Portfolio Optimization in a Downside Risk Framework

September 10

Executive Summary

The present study examines how downside risk measures perform in an investment management context compared to variance or standard deviation. To our knowledge, this paper is the first to include several acknowledged downside risk measures in a thorough analysis where their different properties are compared with those of variance

Risk is an essential factor to consider when investing in the capital markets. The question of how one should define and manage risk is one that has gained a lot of attention and remains a popular topic in both the academic and professional world. This study considers six different downside risk measures and tests their relationship with the cross-section of returns as well as their performance in portfolio optimization compared to variance.

The first part of the analysis suggests that the conditional drawdown-at-risk explains the cross-section of returns the best across methodologies and data frequency. Conditional value-at-risk explains the daily returns the best but the worst in monthly returns. Variance, together with semivariance, perform average in both data frequencies.

The second part of the analysis concludes that conditional value-at-risk and conditional drawdown-at-risk are the two superior risk measures whereas semivariance is the worst performing risk measure – mainly caused by the poor performance during bull markets. Again, variance performs average compared to the downside risk measures in most aspects of this analysis.

Overall, this thesis shows that the choice of risk measure has a significant effect on the portfolio optimization process. The analysis suggests that some downside risk measures outperform variance while others fail to do so. This suggest that downside risk *can* be a better tool in investment management than variance.

Table of Contents

1 Chapter I: Introduction

One of the greatest contributions to the financial theory of today is the establishment of a formal risk/return framework by Nobel laureate, Harry Markowitz, which laid the foundations of what we know as Modern Portfolio Theory (MPT). Markowitz (1952) pioneered the issue of portfolio optimization with a seminal article, which was later expanded into a seminal book (Markowitz, 1959). By quantifying investment risk in the form of the variance or standard deviation of returns, Markowitz gave investors a mathematical approach to asset selection and portfolio management. Markowitz used mean returns, variances and covariances to derive an efficient frontier where every portfolio maximizes the expected return for a given variance (or minimizes variance for a given expected return). This is popularly called the mean-variance (MV) criterion. While MPT a la Markowitz has revolutionized the investment world, it has also received substantial criticism. This criticism is usually centered on two main assumptions of the MV framework: 1) investors' attitude towards risk can be explained by a quadratic utility function, or 2) asset returns can be adequately represented by the normal distribution. Quadratic utility is very unlikely because it implies increasing absolute risk aversion. Furthermore, asset returns often exhibit skewness and/or excess kurtosis, which violates the second assumption of normality and makes the mean-variance approach limited in many cases. In fact, investment returns have unlimited upside potential and can only yield limited losses, making the asymmetrical behavior rather logical.

It should be noted that Markowitz and William Sharpe, the other originator of MPT, from the very beginning acknowledged these limitations and favored another measure of risk that focused on the downside.

"Under certain conditions, the mean-variance approach can be shown to lead to unsatisfactory predictions of behavior. Markowitz suggests that a model based on the semi-variance would be preferable; in light of the formidable computational problems, however, he bases his analysis on the variance and standard deviation." – Sharpe, W. F. (1964)

In other words, MPT is limited by measures of risk that do not always represent the realities of the investment markets, and perhaps a downside risk measure such as semivariance is preferable. Fortunately, advances in portfolio and financial theory, coupled with increased computer power, have overcome these limitations. It is then curious that both practitioners and academics have been optimizing portfolios for more than 50 years using variance (or standard deviation) as a measure of risk.

With substantial evidence that returns are asymmetric and that investors do not exhibit quadratic utility, downside risk has been gaining increasing attention, and numerous magnitudes that capture downside risk are now well known and widely used. Results from previous studies in this field are quite disparate and the question remains whether downside risk measures lead to more efficient allocation than variance.

1.1 Problem Statement

This paper focuses on the differences and similarities between variance and several downside risk measures in a global asset allocation context. We address this issue from both a theoretical and an empirical point of view. The overall aim of this paper is to determine whether or not investors should view and manage risk as a domain of bad outcomes (i.e. downside) in order to efficiently invest in the global capital markets. In order to answer this question, we analyze several properties of a set of risk measures in an attempt to clarify the advantages and drawbacks of each. The questions we seek to answer are summarized below.

How do downside risk measures perform in an investment management context compared to variance?

- How do investors view risk?
- How do the different risk measures explain the cross-section of equity returns?
    - How is this affected by the frequency of returns?
    - How is this affected by market conditions?
- Which effect has the choice of risk measure on portfolio allocation?
    - How do the different risk measures perform over time?
    - How sample-specific are our results?

Previous work on this subject has mostly consisted of comparing variance with a single downside risk measure such as semivariance or value-at-risk (VaR). To our knowledge, this paper is the first to include several of the most acknowledged downside risk measures in a thorough analysis where their different properties are compared with those of variance. Additionally, some of the frameworks and methodologies applied in this paper are our own design, inspired by similar existing theories and heuristics.

1.2 Delimitations

The purpose of this section is to outline some areas that will not be touched upon throughout this paper. We will divide these areas into two parts, namely theoretical delimitations and empirical delimitations. Since the scope of this paper is fairly broad, it is important to keep in mind that the following will not be addressed.

1.2.1 Theoretical Delimitations

The theory employed in this paper is generally based on modern portfolio theory. It is important to remember that, even though we focus mainly on the risk aspect, this is not a study of risk management. Naturally, some elements of risk management will be included since we are dealing with downside risk, but the reader should not expect this to be a key part. Thus, portfolio theory is at the heart of this thesis. In our research and assessment of different downside risk measures, we have selected those we found most intuitive and interesting. Obviously, one could have included additional risk variables but due to limited time and space, we have chosen an amount that fits the scope of our research.

In addition, some introductory utility and prospect theory will be presented. The purpose is to give the reader a basic, yet nuanced, understanding of how investors view risk. Thus, the section addressing investor preferences aims to provide a background for why downside risk can be an important element for an investment manager, and we will not go any deeper into utility theory or prospect theory. That is, the empirical analysis will not deal with investor preferences at all.

Finally, we will not present any econometric theory since all our regressions are ordinary least squares (OLS) and we expect the reader to be familiar with the underlying assumptions. Instead, we will address any problems or violations of these assumptions we may encounter in the empirical analysis.

1.2.2 Empirical Delimitations

The empirical analysis consists of two parts. In the first part, we consider the relationship between the individual risk measures and the cross-section of returns. We will not look into the relationship with a time series of returns because we wish to establish a ranking of the explanatory power of the risk variables in general, i.e. across all our securities. The data used is rather extensive and we do not see any reason to include additional securities, which would only make the analysis intractable. The securities included are stocks, equity indices and government bonds. Thus, we will not deal with options, futures or other derivatives. Likewise, we do not include corporate bonds because we would then have to account for credit risk, which falls outside the scope of the thesis.

Moreover, we have data spanning 30 years back, which is the longest available in Thomson Datastream. We want to get all the data from the same source (Datastream) in order to make it more comparable. Since we want to deal with a globally diversified European investor, all of the data is in Euros. Consequently, exchange rate risk will not be addressed.

In the second part of the analysis, we focus on minimum risk portfolios in order to compare the performance of the different risk measures. This part is not supposed to suggest an optimal portfolio for an investor but rather to analyze the dynamics of using different measures of risk when compared to variance. Furthermore, the paper does not address the financing of the investments, i.e. we do not consider how the capital to be invested has been raised. This is because we want to preserve a focus on how the different risk measures affect the performance of optimized portfolios. Finally, we do not consider taxes and transaction costs, as these are unrelated to market risk.

1.3 Methodology

According to Andersen (2005), the methodology may be viewed as the procedure of theory and data gathering, application, analyses, etc. used in order to generate new knowledge. There exist numerous methodic approaches depending on the knowledge needed, and the choice hereof should always be made so that it fits the subject being assessed in a logical way. Hence, the purpose of this section is to clarify *how* the subject of the paper has been researched and handled.

The main elements of the thesis are the theory and the empiricism, and the interaction between the two conforms the core in the production of new knowledge, which is an iterative process. That is, first, the theory is gathered and assessed and then it is applied in an empirical analysis, which leads us back to assessing the theory and even gathering new theories that better fit the subject at hand. This way, the knowledge generation becomes an iterative and rather dynamic process.

In order to attain a comprehensive theoretical basis for the empirical analysis, we review and apply a broad range of economic and financial literature. Primarily, we use the original articles of many different academics[1], which presents the challenge of connecting the dots while preserving focus on the overall objective. For example, one author may introduce a risk measure that succeeds in explaining the cross-section of stock returns while another author may present a risk measure that works well with other tasks. This means that we must assess the relevance of the given risk measure (or theory) in the context of our thesis subject and, where possible, extend the existing theory to incorporate relevant and useful properties. The manifold sources, however, ensure a thorough and unbiased starting point for the empirical analysis and assessment of results.

In addition to articles from academic journals, we have drawn on several textbooks[2]. These are convenient because they cover topics thoroughly (drawing on the work of many different academics) and are usually unbiased. The theories in the paper include both qualitative and quantitative elements that lay the foundation for the empirical analysis.

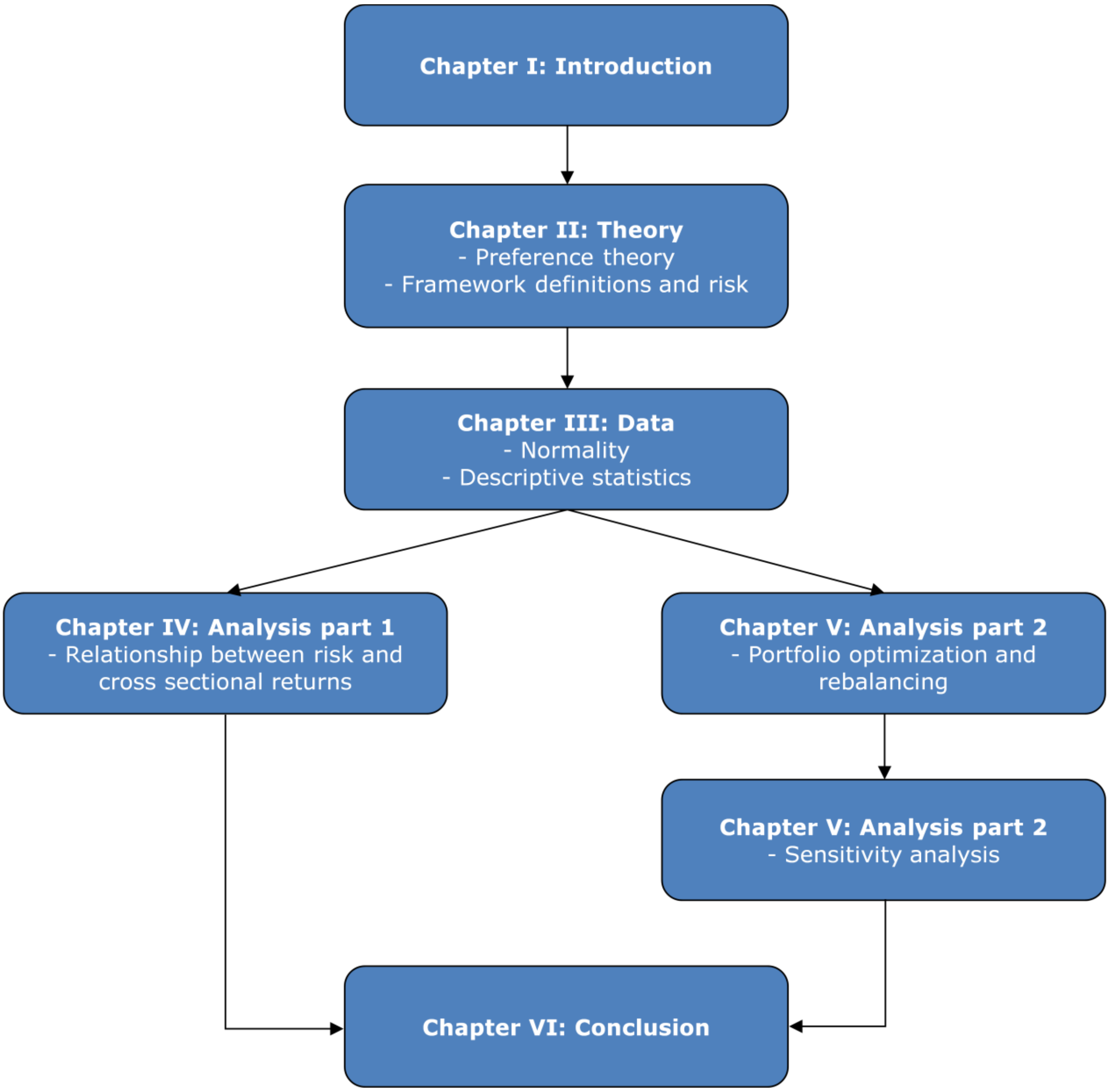
For the empiricism, we gather and apply data from Thomson Datastream exclusively, which contains historical time series of more than two million securities. We believe that the data from Datastream possess the high quality we need to draw valid inferences from the empirical analysis. All the calculations are made in SAS, Excel (and VBA) and R.

1.4 Thesis Structure

This section aims to describe the structure of the thesis. Overall, the thesis consists of six chapters, which are subdivided into a number of sections. We will illustrate the logic behind the structure of the thesis and briefly explain what the reader will find in each chapter.

[1] Including Markowitz (1952), Bawa et al (1977), Estrada (2005)
[2] Including Elton et al (2007)

Figur 1-1: Thesis structure



Chapter 1: Introduction

Chapter 1 aims to give the reader an introduction to the thesis topic and the problem addressed throughout the analysis. Furthermore, the methodology applied throughout the process is presented.

Chapter 2: Theory

The purpose of chapter 2 is twofold. First, investor preferences are reviewed. The purpose is to introduce expected utility and prospect theory and clarify how investors view risk in practice. After reading this section, the reader should have a basic understanding of how utility/value can be quantified and how risk should (and should not) be measured.

The second part of the chapter presents the frameworks and risk measures that will be used in the analysis. It starts out by reviewing the classical mean-variance framework of modern portfolio theory to give the reader a basic understanding of its assumptions, applications, and definitions of risk. Subsequently, alternative frameworks and risk measures are reviewed. These risk measures include lower partial moments and semivariance, the conditional value-at-risk, three measures of drawdowns, and an ad hoc risk measure called the gain-loss spread. The purpose is to give the reader an understanding of what the different risk variables measure, and how we can apply the frameworks in practice.

Chapter 3: Data

This chapter introduces the data used throughout our analysis. The reader is presented to the asset universe, time horizon, return definitions etc. In addition, the asset returns are tested for normality using Jarque-Bera tests and persistency checks. Finally, some descriptive statistics are shown, which should give the reader a general idea of the different characteristics of the securities.

Chapter 4: Analysis part 1

The purpose of chapter 4 is to determine the relationship between risk and the cross-section of returns. The analysis involves an evaluation of correlations, regression analysis, and economic significance. We analyze the differences when using daily versus monthly returns as well as during bear and bull markets. The intention of this chapter is to determine whether downside risk explains returns to a higher degree and therefore could form the basis of a profitable investment strategy.

Chapter 5: Analysis part 2

In this chapter, we assume a highly risk-averse investor and calculate the optimal portfolios by minimizing the different measures of risk. Initially, we show how these portfolios and other efficient portfolios can be derived in order to illustrate the efficient frontiers in the respective mean-risk frameworks. In order to compare the performance of the different portfolios, we do a yearly rebalancing of each assuming an initial investment of €100. This is also helpful when analyzing the dynamics between portfolio performance and market fluctuations, which is not captured when using the whole period. Finally, we consider the impact of sample sensitivity on the optimal weights and thus performance of the constructed portfolios.

Chapter 6: Conclusion

Chapter 6 presents the overall conclusions of the empirical analyses and aims to answer the questions in the problem statement of the thesis.

2 Chapter II: Theory

2.1 Preference Theory

2.1.1 Expected Utility Theory

Today, stock market participants have ample opportunities to construct portfolios corresponding to their individual needs and preferences. Von Neumann & Morgenstern (1944) laid the foundation for the utility theory used to this day, which is based on the key assumption that investors do not choose the alternative that yields the highest expected return but rather choose the alternative that yields the highest expected utility. However, in most situations, such a selection is not possible since complete information about an individual's preference set and hence their utility function is not available. Thus, with imperfect information we want to determine a certain restricted class of utility functions that our individual's utility belongs to. The more restrictive the class of utility functions, the smaller will be the admissible set and thus the more useful will it be in practice. However, more restrictions imply that the admissible set is relevant for a smaller group of individuals and may then lead to a loss in generality. Therefore, we want to determine the admissible set of alternatives for the most restrictive class of utility functions that is consistent with observed economic phenomena.

Arrow (1971) and Pratt (1964) point out that the observations of certain economic phenomena indicate that individual utility functions exhibit decreasing absolute risk aversion (DARA) and to a lesser extent increasing relative risk aversion (IRRA). has raised doubts that IRRA is a plausible assumption, thus it appears that the DARA family is the most restrictive class of utility functions acceptable to most economists. Furthermore, empirical studies by Blume & Friend (1975) and Cohn et al (1975) reveal constant relative risk aversion and decreasing relative risk aversion respectively.

Traditional portfolio theory a la Markowitz (1952) assumes that stock returns can be described by a special case of the location-scale distributions, namely the normal distribution. With the assumption of normality, an investor must only relate to the first two moments of the distribution, that is the mean and variance. Thus, the efficient set of portfolios is obtained by discarding those with a lower mean and a higher variance. Even though this approach has been highly recognized as a valid framework for investment decisions, it has also been known for many years that it is of limited generality since it assumes a quadratic utility function or normality in the return distributions. Arrow (1971) and Hicks (1962) have pointed out that the assumption of quadratic utility is highly implausible because it implies increasing absolute risk aversion (IARA). Also, the assumption that returns on risky investments are normally distributed is unrealistic as it rules out asymmetry or skewness. In fact, several studies including Cootner (1964) and Lintner (1972) show that risky portfolio returns are very unlikely to be normally distributed. Furthermore, there is substantial empirical evidence of the presence of skewness in stock return distributions.

2.1.1.1 Risk Aversion

We can define the risk premium and risk aversion with the help of the derivatives of the utility functions. Markowitz (1959) defines the risk premium as being the maximum amount that an individual is ready to give up to avoid uncertainty. The basis for an investor's decision making is calculated as the difference between the utility of the expected wealth and the expected utility of the wealth.

$$U[E(W)] - E[U(W)]$$

We can illustrate this with a simple example where it is assumed that an investor has a square root utility function of wealth:

$$U(w) = \sqrt{w} = w^{0,5}$$

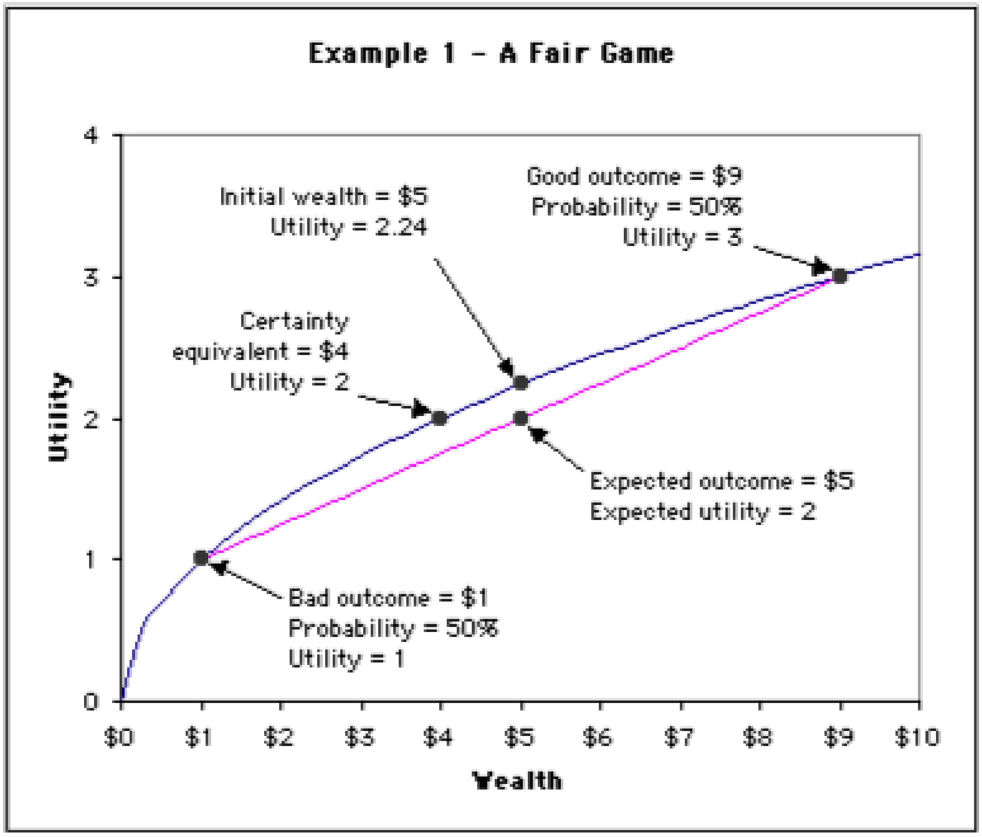
$$\Downarrow$$

$$U'(w) = 0.5w^{-0.5} > 0$$

$$U''(w) - 0.25w^{-1.5} < 0$$

We assume that the investor's current wealth is \$5 and that the investment available is a "fair game", i.e. it is equivalent to a coin toss where she can either win \$4 or lose \$4. Thus, the expected return of this investment is 0%. We assume that our investor only has two choices: she can either refuse to invest, keeping her initial \$5 corresponding to a utility of $\sqrt{5} = 2.24$, or she can invest, which yields an expected utility of $0.5 \cdot \sqrt{1} + 0.5 \cdot \sqrt{9} = 2$. Because our investor is maximizing her utility, and because 2.24 > 2, she will not invest. Figure 2-1 below illustrates our investor's current wealth and utility, the wealth and utility of the two possible outcomes, and the expected outcome and utility of the outcome of the investment.

Figure 2-1: Example of a fair game. Source: Norstad (1999)



The investor in our example is risk-averse since $U[E(W)] > E[U(W)]$, which makes the utility function concave. This means, in general, that she will always refuse to invest where the expected return is 0%. If the expected return had been greater than 0%, the investor may or may not have chosen to invest, depending on the functional form of her utility function.
For example, if the probability of the good outcome in this example were 75% instead of 50%, we would have an expected outcome of $7, corresponding to an expected gain of $2 and an expected return of 40%. This would yield an expected utility of $0.25 \cdot \sqrt{1} + 0.75 \cdot \sqrt{9} = 2.5 > 2.24$ and our investor would be willing to invest. The expected return of 40% is a *risk premium*, which compensates her for undertaking the risk involved with the investment. In our example, the investor attaches greater weight to losses than she does to gains of equal magnitude. The loss of $4 means a decrease in utility of 1.24, while the gain of $4 means an increase in utility of only 0.76.

In the example above, we dealt with a risk averse investor because the utility of her expected wealth was greater than the expected utility of wealth. The sign of this difference allows us to determine the individual's attitude towards risk in the following way:

- If $U[E(W)] > E[U(W)]$, then the utility function is concave and the investor is risk averse;
- If $U[E(W)] = E[U(W)]$, then the utility function is linear and the investor is risk neutral;
- If $U[E(W)] < E[U(W)]$, then the utility function is convex and the investor is risk seeking.

In Markowitz' modern portfolio theory, investors are assumed to be risk averse. Absolute risk aversion (ARA) measures the risk aversion for a given level of wealth, and it is computed as

$$ARA = -\frac{U''(W)}{U'(W)}$$

while relative risk aversion (RRA) is given by

$$RRA = -W\frac{U''(W)}{U'(W)}$$

Constant relative risk aversion (CRRA) means that the loss amount tolerated by an investor increases proportionally to the increase in the investor's wealth.

Example 1: quadratic utility function

Consider a quadratic utility function:

$$U(W) = aW - bW^2$$

Taking its first two derivatives yields:

$$U'(W) = a - 2bW \qquad \textit{and} \qquad U''(W) = -2b$$

We can now deduce the ARA and RRA:

$$ARA = \frac{2b}{a-2bW} \qquad \textit{and} \qquad RRA = \frac{2bW}{a-2bW} = \frac{2b}{\frac{a}{W}-2b}$$

Taking the first derivative of ARA wrt. wealth, W, yields

$$\frac{\partial ARA}{\partial W} = \frac{4b^2}{(a-2bW)^2} > 0$$

Absolute risk aversion is therefore an increasing function of W. In the same way, relative risk aversion is also an increasing function of W.

$$\frac{\partial RRA}{\partial W} = \frac{\frac{a}{W^2}2b}{\left(\frac{a}{W} - 2b\right)^2} > 0$$

These results suggest that an investor becomes more risk averse as her wealth increases, which is counterintuitive. This is exactly one of the disadvantages of the quadratic utility function assumed by Markowitz and applied in the mean-variance framework of modern portfolio theory. As noted in the previous section, we would wish to have a utility function that exhibits a decreasing absolute risk aversion (DARA).

Example 2: logarithmic utility function

$$U(W) = \ln(W)$$

We take its first two derivatives and deduce the ARA and RRA:

$$U'(W) = \frac{1}{W} \qquad and \qquad U''(W) = -\frac{1}{W^2} \qquad \Rightarrow$$

$$ARA = \frac{1}{W} \qquad and \qquad RRA = 1$$

The ARA function is then a decreasing function of wealth and the RRA function is constant. The logarithmic utility function is therefore consistent with the behavior of a risk averse investor.

Table 2-2 summarizes the ARA and RRA of three well known functional forms of utility.

Table 2-1: Utility Functional Forms and Risk Aversion

Utility / Risk Aversion	*Absolute Risk Aversion*	*Relative Risk Aversion*
Quadratic	Increasing	Increasing
Exponential	Constant	Increasing
Logarithmic	Decreasing	Constant

2.1.1.2 Axioms of Expected Utility Theory

Expected utility theory incorporates the following normative descriptions of preferences or axioms.

- Risk aversion: Individuals are risk averse, i.e. they will always prefer a riskless investment to any risky investment yielding the same expected return. This is illustrated in a traditional utility function, which has the characteristic concave shape.

- Transitivity: If A is preferred to B, which in turn is preferred to C, then transitivity governs that A will be preferred to C.
- Substitution: If A is preferred to B, then an even chance to receive A or C is preferred to an even chance to receive B or C.
- Dominance: If A yields outcomes at least as high as B in all states, and higher in at least one state, then A dominates B and is preferred.
- Invariance: The preference order of a series of prospects is invariant of the manner in which the prospects are presented – or framed, which is the term used by Kahneman & Tversky (1984). As a consequence, two versions of a choice problem that are recognized to be equivalent when shown together should elicit the same preference even when shown separately (see below).

2.1.1.3 Violations of the Axioms of Expected Utility Theory

Decision making under risk can be viewed as a choice between prospects or gambles. Following Kahneman & Tversky (1979), we let a prospect $(x_1, p_1; \ldots; x_n, p_n)$ be a contract that yields outcome x_i with probability p_i, where $p_1 + p_2 + \ldots + p_n = 1$. For simplicity, we omit all null outcomes (where either x_i or p_i is zero) and use (x, p) to denote the prospect $(x, p; 0, 1-p)$ that yields x with probability p and 0 with probability $1 - p$.

The application of expected utility theory to choices between prospects is based on the following three tenets.

(i) Expectation: $U(x_1, p_1; \ldots; x_n, p_n) = p_1 u(x_1) + \ldots + p_n u(x_n)$. That is, the overall utility of a prospect equals the expected utility of its outcomes.

(ii) Asset Integration: $(x_1, p_1; \ldots; x_n, p_n)$ is acceptable at asset position w only if $U(w + x_1, p_1; \ldots; w + x_n, p_n) > u(w)$. That is, a prospect is acceptable if the utility resulting from integrating it with one's current assets exceeds the utility of those assets alone.

(iii) Risk Aversion: u is concave $(u'' < 0)$ as explained in the previous section.

In expected utility theory, the utilities of outcomes are weighed by their probabilities. Kahneman & Tversky (1979) illustrate how people's preferences systematically violate this

principle because they overweigh outcomes that are considered certain, relative to outcomes that are merely probable – a phenomenon that they label the *certainty effect*. This effect can be illustrated by the example below, which is
a pair of choice problems where *N* denotes the number of respondents who answered
each problem, and the percentage that chooses each option is given in brackets.

Table 2-2: Pair of Choice Problems (asterisks denote significance at 0.01 level). Source: Kahneman & Tversky (1979)

Problem 1				Problem 2			
Prospect	*Outcome*	*Probability*		*Prospect*	*Outcome*	*Probability*	
A	2500	0.33		C	2500	0.33	
	2400	0.66	(18)		0	0.67	(83)*
	0	0.01					
B	3000	1.00		D	2400	0.34	
	0	0.00	(82)*		0	0.66	(17)
N=72				N=72			

In Problem 1, the expected payoff of prospect A is 2,409 and thus higher than that of prospect B, which is 2,400. In Problem 2, prospect C has an expected payoff of 825 while prospect D has an expected payoff of only 816. The respondents showed a significant preference for prospect B and prospect C in Problem 1 and Problem 2 respectively (in fact, 61% of the respondents chose the a combination of prospects B and C). This pattern of preferences violates expected utility theory, which in Problem 1 would have revealed the following preference order (setting $u(0) = 0$):

$$u(2{,}400) > 0.33u(2{,}500) + 0.66u(2{,}400) \quad \Leftrightarrow \quad 0.34u(2{,}400) > 0.33u(2{,}500),$$

which conflicts with the results from Problem 2, which show a clear preference for prospect C rather than prospect D. It is clear from this example that people overweigh certain outcomes relative to probable outcomes, i.e. the certainty effect obtains. This clearly conflicts with the notion of weighing by absolute probabilities as suggested by expected utility theory.
The substitution axiom in expected utility theory suggests that if one prospect is preferred to another, then any probability combination of the former must be preferred to the combination of the latter. This is does not hold as the certainty effect obtains; apparently, reducing the probability of a certain gain has greater effect than reducing the probability of an uncertain gain.

The certainty effect is not the only type of violation of the substitution axiom as is shown in the problems below.

Table 2-3: Pair of Choice Problems (asterisks denote significance at 0.01 level). Source: Kahneman & Tversky (1979)

Problem 3			
Prospect	*Outcome*	*Probability*	
A	6000	0.45	
	0	0.55	(14)
B	3000	0.90	
	0	0.10	(86)*
N=66			

Problem 4			
Prospect	*Outcome*	*Probability*	
C	6000	0.001	
	0	0.999	(73)*
D	3000	0.002	
	0	0.998	(27)
N=66			

In Problem 3, the probabilities of winning are substantial (45% and 90%), and a significant majority chooses the prospect with the highest probability. In Problem 4, there is a *possibility* of winning but the probabilities are very low. In this situation, most of the respondents choose the prospect that offers the larger gain regardless of its lower probability. Note that the probabilities of winning in both B and D are twice the probabilities of winning in A and C respectively. However there is a clear inconsistency in the decision making, which according to expected utility, should not occur. In summary, the majority of respondents seem to subjectively overestimate low probabilities.

The following set of problems illustrates how the expected utility theory assumptions of dominance, invariance and emphasis on final states are violated[3].

[3] Kahneman & Tversky (1984), pp. 5

Table 2-4: Choice Problems (asterisks denote significance at 0.01 level). Source: Kahneman & Tversky (1984)

Problem 5

Prospect	*Outcome*	*Probability*	
A	240	0.25	
	-760	0.75	(0)
B	250	0.25	
	-750	0.75	(100)
N=86			

Problem 6

Prospect	*Outcome*	*Probability*	
C	240	1.00	
	0	0.00	(84)
D	1000	0.25	
	0	0.75	(16)
N=150			

Prospect	*Outcome*	*Probability*	
E	-750	1.00	
	0	0.00	(13)
F	-1000	0.75	
	0	0.25	(87)
N=150			

It is clear from Problem 5 that prospect B dominates prospect A so that 100% of the respondents made a rational decision. In Problem 6, the majority of respondents (73%) preferred the combination C and F. However, this combination yields the exact same outcome as prospect A in Problem 5[4], which was unambiguously rejected. In fact, the combination B and E is equivalent to the dominant prospect B in Problem 5. This is a violation of several of the axioms in expected utility theory.

(i) The invariance axiom fails dramatically as the Problem 5 and Problem 6 are in fact identical and should therefore elicit the same preferences.

(ii) The dominance axiom is clearly violated as the chosen combination of C and F is dominated by the alternative combination of D and E.

(iii) The axiom that investor cares about final states rather than changes is violated as well because the respondents do not aggregate the outcomes of Problem 6 and thereby reveal the equivalence with Problem 5.

(iv) The axiom of risk aversion does not hold in prospects with negative outcomes. In fact, respondents chose the certain gain in the first decision of Problem 6, which indicates risk aversion. However, the respondents chose the probable loss over the certain loss in the second decision of Problem 6, which indicates a risk seeking behavior.

[4] C+F yields a certain 240 plus a 75% chance of losing 1,000 and a 25% chance of a zero outcome, which is equivalent to a 75% chance of losing 760 and a 25% chance of gaining 240.

Kahneman & Tversky's experiments revealed the failure of expected utility theory as a normative theory due to the many violations of its core axioms. In the next section, we will present the value function of their prospect theory as an alternative to expected utility theory.

2.1.2 The Value Function of Prospect Theory

Kahneman & Tversky (1979) present an alternative to expected utility theory that is not as restricted in its assumptions. In this section, we refrain from looking at their prospect theory in its entirety and focus only on the value function, which is the centerpiece in prospect theory, and is represented by

$$V(x,p;y,q) = \pi(p)v(x) + \pi(q)v(y) \tag{1}$$

The experimental analyses reviewed in the previous section lead Kahneman & Tversky (1979) to suggest three important characteristics of decision makers that have implications for the shape of the value function. These are addressed in the following.

Reference dependence

An essential feature of prospect theory is that the carriers of value are *changes* in wealth rather than final states. This should not be taken to imply that the value of a particular change is independent of the initial position. Value should be treated as a function of the asset position that serves as a reference point, and the magnitude of the change (positive or negative) from that reference point.

Diminishing sensitivity

As we saw in section 2.1.1.3, investors (or more generally, decision makers) are found to be risk averse in the domain of gains and risk seeking in the domain of losses. Diminishing sensitivity explains how the psychological response is a concave function of changes, that is the difference in value between a gain (loss) of 100 and a gain (loss) of 200 appears to be greater than the difference between a gain (loss) of 1,100 and a gain (loss) of 1,200. Thus, the value function for changes of wealth is normally concave above the reference point $\left(v''(x) < 0,\ \mathit{for}\ x > 0\right)$ and often convex below it $\left(v''(x) > 0,\ \mathit{for}\ x < 0\right)$. In other words, the marginal value of both gains and losses generally decreases with their magnitude.

Kahneman & Tversky (1979) present the following problem results to underpin the shape of the value function.

Table 2-1: Pair of Choice Problems (asterisks denote significance at 0.01 level). Source: Kahneman & Tversky (1979)

Problem 7				Problem 8			
Prospect	*Outcome*	*Probability*		*Prospect*	*Outcome*	*Probability*	
A	4000	0.25		A	-4000	0.25	
	2000	0.25	(82)*		-2000	0.25	(30)
	0	0.50			0	0.50	
B	6000	0.25		B	-6000	0.25	
	0	0.75	(18)		0	0.75	(70)*
N=66				N=72			

Applying equation (1) to these problems, we get

$$V(4{,}000, 0.25; 2{,}000, 0.25) > V(6{,}000, 0.25) \quad \Leftrightarrow \quad \pi(0.25)[v(4{,}000) + v(2{,}000)] > \pi(0.25)v(6{,}000)$$

and

$$V(-4{,}000, 0.25; -2{,}000, 0.25) < V(-6{,}000, 0.25) \quad \Leftrightarrow$$

$$\pi(0.25)[v(-4{,}000) + v(-2{,}000)] < \pi(0.25)v(-6{,}000)$$

hence

$v(4{,}000) + v(2{,}000) > v(6{,}000)$ implying concavity in the domain of gains.

$v(-4{,}000) + v(-2{,}000) < v(-6{,}000)$ implying convexity in the domain of losses.

Loss aversion

A salient characteristic of attitudes to changes in wealth is that losses loom larger than gains, i.e. the aggravation experienced in losing a sum of money appears to be greater than the pleasure associated with gaining the same amount. Loss aversion implies that the value function is steeper in the domain of losses than in that of gains. Kahneman & Tversky (1979) find that most people find symmetric bets of the form $(x, 0.50; -x, 0.50)$ unattractive[5]. Moreover, the aversion of this prospect generally increases with the size of the stake. That is, if $x > y \geq 0$, then $(y, 0.50; -y, 0.50)$ is preferred to $(x, 0.50; -x, 0.50)$.

Applying equation (1) once again, we get

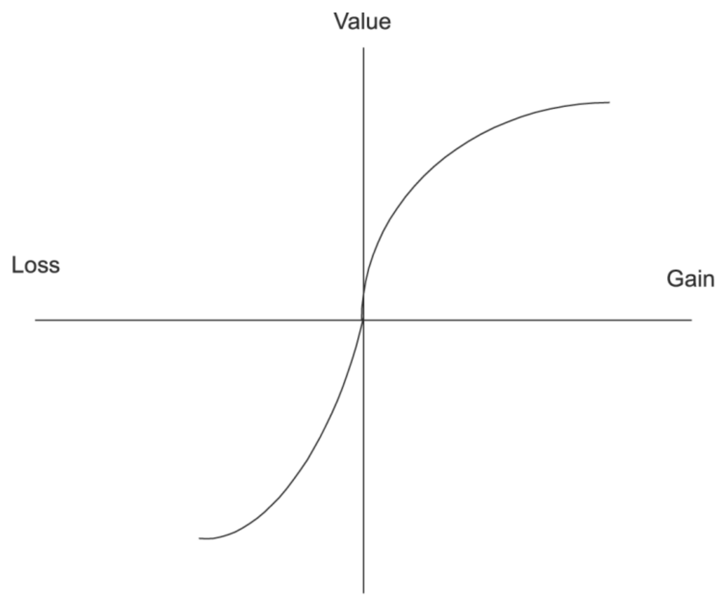
$$v(y) + v(-y) > v(x) + v(-x) \quad \Leftrightarrow \quad v(-y) - v(-x) > v(x) - v(y)$$

[5] Kahneman & Tversky (1979), pp. 279

Setting y = 0 yields $v(x) < -v(-x)$, which shows that the value function is steeper in the domain of losses than in the domain of gains. This can be seen explicitly by letting y approach x to get the relationship between the slopes: $v'(x) < v'(-x)$.

To summarize, the value function is (i) defined on deviations from the reference point; (ii) generally concave for gains and commonly convex for losses; (iii) steeper for losses than for gains. Figure 2-2 depicts Kahneman and Tversky's hypothetical value function.

Figure 2-2: Kahneman and Tversky's hypothetical value function



It can be seen that the value function takes on an asymmetric S-shape, which implies that the steepest point on the curve is at the reference point. Thus, the reference point is where the investor or decision maker is most sensitive to changes in relative outcomes. This contrasts with the utility function postulated by Markowitz (1959), which is relatively shallow in that region.

2.1.3 Summary

We have seen that the traditional assumption of quadratic utility applied in modern portfolio theory is inconsistent with the preferences individuals actually exhibit. Not only is it counterintuitive to exhibit increasing absolute risk aversion as suggested by quadratic utility, but experimental analyses have been undertaken to prove violations of several vital axioms in expected utility theory as well. These axioms include the central tenet that the utilities of outcomes are weighed by their absolute probabilities, which is violated by the certainty effect; the substitution axiom, which is violated by the overweighing of low probabilities; the invariance and dominance axioms, which are violated by how individuals respond to framing; and the axiom that individuals care about final states rather than changes, which is violated again by the framing of prospects. Another drawback of the quadratic utility function is that there is evidence that the preferences of individuals cannot be characterized by one global degree of risk aversion. In fact, we saw that individuals exhibit risk aversion in the domain of gains and risk seeking behavior in the domain of losses.

We saw that individuals are more sensitive to losses than to gains, which means that a symmetric measure such as variance does not capture the risk as it is perceived by investors. Finally, the assumption that returns on risky investments are normally distributed is unrealistic as it rules out asymmetry or skewness. Therefore, it is unsound to assume that investors only relate to the first two moments of the return distribution. In fact, the DARA class of utility functions is valid for all risk averse investors exhibiting skewness preference ($u' > 0$, $u'' < 0$ and $u''' > 0$). This means that a feasible functional form of utility would be logarithmic utility rather than quadratic utility, and that one should focus on the lower partial moments rather than entire distributions where normality is assumed.

2.2 Risk Theory

For decades, there have been several definitions of risk and discussions about how risk should be measured. In the following we will try to define risk according to Frank H. Knight (1921) and Douglas W. Hubbard (2007) and relate it to the purpose of this thesis.

In 1921, Frank H. Knight argued that there is a difference between uncertainty and risk. According to Knight, risk is a combination of the likelihood (probability) of an occurrence of a hazardous event, meaning an event that could cause harm in terms of losses or undesirable outcome, and its magnitude. He also believes that it is possible to calculate the probability of the risk, which makes risk measurable. Uncertainty is characterized as the existence of more than one possibility in the future, but unlike risk, uncertainty is not measureable (also known as Knightian Uncertainty). Hubbard offers a very clear definition of these two:

> *"Uncertainty: The lack of complete certainty, that is, the existence of more than one possibility. The "true" outcome/state/result/value is not know.*
> *(...) Risk: A state of uncertainty where some of the possibilities involve loss, catastrophe or other undesirable outcome."* - *Douglas W Hubbard.*

In this thesis we are only going to focus on risk and not uncertainty. There has been different suggestions on how to calculate risk, e.g. by the variance, introduced by Markowitz in the Modern Portfolio Theory. However, variance is often heavily criticized as a risk measure since it considers gains as much as losses and as shown in the definition above, the emphasis in risk lies in losses (negative outcomes). Risk variables that incorporate this by measuring below a certain point are commonly referred to as *downside risk* measures.

In the next section, we will review the mean variance theory and introduce alternative risk measures that consider downside risk in different ways.

2.2.1 The Mean-Variance Framework And Modern Portfolio Theory

The most applied and recognized investment theory is the Modern Portfolio Theory (MPT) introduced by Harry Markowitz in 1952. The essentials of the theory are not very complicated and easy to apply, which is one of the reasons for its success. The introduction of this theory has been very important for the understanding of the relationship between risk and return as well as purpose of diversifying portfolios. It is often misinterpreted that people did not diversify before 1952, which is not entirely true. What was lacking before 1952 was a formal framework covering the effect of diversification when risks are correlated and the risk/return tradeoff on the portfolio as a whole (H. M. Markowitz 1959). In the following we will review the main ideas of this framework[6].

2.2.1.1 Risk and Return for Single Stocks

One of the most liquid assets are equities. Therefore, there is often focus on stocks in investment theory. When an investor composes an optimal portfolio of stocks, it is necessary to have a measure for each stock's expected return and risk. The future returns are not known with certainty, and therefore an investor considers expected returns, as described below.

$$E(r)=\sum_{i=1}^{I} r_i s_i$$

and the corresponding variance:

$$\sigma^2=\sum_{i=1}^{I}(r_i-E(r))^2 s_i ,$$

r_i: return of incident i

s_i: the probability of obtaining r_i

i: number of possible incidents

The standard deviation, which is defined as risk in this framework is then expressed as following:

$$\sigma=\sigma^2=\sqrt{\sum_{i=1}^{I}(r_i-E(r))^2 s_i}$$

When constructing a portfolio it is not enough to simply look at the risk and return of the individual stocks. We also need to include variables that represent the effect among the stocks. This information can be obtained by calculating the covariance between the stocks. The covariance between two stocks is defined as following:

$$\sigma_{12}=\sum_{i=1}^{I}(r_{1i}-E(r_{1i}))(r_{2i}-E(r_{2i}))s_i$$

where, r_{1i} and r_{2i} are the return possibilities for stock 1 and 2 respectively. It can be interpreted from the covariance equation that the covariance between stock 1 and 2 must be the same as the covariance between stock 2 and 1:

$$\sigma_{12}=\sigma_{21}$$

Another relation between two securities is the correlation, which is expressed as the covariance divided by the product of the standard deviation of the two stocks.

$$\rho_{12}=\frac{\sigma_{12}}{\sigma_1\sigma_2} \tag{2}$$

[6] Elton, et al. (2007)

The correlation will always be between one and minus one. A correlation of one means that the two stocks are perfectly positively correlated while a correlation of minus one means that they are perfectly negative correlated. A correlation of zero on the other hand, means that the stocks are completely uncorrelated and thereby completely independent in their movement relative to each other.

These are the basic definitions and notations needed to understand the mean-variance framework. However, it is important to note that thus far the return and standard deviation are calculated using future returns and the probability for these returns to occur, which can be very hard to estimate. This is an issue we will process later in this section.

2.2.1.2 Risk and Return for a Portfolio

The portfolio return and total risk is calculated slightly different since we need to consider the effect of the correlations between the stocks. The total portfolio return is calculated by simply taking the weighted average of all stock returns included in the portfolio. However, the same method cannot be applied when calculating the total portfolio risk, since the covariance among the stocks has a significant influence on the total risk.

The total expected portfolio return is calculated as following:

$$E(r_p) = \sum_{i=1}^{N} E(r_i) w_i$$

$$\sum w_i = 1$$

The sum of all stock weights needs to be equal to the total portfolio (and thereby one). We could also include another constraint restricting the investor from short selling by requiring positive weight values. However, we choose to allow short selling in order to make this thesis more realistic.

The total risk for the portfolio can be written in the simplest form by the following equation:

$$\sigma_p^2 = \sum_{i=1}^{J} (r_i - E(r_i))^2$$

If we continue to assume two stocks in our portfolio (stock 1 and stock 2), we can substitute the portfolio return with the weighted sum of the two stock returns:

$$\sigma_p^2 = E\left[w_1 r_{1j} + w_2 r_{2j} - (w_1 E(r_1) + w_2(r_2))\right]^2$$
$$= E\left[w_1\left(r_{1j} - E(r_1)\right) + w_2\left(r_{2j} - E(r_2)\right)\right]^2$$

we are able to rewrite the above equation to:

$$\sigma_p^2 = E\left[w_1^2(r_{1j} - E(r_1))^2 + w_2^2(r_{2j} - E(r_2))^2 + 2w_1w_2(r_{1j} - E(r_1))(r_{2j} - E(r_2))\right]$$

Looking at the above equation, it is clear that the standard deviation for the two stocks is expressed by returns. We can therefore simplify the equation by substituting $(r_{ij} - E(r_j))^2$ with the variance of stock j. The expression $(r_{1j} - E(r_1))(r_{2j} - E(r_2))$ is also known as the covariance between the two stocks and is denoted as σ_{ij}.

$$\sigma_p^2 = w_i^2\sigma_i^2 + w_i^2\sigma_j^2 + 2w_iw_i\sigma_{ij} \qquad (3)$$

Equation (3) can be generalized to include an arbitrary number of securities:

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j=1}^{N} w_iw_j\sigma_{ij} \qquad (4)$$

2.2.1.3 Correlations and Diversification

As equation (2) suggests, the correlation between two stocks is partly explained by the covariance. It is therefore possible to express the variance of a portfolio by substituting the covariance with $\sigma_1\sigma_2\rho_{12}$ in equation (3).

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12}$$

It is interesting to include the correlation in the calculation of portfolio variance in order to see the correlation effect on total risk.

In the extreme case that the correlation between stock 1 and 2 is zero, the total variance of the portfolio will simply be the weighted sum of the variances.

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2 0 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2$$

Since the standard deviation is the square root of the variance, the total standard deviation of the portfolio will be:

$$\sigma_p = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2}$$

If we examine the second extreme case, where the two securities are perfectly positively correlated, meaning $\rho_{12} = 1$, we are able to rewrite the portfolio variance as following:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2$$

It is now notable that the equation is in the form of $a^2 + b^2 + 2ab$ and can therefore be reduced to:

$$\sigma_p^2 = (w_1\sigma_1 + w_2\sigma_2)^2$$

And the standard deviation of the portfolio will then simply be the weighted sum of the two stocks:

$$\sigma_p = \sqrt{(w_1\sigma_1 + w_2\sigma_2)^2} = w_1\sigma_1 + w_2\sigma_2$$

Thus, when the two stocks are perfectly positively correlated, we do not reduce the standard deviation (risk) by diversifying the portfolio into two stocks. In this case the effect of diversifying is zero.

On the other hand, if we consider the other extreme case where the stocks are perfectly negatively correlated, the equation for the portfolio standard deviation would be reduced to:

$$\sigma_p = w_1\sigma_1 - w_2\sigma_2 \text{ or } -w_1\sigma_1 + w_2\sigma_2$$

The standard deviation of the portfolio is always smaller when the correlation is negative compared to a positive correlation. In theory, if we could find two securities that are perfectly negatively correlated, it should always be possible to find a combination that would have zero in standard deviation (risk). This is exactly the purpose of diversification.

The purpose of this section is to emphasize that the correlation between two stocks can have a big impact on the standard deviation of the portfolio. It is not possible to reduce risk if the securities are perfectly positively correlated, since this would simply generate the weighted sum of the individual securities standard deviation. However, we are able to reduce the standard deviation if the correlation is below 1. In other words, diversifying the portfolio would reduce the total risk.

2.2.1.4 The Minimum Variance Portfolio

For simplicity, we continue our two stock example in the following. The portfolio can be constructed in several different ways by adjusting the weights in the two different securities. The investor is interested in a portfolio that generates the highest return given a certain standard deviation (or the lowest standard deviation given a certain return). These portfolios are referred to as efficient. The efficient portfolio that exhibits the lowest risk is called the minimum variance portfolio.

We can derive the minimum variance portfolio by differentiating the portfolios' standard deviation with respect to one of the weights for the security (the other weight being $w_2 = 1 - w_1$).

$$\frac{\partial \sigma_p}{\partial w_1} = \left(\frac{1}{2}\right) \frac{\left[2w_1\sigma_1^2 - 2\sigma_2^2 + 2w_1\sigma_2^2 + 2\sigma_1\sigma_2\rho_{12} - 4w_1\sigma_1\sigma_2\rho_{12}\right]}{\left[w_1^2\sigma_1^2 + (1-w_1)^2\sigma_2^2 + 2w_1(1-w_1)\sigma_1\sigma_2\rho_{12}\right]^{1/2}}$$

Setting this equal to zero and solving for w_1 yields the following:

$$w_1 = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}$$

The above equation generates the weight in one of the two securities (the other weight being $w_2 = 1 - w_1$) that will conduct the minimum variance portfolio.

The equation becomes somewhat more complicated if we include more than two securities, which makes it difficult to calculate without a computer. Nonetheless, the intuition and idea remains the same.

2.2.1.5 The Efficient Frontier

As mentioned before, there are many different ways to construct a portfolio by changing the weights in the securities. The universe of efficient portfolios is called the *efficient frontier*. We can calculate this frontier by solving a maximization (minimization) problem with linear constraints. Markowitz discovered that only the portfolios *above* the minimum variance portfolio were *efficient*. The investor is able to gain the highest return for a given standard deviation when selecting a portfolio from the efficient frontier. We can easily derive the efficient frontier by maximizing the return for all levels of standard deviation:

$$\max \quad E(r_p) = \sum_{i=1}^{N} E(r_p) w_i$$

$$s.t. \quad \sigma_p = \sqrt{\sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}}$$

$$\sum_{i=1}^{N} w_i = 1$$

On the other hand, minimizing risk for all levels of returns can derive the efficient frontier as well:

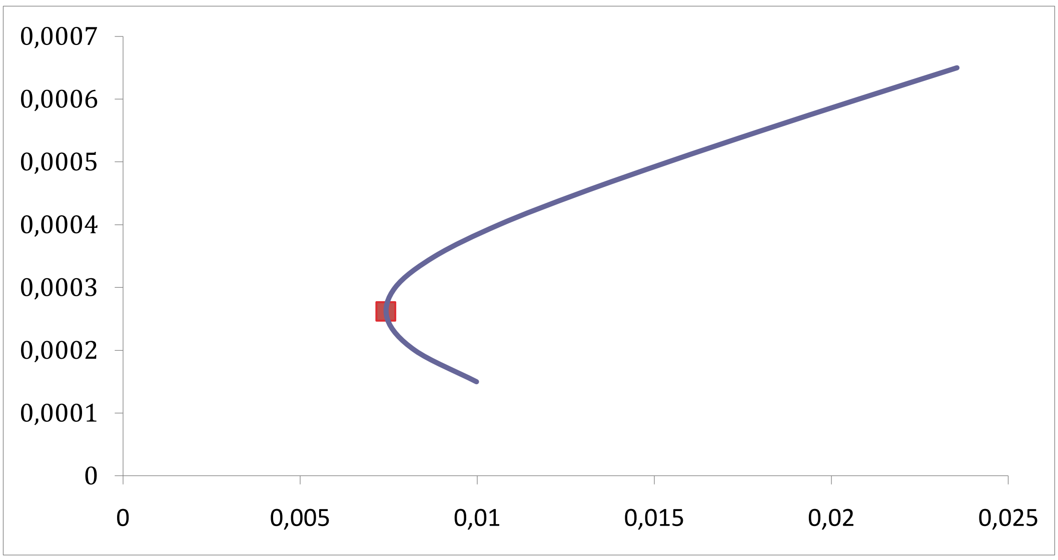
$$\min \quad \sigma_p = \sqrt{\sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}}$$

$$s.t. \quad E(r_p) = \sum_{i=1}^{N} E(r_p) w_i$$

$$\sum_{i=1}^{N} w_i = 1$$

The resulting frontier of portfolios is depicted in figure 2-3 below. Note that the *efficient* frontier does not include the portfolios below the minimum variance portfolio.

Figure 2-3: Frontier & Minimum Variance Portfolio



The red dot indicates the minimum variance portfolio and the blue line above indicates the most optimal portfolios that an investor can obtain by combining the two stocks. All the portfolios on the efficient frontier fulfil the mean-variance criterion above.

2.2.1.6 The Capital Market Line (CML)

So far, we have only considered portfolios that entirely consist of risky assets. When a risk free asset is included in the portfolio, the shape of the different options for the investors looks somewhat simpler. This efficient frontier is called the Capital Market Line (CML).

The risk free asset is an asset with unlimited possibilities for lending and borrowing at a risk free rate, with the expected return of $E(r_f) = r_f$. Since the return is risk free, the following conditions must hold:

$$\sigma_f = 0, \quad \sigma_{if} = 0$$

where σ_f represents the standard deviation of the risk free returns and σ_{if} is the covariance between the risk free asset and a risky security, *i*.

Let w_a denote the stake in portfolio *a*, then $1 - w_a$ will be the portion invested in the risk free asset. The total expected return is then:

$$E(r_p) = (1 - w_a)r_f + w_a E(r_a)$$

It is clear from the equation above that if w_a is equal zero, then the investor only holds the risk free asset. If w_a on the other hand is between zero and one, the investor will hold a combination of the risk free asset and the risky portfolio, *a*. However, if w_a is greater than one, the investor would be selling the risk free asset (borrowing) in order to invest more in portfolio *a*.

In order to calculate the standard deviation of the portfolio, we use equation (4):

$$\sigma_p = \sqrt{\sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}} \Rightarrow$$

$$\sigma_p = \sqrt{(1 - w_a)^2 \sigma_f^2 + w_a^2 \sigma_a^2 + 2(1 - w_a) w_a \sigma_{af}}$$

and since the standard deviation of the risk free asset as well as the correlation between the risk free asset and the portfolio *a* is equal zero, we can reduce the above equation to the following:

$$\sigma_p = \sqrt{w_a^2 \sigma_a^2} = w_a \sigma_a$$

In order to see the relation between risk and return in this scenario, we can rearrange the equation above and substitute it in the total portfolio return:

$$w_a = \sigma_p / \sigma_a$$

The total portfolio return is:

$$E(r_p) = (1 - w_a) r_f + w_a E(r_a) \tag{5}$$

By substituting w_a in equation (5), we get:

$$E(r_p) = (1 - \sigma_p / \sigma_a) r_f + (\sigma_p / \sigma_a) E(r_a) \Rightarrow$$
$$E(r_p) = r_f - (\sigma_p / \sigma_a) r + (\sigma_p / \sigma_a) E(r_a) \Rightarrow$$
$$E(r_p) = r_f + (E(r_a) - r_f)(\sigma_p / \sigma_a) \Rightarrow$$
$$E(r_p) = r_f + \left(\frac{E(r_a) - r_f}{\sigma_a} \right) \sigma_p \tag{6}$$

Looking at the graphical relation between risk and return, it can be seen that all combinations of the risk free and the risky assets lie on a straight line with the slope being $\frac{E(r_a) - r_f}{\sigma_a}$ and the intercept, r_f.

We know that the investor tries to maximize the return relative to risk, which means that she will try to maximize the slope of the above mentioned equation. The maximum slope is obtained at the tangency point between the straight line from equation (6) and the efficient frontier. This tangency point is called the *market portfolio*, and all investors in equilibrium hold a combination of this portfolio and the risk-free asset in order to achieve the most efficient combination of expected risk and return. This phenomenon is called *two-fund separation*. The market portfolio has an expected return of $E(r_m)$ and a standard deviation of σ_m. We can therefore express the tangency line as following:

$$E(r_p) = r_f + \left(\frac{E(r_m) - r_f}{\sigma_m} \right) \sigma_p$$

The above equation is also known as the Capital Market Line (CML), since it provides all the efficient allocations across the capital market.

Note, that CML only describes the risk and return of *portfolios* rather than individual stocks. In the following, we will review a relationship that does so.

2.2.1.7 Systematic Risk: Beta

Random observations of the stock market reveals that the movements in stock prices are often related to the movement in the market in general. If the market goes down, it is very likely that stocks also decline in prices and vice versa. This indicates that some of the correlation among securities is caused by the mutual response to the market and it is therefore worth investigating the relation between individual security returns and market returns. This relationship can be expressed by the following equation:

$$r_i = a_i + \beta_i r_m \tag{7}$$

where,

a_i is the part of security i return that is independent of the market

r_m is the return on the market index

β_i is a constant that measures the effect on r_i given a change in the return of the market index.

β_i in the equation above measures how sensitive the security is towards market movements. E.g. if β_i = 3 it means that if the market increases by 1%, the security will increase by 3%. If β_i is below 1, it means that the security is not very sensitive to market movement.

a_i represents the security return that is independent from market movements. There is some uncertainty to this element and it can therefore be written as:

$$a_i = \alpha_i + e_i$$

where e_i represents a stochastic error.

Substituting a_i with $\alpha_i + e_i$ in equation (7), we get:

$$r_i = \alpha_i + \beta_i r_m + e_i \tag{8}$$

We are interested in the expected return, which can be written as following:

$$E(r_i) = E(\alpha_i + \beta_i r_m + e_i)$$

Since α_i and β_i are constants and the expected value of e_i = 0, we have that:

$$E(r_i) = \alpha_i + \beta_i E(r_m) \tag{9}$$

We can also express the variance of a portfolio by using its beta. Recall that the variance for any security is expressed as following:

$$\sigma_i^2 = (r_i - E(r_i))^2$$

Inserting equation (8) and (9) in the above, we get:

$$\sigma_i^2 = [(\alpha_i + \beta_i E(r_m) + e_i) - (\alpha_i + \beta_i E(r_m))]^2$$

and simplifying, we get that the portfolio variance can be expressed as[7]:

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2$$

We can also express the covariance between two securities with beta by inserting equation (8) and (9) in the covariance equation shown before.

The covariance between two securities can thus be written as:

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2 \tag{10}$$

2.2.1.7.1 Estimating Beta

In previous section we saw that beta can explain the expected return of a given security and it is therefore relevant to look at how to estimate beta. It has been shown that the historical beta provides significant information about future beta values. Many analysts start with estimating the historical beta and then correct it from future influences that could be expected to change beta.

We can derive a simple equation to calculate beta by using the covariance equation (10). Instead of looking at covariance between two securities, we look at the covariance between the market index and security *i*. Since the beta for the market is equal 1, we are able to reduce the equation to the following:

$$\sigma_{im} = \beta_i \sigma_m^2$$

We can now rewrite the equation to express beta:

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$$

---

[7] Assuming that $e_i(r_m - E(r_m)) = 0$

Now, it is possible to estimate the beta of a stock via regression analysis of historical return data:

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{\sum_{t=1}^{N} (r_{it} - E(r_{it}))(r_{mt} - E(r_{mt}))}{\sum_{t=1}^{N} (r_{mt} - E(r_{mt}))} \tag{11}$$

2.2.1.8 The Security Market Line (SML)

Now that we have defined beta, it is interesting to see the relationship between beta and expected return in the return-beta space.

The relationship between expected return and beta is linear and can thus be depicted as a straight line. Any security placed above or below this line would create arbitrage possibilities, which would over time push the security back towards the straight line. Since the relationship is linear, we can write the following equation:

$$E(r_i) = a + b\beta_i$$

In order to estimate the slope and intercept of a straight line, we need two points on the line. The first point being the risk free asset, which is entirely independent of market movements and therefore has a beta of zero. We can therefore estimate the intercept as:

$$r_f = a + b \cdot 0 \Rightarrow$$
$$r_f = a$$

The second point on the line being the market portfolio with a beta of 1 and an expected return of $E(r_m)$. With this second point we can estimate the slope as following:

$$E(r_m) = a + b \cdot 1 \Rightarrow$$
$$E(r_m) = r_f + b \cdot 1 \Rightarrow$$
$$b = \left(E(r_m) - r_f\right)$$

Combining the two points yields the following relationship between stock returns and beta:

$$E(r_i) = r_f + \beta_i\left(E(r_m) - r_f\right) \tag{12}$$

This is commonly known as the Security Market Line (SML). We see that the intercept is equal to the risk free rate and the slope is the difference between expected market return and the risk free rate (i.e. the market premium).

This is the most simple way derive SML. The interested reader is referred to Elton et al (2007) for a more rigorous analysis.

2.2.1.9 Criticism of the Mean-Variance Framework

The mean-variance framework makes many assumptions about investors and capital markets that are not very realistic. In this section, we summarize the shortcomings of the widely used framework and explain why other frameworks might make more investment sense than the mean-variance.

1. Normality

One of the central assumptions in the mean-variance framework is that asset returns are (jointly) normally distributed. In reality, it is frequently observed that returns in equity as well as other markets are not normally distributed, but rather they exhibit skewness and excess kurtosis. This conflicts with the notion that a symmetrical risk measure such as variance captures the entire distribution of returns.

2. Quadratic Utility

When returns are not normally distributed, another way to justify the mean-variance criterion is by assuming a quadratic utility function. As we noted earlier, quadratic utility implies that investors care about mean and variance only, even when returns exhibit asymmetry or fat tails. There are several limitations to this assumption. First, a quadratic utility implies that investors exhibit increasing absolute risk aversion, which is not very reasonable. Second, a quadratic function will ultimately become negative over some return interval, which means that the nonsatiation[8] condition is violated. Finally, this form of utility fails to capture loss aversion, i.e. investors care more about losses than gains.

3. Variance as a risk measure

The two assumptions above justify the use of variance as a valid risk measure. Violations of these assumptions then raise the question of the soundness of using variance to define risk. Variance is a symmetric measure suggesting that abnormally high returns are just as risky (and unwanted) as abnormally low returns. In reality, investors are more concerned about losses than the dispersion or tightness of high (e.g. above-average) returns. A more sensible

[8] More wealth is preferred to less wealth, i.e. $U'(w) > 0$

way to describe variance is that it measures uncertainty rather than risk whereas asymmetric risk measures are intuitively a better approximation of investment risk.

2.2.2 The Mean-Lower Partial Moment Framework

Bawa (1975) presented a formal analysis of downside risk measures for various utility functions in terms of the lower partial moment (LPM) of return distributions. The appeal of these risk measures partly stems from their consistency with the way individuals actually perceive risk. The LPM measure liberates the investor from a constraint of having only one utility function, which is fine if investor utility is best represented by a quadratic function. LPM represents a significant number of Von Neumann-Morgenstern utility functions and thus the whole gamut of human behavior from risk seeking to risk neutral to risk averse.

In addition, the MLPM approach to portfolio selection accommodates the asymmetric nature of risk attitude found in the behavioral finance literature, i.e. that investors weigh losses more heavily than gains. Thus, the MLPM approach is more general than the mean-variance approach both in terms of the assumptions imposed on the investor's utility function and/or on the probability distributions of security returns. Another interesting implication of the MLPM-framework is that it provides a theoretical basis for various classes of risk measures used in the financial economics literature.

In this section, we analyze the investor's portfolio selection problem and the capital market equilibrium in an MLPM framework where it is clear that the two-fund separation property holds for investors' optimal portfolio choices and that a linear risk-return relationship exists. We then briefly discuss the special cases of this framework where we show that the new Capital Asset Pricing Model (CAPM) contains the traditional CAPM's as special cases under the usual distributional assumptions. Finally, we elaborate on the special case of $MLPM_2$ or mean-semivariance framework, which is the one we will use throughout our analyses.

2.2.2.1 The Investor's Portfolio Choice Problem

We consider a downside risk averse investor with portfolio allocation X across k assets with $X' = (X_1, \ldots, X_k)$, and let F_X denote the probability distribution of returns on the portfolio. The set of feasible portfolios is

$$C = \begin{Bmatrix} X \middle| \sum_i X_i = 1, X_i \geq 0 & \textit{without short sales, or} \\ X \middle| \sum_i X_i = 1 & \textit{with short sales} \end{Bmatrix}$$

An investor will choose a portfolio among this set, which maximizes the expected risk adjusted return of her portfolio. We let *R* represent the vector of returns with $R' = (R_1, ..., R_k)$, and let *F* denote their joint distribution. We can now define the *n*th order lower partial moment of the distribution of returns under allocation *X*, computed at point τ (target rate), $LPM_n(\tau; X)$, where R_X is the return on the portfolio, as

$$LPM_n(\tau; X) = \int_{-\infty}^{\tau} (\tau - R_X)^n dF_X(R_X) = \int_{-\infty}^{\tau} (\tau - X'R)^n dF(R) \tag{13}$$

Bawa and Lindenberg (1977) use the risk free rate as target rate in the optimization problem, and the investor's optimization problem becomes to minimize $LPM_n(r_F; X)$ subject to $\left\{\sum_i X_i E(R_i) = \mu\right\}$ and $\{X \in C\}$.

We follow Harlow and Rao (1989)'s generalization of the MLPM and include any possible target rate in order to account for different investor preferences. We introduce the risk free asset with return R_F in a proportion X_0, and redefine portfolio *X* to include both risky and risk free assets. The investor's optimization problem becomes

$$\min_X LPM_n(\tau; X) = \int_{-\infty}^{\tau} (\tau - X'R)^n dF(R) \tag{14}$$

subject to $\left\{X_0 R_F + \sum_{i \neq 0} X_i E(R_i) = \mu\right\}$ and $\left\{X_0 + \sum_{i \neq 0} X_i \in C\right\}$

Equation (14) shows that the downside risk averse investor selects the optimal portfolio weights such that the relevant risk measure (LPM_n) is minimized for a specific value of the expected return (μ) on the portfolio. In the special case of Bawa and Lindenberg (1977), linear combinations of a portfolio *X* of risky assets and the risk free asset lie along a straight line in the mean-$LPM_n^{1/n}$ space. Because of the convexity of the $LPM_n(\mu)$ function, this implies that the risky portfolio, which in combination with the riskless asset yields the minimum LPM for all mean returns, is found by drawing a tangent from R_F to the $LPM_n(\mu)$ function (see figure 2-3). Thus, the two-fund separation of traditional portfolio theory still

obtains and *M*, the tangency point in figure 2-3, can be referred to as the market portfolio of risky assets. Harlow and Rao (1989) obtain a similar result for arbitrary target returns.
To recap, the two-fund separation suggests the following. In an economy in which investors view risk as below-target deviations, the optimal portfolio choice in a $MLPM_n$ framework for n = 1 and 2 involves the allocation of investor wealth between the riskless asset and the "market" portfolio of risky assets.

Bawa and Lindenberg (1977) clearly illustrate this by plotting the MLPM efficient frontier and the Capital Market Line (CML) in mean-$LPM_n^{1/n}$ space as in figure 2-3 below. The reason they use $LPM_n^{1/n}$ rather than LPM_n is to depict the CML as a straight line tangent to the efficient frontier with the risk free rate as its intercept. This makes the two-fund separation more obvious than it is in mean-LPM_n space and we draw a direct parallel to the mean-variance framework.

Figure 2-4: Two-Fund Separation. Source: Bawa & Lindenberg (1977)

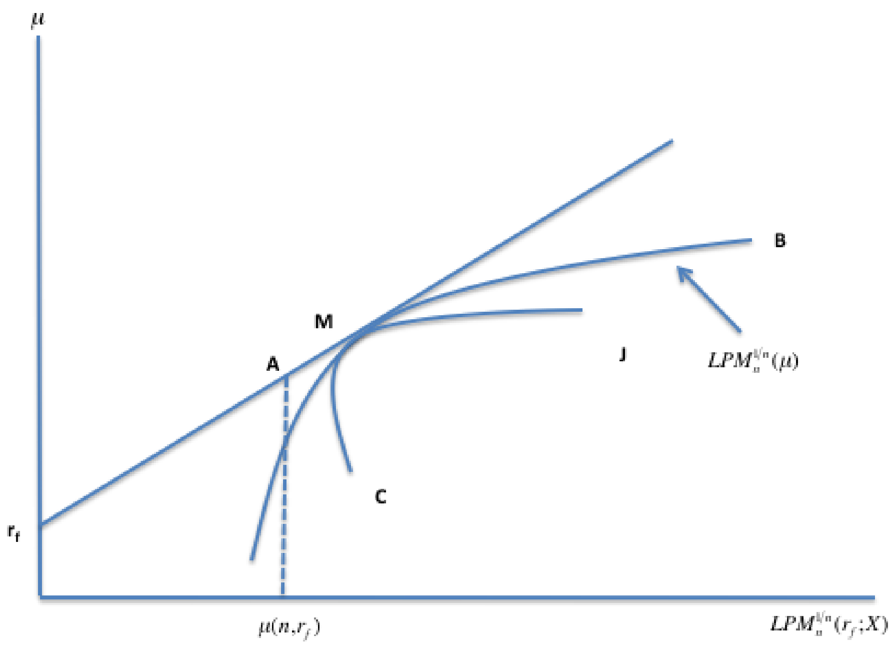


Figure 2-4 clearly shows that two-fund separation obtains in the MLPM framework. However, it may be more intuitive to show the market equilibrium in mean-LPM_n space, as LPM_n is our relevant risk measure. Harlow and Rao (1989) show that the CML is not necessarily a straight line in mean-LPM_n space. In fact, convex combinations of a portfolio of risky assets and the riskless asset are convex in mean-LPM_n space, i.e. we get a convex Capital Market Line.[9]

Figure 2-5: The Mean-LPM Framework. Source: Harlow & Rao (1989)

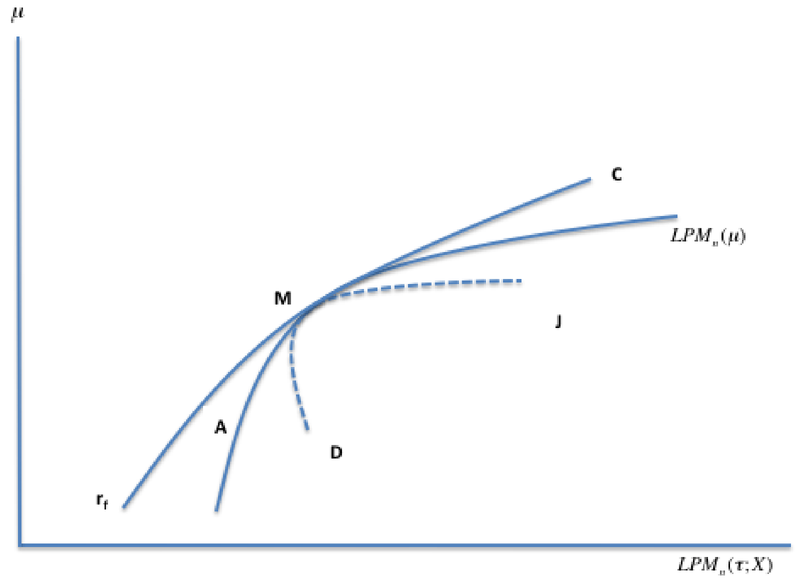


Figure 2-5 shows the convex relationship from combining the risk free asset with any risky portfolio in mean-LPM_n space. The point of tangency (*M*) of the convex CML (from R_f through *M* to *C*) and the efficient frontier ($LPM_n(\mu)$ function) corresponds to the market portfolio in mean-LPM_n space.

2.2.2.2 Capital Market Equilibrium

In this section we use the $MLPM_n$ framework to develop a capital asset pricing model (CAPM) with no restrictions on the distributional form. Under the standard assumptions employed in the Sharpe (1964), Lintner (1965), Mossin (1966) CAPM model, the above findings indicate that we can apply the analytical methodology used in Sharpe (1964) to derive the market equilibrium pricing relationship.

In figure 2-4, any portfolios consisting of a fraction X_j in security *j* (point *J*) and $(1 - X_j)$ in the market portfolio lie along the curve *JMD*, which is continuous and is, in equilibrium, tangent to the efficient frontier at point *M* and thus in turn is tangent to the CML.

[9] For proof see Harlow and Rao (1989)

The slope of the curve *JMD* at *M* is

$$\left[\frac{\partial LPM_n(\tau;R_p)}{\partial E(R_p)}\right]_M = \frac{\int_{-\infty}^{\tau}\int_{-\infty}^{\infty} n(\tau-R_m)^{n-1}(R_m-R_j)dF(R_j,R_m)}{E(R_j)-E(R_m)}$$

where $E(R_m)$, $E(R_j)$ and $E(R_p)$ are the expected returns on the market portfolio, security *j* and our combined portfolio respectively.

Similarly, the slope of the curve R_fMC, i.e. the capital market line, at *M* is given by

$$\left[\frac{\partial LPM_n(\tau;X)}{\partial E(R_p)}\right]_M = \frac{\int_{-\infty}^{\tau} n(\tau-R_m)^{n-1}(R_m-R_j)dF(R_j,R_m)}{R_f-E(R_m)}$$

Since the two equations above denote the slopes of the *JMD* frontier and the capital market line at *M*, these naturally should be equal such that $Slope_{JMD} = Slope_{R_fMC}$, which yields the security market line (SML) in the mean-lower partial moment framework for arbitrary τ[10].

$$\frac{\int_{-\infty}^{\tau}\int_{-\infty}^{\infty} n(\tau-R_m)^{n-1}(R_m-R_j)dF(R_j,R_m)}{E(R_j)-E(R_m)} = \frac{\int_{-\infty}^{\tau} n(\tau-R_m)^{n-1}(R_m-R_j)dF(R_j,R_m)}{R_f-E(R_m)}$$

$$\Rightarrow \qquad E(R_j) = R_f + \beta_j^{mlpm_n(\tau)}\left[E(R_m)-R_f\right] \tag{15}$$

where

$$\beta_j^{mlpm_n(\tau)} = \frac{\int_{-\infty}^{\tau}\int_{-\infty}^{\infty} (\tau-R_m)^{n-1}(R_f-R_j)dF(R_j,R_m)}{\int_{-\infty}^{\infty} (\tau-R_m)^{n-1}(R_f-R_m)dF(R_m)} \tag{16}$$

Following Harlow and Rao (1989), we define the *n*th order-generalized co-lower partial moment between two assets *X* and *Y* about τ and R_f, $GCLPM_n(\tau,R_f;X,Y)$ as

[10] For detailed proof see Harlow & Rao (1989)

$$GCLPM_n(\tau, R_f; X, Y) = \int_{-\infty}^{\tau}\int_{-\infty}^{\infty} (\tau - R_X)^{n-1}(R_f - R_Y)dF(R_X, R_Y),$$

and a nth order-generalized lower partial moment for asset X about τ and R_f, $GLPM_n(\tau, R_f; X)$ as

$$GLPM_n(\tau, R_f; X) = \int_{-\infty}^{\tau} (\tau - R_X)^{n-1}(R_f - R_X)dF(R_X), \tag{17}$$

which means that we can represent $\beta_j^{mlpm_n(\tau)}$ in (16) as

$$\beta_j^{mlpm_n(\tau)} = \frac{GCLPM_n(\tau, R_f; M, j)}{GLPM_n(\tau, R_f; M)} \tag{18}$$

As can be seen, equation (18) is very similar to equation (11) in the mean-variance framework. In both cases, the beta is defined as the co-variance/LPM between a security and the market portfolio divided by the variance/LPM of the market portfolio. As equation (18) suggests, the market has risk when $GLPM_n(\tau, R_f; M)$ is positive, which is only the case if the return on the market has a positive probability of falling below the target rate τ. The definition of $\beta_j^{mlpm_n(\tau)}$ in equation (16) states that a security j contributes to the market's risk when its return, as well as the market's return, are below τ. When the return of security j exceeds τ while the return on the market is below τ, security j reduces the risk of M. Thus, the premium paid for risk is positive whenever $\beta_j^{mlpm_n(\tau)}$ is larger than zero, and negative whenever it is smaller than zero. Finally, if the return on the market exceeds τ, it is not risky by definition, and individual security returns contribute nothing to it regardless of whether these are below or above τ.

As can be seen, equation (15) is very similar to the traditional mean-variance CAPM defined by equation (12). The difference in this framework lies in the measure of systematic risk, $\beta_j^{mlpm_n(\tau)}$. We want to stress again that the SML employed in this section is valid for a very general class of utility functions that displays the standard properties of rational economic behavior, including nonsatiation ($u' > 0$), risk aversion ($u'' < 0$), and skewness preference ($u''' > 0$). In addition, it allows for any pre-specified target rate of return τ.

2.2.2.3 Special Cases of the Generalized MLPM Framework

The type of "moment", *n*, specified in equation (17) captures the investor's preferences by determining the type of utility function consistent with that risk measure. Thus, by setting *n* equal to 0, 1 or 2, and by imposing restrictions on the target rate or the distribution of returns, we can derive many popular notations of risk.

For $n = 0$, the risk measure becomes a 0^{th}-order LPM (denoted LPM_0), which is an expression of the probability of falling below the target rate. This is equivalent to the *shortfall probability* given by Roy's Safety First criterion with a disaster level of τ[11]. In this case, the probability of loss ($\tau = 0$) and probability of ruin (τ = critical value) framework results[12]. LPM_0 is consistent with all utility functions that prefer more to less wealth, i.e. satisfying the assumption of nonsatiation ($u' > 0$). However, note that the LPM_0 efficient set is not convex, and can therefore not be used to derive equilibrium models.

For $n = 1$, LPM_1 becomes the expected deviation of returns below the target, or the *target shortfall*. This measure captures the severity of not achieving the target return, i.e. the mean of the deviations from τ. LPM_1 is consistent with all risk averse utility functions ($u' > 0$ and $u'' < 0$).

For $n = 2$, LPM_2 becomes what is popularly referred to as the *target semivariance*, which measures the dispersion of returns below the target rate. This measure is valid for all risk averse functions displaying skewness preference, i.e. the DARA class of utility functions ($u' > 0$, $u'' < 0$ and $u''' > 0$). If we impose the restriction that $\tau = R_f$, we directly obtain the Bawa-Lindenberg model, and with $\tau = E(R)$, the more traditional definition of semivariance results where risk is measured as worse than expected rates of return. Finally, if we set $\tau = R_f$ in the LPM_2 and we further assume normal (or symmetric) distributions, the LPM_2 measure becomes proportional to variance, and would result in the same ranking of risky assets as in the mean-variance framework.

The $MLPM_n$ framework clearly represents a wide range of asset pricing models depending on the assumptions about *n*, the target rate τ, and the distribution of returns. In our analysis, we

[11] Roy, A. (1952)
[12] The probability of loss has been studied by Kataoka (1963), Hanssman (1968), Peterson and Laughhunn (1971), and Laughhunn and Sprecher (1977).

confine ourselves to second-order lower partial moments (LPM_2) for empirical testing since these are most consistent with statistical methodologies.

2.2.2.4 The Mean-Semivariance Framework

In the following, we will review the special case of the LPM that is mostly applied as a tool for investment decision making. Throughout this section, we follow Estrada (2004, 2007, 2008) in presenting the mean-LPM_2 framework. Setting the restriction n = 2 on our model above, we get what is popularly known as the target semivariance. For application purposes, we compute the semivariance of an empirical *discrete* distribution of portfolio returns as shown below.

$$\Sigma_{i\tau}^2 = E\left\{\min(R_i - \tau, 0)^2\right\} = \frac{1}{T}\sum_{i=1}^{T} \min(R_{it} - \tau, 0)^2 \tag{19}$$

The semideviation is then

$$\Sigma_{i\tau} = \sqrt{\Sigma_{i\tau}^2} = \sqrt{\frac{1}{T}\sum_{i=1}^{T} \min(R_{it} - \tau, 0)^2}$$

which measures the volatility below the target rate of return, τ.

We define the investor's problem as below:

$$\min_{x_1, x_2, \ldots, x_n} \ \Sigma_{p\tau}^2 = \frac{1}{T}\sum_{i=1}^{T} \min(R_{pt} - \tau, 0)^2$$

$$s.t.$$

$$\sum_{i=1}^{n} x_i E_i = E^T \ , \quad and \quad \sum_{i=1}^{n} x_i = 1$$

where R_{pt} denotes the returns of the portfolio and $\Sigma_{p\tau}^2$ the portfolio semivariance. The major obstacle to the solution of this problem is that the semicovariance matrix is endogenous; that is, a change in weights affects the periods in which the portfolio underperforms the target rate of return, which in turn affects the elements of the semicovariance matrix.

2.2.2.4.1 The Endogeneity of the Semicovariance Matrix

In order to clearly illustrate the problem of an endogenous semicovariance matrix, we have summarized return data from two hypothetical stocks in table 2-5 below.

Table 2-5: Endogenous Semicovariance Matrix from Two Hypothetical Stocks

					Conditional Returns					
					80-20 Portfolio			10-90 Portfolio		
Year	Stock A	Stock B	80-20	10-90	Stock A	Stock B	Product	Stock A	Stock B	Product
1	0,1	-0,11	0,058	-0,00099	0	0	0	0,1	-0,11	-0,011
2	0,21	-0,09	0,15	-0,001701	0	0	0	0,21	-0,09	-0,0189
3	0,09	0,29	0,13	0,002349	0	0	0	0	0	0
4	-0,11	-0,21	-0,13	0,002079	-0,11	-0,21	0,0231	0	0	0
5	-0,13	-0,2	-0,144	0,00234	-0,13	-0,2	0,026	0	0	0
6	-0,22	-0,12	-0,2	0,002376	-0,22	-0,12	0,0264	0	0	0
7	0,24	0,19	0,23	0,004104	0	0	0	0	0	0
8	0,06	0,09	0,066	0,000486	0	0	0	0	0	0
9	0,07	0,23	0,102	0,001449	0	0	0	0	0	0
10	0,11	0,05	0,098	0,000495	0	0	0	0	0	0

Table 2-5 displays the annual returns of two hypothetical stocks, Stock A and Stock B, as well as the annual return of two portfolios: one invested 80% in Stock A and 20% in Stock B, and the other invested 10% in Stock A and 90% in Stock B. We first consider the 80-20 portfolio and assume a target rate of return of 0%. We calculate the semivariance of the portfolio returns using equation (19). That obtains a portfolio semivariance with respect to 0% equal to 0.0078, corresponding to a portfolio semideviation of 8.81%.

Accordingly, for any given portfolio, the semivariance/-deviation can always be correctly calculated as above. The problem arises when, instead of the semivariance of one portfolio, we want to find the portfolio with the lowest semivariance from a set of many feasible portfolios. The approach above would be inadequate since we would have to calculate the returns and semivariance from each portfolio, and then select the portfolio with the lowest semivariance. Clearly, as the number of securities available increases, and the number of feasible portfolios increases even more, determining the optimal portfolio with this procedure becomes troublesome. If the elements of the semicovariance matrix were exogenous, then we could formally solve the given optimization problem and obtain a closed-form solution. We could then obtain the weights that minimize the portfolio semivariance in the same manner as we saw in the mean-variance framework in section 2.2.1. Unfortunately, this is not the case.

Markowitz (1959) suggests estimating the portfolio semivariance as below:

$$\Sigma_{p\tau}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \Sigma_{ij\tau} \tag{20}$$

where

$$\Sigma_{ij\tau} = \frac{1}{T} \sum_{t=1}^{K} (R_{it} - \tau)(R_{jt} - \tau) \tag{21}$$

where periods 1 through K are periods in which the portfolio *underperforms* the target return, τ. This definition has one advantage and one drawback. The advantage is that it provides an exact estimation of the portfolio semivariance, and the drawback is that the semicovariance matrix is endogenous. We first look into the advantage of using this definition by considering again the 80-20 portfolio from table 2-5 with a target rate of return equal to 0%. The sixth and seventh columns display the conditional returns of Stock A and B respectively; that is, applying equation (21) we get 0% when the return of the 80-20 portfolio is positive (outperforming the target return), and the return of the given stock when the return of the 80-20 portfolio is negative (underperforming the target). To illustrate, the conditional return of Stock A in year 4 is -11% because the 80-20 portfolio delivered a negative return, and 0% in year 1 because the 80-20 portfolio delivered a positive return.

We can now calculate the four terms of the semicovariance matrix by applying equation (21). We square the conditional returns in the sixth column and take their average, thus obtaining $\Sigma_{A,A,0} = 0{,}0077$; we do the same with the conditional returns in the seventh column and obtain $\Sigma_{B,B,0} = 0{,}0099$; we then take the average of the product in the eighth column and get $\Sigma_{A,B,0} = 0{,}0076$. Finally, we follow equation (20) in order to get the semivariance of the 80-20 portfolio:

$$\Sigma_{p,0}^2 = 0{,}8^2 \cdot 0{,}0077 + 0{,}2^2 \cdot 0{,}0099 + 2 \cdot 0{,}8 \cdot 0{,}2 \cdot 0{,}0076 = 0{,}0078$$

corresponding to a portfolio semideviation of 8,8%, which is exactly the same number obtained before. Therefore, the expression suggested by Markowitz (1959) does indeed provide an exact estimation of the portfolio semivariance. The problem is that, in order to estimate this variable, we need to know whether the *portfolio* itself yields returns higher or lower than our target rate of return. We then run into the problem mentioned earlier: a change in weights affects when the portfolio underperforms, which in turn affects the elements in the semicovariance matrix; i.e. the semicovariance matrix is endogenous.

This can be illustrated by considering the 10-90 portfolio in table 2-5 where 10% is invested in Stock A and 90% in Stock B, and again assuming a target return of 0%. The returns of this portfolio are shown in the fifth column, the conditional returns of the two stocks are shown in the ninth and tenth columns, and the product of the conditional returns in the eleventh column. The first thing to note is that, while the conditional returns are calculated by using equation (21) as before, they differ from those of the 80-20 portfolio. We calculate the four elements of the semicovariance matrix exactly as before and get:

$$\Sigma_{A,A,0} = 0{,}0054\,, \quad \Sigma_{B,B,0} = 0{,}0020\,, \quad and \quad \Sigma_{A,B,0} = -0{,}0030$$

Note that all these numbers are different from those calculated for the 80-20 portfolio, which clearly shows that the semicovariance matrix is endogenous because its elements depend on the asset weights. For the sake of completeness, we calculate the semivariance of the 10-90 portfolio using equation (20):

$$\Sigma_{p,0}^2 = 0{,}1^2 \cdot 0{,}0054 + 0{,}9^2 \cdot 0{,}0020 + 2 \cdot 0{,}1 \cdot 0{,}9 \cdot (-0{,}0030) = 0{,}0012$$

implying a semideviation of 3,39%.

2.2.2.5 A Heuristic Approach

Many authors have proposed solutions to the problem of the endogenous semicovariance matrix in solving the investor's portfolio selection problem[13]. In order to overcome this problem, we follow a heuristic proposed by Estrada (2008). We define the semicovariance between assets *i* and *j* with respect to a target return, τ, as

$$\Sigma_{ij\tau} = E\left\{\min(R_i - \tau, 0) \cdot \min(R_j - \tau, 0)\right\} = \frac{1}{T}\sum_{i=1}^{T}\left[\min(R_{it} - \tau, 0) \cdot \min(R_{jt} - \tau, 0)\right] \qquad (22)$$

This definition can be tailored with any desired τ and generates a symmetric $(\Sigma_{ij\tau} = \Sigma_{ji\tau})$ as well as an exogenous semicovariance matrix. Recall that with equation (21), knowledge of whether the *portfolio* underperforms the target is needed. With equation (22), however, knowledge of whether the *asset* underperforms the target is needed, which means that the elements in the semicovariance matrix are invariant to the weights of the portfolio considered and are, therefore, exogenous.

If we divide equation (22) by the product of asset *i* and asset *j*'s semideviation of returns, we obtain their downside correlation (Θ_{ij}), which is given by

$$\Theta_{ij\tau} = \frac{\Sigma_{ij\tau}}{\Sigma_{i\tau} \cdot \Sigma_{j\tau}} = \frac{E\left\{\min(R_i - \tau, 0) \cdot \min(R_j - \tau, 0)\right\}}{\sqrt{E\left\{\min(R_i - \tau, 0)^2\right\} \cdot E\left\{\min(R_j - \tau, 0)^2\right\}}}$$

Additionally, the semicovariance between an asset *i* and the market portfolio can be divided by the market's semivariance of returns, thus obtaining asset *i*'s downside beta (β_i^D), which is given by

[13] See Hogan and Warren (1972), Ang (1975), Nawrocki (1983), Harlow (1991), Grootveld and Hallerbach (1999), among others.

$$\beta_i^D = \frac{\Sigma_{im\tau}}{\Sigma_{m\tau}^2} = \frac{E\{\min(R_i - \tau, 0) \cdot \min(R_m - \tau, 0)\}}{E\{\min(R_m - \tau, 0)^2\}}$$

and which can also be expressed as $\beta_i^D = (\Sigma_{i\tau}/\Sigma_{m\tau})\Theta_m$. The corresponding SML is then defined by:

$$E(R_i) = R_f + \beta_i^D \left[E(R_m) - R_f\right] \tag{23}$$

As can be seen by a straightforward comparison of (11) and (23), this model replaces the beta of the CAPM by the downside beta, which is the appropriate measure of systematic risk in a downside risk framework.

2.2.3 The Mean-Gain-Loss Spread Framework

Estrada (2009) presented an alternative ad hoc risk measure called the gain-loss spread (GLS) in an effort to explain the cross-section of stock returns. The GLS takes into account the probability of a loss, the average loss and the average gain – information that investors consider relevant when assessing investments. Investors are typically concerned about the probability of suffering a loss, a percentage that can be estimated as the proportion of periods in which an asset generated negative returns. In addition, investors care about the size of these losses, a quantity that can be estimated with the mean return over the periods in which the asset generated negative returns. These two downside variables lead to the expected loss. In the same manner, one can calculate the expected gain. The GLS is then calculated as the difference between the expected gain and the expected loss of the asset, reflecting the spread between the upside and the downside, thus providing an insightful metric for investors.

The GLS is very simple to calculate, as is shown in the following analytical framework. We consider an asset with returns R_t, where t indexes time. Assume that out of the T periods for which we have return data, the asset delivers a loss $L_t = R_t < 0$ in N periods and a gain $G_t = R_t < 0$ in M periods. Then we have that $N + M = T$. The probability of a loss (p_L) and the probability of a gain (p_G) is then defined as:

$$p_L = \frac{N}{T} \qquad , \qquad p_G = \frac{M}{T} = 1 - p_L$$

We implicitly assume that the asset delivers either positive or negative returns; hence there are no periods in which the returns are zero. It is, however, simple to accommodate the model to include such returns.

The average loss (A_L) and average gain (A_G) are defined as the mean return over the N and M periods in which the asset generated a loss/gain:

$$A_L = \frac{1}{N}\sum_{t=1}^{N} L_t \qquad , \qquad A_G = \frac{1}{M}\sum_{t=1}^{M} G_t$$

The expected loss and the expected gain are then defined as:

$$E_L = p_L \cdot A_L = \frac{1}{T}\sum_{t=1}^{N} L_t \qquad , \qquad E_G = p_G \cdot A_G = \frac{1}{T}\sum_{t=1}^{M} G_t$$

Finally, we can calculate the GLS as the spread between the expected gain and the expected loss:

$$GLS = E_G - E_L = \frac{1}{T}\sum_{t=1}^{M} G_t - \sum_{t=1}^{N} L_t$$

As the above expression shows, it is very straightforward to calculate the GLS. It is notable that even though $E_G - E_L$ yields the GLS, $E_G + E_L$ yields the arithmetic mean return.

The portfolio selection problem can be set up as a linear programming problem like before.

$$\min \; GLS = E_G - E_L = \frac{1}{T}\sum_{t=1}^{M} G_t - \sum_{t=1}^{N} L_t$$

$$s.t. \quad E(r_p) = \sum_{i=1}^{N} E(r_p) w_i$$

$$\sum_{i=1}^{N} w_i = 1$$

One could argue that the GLS is not a *downside* risk measure per se since it includes the expected gains as well. However, it has the advantage that it does not depend on any distributional assumptions and provides more information than e.g. the standard deviation. Henceforward, we will refer to the GLS as a downside risk measure.

2.2.4 The Mean-Conditional Value-at-Risk Framework

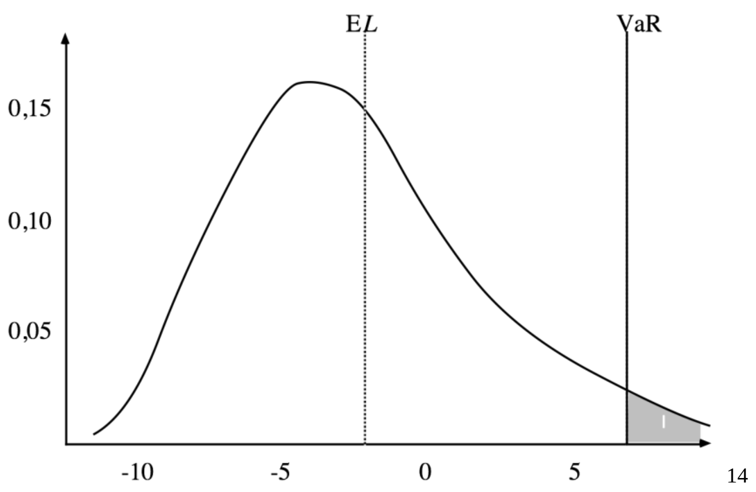
In this section, we examine the conditional value-at-risk as a risk measure in portfolio optimization. There has been a need for new risk measures, alternative to the traditional measures such as standard deviation, by regulation of financial institutions and risk management. One of the first attempts at alternative risk measures has been Value-at-risk, which has gained wide application in many areas of the financial analysis. Throughout this section, we mainly follow Uryasev (2000) supplemented by Rockafellar & Uryasev (2000) and Krokhmal et al (2002).

If we assume a normal distribution, the traditional value-at-risk can be calculated as following:

$$VaR_\alpha = \mu + \sigma t_v^{-1}(\alpha)$$

where μ and σ denote the mean and standard deviation of the loss distribution, given a student t distribution with v degrees of freedom.

Figure 2-6: An Example of the Loss Distribution and VaR



As shown in the figure above, the value-at-risk (VaR) calculates the maximum loss at a specific confidence interval α. If a portfolio has a one-month 5% VaR of $2 million, there is a 5% probability that the portfolio will fall in value by more than $2 million over a one months period, assuming markets are normal. We could also look at it as a loss of $2 million or more on a portfolio is expected in one month out of 20 (100/5) months.

Although, VaR is a commonly used risk measure by banks and other financial institutes, it has received a great amount of criticism from academics, stating its many shortcomings[15]:

1. An investor who subscribes to VaR is implicitly stating that she is indifferent between very small losses exceeding VaR and very high losses. This is an assumption that is far from reality.
2. VaR is not sub-additive. That means that the VaR of a portfolio with two instruments can be larger than the sum of VaR of these two instruments. Thus, VaR could indicate that concentrated portfolios where less risky than diversified portfolios.
3. VaR also has some mathematical drawbacks. It is a non-smooth, non-convex and multi-extremum (many local minima) function that makes it difficult to use in portfolio optimization.
4. VaR further relies on a linear approximation of risk and assumes a normal distribution (or t-distribution) of the underlying market data.

Due to the above-mentioned issues, the conditional value-at-risk (CVaR)[16] was created as an extension to VaR. It provides information that can be considered complementary to that given by VaR, as it measures the expected excess loss above the VaR, if a loss larger that VaR actually occurs. In order words, it calculates the average of the worst (1-β) losses.

CVaR is more sensitive to the shape of the loss distribution in the tail of the distribution whereas VaR only considers a t-distribution. If the underlying loss-distribution is normal, then both risk measures will provide the same optimal portfolio. CVaR is still a relatively new risk measure and is therefore not as widely used in the financial industry as VaR. However, due to enhanced computer power as well as its superiority in terms of relaxed assumptions and better information, it is gaining more attention and application.

The relationship between VaR and CVaR can be explained as following[17]:

The formal definition of VaR is an α-percentile of loss distribution such that the probability that losses exceed or equal this value is greater or equal to α. When looking at CVaR, we can look at the "upper" and "lower" CVaR. The "upper" CVaR determines the expected losses

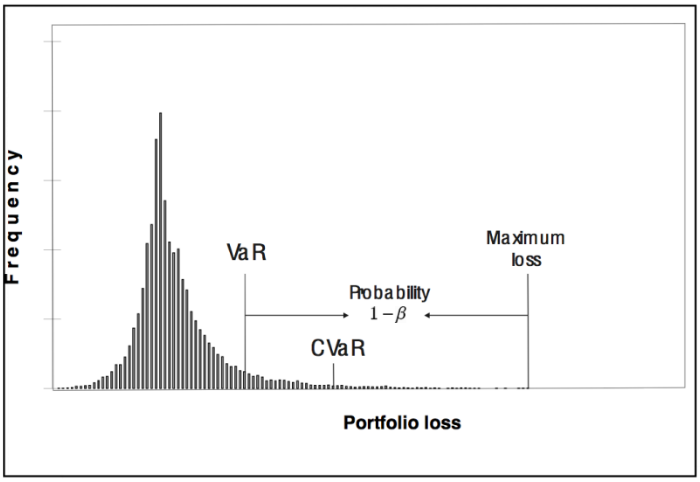
[15] Scherer (2007)
[16] Also called Mean Excess Return, Mean Shortfall, Expected Shortfall or Tail VaR
[17] Uryasev presentation

strictly exceeding VaR and the "lower" CVaR are the expected losses that are equal to or exceed VaR. CVaR is then a weighted average of VaR and the "upper" CVaR:

$$CVaR = \lambda \cdot VaR + (1-\lambda) \cdot CVaR^{+}$$

This way CVaR will always be larger than VaR – its magnitude depending on the distribution of the underlying data.

Figure 2-7: VaR and CVaR. Source: Uryasev (2000)



2.2.4.1 The Investor's Portfolio Choice Problem

CVaR is a very different risk measure from variance or semivariance. It is therefore not possible to use the basic ideas of the mean-variance framework to find the optimal portfolio using CVaR. We need to find the optimal portfolio using a slightly different approach. The basic idea of this approach is to construct a loss function that is then used to define CVaR. The actual optimization problem can be solved using linear programming as shown by Uryasev (2000).

2.2.4.1.1 The Loss Function

We define a loss function depending on a decision vector *x* (where x belongs to a feasible set of portfolios, *X*) and a random vector *y:*

$$f(\mathbf{x},\mathbf{y}_j),\ \ x \in X$$

Consider a portfolio consisting of two instruments. We then have a vector $x=(x_1,x_2)$, which is the position in the two instruments, their corresponding initial prices $m=(m_1,m_2)$ and $y=(y_1,y_2)$ a vector of uncertain prices in the next period. The loss function is then the difference between the current value of the portfolio and the uncertain value of the portfolio in the next period:

$$f(x,y) = (x_1 m_1 + x_2 m_2) - (x_1 y_1 + x_2 y_2) \Rightarrow$$
$$f(x,y) = x_1(m_1 - y_1) + x_2(m_2 - y_2)$$

The same method is applied if the portfolio consists of more than just two instruments. In order to keep this approach convenient, we assume vector *y* has a probability density function *p(y)*. This assumption is not critical for the considered approach since we can easily relax the assumption by looking at the uncertain value as demonstrated above.

2.2.4.1.2 Analytical Representation

The VaR function is denoted as $\alpha(x,\beta)$, which is the percentile of the loss distribution with confidence level β. CVaR is denoted as $\Phi_\beta(x)$ and indicates the conditional expected loss and is defined by:

$$\Phi_\beta(x) = (1-\beta)^{-1} \int_{f(x,y) > \alpha(x,\beta)} f(x,y)p(y)dy \tag{24}$$

Equation (24) above is not exactly intuitive, which makes it hard to apply in the portfolio selection problem. Therefore, it is necessary to have an analytical representation of VaR. Uryasev (2000) uses the following approach to define a more simple function that can be used instead of equation (24):

$$F_\beta(x,\alpha) = \alpha + (1-\beta)^{-1} \int_{f(x,y) > \alpha} (f(x,y) - \alpha)p(y)dy \tag{25}$$

The use of equation (25) rather than (24) can be justified as follows:

1. The function $F_\beta(x,\alpha)$ is convex with respect to w.r.t. α
2. VaR is a minimum point of the function w.r.t. α
3. $\phi_\beta(x) = F_\beta(x,\alpha(x,\beta)) = \min_\alpha F_\beta(x,\alpha)$, i.e. minimizing $F_\beta(x,\alpha)$ w.r.t. α yields CVaR.

In other words, we are able to minimize CVaR by minimizing $F_\beta(x,\alpha)$ with respect to α.

$$\min_{x \in X} \Phi_\beta(x) = \min_{x \in X, \alpha} F_\beta(x,\alpha)$$

Minimizing equation (25) yields the optimal CVaR with vector *x** and the corresponding VaR.

2.2.4.1.3 Minimizing CVaR with LP

It is often the case that the density function $p(y)$ is not available or is very hard to construct for a given instrument, but it is very easy to find historical data for the instruments of the portfolio. We can use these historical prices instead of the density function, $p(y)$. In this case the $F_{\beta}(x,\alpha)$ distribution will be discrete rather than continuous.

$$\tilde{F}_{\beta}(x,\alpha) = \alpha + v\sum_{j=1}^{J}(f(\mathbf{x},\mathbf{y}_j) - a)^{+} \tag{26}$$

In order to simplify equation (26), we use the auxiliary variable, *v*, which is equal to $v = ((1-\beta)J)^{-1}$. If the loss function is linear with respect to *x*, we are able to solve this optimization problem by using linear programming (LP).

In order to set the LP problem, we need another auxiliary variable that helps define the constraints.

$$z_j \geq f(\mathbf{x},\mathbf{y}_j) - \alpha\ , \qquad z_j \geq 0$$

The above constraints are equivalent to the function: $\max(0, f(\mathbf{x},\mathbf{y}_j) - a)$

This lead to the following LP optimization problem:

$$\min \qquad \alpha + v\sum_{j=1}^{J} z_j$$

S.t.

$$z_j \geq f(\mathbf{x},\mathbf{y}_j) - \alpha, \qquad z_j \geq 0, \qquad j = 1,\ldots,J$$

Other constraints can be added if e.g. the investor desires no short sales, $x_j \geq 0$ or she does not accept returns below 8%. These constraints can easily be added to the general linear programming optimization problem.

If the returns are normally distributed and the same return constraints are active, we get the same solution for the optimal portfolio when minimizing variance, semivariance, VaR or CVaR.

In this section, we introduced the conditional value-at-risk as a new risk measure due to the shortcomings of VaR. The conditional value-at-risk calculates the *average* of the highest (1-β) losses whereas VaR only focuses on the maximum loss at a specific confidence interval (1-β). We are able to efficiently minimize CVaR using linear programming in order to find the optimal portfolio and at the same time calculate VaR for that portfolio. The method applied to find the optimal portfolio is efficient and can easily handle a large amount of data.

2.2.5 The Mean-Drawdown Framework

From a fund manager's point of view, someone who manages capital on behalf of clients or banks and whose only source of income takes the form of management fees or incentive remuneration, losing this capital can have very serious repercussions for her business. It is therefore very important for the manager to consider the maximum consecutive loss she can tolerate. A particular client or bank will choose their fund manager by reviewing their previous historical track record and often decide based on their accounts' maximum drawdowns and the duration of these drawdowns. Most clients also give their fund managers a warning drawdown level, at which he is able to work within (e.g. 15%). These issues make it very important for the fund manager to consider the worst drawdowns as well as the duration of these.

In the following, we will review how an investor can use the worst drawdown measure to select the optimal portfolio allocation. The theory is based on Cheklov et al (2003) who introduced a one-parameter family of risk functions called drawdowns.
The definition of drawdown according to Cheklov et al (2003) is as follows:

> "*The drop in the portfolio value compared to the maximum achieved in the past.*"

Drawdowns can be defined in absolute and relative terms, e.g. if the current value of a portfolio is €19m and the maximum value of the portfolio within the relevant time frame was €20m, the absolute drawdown would be €1m and the relative drawdown would be 5%. The different drawdown measures can be generalized in one variable called the *conditional drawdown-at-risk (CDaR)*. The CDaR is based on a confidence parameter α, where the α-CDaR is defined as the average of the 1-α worst drawdowns experienced over a certain period of time. If α is set to be equal to zero, we would be looking at 100% of the drawdowns over the period of time and thereby calculate the *average drawdown*. On the other hand, as α moves towards one, the CDaR move towards the *maximum drawdown*. The CDaR risk function therefore has the average drawdown and maximum drawdown as its upper and lower limits.

The CDaR optimization model is based on a similar concept as the CVaR. We can view CDaR as a modification of CVaR, where the loss function is replaced by a drawdown function. We can use the same approach as with the remaining risk measures and reduce the optimization problem to a linear programming problem.

2.2.5.1 Definition of Drawdown Measures

We are going to consider three drawdown measures:

1. Maximum drawdown
2. Average drawdown
3. Conditional drawdown-at-risk

We denote a function $w(\mathbf{x},t)$ to be the uncompounded portfolio return at time *t* with a vector $x=(x_1,x_2,\ldots,x_m)$ representing the portfolio weights of *m* instruments. The *drawdown function* is defined as the difference between the maximum return in period $\tau \in [0,T]$ and the return at time *t*:

$$D(\mathbf{x},t)=\max_{0\le\tau\le t}\{w(\mathbf{x},\tau\}-w(\mathbf{x},t) \tag{27}$$

Figure 2-8: Drawdown Function

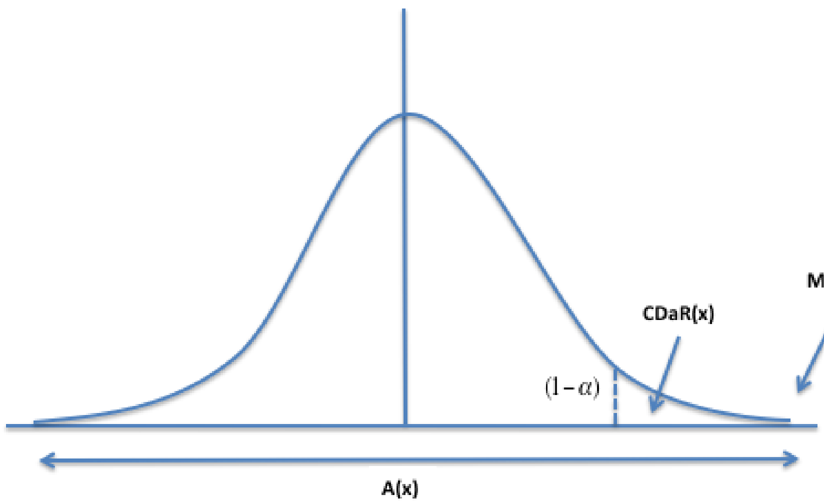


Figure 2-8 depicts the density of the drawdown function as defined by equation (27) as well as the three drawdown measures, which will be described in the following. CDaR is a family of risk functions that depend on the value of α, where the maximum drawdown and average drawdown can be considered special cases of CDaR. The maximum drawdown in the period can be calculated by maximizing equation (27):

$$M(x)=\max_{0\le\tau\le t}\{D(\mathbf{x},t)\}$$

The average drawdown is then the entire area below the drawdown function, i.e. the sum of all drawdowns over the period τ, divided with the length of the period *T*:

$$A(x) = \frac{1}{T}\int D(\mathbf{x},t)dt$$

If we define N as the number of sub-periods in the time interval $[0,T]$ and $(1-\alpha)N$ as an integer number (i.e. we are able to count the number of drawdowns) then the CDaR is calculated as the average drawdown over $(1-\alpha)N$ sub-periods:

$$CDaR(x) = \frac{1}{(1-\alpha)T}\int_{\Omega} D(\mathbf{x},t)dt$$

Let $\zeta_{\upsilon}(x)$ denote a threshold where all $(1-\alpha)N$ of the drawdowns exceed this threshold, and define Ω as $\Omega = \{t \in [0,T] : D(\mathbf{x},t) \geq \zeta_{\upsilon}(x)\}$. However, if $(1-\alpha)N$ is not an integer number, then CDaR is expressed as a linear combination of the threshold and the drawdowns strictly exceeding this threshold, similar as it was done for CVaR:

$$CDaR(x) = \min_{\zeta}\left\{\zeta + \frac{1}{(1-\alpha)T}\int_{0}^{T}[D(\mathbf{x},t)-\zeta]^{+} dt\right\}$$

where $[g]^{+} = \max(0, g)$.

We have now defined the three drawdown measures and can continue to the optimization problem. As mentioned before, we are interested in simplifying this optimization problem to a linear programming problem.

2.2.5.2 The Investor's Portfolio Choice Problem

In the same manner as previously, we can solve the portfolio selection problem by minimizing the drawdown measures for different levels of return. The formal setup includes minimizing the following expressions subject to the same linear constraints:

$$CDaR(x) = \min_{\zeta}\left\{\zeta + \frac{1}{(1-\alpha)T}\int_{0}^{T}[D(\mathbf{x},t)-\zeta]^{+} dt\right\} \tag{28}$$

$$s.t. \quad E(r_p) = \sum_{i=1}^{N} E(r_p) w_i$$

$$\sum_{i=1}^{N} w_i = 1$$

When considering the average drawdown, $\alpha = 0$, which means that equation (28) is reduced to the following problem:

$$A(x) = min \left\{ \frac{1}{T} \int_{0}^{T} [D(\boldsymbol{x}, t - \zeta]^{+} dt \right\}$$

$$s.t. \quad E(r_p) = \sum_{i=1}^{N} E(r_p) w_i$$

$$\sum_{i=1}^{N} w_i = 1$$

Furthermore, the maximum drawdown implies that $\alpha = 1$ and the threshold takes on the highest value of the drawdown function, which means that there are no drawdowns exceeding this threshold $\left[\frac{1}{(1-1)T} \int_0^T [D(\boldsymbol{x}, t - \zeta]^{+} dt\right] \rightarrow 0$. Equation (28) is therefore reduced to:

$$M(x) = \min \{\zeta\}$$

$$s.t. \quad E(r_p) = \sum_{i=1}^{N} E(r_p) w_i$$

$$\sum_{i=1}^{N} w_i = 1$$

The three drawdown measures are fairly new magnitudes that define risk in a different way. Some people may argue that the maximum drawdown is not a complete risk measure since it is only based on two observations whereas the CDaR allows the risk manager to focus on the worst (1-α)*100% drawdowns and is by definition more flexible.

2.2.6 Summary

We have now introduced the theory behind the following seven risk measures:

- Variance
- Semivariance
- Gain-Loss Spread
- Conditional Value-at-Risk

- Conditional Drawdown-at-Risk
- Maximum Drawdown
- Average Drawdown

The variance was introduced in modern portfolio theory and is the most applied risk measure among practitioners today. The semivariance has a somewhat similar underlying theory with the main exception that it concentrates on the lower partial moment of the return distributions. The LPM and semivariance have received a lot of attention since Markowitz introduced his modern portfolio theory. Even then, Markowitz recognized that semivariance is a more adequate risk measure than variance but chose not to use it due to the limited computer power that made it hard to calculate at the time.

The third risk measure reviewed in this chapter is the conditional value-at-risk, which has its roots from the popular risk variable, value-at-risk. Even though value-at-risk is widely used within investment management, it has received substantial criticism for neglecting the importance of fat tails. We have chosen to take this criticism into consideration and instead present the conditional value-at-risk, which takes the losses exceeding value-at-risk into account.

The next three risk measures introduced by Uryasev et al (2000) are based on a drawdown function, which takes extreme values of return distributions into account. The theory behind the conditional drawdown-at-risk is fairly similar to the idea behind the conditional value-at-risk. Conditional drawdown-at-risk also requires a predefined quantile (the most common being 0.9, 0.95 or 0.99), where the average and maximum drawdowns are the two special cases of the conditional drawdown-at-risk (the quantile being 0 or 1 respectively).

The last risk measure reviewed is the gain-loss spread, which is a new risk measure introduced by Estrada (2009). The GLS is the only other risk measure (variance being the first) that considers both downside as well as upside potential. The GLS calculates the spread between the expected gain and the expected loss of a portfolio, reflecting the spread between the upside and the downside, thus providing an insightful metric for investors.

3 Chapter III: Data

3.1 Presentation of data

All the data used in the analysis is extracted from Thomson Datastream, which is the largest financial database available. We believe that it is an advantage that all the data comes from the same source and that it increases the strength of comparability of each time series.

Our data consist of eight stock indices, 29 stocks that have been randomly selected from the stock indices, and the MSCI global government bonds index. In Chapter 4, we will only use the stock indices and stocks whereas the bonds will be used in Chapter 5. The selected data is summarized in table 3-1.

Table 3-1: Securities

Stocks		Index	Bonds
Jyske Bank	Societe Generale	MSCI World	FTSE Global Govt. All
A. P. Møller	Total	DAX 30	
Danske Bank	AXA	Dow Jones US	
Novo Nordisk	Sanofi-Aventis	MSCI EM	
Tesco	Siemens	FTSE 100	
Aviva	BASF	CAC 40	
BP	Allianz	NIKKEI 225	
Royal Bank of Scotland	Deutsche Bank	OMX	
Barclays	Volkswagen		
Nippon	Chevron		
Canon	Citigroup		
Toyota	General Electric		
Mitsubishi	Bank of America		
Honda	AT&T		
Carrefour			

All equity data is reported as total return indices. This way, we will capture the total equity fluctuations where dividends are included. With price indices, a potential dividend payment will reduce the market price of a stock by an amount equivalent to that dividend. This is misleading because such price reduction is not due to real market fluctuations, and the stock might as well be perfectly stable. E.g. imagine that a stock price is €100 on the day before a dividend payment of €7. The price will then be €93 the day after the dividend payment, which merely represents the reallocation of capital.

We have chosen to express all the return data in Euros rather than in local currencies because we wish to conduct the analysis from a European investor's point of view. This choice will have some implications during the first part of the analysis because fluctuations in equity prices caused by exchange rate risk will be incorporated in the total return index. We believe, however, that the noise from exchange rate risk is negligible in relation to our analysis.

3.1.1 Return Frequency and Period

The collected data spans over a 30-year period; from May 21, 1980 to May 21, 2010. We have chosen this period because we want to include the recent developments and at the same time analyze over a longer period.

We include two different return frequencies within the selected 30-year period when examining the relation between risk and return. These frequencies are daily and monthly returns and the respective number of observations in the 30-year period can be found in table 3-2.

Table 3-2: Number of Observations

	Daily	Monthly
Number of obs.	7827	360

We could also have included quarterly and yearly returns, but this would have resulted in fewer observations that would be critically low for some of the statistical tests used throughout our analysis. Furthermore, using quarterly or yearly returns would also contain less information since stock prices may exhibit high fluctuations within short time intervals, which would then not be captured.

3.1.2 Outliers

There has been considerable debate in the empirical finance literature on how to address "outliers" in an empirical study. The big downturns in October 1987 or in the fall 2008 have been particularly interesting. Some believe that all outliers could be included as long as the analysis is done correctly. Others believe that it is important to eliminate the largest outliers from the analysis because they do not fit into the general framework. In the present study, we have decided to not reject any abnormal returns since we wish to include all fluctuations. This is mainly because we are analyzing the downside risk of returns where extreme values may account for an important part.

3.1.3 Return Calculation

The total return indices are based on daily (or monthly) closing prices obtained from Thomson Datastream. It is worth noting that there are two ways to calculate returns from these indices. Equity returns can be calculated as arithmetic returns and logarithmic returns.

An advantage of the logarithmic returns is that the log normal distribution of returns has infinite upside potential while losses cannot exceed 100%.

The arithmetic returns are particularly useful when mutual funds and other financial players must determine the performance of their investments. The reason why the percentage rates are used in this context is that the percentage return corresponds naturally with the market prices and since we are analyzing the relation between risk and return and the performance of the different risk measures, we use the percentage returns.

3.2 Normality Issue

In this section, we will examine whether the returns are normally distributed by applying different statistical techniques. Many financial models rely heavily on the normality assumption, including the mean-variance criterion. It is an important assumption, since the assumption makes calculations involving risk more straightforward than they are with non-normal distributions. The normality assumption in asset returns has been tested many times, reaching the same conclusion that the historical returns are usually *not* normally distributed. In order to make our analysis complete, we test the normality assumption with our specific data for later reference.

One of the most commonly used methods to test for normality is to look at the density function of the returns.

Figure 3-1: Normal Distribution

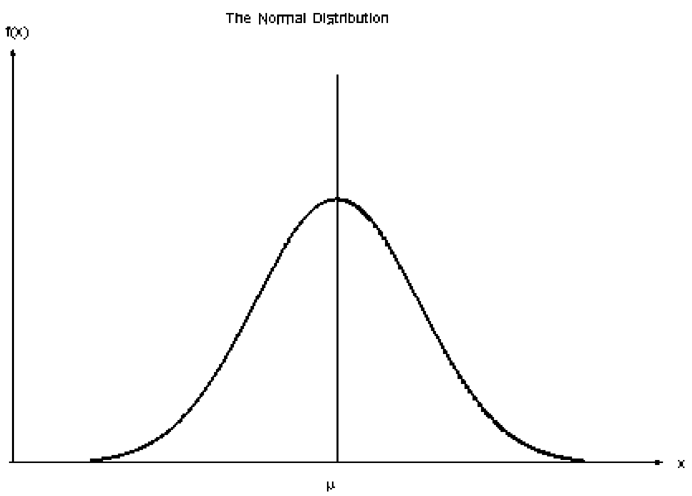


Figure 3-1 depicts a normal distribution, which has the characteristic bell shape and is symmetrical around the mean. If the return distribution differs much from the bell shaped distribution, one can question the normality assumption. We can therefore look at the distribution of the different securities in order to determine any potential deviation from normality.

Figure 3-2: Histogram of DAX 30 Returns

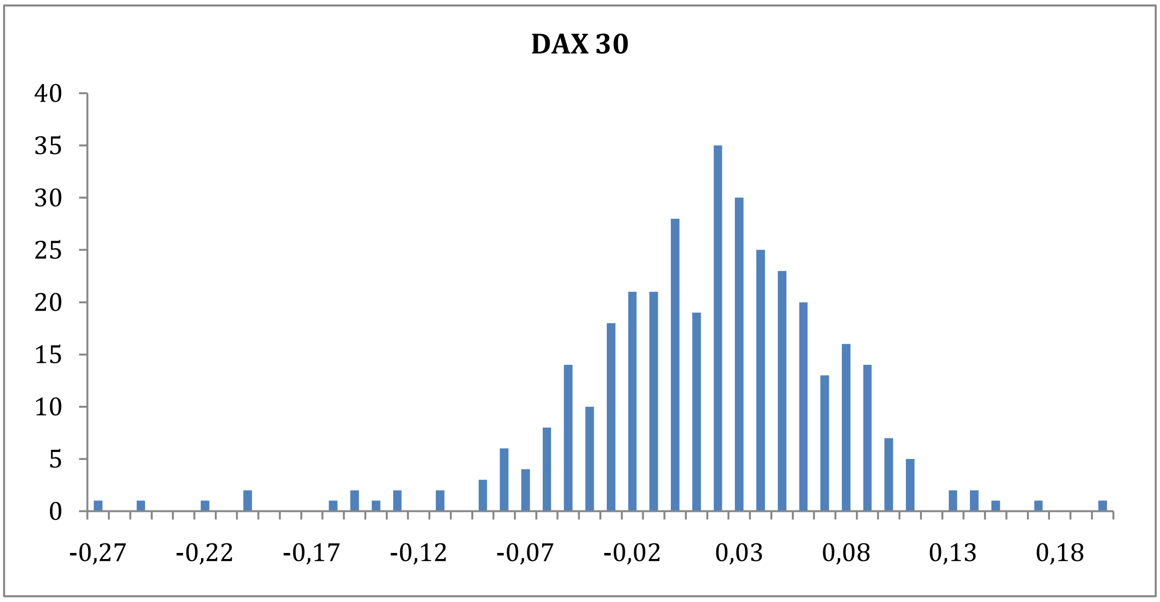


Figure 3-2 shows the histogram of historical returns of the German DAX 30 index. The distribution seems somewhat bell shaped but is not completely symmetric around the mean. We see that there are some values in the left tail of the distribution that cause the histogram to "lean" and is known as a negatively skewed distribution or a left tailed distribution.

When using empirical data samples like returns time series, we would not expect the data to exhibit a form of a perfectly normal distribution. However, we need a different way to determine whether the distribution is significantly different from the normal distribution.

This can be tested numerically by calculating the third and fourth moments of the empirical distribution. The third and fourth moments take specific values in a normal distribution and therefore, it is possible to identify and calculate the deviation from these values[18].

[18] Gujarati (2003)

The third moment in normality represents the skewness (asymmetry) of a density function around the mean. A skewness value different from zero suggests that there is a deviation from normality in the return series. The skewness is defined as:

$$skew = \frac{1}{T}\sum_{t=1}^{T}\frac{(r_i-\mu)^3}{\hat{\sigma}^3} \sim N(0,\sqrt{6/T}$$

where T is the number of observations, r_i is the return in period i, μ is the mean return, σ is the standard deviation and N represents the normal distribution. Negative skewness implies that the distribution is left tailed and the mass of the distribution is concentrated on the right. On the other hand, positive skewness means that the distribution is right tailed and the mass of the distribution is concentrated on the left.

The fourth moment of a distribution is the kurtosis, which describes the “peakedness” of the distribution. Thus, the kurtosis is an expression of the degree to which stock returns are concentrated around the mean. In statistical terminology, it is most common to talk about the "excess kurtosis", which is simply defined as kurtosis minus three and can be expressed as following:

$$kurtosis = \frac{1}{T}\sum_{t=1}^{T}\frac{(r_i-\mu)^4}{\hat{\sigma}^4} - 3 \sim N(0,\sqrt{24/T}$$

There are several statistical tests that are deemed as suitable to perform normality tests. The most widely used is the Jarque-Bera test, which uses the third and fourth moments of the empirical distribution to test for normality. In order for a distribution to be perfectly normally distributed, the series needs a skewness of zero and a kurtosis of three. This test determines whether the distribution exhibits significant skewness and kurtosis significantly different from three. The Jarque-Bera test is an asymptotic test in which the reliability of test results increases with number of observations. The Jarque-Bera test is defined as follows:

$$JB = T\left(\frac{skewness^2}{6} + \frac{Kurtosis^2}{24}\right) \sim \chi^2(2)$$

which follows a chi-square distribution with two degrees of freedom.

Appendix I lists the results of this test for the different stocks and indices at both monthly and daily return frequencies. For the daily returns, it is clear that all stocks and indices show a statistically significant deviation from normality at the 1% level, indicated by a p-value below 0.01. Normality is also rejected for all monthly returns except Sanofi (FR), Chevron (US) and OMXC20 (DK). OMXC20 is the index with the fewest observations, which could question the reliability of the test result as the test is sensitive to the number of observations.

3.2.1 Stability of Deviation "Persistency"

In order to justify the effort to model non-normality, we need to be sure that the statistically significant non-normality is persistent over time. There are several ways to check the persistency but not all academics agree about the reliability of the different tests. Taking this uncertainty into account, we use a test for persistency of skewness presented by Kahn and Stefek (1996). This test can be easily applied to determine whether the non-normality of returns is persistent over time. The test is conducted as follows:

1. Split the data set into two non-overlapping time periods of equal length
2. Calculate skewness, kurtosis or excess semivariance for each time period
3. Run a regression of the form:

$$skew-measure_{t+1} = \beta_0 + \beta_1 \cdot skew-measure_t + \varepsilon_t$$

The persistency is then indicated by a significant β_1 and a high R^2.

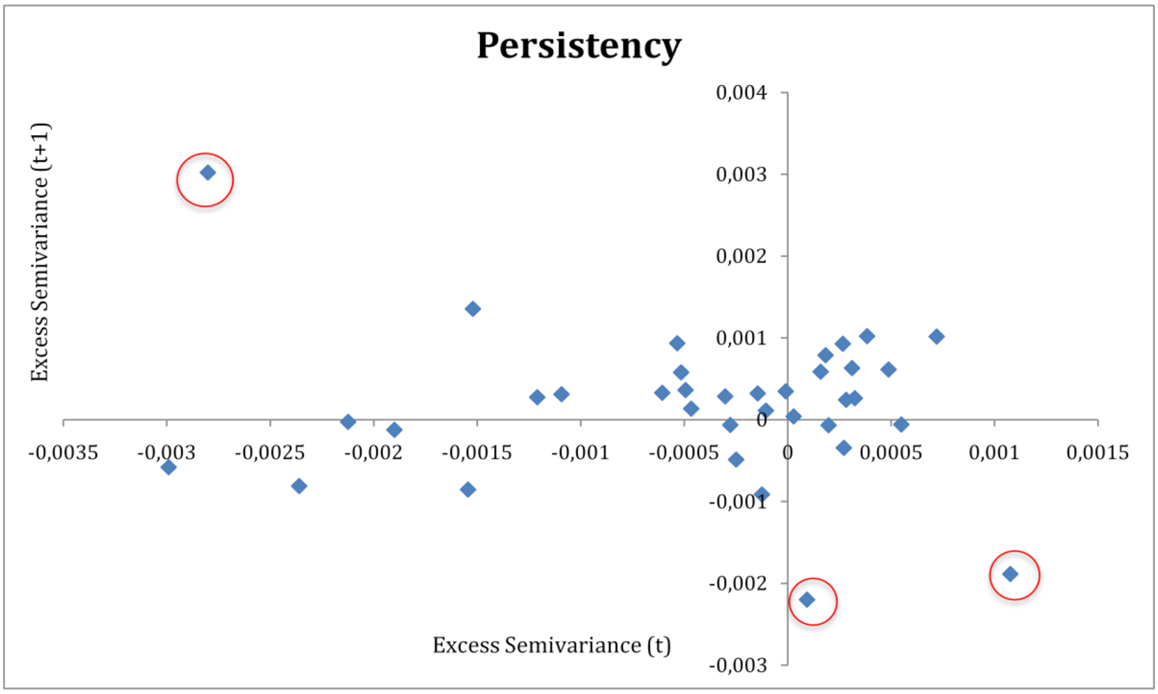
We have decided to use excess semivariance as our skew-measure, which can be calculated as[19]:

$$excess\ semi\operatorname{var}iance = 2 \cdot LPM_{2,\mu} - \sigma^2$$

We have only included the stocks and indices that indicated non-normality using the JB-test. However, our first attempt showed no overall persistency in non-normality since both β_1 was insignificant and R^2 was low. This indicates that there are some stocks in our analysis that are not persistent in their distribution over time. In order to identify the non-persistence stocks, we look at the scatter plot from our regression and see which stocks perform as outliers.

[19] Scherer (2002)

Figure 3-3: Persistency Plot



As can be seen in figure 3-3, there are three stocks that could be performing as outliers and thereby do not have a persistent distribution over time. When running the regression again after excluding these outliers, we find that β_1(0.0014) is significant and R^2 is high (0.3092), meaning that the remaining stocks are persistent over time. Although we are not able to conclude persistency for all stocks, it is safe to say that the vast majority of our securities are persistent in their distributions and we can assume non-normality. For SAS output, see Appendix CD 1.1.

We have shown that returns are not normally distributed in both daily and monthly data with the exception of a few stocks on monthly data. The conclusion on non-normality has also proven to be more statistically significant on daily data than on monthly data. With the results from this section, it is clear that almost all stocks and indices are not normally distributed and that this also holds over time.

3.3 Descriptive Statistics

This section introduces the different risk measures that have been calculated for the each security. We will clarify some of the assumptions that have been made in order to calculate the risk measures. The purpose of this section is also to give the reader a better understanding and feeling of the magnitude of the different risk measures used in the analysis.

The theoretical background for the risk measures has been reviewed in Chapter 2 where we clarified how they are calculated using historical returns. We present the following nine risk measures:

- Standard Deviation
- Semideviation
- Beta
- Downside Beta
- Gain-loss spread (GLS)
- Maximum Drawdown (MaxDD)
- Average Drawdown (AvgDD)
- Conditional Drawdown (CDaR)
- Conditional Value of Risk (CVaR)

Some of these risk measures include variables that can be set and adjusted according to preferences or strategies (e.g. quantiles, target rates of return etc). In the following, we will present our choice of variables included in the risk measures.

The semivariance is a special case of the lower partial moment. The LPM allows one to specify a target rate of return, which distinguishes (downside) risk from upside potential. The most commonly used target rates are zero, the risk free rate and the mean return, but essentially any other return value could be used. Harlow and Rao (1989) investigate the implications of the choice of target rate in the mean-LPM framework using regression analysis and find that the optimal target rate is the mean return. Based on their research paper, we choose to use the mean of the individual securities as our target rate of return. Consequently, this not only affects the semivariance but also the downside beta.

The Gain-loss spread uses a target rate of zero by design, but could also be customized to use any other rate of return. We choose zero as our target in order to separate actual gains from actual losses. In addition, the GLS is a fairly new risk measure and has therefore not seen much application on empirical data. Hence, we choose to follow Estrada's own definition and set the target rate to zero.

The conditional drawdown-at-risk and conditional value-at-risk both require a pre-specified quantile, which is usually set at 0.99, 0.95 or 0.9. The choice of quantile depends on the desired level of confidence as well as the size of the underlying data sample. We have chosen a quantile of 0.95 because we believe that we have the necessary amount of observations in both daily and monthly frequencies. One could argue that a different quantile should be used for monthly data due to the fewer observations. However, we want to use the same quantile in order to keep the results comparable across return frequencies.

Tables 3-3 and 3-4 below summarize the mean returns and risk of the different securities included in the analysis.

Table 3-3: Key Ratios, Monthly (for calculations, see appendix CD 1.2)

	MR	σ	Σ	β	β^D	GLS	AvgDD	MaxDD	CDaR	CVaR
A P Møller Mærsk B	1.70%	9.47%	6.33%	0.56	0.91	7.03%	34.13%	69.09%	56.16%	17.99%
Allianz	1.20%	9.59%	6.78%	0.69	1.09	7.04%	26.58%	66.74%	56.60%	21.35%
AT&T	1.05%	7.31%	5.25%	0.49	0.70	5.69%	20.96%	53.29%	40.16%	15.14%
Aviva	1.20%	9.73%	6.91%	0.75	1.09	7.31%	29.62%	86.90%	58.47%	21.30%
AXA	1.57%	11.53%	8.13%	0.91	1.30	8.21%	40.33%	93.70%	71.40%	26.47%
Bank of America	1.68%	11.39%	7.94%	0.89	1.14	7.89%	36.27%	89.07%	65.07%	24.74%
Barclays	1.68%	11.41%	7.74%	0.87	1.13	7.41%	25.50%	119.11%	89.45%	23.97%
BASF	1.26%	6.90%	5.09%	0.50	0.79	5.21%	15.47%	48.58%	34.52%	16.06%
BP	1.30%	7.67%	5.42%	0.59	0.79	5.80%	30.55%	53.07%	48.06%	16.04%
Canon	1.58%	9.43%	6.47%	0.77	0.96	7.28%	45.96%	78.54%	66.95%	18.45%
Carrefour	1.39%	7.16%	5.23%	0.35	0.66	5.68%	17.95%	44.46%	36.38%	15.49%
Chevron	1.29%	7.16%	4.98%	0.57	0.75	5.65%	23.99%	44.19%	39.22%	13.85%
Citi Group	1.35%	13.11%	9.23%	1.28	1.59	8.69%	28.22%	112.82%	85.75%	30.16%
Danske Bank	1.64%	8.64%	5.90%	0.50	0.79	6.03%	22.16%	77.97%	57.46%	17.63%
DAX 30	0.88%	6.06%	4.68%	0.58	0.82	4.62%	14.84%	44.02%	32.82%	15.13%
Deutshe Bank	1.06%	9.46%	6.71%	0.68	1.06	6.47%	21.11%	75.05%	56.23%	21.88%
Dow Jones	0.84%	6.06%	4.24%	0.81	0.90	4.43%	13.48%	35.80%	29.80%	13.59%
CAC 40	0.92%	5.54%	4.57%	0.65	0.87	4.46%	17.23%	42.60%	33.52%	15.37%
FTSE 100	0.83%	7.81%	4.23%	0.65	0.84	4.27%	13.23%	33.52%	27.50%	13.40%
General Electric	1.45%	8.18%	5.83%	0.82	0.98	6.40%	20.54%	53.32%	40.70%	16.84%
Honda Motors	1.55%	9.65%	6.12%	0.75	0.90	7.36%	46.35%	72.89%	64.97%	16.19%
Jyske Bank	1.51%	7.86%	5.65%	0.39	0.64	5.66%	24.90%	75.60%	44.58%	17.01%
Mitsubishi	1.23%	9.67%	6.79%	0.67	0.99	7.70%	32.84%	75.16%	57.81%	19.35%
MSCI EM	1.36%	5.75%	5.90%	1.08	1.25	6.16%	24.27%	60.00%	43.89%	17.49%
MSCI Worls	1.03%	5.52%	4.14%	1.00	1.00	4.30%	16.76%	38.96%	32.08%	12.75%
NIKKEI 225	0.64%	5.97%	4.58%	0.54	0.70	5.10%	18.87%	39.50%	33.47%	13.09%
Nippon	0.04%	9.02%	5.67%	0.52	0.74	7.47%	35.47%	61.11%	54.66%	16.38%
Novo Nordisk	2.11%	9.10%	5.99%	0.39	0.62	6.43%	56.03%	94.23%	75.46%	17.21%
OMXC20	0.76%	6.49%	4.52%	0.66	0.87	4.33%	11.87%	37.27%	28.63%	14.91%
RBS	1.68%	12.47%	8.88%	0.83	1.24	8.38%	45.79%	119.04%	79.45%	28.03%
Sanofi-Aventis	1.33%	7.66%	5.53%	0.40	0.62	6.13%	19.53%	42.77%	35.88%	15.19%
Siemens	1.17%	8.82%	6.44%	0.74	1.06	6.46%	21.69%	65.51%	51.94%	20.85%
Societe Generale	1.24%	10.77%	7.67%	0.93	1.40	7.72%	29.72%	77.39%	57.87%	24.68%
Tesco	1.68%	7.81%	5.43%	0.40	0.67	6.19%	31.31%	56.80%	48.65%	14.77%
Total	1.58%	7.84%	5.52%	0.39	0.63	5.99%	24.34%	55.67%	41.34%	15.45%
Toyota Motor	1.37%	8.81%	5.72%	0.65	0.83	6.68%	37.82%	65.65%	55.43%	15.43%
Volkswagen	1.27%	10.21%	7.07%	0.46	0.91	7.73%	31.46%	90.49%	63.88%	21.26%

Table 3-4: Key Ratios, Daily (for calculations, see appendix CD 1.3)

	MR	σ	Σ	β	β^D	GLS	AvgDD	MaxDD	CDaR	CVaR
A P Møller Mærsk B	0.08%	2.00%	1.33%	0.31	0.65	1.26%	28.25%	46.10%	32.84%	4.39%
Allianz	0.05%	2.01%	1.38%	0.52	0.86	1.36%	18.68%	35.77%	25.83%	4.64%
AT&T	0.05%	1.80%	1.26%	0.62	0.85	1.28%	19.65%	34.29%	26.02%	3.87%
Aviva	0.06%	2.25%	1.56%	0.54	0.94	1.53%	11.47%	60.24%	27.56%	5.04%
AXA	0.07%	2.36%	1.57%	0.62	0.97	1.57%	44.08%	64.18%	50.92%	5.24%
Bank of America	0.08%	2.56%	1.75%	0.85	1.19	1.58%	11.71%	59.12%	34.68%	5.57%
Barclays	0.08%	2.48%	1.64%	0.57	0.96	1.56%	13.33%	96.02%	66.94%	5.26%
BASF	0.06%	1.58%	1.11%	0.39	0.66	1.13%	8.16%	25.65%	14.75%	3.64%
BP	0.06%	1.71%	1.19%	0.40	0.70	1.25%	8.87%	21.89%	14.70%	3.73%
Canon	0.08%	2.33%	1.29%	0.28	0.73	1.72%	23.19%	41.05%	28.80%	4.90%
Carrefour	0.07%	1.78%	1.23%	0.33	0.66	1.29%	13.68%	26.76%	19.46%	4.05%
Chevron	0.06%	1.77%	1.24%	0.58	0.85	1.30%	11.09%	32.31%	20.48%	3.79%
Citi Group	0.07%	3.05%	2.07%	1.11	1.54	1.87%	22.36%	92.67%	55.93%	5.96%
Danske Bank	0.07%	1.68%	1.17%	0.29	0.60	1.10%	10.15%	27.20%	16.06%	3.88%
DAX 30	0.04%	1.35%	0.98%	0.43	0.66	0.96%	6.68%	20.24%	12.26%	3.22%
Deutshe Bank	0.05%	1.94%	1.34%	0.55	0.89	1.28%	11.02%	36.30%	25.12%	4.54%
Dow Jones	0.04%	1.32%	0.96%	0.73	0.91	0.96%	8.67%	21.97%	12.32%	3.04%
CAC 40	0.04%	1.37%	0.97%	0.55	0.77	0.99%	6.93%	18.31%	12.27%	3.16%
FTSE 100	0.04%	1.25%	0.90%	0.50	0.70	0.89%	5.50%	18.45%	11.17%	2.89%
General Electric	0.07%	1.87%	1.30%	0.69	0.98	1.32%	10.64%	26.13%	18.84%	4.16%
Honda Motors	0.08%	2.32%	1.58%	0.26	0.72	1.83%	15.46%	34.92%	20.96%	4.93%
Jyske Bank	0.07%	1.59%	1.08%	0.20	0.48	0.98%	17.89%	34.65%	22.25%	3.65%
Mitsubishi	0.06%	2.31%	1.46%	0.27	0.70	1.93%	12.74%	28.71%	19.71%	5.01%
MSCI EM	0.06%	1.51%	1.05%	0.89	1.07	0.80%	18.57%	48.63%	22.32%	3.15%
MSCI Worls	0.05%	1.12%	0.75%	1.00	1.00	0.67%	12.90%	31.04%	16.58%	2.39%
NIKKEI 225	0.03%	1.48%	1.05%	0.26	0.53	1.05%	10.71%	22.92%	16.74%	3.41%
Nippon	0.02%	2.42%	1.60%	0.24	0.66	2.77%	33.31%	46.57%	-31.13%	4.55%
Novo Nordisk	0.10%	1.91%	1.34%	0.20	0.52	1.23%	19.92%	43.09%	24.33%	4.14%
OMXC20	0.03%	1.18%	0.84%	0.38	0.60	0.83%	5.14%	17.87%	10.94%	2.79%
RBS	0.08%	2.71%	1.91%	0.53	0.97	1.62%	41.03%	109.13%	47.87%	5.77%
Sanofi-Aventis	0.07%	1.91%	1.32%	0.30	0.63	1.41%	13.45%	27.73%	18.67%	4.23%
Siemens	0.05%	1.83%	1.29%	0.51	0.84	1.27%	10.48%	32.03%	21.08%	4.31%
Societe Generale	0.06%	2.28%	1.57%	0.66	1.05	1.62%	24.65%	41.01%	30.66%	4.72%
Tesco	0.08%	1.79%	1.24%	0.28	0.62	1.32%	14.54%	27.76%	19.00%	3.81%
Total	0.08%	1.88%	1.32%	0.35	0.69	1.37%	9.83%	29.48%	14.99%	4.15%
Toyota Motor	0.07%	2.05%	1.40%	0.24	0.62	1.54%	16.64%	34.13%	21.69%	4.26%
Volkswagen	0.07%	2.77%	1.57%	0.33	0.71	1.48%	16.95%	167.77%	124.57%	4.84%

3.3.1 Performance

Table 3-5 shows how the different securities are ranked according to risk adjusted return based on daily observations. The risk adjusted return is calculated as following:

$$RAR = \frac{R_i}{RV_i}$$

where R_i is the mean return and RV_i is the risk variable. It is clear that the ranking for standard deviation, semideviation and CVaR are very similar. The similarity in the ranking of the standard deviation and semideviation was expected due to their similar theoretical frameworks. CVaR is somewhat different but as the table shows still quite comparable with standard deviation and semi-deviation. MaxDD, AvgDD and CDaR all originate from the same theory of drawdown functions and are, as expected, very similar in their ranking of the securities. Beta and downside beta also rank the securities similarly, which is not surprising after seeing the same trend for standard deviation and semideviation. However, it is important to remember that beta and downside beta only measure the *systematic* risk rather than the total risk, which explains the differences between the standard deviation and beta as well as the semideviation and downside beta respectively.

Table 3-5: Ranking According to Risk Adjusted Return

Name	SD	Semi D	GLS	Beta	Beta-	MaxDD	AvgDD	CDaR	CVaR
Novo Nordisk	1	1	1	1	1	4	7	4	1
Tesco	2	2	7	5	3	1	1	1	2
Danske Bank	3	4	5	8	4	2	4	3	5
MSCI WORLD	4	3	3	37	34	22	27	26	3
Jyske Bank	5	6	4	2	2	11	9	8	7
MSCI EM	6	8	2	34	28	26	32	31	4
Total	7	9	8	11	7	7	20	15	6
A. P. Møller	8	7	6	7	5	17	5	6	8
Carrefour	9	10	9	12	10	10	18	14	10
BASF	10	12	11	15	13	3	2	2	13
General Electric	11	11	10	24	19	12	12	10	11
Chevron	12	13	14	19	18	18	16	12	9
BP	13	14	15	14	15	6	6	7	14
Canon	14	5	17	4	6	16	14	13	12
Sanofi-Aventis	15	16	18	10	11	9	10	11	17
Honda	16	15	24	3	8	5	8	9	15
Toyota	17	17	22	6	9	14	21	19	16
Dow Jones	18	24	23	36	35	24	30	29	21
Barclays	19	18	13	17	16	33	24	24	18
FTSE 100	20	23	21	32	30	15	22	20	23
CAC 40	21	20	25	33	31	8	3	5	24
Bank of America	22	19	12	26	21	27	26	27	19
DAX 30	23	26	26	25	25	19	23	23	27
OMX	24	29	28	27	29	20	25	25	29
AT&T	25	27	29	30	26	28	29	30	22
AXA	26	21	20	20	20	31	15	17	26
Siemens	27	28	27	23	22	21	19	21	30
Royal Bank of Scotland	28	30	16	16	17	35	37	37	25
Mitsubishi	29	22	35	9	14	13	11	16	28
Allianz	30	31	30	22	23	23	13	18	33
Aviva	31	32	31	21	24	32	33	33	32
Societe Generale	32	33	34	29	32	25	17	22	31
Volkswagen	33	25	19	13	12	37	36	35	20
Deutsche Bank	34	34	32	31	33	29	28	28	35
Citigroup	35	35	33	35	36	34	34	34	34
NIKKEI 225	36	36	36	18	27	30	31	32	36
Nippon	37	37	37	28	37	36	35	36	37

4 Chapter IV: Analysis part I

In this chapter, we wish to test the relationship between the different downside risk measures and the cross-section of stock returns in order to determine whether these exhibit a higher explanatory power than the traditional risk measures from the mean-variance framework. We follow the general methodology of Estrada (2007, 2009), taking into account the statistical and economic inadequacies of his approach. We start out by looking at the correlations between the different risk measures and the cross-sectional return. Hereafter, we conduct a more in-depth regression analysis in order to check the significance of the correlations. Finally, we check the robustness of our results by forming portfolios ranked by their risk and taking the spreads in mean returns between the riskiest and the least risky portfolio. We conduct the analysis of both monthly and daily returns and continuously compare these to see how the data frequency affects the correlations. In addition, we compare historical bull markets with bear markets again in both monthly and daily frequencies.

4.1 Correlations

The first step of the analysis consists of calculating the monthly and daily (arithmetic) means and the respective risk variables for each stock/index over the whole sample period. The nine risk measures considered in the analysis are summarized for all equities in tables X and Y. We start out by computing a correlation matrix between the ten variables (the nine risk variables and the mean return) in order to provide a preview of some results analyzed in more detail later on.

Table 4-1: Correlation Matrix (monthly data) [20]

	MR	σ	Σ	*GLS*	β	β^D	*MaxDD*	*AvgDD*	*CDaR*	*CVaR*
MR	1									
σ	0.470483	1								
Σ	0.458023	0.982401	1							
GLS	0.409632	0.964922	0.928703	1						
β	0.005159	0.423798	0.486936	0.348015	1					
β^D	0.107336	0.660818	0.737876	0.578981	0.904817	1				
MaxDD	0.551072	0.929344	0.914073	0.847235	0.363333	0.575378	1			
AvgDD	0.5417	0.665111	0.569601	0.717829	0.070606	0.171342	0.664114	1		
CDaR	0.538431	0.932625	0.886181	0.876304	0.366802	0.547175	0.962539	0.753141	1	
CVaR	0.348415	0.918516	0.968127	0.825481	0.561322	0.815954	0.865367	0.421064	0.81174	1

[20] MR: mean return, σ: standard deviation, Σ: semideviation, GLS: gain-loss spread, β: beta (with respect to MSCI WRLD), β^D: downside beta (with respect to MSCI WRLD), MaxDD: maximum drawdown, AvgDD: average drawdown, CDaR: conditional drawdown-at-risk, CVaR: conditional value-at-risk.

Table 4-2: Correlation Matrix (daily data)

	MR	σ	Σ	*GLS*	β	β^D	*MaxDD*	*AvgDD*	*CDaR*	*CVaR*
MR	1									
σ	0.456973	1								
Σ	0.441018	0.968224	1							
GLS	0.139451	0.805054	0.781899	1						
β	-0.13054	0.069643	0.129276	-0.16759	1					
β^D	0.049626	0.441184	0.496647	0.175377	0.914097	1				
MaxDD	0.315165	0.737936	0.661633	0.355249	0.180528	0.377203	1			
AvgDD	0.249936	0.584518	0.576326	0.525195	0.032318	0.224805	0.461889	1		
CDaR	0.427063	0.559435	0.461925	0.045823	0.175657	0.31109	0.861557	0.230849	1	
CVaR	0.499596	0.96264	0.963353	0.757735	0.052676	0.433891	0.619627	0.55604	0.48645	1

As can be seen in table 4-1, all three drawdown functions outperform the other risk measures for monthly data, closely followed by the standard deviation. The CAPM beta has the lowest correlation with the mean return, followed by the downside beta. This is somewhat intuitive as they are both measures of *systematic* risk as opposed to the other variables, which measure total risk. However, it is interesting that the correlation of the CAPM beta is only about 0.005 while that of the downside beta is about 0.11 or 22 times higher.

It is also interesting to look at the correlations between the different risk measures. Naturally, the standard deviation and the semideviation are almost perfectly correlated. In addition, it makes sense that the GLS and the standard deviation are highly correlated as well (0.96), since the GLS provides basically the same information about risk. In short, the GLS provides a more insightful risk measure and is at the same time almost as correlated with returns as the standard deviation. In addition, the correlation between CVaR and the semideviation is 0.97, which makes sense because a larger semideviation implies larger values beneath the 5% quantile. Finally, it is also reasonable that the different drawdown metrics have high mutual correlations.

In table 4-2, we have reported the correlations between the same ten variables based on daily observations. As can be seen, the dominant risk measure is now CVaR, followed by the standard deviation and the semideviation. The risk variable that correlates with mean return the least is now the downside beta with 0.05, followed by the CAPM beta, which is now negatively correlated with returns by -0.13. The correlations between the drawdown measures and the mean return have decreased from monthly to daily data. The correlation in the two extreme cases, MaxDD and AvgDD, is significantly reduced whereas that of CDaR has had a relatively smaller reduction. Additionally, the standard deviation has decreased slightly due to the higher divergence from normality in daily return data. Finally, it is remarkable that the GLS correlation has decreased dramatically from 0.41 in monthly data to 0.14 in daily data.

4.2 Statistical Significance: The Full Sample

While the above results imply how the different risk variables correlate with the mean return, we want to determine whether these relationships are significant or not. More detailed results about the relationship between risk and return across equities can be obtained from cross-section regressions. We begin by running a simple linear regression model relating mean returns to each of the nine risk variables considered. More precisely:

$$\mu_i = \beta_0 + \beta_1 RV_i + u_i \tag{29}$$

where μ_i and RV_i represent the mean return and risk variable, respectively, β_0 and β_1 are coefficients to be estimated, u_i is an error term, and i indexes equities. Subsequently, we run multiple regressions with the standard deviation and one of the downside risk variables in order to see their relative performance.

$$\mu_i = \beta_0 + \beta_1 \sigma_i + \beta_2 DRV_i + u_i \tag{30}$$

where σ_i is the standard deviation and DRV_i is the downside risk variable. The same form of regressions is run where we include the CAPM beta instead of the standard deviation. Table 4-3 presents our regression results for monthly data[21]. For SAS output, see Appendix CD 2.1.

[21] Asterisks denote significance level: 0.05(*), 0.01(**), 0.001(***)

Table 4-3: Results from regression analysis on monthly data (significance based on White's heteroskedasticity-consistent standard errors)

	Constant	*σ*	*β*	*Σ*	*GLS*	*$β^D$*	*Max DD*	*Avg. DD*	*CDaR*	*CVaR*	*R^2*	*VIF*
	0,005040**	0,090840***									0,2214	
	0,012760***		0,000091								0,0000	
	0,004650*			0,135370***							0,2098	
	0,004870*				0,124980**						0,1678	
Panel A	0,011200***					0,001750					0,0115	
	0,008410***						0,006130***				0,2373	
	0,008440***							0,012430**			0,2027	
	0,007440***								0,009840***		0,2273	
	0,007350**									0,030000**	0,1214	
	0,005230	0,113560		-0,035410							0,2219	28,66
	0,007260*	0,210700			-0,196310						0,2499	14,51
	0,006540***	0,136940***				-0,005890					0,2949	1,76
Panel B	0,006870**	0,034890					0,004110				0,2443	4,66
	0,005800*	0,059910						0,005350			0,2333	3,16
	0,006160*	0,040110							0,005900		0,2341	6,44
	0,005310***	0,18582*								-0,046110	0,2662	6,40
	0,005550**		-0,005060	0,176470***							0,2720	1,31
	0,005660**		-0,002770		0,141580**						0,1893	1,41
	0,010280***		-0,00899*			0,009230*					0,0582	5,52
Panel C	0,010280***		-0,004050				0,007290***				0,2811	1,19
	0,010250***		-0,004220					0,015310***			0,2486	1,24
	0,009170		-0,004420						0,012080		0,2778	1,23
	0,008190		-0,004930							0,043430	0,1743	1,46

Panel A shows the results of our simple regressions from equation (29), where the coefficients of all the risk measures have the expected signs. Beta and downside beta are the only insignificant variables (with very low R^2 values), which is in accordance with the low correlations we saw earlier. As can be seen, the maximum drawdown has the highest explanatory power with an R^2 value of 23.73% and is significant at the 0.001 level. The CDaR and standard deviation are also significant at the 0.001 level. Panel B shows the regression results from running equation (30) with the standard deviation and a downside risk measure as explanatory variables. Estrada (2007, 2009) runs similar multiple regressions but does not take into account the presence of multicollinearity, i.e. the explanatory variables are inter-correlated. We saw in the correlation tables above that there is a high correlation between several of our risk measures and the standard deviation and beta respectively. A rule of thumb says that severe multicollinearity obtains when the correlation between the explanatory variables exceeds 0.75. We can see that this is the case for some variables in both monthly and daily frequencies. Luckily, we can still obtain correct estimates in spite of the presence of multicollinearity, however the consequence is inflated standard errors, which can lead to the acceptance of the "zero null hypothesis" (i.e. the true population coefficient is zero) due to wider confidence intervals[22]. In fact, Panel B shows that running the multiple regressions leads to insignificant coefficients in almost every case. We have computed for each multiple regression the variance-inflating factor (VIF), which measures the speed with which variances and covariances increase, as seen below:

$$VIF = \frac{1}{\left(1-\rho_{ij}^{2}\right)}$$

where ρ_{ij}^{2} is the coefficient of correlation between X_i and X_j. VIF shows that the variance of an estimator is inflated by the presence of multicollinearity. A VIF value of more than 2 represents severe multicollinearity, and a VIF value of more than 10 represents destructive multicollinearity. Panel B shows that the only regression without severe multicollinearity includes downside beta as the second risk variable, which means that we can trust the significance of the coefficients. In this case only the standard deviation is significant, which makes this regression less interesting. The remaining regressions have either severe or destructive multicollinearity, which means that we cannot be sure of whether the coefficients (relationships) are significant or not. Panel C shows the multiple regression results where we

[22] Gujarati (2003)

include the CAPM beta together with a downside risk measure. It can be seen that in almost every case, the beta is insignificant while the downside risk measure is significant. As we noted earlier, the beta measures only the systematic risk while the other variables measure the total risk, which makes the beta's contribution to the model relative to the other risk variable insignificant. Intuitively, there is no sign of multicollinearity in Panel C (except for when we use the CAPM beta with the downside beta) as opposed to Panel B. Finally, our model becomes overall insignificant when including beta with CDaR and CVaR respectively.

Table 4-4: Results from regression analysis on daily data (significance based on White's heteroskedasticity-consistent standard errors)

	Constant	*σ*	*β*	*Σ*	*GLS*	$β^D$	*Max DD*	*Avg. DD*	*CDaR*	*CVaR*	R^2	*VIF*
	0,000305**	0,015920**									0,2088	
	0,000658***		-0,000094								0,0170	
	0,000287**			0,024800**							0,1945	
	0,000534**				0,005850						0,0194	
Panel A	0,000582***					0,000038					0,0025	
	0,000532***						0,000176*				0,0981	
	0,000495***							0,000573**			0,1446	
	0,000468***								0,000578**		0,1644	
	0,000215*									0,009480***	0,2496	
	0,000307**	0,016690		-0,001290							0,2089	15,99
	0,000320**	0,034130***			-0,027230*						0,3571	2,84
	0,000364***	0,018820**				-0,000145					0,2375	1,24
Panel B	0,000292*	0,017250*					-0,000029				0,2100	2,19
	0,000322**	0,012550						0,000233			0,2233	1,65
	0,000323**	0,011770							0,000248		0,2248	1,89
	0,000185	-0,011380								0,015450	0,2574	13,64
	0,000334**		-0,000137	0,026190**							0,2303	1,02
	0,000582**		-0,000079		0,005070						0,0313	1,03
	0,000354**		-0,000769**			0,000788**					0,1906	6,08
Panel C	0,000590***		-0,000141				0,000197*				0,1352	1,04
	0,000560***		-0,000177					0,000670**			0,2013	1,07
	0,000531***		-0,000179						0,000667**		0,2224	1,07
	0,000263*		-0,000113							0,00964***	0,2743	1,00

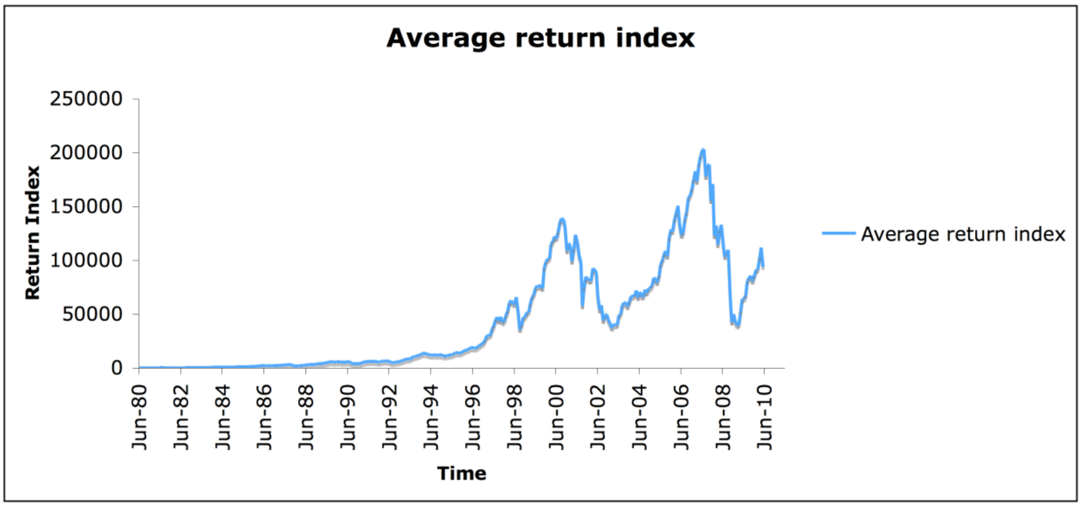
Table 4-4 shows our regression results based on daily return data. Panel A shows that beta and downside beta are insignificant like before but that GLS now also becomes insignificant. Panel B shows more or less the same trend as with monthly data, only here the GLS is significant together with the standard deviation but has the wrong sign. Finally, Panel C shows that all the downside risk variables but the GLS are significant and have the expected sign. For SAS output, see Appendix CD 2.2.

The reason why it is interesting to compare the results of monthly data with that of daily data is twofold. First, we have many more observations when using daily data. This means that we expect a measure such as MaxDD to have a lower explanatory power than before since it is based on merely two observations out of the whole sample. This also seems to be the case as the R^2 value has dropped from 24% to 10% and the relationship is now less significant than previously. Overall, we would expect the downside risk measures in general to perform better than before since daily return data exhibits a greater deviation from normality than monthly return data. Strangely, we see the opposite tendency with more insignificant coefficients and generally lower R^2 values.

4.3 Statistical Significance: Bear vs. Bull Markets

Our equity data spans 30 years back and thus includes several economic up- and downturns, which is reflected in the return development in figure 4-1.

Figure 4-1: Average Return Index



Thus, it is interesting to analyze the performance of our risk measures in up (bull) markets compared to down (bear) markets. In this section, we consider two subsamples of our data, one bull market (1996 – 2000) and one bear market (2000 – 2003), and re-assess the significance and explanatory power of each of our risk variables. In theory, the downside risk frameworks make more sense the more skewed the distributions of returns are, and it is fair to assume that bear/bull markets exhibit more skewness than the full sample. Therefore, we expect the downside risk variables to outperform in terms of explaining the fluctuations in equity returns.

Table 4-5: Results from regression analysis on monthly data (significance based on White's heteroskedasticity-consistent standard errors)

		Intercept	*β*	*Max DD*	*Avg. DD*	*CDaR*	*CVaR*	*R^2*
Daily	*Bear Market*	-0,000040	-0,000490*					0,0720
	Bull Market	0,000720***					0,010250	0,0731
Monthly		0,015640***		0,018810*				0,0729
	Bull Market	0,015390***			0,036520*			0,1125
		0,015260***				0,020690*		0,0765
	Bear Market	0,007650					-0,077030**	0,1234

Table 4-5 summarizes those regression results from equation (29) that are significant[23]. First of all, note that we have reduced our sample substantially in terms of observations, which is why we include the coefficients that are significant at the 10% level. Again, we use both daily and monthly frequencies in the analysis. It is clear that, despite the majority of our regressions being insignificant, the downside risk variables dominate the picture. Although we are testing seven downside measures versus two traditional measures, it should be noted that the MaxDD, the AvgDD and the CDaR are very similar metrics and thus are highly inter-correlated. Surprisingly, with daily observations the CAPM beta has a significant relationship with the mean return of the bear market. However, it has a negative coefficient and only explains about 7% of the variation in returns. The negative coefficient on beta can be explained by the following reasoning. We are using equity return data from a bear market, which implies that the market in general has declined. Since the CAPM beta measures the volatility associated with aggregate market returns, we must have that

$$\beta_i^+ \quad \Rightarrow \quad R_M \downarrow \quad \rightarrow \quad R_i \downarrow$$
$$\beta_i^- \quad \Rightarrow \quad R_M \downarrow \quad \rightarrow \quad R_i \uparrow$$

[23] For SAS output, see Appendix CD 2.3

where β_i^+ and β_i^- denote a positive and negative CAPM beta of stock *i* respectively, and R_M and R_i represent the market return and stock return respectively. Thus, a positive (negative) beta implies that a given stock goes up (down) when the market goes up. In order for the above relationship to hold, we must have a negative coefficient on beta, since we know that R_M declines in a bear market. This means that the model

$$R_i = \gamma_0 + \gamma_1 \beta_i^{CAPM} + u_i$$

should yield a negative value for γ_1. These results should, however, be treated with caution since beta only reflects the systematic volatility and it is therefore curious that the coefficient is significant at all.

Proceeding with the other variables, we see that CVaR is borderline significant at the 10% level with daily bull market observations. With monthly data, the bull market mean return is significantly correlated with the three drawdown measures, where the AvgDD has the highest explanatory power. Finally, it can be seen that the CVaR explains the monthly bear market returns the most (R^2 = 12.34%) and is significant at the 5% level. However, we did not expect it to have a negative sign.

4.4 Economic Significance: Return Spreads

In order to check for the robustness of the results discussed in the previous section, we divide all equities into three equally weighted portfolios ranked by the different risk measures; the top third riskiest equities (P1) and the bottom third least risky equities (P3). This process is repeated for all of our risk measures. Finally, we calculate the spread in mean returns between P1 and P3. The results are summarized in table 4-6 below.

Table 4-6: Spreads in Monthly and Daily Mean Returns

Panel A: Daily									
	σ	Σ	GLS	β	β^D	MaxDD	AvgDD	CDaR	CVaR
P1	0.050%	0.050%	0.053%	0.066%	0.066%	0.052%	0.047%	0.047%	0.049%
P2	0.067%	0.069%	0.065%	0.057%	0.060%	0.067%	0.065%	0.064%	0.064%
P3	0.066%	0.065%	0.066%	0.060%	0.063%	0.066%	0.072%	0.072%	0.069%
Spread	**0.016%**	**0.015%**	**0.013%**	**-0.005%**	**-0.003%**	**0.014%**	**0.025%**	**0.025%**	**0.020%**

Panel B: Monthly									
	σ	Σ	GLS	β	β^D	MaxDD	AvgDD	CDaR	CVaR
P1	1.10%	1.01%	1.03%	1.35%	1.26%	1.04%	1.00%	1.08%	1.05%
P2	1.34%	1.40%	1.47%	1.14%	1.20%	1.32%	1.36%	1.29%	1.41%
P3	1.38%	1.40%	1.34%	1.43%	1.37%	1.50%	1.46%	1.50%	1.36%
Spread	**0.28%**	**0.38%**	**0.31%**	**0.08%**	**0.11%**	**0.46%**	**0.46%**	**0.42%**	**0.31%**

Panel A shows the spreads in mean monthly returns between the riskiest and the least risky portfolios. It can be seen that this spread is highest when constructing portfolios ranked by MaxDD (0.46%) and lowest when ranked by the downside beta. This is in accordance with the previous results from our regression analysis where the downside beta failed to significantly explain the monthly mean returns while the MaxDD was the dominant risk variable.

Panel B shows the spreads in mean daily returns between the different portfolios. The results show that the AvgDD portfolios exhibit the highest spread in mean returns (0.02%). Again, this difference stems from fewer observations in monthly data together with the fact that MaxDD only incorporates two values while AvgDD is based on the whole sample. Interestingly, when ranking the portfolios by the CAPM beta, we get that the riskiest portfolio yields a lower return than the least risky portfolio while its risk is more than 2.5 times higher. This supports our hypothesis that the CAPM beta is inadequate as a tool for investment decisions.

4.5 Summary

In this chapter, we have attempted to clarify how the different risk measures explain the cross-section of equity returns. As a starting point, we computed a correlation matrix for monthly and daily data. With monthly data, the maximum drawdown appeared to have the highest correlation with returns, namely 49%. With daily data, CVaR was the dominant risk measure with a correlation with returns of 50%.

We started the analysis by regressing the individual risk variables on the cross-sectional returns and found that the maximum drawdown and CVaR had the highest explanatory power when using monthly and daily data respectively. This supported the initial observation that these were the most correlated with the mean return. Subsequently, we ran multiple regressions in order to test the significance and explanatory power of the downside risk measures while being regressed together with the traditional risk measures. While we did not get much information from regressing the CAPM beta, we found that all our downside risk measures were insignificant when regressed together with the standard deviation (with monthly data). Using daily data, we saw that our model became significant when including the standard deviation and GLS. This regression also yielded the highest explanatory power of all, namely 36%, without the presence of harmful multicollinearity. When looking at bear and bull markets separately, we regressed the individual risk measures independently. We found that some of our risk variables were insignificant in both monthly and daily data. The most significant variable was CVaR, which also had the highest explanatory power, though with a negative sign.

Finally, we checked the robustness of our results by forming portfolios ranked by their risk and taking the spreads in mean returns between the riskiest and the least risky portfolio. With monthly data, we found that the maximum drawdown was dominant, which was in accordance with the previous results. However when using daily data, the absolute spread was greatest between the portfolios ranked by the average drawdown. While the AvgDD performed well throughout our analysis, we had expected the CVaR to outperform AvgDD.
Overall, our results were fairly robust. Throughout the entire analysis, the MaxDD seemed to explain the cross-sectional returns the best when using monthly data. When we used daily data, the two prevailing risk measures were the CVaR and the average drawdown. In order to get an overview, we have ranked the different risk measures in table table 4-7.

Table 4-7: Ranking of Risk Measures

Monthly			
	Regression Analysis	Return Spreads	*Total*
MaxDD	1	1	*2*
CDaR	2	3	*5*
AvgDD	5	2	*7*
Σ	4	4	*8*
σ	3	7	*10*
GLS	6	6	*12*
CVaR	7	5	*12*

Daily			
	Regression Analysis	Return Spread	*Total*
CVaR	1	3	*4*
CDaR	4	2	*6*
AvgDD	5	1	*6*
σ	2	4	*6*
Σ	3	6	*9*
MaxDD	6	5	*11*
GLS	7	7	*14*

As we can see, the CDaR is one of the dominant variables in explaining variations in cross-sectional returns when looking at both monthly and daily data. MaxDD seems to explain monthly returns relatively well, but fails with daily returns. This could be explained by the fact that MaxDD only considers two observations and we have significantly more observations with daily data. The most dominant risk variable for daily data turned out to be CVaR closely followed by CDaR. However, CVaR had the worst performance for monthly data but since our further analysis will be based on daily returns, we expect CVaR to do fairly well. Variance and semivariance performed average in both daily and monthly return data relative to the other risk measures.

The overall analysis suggests that downside risk measures could in fact be a relevant alternative to variance.

5 Chapter V: Analysis part II

In this chapter, we will look at the portfolio optimization problem and analyze the dynamics between (downside) risk and performance across market conditions. Throughout the analysis we assume a highly risk-averse investor and therefore focus on the minimum risk portfolios, which are optimal for such an investor. In order to make the analysis tractable, we reduce the asset universe to seven securities. Since we are dealing with a European investor, we assume that she has knowledge about the European market and we therefore include individual European equities and indices for foreign markets. We begin the analysis by demonstrating how to derive the minimum risk portfolio for each risk measure and depict its position on the respective efficient frontier. Since each portfolio is constructed by minimizing different measures of risk, we cannot directly compare them by means of risk-adjusted returns. To overcome this problem, we will rebalance the minimum risk portfolios on a yearly basis and compare the terminal value of each portfolio after 17 years. Another useful feature about rebalancing is that we can better analyze the dynamics when the market goes up and when it goes down. Finally, we will illustrate how uncertainty in the input parameters (means and risk) affects the optimal weights and performance of the portfolios. This will provide a general idea of the sensitivity of the different optimization processes towards the input parameters.

5.1 Efficient Frontiers and Minimum Risk Portfolios

The purpose of this section is to illustrate how the frontiers and minimum risk portfolios from each theoretical framework can be derived. The method applied throughout this process is linear programming.

5.1.1 The Mean-Variance Framework

We follow the procedure presented in section 2.2.1 in order to find the minimum variance portfolio based on our seven securities. As we saw, it is possible to define the covariance matrix from our securities, which makes it straightforward to find this portfolio applying matrix calculus. Below we have presented the individual mean returns and the covariance matrix of our securities.

$$\mu = \begin{pmatrix} Novo\ Nordisk \\ RBS \\ AXA \\ Deutsche\ Bank \\ Dow\ Jones \\ Nikkei225 \\ Bonds \end{pmatrix} = \begin{pmatrix} 0{,}0861\% \\ 0{,}0604\% \\ 0{,}0569\% \\ 0{,}0433\% \\ 0{,}0092\% \\ 0{,}0384\% \\ 0{,}0237\% \end{pmatrix}$$

$$S = \begin{pmatrix} 0{,}0312\% & 0{,}0077\% & 0{,}0095\% & 0{,}0080\% & 0{,}0016\% & 0{,}0038\% & -0{,}0001\% \\ 0{,}0077\% & 0{,}0929\% & 0{,}0361\% & 0{,}0325\% & 0{,}0060\% & 0{,}0118\% & -0{,}0007\% \\ 0{,}0095\% & 0{,}0361\% & 0{,}0629\% & 0{,}0345\% & 0{,}0066\% & 0{,}0128\% & -0{,}0012\% \\ 0{,}0080\% & 0{,}0325\% & 0{,}0345\% & 0{,}0514\% & 0{,}0053\% & 0{,}0135\% & -0{,}0011\% \\ 0{,}0016\% & 0{,}0060\% & 0{,}0066\% & 0{,}0053\% & 0{,}0260\% & 0{,}0032\% & 0{,}0015\% \\ 0{,}0038\% & 0{,}0118\% & 0{,}0128\% & 0{,}0135\% & 0{,}0032\% & 0{,}0181\% & 0{,}0012\% \\ -0{,}0001\% & -0{,}0007\% & -0{,}0012\% & -0{,}0011\% & 0{,}0015\% & 0{,}0012\% & 0{,}0014\% \end{pmatrix}$$

We then define the portfolio variance as:

$$\sigma_P^2 = \omega^T S \omega \tag{31}$$

where ω is a vector of weights or fractions to be invested in the different securities. We find these weights by minimizing the portfolio variance, thus obtaining the minimum variance portfolio. This portfolio has the following properties:

Table 5-1: Minimum Variance Resutls

	Novo Nordisk	RBS	AXA	Deutshe Bank	Dow Jones	Nikkei 225	Bonds
MR	0.086%	0.061%	0.059%	0.044%	0.009%	0.039%	0.023%
Weights	3.992%	0.468%	2.313%	3.559%	-2.401%	-4.300%	96.370%

	MR	Variance	SD	RAR
Min MV	0.028%	0.0012%	0.349%	7.877%

As can be seen, almost the entire investment should be placed in bonds (96%) in order to minimize the variance. This makes sense since bonds exhibit the lowest volatility (0.0014%) of all the securities, and we did not define any constraint on the magnitude of our portfolio return[24]. Furthermore, the fractions to be invested in Dow Jones and Nikkei225 are -2.40% and -4.30% respectively, which means that we should take a short position on these indices. Finally, if we want some action in our portfolio, RBS seems to be the right stock with an expected 3% change in value per day. However since we are minimizing the volatility, it is reasonable that only 0.47% should be invested in RBS.

Now that we have determined the minimum variance portfolio, we can compute the efficient frontier in the mean-variance framework. We do this by means of linear programming where we solve for minimum variances given different levels of portfolio return, thus identifying several sets of weights that all compose efficient portfolios.

Figure 5-1: MV Frontier and Minimum Variance Portfolio

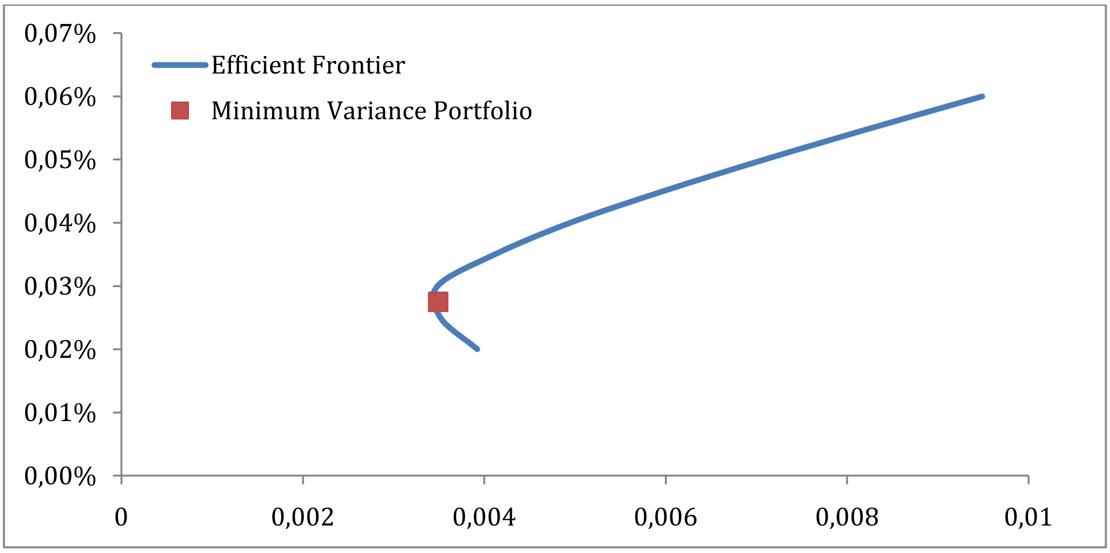


Figure 5-1 illustrates the mean-variance frontier of investing in our seven securities. Note that the *efficient* frontier does not include the portfolios below the minimum variance portfolio, as these are clearly inferior. As we increase the return constraint, the weights gradually shift from bonds to Novo Nordisk since bonds yield the lowest return while Novo Nordisk yields the highest. Additionally, the two stock indices behave rather differently. While Nikkei 225 obtains a positive weight after a certain level, the weight on Dow Jones keeps decreasing (in fact, the only time we would take a long position in Dow Jones is when the required return is less than that of the minimum variance portfolio). The weights and properties of the computed portfolios can be found in appendix CD 3.1.

[24] Note that equation (31) does not include the return vector. The portfolio variance is independent of the individual returns.

5.1.2 The Mean-Semivariance Framework

In section 2.2.2.4 we saw that the semicovariance matrix as defined by Hogan and Warren (1972) suffered from endogeneity and was asymmetric, i.e. $(\Sigma_{ij\tau} \neq \Sigma_{ji\tau})$. This problem was overcome by applying Estrada (2002, 2007)'s definition of the semicovariance, which leads to a symmetric, exogenous semicovariance matrix. Therefore, we can apply a heuristic approach to the mean-semivariance optimization problem that follows the exact same procedure we used in the mean-variance framework[25]. By using equation (21), we compute the semicovariance matrix below:

$$\Phi = \begin{pmatrix} 0{,}0153\% & 0{,}0085\% & 0{,}0085\% & 0{,}0078\% & 0{,}0042\% & 0{,}0043\% & 0{,}0009\% \\ 0{,}0085\% & 0{,}0488\% & 0{,}0205\% & 0{,}0198\% & 0{,}0083\% & 0{,}0092\% & 0{,}0015\% \\ 0{,}0085\% & 0{,}0205\% & 0{,}0291\% & 0{,}0185\% & 0{,}0078\% & 0{,}0087\% & 0{,}0012\% \\ 0{,}0078\% & 0{,}0198\% & 0{,}0185\% & 0{,}0246\% & 0{,}0070\% & 0{,}0087\% & 0{,}0011\% \\ 0{,}0042\% & 0{,}0083\% & 0{,}0078\% & 0{,}0070\% & 0{,}0129\% & 0{,}0042\% & 0{,}0013\% \\ 0{,}0043\% & 0{,}0092\% & 0{,}0087\% & 0{,}0087\% & 0{,}0042\% & 0{,}0093\% & 0{,}0012\% \\ 0{,}0009\% & 0{,}0015\% & 0{,}0012\% & 0{,}0011\% & 0{,}0013\% & 0{,}0012\% & 0{,}0007\% \end{pmatrix}$$

Now we can apply the equivalent to equation (31) to find the portfolio *semi*variance in order to find the weights that minimize it.

$$\Sigma_P^2 = \omega^T \Phi \omega$$

Thus, the result of the minimum semivariance portfolio can be seen in table 5-2:

Table 5-2: Minimum Semivariance Portfolio Results

	Novo Nordisk	RBS	AXA	Deutshe Bank	Dow Jones	Nikkei 225	Bonds
MR	0.086%	0.061%	0.059%	0.044%	0.009%	0.039%	0.023%
Weights	0.376%	-1.049%	0.018%	2.058%	-5.098%	-6.047%	109.744%

	MR	$\sum^2$	$\sum$	RAR
Min MSV	0.024%	0.001%	0.249%	9.531%

As the results imply, minimizing semivariance yields a similar overall allocation as when minimizing variance. However, the individual weights are more emphasized than before, with

[25] Estrada (2008)

a larger portion being invested in bonds while less is invested in equities. This could be an indication of some downside risk (perhaps in the form of skewness or fat tails) that was not captured in the mean-variance framework. Since we are using the respective securities' mean returns as our target rates of return, it could mean that the lower partial moments of the equities are greater than their higher moments. We are now investing more than 100% in bonds, which could indicate that they are exhibiting a positive volatility. If normality obtained, we would have that $\Sigma^2 = \frac{1}{2}\sigma^2$. It can be calculated that the bonds exhibit a lower semivariance than half their variance[26], which means that a semivariance minimizing investor would in fact allocate a higher part of their investment to bonds. The weights on Novo Nordisk, AXA and Deutsche Bank are lower than before, and we are now taking a short position in RBS[27]. In addition, our short positions in the two indices are also of greater value than before.

The next step is to compute the mean semivariance frontier. As before, we construct minimum semivariance portfolios for a range of pre-specified rates of return.

Figure 5-2: MSV Frontier and Minimum Semivariance Portfolio

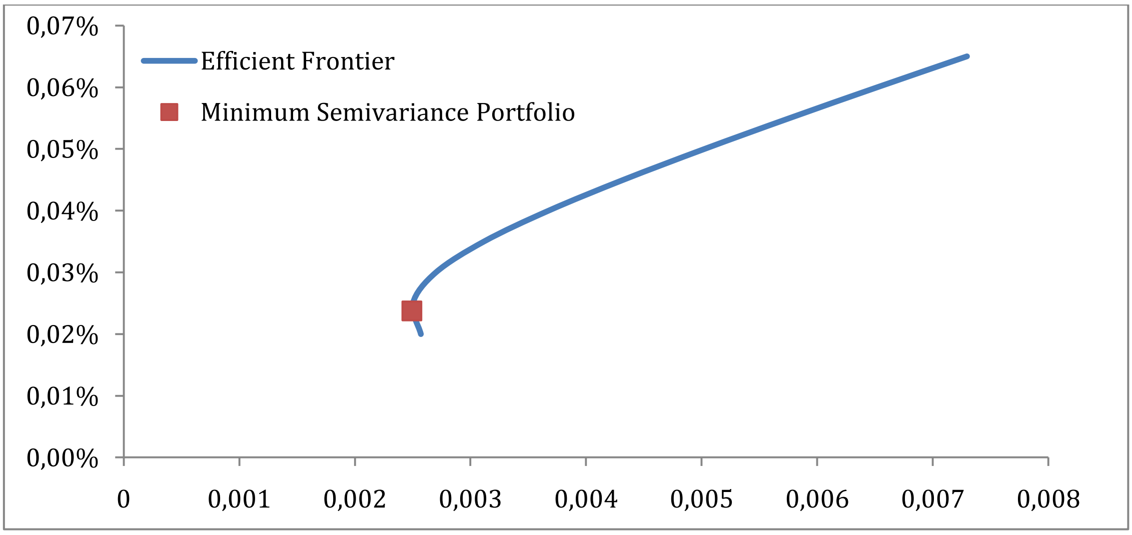


Figure 5-2 shows the portfolios of our seven securities that lie on the frontier (see Appendix CD 3.2 for details and calculations). Again, only those above the minimum semivariance portfolio are efficient. As we increase the return constraint, we get a similar trend in the

26 $0.0014107\%/2 = 0.0007053\% \quad > \quad 0.0006841\%$
27 RBS exhibits a higher semivariance than half its variance.

portfolio weights as in the mean-variance framework with the weight in bonds shifting toward Novo Nordisk. However, the speed in which the weights change varies a lot. In this framework, the weight on bonds falls at a slower rate than before while the weight on Novo Nordisk rises faster than before. Most remarkable is that the proportion invested in AXA increases more than twice as fast as before (from -0.69% to 7.73% as opposed to from 1.21% to 7.85% over the same range of returns). However, the overall picture suggests that the two frameworks are somewhat similar to each other. This was expected since the two risk measures are highly correlated, as we saw in chapter 4.

5.1.3 The Mean-GLS Framework

The gain-loss spread is an ad hoc risk measure that was introduced by Estrada (2009) in an effort to explain the cross sectional returns across equities, countries and industries. Thus, it has not yet been applied to portfolio optimization problems and there is no formal framework to date. It is not as straightforward to compute a co-GLS matrix in order to calculate the portfolio GLS as we saw with the mean-variance and -semivariance frameworks in the previous sections. Therefore, we choose to address this heuristically by applying linear programming tools in order to determine the minimum GLS portfolio.

$$\min GLS_P = \frac{1}{M}\sum_{t=1}^{M} G_{P,t} - \frac{1}{N}\sum_{t=1}^{N} L_{P,t}$$
$$s.t. \quad \sum \omega = 1$$

where G_P and L_P depend on the weight vector, ω.

Thus, the results of the minimum semivariance portfolio can be seen in table 5-3:

Table 5-3: Minimum GLS Portfolio Results

	Novo Nordisk	RBS	AXA	Deutshe Bank	Dow Jones	Nikkei 225	Bonds
MR	0.086%	0.061%	0.059%	0.044%	0.009%	0.039%	0.023%
Weights	2.824%	0.308%	1.480%	3.802%	-1.986%	-5.242%	98.814%

	MR	GLS	RAR
Min GLS	0.000263	0.00261	0.100705

Like in the other frameworks, the minimum GLS portfolio suggests that almost the entire investment should be allocated to bonds. Again, it suggests going short on the two indices and placing the lowest weight on RBS. As we saw in chapter 4, GLS is almost perfectly correlated with the standard deviation and therefore we expect similar results, which seems to be the case. This implies that the GLS can in fact be applied as a tool in asset allocation decisions.

We compute the mean-GLS frontier in the exact same manner as in the previous sections.

Figure 5-3: MGLS Frontier and Minimum GLS Portfolio

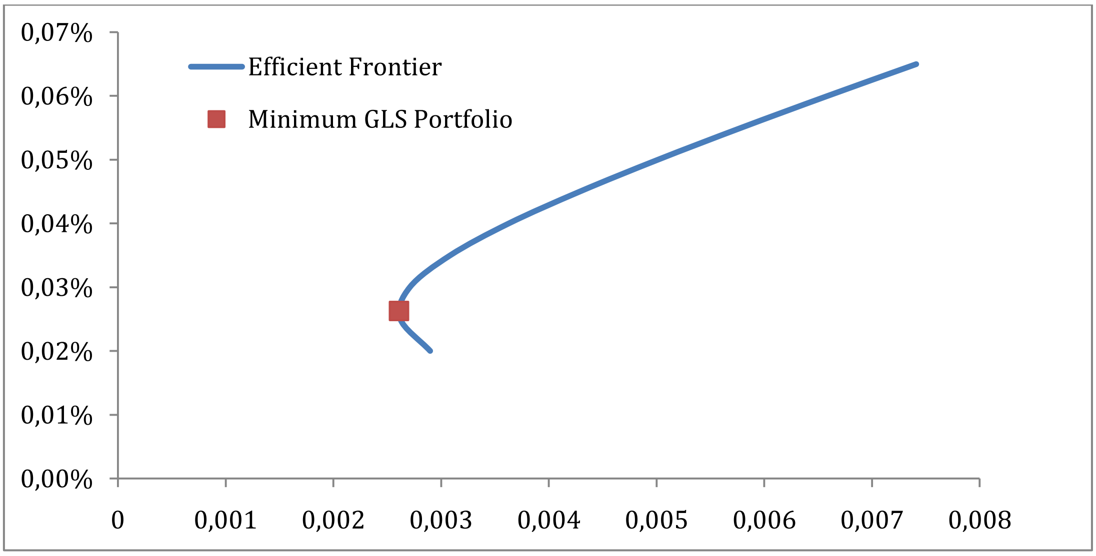


Figure 5-3 shows the mean-GLS frontier for our seven securities. As we increase the return constrain, the weights behave in the same manner as in the other cases. For details and calculations, see Appendix CD 3.3.

5.1.4 The Mean-CVaR Framework

In the following we will calculate and illustrate the minimum CVaR portfolio and efficient frontier. As mentioned before, we are using 4 stocks, 2 indices and one global government bond. Since the bond is the least risky asset, it is expected that the minimum CVaR portfolio will consist mostly of bonds.

In section 2.2.3 we examined CVaR and the theory on how to calculate the minimum CVaR portfolio. As mentioned in the theory, the minimum CVaR portfolio can be obtained using linear programming:

$$\min \quad CVaR = VaR + \frac{\frac{1}{S}\sum_{s=1}^{S} \max\left(VaR - \omega' R_s, 0\right)}{1-\beta}$$

Minimizing CVaR also provides us with the corresponding VaR. It should be noted that VaR calculated using this procedure is *not* the minimum VaR, though in most cases it is not far from it.

Table 5-4: Minimum CVaR Portfolio Results

	Novo Nordisk	RBS	AXA	Deutshe Bank	Dow Jones	Nikkei 225	Bonds
MR	0.086%	0.061%	0.059%	0.044%	0.009%	0.039%	0.023%
Weights	2.383%	0.464%	2.556%	3.617%	-2.093%	-6.787%	99.861%

	MR	VaR	CVaR	RAR
Min CDaR	0.026%	0.527%	0.742%	3.527%

The solution is given in table 5-4 where we see that the bonds again have received the highest weight in the minimum CVaR portfolio, namely 99.9% (see Appendix CD 3.4 for detailed calculations). Again, we take a short position in Dow Jones and Nikkei 225 with -2.09% and -6.79% respectively. The minimum CVaR portfolio yields a CVaR of 0.742% and a corresponding VaR of 0.527%. CVaR is always larger than VaR since it also considers losses exceeding VaR. The gap between the two depends on how far we move into the tail of the distribution and since we are looking at a 95% confidence level, this gap is relatively small. We could also have performed this exercise with a different confidence level but since we are using daily observations since 1993, there is enough data to set a 95% confidence level.

Figure 5-4: MCVaR Frontier and Minimum CVaR Portfolio

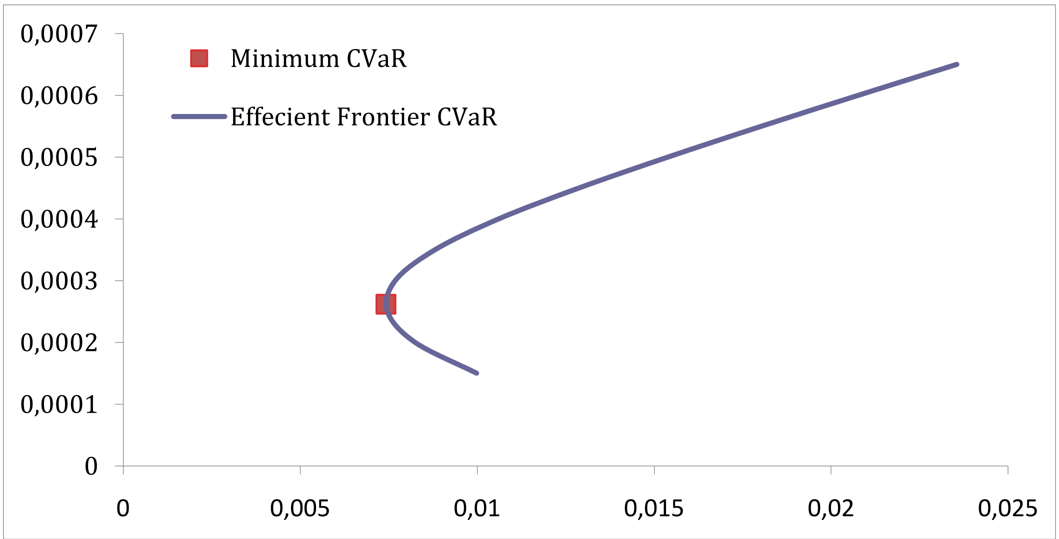


Figure 5-4 illustrates the mean-CVaR frontier including the minimum CVaR portfolio. Again, note that only the portfolios above the minimum CVaR are the efficient portfolios.
Again, the weights shift from bonds to stocks, as we require higher return. It is specifically Novo Nordisk that becomes a favorable stock, which also was the highest ranked security when looking at the return/CVaR ratio (see section 3.3.1).

Linear programming tends to produce portfolios that are concentrated in a few holdings, which we also observe in our example. However, the computational ease and theoretical superiority of CVaR as a risk measure still makes it an interesting and important alternative to the traditional mean-variance approach.

5.1.5 The Mean-Drawdown Framework

In the following, we will calculate the minimum risk portfolio and efficient frontier for the three different drawdowns: conditional, average and maximum drawdown. Since the three drawdown measures stem from the same theoretical framework, the approach will be very similar.
The conditional drawdown-at-risk (CDaR) is very similar to CVaR while the average and maximum drawdowns are special cases of CDaR.

$$CDaR = \zeta + \frac{\frac{1}{S}\sum_{s=1}^{S}\max(D(x,t)-\zeta,0)}{1-\beta}$$

For CDaR we define β as a 95% confidence level like we did with CVaR. By setting β to zero, we get the average drawdown, and β = 1 generates the maximum drawdown. This way we can easily calculate the minimum risk portfolio for all three risk measures by minimizing the above equation for β = 0.95, 1 and 0. ζ can be considered as VaR for the portfolio's *drawdown* function that will be equal to zero for the average drawdown, and equal to the maximum drawdown when β is set to 1. This is also in line with the previous explanation of the gap between VaR and CVaR (ζ and CDaR), which stated that the gap is smaller the more we move into the tail of the distribution (and the gap is zero at the end of the tail distribution).

Table 5-5 show the security weights for the minimum CDaR, AvgDD and MaxDD portfolios respectively. For all three risk measures, we see that bonds again are the most dominating security in the portfolio. However, the weights in bonds are less extreme than in the other frameworks, which results in a higher mean return for the minimum risk portfolio. The most notable difference among the three portfolios is that the maximum drawdown has a much higher weight in bonds than the other two. This makes sense since bonds have less extreme fluctuations than equities. For all drawdowns, Deutsche Bank is the dominant stock although it has a low ranking for all three risk measures.

Overall, CDaR and AvgDD yield the most similar weights while MaxDD shorts additional securities. Even though the three risk measures are generated from the same idea and can be calculated from the same equation by changing β, it is important to remember that they still represent very different risk measures.

Table 5-5: Minimum Drawdown Portfolio Results

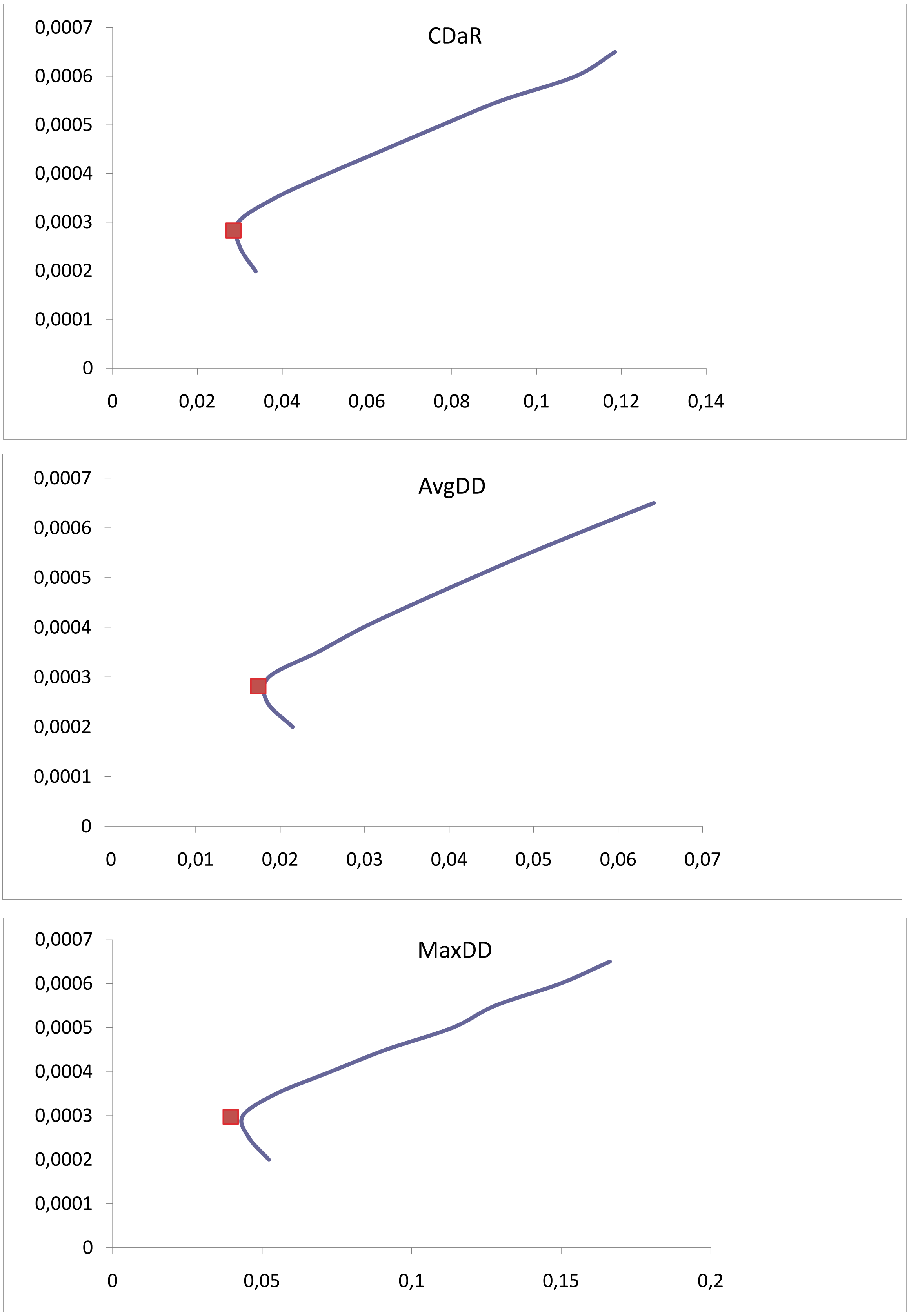
	Novo Nordisk	RBS	AXA	Deutshe Bank	Dow Jones	Nikkei 225	Bonds
Mean Vector	0.086%	0.061%	0.059%	0.044%	0.009%	0.039%	0.023%
Min CDaR	7.309%	1.913%	-9.067%	9.744%	5.968%	10.349%	73.783%
Min MaxDD	7.036%	-0.816%	-0.766%	7.985%	-3.430%	2.211%	87.780%
Min AvgDD	8.278%	2.818%	-10.302%	10.151%	8.478%	7.997%	72.579%

	MR	ζ	CDaR	RAR
Min CDaR	0.028%	2.502%	2.850%	0.993%
Min MaxDD	0.030%	3.949%	3.949%	0.753%
Min AvgDD	0.028%	0.000%	1.741%	1.618%

The mean returns are at around the same level for CDaR and AvgDD, namely 0.028%, and somewhat higher for MaxDD at approximately 0.03%.

Figure 5-5 shows the frontiers for all three risk measures (for calculations on CDaR, MaxDD and AvgDD, see Appendix CD 3.5, Appendix CD 3.6, and Appendix CD 3.7 respectively).

Figure 5-5: Drawdown Frontiers and Minimum Drawdown Portfolios



5.2 Rebalancing Portfolios

In this section, we will try to determine which of the seven risk measures performs best empirically. We have chosen to continue using the same securities and data as in the previous section in order to keep it simple for the reader.

In order to compare the performance of the different risk measures, we cannot simply use a universal performance ratio as the different performance ratios are biased towards specific risk measures. The main challenge is that the risk variables measure risk differently, which makes it difficult to compare them using a single performance measure. Most performance measures are developed based on a specific risk measure or risk profile, which obviously makes the performance measures favorable to that specific risk measure. For example, the Sharpe ratio favors the standard deviation while the Sortino ratio favors the semideviation. Moreover, risk adjusted returns are not comparable in size either. An example is the maximum drawdown measure, which has a much greater value than, say, the standard or semideviation. In the preceding analysis, we had a risk adjusted return of 9.53% based on the minimum semivariance portfolio while the minimum MaxDD portfolio had 0.75% in risk adjusted return. This is due to the size of the maximum drawdown, which is not comparable to that of semideviation, and thus the denominator is larger in the performance measure. The above results imply that the minimum semivariance portfolio is substantially superior, which may not necessarily be the case.

Since we have not been able to find an unbiased performance measure, we have chosen to investigate the empirical performance in a different way. Our idea is to look at the seven risk measures as different investment strategies and then use these strategies to rebalance a portfolio every year in the period 1993-2010. By considering seven highly risk averse investors (representing each risk measure) with an initial wealth of €100 each and who wish to minimize risk, we will be able to compare the investors' terminal wealth and thereby determine the performance of the seven risk measures. We consider this method completely unbiased since it does not consider any specific risk behavior (other than minimizing risk, which is done for all seven risk measures) or other factors that could favor a specific risk measure.

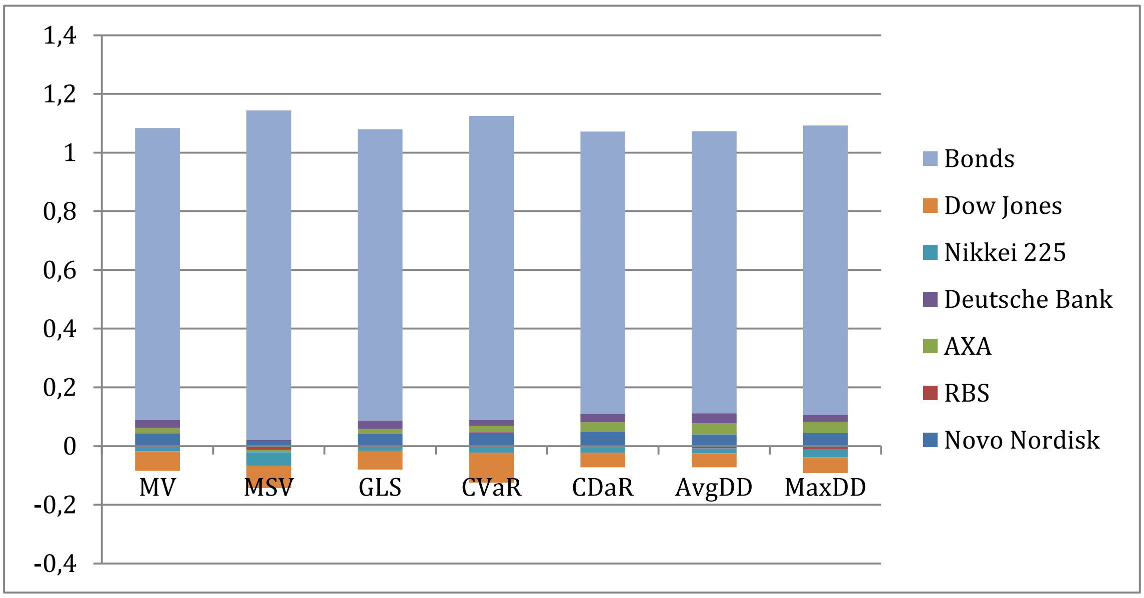
On the other hand, the conclusion from this method relies heavily on the underlying data. This means, that the result could change if we had used a different set of securities or a different time period, which is also an issue when testing the traditional mean-variance framework.

However, taking these drawbacks into account, we still believe that this is the best method to test and compare the performance across risk measures.

In the previous section, we demonstrated how the minimum risk portfolio is calculated for the seven risk measures. We use the exact same method to find the minimum risk portfolio when rebalancing, but instead of using the data for the entire period, we use the daily returns to rebalance the portfolio on a yearly basis. Having daily returns from 1993-2010 will then result in 17 rebalancings for each risk measure. For detailed calculations, see Appendix CD 3.8.

Figure 5-6 shows the average allocation over the 17 years for all seven risk measures.

Figure 5-6: Average Allocation Over the Rebalancing Period



Since we are using the minimum risk portfolio, all risk frameworks have about 90% represented in bonds. However, there seems to be differences in the allocation of stocks. Semivariance and conditional value-at-risk are the two risk strategies that have the highest short positions, whereas the three drawdowns have the lowest.

Table 5-6 shows the results of this exercise. The terminal value (the investors capital in 2010, when invested €100 in 1993), the average return and the average risk are reported.

Table 5-6: Rebalancing Results

	σ	Σ	GLS	CVaR	CDaR	AvgDD	MaxDD
Terminal value	258.31	232.81	251.78	281.53	292.24	271.34	276.80
Average return	0.0225%	0.0201%	0.0219%	0.0247%	0.0254%	0.0236%	0.0241%
Average risk	0.3183%	0.2391%	0.2376%	0.6021%	1.4385%	0.6772%	1.6224%
RAR	7.07%	8.43%	9.20%	4.09%	1.77%	3.49%	1.49%

Looking at the terminal value, we end up with the following performance ranking of the investment strategies:

1. Conditional Drawdown-at-risk
2. Conditional Value-at-Risk
3. Maximum Drawdown
4. Average Drawdown
5. Variance
6. Gain-Loss-Spread
7. Semivariance

The biggest surprise with this ranking is semivariance, which we expected to perform better since it had a decent correlation and high significance with returns as demonstrated in chapter 4. Furthermore, we expected semivariance to perform better than variance because of the non-normality issue, which is a clear strength for semivariance and a weakness for variance. Conditional drawdown-at-risk came out as the strongest strategy in this exercise, which is not surprising since it also was the best to explain returns. The same argument goes for Conditional Value-at-Risk and Maximum drawdown, since they both explained cross sectional returns better than variance. Although CVaR and CDaR are very similar risk measures, they still have their differences. CDaR is a forward looking risk measure in the sense that drawdowns are defined as the difference between the highest return point in the past and the current return value. In other words, it does not consider a significant loss in the past unless it followed a higher return. To exemplify, consider a hypothetical security with a positive linear return development. In this case, there will be no drawdowns and the CDaR strategy would consider this security risk free. On the other hand, the CVaR strategy would minimize the 5% worst outcomes, thus considering the security somewhat risky despite its positive development. Now, consider a second hypothetical security with a similar positive linear development, but with higher returns. CDaR would consider both securities equally risky (risk free), whereas CVaR would favor the second security. This is a theoretical disadvantage of the drawdown measures in general. The fact that CDaR does not always distinguish between different types of securities could indicate that the CDaR strategy, in our case, is exposed to higher risk (despite minimizing CDaR) and thereby obtain a higher return than CVaR.

The gain-loss-spread which calculates the difference between average gains and average losses, is the closest one to variance. This outcome was also expected since GLS had a very high correlation with variance and performed very similar to variance in our previous analysis.

Figure 5-7: Index Return

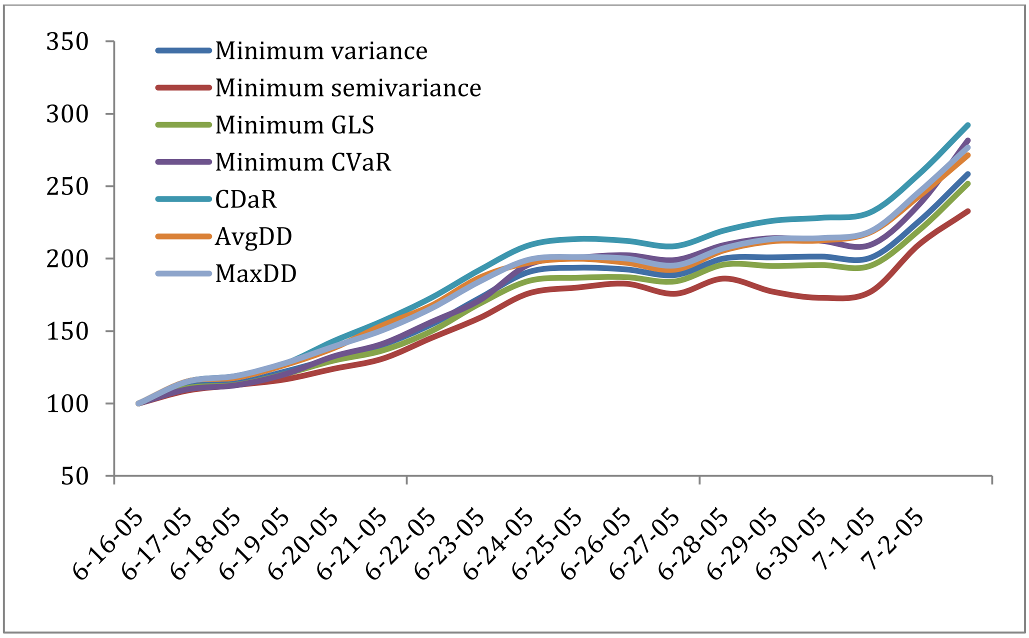


Figure 5-7 shows the development in accumulated returns for the different strategies (risk measures). It is clear that the development for the different risk strategies is very similar and since the figure shows the accumulated returns we see an increase in the spread. There is a somewhat steady increase in returns between 1995-2001, where after the returns stagnate until 2008 with some drops in 2003-04 and 2007-08. The returns then again increase in the last couple of years.

We expected the accumulated returns to be somewhat similar among the seven risk measures, but since it is clear that a spread does obtain over time, it could indicate that the seven strategies differ across bear and bull markets or market movements in general.

Figure 5-8: Average Daily Return

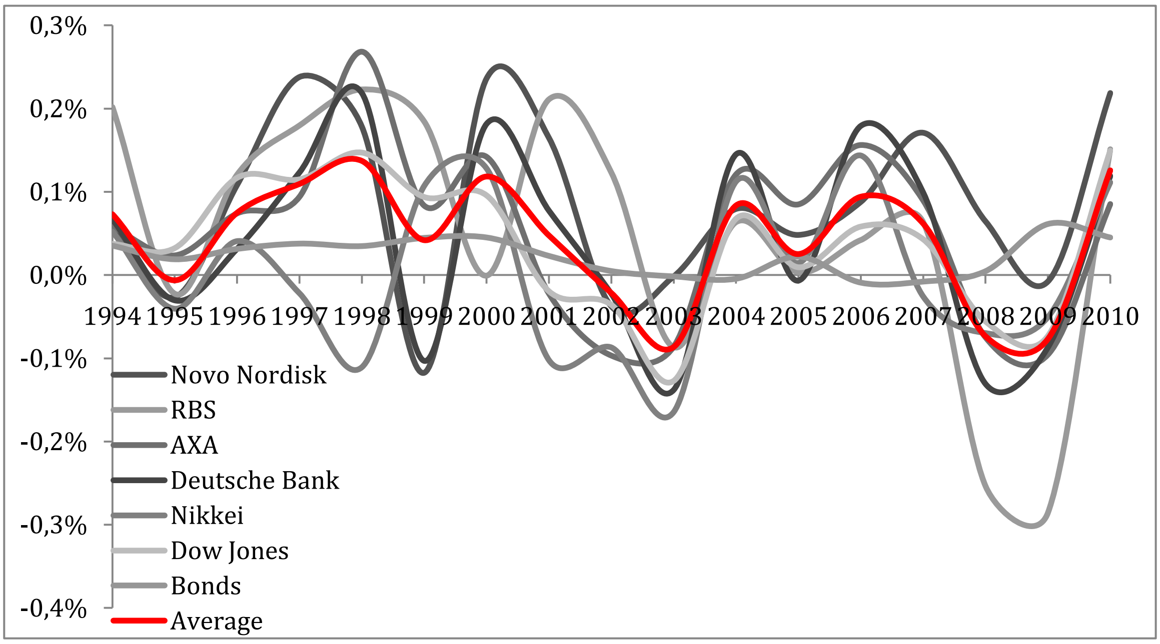


Figure 5-8 shows the general movement of average daily returns over the period 1993-2010 for all stocks, indices and bonds used in the portfolio allocation, where the red curve represent an average of all securities.

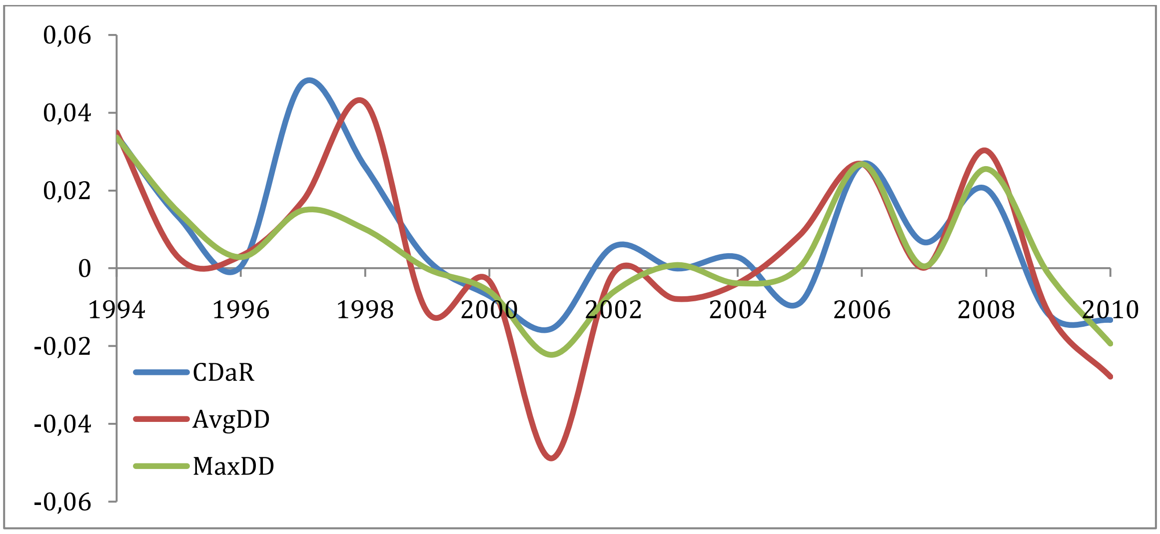
In order to investigate the effect of market movements, we look at the average daily returns for the six risk measures relative to variance ($R_i - R_{Variance}$). This will also allow us to make a better comparison between the risk measures over time. Figure 5-9 shows the average daily returns for the six risk measures relative to variance (for convenience, we have divided the graph into two).

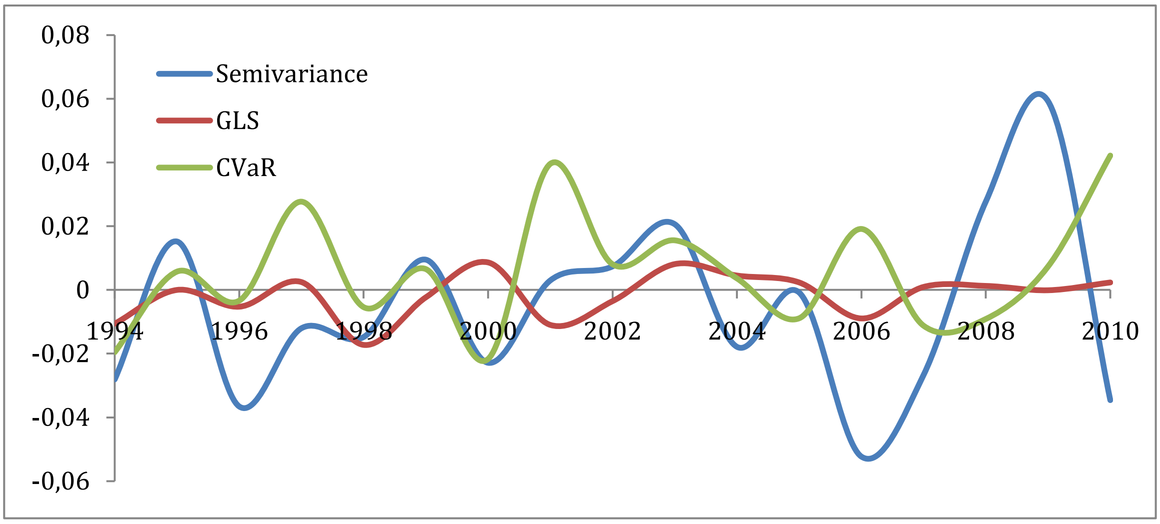
As figure 5-9 shows, the three drawdown measures have similar developments throughout the period. CDaR seems to outperform variance throughout most of the period. Moreover, it performs relatively well compared to MaxDD and AvgDD in the periods where variance outperforms the drawdown measures (e.g. 2001). Looking at the development of AvgDD, it seems to be the most volatile compared to variance – especially when variance outperforms the drawdowns – whereas MaxDD seems to follow variance the most. In general, there does not seem to be any pattern in terms of the drawdowns' performance relative to variance and the underlying market movements. For example, the drawdowns outperform variance in 2006 when the market does well but also in 2008 when the market is down. It is noteworthy that during market downturns, the drawdown measures perform either similarly or better than variance (e.g. 2008). For example, the CDaR strategy *never* underperforms the variance strategy during market downturns. This is interesting because it suggests that drawdown measures capture the downside movements to a greater extent than the variance, and adjust the optimal weights accordingly.

We previously discovered that there is a high correlation between variance and GLS, which is why it is not surprising that GLS performs very similar to variance. However, we also discovered high correlation between variance and semivariance, but semivariance has a fairly different movement in returns. Semivariance performs significantly better when the market is down. Throughout our time period we have two incidents with negative average daily return, 2002-03 and 2008-09. In both cases, semivariance manages to outperform all risk measures. This is line with our prior expectations that semivariance performs well during bear markets, since it only considers the lower partial moment of return distributions. On the other hand, semivariance performs very poorly when the market is up and since the general movement in the market between 1993-2010 has been positive, semivariance fails to outperform the other risk strategies.

CVaR generally outperforms variance when the market does well and only underperforms in seven out of 17 years. The overall development in CVaR is somewhat similar to variance, but it is important to remember that CVaR in the end yields the second highest terminal value.

Figure 5-9: Average Daily Return Relative To Variance (Two Figures)





Throughout this analysis we have found that CDaR and CVaR are the two best performing risk measures and, surprisingly, that semivariance is the worst. It was also clear that the performance of the different risk measures depends on the market movement. During bear markets, the downside risk measures generally outperform variance with semivariance at the top. In our analysis we used the time period 1993-2010, which mostly consist of bull market. This mean that another time period could change the ranking of the seven risk measures.

5.3 Rebalancing Portfolios Without Bonds

It is obvious that minimizing risk will result in a portfolio highly concentrated in bonds, regardless of risk measure. In the previous section we saw that the minimum risk portfolios for all risk measures were above 80% in bonds in most cases. This raises the question of whether the conclusion would change if we exclude bonds. In the following section we will look at the effect and changes of yearly rebalancing using only equities. For detailed calculations, see Appendix CD 3.9).

Figure 5-10: Allocation Without Bonds

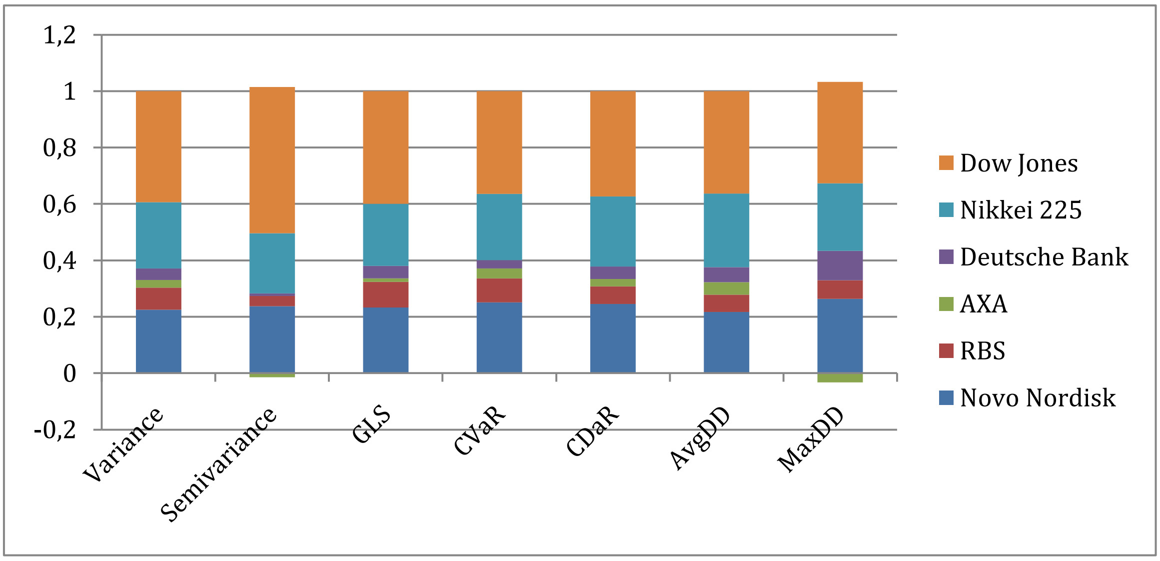


Figure 5-10 shows the resulting allocation across our equities. Excluding bonds results in a higher risk portfolio, where we in our case, see high concentration in Dow Jones, Nikkei 224 and Novo Nordisk for all risk measures. The only two risk strategies taking short position are semivariance and maximum drawdown, although they are not very significant.

Table 5-7: Rebalancing Results Without Bonds

	σ	Σ	GLS	CVaR	CDaR	AvgDD	MaxDD
Terminal value	497.373625	488.989399	527.030129	566.852115	501.753946	423.141058	459.866243
Average return	0.045%	0.045%	0.046%	0.048%	0.046%	0.044%	0.044%
Average risk	0.916%	0.743%	0.667%	1.783%	3.947%	1.892%	4.720%
RAR	4.904%	6.036%	6.888%	2.685%	1.162%	2.312%	0.943%

Naturally, excluding bonds will result in higher risk, return and volatility over the rebalancing period. Table 5-7 shows the terminal value, average return and risk. It is clear that the terminal value is much higher for all risk measures compared to the previous rebalancing.
It is interesting to look at the ranking of terminal value in order to see the change in performance of the different risk measures when only considering equities.

1. Conditional Value-at-Risk
2. Gain Loss Spread
3. Conditional Drawdown-at-Risk
4. Variance
5. Semivariance

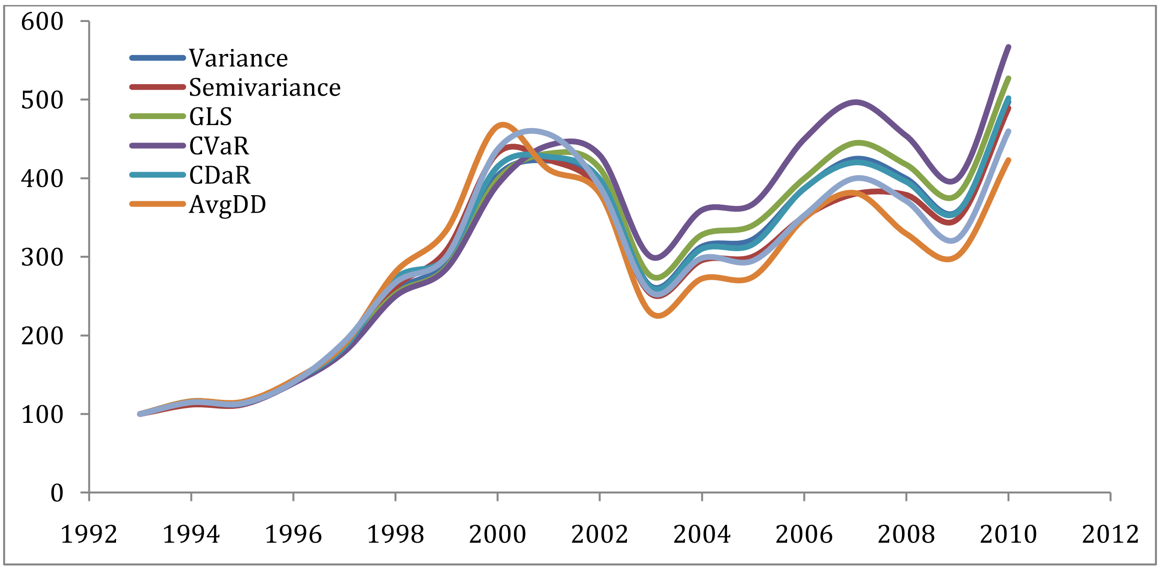
6. Maximum Drawdown
7. Average Drawdown

The most notable change is GLS, which is now the second best performing risk measure. Furthermore, the average and maximum drawdown strategies are the two worst performing, which is curious because of their good performance in the previous analysis. While these changes may seem somewhat random, we see a pattern in remaining strategies. CVaR and CDaR are still in the top three with CVaR being the best performing risk strategy.

Since GLS is calculated as the spread between expected gains and expected losses, minimizing GLS could in theory mean minimizing expected gains, expected losses or both. In this case, the minimum GLS portfolio exhibits high return, which could indicate that the spread has been minimized mainly on the domain of losses. However when we included bonds, GLS turned out to be the second worst performing risk strategy. This makes it hard to determine whether GLS is minimizing expected gains, losses or both under different conditions.

Figure 5-11 shows the movement in accumulated return for all risk measures. It is clear that the movement in return is much more volatile than with bonds. However, the overall pattern in the movement is the same for all risk measures, although the gap between accumulated return is increasing over time.

Figure 5-11: Index Return Without Bonds

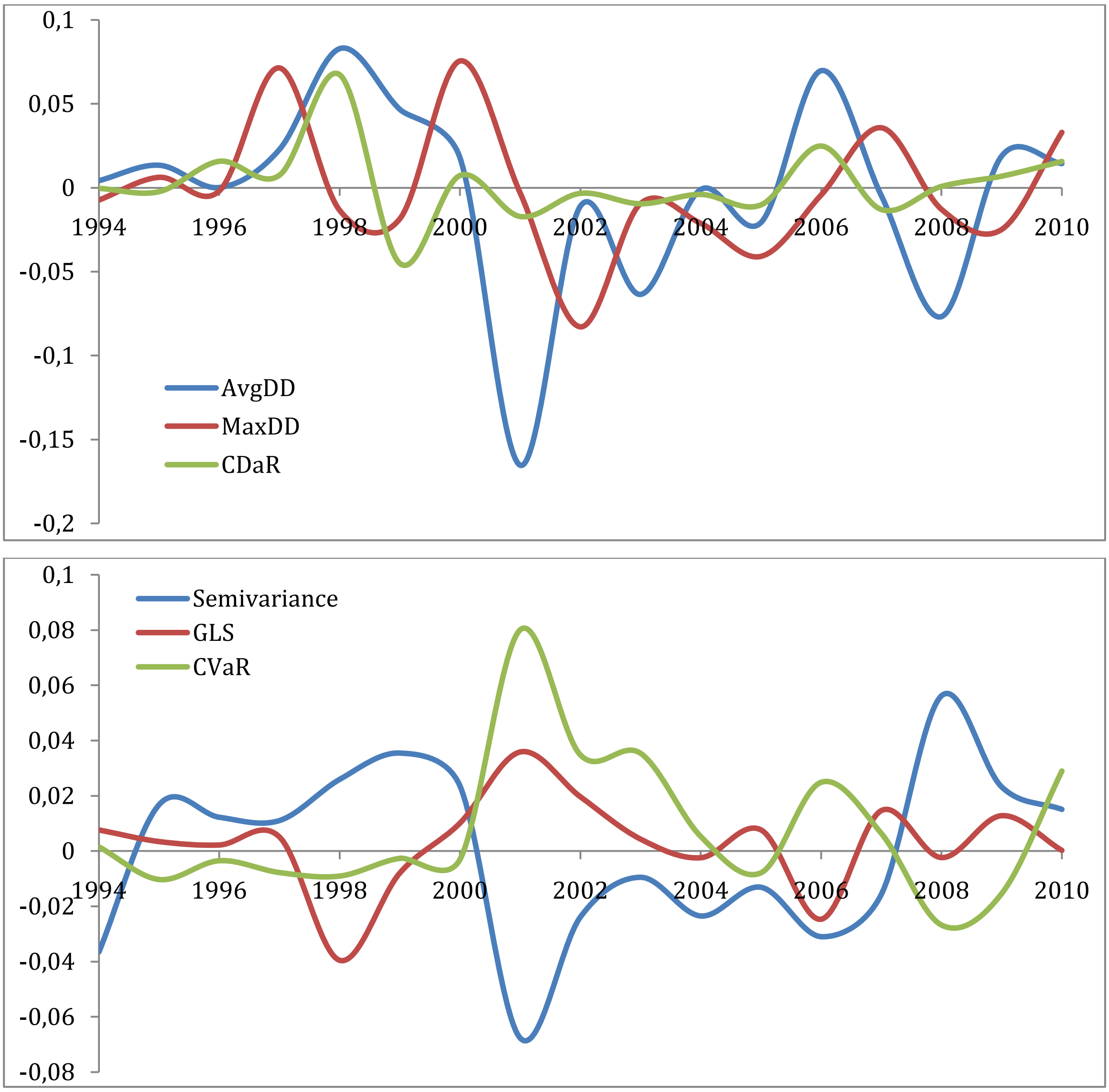


Since the movement is very similar for all seven risk measures, it is really hard to distinguish between the differences. In order to do so, we look at the development of the downside risk measures compared to variance, as we did the previous section. This relationship can be seen in figure 5-12 where we have split the graph into two, in order to make it clearer. Again, the first graph shows all three drawdown risk measures compared to variance whereas the second graph shows the remaining three risk measures. Most of the risk measures perform very similar to their performance with bonds, just with higher volatility. However, there are some deviations that are worth noticing. First of all, CDaR is the drawdown measure that resembles the development of variance the most as opposed to the previous case where it was MaxDD. We have seen that the average drawdown strategy is very sensitive to market movements, which is seen to an even greater extent when excluding bonds. During market downturns, the minimum AvgDD portfolio performs very poorly compared to the other risk strategies. For example we see that in 2001, it performs 17% worse than the minimum variance portfolio, which very well could be the reason for the lowest terminal value.

CVaR seems to outperform the remaining risk strategies when the market becomes more volatile. Compared to variance, we see clear superiority, as its underperformance is borderline while its outperformance is substantial.

The surprising development in GLS can be attributed to the higher volatility in the market. As opposed to before, its performance over time is significantly different from that of variance. This is curious because of the high correlation between GLS and standard deviation. GLS measures risk in more or less the same way as standard deviation, only without the assumption of normality. The larger difference in the allocation weights could be rooted in potentially higher skewness or fatter tails of the return distribution.

Figure 5-12: Average Return Relative To Variance, Without Bonds



5.4 Sensitivity Analysis

In the preceding analysis, we have treated the estimated distribution parameters (i.e. mean and risk variable) as the *true* parameters even though these were estimated from historical return data, i.e. using only one possible realization of return series. The different frameworks presented give the right way to invest given that the investor has exactly the correct parameter inputs. In reality, risk-return estimates are highly uncertain and sensitivity to changes in optimization inputs leads to portfolio optimality ambiguity. Even if stationarity (constant mean, non-time-dependent covariances) is assumed, only in very large samples can the point estimates for risk and return inputs equal the *true* distribution parameters. Therefore, portfolio optimizers often construct inefficient portfolios based on inadequate inputs. This uncertainty may result from uncharacteristically high or low recent returns of the underlying securities, which then results in much higher or lower allocations. Thus, small changes in the optimization inputs often lead to very different portfolio weights and accordingly diverging efficient frontiers.

Michaud (1998) introduced Monte Carlo resampling and bootstrapping methods[28] into MV optimization in order to reflect the uncertainty in investment information. The end result was generally more stable, realistic, and investment effective MV optimized portfolios.

Inspired by Michaud's resampling technique, we seek to determine the effects of this uncertainty in the optimizing process when using the different risk measures applied. We will not try to determine a more efficient allocation as the methodology proposes because the underlying assumptions differ from the ones we have made throughout this thesis. Rather we will analyze the size and effect of parameter uncertainty across our minimum risk portfolios in order to determine how sample specific our previous results are. That is, can we trust our results or do they only apply for this specific data set?

In the following, we briefly review Michaud's methodology and illustrate this sensitivity in an example from the MV framework.

5.4.1 A Resampling Approach

Resampled efficiency optimization, introduced by Richard Michaud and Robert Michaud, addresses information uncertainty in risk-return estimates. The procedure applies Monte Carlo simulation methods to produce multiple sets of *statistically equivalent* risk-return estimates based on the original estimates. These estimates are then used to compute multiple efficient frontiers that represent the many possible ways in which assets may perform relative to uncertainty in the inputs.

The five steps below describe the procedure more methodically:

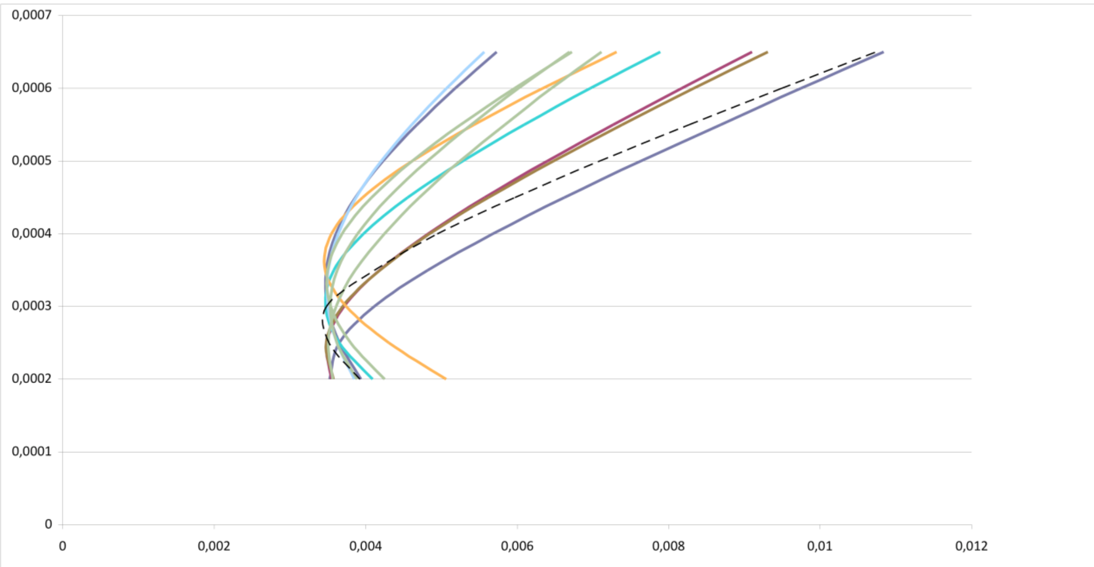
[28] *Bootstrapping* generally refers to the technique of redrawing historical observations with replacement. *Resampling* typically refers to recreating a simulation of the historical data from an assumed probability distribution such as multivariate normality. In practice, resampling is usually convenient since few investment strategies are based solely on risk-return estimates from historical return data. Thus, we will apply the resampling methodology in the analysis.

1. Estimate the covariance matrix and the mean vector of the historical returns.
2. Estimate the mean-variance efficient frontier as we did in section 5.1.1.
3. Now begin the Monte Carlo analysis. Assume a multivariate normal distribution with the mean vector, μ, and the covariance matrix, Σ, estimated in step 1, and draw as many returns as necessary to approximate the historical return distribution. With the generated data sample, compute the simulated mean vector, μ^*, and covariance matrix, Σ^*.
4. Now, Σ and Σ^* are statistically equivalent. Using the inputs derived in step 3, compute a new efficient frontier.
5. Repeat step 3 and 4 a number of times to visualize the sample sensitivity in the form of disperse efficient frontiers.

Michaud takes it further and develops a resampled efficient frontier, which he claims outperforms the sample specific efficient frontier. While this part is irrelevant for our analysis, the interested reader is referred to Michaud, R. O. (1998) *"Efficient Asset Management"*.

Following the five steps above, we simulate 10 statistically equivalent data sets and illustrate the consequence of sample specificity in the mean-variance efficient frontiers below:

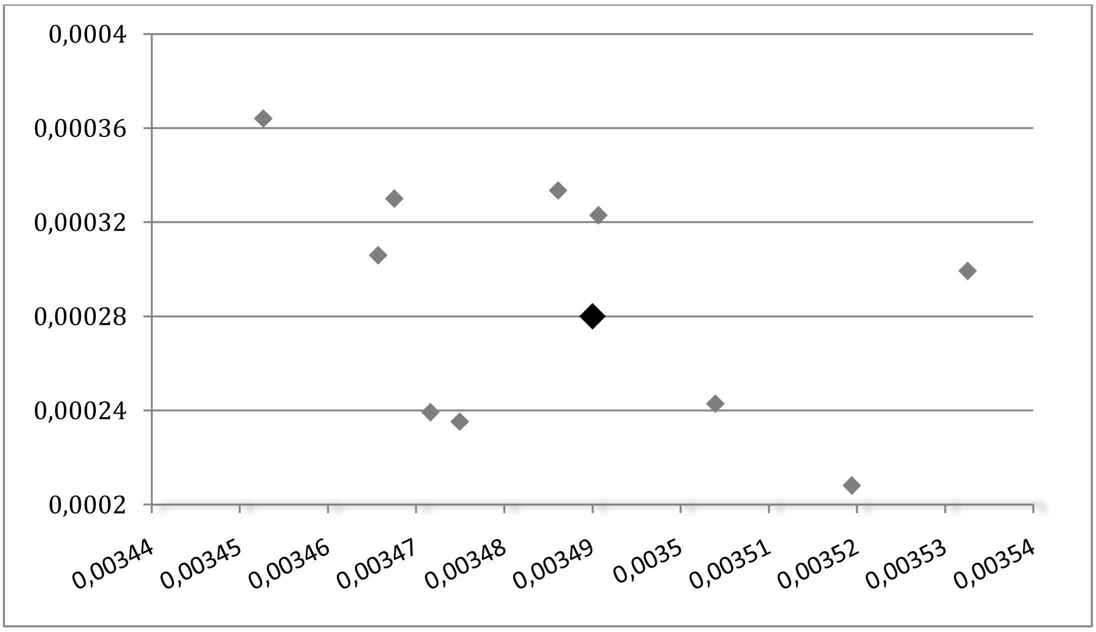
Figure 5-13: Original and Simulated Frontiers



As can be seen, there is dispersion between the simulated frontiers, which are all equally likely. The black dashed curve is the original frontier as of section 5.1.1 and the 10 colored frontiers are based on the 10 simulations from the multivariate normal distribution. The difference between the frontiers illustrates the sample sensitivity of the mean-variance optimization. Holding the upper return limit of the frontiers constant (0.00065), we see that the standard deviation ranges from 0.0056 to 0.0108, equivalent to an increase of approximately 95%. Similarly at a standard deviation of 0.0056, it is equally likely to obtain a return of 0.00065 and one just below 0.0004. According to Michaud, any two portfolios corresponding to the same return (risk) level on the original MV efficient frontier are statistically equivalent. As we can see, these statistically equivalent portfolios have similar risk and return levels around the minimum variance portfolio, but as we venture towards higher returns, such similarities decrease.

Since we have considered a highly risk averse investor throughout the paper, the minimum risk portfolio (in this example, the minimum variance portfolio) is the optimal portfolio for any given efficient frontier. Therefore, we are more interested in visualizing the sample sensitivity of the minimum risk portfolios rather than the entire efficient frontiers.

Figure 5-14: Minimum Variance Portfolio For Original and Simulated Data



It is clear from figure 5-14 that there is also huge dispersion between the simulated minimum variance portfolios in relative terms. The highest and lowest risk adjusted returns are 0.1055 and 0.0591 respectively, equivalent to a difference of approx. 78%. Again, the black square represents the original minimum variance portfolio as of section 5.1.1 while the gray ones are based on the 10 simulated samples.

The above mean-variance example illustrates the importance of sample sensitivity. Initially, the resampling procedure described above was aimed to minimize the impact of extreme historical observations by simulating statistically equivalent, though more stable (i.e. fewer outliers) return data in order to enhance out-of-sample performance of optimal portfolios. In other words, the resampled return data fluctuated less than the historical data and was simply based on the given mean vector and covariance matrix, i.e. there is not the same correlation effect that causes the uncharacteristically high or low returns.

In our analysis, however, we want to preserve these extreme fluctuations in order to analyze the dynamics between the risk measures and illustrate the impact of their respective sensitivity to the underlying data. Therefore, we wish to simulate return data on a yearly basis rather than over the entire period as in above example. This way, the simulations will capture the correlation effects in the bull and bear markets respectively.

5.4.2 Analysis of Sample Sensitivity

In this section, we seek to clarify where sample sensitivity has its greatest impact. We saw in section 5.2 that the conditional drawdown-at-risk and conditional value-at-risk had the best performance when including bonds. Thus, it is interesting to see how much value these estimates have out of sample. While we will not test the out-of-sample performance of the different risk variables, we wish to illustrate where sample sensitivity has its deepest impact and thus which of the estimated risk variables are the most reliable for decision making. The analysis is based on yearly simulations following a multivariate normal distribution, i.e. 17 x 10 simulations of daily returns[29]. Even though we have seen that the asset returns do not follow a normal distribution, it is a fair approximation to use the multivariate normal distribution because we are not using it to compare performance[30]. Besides, dividing the simulations into years means that we capture the extreme fluctuations that are otherwise disregarded when resampling over the whole period. As in section 5.2, we will rebalance the minimum risk portfolios on a yearly basis in order to compare their sample sensitivity.

[29] All simulations are done in the statistics program, R. See Appendix CD 3.10.

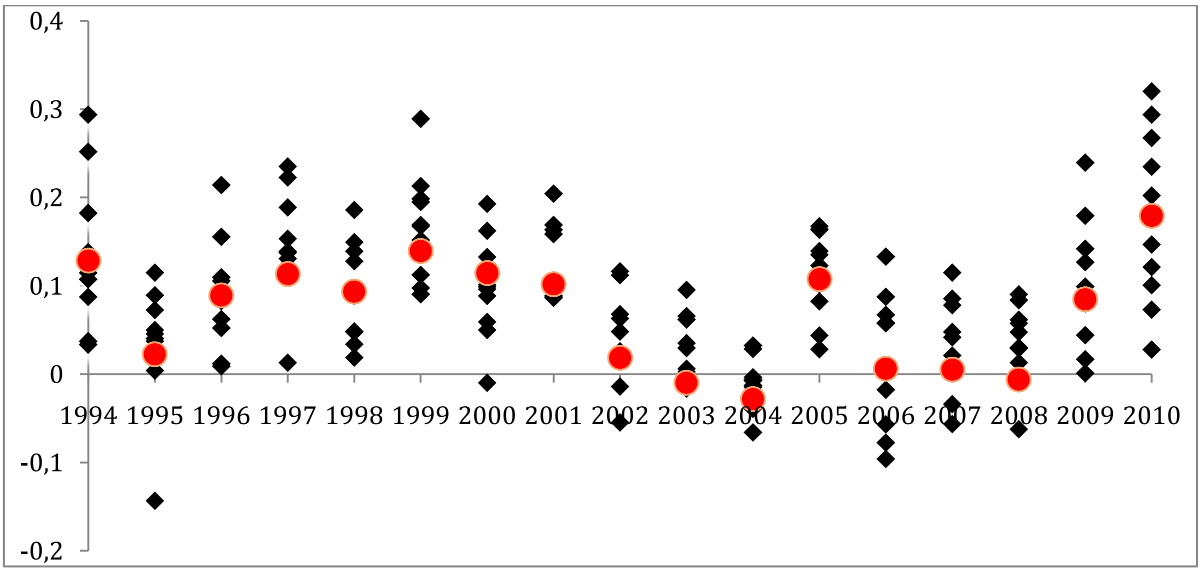
[30] It would not make sense to compare the performance of variance with that of downside risk based on normally distributed data

As figure 5-14 shows, the simulated minimum variance portfolios diverge in terms of returns as well as risk. In order to compare the sample sensitivity across different risk measures, we use the risk-adjusted rate of return. This way we get a fair indication of how sensitive the different optimization techniques are to input parameters (i.e. respective risk measures), and we can compare the effect directly. However, the differences in risk-adjusted returns do not provide any information on how the results could change in the rebalancing part. In order to detect the effect there, we need to simply calculate the terminal value for the 10 simulations.

5.4.2.1 Sample Sensitivity – Variance

In order to visualize the sample sensitivity of the seven risk strategies, we calculate the minimum risk portfolio for the 10 different simulations and compare the risk adjusted returns. Figure 5-15 shows the risk adjusted returns of both the original and the 10 simulated variance strategies for the past 17 years. It is clear that the sample sensitivity is not constant over time, but is affected by the trend of the underlying data. The red dots represent the results from section 5.2 while the black dots represent the simulated results.

Figure 5-15: Sample Sensitivity of The Minimum Variance Portfolios



There seems to be a positive correlation between the size of the spread in values and the original values. In periods with high risk adjusted return we see an above average spread whereas periods with low risk adjusted return exhibit a lower spread.

The sample sensitivity of the remaining six risk strategies can be found in Appendix B.

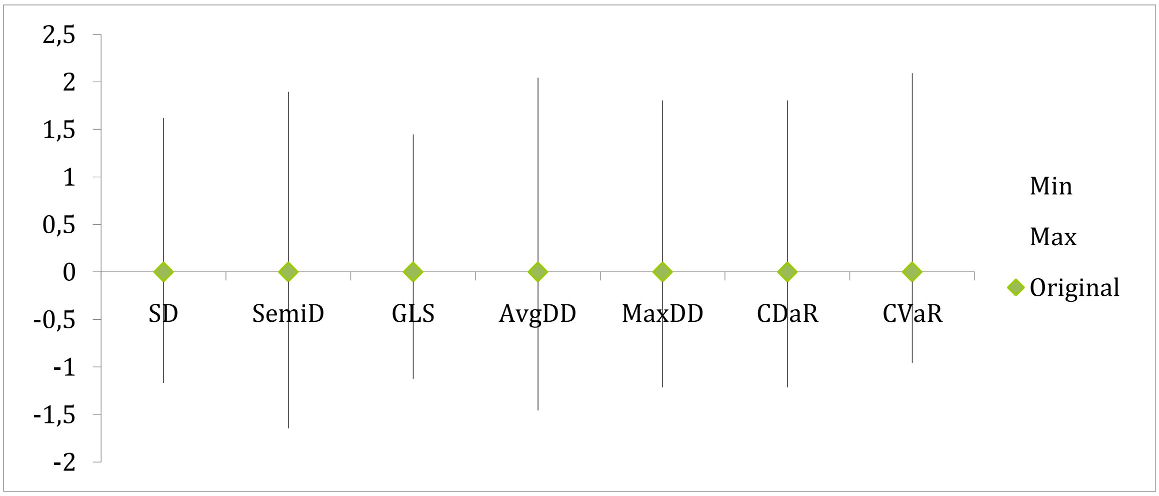
5.4.3 Comparing Sample Sensitivity

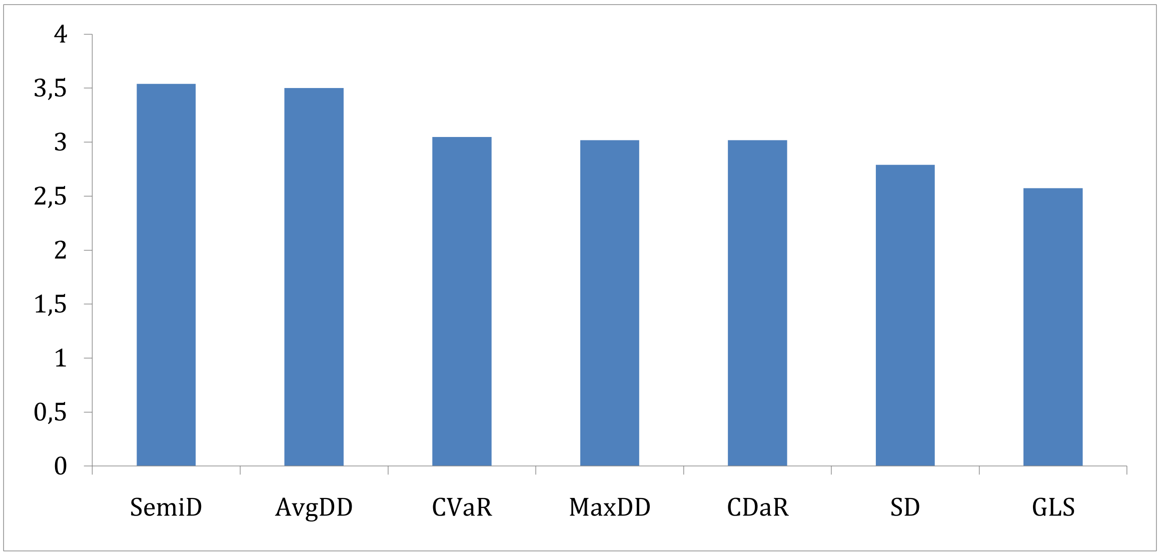
Appendix B shows that there is a difference in sample sensitivity depending on the risk measure applied. In order to make a fair comparison of this sensitivity across the risk strategies, we look at the difference between the percentage deviation of the maximum value from the original value and that of the minimum value from the original value.

$$\frac{RAR_{max} - RAR_{org}}{RAR_{org}} - \frac{RAR_{min} - RAR_{org}}{RAR_{org}}$$

Figure 5-16 shows the results of these calculations where we have normalized the values so that the horizontal axis represents the original value and the lines above and below the horizontal axis represent the sample sensitivity. It is notable that the sample sensitivity leans towards higher risk adjusted returns for all risk strategies with the clearest tendency in CVaR. This suggests that CVaR might potentially achieve an even higher out-of-sample performance than it is the case in our analysis.

Figure 5-16: Sample Sensitivity Across Strategies





The second diagram in figure 5-16 illustrates the ranking of the seven risk strategies by their total sample sensitivity. GLS seems to be the least sample specific strategy followed by the standard deviation. This is not surprising since both risk measures are based on all the historical observations. However, we did expect the maximum drawdown to exhibit a higher spread since it only requires two values of the historical data. These values are the most extreme values (i.e. highest and subsequently lowest return), which we expected to be less extreme in the simulated data. Nonetheless, this effect has been somewhat mitigated by simulating on a yearly basis rather than over the whole period.

Semivariance has the highest spread, which questions the reliability of our previous results, namely that it was ranked as the worst performing risk strategy in terms of terminal value. This is a clear disadvantage for its out-of-sample performance because the optimal weights are more sensitive towards the input parameters.

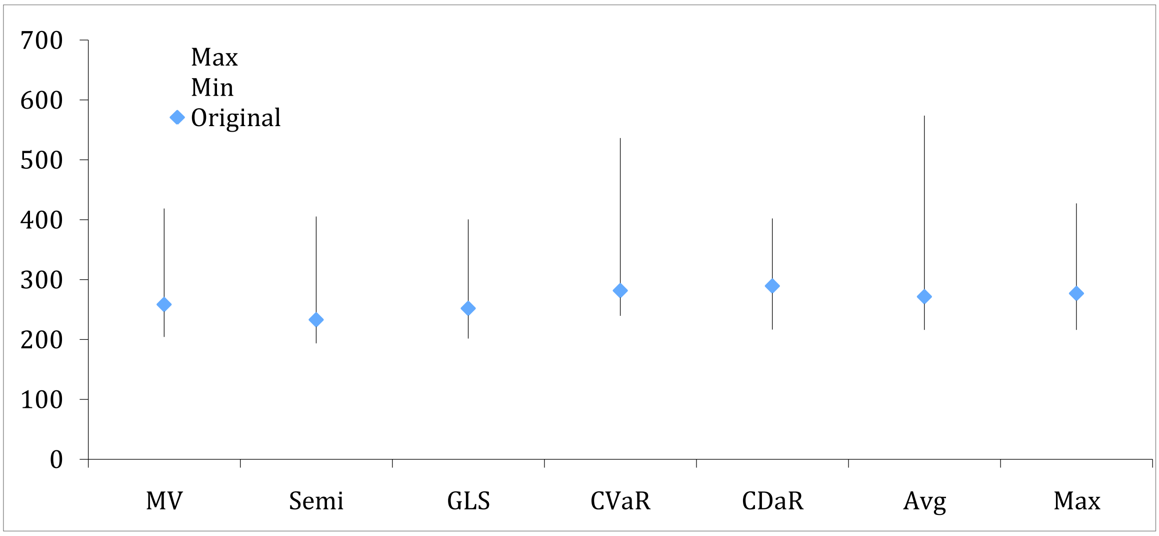
It is important to remember that we have been looking at the risk-adjusted return, which means that it is not possible to compare the *real effect* of sample sensitivity across the different risk strategies. In order to do so, we calculate the terminal value of the minimum risk portfolios based on the 10 simulations as we did in section 5.2. The results can be seen in figure 5-17, where the maximum, minimum and original terminal values are illustrated. The results indicate that the effect of sample sensitivity across all the risk strategies is great enough to potentially change the rankings based on the original terminal values. Furthermore, most of the original values are much closer to the minimum scenario with the exception of

CDaR and to some extent MaxDD. Semivariance has the lowest minimum terminal value, which is in accordance with its low ranking in section 5.2. On the other hand, AvgDD has the highest maximum terminal value, which suggests that the AvgDD strategy might perform better out-of-sample than suggested in section 5.2. However, there is high uncertainty regarding AvgDD.

We previously saw that semivariance had the highest spread in risk adjusted return closely followed by AvgDD. Its effect on the terminal value, on the other hand, is substantially larger for the minimum AvgDD portfolio than the minimum semivariance portfolio. More remarkable, the terminal value of the minimum CVaR portfolio seems to be rather sensitive to its inputs, despite the relatively lower spread in risk adjusted return. In fact, it seems to be more sensitive than semivariance, which had the highest spread in figure 5-16.

Finally, the minimum CDaR portfolio seems to be the least sample sensitive as the spread of possible terminal values is the lowest.

Figure 5-17: Sample Sensitivity of Terminal Values



Throughout this section, we have seen that all risk measures (strategies) are significantly sensitive towards the historical data sample and thus parameter inputs. This is not to be ignored as it could have an impact on their performance ranking. However, we have been able to observe a general pattern that favors risk measures as CVaR and CDaR and creates significant uncertainty about average drawdown.

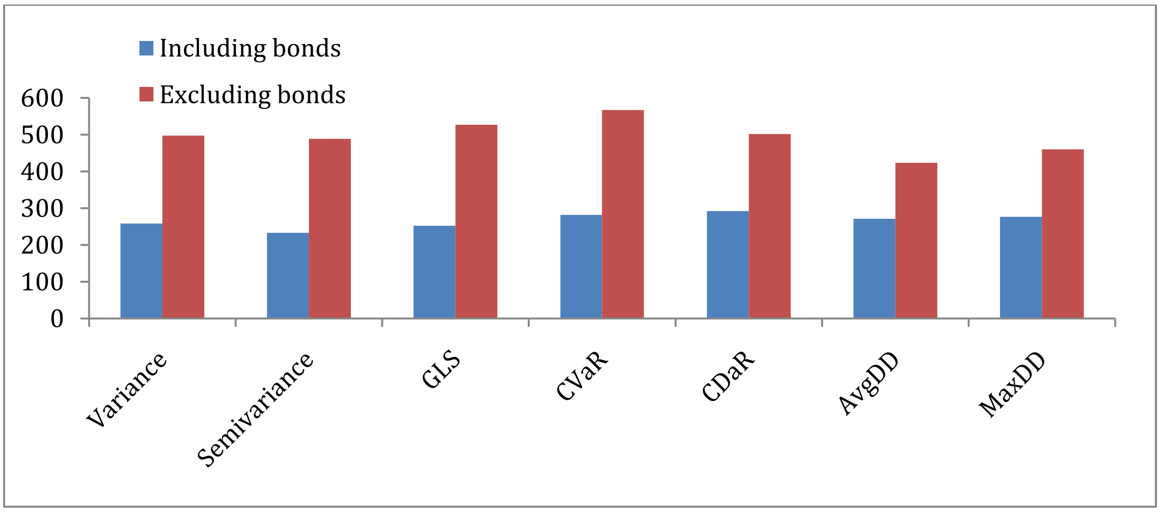
5.5 Summary

After looking at the relationship between the risk and return, we moved focus to the implications of using the different risk measures in portfolio allocation.

We started out by demonstrating how to compute the efficient frontiers and calculate the different minimum risk portfolios. The common factor for these calculations was the use of linear programming, which we used to both calculate the efficient frontiers as well as the minimum risk portfolios.

We were then faced with the challenge of comparing the performance of these minimum risk portfolios. The main challenge is that the risk variables measure risk differently, which makes it difficult to compare them using a single performance measure (e.g. Sharpe ratio, Sortino ratio, Omega etc). Most performance measures are developed based on a specific risk measure or risk behavior, which obviously will make the performance measures favorable to that specific risk measure. In order to solve this problem, we rebalance the portfolios on a yearly basis. The idea behind the rebalancing is to divide the total period into shorter periods and then rebalance according to the minimum risk portfolios for each period. This way, we are able to use the terminal value (end value when investing €100) to compare the performance of the different risk measures. We repeated this exercise with bonds and without bonds in order to see the effect of excluding low risk securities. The rebalancing generated the following results:

Figure 5-18: Terminal Values



We use these results to rank the different risk measures according to performance and find that there are several risk measures that outperform variance. It is especially CDaR and CVaR that perform well with and without bonds. Average drawdown does well with bonds, but is the worst performing risk measure without bonds. This could be explained by its high sensitivity towards bear markets.

After looking at the performance, we wanted to investigate how sample specific our results are. In order to do so, we used a method inspired by Michaud where we calculated the minimum risk portfolios based on 10 simulated datasets. Throughout the sensitivity analysis, we are able to conclude that all risk measures are somewhat sensitive towards the parameter inputs. The analysis also showed a certain pattern that supported our previous favoring of the conditional value-at-risk as well as the conditional drawdown-at-risk, and created significant uncertainty about the average drawdown.

Overall, we were surprised to see the poor performance of semivariance and the degree of uncertainty it was associated with. Finally, variance seems to perform average compared to the other risk measures in most aspects of our analysis.

6 Chapter VI: Conclusion

Risk is an essential factor to consider when investing in the capital markets. The question of how one should define and manage risk is one that has gained a lot of attention and remains a popular topic in both the academic and professional world. Since the dawn of modern portfolio theory, there has been a consensus that investors should not minimize uncertainty in general but rather minimize "bad" uncertainty. In other words, investors should minimize *downside* risk. In order to understand the way investors think of risk, expected utility theory has been in the forefront for decades. However, research has proved that expected utility theory generally fails to capture the way investors think in reality. Quadratic utility as suggested by the mean-variance framework implies increasing absolute risk aversion, which has unrealistic behavioral implications. In addition, quadratic utility suggests that investors care only about the mean return and the variance of their portfolio – the latter being equivalent to uncertainty. Experimental research has proven that investors are more sensitive to losses than to gains, which suggests asymmetry in the domains of gains and losses respectively. Nevertheless, the mean-variance framework can still be reasonably applied if the asset returns follow a normal distribution. However, this assumption has also been discredited as it rules out skewness and excess kurtosis. The empirical evidence seems to suggest that in either case, the mean-variance framework is inadequate and that an asymmetric risk measure would better fit the way investors view risk as well as the empirical distributions of asset returns.

In chapter 4, we analyzed the relationship between risk (in several forms) and the cross-section of returns. The correlation tables showed that the three drawdown measures had the highest correlations with mean return when using monthly data. With daily data, the conditional value-at-risk was the most correlated, followed by the standard deviation. These results were confirmed by the regression analysis where all four risk measures were highly significant and explained between 20 and 25% of the variation in mean returns. Furthermore, we saw an overrepresentation of downside risk measures in the significant regressions when dividing the period into a bear and a bull market. This is in line with the logic that the further the return distribution deviates from normality, the better do downside risk measures perform. Here, the conditional value-at-risk was the most significant and had the greatest explanatory power. Our overall results were somewhat supported when we created portfolios ranked by risk and looked at the return spreads between the most and the least risky. With monthly data, we can conclude that the maximum drawdown explains the cross-section of returns to the greatest extent. However, with daily data the average drawdown portfolios yield the highest spread in returns where we had expected the conditional value-at-risk to do so. Nonetheless, the average drawdown performed well throughout the entire analysis, as did the conditional drawdown-at-risk. The overall conclusion of the analysis is that the maximum drawdown explains the cross-section of *monthly* returns the best while the conditional value-at-risk explains the cross-section of *daily* returns the best. Additionally, the conditional drawdown-at-risk performs second-best across data frequency. This suggests that downside risk measures *can* explain the cross-section of equity returns better than variance (or standard deviation).

In chapter 5, we analyzed the implications of using different risk measures on the portfolio optimization problem. We discovered that the allocation was very different depending on the risk measure. In order to determine the performance of the different risk measures, we used an alternative approach by rebalancing the minimum risk portfolio every year for all seven risk measures and compared the terminal values of this exercise. The results showed that the best performing risk measures are the conditional value-at-risk and the conditional drawdown-at-risk – both when we included and excluded bonds. All three drawdown measures, together with conditional value at risk performed well when we included bonds. However, the picture somewhat changed when we excluded bonds where both the maximum drawdown and the average drawdown became the worst performing risk measures. Average drawdown is very market sensitive, especially during down markets, relative to the other risk measures and since the market volatility increased by excluding bonds, average drawdown failed to outperform the other risk measures. We were surprised to see the poor performance of semivariance when the rebalancing included bonds. However, the performance of semivariance improved somewhat when bonds were excluded. In our in-depth analysis, we discovered that semivariance was the best performing risk measure during down markets, but failed significantly to outperform during bull markets (hence the overall low ranking). Finally, variance seems to perform average compared to the other risk measures in most aspects of our analysis. In order to measure the robustness of our results, we looked at how sample sensitive the different risk measures are. Throughout the sensitivity analysis, we were able to conclude that all risk measures have significant spreads in risk adjusted returns when based on simulated, though statistically equivalent, datasets. The analysis also showed a certain pattern that supported our previous favoring of the conditional value-at-risk as well as the conditional drawdown-at-risk, and created significant uncertainty about the average drawdown.

Overall, we have found that the choice of risk measure has a significant effect on portfolio allocation. Our analysis shows that there are some downside risk measures that outperform variance while others fail to do so. This suggest that downside risk can be a better tool in investment management than variance.

7 Bibliography

Andersen, Ib. *Den Skindbarlige Virkelighed - Vidensproduktion indenfor samfundsvidenskaberne.* Forlaget Samfundslitteratur, 2005.

Ang, J. "A Note on the E, SL Portfolio Selection Model." *Journal of Financial and Quantitative Analysis, 10*, 1975.

Arrow, K J. *Essays on the Theory of Risk Bearing.* Markham Economics Series, 1971.

Arzac, E R, and V S Bawa. "Portfolio Choice and Equilibrium in Capital Markets with Safety-First Investors." *Journal of Financial Economics, 4*, 1977.

Bawa, Vijay S. "Optimal rules for ordering uncertain prospects." *Journal of Financial Economics Vol 2,1*, 1975.

Bawa, Vijay S., and Eric B. Lindenberg. "Capital Market Equilibrium in a Mean-Lower Partial Moment Framework." *Journal of Financial Economics 5*, 1977.

Cheklov, A, S Uryasev, and M Zabarankin. "Drawdown Measure In Portfolio Optimization." *International Journal of Theoretical and Applied Finance, 8*, 2003.

Cohn, R, W Lewellen, R Lease, and G. & Sclarbaum. "Individual Investor Risk Aversion and Investment Portfolio Composition." *Journal of Finance, 30*, 1975.

Cootner, P H. "The Random Character of Stock Market Prices." *M.I.T. Press*, 1964.

Darst, D M. *The Art of Asset Allocation.* McGraw Hill, 2008.

Elton, Edwin J, Martin J Gruber, Stephen J Brown, and William N Goetzmann. *Mordern Portfolio Theory and Investment Analysis, 7th Edition.* John Wiley & Sons, Inc., 2007.

Estrada, J. "Downside Risk in Practice." *Journal of Applied Corporate Finance, 18*, 2006.

Estrada, J. "Mean-Semivariance Behavior: An Alternative Behavioral Model." *Journal of Emerging Market Finance, 3*, 2004.

Estrada, J. "Mean-Semivariance Behavior: Downside Risk and Capital Asset Pricing." *International Review of Economics and Finance, 16*, 2007.

Estrada, J. "Mean-Semivariance Optimization: A Heuristic Approach." *Journal of Applied Finance*, 2008.

Estrada, J. "The Gain-Loss Spread: A New and Intuitive Measure of Risk." *Journal of Applied Corporate Finance, 21*, 2009.

Estrada, Javier. "Mean-Semivariance Behaviour: Downside Risk and Capital Asset Pricing." *International Review of Economics and Finance 16*, 2005.

Fishburn, P C. "Mean-Risk Analysis with Risk Associated with Below-Target Returns." *The American Economic Review, 67*, 1977.

Friend, I, and M. E. Blume. "The Demand for Risky Assets." *The American Economic Review, 65*, 1975.

Grootveld, H, and W Hallerbach. "Variance vs. Downside Risk: Is There Really that Much Difference?" *European Journal of Operational Research, 114*, 1999.

Gujarati, D N. *Basic Econometrics, Fourth Edition.* McGraw Hill, 2003.

Hanssman, F. "Probability of Survival as an Investment Criterion." *Management Science, 15*, 1968.

Harlow, V. "Asset Allocation in A Downside Risk Framework." *Financial Analysts Journal, 47*, 1991.

Harlow, W V, and R K S Rao. "Asset Pricing in a Generalized Mean-Lower Partial Moment Framework: Theory and Evidence." *Journal of Financial and Quantitative Analysis, 24*, 1989.

Hicks, J R. "Liquidity." *Economic Journal*, *72*, 1962.

Hogan, W, and J Warren. "Computation of the Efficient Boundary in the E-S Portfolio Selection Model." *Journal of Financial and Quantitative Analysis*, *7*, 1972.

Hogan, W, and J Warren. "Toward the Development of an Equilibrium Capital-Market Model Based on Semivariance." *The Journal of Financial and Quantitative Analysis, 9*, 1974.

Hubbard, D W. *How to Measure Anything: Finding the Value of Intangibles in Business and The Failure of Risk Management: Why It's Broken and How to Fix It.* John Wiley & Sons, Inc., 2007.

J, Lintner. "Equilibrium in a Random Walk and Lognormal Security Market." *Harvard Insititute of Economic Research*, 1972.

Kahn, R, and D Stefek. *Heat, Light and Downside Risk.* Barra Research Paper, 1996.

Kahneman, D, and A Tversky. "Choices, Values, and Frames." *American Psychologist, 39*, 1984.

Kahneman, D, and A Tversky. "Prospect Theory: An Analysis of Decision under Risk." *Econometrica*, 1979.

Kataoka, S. "A Stochastic Programming Model." *Econometrica, 31*, 1963.

Knight, F. *Risk, Uncertainty, and Profit.* Boston, MA: Hart, Schaffner, 1921.

Krokhmal, P, J Palmquist, and S Uryasev. "Portfolio Optimization with Conditional Value-at-Risk Objective and Constraints." *Journal of Risk*, *4*, 2002.

Laughhunn, D, and C Sprecher. “Probability of Loss and the Capital Asset Pricing Model.” *Financial Management, 6*, 1977.

Markowitz, H M. *Portfolio Selection, Efficient Diversification of Investment.* John Wiley & Sons, Inc., 1959.

Markowitz, Harry. *Portfolio selection.* The Journal of Finance, Vol. 7, No. 1, 1952.

Michaud, R. *Efficient Asset Management.* Harvard Business School Press, 1998.

Michaud, R, and R Michaud. “Estimation Error and Portfolio Optimization: A Resampling Solution.” *Journal of Investment Management, 6,* 2008.

Nawrocki, D. “A Comparison of Risk Measures When Used in A Simple Portfolio Selection Heuristic.” *Journal of Business Finance and Accounting, 10*, 1983.

Norstad, J. “An Introduction to Utility Theory.” *Research Paper,* 1999.

Pavabutr, P. “An Evaluation of MLPM Allocation Rules on Emerging Markets Portfolios.” *Emerging Markets Review, 4*, 2003.

Peterson, D, and D Laughhunn. “Capital Expenditure Planning and Some Alternative Approaches to Risk.” *Management Science, 17*, 1971.

Pratt, J W. “Risk Aversion in the Small and in the Large.” *Economitrica, 32*, 1964.
Rockafellar, R T, and S Uryasev. “Optimization of Conditional Value-at-Risk.” *Journal of Risk, 2*, 2000.

Roy, A. “Safety-First and the Holding of Assets.” *Econometrica, 20,* 1952.

Scherer, B. *Portfolio Construction and Risk Budgetting.* Financial Publishing Ltd., 2002.

Sortino, F, and S Satchell. *Managing Downside Risk in Financial Markets.* Elsevier Science, 2001.

Stiglitz, J.E. “Review of some aspects of theory of risk-bearing by K.J. Arrow.” *Econometrica*, *38*, 170.

Szegö, G. “Measures of Risk.” *Journal of Banking and Finance, 26*, 2002.
—. *Risk Measures for the 21th Century*. John Wiley & Sons, Ltd., 2004.

Uryasev, S. “Conditional Value-at-Risk (CVaR): Algorithms and Applications (PowerPoint presentation).” *University of Florida.* Risk Management and Financial Engineering Lab.

Uryasev, S. “Conditional Value-at-Risk: Optimization Algorithms and Applications.” *Financial Engineering News, 14*, 2000.

Von Neumann, John, and Oskar Morgenstern. *Theory of Games and Economic Behaviour*. Princeton University Press, 1944.

8 Appendices

8.1 Appendix A

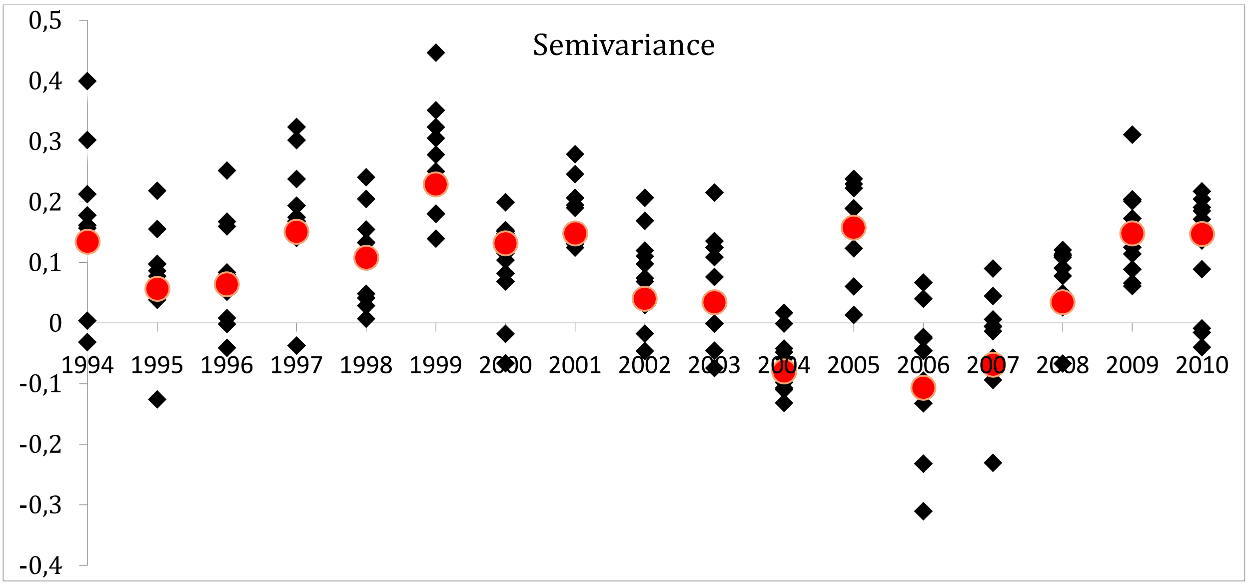
JB-test, Monthly data:

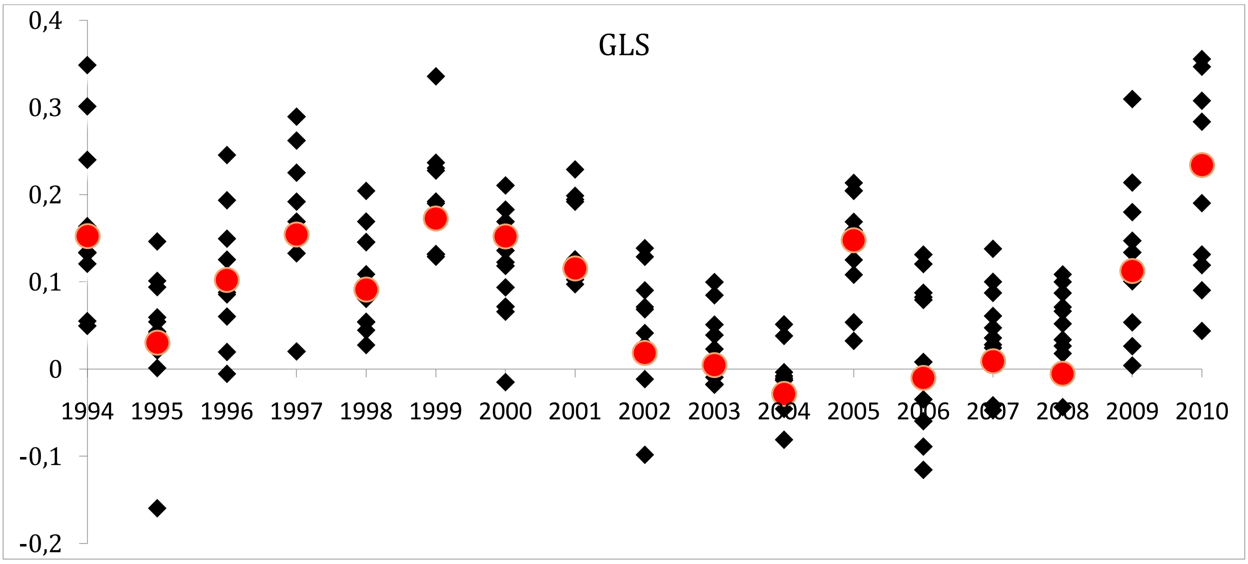
Name	Mean	Std	Skewness	Kurtosis	JB test	p-value	No. Obs.
A P Møller Mærsk B	1.70%	9.47%	0.40	1.69	52.88	0.00000	360
Allianz	1.20%	9.59%	0.07	3.46	179.75	0.00000	360
AT&T	1.05%	7.31%	-0.10	1.11	16.68	0.00024	318
Aviva	1.20%	9.73%	-0.12	2.40	87.37	0.00000	360
AXA	1.57%	11.53%	0.04	2.90	125.90	0.00000	360
Bank of America	1.68%	11.39%	0.05	4.50	304.55	0.00000	360
Barclays	1.68%	11.41%	1.36	19.33	5717.66	0.00000	360
BASF	1.26%	6.90%	-0.50	2.32	95.73	0.00000	360
BP	1.30%	7.67%	0.08	1.48	33.23	0.00000	360
Canon	1.58%	9.43%	0.33	1.98	65.07	0.00000	360
Carrefour	1.39%	7.16%	-0.28	0.59	9.96	0.00688	360
Chevron	1.29%	7.16%	0.10	0.31	2.14	**0.34369**	360
Citi Group	1.35%	13.11%	0.18	9.02	957.43	0.00000	282
Danske Bank	1.64%	8.64%	0.61	8.61	1134.80	0.00000	360
DAX 30	0.88%	6.06%	-0.94	3.00	187.08	0.00000	360
Deutshe Bank	1.06%	9.46%	0.04	6.85	704.90	0.00000	360
Dow Jones	0.84%	5.75%	-0.38	1.13	21.69	0.00002	280
CAC 40	0.92%	5.97%	-0.82	2.66	20.76	0.00003	268
FTSE 100	0.83%	5.54%	-0.69	1.22	118.82	0.00000	292
General Electric	1.45%	8.18%	-0.04	0.95	13.67	0.00107	360
Honda Motors	1.55%	9.65%	0.99	3.26	218.42	0.00000	360
Jyske Bank	1.51%	7.86%	-0.45	5.05	394.30	0.00000	360
Mitsubishi	1.23%	9.67%	0.10	1.16	20.87	0.00003	360
MSCI EM	1.36%	7.81%	-0.52	1.76	15.54	0.00042	268
MSCI Worls	1.03%	5.52%	-0.54	1.76	63.59	0.00000	360
NIKKEI 225	0.64%	6.49%	0.04	0.51	63.59	0.00000	360
Nippon	0.04%	9.02%	0.89	2.01	83.43	0.00000	279
Novo Nordisk	2.11%	9.10%	0.91	6.77	736.87	0.00000	360
OMXC20	0.76%	5.93%	-0.75	2.77	501.48	**0.00000**	245
RBS	1.68%	12.47%	-0.35	7.26	797.97	0.00000	360
Sanofi-Aventis	1.33%	7.66%	-0.14	0.14	1.54	**0.46386**	358
Siemens	1.17%	8.82%	-0.30	2.54	101.91	0.00000	360
Societe Generale	1.24%	10.77%	-0.23	2.35	65.16	0.00000	274
Tesco	1.68%	7.81%	0.20	0.94	15.51	0.00043	360
Total	1.58%	7.84%	-0.13	1.84	51.57	0.00000	360
Toyota Motor	1.37%	8.81%	0.67	1.66	67.91	0.00000	360
Volkswagen	1.27%	10.21%	0.23	2.25	79.11	0.00000	360

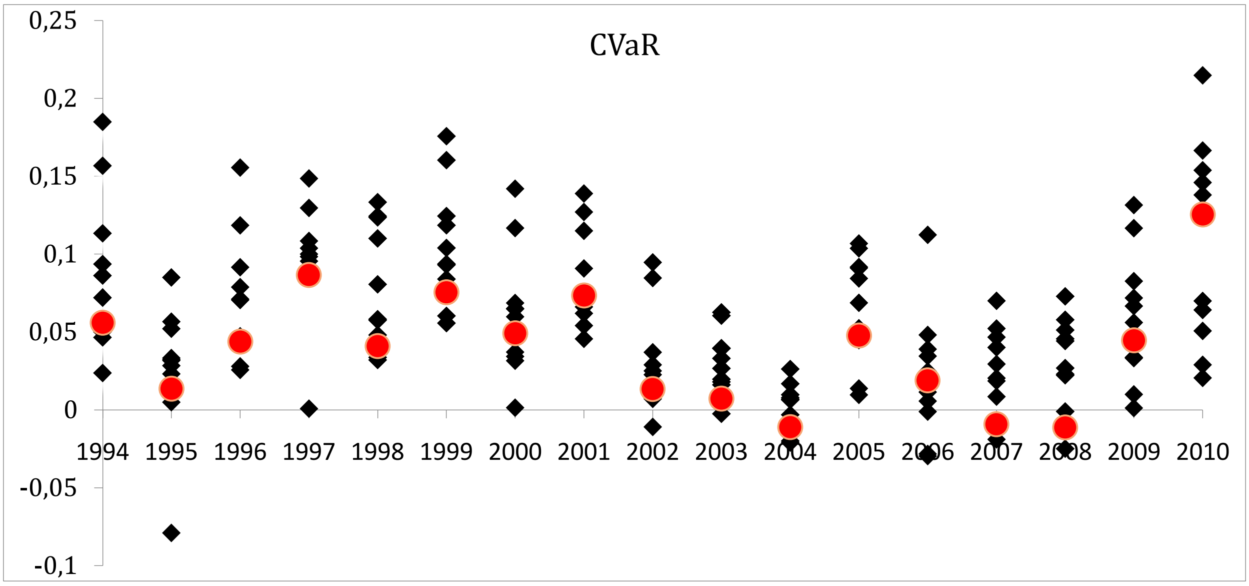
JB-test, Daily data

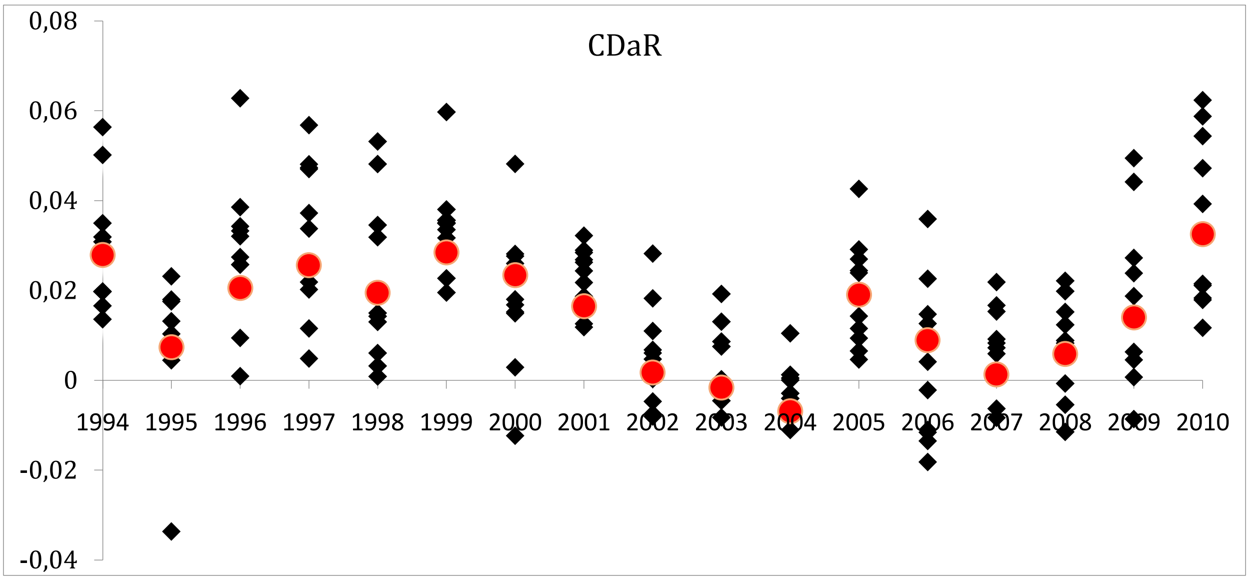
Name	Mean	Std	Skewness	Kurtosis	JB test	p-value	No. Obs.
A P Møller Mærsk B	0.079%	1.995%	1.2	19.4	124621.7	0.00000	7827
Allianz	0.054%	2.014%	0.6	11.8	45569.0	0.00000	7827
AT&T	0.052%	1.804%	0.1	10.5	31562.8	0.00000	6909
Aviva	0.059%	2.253%	0.0	14.4	67928.8	0.00000	7827
AXA	0.068%	2.361%	1.5	26.7	235283.5	0.00000	7827
Bank of America	0.079%	2.562%	0.8	27.8	252133.5	0.00000	7827
Barclays	0.077%	2.480%	3.3	102.5	3438339.3	0.00000	7827
BASF	0.060%	1.585%	0.1	6.3	13019.7	0.00000	7827
BP	0.061%	1.706%	0.1	3.9	5027.1	0.00000	7827
Canon	0.080%	2.327%	0.5	6.2	12867.2	0.00000	7827
Carrefour	0.068%	1.782%	-0.1	6.1	12248.7	0.00000	7827
Chevron	0.064%	1.769%	0.2	7.1	16437.9	0.00000	7827
Citi Group	0.066%	3.046%	1.1	36.6	344051.3	0.00000	6142
Danske Bank	0.073%	1.676%	0.1	8.5	23794.5	0.00000	7827
DAX 30	0.041%	1.354%	-0.1	7.2	23794.5	0.00000	7827
Deutshe Bank	0.046%	1.941%	0.5	14.0	64488.3	0.00000	7827
Dow Jones	0.042%	1.320%	-0.5	11.0	50218.0	0.00000	6095
CAC 40	0.042%	1.367%	0.1	5.6	29595.0	0.00000	5836
FTSE 100	0.039%	1.250%	-0.2	7.5	8256.8	0.00000	6358
General Electric	0.070%	1.867%	0.2	7.4	18027.9	0.00000	7827
Honda Motors	0.078%	2.319%	0.4	4.7	7390.0	0.00000	7827
Jyske Bank	0.067%	1.593%	0.5	12.7	53365.0	0.00000	7827
Mitsubishi	0.063%	2.309%	0.2	3.8	4691.1	0.00000	7827
MSCI EM	0.062%	1.506%	-0.4	55.1	3497.8	0.00000	5836
MSCI Worls	0.047%	1.123%	0.7	27.7	250411.7	0.00000	7827
NIKKEI 225	0.032%	1.481%	0.1	6.1	12251.2	0.00000	7827
Nippon	0.022%	2.419%	0.9	9.0	21303.7	0.00000	6069
Novo Nordisk	0.097%	1.909%	-0.3	17.9	105178.8	0.00000	7827
OMXC20	0.034%	1.175%	-0.1	6.4	9191.3	0.00000	5334
RBS	0.077%	2.705%	-1.2	76.7	1921824.9	0.00000	7827
Sanofi-Aventis	0.066%	1.908%	0.1	4.4	6178.7	0.00000	7779
Siemens	0.053%	1.830%	0.1	8.2	22100.8	0.00000	7827
Societe Generale	0.056%	2.283%	0.6	10.2	26419.8	0.00000	5962
Tesco	0.080%	1.791%	0.3	3.8	4888.3	0.00000	7827
Total	0.076%	1.879%	-0.2	7.8	19757.1	0.00000	7827
Toyota Motor	0.067%	2.051%	0.5	6.1	12448.1	0.00000	7827
Volkswagen	0.067%	2.765%	16.0	700.2	160227393.9	0.00000	7827

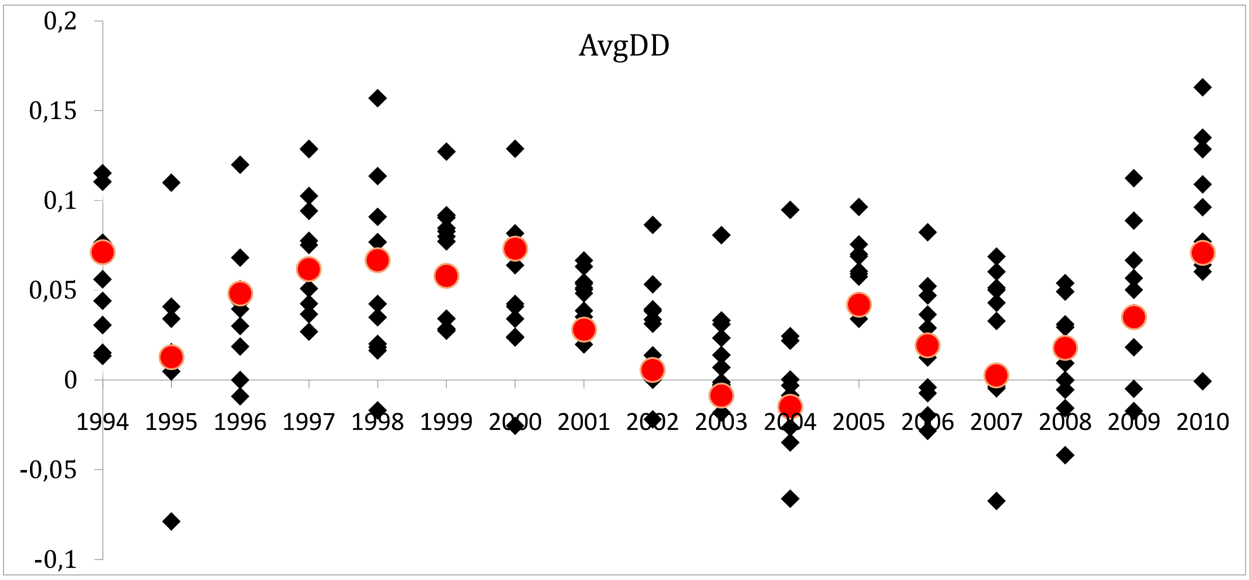
8.2 Appendix B

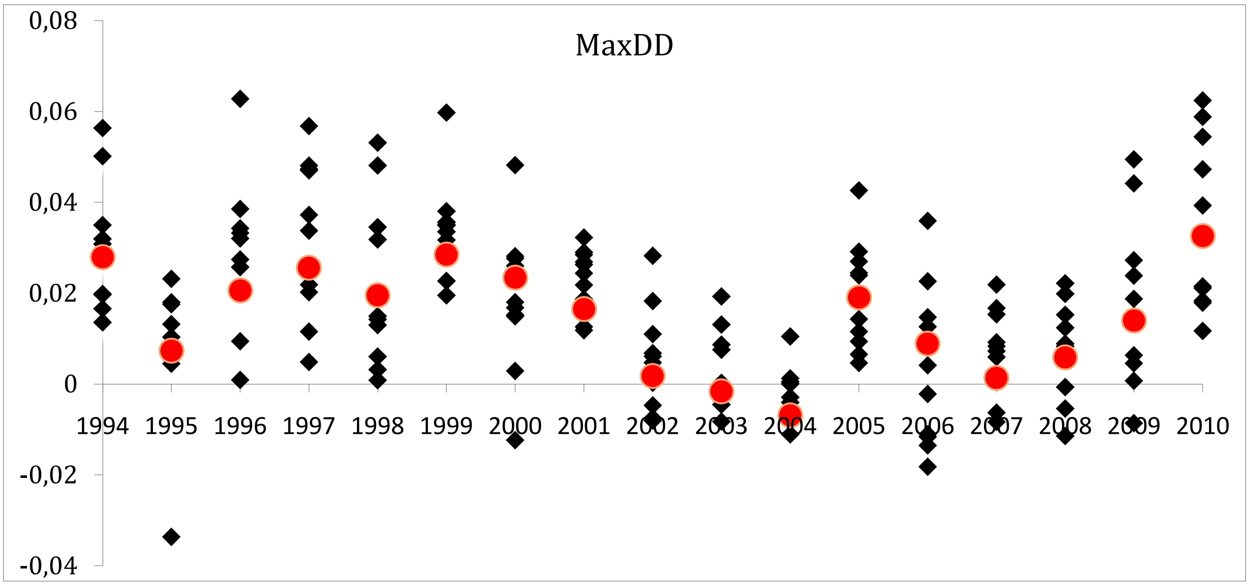





CVaR
0,25
0,2
0,15
0,1
0,05
0
-0,05
-0,1
1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010


CDaR
0,08
0,06
0,04
0,02
0
-0,02
-0,04
1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010


AvgDD
0,2
0,15
0,1
0,05
0
-0,05
-0,1
1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010


MaxDD
0,08
0,06
0,04
0,02
0
-0,02
-0,04
1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010