.3865$) results in an appreciation of the option price of 28.5% (to approximately \$0.11). We can clearly see the impact the convenience yield has on financial products based on commodities.

A complementary way to understand how critical this parameter is for an accurate option pricing is to evaluate how would our implemented Monte Carlo model behave using a value for this parameter that is aligned with the findings of other authors. The reasoning behind this is that other academics have estimated values for the volatility of the convenience yield that are notably different from our findings (Schwartz, 1997), (Cortazar & Schwartz, 2003). It is important to mention again that those studies were based on different models and use different periods of analysis than the one applied to this work. Nevertheless, we believe that they still provide a suggestion of the value of the volatility of the convenience yield we should have expected to find. While Cortazar and Schwartz (2003) estimate values for σ_y that range from 0.623 to 0.717, depending on the period of analysis, (Casassus & Collin-Dufresne, 2005). Using, for example, $\sigma_y = 0.67$, which is the average between the maximum and minimum values estimated by Cortazar and Schwartz, we can see that the Monte Carlo prices of the option considerably change to a value that is extremely close to the observed market price. For instance for the first trading day of 2011, we estimated a spread option price of \$0.48 and the observed price was \$0.46. For the same option and using this level of volatility we performed the analysis to some other random days. We observed that the estimated prices are much closer to the ones observed on the market than the results with the original value of σ_y used in our model. While our initial results imply estimated prices that are generally much lower than observed prices, the results with the new volatility figure, despite being apparently more accurate, specify a general propensity for overpricing.

We decided to perform another analysis in order to support even more the suggestion that the model implemented in Chapter 3 underestimated the parameter $\sigma_{\hat{y}}$. At the beginning of this chapter we introduced the basic concepts related to option pricing. We dedicated few paragraphs to the information that these financial products can provide to financial practitioners. In particular, we pointed out that it is possible to extrapolate the volatility estimate of the underlying assets simply using the market value and the model used to price the option. Therefore we decided to modify our Monte Carlo simulation model in order to be able to find the value that the market implies for the volatility of the demeaned convenience yield. The only modification we have to make is to create an objective cell. In our model this cell consist in the sum of the squared errors, between the observed prices of the three options and their respective estimations. Then to find the implied volatility we ask Solver® to minimize this sum changing only the value of the variance of the demeaned convenience yield.

We applied this technique to the first trading day of 2011. We asked Solver® to find the value of the parameter $\sigma_{\hat{y}}^2$ that minimize the sum of squared errors. We need to stress that to implement this procedure, we have to be once again careful regarding the circular references, as we had to be in the estimation of these parameters (Chapter 3).

This estimation provided us with a value for $\sigma_{\hat{y}}^2$ of 0.502461,which is equivalent to $\sigma_{\hat{y}} = 0.708845$. This is in line with our suggestions we discussed in the previous paragraphs. For other days of the year 2011, the $\sigma_{\hat{y}}$ value found usually does not provide an estimated price equal to the market price, suggesting the possibility that the volatility of the convenience yield is not constant over the period considered (Routledge, Seppi and Spatt (2000) in Lutz 2010). For instance, for the 31st of March, the implied standard deviation of the convenience yield is 0.707636 and on the 28th of September it is only 0.587134. It might also be that, in case the convenience yield volatility is constant, the differences in estimation errors among different periods of time is due to inaccuracies of other estimated parameters.

According to this intuition we performed the sensitivity analysis for the other first-order inputs that influence volatilities and correlations, meaning: σ_s ; $\sigma_{\hat{v}}^2$; $\sigma_{s\hat{y}}$; $\sigma_{\hat{y}\hat{v}}$ and $\rho_{s\hat{v}}$. Regarding these parameters, we gave particular attention to the correlation between the first and the third state variables ($\rho_{s\hat{v}}$). The fact that the estimated value of this parameter is not only negative but also quite low ($\rho_{s\hat{v}} \cong -0.67$) seems to have a downward influence on the estimation of option prices. As we already pointed out when discussing the finding of the oil pricing model, we highlighted the fact that this parameter seems to be too low and it might have an influence on the mean reversion scheme of the three factor model (Lutz, 2010). A less extreme value for this parameter would imply higher estimated option prices. For example for the value $\rho_{s\hat{v}} = -0.05$, *ceteris paribus*, the price of the WAF2C G2 .00 option would be approximately 34% higher for the first day of analysis. However, using our Monte Carlo model, it is easy to see that this potential error on the estimation of this parameter is not sufficient to explain, by itself, the important gap between estimated and observed option prices. It is difficult to find an acceptable value for this parameter that would imply estimate option prices similar to the ones observed in the market. As we did for the volatility of the convenience yield we tried to find the value of the implied correlation between the first and the third state variable. Unfortunately, we were not able to find a reasonable value for this parameter.

Therefore, according to our analyses, the main reason for the underestimation of option prices comes from a very low value of the convenience yield's volatility. Additionally an important parameter also influencing the low estimated option prices is the correlation between the first and the third state variables ($\rho_{s\hat{v}}$). However, a potential inaccuracy in the estimation of this last figure is not sufficient to explain the model's underestimations.

We are aware that the analysis we implemented in this section has some limitations, since it was not performed for the entire period and addressed only one specific calendar spread option. However, we believe these results are representative of the general influence that the variance and covariance inputs used in the Monte Carlos simulation have on option pricing.

This numerical technique is a very flexible and intuitive method for option pricing. However, if the input parameters that describe the expected evolution of the underlying products lack accuracy, the payoffs generated thanks to the Monte Carlo simulation, will suffer from the same disease, mispricing the financial product.

When we estimated the parameters, we decided to use as input the prices of the Futures contracts that were traded in that period. According to our model, we were able to capture the majority of the information from the term structure of Futures prices. The fact that the RMSE measure for Futures prices estimation was acceptable provided us with the idea that our parameters would have been able to well replicate the behaviour of options, too. Unfortunately, we can see from our last results that this is not the case. We believe that option prices contain some information that the Futures prices are not able to provide. The information that comes from the underlying assets is not sufficient to describe the evolution of option prices. However, we do not consider the lack of information to be the only reason for our underestimated option prices. As we already pointed out, volatilities and correlations of the pricing parameters have also a great impact on estimations. Moreover the estimation technique is not fully explained in the original article of 2003. Therefore, it might be the case that there is a misinterpretation of the implementation of the Cortazar and Schwartz technique, which consequently distorts the value of the pricing parameters.

Having highlighted these elements, it comes to mind a relative easy improvement that could be applied to the Cortazar and Schwartz estimation model. If the purpose of the model is not limited to the estimation of the term structure of Futures prices, but it is enlarged to option pricing, we could include in the estimation procedure the prices of some relevant options, in order to include more market pertinent information. This should permit to increase the performance of the estimation of Futures prices and definitely the one of options at the same time. Cortazar and Schwartz suggested that this improvement can be done in their estimation technique (Cortazar & Schwartz, 2003). In fact their method to estimate parameters does not require the payoffs to be linear. This is an advantage compared to the Kalman filter technique that necessitates linearity in the pricing expression (Cortazar & Schwartz, 2003). Since the non-linearity of the payoff can be handled by the model, we can incorporate observed option prices in our estimation procedure. A way to integrate this improvement in the model would be to compute the estimation error for each option, as it was done for Futures contracts. This error could be summed to the one of the estimated Futures prices and then minimized in order to find the correct crude oil parameters. However, this enlargement of the model requires a greater effort in building it and it might beneficial to implement it in another spreadsheet-like application.

In this section we deeply discussed the results we obtained and the possible reason that explain the unsatisfactory results of our model. Summing up we conclude that the meagre performance of our model is due to the estimated parameters we obtained in Chapter 3. Although these parameters permitted to accurately describe the dynamics of oil price, they did not provide the same results for option pricing. Lastly, we also suggested a method that could improve our findings.

Conclusion

In this study we develop a method to forecast the prices of crude oil one-month calendar spread options traded on NYMEX. We concretely implement and analyse the estimation technique proposed by Cortazar and Schwartz and we develop a related Monte Carlo simulation model for pricing the calendar spread options.

There is no doubt about the importance that commodity markets have for the trading of raw materials in the industry. These markets permit the exchange of products that are vital in our everyday life. Crude oil is widely known for being an effective energy commodity but it is also known for its limited availability. It is definitely one of the world's most important resources, especially regarding the production of power. We showed and largely discussed the strong relation between the price of petroleum and global macro indicators. In our work we recognize the evidence of other studies that GDP and dollar appreciation are closely related to the return in price of crude oil. This is one of the reasons that convinced us to focus on this specific energy market.

At the beginning of this study we talked about the relevance of energy markets and we also performed a comparison between them and other financial markets. We pointed out the particularities of investing in energy commodities. We also highlighted the relevance that these markets have in the composition of investment portfolios. In effect, commodities have been widely used as investment assets because of their good diversification proprieties and their respectable returns. We also explained the different strategies that investors can employ to enhance their exposure towards the energy sector. Futures contracts and options, which are the main focuses in this study, are some of the possible instruments that can be used to directly invest in these markets.

The energy sector in the last decades has been growing in importance. One of the causes is the energy deregulation phase that many of the most developed countries are facing, which strongly interests the industry's stakeholders. Firms active in the energy sector need to secure their input and output in order to sustain their business. This is the reason why in the majority of the time they have to recourse to financial markets to hedge their risks, protecting the company's cash flows.

Trading in financial markets has been increasing, as well as the demand for more performing pricing models of commodities. We deeply analyse the different characteristic that commodity prices possess, with particular emphasis in energy commodities A first feature that can be noticed when analysing such prices is their mean reversion behaviour. In fact, the prices of commodities usually revert to a long-term price level. This is a substantial difference with respect to stock prices, which can in theory grow infinitely. A second characteristic that we decided to include in our pricing model is the convenience yield. This element value the benefit of physically owning the commodity and it help us understanding the dynamics of the price. We reviewed the importance of these elements, supported by the theory of storage and also by empirical analysis, when modelling prices of commodities.

We were highly inspired by the study conducted by Cortazar and Schwartz in 2003. We found their model to be really intuitive to implement. The *parsimonious* three-factor model proposed by these two authors has a solid background since it is based on the well-known two-factor model of Schwartz (1997). We dedicated particular consideration to the estimation technique of the relevant parameters of the three-stochastic equations, individuating some advantages and drawbacks. This technique is extremely similar to the more complex Kalman filter. This definitely increases the reliability of this procedure.

The Cortazar and Schwartz estimation technique allowed us finding the value of the different parameters we needed to estimate prices according to the *parsimonious* three-factor model. In our study, the Cortazar and Schwartz's model showed a decent performance, as suggested by the Root Mean Square Error analysis, and it definitely met the purpose of replicating the term structure of Futures contracts. Our findings suggest that in recent years the mean reversion of the demeaned convenience yield has accelerated. In fact, compared to other studies, our model suggests that this state variable reverts to its long-term value (in our model this value equals to zero) quicker than in previous empirical analysis. This might have consequences on the drift of the spot price. The positive correlation between the spot price and the demeaned convenience yield supports the mean reversion behaviour of the crude oil price for the period we decided to analyse (i.e. 2006-2011); this is consistent with our assumptions. This level of correlation is a bit lower than precedent empirical analysis, this could be attributed to the particular period analysed. Our results also showed a low value for the volatility of the convenience yield, which had an impact on the estimated price of calendar spread options. Another interesting result is the significant negative correlation between S_t and the price appreciation of oil (v), which is not suggested in other studies on oil. Regarding the other results we obtained, they were in line with the ones proposed by former studies.

In the last chapter we focus on the main aim of this study, meaning pricing calendar spread options. We developed a model for estimating the price of these particular financial instruments available on NYMEX and Over-the-Counter. Our model is based on the well-known Monte Carlo simulation. We decided to not implement closed-form solutions, because of the low academic consensus on the most accurate analytical model for pricing these options. In fact, the complexity of the closed-form solution derives from the non-linearity of the expected payoff and in the particular case of spread options because of the two-dimensional integration and the consequent non-linearity in the exercise boundary

(Deng, et al., 2008). Furthermore, since it was also a purpose of this study to give access to knowledge regarding commodity pricing models to people which are less familiar with the topic, we preferred to use a more intuitive method. Nevertheless, the numerical technique we employed is still a relevant method for pricing options and it is highly flexible.

In the option pricing model we generated the random paths of the three state variables needed to model crude oil prices and we estimated the prices of the underlying Futures contracts at the maturity date of the option. For these purposes, we used the *parsimonious* model of Cortazar and Schwartz and the results – parameters and state variables – we obtained in Chapter 3 for our stochastic pricing process. Afterwards, we computed the expected payoffs of the different options and discounted their values to the day of analysis, in order to find their price. Our model was not able to provide satisfactory estimates of the prices of the options. Although the prices of the options successfully follow the behaviour of the spread and the trend of the market price, our model consistently underestimates its value. As the RMSE measure confirms, our approximation of the price are noticeably off. Nevertheless we believe that the structure of our option pricing model is well definite and complete, which support its use for others analysis.

The meagre performance of our model is due to the quality of the inputs, particularly the parameters we used to compute the path of the Futures prices. In effect, in order to understand the reasons of the poor performance of the Monte Carlo simulation, we devoted ourselves to a sensitivity analysis of some of these parameters. We focused most of our analysis on the level of volatility of the demeaned convenience yield, which we considered to be too low. Interestingly, the sensitivity analysis confirmed our hypothesis. Subsequently, we adjusted our option pricing model in order to compute the implied volatility of the convenience yield. We obtained a result that was in line with our expectations. We reproduced the sensitivity analysis and the optimization of our model also for the correlation between the spot price and the price return of oil. We also found support to this hypothesis that its value is too small. Unfortunately, our model was not able to provide us with a satisfactory implied estimation of this parameter, suggesting that the high interdependence of the different statistical measures cannot be neglected. Therefore, it seems that the Cortazar and Schwartz model and estimation technique were not able to provide us with very precise parameters, leading to poor estimations of calendar spread options in the oil market.

Both models we presented in this study are valid for the purpose they were originally built and allow for a high degree of flexibility. In effect, the Cortazar and Schwartz estimation procedure can account for products that do not have linear payoffs, such as options. With the objective of pricing options and, in order to improve our results, it is possible to consider adding the prices of options as an input to the estimation technique. At the same time, the Monte Carlo simulation permits to deal, in a simply way, with the particularity of certain payoffs as it is in the case of calendar spread options.

One of the main difficulties we encountered in our analysis is the implementation of the models in Microsoft Excel®. In fact, even if the use of this spreadsheet application is rather straightforward, it usually faces hard time when the volume of input is large. An easy way to simplify our model would be to use weekly prices instead of daily figures. This adjustment definitely reduces the number of input but at the same time it provides a less accurate representation of real situation.

To conclude, we offer a new estimation of the parameters commonly used in pricing models. We also hope we provided a clear overview of some of the different characteristics and difficulties related to option pricing in the energy market, especially for crude oil. We must say that even if we were not able to perfectly replicate the price of the one-month calendar spread options as we were expecting, we were able to identify some plausible reasons that led to our breakdown. In case new parameters for simulating the underlying assets would be found, we believe that our model would achieve higher results and provide reasonably good estimation of calendar spread options prices. Financial practitioners have to be particularly careful when estimating the value of financial derivatives. A mispricing of one product can have a significant impact on the portfolio performance of customers or on the strategy of companies.

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Appendices

Appendix I

Total Primary Energy Consumption

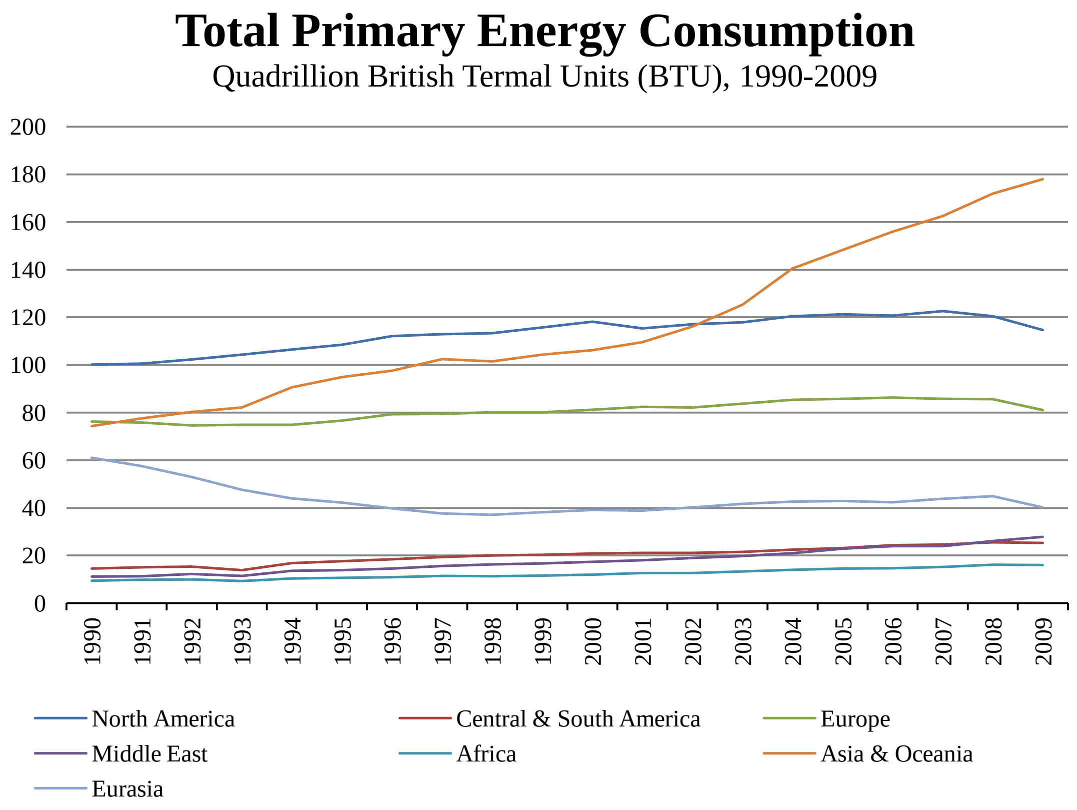


Figure 1 – Total Primary Energy Consumption by World Region. Source: Energy Information Administration

Appendix II

Itô's Lemma: Example

Suppose that $f(x) = \ln(x)$ and x is a Geometric Brownian motion. When we compute the partial derivatives with respect to t and x, we obtain:

$$\frac{\partial f}{\partial t} = 0; \; \frac{\partial f}{\partial x} = \frac{1}{x} \text{ and } \frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}$$

employing Itô's Lemma we have the following stochastic differential equation:

$$df = \frac{1}{x}dx - \frac{1}{2x^2}dx^2$$

$$df = \left[0 + \mu_x(x,t) \cdot x \cdot \frac{1}{x} - \frac{1}{2} \cdot \frac{1}{x^2} \cdot x^2 \sigma_x^2(x,t)\right] dt + \frac{1}{x} \cdot x \cdot \sigma_x^2(x,t) \cdot dW$$

$$df = \left[\mu_x(x,t) - \frac{1}{2} \cdot \sigma_x^2(x,t)\right] dt + \sigma_x^2(x,t) \cdot dW$$

Thus, the change in the natural logarithm of x is normally distributed with a drift equal to $\mu_x(x,t) - \frac{1}{2} \cdot \sigma_x^2(x,t)$ and variance $\sigma_x^2(x,t)$. This is an important result for the purpose of our study.

Appendix III

Buy & Hold Strategy

Definitions:

- $F(t,T)$ = price of a Futures contract at time t with maturity T
- $B(t,T)$ = price of a zero coupon bond at time t with maturity T
- $S(t)$ = spot price at time t
- The underlying assets follows a Geometric Brownian motion as:

$$dS(t) = \big(r(t) - \delta_S(t)\big)S(t)dt + \sigma_S(t)S(t)d\widetilde{W}_t$$

The drift of the process is composed of $r(t)$ and $\delta_S(t)$, both are time depend deterministic functions. We could have assumed both to be constant: r and δ_S too.

In a complete market, without opportunity of arbitrage the following equation must hold:

$$F(t,T) = E_t^Q[S(T)] = e^{\int_t^T (r(s)-\delta_S(s))ds} S(t)$$

The Futures contract at time t that expires at time T is the expect value at time t under risk-adjusted probabilities of the spot price at time T.

Therefore in order to respect the non-arbitrage assumption we have that the following two strategies needs to be identical at time T. The first strategy consists in holding the underlying assets $S(t)$ until maturity reinvesting all the dividends that this asset generates. At expiration date this strategy provides us with: $S(T)$. The second strategy consists in creating a portfolio composed of a long position in the Futures contract $F(t,T)$ and a long position in a zero coupon bond with the same maturity of the Futures. This latter strategy also provide the investor with $S(T)$. Consequently the two strategies must have the same value:

$$F(t,T) \cdot B(t,T) = e^{\int_t^T -\delta_S(s)ds} S(t)$$

$$F(t,T) = \frac{e^{\int_t^T -\delta_S(s)ds} S(t)}{e^{-\int_t^T r(s)ds}}$$

$$F(t,T) = e^{\int_t^T (r(s)-\delta_S(s))ds} S(t) = E_t^Q[S(T)]$$

Appendix IV

Model Modifications

Stochastic differential equation for dy:

$$dy = (-\kappa y - \lambda_y)dt + \sigma_{\hat{y}} d\widetilde{W}^y$$

$$\Leftrightarrow d\hat{y} = \left(-\kappa\left(\hat{y} - \frac{\lambda_y}{\kappa}\right) - \lambda_y\right)dt + \sigma_{\hat{y}} d\widetilde{W}^y$$

$$\Leftrightarrow d\hat{y} = (-\kappa\hat{y} + \lambda_y - \lambda_y)dt + \sigma_{\hat{y}} d\widetilde{W}^y$$

$$\Leftrightarrow d\hat{y} = (-\kappa\hat{y})dt + \sigma_{\hat{y}} d\widetilde{W}^{\hat{y}}$$

Stochastic differential equation for $d\hat{v}$:

$$dv = [a(\bar{v} - v) - \lambda_v] \cdot dt + \sigma_v \cdot d\widetilde{W}^v$$

$$\Leftrightarrow d\hat{v} = \left[a\left(\bar{v}^* + \lambda_S - \frac{\lambda_y}{\kappa} - \left(\hat{v} + \lambda_S - \frac{\lambda_y}{\kappa}\right)\right) - \lambda_v\right] \cdot dt + \sigma_v \cdot d\widetilde{W}^v$$

$$\Leftrightarrow d\hat{v} = [a(\bar{v}^* - \hat{v}) - \lambda_v] \cdot dt + \sigma_v \cdot d\widetilde{W}^v$$

$$\Leftrightarrow d\hat{v} = [a\bar{v}^* - a\hat{v} - \lambda_v] \cdot dt + \sigma_v \cdot d\widetilde{W}^v$$

$$\Leftrightarrow d\hat{v} = [(a\bar{v}^* - \lambda_v) - a\hat{v}] \cdot dt + \sigma_v \cdot d\widetilde{W}^v$$

We know: $\beta = \frac{a\bar{v} - \lambda_v + \sigma_{S,\hat{v}}}{a^2} \Leftrightarrow \beta a^2 - \sigma_{S,\hat{v}} = a\bar{v}^* - \lambda_v$

$$\Leftrightarrow d\hat{v} = \left[\beta a^2 - \sigma_{S,\hat{v}} - a\hat{v}\right] \cdot dt + \sigma_v \cdot d\widetilde{W}^{\hat{v}}$$

Stochastic differential equation for dS:

$$dS = (v - y - \lambda_S) \cdot S \cdot dt + \sigma_S \cdot S \cdot d\widetilde{W}^S$$

$$\Leftrightarrow dS = \left(\hat{v} + \lambda_S - \frac{\lambda_y}{\kappa} - \hat{y} + \frac{\lambda_y}{\kappa} - \lambda_S\right) \cdot S \cdot dt + \sigma_S \cdot S \cdot d\widetilde{W}^S$$

$$\Leftrightarrow dS = (\hat{v} - \hat{y}) \cdot S \cdot dt + \sigma_S \cdot S \cdot d\widetilde{W}^S$$

Parsimonious three-factor model:

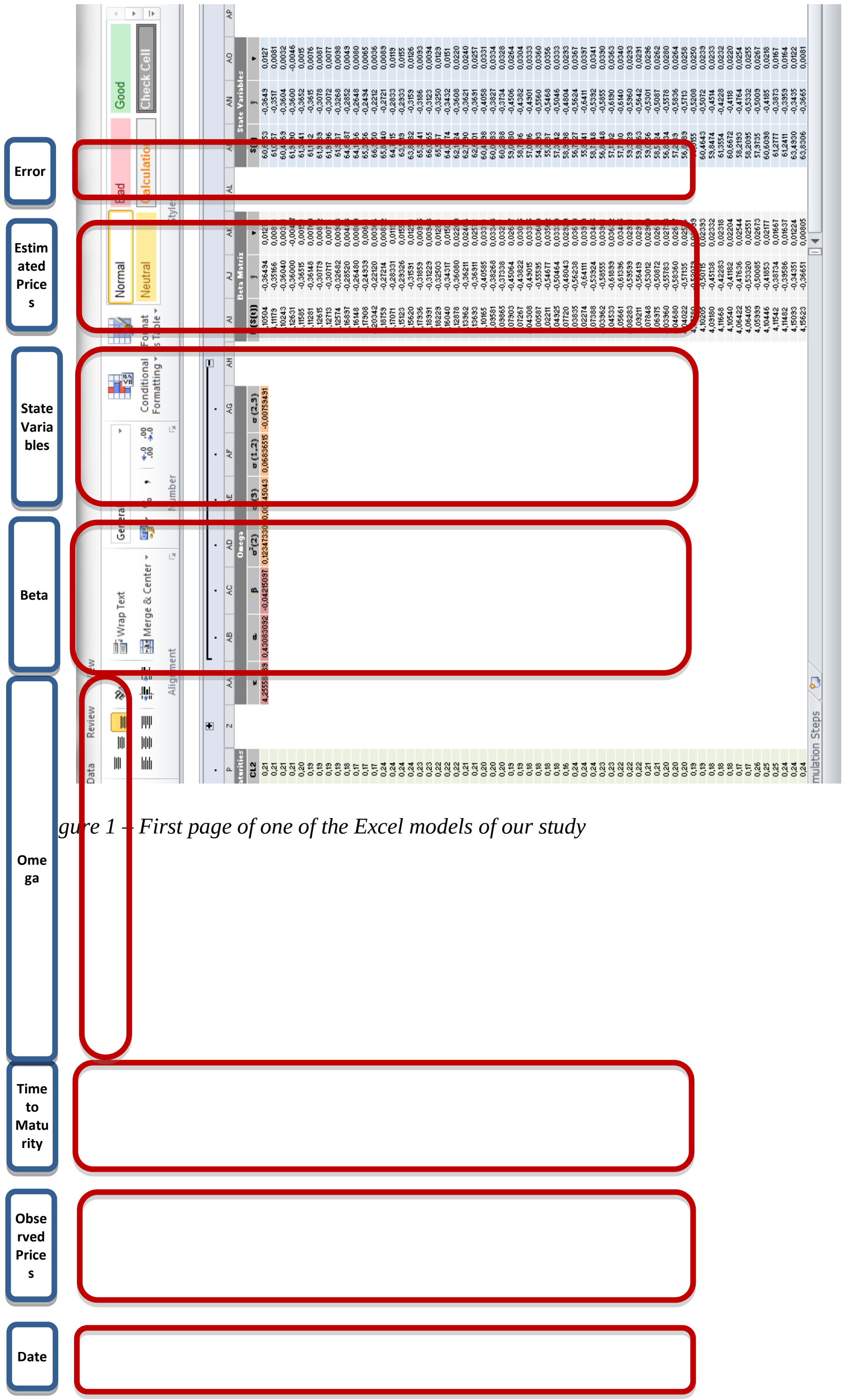
$$dS = (\hat{v} - \hat{y}) \cdot S \cdot dt + \sigma_S \cdot S \cdot d\widetilde{W}^S$$

$$d\hat{y} = (-\kappa\hat{y}) \cdot dt + \sigma_y \cdot d\widetilde{W}^{\hat{y}}$$

$$d\hat{v} = \left[\beta a^2 - \sigma_{S,\hat{v}} - a\hat{v}\right] \cdot dt + \sigma_v \cdot d\widetilde{W}^{\hat{v}}$$

Appendix V

Cortazar & Schwartz Excel Spreadsheet



gure 1 – First page of one of the Excel models of our study

Appendix VI

VBA Codes

```
Option Base 1
Option Explicit

Function First_Term(K, T)
        First_Term = -((1 - Exp(-K * T)) / (K))
End Function

Function Second_Term(A, T)
        Second_Term = ((1 - Exp(-A * T)) / (A))
End Function

Function Constant(K, A, B, var2, var3, p12, p23, T)
        Constant = (-p12 / (K ^ 2)) * (K * T + Exp(-K * T) - 1) + (var2 / (4 * K ^ 3)) * (-
        Exp(-2 * K * T) + 4 * Exp(-K * T) + 2 * K * T - 3) + B * (A * T + Exp(-A * T) - 1) -
        (var3 / (4 * A ^ 3)) * (Exp(-2 * A * T) - 4 * Exp(-A * T) - 2 * A * T + 3) - (p23 / (K
        ^ 2 * A ^ 2 * (K + A))) * (K ^ 2 * Exp(-A * T) + K * A * Exp(-A * T) + K * (A ^ 2) * T
        + K ^ 2 * A * T + K * A * Exp(-K * T) + A ^ 2 * Exp(-K * T) - K * A * Exp(-(K + A) *
        T) - K ^ 2 - K * A - A ^ 2)
End Function

Function Future_Price_Estimation(s, y, v, K, A, B, var2, var3, p12, p23, T)
        Dim w, X, Z As Double
        w = Constant(K, A, B, var2, var3, p12, p23, T)
        X = First_Term(K, T)
        Z = Second_Term(A, T)

        Future_Price_Estimation = s * Exp(w + X * y + Z * v)
End Function
```

```
Function  Cortazar_Schwartz_Future_Price_Estimation(State_Variables  As  Range,  Time_Matrix  As
Range, Omega As Range)
        Dim s, y, v, K, A, B, var2, var3, p12, p23, T, c, l As Double

        c = Time_Matrix.Columns.Count

        Dim Temporary()
        ReDim Temporary(1, c)

        s = State_Variables(1, 1)
        y = State_Variables(1, 2)
        v = State_Variables(1, 3)
        K = Omega(1, 1)
        A = Omega(1, 2)
        B = Omega(1, 3)
        var2 = Omega(1, 4)
        var3 = Omega(1, 5)
        p12 = Omega(1, 6)
        p23 = Omega(1, 7)

        For l = 1 To c
                If Time_Matrix(1, l) = 0 Then
                Temporary(1, l) = 0
                Else
                Temporary(1, l)  = Future_Price_Estimation(s, y, v, K, A, B, var2, var3, p12,
                p23, Time_Matrix(1, l))
                End If
        Next l
        Cortazar_Schwartz_Future_Price_Estimation = Temporary
End Function

Function Difference_Sum_Future_Price(Future_Observed As Range, Future_Estimate As Range)
        Dim i, j As Integer
        Dim q As Double
        q = 0
        j = Future_Observed.Columns.Count
        For i = 1 To j
                If Future_Estimate(i) = 0 Then
                q = q + 0
                Else
                q = q + (Log(Future_Estimate(i)) - Log(Future_Observed(i))) ^ 2
                End If
        Next i
        Difference_Sum_Future_Price = q
End Function
```

```
Function Matrix_Y(Future As Range, Time As Range, Omega As Range)
        Dim i, r, K, A, B, var2, var3, p12, p23, T As Integer
        r = Time.Columns.Count
        Dim matrix()
        ReDim matrix(1, r)

        K = Omega(1, 1)
        A = Omega(1, 2)
        B = Omega(1, 3)
        var2 = Omega(1, 4)
        var3 = Omega(1, 5)
        p12 = Omega(1, 6)
        p23 = Omega(1, 7)

        For i = 1 To r
                If Future(1, i) = 0 Then
                matrix(1, i) = 0
                Else
                matrix(1, i) = Log(Future(1, i)) - Constant(K, A, B, var2, var3, p12, p23,
                Time(1, i))
                End If
        Next i
        _Y = matrix
End Function

Function Matrix_X(Time As Range, Omega As Range)
        Dim j, q, f, h, K, A, B, var2, var3, p12, p23, T As Integer
        q = Time.Columns.Count
        h = 3
        Dim Matrix_2()
        ReDim Matrix_2(q, h)

        K = Omega(1, 1)
        A = Omega(1, 2)
        B = Omega(1, 3)
        var2 = Omega(1, 4)
        var3 = Omega(1, 5)
        p12 = Omega(1, 6)
        p23 = Omega(1, 7)

        For j = 1 To q
                For f = 1 To h
                If f = 1 Then
                If Time(1, j) = 0 Then
                Matrix_2(j, f) = 0
                Else
                Matrix_2(j, f) = 1
                End If
                ElseIf f = 2 Then
```

```
                Matrix_2(j, f) = First_Term(K, Time(1, j))
                Else
                Matrix_2(j, f) = Second_Term(A, Time(1, j))
                End If
                Next f
        Next j
        Matrix_X = Matrix_2
End Function

Function Matrix_B_1(Time As Range, Omega As Range)
        Matrix_B_1    =    Application.MMult(Application.Transpose(Matrix_X(Time,    Omega)),
        Matrix_X(Time, Omega))
End Function

Function Matrix_B_2(Time As Range, Omega As Range)
        Matrix_B_2 = Application.MInverse(Matrix_B_1(Time, Omega))
End Function

Function Matrix_B_3(Time As Range, Omega As Range)
        Matrix_B_3 = Application.Transpose(Matrix_X(Time, Omega))
End Function

Function Matrix_B_4(Time As Range, Omega As Range)
        Matrix_B_4 = Application.MMult(Matrix_B_2(Time, Omega), Matrix_B_3(Time, Omega))
End Function

Function Matrix_B_5(Future As Range, Time As Range, Omega As Range)
        Matrix_B_5 = Application.MMult(Matrix_B_4(Time, Omega),
        Application.Transpose(Matrix_Y(Future, Time, Omega)))
End Function

Function Matrix_B_6(Future As Range, Time As Range, Omega As Range)
        Matrix_B_6 = Application.Transpose(Matrix_B_5(Future, Time, Omega))
End Function
```

Appendix VII

Price Comparison Between Observed Values and Estimated Prices

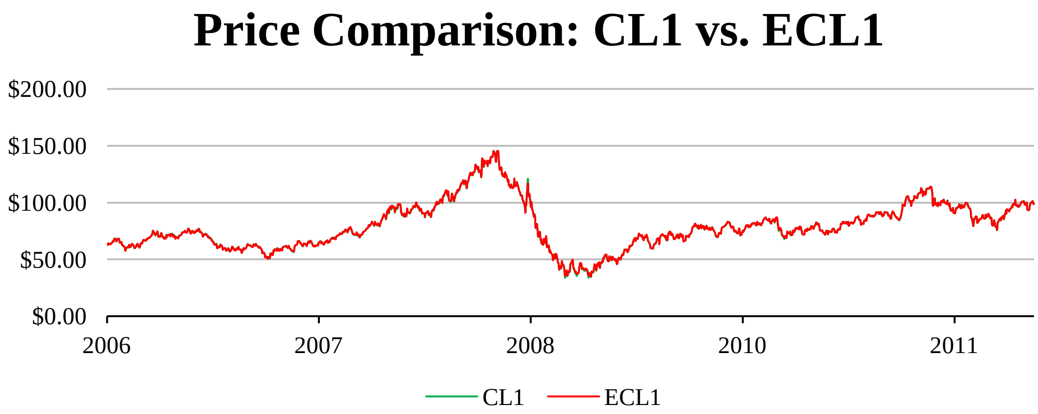


Figure 1 – CL1 versus Estimated CL1

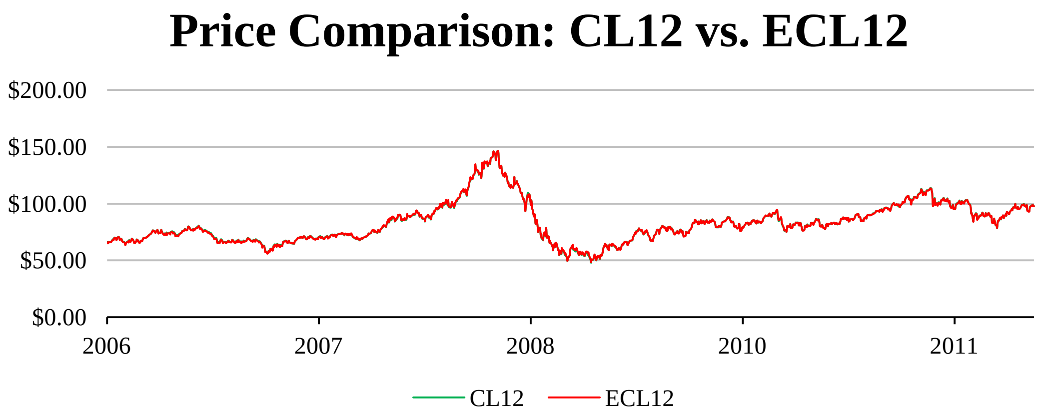


Figure 2 – CL12 versus Estimated CL12

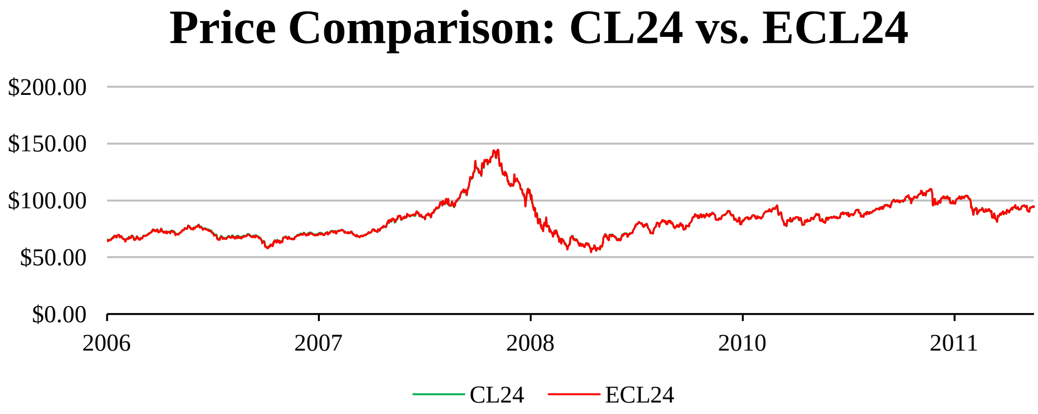


Figure 3 – CL24 versus Estimated CL24

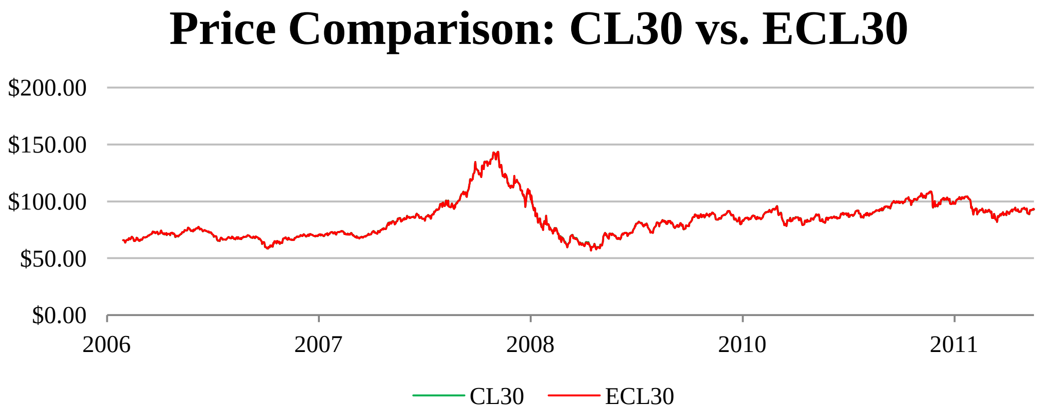


Figure 4 – CL30 versus Estimated CL30

Appendix VIII

Carmona and Durrleman Approximation

Carmona and Durrleman spread option approximation (Carmona & Durrleman, 2003):

$$\hat{c}_t^{CD} = S_{1,t}e^{-\delta_1(T-t)}\Phi(d^* + \sigma_1 \cos(\theta^* + \varphi)\sqrt{T-t}) - S_{2,t}e^{-\delta_2(T-t)}\Phi\left(d^* + \sigma_2 \sin\theta^* \sqrt{T-t}\right) - Ke^{-r(T-t)}\Phi(d^*)$$

Where:

$$d^* = \frac{1}{\sigma \cos(\theta^* - \omega)\sqrt{(T-t)}} \ln\left(\frac{S_{1,t}e^{-\delta_1(T-t)}\sigma_1 \sin(\theta^* + \varphi)}{S_{2,t}e^{-\delta_2(T-t)}\sigma_2 \sin\theta^*}\right) - \frac{1}{2}(\sigma_1 \cos(\theta^* + \varphi) + \sigma_2 \cos\theta^*)\sqrt{T-t}$$

and $\cos(\omega) = \frac{\sigma_2 - \rho\sigma_1}{\sigma}$

The factor θ^* is the optimal solution to:

$$\frac{1}{\sigma_2\sqrt{T-t}\cos\theta} \ln\left(-\frac{\sigma_1\sqrt{T-t}Ke^{-r(T-t)}\sin(\theta+\varphi)}{S_{2,t}e^{-\delta_2(T-t)}\left[\sigma_1\sqrt{T-t}\sin(\theta+\varphi) - \sigma_2\sqrt{T-t}\sin\theta\right]}\right) - \frac{\sigma_2\sqrt{T-t}\cos\theta}{2}$$
$$= \frac{1}{\sigma_1\sqrt{T-t}\cos(\theta+\varphi)} \ln\left(-\frac{\sigma_2\sqrt{T-t}Ke^{-r(T-t)}\sin(\theta)}{S_{1,t}e^{-\delta_1(T-t)}\left[\sigma_1\sqrt{T-t}\sin(\theta+\varphi) - \sigma_2\sqrt{T-t}\sin\theta\right]}\right) - \frac{\sigma_1\sqrt{T-t}\cos(\theta+\varphi)}{2}$$

where $\cos\varphi = \rho$

Appendix IX

Cholesky Decomposition

This example is adapted from the lecture notes (version August 2011) of the elective course Asset Allocation held by Professor M. Marekwica at Copenhagen Business School.

In case of a $n \times n$ covariance matrix $\boldsymbol{\Sigma}$ we have to find the lower triangular matrix $\boldsymbol{A}$, such that the following equation holds:

$$\begin{bmatrix} A_{11} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} \cdot \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ 0 & A_{22} & \dots & A_{n2} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & A_{nn} \end{bmatrix} = \boldsymbol{\Sigma}$$

This leads to the system of equations:

$$A_{11}^2 = \Sigma_{11}$$
$$A_{21}A_{11} = \Sigma_{21}$$
$$\vdots$$
$$A_{n1}A_{11} = \Sigma_{n1}$$
$$A_{21}^2 + A_{22}^2 = \Sigma_{22}$$
$$\vdots$$
$$A_{n1}^2 + \dots + A_{nn}^2 = \Sigma_{nn}$$

The system of equations can be sequentially solved, because in each equation exactly one new entry in the matrix $\boldsymbol{A}$ is defined, so:

$$\Sigma_{ij} = \sum_{k=1}^{j} A_{ik}A_{jk} \text{ for } j \leq i$$

this implies:

$$A_{ij} = \frac{\left(\Sigma_{ij} - \sum_{k=1}^{j-1} A_{ik}A_{jk}\right)}{A_{jj}} \text{ for } j < i$$

and

$$A_{ii} = \sqrt{\Sigma_{ii} - \sum_{k=1}^{i-1} A_{ik}^2} \text{ for } j = i$$

Example:

Assume $\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 13 & 23 \\ 4 & 23 & 77 \end{bmatrix}$ using the algorithm we have just explained we can find the lower triangular matrix $\boldsymbol{A}$:

$$A_{11} = \sqrt{\Sigma_{11}} = \sqrt{1} = 1$$

$$A_{21} = \frac{\Sigma_{21}}{A_{11}} = \frac{2}{1} = 2$$

$$A_{22} = \sqrt{\Sigma_{21} - A_{21}^2} = \sqrt{13 - 2^2} = 3$$

$$A_{31} = \frac{\Sigma_{31}}{A_{11}} = \frac{4}{1} = 4$$

$$A_{32} = \frac{\Sigma_{31} - A_{31}A_{21}}{A_{22}} = \frac{23 - 4 \cdot 2}{3} = 5$$

$$A_{33} = \sqrt{\Sigma_{33} - A_{31}^2 - A_{32}^2} = \sqrt{77 - 4^2 - 5^2} = 6$$

That is, $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$. Now we can use this matrix to generate random numbers that are not anymore independent among them.

Appendix X

Results of Option Pricing

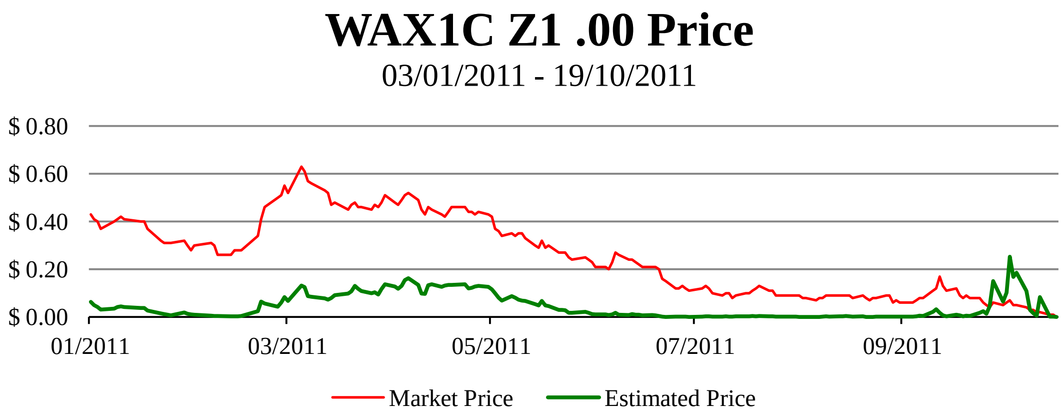


Figure 1 – Comparison between observed price on the market and estimated price of WAX1C Z1 .00

Period: 03/01/2011 - 19/10/2011

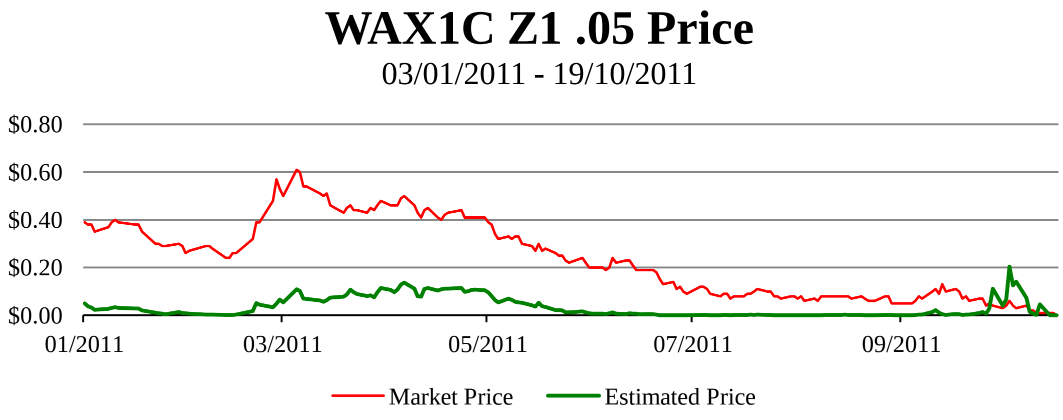


Figure 2 – Comparison between observed price on the market and estimated price of WAX1C Z1 .05

Period: 03/01/2011 - 19/10/2011

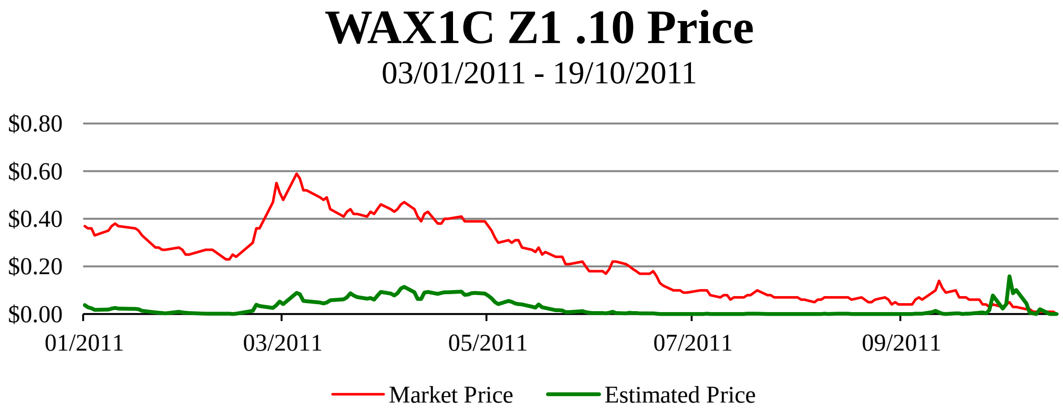


Figure 3 – Comparison between observed price on the market and estimated price of WAX1C Z1 .10
Period: 03/01/2011 - 19/10/2011

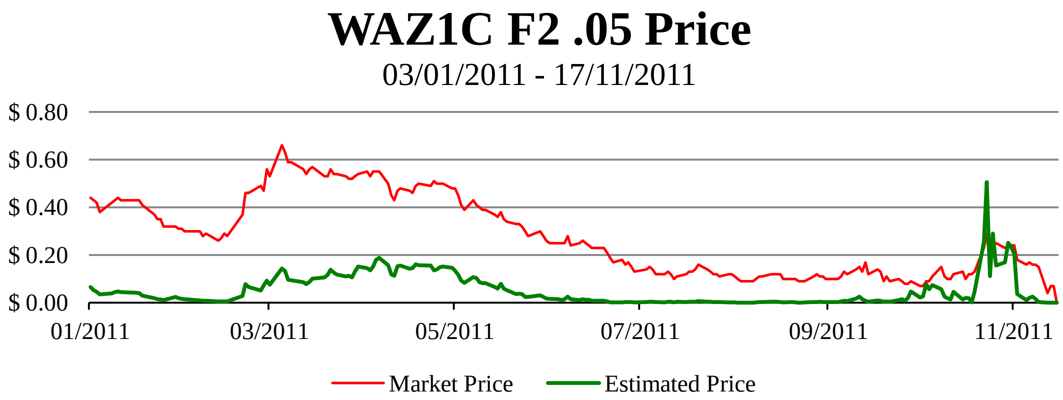


Figure 4 – Comparison between observed price on the market and estimated price of WAZ1C F2 .05
Period: 03/01/2011 - 17/11/2011

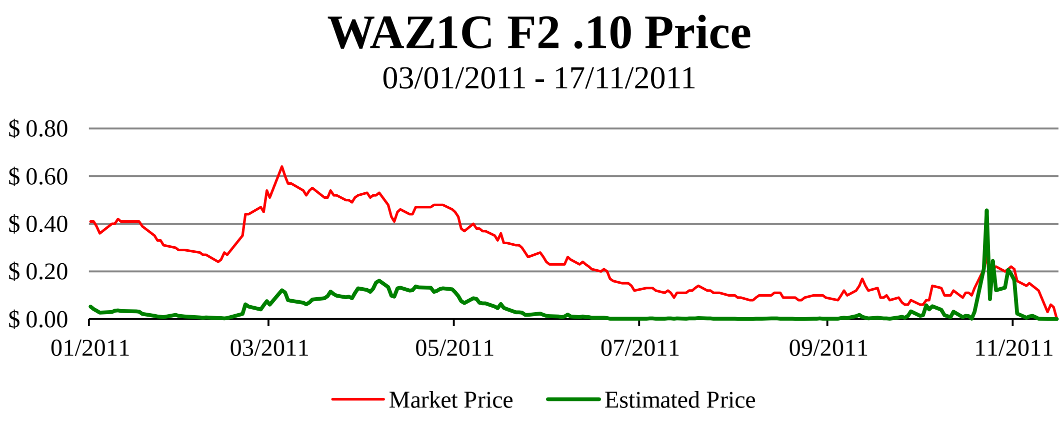


Figure 5 – Comparison between observed price on the market and estimated price of WAZ1C F2 .10

Period: 03/01/2011 - 17/11/2011

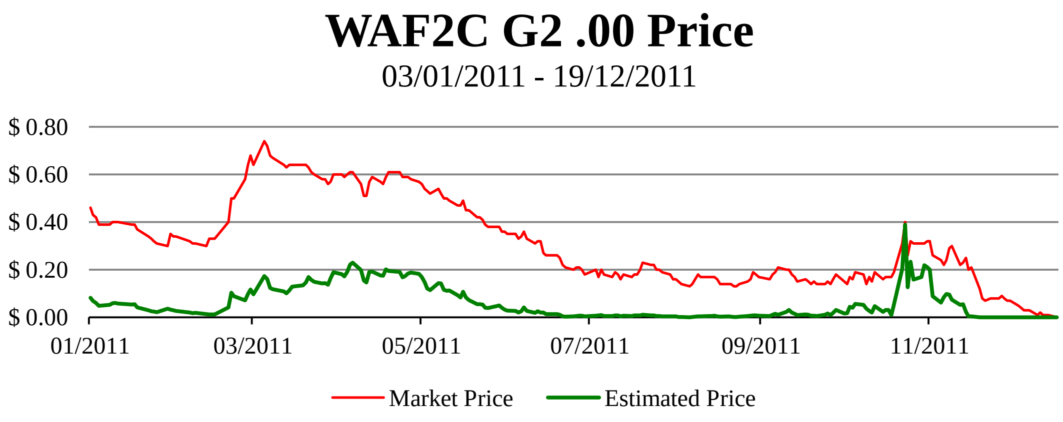


Figure 6 – Comparison between observed price on the market and estimated price of WAF2C G2 .00

Period: 03/01/2011 - 19/12/2011

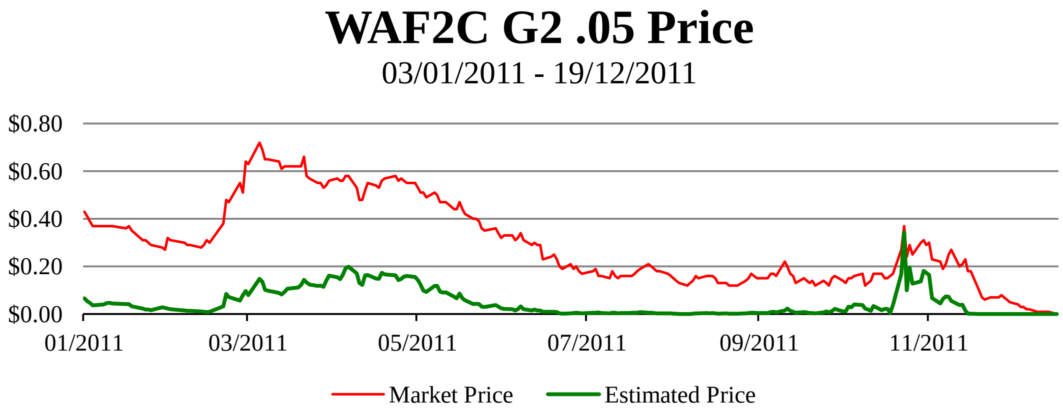


Figure 7 – Comparison between observed price on the market and estimated price of WAF2C G2 .05

Period: 03/01/2011 - 19/12/2011

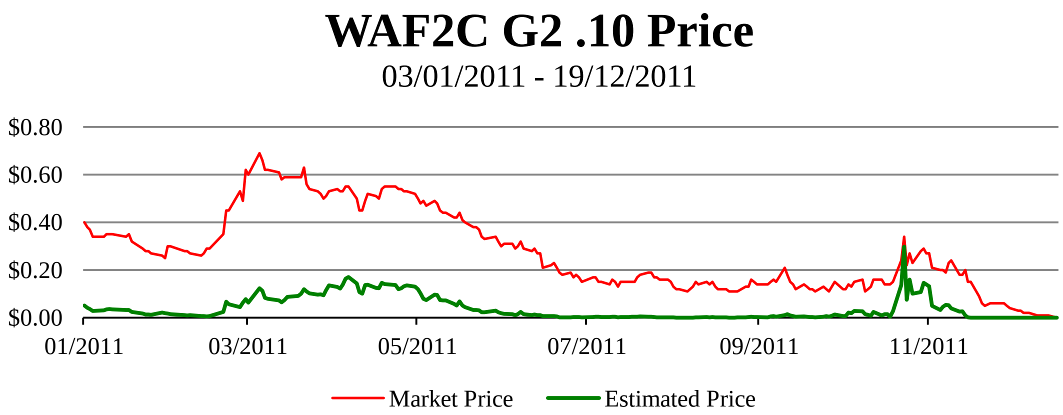


Figure 8 – Comparison between observed price on the market and estimated price of WAF2C G2 .10

Period: 03/01/2011 - 19/12/2011

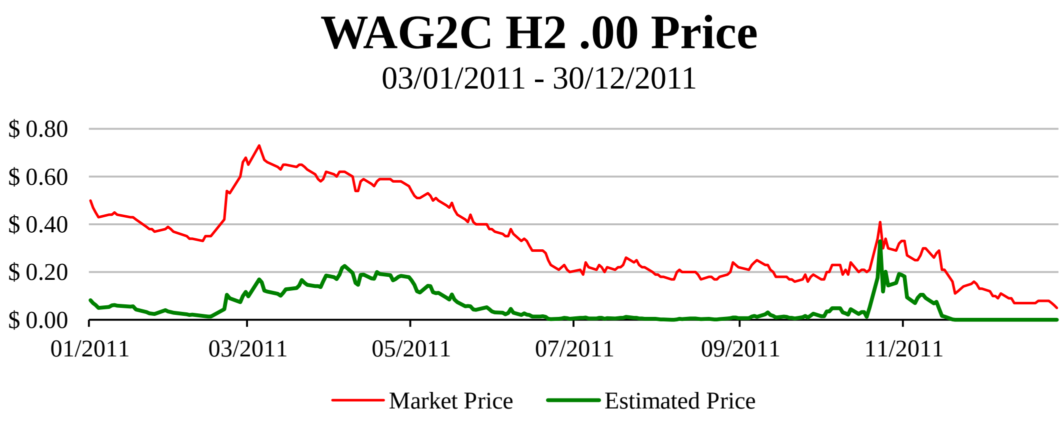


Figure 9 – Comparison between observed price on the market and estimated price of WAG2C H2 .00

Period: 03/01/2011 - 30/12/2011

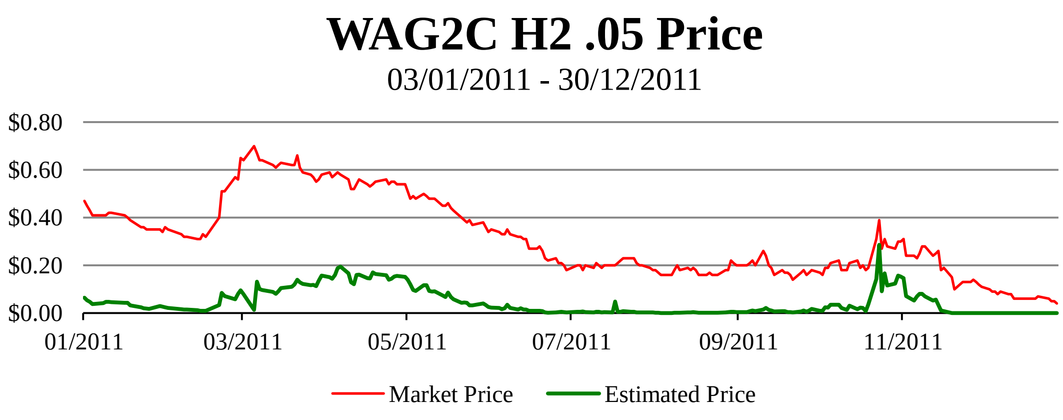


Figure 10 – Comparison between observed price on the market and estimated price of WAG2C H2 .05

Period: 03/01/2011 - 30/12/2011

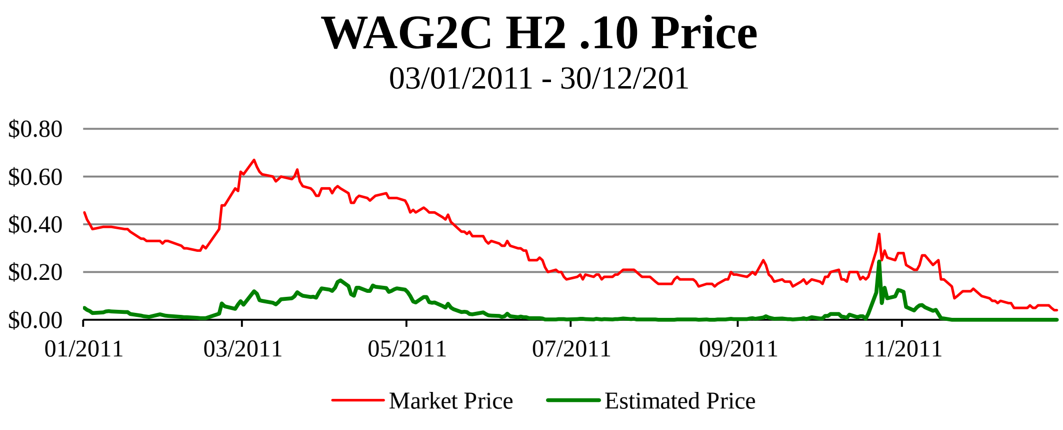


Figure 11 – Comparison between observed price on the market and estimated price of WAG2C H2 .10
Period: 03/01/2011 - 30/12/2011

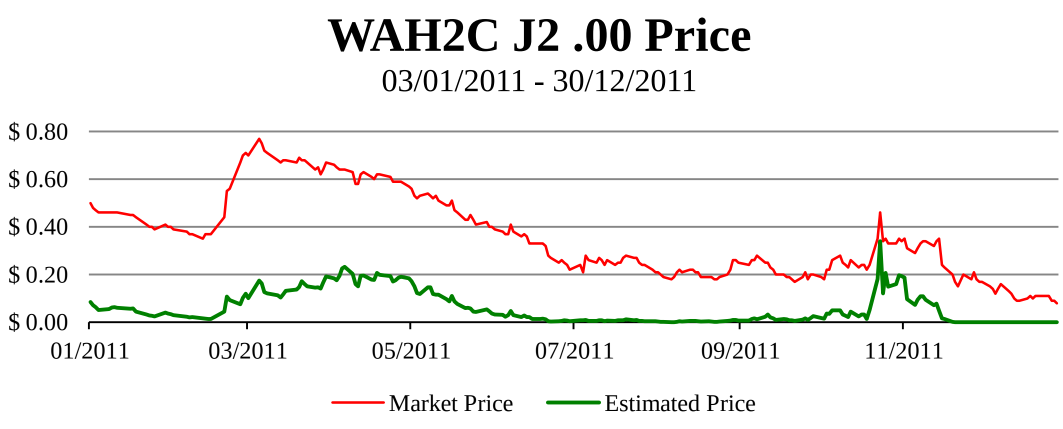


Figure 12 – Comparison between observed price on the market and estimated price of WAH2C J2 .00
Period: 03/01/2011 - 30/12/2011

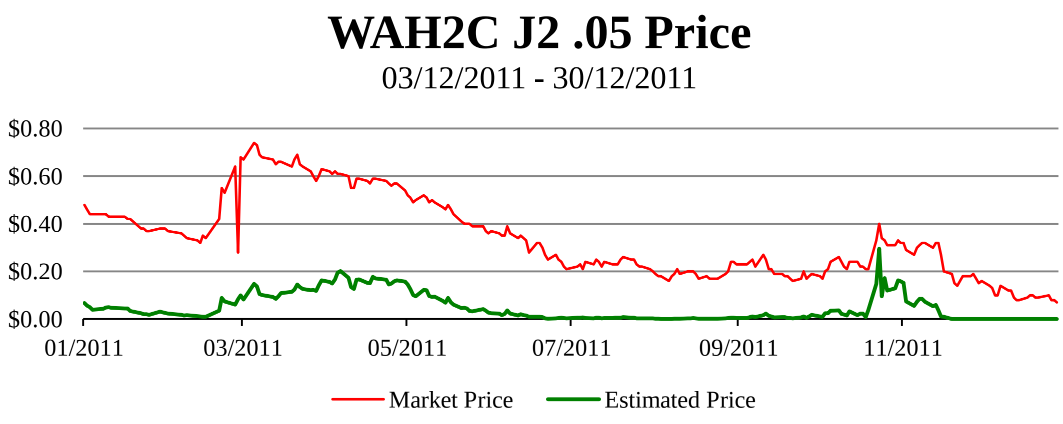


Figure 13 – Comparison between observed price on the market and estimated price of WAH2C J2 .05
Period: 03/01/2011 - 30/12/2011

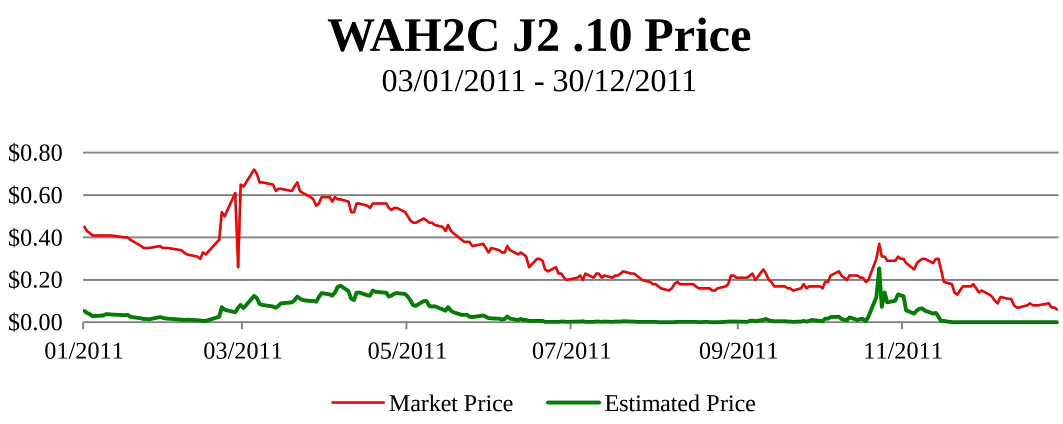


Figure 14 – Comparison between observed price on the market and estimated price of WAH2C J2 .10
Period: 03/01/2011 - 30/12/2011