Copenhagen Business School

MSc Advanced Economics and Finance

Master's Thesis

The Credit Default Swap Option

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Executive Summary

I de seneste år er markedet for eksotiske derivativer heriblandt CDS optioner vokset støt. Det er derfor interessant at undersøge hvilken værdi og nytte disse CDS optioner har for investoren. Dette speciale forsøger at blotlægge om CDS optionen gør en forskel enten i form af profit eller i reducering af risiko for investoren, de finansielle institutioner eller de store virksomheder.

Specialet er struktureret på følgende måde.

Part I beskriver Credit Default Swap Optionen. Jeg redegører for den underliggende CDS kontrakt, for markedet for CDS kontrakter og for markedet for CDS optioner. Derudover beskriver jeg hvordan investorer kan drage nytte af CDS optionen.

I Part II beskriver og udleder jeg den teoretiske prismodel for CDS optioner præsenteret af [Brigo and Morini, 2005]. Denne model tager udgangspunkt i den mest anvendte model til prisfastsættelse af optioner – Black-Scholes modellen. At udvide Black-Scholes modellen til også at kunne prise CDS optioner er derfor et naturligt første skridt.

I Part III præsenterer jeg en empirisk analyse af CDS optionen. Ved at kalibrere overlevelses-sandsynligheder for forskellige lande kan jeg beregne optionsprisen på en CDS kontrakt for netop disse lande. Jeg afslutter Part III med en diskussion af de fundne optionspriser. Dette gøres ved at sammenligne de beregnede priser til optionspriser fundet af andre.

Part IIII indeholder en undersøgelse af brugen af CDS optioner. På den ene side, finder jeg at CDS optioner ofte har en meget kort løbetid. Dette indikerer at CDS optionen bliver brugt af spekulanter for at opnå profit og ikke af investorer som ønsker, at afdække deres risikoeksponering. Jeg undersøger desuden afkastet ved at handle CDS optioner og finder at handel med CDS optioner ikke er profitabelt i den periode jeg har data på. På den anden side, finder jeg at den implicitte volatilitet på CDS optionen forudsiger volatiliteten på CDS spreadet signifikant. Dette resultat gælder uanset kreditværdigheden af det relevante land. På den måde må bevægelser i CDS optionen reflektere bevægelser i det underliggende, hvilket indikerer at CDS optionen også kan blive brugt til at afdække risiko eksponering.

Da CDS optionen er et relativt nyt instrument, hvorom der eksisterer meget lidt litteratur, vil mine undersøgelser være præget af mangel på data. Derfor kan dette speciale ses som et intelligent udgangspunkt for videre undersøgelse af ikke bare CDS optionen, men også af andre eksotiske derivater.

Abstract

During the financial crisis the demand for exotic derivatives decreased. However, the market for exotic derivatives among them CDS options has grown tremendously ever since. This thesis presents a necessary guideline on how to investigate the market of a new instrument. In this thesis we will investigate the market for the Credit Default Swap Option - A credit instrument in which there exists very little literature about.

I discuss the underlying Credit Default Swap, the market for the CDS and the market for the CDS option. Combining extensive literature on valuing CDS options I arrive at an extension of the Black-Scholes formula. Thereby I succeed in pricing the CDS option. Empirically I manage to successfully calibrate the survival probability of a reference entity and by that I am able to calculate the option price on a CDS.

I hope with this thesis that I can discover if the CDS option makes a difference – in terms of money or risk - to investors, financial institutions, companies or even to private individuals. Using data from the CDS option market I expect to find both signs of speculative trading and indications that investors use the CDS option to reduce risk exposure.

Contents

Chapter 1

Intro

In recent years the market for CDS options has grown tremendously forcing people to ask themselves what value we can assign to this product and what purpose the CDS options serve. This thesis attempts to answer these questions by building upon some of the most classical models from modern financial theory. With this thesis I wish to discover if this new derivative makes a difference – in terms of money or risk - to investors, financial institutions, companies or even to private individuals.

The thesis is structured as follows.

Part I describes the new derivative of our interest – the Credit Default Swaption. I discuss the underlying Credit Default Swap, the market for the CDS and the CDS option. Furthermore, I describe how financial institutions, investors and large companies might benefit from this new product.

In Part II I explain the derivation of a pricing model for the Credit Default Swaption. Since 1973 the Black-Scholes has been the most used option pricing formula. Therefore, a natural first step in pricing the CDS option will be to use that formula. Combining the literature from [O'Kane, 2011], [Brigo and Morini, 2005], [Jamshidian, 2004] I arrive at an extension of the Black-Scholes formula and thereby I succeed in pricing the CDS option. However, the Black-Scholes formula has been facing massive critique due to multiple restrictive assumptions and therefore I end Part 2 with a discussion of the drawbacks to the Black-Scholes formula. I then present the most common extension of the Black-Scholes formula and thereby encourage further research.

In Part III of this thesis I will present an empirical analysis of the Credit Default Swaption. By successfully calibrating the survival probability of a reference entity I am able to calculate the option price of a CDS. I end Part III with a discussion of my results and a comparison of the results to the research of others.

Finally, Part IV searches the market for CDS options looking for an answer as to why this derivative is being increasingly traded. On the one hand, I find that the CDS options are often short-term-maturity options indicating that investors trade the CDS option for the short-term profit and not for the long-term reduction in risk exposure. However, the profit from trading the CDS option in the time frame in which I have data is negative. On the other hand, I find that the implied volatility of the CDS option significantly forecasts the volatility of the CDS spread no matter the credit rating of the reference entity. This suggests that the CDS option reflects the underlying asset thereby indicating that the CDS options are traded not only for profiting but also for reducing different risk exposures.

We have only seen the beginning of the CDS option being traded, and we have yet to obtain solid financial data on the CDS option trades. This is not only a result of the CDS option being a new derivative. Rather, it is also a result of the paradox that financial institutions that sell and buy the CDS options have no incitements to publish the data on these transactions. This limits my research and therefore further research when data is available will shed even more light on the Credit Default Swaption. I encourage the reader to see this thesis as an intelligent guideline on how to analyze new derivatives.

Part I

Setup

Chapter 2

CDS

With a new derivative in the market a natural first step will be looking at the different components of the derivative. The Credit Default Swap Option is an option on a Credit Default Swap(CDS). Therefore we will begin with a discussion of the characteristics of the underlying CDS. This chapter will describe the features of the CDS and at the end of this chapter I will briefly discuss the market for CDSs. Understanding the market for CDSs will help us in understanding the market for CDS options. In Chapter 3 we will see that we have limited data on the market for CDS options and therefore an understanding of the underlying CDS will be even more important.

2.1 Credit Default Swap (CDS)

The credit default swap (CDS) contract is the most liquid single-name credit derivatives contract [O'Kane, 2011] with a net notional amount of 2.7 trillion in 2011 (See [ISDA]). As with every other contract or trade the CDS contract involves of a buyer and a seller.*The protection buyer* buys the CDS with the purpose of protecting himself from the loss-given-default(LGD) on a specified bond or loan. This specified bond or loan is known as *the deliverable obligation* and the issuer of this bond or loan is known as *the reference entity*. *The protection seller* receives payments from the buyer until either the protection stops - at a specified maturity - or until a *credit event* (e.g. a default) occurs. There is no initial cost in entering into a CDS, instead the premium payments from the buyer and the protection from the seller can be viewed as two separate legs as illustrate in Figure 2.1 and described below.

Figure 2.1: The Premium Leg and Protection Leg of the CDS contract

The following subsections will be a brief discussion of the various components of a CDS contract.

The Protection Leg

If a credit event occurs before maturity of the contract the protection seller must pay the protection buyer for the loss. The amount that the protection buyer must pay depends on the pre specified recovery rate(REC) and the type of credit event. The different types of a credit event will be discussed later in this chapter, but for now the reader might think of a credit event as a default of the reference entity. The protection leg is one single payment at the time of the credit event given that the credit event occurs before maturity of the contract. There are two ways in which the payment can be settled, either by *physical settlement* or by *cash settlement* [1]. Given that the scope of this thesis is pricing CDS option I will not dwell on the different settlement structures of a CDS instead I will from now on assume that if a credit event occurs, there will be a cash settlement.

The Premium Leg

In return for the protection the protection seller receives payments, typically quarterly, from the protection buyer. The payments stop immediately following a credit event or at the contract's maturity depending on which occurs first. The size of the payments are quoted in the market as annualized spreads called the credit default swap spread (CDS Spread).

It would be unrealistic to assume that a credit event always happens exactly at the payment dates. Therefore, if a credit event occurs in between two payment dates, there will be a smaller last payment corresponding to the time between the second to last payment and the credit event. Later, we will see that treating the credit event as a random time point in between the payment dates will yield some unnecessary complication of the CDS option pricing model. Therefore, one often assumes e.g. in [O'Kane, 2011] that the credit event occurs on average halfway in between to protection payments and unless something else stated, we will use this assumption in the entire thesis.

Credit Events and Recovery Rates

For the sake of simplicity and to avoid shifting the focus of the thesis unnecessarily, I will not discuss the many different types of credit events. Instead I will only briefly discuss the term. The credit event is the legal term that triggers the payment from the protection seller to the protection buyer. This event can have multiple forms among them default, restructuring of debt etc. The settlement after a credit event may depend on the type of the credit event, hence having the credit event pre specified before entering the contract is necessary. However, this thesis will from now on use default and credit event interchangeably unless otherwise stated.

In the event of the reference entity defaulting, the protection seller will pay the protection buyer the cash settlement as shown in Figure 2.1. However, at this point I still need to define the amount of this cash settlement. The cash settlement depends on the *recovery rate* of the reference entity. The recovery rate is the ratio between the amount received upon settlement to the face value of the reference entity and this amount is pre specified before entering the CDS. Typically the recovery rate (REC) will be 0.4 and then the loss given default (LGD) will be $(1 - REC) = 0.6$.

[1] See [O'Kane, 2011] for further explanation

2.2 CDS indices

In the description of the CDS market in the next section we will discuss the entering of the CDS indices in 2002. The entering of the CDS indices changed the entire credit market making it possible to hedge against regions, industries etc. A CDS index consists of a weighted portfolio of reference entities; e.g. the CDX.NA.HY consists of the 100 liquid Northern American (NA) entities with high yield (HY) that trade in the CDS market (See [Markit, a]). In Part II when I will derive a price for the CDS option it will basically be the same pricing model for both single named credit derivatives and multi named credit derivative. However, in the empirical part of the thesis I will choose to price options on CDS indices due to data availability being higher for CDS indices than for CDS single entities in the later years.

2.3 The CDS Market

Based on the information from International Swaps and Derivatives Association (ISDA) and from Bank for International Settlements (BIS) I introduce a brief overview of the CDS market during recent years. The CDS market was created in the mid-1990s by JP Morgan (See [Bloomberg, b]) to reduce a bank's risk exposure to large corporate loans.

At the beginning there were a limited number of parties to the CDS transactions, and in most cases the buyer of the protection also held the underlying credit asset (loan or bond). Then in the early 2000s several changes occurred resulting in a less see-through market but also leading to a lot of new market participants and to a great trading volume. The numbers of different CDS increased and market participants were now buying the CDSs without exposure to the underlying reference entity making the CDS market the speculative market we know today.

As shown in Figure 2.2 from [ISDA, 2013] it is justifiable to separate the period from the first reporting CDS notional by BIS in 2004 up until now in two periods: before and after the financial crisis. In Figure 2.2 we observe the evolution in the CDS market; the y-axis being the gross notional value of outstanding in the market. It is clear that prior to the financial crisis we experienced a doubling in the CDS every year. Between 2007 and 2008 there was a drop of 28 percent of the gross notional amount outstanding. From 2008 up until today we can observe a decrease by 40 percentage in total. To the reader this might indicate that the CDS market is less important today than before the financial crisis. However, according to ISDA the decrease was caused by a portfolio compression reducing the notional amount. Looking at the notional amount of new CDS market risk transaction, ISDA finds an increase in the later years.

In Figure 2.3 we see the amount of traded CDS separated into single-named CDS and CDS indices. It is clear that the market for CDS indices becomes a larger fraction of the entire CDS market. In 2013 and 2014 the traded CDS indices are 50% of the entire market.

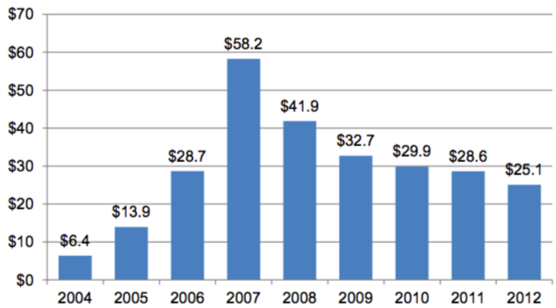

Figure 2.2: The CDS Market, 2004-2012, Source: ISDA

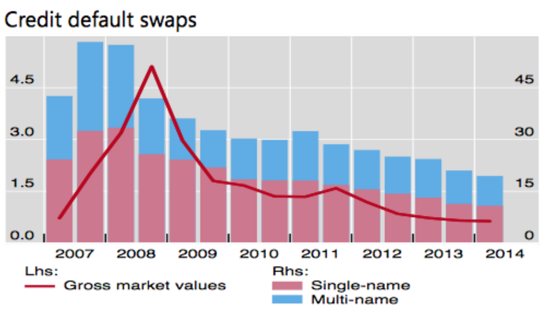

Figure 2.3: The CDS + CDS indices Market, 2007-2014, Source: BIS

Chapter 3

Option theory

In the previous chapter we discussed the characteristics of the CDS contract and the market for CDSs. In this chapter the main focus will be on understanding options and on eventually understanding options on credit default swaps (CDS). The chapter will end with a brief overview of the CDS option market.

3.1 Options and CDS options

An option is a financial contract between two parties. The option gives the buyer the right but not the obligation to buy/sell a specific quantity of a commodity or financial instrument *(the underlying asset)* at a certain point in time *(the expiration date)* at a certain price *(the strike price)*. A *call option* gives the buyer of the option the right to buy at the expiration date. Similarly a *put option* gives the buyer of the option the right to sell. If the buyer chooses to buy or sell at the expiration date it is known as exercising the option. The two most common plain options are the *European option* and the *American option*. The European option can not be exercised before maturity whereas the American option allows that possibility. For the purpose of this thesis it is not necessary to distinguish between the two kinds of options and therefore all options will henceforward be thought of as European options.

The focus of this section will be reaching the understanding of the Black&Scholes formula introduced in [Black and Scholes, 1973]. We wish to extend the Black&Scholes formula such that eventually we will be able to price CDS options. Therefore, this section will not focus on binomial derivation of the option price nor will I describe in depth the basics of option theory as this will be familiar to the reader.

Let K be the strike price of a call option and S_T be the price of the underlying asset at the option expiry date T. Then the value of that option at time T will be given as

$$C_T = \max(S_T - K, 0) \tag{3.1}$$

and the value of a put option is given by

$$\Pi_T = \max(K - S_T, 0) \tag{3.2}$$

Looking at Figure 3.1 we can see the intuition behind the above equation. Buying the right to exercise a call option with strike price K implies that if the true price P_T is above the strike price K, then the buyer will choose to exercise and hence the option will have a positive value - increasing linearly with the price - and the opposite holds for a put option.

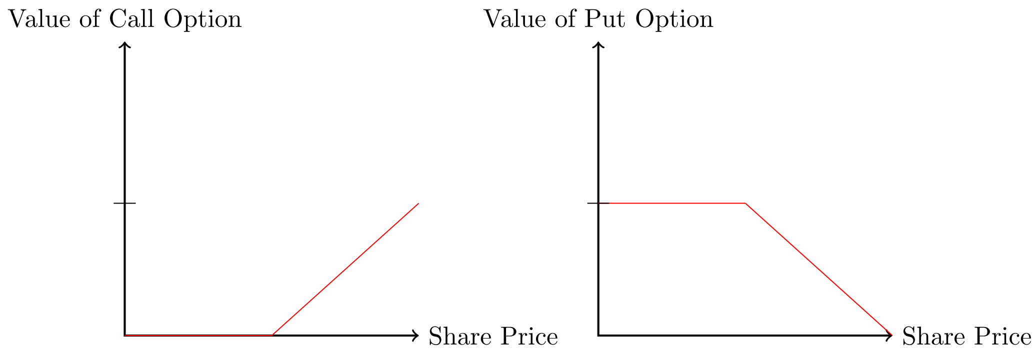

Figure 3.1: European Call and Put option

At this point it is suitable to discuss the benefits of options and the reasons why options are traded widely and extensively. Standing at point t wanting to by e.g. an asset at time $t+1$ you have three options:

1. You can wait until time $t+1$ and buy at the price at that time point
2. You can buy a forward at time t already agreeing on the time $t+1$ price.
3. You can buy an option at time $t+1$ with a pre specified strike price.

Using option 1. you are hoping that the price will not increase before time $t+1$ and thus you are exposed to the fluctuation in the price of the asset. Choosing option 2. you "win" if the price increases since you already agreed on buying the asset at the low price. However, if the price drops, then you "loose". The third possibility would be buying the call option. As Figure 3.1 shows you "earn" a profit when the price increases since you can buy the asset at the pre-specified low strike price. However, the option value is 0 when the price of the asset at time $t+1$ is lower than at time t and you can then buy the asset at the low market price. Buying the option gives you "the better deal" and therefore reduces the risk exposure. Note that the price of the forward contract is zero whereas you pay a premium to obtain the option.

Now that we have discussed the value of the option, we will need to know the price of the option. The price of the option is naturally based on the no arbitrage principle. This means that standing at time t in the above example the buyer of the option should be indifferent between the 3 options. The price of a call option can be calculated using the binomial method. When the distance Δt goes towards zero we arrive the time-continuous option model - the Black&Scholes formula. In [Black and Scholes, 1973], Fischer Black and Myron Scholes present the option pricing formula pricing European options. I will only briefly explain the setting in [Black and Scholes, 1973], as this is already familiar to the reader.

The model assumes that the evolution of the underlying asset can be described by a Itō drift-diffussion process presented in e.g. [Øksendal, 2003] and given by

$$dX_t = \mu_t dt + \sigma_t dZ_t^{\mathbb{P}'} \tag{3.3}$$

with $Z_t^{\mathbb{P}'}$ being a Brownian motion under the relevant probability measure $\mathbb{P}'$.

The assumption stating that $Z_t^{\mathbb{P}'}$ is a Brownian motion - from now on denoted as $W_t^{\mathbb{P}'}$ - implies that

- $W_0 = 0$
- Normally distributed increments: For all t_1,t_2 with $t_1 < t_2 : W_{t_2} - W_{t_1} \sim N\left(0, \sigma^2 t_2 - t_1\right)$.
- Independent increments: For all $0 \leq t_0 < t_1 < ... < t_n$, the random variables $W_{t_1} - W_{t_2}, ..., W_{t_n} - W_{t_{n-1}}$ are mutually independent.
- W has continuous path

In addition we assume that the volatility of the asset is constant $\sigma_t = \sigma \ \forall t$.

Given the above assumptions and assuming we have that X_t is log-normally distributed with mean μ and variance σ^2

$$ln(X_t) \sim N(\mu, \sigma^2) \tag{3.4}$$

Then [Black and Scholes, 1973] states that the solution to

$$\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right] = \mathbb{E}\left[max(S_T - K, 0)\right] \tag{3.5}$$

will be the following option pricing formula

$$C_t = SN(d_1) - e^{-r(T-t)}KN(d_2) \tag{3.6}$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}} \tag{3.7}$$

$$d_2 = \frac{\ln(S/K) - (r + \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}} \tag{3.8}$$

There are several advantages and disadvantages of using Black&Scholes. I will discuss these later in Chapter 9 Section 9.2 where I intend to extend the Black&Scholes-formula to price CDS options.

I will complete this section by discussing the features of an option on a CDS and an option on a CDS index. Buying a CDS option, also called Credit Default Swaption means buying the right to enter a CDS contract at a future point in time. Buying a Call options means buying the right to enter a CDS contract in which the buyer pays the premium thereby getting the protection, hence named *a payer swaption*. Buying a Put options means buying the right to sell a CDS contract at a future time point, receiving the coupon payments, hence named *a receiver swaption*. Almost all CDS option ([O'Kane, 2011]) are European options meaning that the buyer is not allowed to exercise before maturity and therefore cannot enter the CDS contract before maturity. Figure 3.2 shows the payment streams from the buyer of a European Call option given that no defaults occur before the maturity of the underlying CDS contract. At time 0 the buyer acquires the CDS option reflecting the negative income stream in Figure 3.2. Let us assume that the maturity of the option and the first payment day in the CDS contract are equal both being T_a. In the figure the buyer of the option chooses to exercise, and from T_a up until the maturity of the contract T_b he pays the pre-specified protection payments. Figure 3.3 shows the payment streams from the buyer of a Call option given a default before maturity of the CDS contract. The figure shows that the protection payments stop immediately after the default and instead the buyer of the option - the buyer of the protection - receives the protection illustrated by the huge positive payment stream.

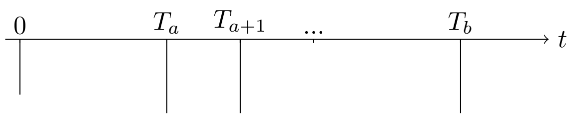

Figure 3.2: The life time of the Call Option, no default

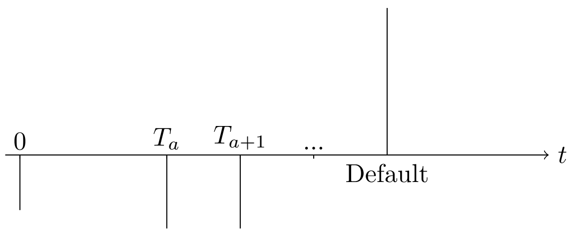

Figure 3.3: The life time of the Call Option, with default

3.2 The Market for CDS Options

Before the financial crisis the market for exotic derivatives, including the market for CDS options, was growing fast as discussed in Section 2.3. However, during the crisis the demand for exotic derivatives among them CDS options declined. We have little data on the notional amount of traded CDS option but several articles e.g. [Bloomberg, a] and [FinancialTimes] suggest that today the market for CDS option is growing tremendously. [FinancialTimes] mentions that especially the market for option on

CDS indices is increasing. Bob Douglas (head of credit electronic trading at Barclays) claims in the article that the increase in traded CDS option is a result of investors worrying about tail-risk despite the relatively historically low volatility. The goal of this thesis is reaching a better understanding of the CDS option and the market for CDS option. However, the research in the rest of the thesis will face several problems due to the limited data. It is not surprising that we lack data when trying to describe a new derivative. Therefore this thesis will serve as a starting point for further research when the financial institutions begin to publish data on CDS options.

Chapter 4

Trading the CDS options

In the previous chapter I discussed the characteristics of the CDS option and the market for CDS options. This chapter will focus on the use of the CDS option. The use of the CDS option often falls in two categories; hedging specific risk exposure or doing speculative trading. Later in Part IV I will analyse the CDS option market and search for a better understanding of why the CDS options are increasingly being traded. Therefore, I will discuss different signals in the market indicating if the CDS option market is mainly a result of speculative trading or a result of hedging risk exposure.

4.1 Hedging

A large company e.g. a multinational company will often be exposed to different risk through their operations as discussed in [Hull, 2012]. Let us use exposure to a given exchange rate as an example. If a company operates in one country - e.g. Germany paying material, wage etc. in Euros but sells a significant part of their output to another country with a different currency e.g. the USA they will be exposed to an exchange risk. If the dollar increases relative to the Euro while the company is producing in Germany, the company will earn a larger than expected profit when selling their products and receiving dollars. However, the exact opposite might also occur resulting in huge losses. Such an exchange can have a huge positive or negative impact on the company's profit. This was the case with e.g. Carlsberg's exposure to the Russian market ([Bloomberg, c]). Consequently, it will be worth hedging against exchange rate risk. Given that the company has no expertise in calculating the perfect hedge it will have a bank or another financial institutions carrying out the process of hedging for them. As this thesis considers options on CDS indices, hedging will be carried out when a company is exposed to industry specific defaults - e.g. buying CDS options on biotechnology - or even country specific defaults e.g. buying/selling option on CDS index on the state of Italy.

In this thesis I will discuss whether or not CDS options traded in the market today are mainly used for hedging or speculative trading. One indicator of CDS options being used as a hedging instrument will be checking if the trading amount in CDS options is comparable to the trading amounts of the underlying CDS or the reference entity as suggested by [Oehmke and Zawadowski, 2014]. However, it will come apparent that we are unable to test this due to limited data. Instead, [Shu and Zhang, 2003] show that the implied volatility of an option serve as a good forecast for the volatility of the underlying asset. We will show that this is also true for the implied volatility of the CDS option. This means that the CDS option reflects the underlying CDS indicating that the CDS option is being used for reducing risk exposure in the underlying CDS.

4.2 Speculative trading

As the name indicates speculative trading occurs when investors speculate in future movements and trade with the purpose of earning a profit and not of hedging risk. In the market for CDS options speculative trading implies that investors trade CDS options on specific companies, indexes or industries without being exposed to changes in those reference entities. [Oehmke and Zawadowski, 2014] suggest that the investors disagreements about a reference entity's future earnings can serve as a proxy for speculative trading. Therefore, it would have been very interesting to test if larger disagreements about the reference entities future earnings are associated with larger notional amount of traded CDS options. This would indicate that the CDS options are being used for speculative trading. However due to data limitation this is not possible. Instead, we will observe the CDS option from an investors point of view testing if he can profit from trading CDS options. We find that this is not the case but since the available data is too small we cannot conclude that this is always the case.

Chapter 5

Modelling the Yield Curve

The CDS Call option gives the buyer of the option the right to enter into a CDS contract at a future point in time. This means that pricing the CDS option will involve valuing the components of the CDS at different points in time. Therefore we will need a method for discounting back those components. There exists many different ways of extracting of discounting and the data being used for this have been discussed widely e.g in[Feldhütter and Lando, 2008]. The scope of this thesis is not to shed light on the discussion of Treasury yield, LIBOR or interest rate swap used for discounting. Instead I choose to use US treasury yields and then I will comment on that choice in the Appendix in Chapter 15.

5.1 Forward Rate, Discount Factor, Yield curve

This section will briefly discuss the relationship between the forward rate, the discount factor and the yield curve. We will later see that we will need those relations for pricing the CDS option.

Let r_t denote the instantaneously risk-free interest rate at time t such that the return over an infinitesimal interval $[t, t+dt]$ is $r_t dt$. We refer to r_t as the short-term interest rate. Let $A = (A_t)_{t \geq 0}$ denote the price process of the bank account. The increment to the bank account over the infinitesimal interval $[t, t+dt]$ is known at time t to be $dA_t = A_t r_t dt$. This means that depositing A_0 at time 0 will, at time t, grow to $A_t = A_0 e^{\int_0^t r_u du}$, when continuous compounding is used.

Let $B_0 = 1$ and let us define the stochastic discount factor as

$$D(t,T) = \frac{B_t}{B_T} = \frac{1 \cdot e^{\int_0^t r_u du}}{1 \cdot e^{\int_0^T r_u du}} = e^{\int_t^T -r_u du} \tag{5.1}$$

This is the price of the zero-coupon bond reflecting the price on a loan between today and a given future time point. The forward rate on the other hand reflects the price on a loan between two different future time points and is denoted by $f_t(S,T)$. Using the results from Munk [2011] we have the following relationship between the forward rate and the discount factor

$$D(t,T) = e^{-\int_t^T f_t(u) du} \tag{5.2}$$

As we will see in the next section we have data on the zero coupon rate $y_t(\tau)$, hence we need the relation between the forward rate and the zero coupon rate, which is given by

$$y_t(\tau) = \frac{1}{\tau - t}\int_t^\tau f_t(u,T)du \tag{5.3}$$

This implies that the discount factor will be given by

$$D(t,T) = e^{-(T-t)y_t(T)} \tag{5.4}$$

This result will be used multiple times in this thesis. In the next section we will discuss the discount factor in a defaultable setting but we will see that with some assumptions we can still use the above relations.

5.2 Discounting in a Defaultable Setting

In Part II when we wish to price the CDS and the CDS option we will need to do so in a defaultable setting. A defaultable environment is a state in which default of the reference entity/underlying is possible. An investor who makes an investment today wants to know the expected future return, hence we will need to know how to discount in a defaultable environment. Consider a credit risky structure paying 1 at the time of default, τ, if $\tau < T$ and nothing if there is no default before time T. From [O'Kane, 2011] we know that at time 0 this payment will be given by

$$D(0,T) = \mathbb{E}\left[exp(-\int_0^\tau r_s ds)\mathbf{1}_{\tau>T}\right] \tag{5.5}$$

τ is stochastic meaning the time of the payment is unknown and therefore the discounting will be more difficult than in the non-defaultable environment.

Let $\mathcal{F}_t$ be the basic filtration containing information about interest rate and other default-free market quantities and let $\sigma\left(\{\tau < u\}, u \leq t\right)$ be the sub filtration generated by τ. Observing only information from $\mathcal{F}_t$ one have information about the probability of default but not if or when default occurs. Let us then define

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma\left(\{\tau < u\}, u \leq t\right) \tag{5.6}$$

With the above filtration $\mathcal{G}_t$ we ensure that observation of default is possible. Let $\mathbb{E}^{\mathbb{Q}}$ be the risk-neutral expectation conditional on the default-free sigma field $\mathcal{F}_t$

In Lando [1998] he shows the following two results

$$\mathbb{E}\left[\mathbf{1}_{\tau>T} \mid \mathcal{F}_T \vee \mathcal{H}_t\right] = \mathbf{1}_{\tau>t} exp\left(-\int_t^T \lambda_s ds\right) \tag{5.7}$$

$$\mathbb{E}\left[exp\left(-\int_t^T r_s + \lambda_s ds\right) X \mid \mathcal{F}_t \vee \mathcal{H}_t\right] = \mathbb{E}\left[exp\left(-\int_t^T r_s + \lambda_s ds\right) X \mid \mathcal{F}_t\right] \tag{5.8}$$

We refer to Chapter 6 for an explanation of the intensity parameter λ. We will use these results and the assumption that the short rate r_s and the intensity parameter λ_s are independent. This implies as described in O'Kane [2011] that

$$\hat{Z}(0,T) = \mathbb{E}\left[exp\left(-\int_t^T r_s + \lambda_s ds\right) \mid \mathcal{F}_t\right] = \mathbb{E}\left[exp\left(-\int_t^T r_s\right) exp\left(-\int_t^T \lambda_s ds\right) \mid \mathcal{F}_t\right] \tag{5.9}$$

$$= \mathbb{E}\left[exp\left(-\int_t^T \lambda_s ds\right) \mid \mathcal{F}_t\right] \cdot \mathbb{E}\left[exp\left(-\int_t^T r_s\right) \mid \mathcal{F}_t\right] \tag{5.10}$$

In Chapter 6 we will return back to this result. Note that as a result of the above relations we can write the defaultable zero coupon bond as

$$\hat{Z}(0,T) = Z(0,T)\mathbb{Q}_0(\tau > T) \tag{5.11}$$

This means that we can empirically find the yield curve as we would in a default-free environment and then afterwards account for the default risk. In the next section we will extract the yield curve and then in Chapter 6 we will adjust for the default risk.

5.3 Extracting the yield curve

From now on we will use US Treasury yield data and I refer to the Appendix in Chapter 15 for a discussion of the chosen data. The federal reserve (Fed) has publicised the treasury yield curve estimates of the Federal Reserve Board at a daily frequency from 1961 to the present and we will use the data from this source.

When extracting the yield curve from data points we can do so in many different ways. In this thesis we will use the Nelson-Siegel-Svensson parameterization of the term structure of interest rates, used e.g. in [Gilli et al., 2010]. The reasons for doing so are 1. The daily data that we use is estimated by Fed based on the Nelson-Siegel-Svensson parameterization and 2. When the Nelson-Siegel-Svensson parameterization was introduced, it quickly become popular due to its simplicity yet ability to present term structure of different forms.

The Nelson-Siegel-Svensson yield curve parameterization is given by

$$\begin{aligned}\bar{y}(t) =& \beta_0 + \beta_1 \frac{1 - e^{-t/\tau_1}}{t/\tau_1} + \beta_2 \left(\frac{1 - e^{-t/\tau_1}}{t/\tau_1} - e^{-t/\tau_1}\right) \\ &+ \beta_3 \left(\frac{1 - e^{-t/\tau_2}}{t/\tau_2} - e^{-t/\tau_2}\right)\end{aligned} \tag{5.12}$$

where the last term is the "Svensson" part of the equation allowing for additional flexibility in the yield curve.

The first term (β_0) in the above equation can be interpreted as the long run levels of interest rates, β_1 the short term component and β_2 the medium term component. τ_1 and τ_2 is the decay factors meaning the rate at which the function goes toward zero. This implies that small values of τ_1 and τ_2 produce slow decay and can better fit the curve at long maturities whereas large values produce fast decay and can better fit the curve at short maturities.

Figure 5.1 shows an example of using this Nelson-Siegel-Svensson parameterization to obtain the yield curve. We observe that the function has the very characteristic yield curve increasing with time.

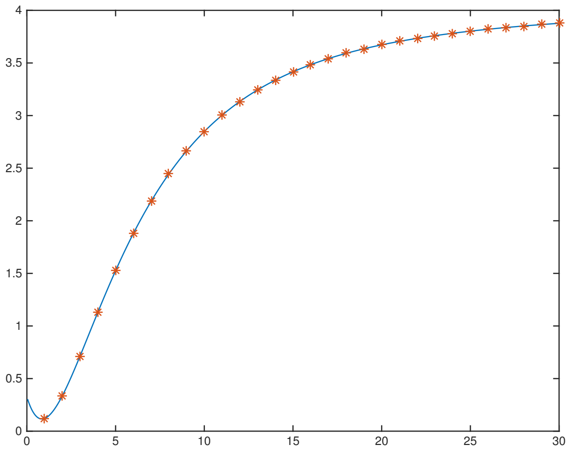

Figure 5.1: The Yield Curve 07-02-2014

However, it is also clear that between 0 and 1 the shape of the yield curve is not very well fitted which result in a weird hump. We know that the yield curve should be 0 in 0 hence I suggest a linear approximation between 0 and 1. This will give some errors when using the yield curve but given the small time interval we will use this approximation anyway. Figure 5.2 shows the yield curve with a linear approximation between 0 and 1.

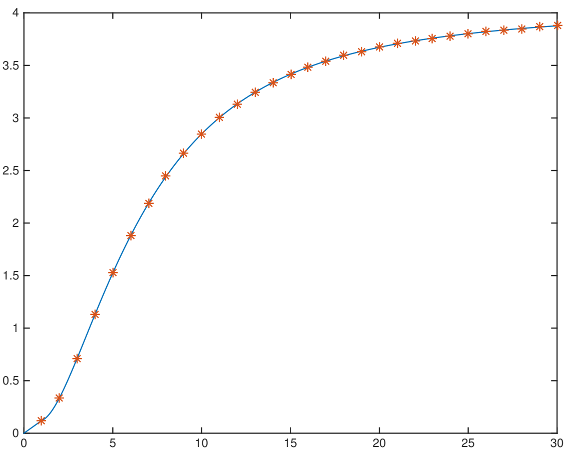

Figure 5.2: The Yield Curve 07-02-2014, with linear approximation between 0 and 1

Chapter 6

Credit Risk Modelling

In Chapter 2 and Chapter 3 we discussed the construction of a CDS and the CDS option. It is clear that the credit event - the default of the reference entity - plays an important role in pricing both the CDS and the CDS option. This chapter will focus on the way to model credit risk. Previous literature can be divided into two classes of modelling credit risk; 1. The structural approach, and 2. The reduced form approach. For several reasons which we will describe in Subsection 6.1.2 we have chosen to use the reduced form approach. This chapter will begin with an explanation of the reduced form approach and of the advantages and disadvantages by this approach. Next we will discuss the theory of modelling default intensity and we will end this chapter with simplifying assumptions and finally with a calibration of the survival probability curve.

6.1 Reduced Form Approach

The reduced form approach is less intuitive than the structural approach given that it does not relate the company defaulting to the balance sheet of the company. Instead, it considers the credit event as a stochastic event which we will see has some limitation but most importantly also many advantages.

The class of reduced form models is large but a very common model presented by [Lando, 1998]. In [Lando, 1998] he assumes that the default can be modelled as the first jump of a Poisson process. The Poisson process is used to model rare events, and since default happens only once in a company's life except for a few rare cases, it is suitable to use the features of the Poisson process.

We assume that default can happen at any time τ between now ($t = 0$) and maturity T. The following results presented in Tankov [2004] will help us understand how credit risk can be modelled. Recall the definition of a Poisson process

> **Definition:** Let $(\tau_i)_{i\geq 1}$ be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^{n} \tau_i$. Then process $(N_t)_{t\geq 0}$ defined by
>
> $$N_t = \sum_{n\geq 1} \mathbf{1}_{t\geq T_n} \tag{6.1}$$
>
> is called a Poisson process with intensity λ.

The above definition means that the Poisson process is a counting process, counting the number of random times T_n that occur between 0 and t, where $(T_n - T_{n-1})_{n\geq 1}$ is an independent and identically distributed (i.i.d) sequence of exponential variables. The above definition is equivalent to the following statement:

> A Poisson process N_t with intensity $\lambda > 0$ is a non-decreasing, integer valued process with initial value $N_0 = 0$ whose increments are independent and for all $0 \leq t < T$ they satisfy
>
> $$\mathbb{P}\left(N_T - N_t = n\right) = \frac{1}{n!}(T-t)^n \lambda^n e^{-\lambda(T-t)} \tag{6.2}$$

The intensity parameter λ is interesting to us since this is the term determining the jumps and thereby the defaults.

One can use different approaches in treating the intensity but in this thesis we will need an extension of the Poisson process. Next subsection will present the Cox-process which allows a stochastic intensity.

6.1.1 Cox Process

Following [Lando, 1998] one can simulate the first jump τ of the Poisson process by e.g. letting E be a unit exponential random variable and define:

$$\tau = \inf\left\{t \ : \ \int_0^t \lambda(u)du \geq E\right\} \tag{6.3}$$

As mentioned, the Cox process is a generalization of the Poisson process by allowing for random intensity. Denote the random intensity by $\lambda(X_s)$, X being an R^d-valued stochastic process and $\lambda : R^d \to [0, \infty)$ a non-negative, continuous function. [Lando, 1998] then define the default time τ as follows:

$$\tau = \inf\left\{t \ : \ \int_0^t \lambda(X_s)ds \geq E\right\} \tag{6.4}$$

The above expression implies that the default time can be thought of as the first jump of a Cox process with intensity process $\lambda(X_s)$ instead of the former definition of jump with a deterministic intensity function in Equation 6.3. When $\lambda(X_s)$ is very large the integral grows faster, reaching the value E faster implying that the probability of τ being small becomes higher. Following [Lando, 1998] we get the following relationships:

$$\mathbb{P}\left(\tau > t \mid (X_s)_{0\leq s<t}\right) = exp\left(-\int_0^t \lambda(X_s)ds\right) \tag{6.5}$$

$$\mathbb{P}\left(\tau > t\right) = Eexp\left(-\int_0^t \lambda(X_s)ds\right) \tag{6.6}$$

The above expression will become useful in the next chapter where we model the intensity and end up with the risk-neutral probability of a company surviving up until a specific point in time. Notice however that if we set the maturity $T = \infty$ then the integral $\int_0^t \lambda(X_s)ds$ will eventually become larger than E hence the theory claims that all reference entity will eventually default. This is not a problem since a reference entity with a very healthy economy will have a very slowly growing intensity function $\lambda(X_s)$ meaning that the reference entity defaults in the very far future[1].

[1] See [Lando, 1998] for further explanation

6.1.2 Advantages and Disadvantages of the Structural Approach

The reduced form approach of modelling credit risk has several advantages compared to the structural approach. First, it allows for unexpected default and not only for default at maturity which is more realistic. Second, the reduced form approach is not relying on unobserved information information known inside the company but unobservable to the shareholders. Instead, default is stochastic and can therefore be perceived by any market participant. One disadvantage of the reduced form approach is the lack of intuition as mentioned earlier. It would be more intuitive if the default of the company depended on the balance-sheet as with the structural approach but this is not the case for the reduced form approach. However, the intensity function $\lambda(X_s)$ is different for every reference entity hence $\lambda(X_s)$ implicitly maps the information in the market about the company and their balance sheet.

6.2 Theory of Modelling Intensity

The following sections will attempt to model the intensity. We will begin with a theoretical derivation of the way to model the intensity using the Cox-Ingersoll-Ross(CIR) process. In the next section we will empirically model the survival probability of three countries with different credit rating. This empirical derivation will be necessary when we wish to price the CDS option.

In Chapter 5 we discussed the defaultable zero coupon bond paying 1 as long as the reference entity has not yet defaulted, which was given by

$$D(0,T) = \mathbb{E}^{\mathbb{Q}}\left[exp(-\int_0^T r_s ds)1_{\tau>T}\right] \tag{6.7}$$

In that chapter we argued that due to independence of the short rate r_t and the default intensity λ_t we were allowed to write the zero coupon bond as

$$D(0,T) = \mathbb{E}^{\mathbb{Q}}\left[exp(-\int_0^T r_s ds)\right]\mathbb{E}^{\mathbb{Q}}\left[exp(-\int_0^T \lambda_s ds)\right] \tag{6.8}$$

The only thing we still need to model is the last term in the above equation, which is the risk neutral expectation of the default intensity also known as the risk neutral probability of default. Using the work of [Lando, 1998] and the results presented in section 6.1 the risk neutral probability of surviving (no default) between time t and T is given by:

$$\mathbb{Q}(N_T - N_t = 0) = \mathbb{Q}(t,T) = 1_{\tau>t}\mathbb{E}^{\mathbb{Q}}\left[e^{(-\int_t^T \lambda_s ds)} \mid \mathcal{G}_t\right] \tag{6.9}$$

where $\lambda_s = \lambda(X_s)$ and $\mathcal{G}_t = \sigma(X_s : 0 < s < t)$ is the sigma-algebra generated by the state variables. Notice that the above equation suggests the same relationship between the intensity and the survival probabilities as with the zero coupon bond and the short rate (See [O'Kane, 2011]). In Chapter 5 we discussed different term structure models and due to the described similarities it is common to use the CIR process to model the intensity. The next subsection will explain a theoretical method to model the default intensity using the (CIR) process.

6.2.1 CIR

The Cox-Ingersoll-Ross model (CIR) described in [Cox et al., 1985] is a one-factor diffusion model which is often used when modelling the short rate. We wish to model the intensity and due to the above described similarities we can use the CIR-model. The model assumes that the intensity follows a square root process:

$$d\lambda_t = \kappa(\bar{\lambda} - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t \tag{6.10}$$

where W_t is a Geometric Brownian Motion and κ, $\bar{\lambda}$ and $\sigma > 0$. We see that the first term make the intensity mean reverting meaning that when λ is below/above its "mean" $\bar{\lambda}$ it will be pushed up/down by κ. σ is the volatility parameter but the volatility will also depend on the level of intensity through the square-root term. From the theory of the CIR-process when modelling the short rate [Munk, 2011] we know that the model has a closed form solution given by

$$\mathbb{Q}_t(\tau > T) = \mathbf{1}_{\tau>t}A(t,T)e^{B(t,T)\lambda_t} \tag{6.11}$$

$$A(t,T) = \left(\frac{2\gamma e^(\kappa+\gamma)(T-t)\frac{1}{2}}{(\kappa+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}\right)^{\frac{2\kappa\bar{\lambda}}{\sigma^2}} \tag{6.12}$$

$$B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(\kappa+\gamma)(e^{\gamma(T-t)}-1)+2\gamma} \tag{6.13}$$

$$\gamma = \sqrt{\kappa^2 + 2\sigma^2} \tag{6.14}$$

This is how we would model the intensity if it was necessary to know the exact intensity paramter λ. We will see in the next section that in our case this is not necessary since we are only interested in the survival probability. Nevertheless the CIR-model will give us some intuition to the way we will find the survival probability. One might argue that we could have modelled the intensity in many different ways e.g. by using the Ornstein-Uhlenbeck process as described in Cariboni and Schoutens [2009]. However, since we are not directly modelling the intensity but only use this section for intuition we will without discussion continue with the CIR process.

6.3 Empirically Modelling - Survival Probability

In this section we will empirically extract the survival probability $\mathbb{Q}_t(\tau > t)$ from CDS spreads by calibration. This is a necessary step in the empirical derivation of the CDS option prices given that the option price model in Chapter 8 is calculated under the risk-neutral probability measure. Therefore the CDS option price depends on the risk neutral probability of the credit entity surviving up until the relevant time point. We will use market data on CDS spreads resulting in the best fit survival-curve for those entities.

In the previous section we presented the survival probability function:

$$\mathbb{Q}_t(\tau > T) = 1_{\tau>t}\mathbb{E}^{\mathbb{Q}}\left[e^{(-\int_t^T \lambda_s ds)} \mid \mathcal{G}_t\right]. \tag{6.15}$$

If we were interested in the specific intensity function it would be natural to use the CIR-model as presented in the above section. However, there are several reasons why we will only use the CIR-model as an inspiration when we extract the survival probability curve. Firstly, having data on the intensity requires that the reference entity default, which only happens once in the life of the reference entity. This implies that modelling the time in between jump (defaults) would be very difficult and would probably result in vague effects. Secondly, this thesis is only interested in the survival probability function so that we can price the CDS. Therefore, it is not that important when the jumps occur but rather what is the probability of the jumps occurring. This means that we can merely calibrate the best-fit survival probability function and not worry above the intensity parameter which is a lot simpler.

6.3.1 Calibration of Survival Probability

We will calibrate the survival probability by using the same method as in [Feldhütter and Nielsen, 2012]. Given that neither this calibration nor this section in general are the keystones in this thesis, we will rely on some derivation from [Feldhütter and Nielsen, 2012] and refer to further analysis in [O'Kane, 2011]

In the market we observe the CDS spreads given by

$$S(t,T) = \frac{Prot(t,T)}{Prem(t,T)} \tag{6.16}$$

In the above equation $Prot(t,T)$ and $Prem(t,T)$ refers to the protection leg and premium leg respectively. It is shown in [Feldhütter and Nielsen, 2012] that given our assumption of independence between the risk free interest rate r_s and the default time τ plus the additional assumption of constant recovery rate, δ, the protection and premium leg are given by

$$Prot(t,T) = (1-\delta)\sum_{j=1}^{M} P(t, \frac{t_{j-1}+t_j}{2}) \cdot (\mathbb{Q}_t(\tau > t_{j-1}) - \mathbb{Q}_t(\tau > t_j)) \tag{6.17}$$

$$\begin{aligned} Prem(t,T) = &\sum_{j=1}^{M} P(t, \frac{t_{j-1}+t_j}{2}) \cdot \frac{t_j - t_{j-1}}{2} \cdot (\mathbb{Q}_t(\tau > t_{j-1}) - \mathbb{Q}_t(\tau > t_j)) \\ &+ \sum_{j=1}^{M} P(t, t_j) \cdot (t_j - t_{j-1}) \end{aligned} \tag{6.18}$$

From the above expression we clearly see that the survival probability plays an important role in both the protection and the premium leg. Fortunately, not much else is necessary to describe the premium and the protection leg and knowing the yield curve and observing the CDS spreads will therefore make it possible to calibrate the survival probability.

Assume, as in [Feldhütter and Nielsen, 2012], that the probability at time t of an entity surviving up until time s will take the form

$$\mathbb{Q}_t(\tau > s) = \frac{1}{1+\alpha_2+\alpha_4}(e^{-\alpha_1(t-s)} + \alpha_2 e^{-\alpha_3(t-s)^2} + \alpha_4 e^{-\alpha_5(t-s)^3}) \tag{6.19}$$

The above equation has several features that justify using this formula for the survival probability function. First, note that since the above expression describes a probability $\mathbb{Q}_t(\tau > s)$, the value should lie between 0 and 1, which will be the result when we restrict the parameters $\alpha_1, ..., \alpha_5$ to be non-negative. Secondly, the model presented by [Feldhütter and Nielsen, 2012] is more flexible than the CIR-model allowing for a better estimate of the survival probability.

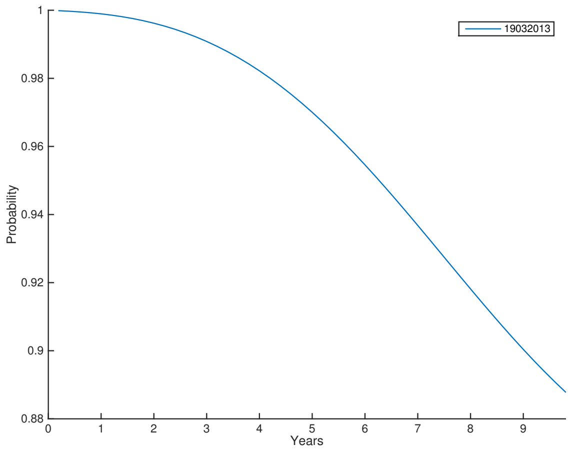

Figure 6.1: Survival probability for Germany 19-03-13

The calibrated survival probability function $s \mapsto \mathbb{Q}_t$ is then the function that minimizes the following problem:

$$\min_T \left(\frac{Prot(t,T)/Prem(t,T) - S_{obs}}{S_{obs}} \right)^2 \tag{6.20}$$

It is necessary to perform the above calibration for each reference entity and for each time point. To illustrate this the following subsection will show three examples of the calibrated survival probability function. We will use Germany, Italy and Egypt as they are of different credit-rating; high, medium and low respectively.

Example: High credit rating: Germany

Consider the setting in which a buyer deliberates acquiring a CDS option with reference entity the Federal Republic of Germany on 19/03/2013. To price this option the buyer will need to know the probability of survival of the Germany at specific future time points, the future time points being the ones at which the premium payments are made. By the probability of survival of Germany we mean the probability of Germany not defaulting on their debt.

Let us for a brief moment discuss our expectations to the evolvement of the survival probability function for Germany. Both Standard&Poors and Moody give Germany the highest credit rating, AAA and Aaa respectively (See Tra). This implies that the economy of Germany is very stable and we would therefore expect a very high survival probability looking into the future. Recent data (See Tra) also indicates a very stable economy having a high government budget value of 6.17 EUR Billion and a relatively low Debt-to-GDP ratio of 74.7 percentage compared to e.g. Egypt which will be discussed later. Looking at the survival probability curve from 19/03/2013 and 10 years ahead in figure 6.1 we see that the survival probability is indeed high. 10 years from 19/03/2013 the probability of Germany not defaulting on their loans is 0.88.

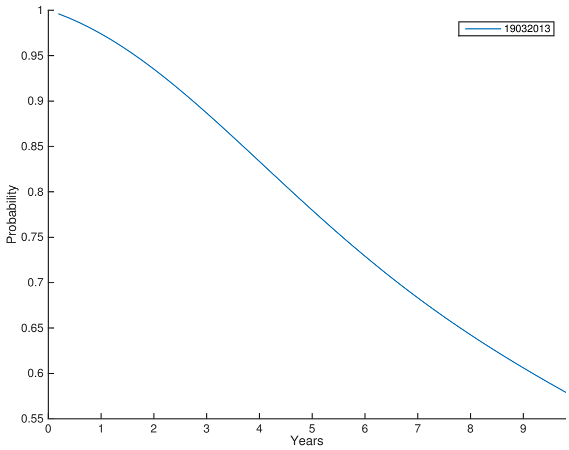

Figure 6.2: Survival probability for The republic of Italy 19-03-13

Example: Medium credit rating: The republic of Italy

Consider the setting in which a buyer deliberates acquiring a CDS option of the Italian Republic at 19/03/2013. In the last couple of years Italy has been announced as "the new Greece" meaning that they face several of the same economic problems as Greece is now experiencing(See Tra). Italy is rated BBB- by Standard&Poors which is the fifth credit rating indicating that they are in fact facing severe economic problems. Comparing to Germany we also see that Italy has a Debt-to-GDP rate of 123 percent and their government budget value is -12.45 billion EUR implying an unhealthy economy. Therefore we would expect a low survival probability at least compared to Germany. In Figure 6.2 we see that the survival probability looking 10 years into the future is around 0.57 percent indicating a medium high probability of default.

Example: Low credit rating: The republic of Egypt

Finally, we consider the setting in which a buyer deliberates acquiring a CDS option of the Arab Republic of Egypt on 19/03/2013. With a credit rating of B- by Standard&Poors, a Debt-to-GDP ratio of 81.7 percent and a government budget value of -22.44 billion EUR we would expect to see a very low survival probability curve. 10 years from 19/03/2013 we see in Figure 6.3 that the probability of Egypt not having to restructure their debt is below 0.33. The reader might wonder why the survival probability and credit rating is higher for Italy than for Egypt considering the higher Debt-to-GDP in Italy. This can be explained by a lot of circumstances. By way of example the fact that Italy is part of EU and thereby can receive help from e.g. Germany will be mirrored in the survival probability curve.

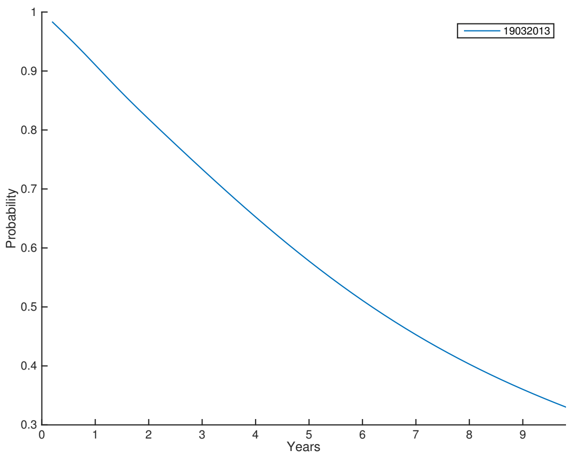

Figure 6.3: Survival probability for The republic of Egypt 19-03-2013

Part II

Pricing CDS and CDS options

Chapter 7

Pricing a CDS

In the second part of this thesis we will discuss the price of the CDS contract and afterwards the price of the CDS option based on [Brigo and Morini, 2005] and [Brigo, 2005]. In this chapter I will describe the payout from the CDS contract and afterwards price the CDS contract as the risk-neutral expectation of the payout. At the end of this chapter I will find the fair CDS rate/spread $R_a, b(t)$ from the buyer of the protection to the seller making the value of the CDS contract equal to 0 at time t. This will lead up to the subsequent chapter in which I will use that the CDS spread is the underlying of the CDS option.

7.1 Payout from a CDS

This section will determine the payout from the CDS based on the definition of a CDS contract in Chapter2 We consider a forward starting CDS with a protection buyer A, a protection seller B and a reference entity named C. Buyer A pays the premium leg in rates R at times $T_{a+1}, ..., T_b$, $[a, b]$ being the time interval in which A seeks protection from B. In exchange of these payments the buyer A receives the protection leg which is a single payment LGD (loss given default). In this case LGD is assumed to be deterministic and $LGD = 1 - REC$ where REC is the recovery rate which is assumed to be deterministic and the notional is set to 1. A receives this protection if the time of default, τ, occurs in the time interval $[a, b]$ and as a consequence the rates paid by A will stop. We saw an implicit illustration of this contract in Chapter 2 Figure 3.2 and Figure 3.3.

Let us now describe the discounted payoff from this CDS seen from the protection seller's point of view. The discounted payoff can be defined as the discounted premium leg subtracting the discounted protection leg. Formally, the discounted payoff of this CDS contract at time t will be given as

$$\Pi_{CDS_{a,b}}(t) = \sum_{i=a+1}^{b} D(t, T_i)\alpha_i R \mathbf{1}_{\{\tau \geq T_i\}} + D(t, \tau)(\tau - T_{\beta(\tau)-1}) R \mathbf{1}_{\{T_a < \tau < T_b\}} - \mathbf{1}_{\{T_a < \tau \leq T_b\}} D(t, \tau) LGD \tag{7.1}$$

The above equation consists of three parts. The first two terms are the premium leg. The first term consists of all the payments from A to B up until the point of default. Here α_i denotes the year fraction between T_{i-1} and T_i. The second term of the premium leg captures the fact that the default will often not occur exactly on one of the payment dates. Therefore, the second term describes the last payment from A just before the default hence $t \in [T_{\beta(t)-1}, T_{\beta(t)}]$ for all t and therefore $T_{\beta(\tau)-1}$ is the last time period before the default. The last part of the equation is the discounted protection leg where A receives LGD at time τ from B.

For simplicity we can sometimes consider a different payout structure for the CDS. In that case, the protection payment LGD will not be paid at the exact default time τ but will instead be postponed until the subsequent time point, in this case $T_{\beta(\tau)}$. The difference between this approach and the previous method in Equation 7.1 depends on the length between the times of payment and if the payments occur every three or six month, the difference will be at most a few months. By taking this approach the first part of Equation 7.1 can be removed and the discounted payout from the CDS will be given as

$$\Pi_{PCDS_{a,b}}(t) = \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau \geq T_i\}} - \sum_{i=a+1}^{b} \mathbf{1}_{\{T_{i-1} \leq \tau \leq T_i\}} D(t,T_i) LGD \tag{7.2}$$

where PCDS denotes that this approach is called the postponed CDS. Later in Chapter CDS option price we will use the postponed payout due to its simplicity when deriving CDS option prices.

7.2 The Price of a CDS

Now that we have defined the discounted payout of a CDS, the next step will be pricing this payout.

Let $CDS_{a,b}(t,R,LGD)$ be the price at time t of the standard CDS flow to the protection seller B described above. This price depends on assumptions about interest rate dynamics and the default time τ as discussed in Chapter 6. In this setting the intensity is a stochastic $\mathcal{F}_t$-adapted continuous process where $\mathcal{F}_t$ is all relevant information without default. Default is then modelled as the first jump time of a COX process with the given intensity process mentioned in Chapter 6.

Following Brigo [2005], using the no-arbitrage pricing strategy, the price of the CDS is given by the risk neutral expectation of its discounted payout:

$$CDS_{a,b}(t,R,LGD) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_{CDS_{a,b}} \mid \mathcal{G}_t\right] \tag{7.3}$$

In the above expression $\mathbb{Q}$ is the risk-neutral equivalent martingale measure and the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \leq t)$ represent all available information up to t. Default is then modelled as a $\mathcal{G}_t$-stopping time.

By arguing that $\Pi_{CDS_{a,b}}$ is measurable with respect to $\mathcal{G}_t$ we can use Bielecki and Rutkowski [2002] and rewrite the above expression as

$$CDS_{a,b}(t,R,LGD) = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t(\tau > t)}\mathbb{E}\left[\Pi_{CDS_{a,b}}(t) \mid \mathcal{F}_t\right] \tag{7.4}$$

The argument for $\Pi_{CDS_{a,b}}(t)$ being measurable with respect to $\mathcal{G}_t$ is as follows: The first part of Equation 7.1 is measurable w.r.t $\mathcal{F}_t$ since default is always greater than T_i. The second part is measurable with respect to $\sigma(\{\tau < u\}, u \leq t)$ since default occurs before t.

To make the price of the CDS more explicit we can substitute the payouts into the equation, resulting in

$$\begin{aligned} CDS_{a,b}(t,R,LGD) =& \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t(\tau > t)} \\ & \mathbb{E}\left[D(t,\tau)(\tau - T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R\mathbf{1}_{\{\tau\geq T_i\}} - \mathbf{1}_{\{T_a<\tau\leq T_b\}}D(t,\tau)LGD\right] \end{aligned} \tag{7.5}$$

Using the fact that LGD and R are deterministic and that there exists linearity of expectations we get

$$\begin{aligned}CDS_{a,b}(t,R,LGD) =& \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t(\tau>t)}\\ &\Big(-LGD\mathbb{E}_t\left[\mathbf{1}_{\{T_a<\tau\leq T_b\}}D(t,\tau)\right]\\ &+R\mathbb{E}_t\left[D(t,\tau)(\tau-T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}}\right]+R\sum_{i=a+1}^{b}\mathbb{E}_t\left[D(t,T_i)\alpha_i\mathbf{1}_{\{\tau\geq T_i\}}\right]\Big)\end{aligned} \tag{7.6}$$

By the same arguments we can obtain the price of the postponed CDS given as

$$\begin{aligned}PCDS_{a,b}(t,R,LGD) =& \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t(\tau>t)}\cdot\\ &\left\{-LGD\sum_{i=a+1}^{b}\mathbb{E}_t\left[\mathbf{1}_{\{T_{i-1}\leq\tau\leq T_i\}}D(t,T_i)\right]+R\sum_{i=a+1}^{b}\mathbb{E}_t\left[D(t,T_i)\alpha_i\mathbf{1}_{\{\tau\geq T_i\}}\right]\right\}\end{aligned} \tag{7.7}$$

7.3 The Fair Rate $R_{a,b}(t)$ also Called the Par CDS spread

The purpose of the next calculations is finding the fair rate for a CDS meaning the rate that makes the price of the CDS to the protection seller equal to zero. Remember that the CDS contract costs nothing to enter. Let us consider the CDS from equation 7.6 and let us define the CDS forward rate as $R_{a,b}(t)$. Then the fair rate of a CDS at time t is the forward rate $R_{a,b}(t)$ that results in

$$CDS_{a,b}(t,R_{a,b}(t),LGD)=0 \tag{7.8}$$

Solving for the fair rate the following must hold

$$\frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t(\tau>t)}\mathbb{E}\left[\Pi_{RCDS_{a,b}}(t)\mid\mathcal{F}_t\right]=0 \tag{7.9}$$

The indicator function in front of the conditional expectation is 0 when $\tau<t$ and 1 when $\tau>t$. This means that when $\tau<t$ the equation holds no matter the forward rate $R_{a,b}$. Therefore, the following fair forward rate will strictly speaking only hold when $\tau>t$. Given that $\tau>t$ and using 7.6 the following must hold [1]

$$\begin{aligned}0 =& \mathbb{E}\left[\prod_{RCDS_{a,b}}(t)\mid\mathcal{F}_t\right]\\ &\Rightarrow\\ 0 =& -LGD\mathbb{E}_t\left[\mathbf{1}_{\{T_a<\tau\leq T_b\}}D(t,\tau)\right]\\ &+R_{a,b}(t)\sum_{i=a+1}^{b}\mathbb{E}_t\left[D(t,T_i)\alpha_i\mathbf{1}_{\{\tau\geq T_i\}}\right]\\ &+R_{a,b}(t)\mathbb{E}_t\left[D(t,\tau)(\tau-T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}}\right]\end{aligned}$$

[1] $\mathbb{E}_t[.]=\mathbb{E}[.\mid\mathcal{F}_t]$ to ease notation

Then simple rearranging yields

$$\begin{aligned}&LGD\mathbb{E}_t\left[\mathbf{1}_{\{T_a<\tau\le T_b\}}D(t,\tau)\right]\\&=R_{a,b}(t)\cdot\left(\sum_{i=a+1}^{b}\mathbb{E}_t\left[D(t,T_i)\alpha_i\mathbf{1}_{\{\tau\ge T_i\}}\right]+\mathbb{E}_t\left[D(t,\tau)(\tau-T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}}\right]\right)\end{aligned}\tag{7.10}$$

This implies that the fair rate $R_{a,b}(t)$ is given by

$$R_{a,b}(t)=\frac{LGD\mathbb{E}_t\left[\mathbf{1}_{\{T_a<\tau\le T_b\}}D(t,\tau)\mid\mathcal{F}_t\right]}{\sum_{i=a+1}^{b}\mathbb{E}_t\left[D(t,T_i)\alpha_i\mathbf{1}_{\{\tau\ge T_i\}}\mid\mathcal{F}_t\right]+\mathbb{E}_t\left[D(t,\tau)(\tau-T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}}\mid\mathcal{F}_t\right]}\tag{7.11}$$

To simplify the above equation denote $P(t,T)$ as the price at time t of a default-free zero coupon bond maturating at time T and let $\bar{P}(t,T)$ be the corresponding price of a defaultable zero-coupon bond. Using that $\mathbf{1}_{\{\tau>t\}}$ is measurable wrt. $\mathcal{G}_t$ and the [Bielecki and Rutkowski, 2002] we have the following equations, which will be useful in simplifying the above equation

$$\begin{aligned}\bar{P}(t,T)=&\mathbb{E}\left[D(t,T)\mid\mathcal{G}_t\right]\\\bar{P}(t,T)\mathbf{1}_{\{\tau>T\}}=&\mathbb{E}\left[D(t,T)\mathbf{1}_{\{\tau>T\}}\mid\mathcal{G}_t\right]\\=&\frac{\mathbb{E}\left[D(t,T)\mathbf{1}_{\{\tau>t\}}\mid\mathcal{F}_t\right]}{\mathbb{Q}\left(\tau>t\mid\mathcal{F}_t\right)}\end{aligned}\tag{7.12}$$

This implies that the fair forward rate of a CDS is given by

$$R_{a,b}(t)=\frac{LGD\mathbb{E}_t\left[\mathbf{1}_{\{T_a<\tau\le T_b\}}D(t,\tau)\mid\mathcal{F}_t\right]}{\sum_{i=a+1}^{b}\alpha_i\mathbb{Q}\left(\tau>t\mid\mathcal{F}_t\right)\bar{P}(t,T_i)+\mathbb{E}_t\left[D(t,\tau)(\tau-T_{\beta(\tau)-1})R\mathbf{1}_{\{T_a<\tau<T_b\}}\mid\mathcal{F}_t\right]}\tag{7.13}$$

Doing the exact same calculations but instead using the postponed CDS yields the following fair forward rate

$$R^P_{a,b}(t)=\frac{LGD\sum_{i=a+1}^{b}\mathbb{E}\left[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}\mid\mathcal{F}_t\right]}{\sum_{i=a+1}^{b}\alpha_i\mathbb{Q}\left(\tau>t\mid\mathcal{F}_t\right)\bar{P}(t,T_i)}\tag{7.14}$$

Chapter 8

Pricing CDS Option

In this chapter I will discuss a theoretic formula for pricing a CDS option based on the derivation of [Brigo and Morini, 2005]. This pricing formula will be the theoretical center of my thesis and the following chapters will verify the pricing method empirically. When pricing an option on an asset or even, on as in this case, another derivative the first step will be to look at the underlying asset or derivative. We aim to price CDS option and we should therefore be using the CDS as the underlying "asset". Based on [Brigo, 2005] we implicitly use the CDS contract as the underlying by using the fair rate R as the underlying asset. As discussed earlier in Chapter 3, buying a Call option on a CDS the buyer obtains the right to buy the CDS at a pre-specified time point. At the time of the acquisition of the CDS option the fair rate is set such that the expected value of the CDS should be zero thus eliminating the opportunity for arbitrage. As the value of the CDS evolves after the acquisition of the option, the buyer will exercise the option when it is In The Money. This is equivalent to the fair rate at the expiration date of the option being larger than the pre-specified fair rate. Therefore, the value of the option will implicitly depend on the dynamics of the fair rate R.

8.1 The Payout from a CDS Option

As described in Chapter 3, a CDS option is the right of A to enter into (or sell) a CDS at the option maturity, time $T_a > t$, paying to (receiving from) B a pre-specified strike rate $R = K$ for a protection payment LGD to be received/paid in case of default of C. For simplicity this section will focus on the call option hence A has the right to enter into a CDS at its first reset time $T_a > t$. It is clear that the option will be exercised only when the expected payout to A is positive at T_a.

In section 7.6 the differences between postponed and not-postponed CDS were described and given that both types of CDSs can be used as the underlying asset they are both relevant. However, this section will use the postponed version because of the simpler notation which means that from this point $R^P_{a,b} = R_{a,b}$ omitting the P.

Given the described characteristics of a CDS option from Chapter 3 and Chapter 7 the discounted payout at time t of an option on a CDS is

$$\Pi_{Call,PCDS_{a,b}}(t, K) = D(t, T_a)\left[-CDS_{a,b}\left(T_a, K, LGD\right)\right]^+ \tag{8.1}$$

In the above expression $D(t, T_a)$ is the discount factor from T_a - the point where A can exercise -

back to t. $[-CDS_{a,b}(T_{a,k,LGD})]^+$ is the max of the pay-off from the postponed CDS and 0. When describing the value of the option we are observing the value from the buyer's point of view hence buyer A will (if exercising) pay the rate $R_{a,b}$ to B in return for the protection if a credit event occurs. This means that the negative sign in the maximum comes from the fact that the pay-off to the seller of the CDS B would be negative in the case where A chooses to exercise. Therefore the value of this maximum determines whether or not A will exercise.

The fair rate of a CDS, $R_{a,b}(T_a)$, at time T_a makes the underlying CDS value exactly equal to 0 by definition, hence we can rewrite the above expression without changing the value resulting in

$$\prod_{Call,CDS_{a,b}}(t,K) = D(t,T_a)\left[CDS_{a,b}\left(T_a, R_{a,b}(T_a), LGD\right) - CDS_{a,b}\left(T_a, K, LGD\right)\right]^+ \tag{8.2}$$

Next step will be to insert the explicit formula of the postponed CDS payout and simplify the expression. To achieve this simplification consider first $CDS_{a,b}(T_a, R_{a,b}(T_a), LGD)$ and $CDS_{a,b}(T_a, K, LGD)$ separately

$$\begin{aligned} CDS_{a,b}(T_a, R_{a,b}, LGD) =& \frac{\mathbf{1}_{\{\tau>T_a\}}}{\mathbb{Q}_{T\,a}(\tau > T_a)}\{-LGD\sum_{i=a+1}^{b}\mathbb{E}_{T\,a}\left[\mathbf{1}_{T_{i-1}\leq\tau\leq T_i}\right] \\ &+ R_{a,b}(T_a)\sum_{i=a+1}^{b}\mathbb{E}_{T\,a}\left[D(T_a,T_i)\alpha_i 1_{\{\tau\geq T_i\}}\right]\} \end{aligned} \tag{8.3}$$

$$\begin{aligned} CDS_{a,b}(T_a, K, LGD) =& \frac{\mathbf{1}_{\{\tau>T_a\}}}{\mathbb{Q}_{T\,a}(\tau > T_a)}\{-LGD\sum_{i=a+1}^{b}\mathbb{E}_{T\,a}\left[\mathbf{1}_{T_{i-1}\leq\tau\leq T_i}\right] \\ &+ K\sum_{i=a+1}^{b}\mathbb{E}_{T\,a}\left[D(T_a,T_i)\alpha_i 1_{\{\tau\geq T_i\}}\right]\} \end{aligned} \tag{8.4}$$

From the two expressions above it is clear that the difference between those two will be the parts containing $R_{a,b}(T_a)$ and K respectively. Additionally, the other terms will cancel out yielding

$$\begin{aligned} &CDS_{a,b}(T_a, R_{a,b}(T_a), LGD) - CDS_{a,b}(T_a, K, LGD) \\ &\quad= \frac{\mathbf{1}_{\{\tau>T_a\}}}{\mathbb{Q}_{T\,a}(\tau > T_a)}\{R_{a,b}(T_a)\sum_{i=a+1}^{b}\mathbb{E}_{T\,a}\left[D(T_a,T_i)\alpha_i 1_{\{\tau\geq T_i\}}\right] \\ &\quad - K\sum_{i=a+1}^{b}\mathbb{E}_{T\,a}\left[D(T_a,T_i)\alpha_i 1_{\{\tau\geq T_i\}}\right]\} \end{aligned} \tag{8.5}$$

$$= \frac{\mathbf{1}_{\{\tau>T_a\}}}{\mathbb{Q}_{T\,a}(\tau > T_a)}\{(R_{a,b}(T_a) - K)\sum_{i=a+1}^{b}\mathbb{E}_{T\,a}\left[D(T_a,T_i)\alpha_i 1_{\{\tau\geq T_i\}}\right]\} \tag{8.6}$$

This implies that the payout from a CDS Call option will be given by

$$\Pi_{Call,RCDS,a,b}(t,K) = \frac{\mathbf{1}_{\{\tau>T_a\}}}{\mathbb{Q}_{T_a}(\tau > T_a)}\left[D(t,T_a)\left(R_{a,b}(T_a) - K\right)^+\sum_{i=a+1}^{b}\mathbb{E}_{T_a}\left[D(T_a,T_i)\alpha_i\mathbf{1}_{\{\tau\geq T_i\}}\right]\right] \tag{8.7}$$

Note that this is the payout to the buyer of the call option A. The buyer A obtains value from the option agreement only in the case where the value of the CDS at T_a is higher than the expected value of the CDS at time t. This means that buying the CDS at time T_a at the time t price will imply that the buyer A will pay a fixed rate $R = K$ which is lower than the fair rate $R_{a,b}$ calculated at time T_a.

8.2 Arriving at a Price of a CDS Option

In the last section we looked at the payout from the CDS option to the buyer of the option A. The purpose of this section is to derive a price of a CDS option. Using the fact that the price of a payout is the risk-neutral expectation to this payout, the price of a CDS option will be given by

$$\begin{aligned} CDSO(t, K, LGD) =& \mathbb{E}^{\mathbb{Q}}\left[\prod_{Call}(t, K) \mid \mathcal{G}_t\right] \\ =& \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t(\tau > t)}\mathbb{E}^{\mathbb{Q}}\left[\prod_{Call}(t, K) \mid \mathcal{F}_t\right] \end{aligned} \tag{8.8}$$

The above rearrangement is identical to the rearrangement in Chapter 7 and consequently there is no need for further clarification. Our next step will be to insert the payout from the CDS option explicitly.

$$CDSO(t, K, LGD) = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t(\tau > t)}\mathbb{E}^{\mathbb{Q}}\left[\prod_{Call}(t, K) \mid \mathcal{F}_t\right] \tag{8.9}$$

$$= \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t(\tau > t)} \cdot \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbf{1}_{\{\tau>T_a\}}}{\mathbb{Q}_{T_a}(\tau > T_a)}\left[D(t, T_a)\left(R_{a,b}(T_a) - K\right)^+ \sum_{i=a+1}^{b} \mathbb{E}_{T_a}\left[D(T_a, T_i)\alpha_i \mathbf{1}_{\{\tau \geq T_i\}}\right]\right] \mid \mathcal{F}_t\right] \tag{8.10}$$

Notice that from this point onwards we will write $\mathbb{E}_t^{\mathbb{Q}}$ where t is suppressing that the expectation is taking on $\mathcal{F}_t$ unless another notation will be more informative. We can simplify the above expression by using some relatively straight forward conditional mean calculations. First, we condition on $\mathcal{F}_{T_a}$ which is bigger than $\mathcal{F}_t$ given that $t < T_a$ and then we perform the following suitable rearrangements.

$$\begin{aligned} & CDSO(t, K, LGD) \\ =& \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t(\tau > t)} \cdot \\ & \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{E}_{T_a}^{\mathbb{Q}}\left[\frac{\mathbf{1}_{\{\tau>T_a\}}}{\mathbb{Q}_{T_a}(\tau > T_a)}\left[D(t, T_a)\left(R_{a,b}(T_a) - K\right)^+ \sum_{i=a+1}^{b} \mathbb{E}_{T_a}\left[D(T_a, T_i)\alpha_i \mathbf{1}_{\{\tau \geq T_i\}}\right]\right]\right]\right] \end{aligned} \tag{8.11}$$

$$\begin{aligned} =& \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t(\tau > t)} \cdot \\ & \mathbb{E}_t^{\mathbb{Q}}[\frac{1}{\mathbb{Q}_{T_a}(\tau > T_a)} D(t, T_a)\left(R_{a,b}(T_a) - K\right)^+ \mathbb{E}_{T_a}^{\mathbb{Q}}\left[\mathbf{1}_{\{\tau>T_a\}}\left[\sum_{i=a+1}^{b} \mathbb{E}_{T_a}\left[D(T_a, T_i)\alpha_i \mathbf{1}_{\{\tau \geq T_i\}}\right]\right]\right]] \end{aligned} \tag{8.12}$$

$$=\frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t\left(\tau>t\right)}\cdot$$
$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\mathbb{Q}_{T_a}\left(\tau>T_a\right)}D(t,T_a)\left(R_{a,b}(T_a)-K\right)^{+}\sum_{i=a+1}^{b}\mathbb{E}_{T_a}\left[D(T_a,T_i)\alpha_i\mathbf{1}_{\{\tau>T_i\}}\right]\cdot\mathbb{E}^{\mathbb{Q}}_{T_a}\left[\mathbf{1}_{\{\tau>T_a\}}\right]\right] \tag{8.13}$$

$$=\frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t\left(\tau>t\right)}\cdot$$
$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\mathbb{Q}_{T_a}\left(\tau>T_a\right)}D(t,T_a)\left(R_{a,b}(T_a)-K\right)^{+}\sum_{i=a+1}^{b}\mathbb{E}_{T_a}\left[D(T_a,T_i)\alpha_i\mathbf{1}_{\{\tau>T_i\}}\right]\cdot\mathbb{Q}_{T_a}\left(\tau>T_a\right)\mid\mathcal{F}_t\right] \tag{8.14}$$

$$=\frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t\left(\tau>t\right)}\cdot$$
$$\mathbb{E}^{\mathbb{Q}}_t\left[D(t,T_a)\left(R_{a,b}(T_a)-K\right)^{+}\sum_{i=a+1}^{b}\mathbb{E}_{T_a}\left[D(T_a,T_i)\alpha_i\mathbf{1}_{\{\tau>T_i\}}\right]\right] \tag{8.15}$$

In the above calculations we use the fact that the sum $\sum_{i=a+1}^{b}\mathbb{E}^{\mathbb{Q}}_{T_a}\left[D(T_a,T_i)\alpha_i\mathbf{1}_{\{\tau>T_i\}}\right]$ and $\mathbb{Q}_{T_a}\left(\tau>T_a\right)=\mathbb{E}_{T_a}\left[\mathbf{1}_{\{\tau>T_a\}}\right]$ are measurable with respect to $\mathcal{F}_{T_a}$, given that they are already expectation at that time point. Using the definition of $\bar{P}(t,T)$ from Equation 7.12 we can simplify the above expression. Consequently the price of a CDS option is then given by

$$\begin{aligned} CDSO(t,K,LGD)=&\frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t\left(\tau>t\right)}\mathbb{E}^{\mathbb{Q}}_t\left[D(t,T_a)\left(R_{a,b}(T_a)-K\right)^{+}\sum_{i=a+1}^{b}\mathbb{E}_{T_a}\left[D(T_a,T_i)\alpha_i\mathbf{1}_{\{\tau>T_i\}}\right]\right] && (8.16)\\ =&\frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t\left(\tau>t\right)}\mathbb{E}^{\mathbb{Q}}_t\left[D(t,T_a)\left(R_{a,b}(T_a)-K\right)^{+}\mathbb{Q}_{T_a}(\tau>T_a)\sum_{i=a+1}^{b}\alpha_i\bar{P}(T_a,T_i)\right] && (8.17)\end{aligned}$$

Notice that since $\bar{P}(t,T)$ is the price of a defaultable zero coupon bond, the summation in the above equation can be thought of as a portfolio of defaultable bonds with zero recovery and different maturities. This is used in the following definition of defaultable present value per basis point.

> **Definition:** Let $\bar{C}_{a,b}(t)=\sum_{i=a+1}^{b}\alpha_i\bar{P}(t,T_i)$ and define the defaultable present value per basis point (DPVBP) as
> $$\hat{C}_{a,b}(t)=\mathbb{Q}(\tau>t\mid\mathcal{F}_t)\bar{C}_{a,b}(t) \tag{8.18}$$

Finally, all of the above simplification and derivations result in the following formula for the price of a CDS option

$$CDSO(t,K,LGD)=\frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t\left(\tau>t\right)}\mathbb{E}^{\mathbb{Q}}_t\left[D(t,T_a)\left(R_{a,b}(T_a)-K\right)^{+}\hat{C}_{a,b}(T_a)\right] \tag{8.19}$$

Let us for a moment discuss the intuition of the above equation. The first term is a result of changing the expectation to be conditioned on the filtration $\mathcal{F}_t$ instead of $\mathcal{G}_t$. At this point in time, the above filtration notation might not yield much intuition to the reader. However, one can think of the equation as a risk-neutral expectation conditioned on all the relevant information that is available at the relevant point in time. We are also capable of assigning some intuition to the terms inside the expectation. The maximum in the CDS option is similar to the maximum in a regular option but in this case the underlying asset is the fair rate from the protection leg in the CDS. Given the fact that we are looking at the right for the buyer of the option to enter a CDS at time T_a but buying the right to enter at time t, the discounted factor $D(t, T_a)$ represents the time difference between buying the right and actually deciding whether or not to enter into a CDS contract. The last term $\hat{C}_{a,b}(T_a)$ is given by the above definition and is called the defaultable present value per basis point.

The above equation has given us some intuition as to how we can think of a CDS option. Still it is not possible or easy to calculate an actual price from this expression. This is because we have no idea of the dynamics of $R_{a,b}(T)$ and $\hat{C}_{a,b}(T_a)$ under the risk-neutral expectation $\mathbb{E}_t^{\mathbb{Q}}$. Therefore, the next section will use the very common technique of changing the probability measure. This is not necessarily a change into a more intuitive probability measure. However, by using this technique we arrive at a specific format of the above option price which enables us to calculate actual option prices under some assumptions.

8.3 Simplifying Pricing Formula

From the previous section we know that the price of a CDS option is given by

$$CDSO(t, K, LGD) = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t\,(\tau > t)} \mathbb{E}_t^{\mathbb{Q}} \left[D(t, T_a)\,(R_{a,b}(T_a) - K)^+\,\hat{C}_{a,b}(T_a) \right] \tag{8.20}$$

In the derivation of the above equation we discussed the intuition behind this formula and concluded that at this point there is little to say about an explicit price of a CDS option. However, we can change the probability measure and thereby derive at something that might not be "prettier" but at least easier to estimate. This is a very common technique that will be familiar to the reader, although the different steps will be clarified in this section. We refer to [Jamshidian, 2004].

Let $\mathbb{C}_{a,b} = \hat{C}_{a,b}(T_a)\frac{B}{B_{T_a}}$, B being the bank account at a terminal date T. Then we can rearrange the pricing formula of the CDS option yielding

$$CDSO(t, K, LGD) = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}_t\,(\tau > t)} \mathbb{E}_t^{\mathbb{Q}} \left[D(t, T_a)\,(R_{a,b}(T_a) - K)^+\,\mathbb{C}_{a,b}\frac{B_{T_a}}{B} \right] \tag{8.21}$$

Let the probability measure $\mathbb{P}^{\mathbb{C}_{a,b}}$ associated with the numeraire $\mathbb{C}_{a,b}$ be defined by the Radon-Nikodym derivative:

$$\frac{d\mathbb{P}^{\mathbb{C}_{a,b}}}{d\mathbb{Q}} = \frac{B_t\mathbb{C}_{a,b}}{\mathbb{C}_{a,b}(t)B} \tag{8.22}$$

From [Brigo and Morini, 2005] we know that $(B_t\mathbb{C}_{a,b})/(\mathbb{C}_{a,b}(t))$ is a traded asset. From e.g. [Björk, 2004] we know that this implies that the above defined Radon-Nikodym derivative will be a martingale under the risk neutral probability measure $\mathbb{Q}$ resulting in

$$\mathbb{E}^{\mathbb{Q}}[\frac{d\mathbb{P}^{\mathbb{C}_{a,b}}}{d\mathbb{Q}} \mid \mathcal{F}_t] = 1 \tag{8.23}$$

We will now use this when changing the probability measure in the price of the CDS option.

Firstly, we can establish that

$$\frac{d\mathbb{P}^{\mathbb{C}_{a,b}}}{d\mathbb{Q}} = \frac{B_t \mathbb{C}_{a,b}}{\mathbb{C}_{a,b}(t) B} \tag{8.24}$$

$$\Rightarrow$$

$$\frac{B_t \mathbb{C}_{a,b}}{B} = \frac{d\mathbb{P}^{\mathbb{C}_{a,b}}}{d\mathbb{Q}} \mathbb{C}_{a,b}(t) \tag{8.25}$$

Secondly, we can use the above rearrangement and substitute the term into our price of the CDS option yielding

$$CDSOption(t, K, LGD) = \frac{\mathbf{1}_{\{\tau > t\}}}{\mathbb{Q}_t(\tau > t)} \mathbb{E}_t^{\mathbb{Q}} \left[D(t, T_a) \left(R_{a,b}(T_a) - K \right)^+ \mathbb{C}_{a,b} \frac{B_{T_a}}{B} \right] \tag{8.26}$$

$$= \frac{\mathbf{1}_{\{\tau > t\}}}{\mathbb{Q}_t(\tau > t)} \mathbb{E}_t^{\mathbb{Q}} \left[\frac{B_t}{B_{T_a}} \left(R_{a,b}(T_a) - K \right)^+ \mathbb{C}_{a,b} \frac{B_{T_a}}{B} \right] \tag{8.27}$$

$$= \frac{\mathbf{1}_{\{\tau > t\}}}{\mathbb{Q}_t(\tau > t)} \mathbb{E}_t^{\mathbb{Q}} \left[\frac{B_t}{B} \left(R_{a,b}(T_a) - K \right)^+ \mathbb{C}_{a,b} \right] \tag{8.28}$$

$$= \frac{\mathbf{1}_{\{\tau > t\}}}{\mathbb{Q}_t(\tau > t)} \mathbb{E}_t^{\mathbb{Q}} \left[\frac{d\mathbb{P}^{\mathbb{C}_{a,b}}}{d\mathbb{Q}} \mathbb{C}_{a,b}(t) \left(R_{a,b}(T_a) - K \right)^+ \right] \tag{8.29}$$

Finally, using Theorem 10 in [Brigo and Morini, 2005] and the above martingale argument we can change the probability measure resulting in the following pricing formula for the CDS option

$$CDSO(t, K, LGD) = \frac{\mathbf{1}_{\{\tau > t\}}}{\mathbb{Q}_t(\tau > t)} \mathbb{E}_t^{\mathbb{C}_{a,b}} \left[\mathbb{C}_{a,b}(t) \left(R_{a,b}(T_a) - K \right)^+ \right] \tag{8.30}$$

Using the definition of $\mathbb{C}_{a,b}(t)$, noticing that it is measurable with respect to $(F)_t$ we obtain

$$CDSO(t, K, LGD) = \frac{\mathbf{1}_{\{\tau > t\}}}{\mathbb{Q}_t(\tau > t)} \mathbb{C}_{a,b}(t) \mathbb{E}_t^{\mathbb{C}_{a,b}} \left[\left(R_{a,b}(T_a) - K \right)^+ \right] \tag{8.31}$$

With this final derivation we are not left with a more intuitive pricing formula given that the probability measure $\mathbb{P}^{\mathbb{C}_{a,b}}$ is even more complicated than the risk neutral measure. However, we notice that the above pricing formula for the CDS option is very similar to the most famous Black-Scholes formula for a European option. This is exactly what we will use in the next chapter and we will see that the above derivation is essential to empirically calculating prices for CDS options.

Chapter 9

Derivation of a Market Model

This chapter will demonstrate that under the new probability measure defined in the previous chapter, the fair rate $R_{a,b}(T_a)$ is a martingale. This implies that $R_{a,b}(T_a)$ has no drift under that probability measure, which will be preferable in pricing the CDS option. With some assumptions we will see how the martingale-property can be used to price the CDS option with results from [Black and Scholes, 1973].

The theory from the previous sections leaves us with the following pricing model of CDS

$$CDSO(t, K, LGD) = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}(\tau > t \mid \mathcal{F}_t)} \mathbb{C}_{a,b}(t) \mathbb{E}_t^{\mathbb{C}_{a,b}} \left[(R_{a,b}(T_a) - K)^+ \right] \tag{9.1}$$

where $\mathbb{E}^{\mathbb{C}_{a,b}}$ denotes expectation related to the probability measure $\mathbb{Q}^{a,b}$ which is equivalent to the risk-neutral probability measure as discuss previously.

To simplify the above equation we will apply the definition of *martingale invariance* introduced by [Jeanblanc and Rutkowski, 2002] also named conditional independence of subfiltrations by [Jamshidian, 2004].

> **Definition:** Given the numeraire α, $\mathcal{F}_t$ is a $\mathbb{P}^\alpha$-conditionally independent subfiltration of $\mathcal{G}_t$ if under $\mathbb{P}^\alpha$ every process which is a martingale when conditioning on $\mathcal{F}_t$ is a martingale also when conditioning on $\mathcal{G}_t$. This means if X is a bounded $\mathcal{F}$-measurable stochastic process then,
>
> $$\mathbb{E}^{\mathbb{P}^\alpha}[X \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{P}^\alpha}[X \mid \mathcal{G}_t] \forall t \tag{9.2}$$

From now on we assume that $\mathcal{F}_t$ is a $\mathbb{Q}$-conditionally independent subfiltration of $\mathcal{G}_t$ and refer to the Appendix in [Brigo and Morini, 2005] for a proof. We will then prove that if the above statement holds then

$$\mathbb{C}_{a,b}(t) = \bar{C}_{a,b}(t) \tag{9.3}$$

$$\tag{9.4}$$

Assuming that $\mathcal{F}_t$ is a $\mathbb{Q}$-conditionally independent subfiltration of $\mathcal{G}_t$ the following relationship holds:

$$\mathbb{C}_{a,b}(t) = B_t \mathbb{E}^{\mathbb{Q}}[\frac{\mathbb{C}_{a,b}}{B} \mid \mathcal{G}_t] \tag{9.5}$$

$$= B_t \mathbb{E}^{\mathbb{Q}}[\frac{\hat{C}_{a,b}(T_a)}{B} \frac{B}{B_{T_a}} \mid \mathcal{G}_t] \tag{9.6}$$

$$= \mathbb{E}^{\mathbb{Q}}[\hat{C}_{a,b}(T_a) \frac{B_t}{B_{T_a}} \mid \mathcal{G}_t] \tag{9.7}$$

$$= \mathbb{E}^{\mathbb{Q}}[\hat{C}_{a,b}(T_a) \frac{B_t}{B_{T_a}} \mid \mathcal{F}_t] \tag{9.8}$$

$$= \mathbb{E}^{\mathbb{Q}}[\frac{B_t}{B_{T_a}} \mathbb{Q}(\tau > T_a \mid \mathcal{F}_{T_a}) \bar{C}_{a,b}(T_a) \mid \mathcal{F}_t] \tag{9.9}$$

$$= \mathbb{E}^{\mathbb{Q}}[\frac{B_t}{B_{T_a}} \mathbb{Q}(\tau > T_a \mid \mathcal{F}_{T_a}) \sum_{i=a+1}^{b} \alpha_i \bar{P}(T_a, T_i) \mid \mathcal{F}_t] \tag{9.10}$$

$$= \mathbb{E}^{\mathbb{Q}}[D(t, T_a) \frac{\mathbb{Q}(\tau > T_a \mid \mathcal{F}_{T_a})}{\mathbb{Q}(\tau > T_a \mid \mathcal{F}_{T_a})} \sum_{i=a+1}^{b} \alpha_i \mathbb{E}^{\mathbb{Q}}[D(T_a, T_i) \mathbf{1}_{\tau > T_i} \mid \mathcal{F}_{T_a}] \mid \mathcal{F}_t] \tag{9.11}$$

$$= \mathbb{E}^{\mathbb{Q}}[\sum_{i=a+1}^{b} \alpha_i \mathbb{E}^{\mathbb{Q}}[D(t, T_a) D(T_a, T_i) \mathbf{1}_{\tau > T_i} \mid \mathcal{F}_{T_a}] \mid \mathcal{F}_t] \tag{9.12}$$

$$= \mathbb{E}^{\mathbb{Q}}[\sum_{i=a+1}^{b} \alpha_i [D(t, T_i) \mathbf{1}_{\tau > T_i} \mid \mathcal{F}_t] \tag{9.13}$$

$$= \bar{C}_{a,b}(t) \tag{9.14}$$

In the above derivation we use the martingale invariance going from Equation 9.7 to Equation 9.8. From Equation 9.12 to Equation 9.13 we use that $T_a > t$. This implies that when $\mathcal{F}_t$ is a $\mathbb{Q}$-conditionally independent subfiltration of $\mathcal{G}_t$, the price of the CDS option is given by

$$CDSO(t, K, LGD) = \mathbf{1}_{\{\tau > t\}} C_{a,b}(t) \mathbb{E}_t^{\mathbb{C}_{a,b}} \left[(R_{a,b}(T_a) - K)^+ \right] \tag{9.15}$$

$$CDSO(t, K, LGD) = \mathbf{1}_{\{\tau > t\}} \sum_{i=a+1}^{b} \alpha_i \bar{P}(t, T_i) \mathbb{E}_t^{\mathbb{C}_{a,b}} \left[(R_{a,b}(T_a) - K)^+ \right] \tag{9.16}$$

Looking at the above price formula for the CDS option we clearly see some similarity between this price formula and the standard European option discussed in Chapter 3.

$$C_t = D(t, T) E_t^{\mathbb{Q}} \left[(S_T - K)^+ \right] \tag{9.17}$$

Black-Scholes uses the no-drift property of a martingale to price the European option. They show that if S_t is a martingale under a specific probability measure - in [Black and Scholes, 1973] $\mathbb{Q}$ - there exists a replicating strategy and it is then possible to price the call option by

$$C(P,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)} \tag{9.18}$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \tag{9.19}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{9.20}$$

In the attempt of pricing options on CDS a naturally first step will therefore be to rely on Black-Scholes. In the case of the CDS option the underlying asset will be the fair rate $R_{a,b}(t)$ hence we wish to show that it is a martingale under the relevant probability measure, which in our case is $\mathbb{Q}^{\mathbb{C}_{a,b}}$. This is exactly what we will show in the next section using the martingale invariance theorem once more.

9.1 The Dynamics of the Underlying Spread

With the above description of the Black-Scholes method in mind consider the fair spread $R_{a,b}$ given by

$$R_{a,b}(t) = \frac{LGD \sum_{i=a+1}^{b} \mathbb{E}\left[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} \mid \mathcal{F}_t\right]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{Q}\left(\tau > t \mid \mathcal{F}_t\right) \bar{P}(t,T_i)} \tag{9.21}$$

$$\Rightarrow$$

$$R_{a,b}(t) = \frac{LGD \sum_{i=a+1}^{b} \mathbb{E}\left[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} \mid \mathcal{F}_t\right]}{\bar{C}_{a,b}(t)} \tag{9.22}$$

where the last equation uses the definition of $C_{a,b}$. If we can show that this fair spread is a martingale under $\mathbb{Q}^{\mathbb{C}_{a,b}}$, it follows implicitly that the fair spread has no drift under this probability measure. To show that $R_a, b(t)$ is a martingale under $\mathbb{Q}^{\mathbb{C}_{a,b}}$ we will need to show

$$\mathbb{E}^{\mathbb{C}_{a,b}}[R_{a,b}(s) \mid \mathcal{F}_t] = R_{a,b}(t) \;\; for\; t < s \tag{9.23}$$

In the following proof we will use the fact that $\mathcal{F}_t$ is a $\mathbb{Q}$-conditionally independent subfiltration of $\mathcal{G}_t$ and that $\mathcal{F}_t$ is a $\mathbb{Q}^{\mathbb{C}_{a,b}}$-conditionally independent subfiltration of $\mathcal{G}_t$. Again, we refer to the Appendix in [Brigo and Morini, 2005] for a proof of both statements.

Consider the following claim defined in [Brigo and Morini, 2005]

$$R^C = \sum_{i=a+1}^{b} \mathbb{E}^{\mathbb{Q}}[D(T_a,T_i)\mathbf{1}_{T_{i-1}<\tau<T_i} \mid \mathcal{F}_{T_a}]\frac{C_{a,b}}{C_{a,b}(T_a)} \tag{9.24}$$

We know from Chapter 8 that using $\mathbb{C}_{a,b}$ as numeraire all asset will be martingales under $\mathbb{Q}^{\mathbb{C}_{a,b}}$. When $\mathcal{F}_t$ is a $\mathbb{Q}$-conditionally independent subfiltration of $\mathcal{G}_t$ and when $\mathcal{F}_t$ is a $\mathbb{Q}^{\mathbb{C}_{a,b}}$-conditionally independent subfiltration of $\mathcal{G}_t$ we showed that $\mathbb{C}_{a,b}(t) = \bar{C}_{a,b}(t)$. Consequently we can write

$$R_t^C = C_{a,b}(t)\mathbb{E}^{\mathbb{C}_{a,b}}[\frac{R^C}{C_{a,b}} \mid \mathcal{G}_t] \tag{9.25}$$

$$= C_{a,b}(t)\mathbb{E}^{\mathbb{C}_{a,b}}[\sum_{i=a+1}^{b} \mathbb{E}^{\mathbb{Q}}[D(T_a,T_i)\mathbf{1}_{T_{i-1}<\tau<T_i} \mid \mathcal{F}_{T_a}] \mid \mathcal{G}_t] = \quad C_{a,b}(t)\mathbb{E}^{\mathbb{C}_{a,b}}[R_{a,b}(T_a) \mid \mathcal{G}_t] \tag{9.26}$$

We know from [Black and Scholes, 1973] that all assets are martingales under the risk neutral probability measure with the bank account as the numeraire. This implies that we can also write the claim R^C as

$$
\begin{aligned}
R_t^C =& B_t \mathbb{E}^{\mathbb{Q}}[\frac{R^C}{B} \mid \mathcal{G}_t] && (9.27)\\
=& B_t \mathbb{E}^{\mathbb{Q}} \left[\frac{\sum_{i=a+1}^{b} \mathbb{E}^{\mathbb{Q}}[D(T_a,T_i)\mathbf{1}_{T_{i-1}<\tau<T_i} \mid \mathcal{F}_{T_a}] \frac{C_{a,b}}{C_{a,b}(T_a)}}{B} \mid \mathcal{G}_t \right] && (9.28)\\
=& B_t \mathbb{E}^{\mathbb{Q}} \left[\frac{\sum_{i=a+1}^{b} \mathbb{E}^{\mathbb{Q}}[D(T_a,T_i)\mathbf{1}_{T_{i-1}<\tau<T_i} \mid \mathcal{F}_{T_a}] C_{a,b}}{B C_{a,b}(T_a)} \mid \mathcal{G}_t \right] && (9.29)
\end{aligned}
$$

Once again we use that when $\mathcal{F}_t$ is a $\mathbb{Q}$-conditionally independent subfiltration of $\mathcal{G}_t$ then $\mathbb{C}_{a,b}(t) = C_{a,b}(t)$. This means that $C_{a,b} = \mathbb{C}_{a,b} = C_{a,b}(T_a)B/B_{T_a}$ resulting in

$$
\begin{aligned}
R_t^C =& B_t \mathbb{E}^{\mathbb{Q}} \left[\frac{\sum_{i=a+1}^{b} \mathbb{E}^{\mathbb{Q}}[D(T_a,T_i)\mathbf{1}_{\{T_{i-1}<\tau<T_i\}} \mid \mathcal{F}_{T_a}] C_{a,b}(T_a)B/B_{T_a}}{B C_{a,b}(T_a)} \mid \mathcal{G}_t \right] && (9.30)\\
=& B_t \mathbb{E}^{\mathbb{Q}} \left[\frac{\sum_{i=a+1}^{b} \mathbb{E}^{\mathbb{Q}}[D(T_a,T_i)\mathbf{1}_{\{T_{i-1}<\tau<T_i\}} \mid \mathcal{F}_{T_a}]}{B_{T_a}} \mid \mathcal{G}_t \right] && (9.31)\\
=& B_t \mathbb{E}^{\mathbb{Q}} \left[\sum_{i=a+1}^{b} \mathbb{E}^{\mathbb{Q}}[\frac{D(T_a,T_i)\mathbf{1}_{\{T_{i-1}<\tau<T_i\}}}{B_{T_a}} \mid \mathcal{F}_{T_a}] \mid \mathcal{G}_t \right] && (9.32)\\
=& B_t \mathbb{E}^{\mathbb{Q}} \left[\sum_{i=a+1}^{b} \mathbb{E}^{\mathbb{Q}}[\frac{\frac{B_{T_a}}{B_{T_i}}\mathbf{1}_{\{T_{i-1}<\tau<T_i\}}}{B_{T_a}} \mid \mathcal{F}_{T_a}] \mid \mathcal{G}_t \right] && (9.33)\\
=& B_t \mathbb{E}^{\mathbb{Q}} \left[\sum_{i=a+1}^{b} \mathbb{E}^{\mathbb{Q}}[B_{T_i}\mathbf{1}_{\{T_{i-1}<\tau<T_i\}} \mid \mathcal{F}_{T_a}] \mid \mathcal{G}_t \right] && (9.34)\\
=& \mathbb{E}^{\mathbb{Q}} \left[\sum_{i=a+1}^{b} \mathbb{E}^{\mathbb{Q}}[\frac{B_t}{B_{T_i}}\mathbf{1}_{\{T_{i-1}<\tau<T_i\}} \mid \mathcal{F}_{T_a}] \mid \mathcal{G}_t \right] && (9.35)\\
=& \sum_{i=a+1}^{b} \mathbb{E}^{\mathbb{Q}} \left[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau<T_i\}} \mid \mathcal{G}_t \right] && (9.36)\\
=& \sum_{i=a+1}^{b} \mathbb{E}^{\mathbb{Q}} \left[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau<T_i\}} \mid \mathcal{F}_t \right] && (9.37)\\
=& R_{a,b}(t) C_a, b(t) && (9.38)
\end{aligned}
$$

In the above calculations we use that $t < T_a$ going from Equation 9.34 to 9.36 and we use that $\mathbb{Q}$-conditionally independent subfiltration of $\mathcal{G}_t$ going from Equation 9.36 to Equation 9.37. Combining all of the above calculation we have the following relationship

$$C_{a,b}(t)\mathbb{E}^{\mathbb{C}_{a,b}}[R_a, b(T_a) \mid \mathcal{G}_{\sqcup}] = C_{a,b}(t)R_{a,b}(t) \tag{9.39}$$

$$\mathbb{E}^{\mathbb{C}_{a,b}}[R_a, b(T_a) \mid \mathcal{G}_{\sqcup}] = (t)R_{a,b}(t) \tag{9.40}$$

This means that $R_{a,b}(t)$ is a $\mathcal{G}_t$-martingale under $\mathbb{Q}^{\mathbb{C}_{a,b}}$ for $t < T_a$. As mentioned before we know from [Brigo and Morini, 2005] that $\mathcal{F}_t$ is also $\mathbb{Q}^{\mathbb{C}_{a,b}}$-conditionally independent. It can be shown (See Appendix in [Brigo and Morini, 2005]) that this implies that $R_a, b(t)$ also is $\mathcal{F}_t$-martingale under $\mathbb{Q}^{\mathbb{C}_{a,b}}$. We end this section by summarizing our findings:

1. The price of the CDS option is given by

$$CDSO(t, K, LGD) = \frac{\mathbf{1}_{\{\tau > t\}}}{\mathbb{Q}\left(\tau > t \mid \mathcal{F}_t\right)} \mathbb{C}_{a,b}(t)\mathbb{E}_t^{\mathbb{C}_{a,b}}\left[(R_{a,b}(T_a) - K)^+\right] \tag{9.41}$$

2. $R_{a,b}(t)$ is a martingale under $\mathbb{P}^{\mathbb{C}_{a,b}}$

This is how far we can go without making any assumptions about the dynamics of the underlying asset $R_{a,b}(t)$. The next section will assume that the dynamics of $R_{a,b}(t)$ is a diffusion process, which will lead us to a similar-to-Black-Scholes option pricing formula for CDS option.

9.2 Black Scholes

In the previous subsection we showed that $R_{a,b}(t)$ is a martingale under $\mathbb{Q}^{\mathbb{C}_{a,b}}$ and therefore has no drift under this probability measure. We know that a stochastic process evolves as

$$dX_t = \mu_t dt + \sigma_t dZ_t \tag{9.42}$$

The no-drift property of a martingale implies that the “only” thing we need to consider is the volatility σ_t and the type of process Z_t. Now, we are at a turning point where we will need to make additional general assumptions with the purpose of deriving a full market model. Here it will be natural to begin with Black-Scholes.

In the last part of the section we will look at the drawback of this assumption but for now we will assume that the fair spread has the following dynamics:

$$dR_{a,b}(t) = \sigma_{a,b}R_{a,b}(t)dW^{a,b}(t) \tag{9.43}$$

where $W^{a,b}(t)$ is a Brownian motion under the probability measure $\mathbb{Q}^{\mathbb{C}_{a,b}}$. This assumption implies that we can use the Black-Scholes formula to price the CDS option, and in this case the price is given by

$$\begin{aligned} Call_{a,b}(t, K, LGD) =& \mathbf{1}_{\{\tau > t\}} C_{a,b}(t)\left[R_{a,b}(t)N(d_1(t) - KN(d_2(t)\right] \\ d_{1,2} =& \frac{\left(ln\left(\frac{R_{a,b}(t)}{K}\right) \pm \left(T_a - t\right)\sigma_{a,b}^2/2\right)}{\sigma_{a,b}\sqrt{T_a - t}} \end{aligned} \tag{9.44}$$

This means that given the fair spread $R_{a,b}(t)$ of a CDS starting at time T_a it is possible to price the option once we know the volatility. We will end this section with a discussion of the limitations of the Black-Scholes formula. With the limitations of the Black-Scholes formula in mind we are finally ready to empirically calculate the price of the CDS option in Part III.

9.2.1 Critique of Black-Scholes

This section will discuss the most common critique of the Black-Scholes model. The general Black-Scholes formula price options on assets. In this thesis the underlying product is the CDS spread with no price to enter. There exists minimal literature on how the critique of the general Black-Scholes formula can be transferred to the case of the CDS option. Therefore, I will present the critique and for further research we will have to investigate the consequences of the assumption in the Black-Scholes formula when pricing CDS options.

Firstly, In the Black-Scholes model it is assumed that the percentage change in the underlying price is normally distributed which also implies that the underlying price is lognormally distributed (see [Hull, 2012]). Extensive literature e.g. [Bradley and Taqqu, 2003] has shown that asset returns tends to have heavy-tails making the assumption about the dynamics of the underlying questionable.

Secondly, the Black-Scholes formula assumes constant volatility σ. From [Teneng, 2011] we know that this is not a realistic assumption and in Chapter 11 we will see several examples of volatility of the underlying asset changing over time.

Finally it is worth mentioning that the Black-Scholes model also assumes no transaction cost and that the market is perfectly liquid. Again these assumptions does not fit very well with the reality as discussed in [Munk, 2011].

The Black-Scholes formula can be extended to account for some of the above mentioned drawbacks. This is attempted in e.g. [Kim and Kunitomo, 1999] by incorporating stochastic interest rate and in [Fouque et al., 2000] by incorporate stochastic volatility.

In this thesis I will use the Black-Scholes formula without any extensions. This is for the same reasons that the Black-Scholes formula still is the most used option pricing formula in the financial world. It is easy to calculate and a good approximation.[1]

[1] See [Guardian] for further discussion of the Black-Scholes Formula

Part III

Empirical Results

Chapter 10

Empirical Implementation

In Part I I described the different components of the CDS contract and those of the CDS option. In Part II I based my calculation on [Brigo and Morini, 2005] and derived a pricing formula for the CDS call option. In the end of II I made some simplifying assumption about the fair rate $R_{a,b}(t)$ resulting in a Black-Scholes-like formula for the CDS call option. In this part of the thesis I will use the Black-Scholes formula for the CDS option to calculate multiple prices. I will then compare the results to the results of [White, 2014] making sure that my empirical implementation is correct. With this implementation we are ready to discuss the use of the CDS option which I will do in Part IV.

This chapter will empirically implement the Black-Scholes formula and compare the results to the results of [White, 2014]. We will end this chapter with a discussion of the drawbacks in this model

10.1 The Setup

From Chapter 9 we derived a market model for pricing CDS option given by

$$CDSOption(t, K, LGD) = \frac{\mathbf{1}_{\{\tau > t\}}}{\mathbb{Q}(\tau > t|\mathcal{G}_t)} \mathbb{C}_t^{a,b} \bar{\mathbb{E}}^{a,b} \left[(R_{a,b}(T_a) - K)^+ | \mathcal{G}_t \right] \tag{10.1}$$

Assuming that $R_{a,b}(t)$ follows an Itō-diffusion process with no drift under $\mathbb{Q}^{\mathbb{C}_{a,b}}$, relying on the Black-Scholes formula we can price the CDS option by the following equation

$$\begin{aligned} Call_{a,b}(t, K, LGD) =& \mathbf{1}_{\{\tau > t\}} C_{a,b}(t) \left[R_{a,b}(t) N(d_1(t) - K N(d_2(t)) \right] \\ d_{1,2} =& \frac{\left(ln\left(\frac{R_{a,b}(t)}{K}\right) \pm (T_a - t)\, \sigma_{a,b}^2 / 2 \right)}{\sigma_{a,b}\sqrt{T_a - t}} \end{aligned} \tag{10.2}$$

This section will prepare us to empirically price the CDS option by a walk-through of what we have

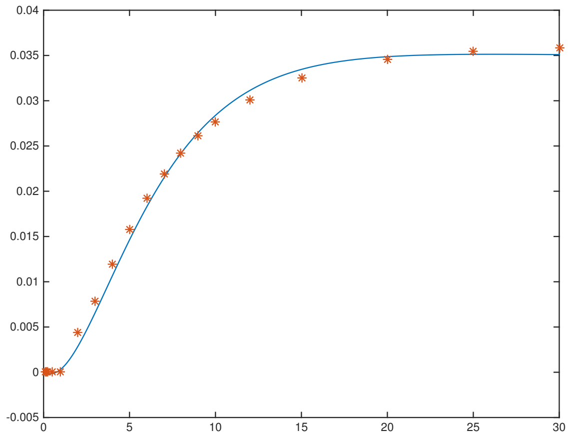

Figure 10.1: Yield Curve 05-02-2014

already estimated. First, we notice that

$$C_{a,b}(t) = \sum_{i=a+1}^{b} \alpha_i \bar{P}(t, T_i) \tag{10.3}$$

$$= \sum_{i=a+1}^{b} \alpha_i \frac{\mathbb{E}\left[D(t, T_i)\mathbf{1}_{\{\tau > T_i\}} \mid \mathcal{F}_t\right]}{\mathbb{Q}\left(\tau > t \mid \mathcal{F}_t\right)} \tag{10.4}$$

This implies that to empirically price the CDS option we will need the survival probability for different future time points for a specific firm or for an index standing at a specific time point. This is exactly what we calibrated in Chapter 6. In this section we will price a CDS option using the model from Chapter 8 and perform the implementation on data used in the article by [White, 2014]. By doing this we ensure that our model and implementation yield approximately the same prices as found by others, hence we can trust our results. The next section will use the empirical model to price different CDS options and by that we are ready to discuss the use of CDS options in the market.

In the article by [White, 2014] they price a 5Y CDS Call option with the republic of Italy as reference, trading on 5 February 2014 with option expiry on 20 March 2014. On 20 March 2014 the buyer of the call option can enter a CDS starting with premium payments on 21 March 2014 and maturity 20 June 2019. As mentioned earlier in this section we wish to replicate the option prices found in [White, 2014] using our model so that further analysis will be correct. Using the data from the article we fit the yield curve as described in chapter 5 resulting in the yield curve shown in Figure 10.1.

After fitting the yield curve we will also need to calibrate the survival probability curve. This is done by the method described in Chapter 6 using data from the article by [White, 2014] resulting in the survival probability curve shown in Figure 10.2

In the above calibration of the survival curve we use the formula for the CDS spread

$$S(t, T) = \frac{Prot(t, T)}{Prem(t, T)} \tag{10.5}$$

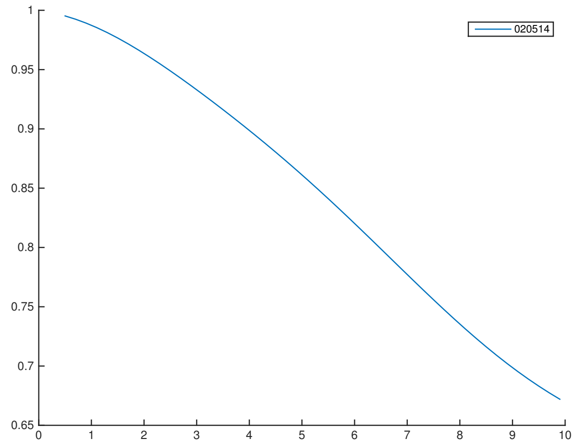

Figure 10.2: Survival probability curve 05-02-2014, The republic of Italy

specified in chapter 6. By using the found parameters in the survival probability curve it is possible to calculate CDS spreads with different yields. We can then compare the calculated spreads to the observed spreads and verify the appropriateness of our fit as shown in Figure 10.3.

From this figure we observe a well-fitted curve indicating that so far our implementations are correct.

10.2 Calculating the Fair Rate $R_{a,b}(t)$

Next step in implementing our theoretical model for option pricing will be calculating the fair premium rate $R_{a,b}(t)$ on 5 February 2014. This price will be the rate from the buyer of the option to the protection seller if the buyer chooses to exercise the option and thereby enter the CDS contract. In the article by [White, 2014] they calculate the fair premium on 5 February 2014 to the price of 182.764 basis points (bps). They compare the result to the price in Bloomberg CDSO Calculator [1] and find that Bloomberg calculates the fair premium to 182.767 bps. For our pricing model to be correct both theoretically and when we empirically implement the model, we need a fair premium $R_{a,b}(t)$ in the range of the two above-mentioned prices. Using the derived equation for the fair spread

$$R_{a,b}^{POST}(t) = \frac{LGD \sum_{i=a+1}^{b} \mathbb{E}\left[D(t,T_i)\mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} \mid \mathcal{F}_t\right]}{\sum_{i=a+1}^{b} \alpha_i \mathbb{Q}\left(\tau > t \mid \mathcal{F}_t\right) \bar{P}(t,T_i)} \tag{10.6}$$

we obtain a price 183.696 bps [2]. This fair rate is 0.5099% and 0.5082% away from the price in the article and from the price calculated in Bloomberg CDSO respectively. One might argue that this price is too far away given that the difference between prices in Bloomberg CDSO and the article is only 0.00164%. However, there are several small changes that we could be make that would have a great impact on the result. One drawback when we try to use the same data as in the article would be the day count - the way of counting the days in between the trade date, the expiry date and the date of

[1] See Appendix, Chapter 15
[2] See Appendix, Chapter 15

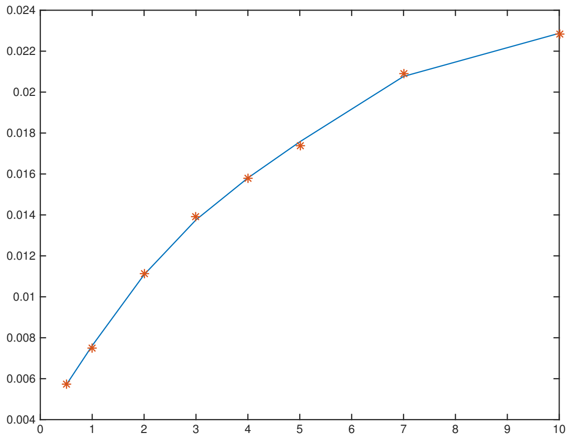

Figure 10.3: Observed and Calculated CDS spreads, The republic of Italy

maturity for the CDS contract. In the article they use the ISDA ACT/ACT day count to calculate the days between the trade date (05-02-2014) and the option expiry date (20-03-2014). However, it is not clearly stated how they calculated the days between the start of the CDS contract (20-03-2014) and the CDS maturity (20-06-2019). Nor is it clear how they manage the leap year 2016. We will discuss the drawbacks of our model later in this chapter but for our purpose we will accept the results from the empirical implementation that we have made.

10.3 The Price of the CDS Option

The final calculation will be the price of the CDS option. Given the differences in the fair rate we should expect some differences in the option price as well. However, we will still need the option price to be acceptable in order to use our empirical implementation for further analysis. In the article by [White, 2014] they attempt to price the above-described option with different strikes between 100 bps and 300 bps. Using the strike price of 140 bps they obtain an option premium ranging from 2.078-2.079% depending on whether they use the Bloomberg CDSO fair rate, the Bloomberg CDSO option price or their own fair rate and their own option pricing formula. If we use the option pricing formula derived in Chapter 9 then

$$\begin{aligned} Call_{a,b}(t,K,LGD) =& \mathbf{1}_{\{\tau>t\}} C_{a,b}(t)\left[R_{a,b}(t)N(d_1(t) - KN(d_2(t)\right] \\ d_{1,2} =& \frac{\left(ln\left(\frac{R_{a,b}(t)}{K}\right) \pm (T_a - t)\,\sigma^2_{a,b}/2\right)}{\sigma_{a,b}\sqrt{T_a - t}} \end{aligned} \tag{10.7}$$

Setting the strike price to 140 bps and using the above-calculated fair spread, we obtain a CDS option premium of 2.059%. When we use their fair rate but our model we obtain a CDS option premium of 2.078% [3]. This implies that the differences in the option prices stems from the differences in the fair rate. Nevertheless, mentioned earlier we will accept the results from the implementation. The reader

[3]See Appendix, Chapter 15

might question this acceptance. Note, however, that the purpose of this thesis is not to find the best pricing model, but rather to use this pricing model in a discussion of the purpose of the CDS option. We refer to the note on further research in Chapter 14.

In the Appendix in Chapter 15 we compare the results found using the Bloomberg CDSO calculator to our the results of our implementation. We make this comparison for many different strike prices. As discussed earlier we observe some differences but not in a sudden range that we need to reject our discussions and results from further analysis.

10.3.1 Implied Volatility

Later in Chapter 11 we will use the implied volatility of the CDS option and compare it to the implied volatility of the CDS Spread. This serve as a starting point for the discussion of speculative trading in the CDS option market, hence a natural final step in this chapter will be calculating the implied volatility of the CDS option. Then compare I will compare the results to the implied volatility calculated in the article by [White, 2014].

The implied volatility is the volatility in the Black-Scholes model. Knowing the option premium the word "implied" means the volatility that is necessary for the Black-Scholes formula to hold for the given option premium. We will not dwell with the definition of the implied volatility as this will be familiar to the reader. We refer to [Christensen and Prabhala, 1998] for a thorough description of the implied volatility.

Given that the CDS option formula is not equal to the European stock option formula we cannot use the implied volatility formula **blsimpv** in Matlab without doing some rearrangements. Using our CDS option pricing formula and dividing by $C_{a,b}(t)$ (Note $1_{\{\tau>t\}} = 1$) we obtain the following formula

$$\frac{Call_{a,b}(t, K, LGD)}{C_{a,b}(t)} = [R_{a,b}(t)N(d_1(t) - KN(d_2(t)] \tag{10.8}$$

$$\tag{10.9}$$

The above right hand side is now equivalent to the European stock option pricing formula discussed in Chapter 3. This means that now we are able to use the implied volatility formula in Matlab. We will use this method when comparing the implied volatility of the CDS option to the volatility of the underlying CDS spread.

Part IV

Trading the CDS option

Chapter 11

CDS Volatility vs. CDS Option Volatility

In Part II I developed a theoretical pricing model for CDS options and in Part III I compared the results of my pricing model to the results from the Bloomberg calculator CDSO and to results from the article by [White, 2014]. I demonstrated that despite a few differences our pricing methods resulted in similar option prices. Until now we have discussed the characteristics of the CDS option, the CDS option market and the way to price the CDS option. However, we have yet to discuss how the market uses the CDS options. Are the CDS options used for reducing risk exposure or do traders trade the CDS option merely for profit? In this chapter we will look for a correlation between the implied volatility of the CDS option and the volatility of the underlying asset - the CDS contract. We know from Chapter 9 that the implied volatility of the CDS option is a reflection of the price of the underlying asset. This implies, as discussed by e.g. [Shu and Zhang, 2003], that the implied volatility option can be used as a forecast for the realized volatility on the CDS contract if the market is efficient. This chapter will test if this relationship also exists in the CDS option market. I will end this chapter with a discussion to why this relationship might hold.

11.1 The Volatility of CDS Spread

In this section I will estimate the volatility of the CDS spread. I will base my calculation of the volatility on the method described in O'Kane [2011]. This method builds upon some of the assumptions about Black Scholes discussed in Chapter 9. Assume, as we already did in Chapter 9, that the dynamics of the CDS spread $R_{a,b}(t)$ is a lognormal process meaning that

$$dR_{a,b}(t) = \sigma R_{a,b}(t) dW(t) \tag{11.1}$$

where $W(t)$ is a brownian motion as discussed in Chapter 3. The lognormal assumption is an attractive choice for several reasons. First, the assumption ensures that the spread $R_{a,b}(t)$ can never be negative. Given that the spread is equal to the protection leg divided by the premium leg as discussed in Chapter 6, it would be unrealistic to expect negative values. Second, the lognormal distribution is skewed assigning larger probability to higher future spread value than to lower future spread value consistent with the statements written by [O'Kane, 2011].

Following the methods from O'Kane [2011] we collect multiple daily CDS Spread $R(i)$, where $i = 1, ..., N$, where N is the number of observation. Then continuously compounded daily return is given by

$$u_i = \ln\left(\frac{R(i)}{R(i-1)}\right) \tag{11.2}$$

Following O'Kane [2011] the daily volatility is then given by

$$\sigma_{day} = \sqrt{\frac{1}{N-1}\sum_i (u_i - \bar{u})^2} \tag{11.3}$$

In the above equation $\bar{u}$ denotes the mean of the compounded daily returns. The following two subsections will perform the above calculation for data on CDS Spreads for Italy and Germany. Then we will compare the daily volatility of the CDS Spreads to the daily implied volatility of the CDS option, which we have obtained from the Bloomberg CDSO calculator.From the results in [Shu and Zhang, 2003] we would expect the volatility of the CDS option to be a good forecast for the realized volatility of the CDS Spread and consequently we would expect the two diffent volatilities to be somewhat correlated. If this is not the case, it might be an indicator that the CDS option is traded mainly as a stand-alone product and has little to do with the underlying CDS contract. No correlation between the implied volatility and the CDS Spread volatility might suggest that the CDS option is mainly used for speculative trading, but we will discuss this in the end of this chapter while also discussing the results of the following two subsections.

Example: Italy

From [Markit, b] we have data on CDS Spread with yield 0.5,1,2,3,4,5,7,10,15,20 and 30 years ranging from the 8 January 2013 to 22 April 2015. Figure 11.1 shows a plot of CDS Spread with the above mentioned different yield. We have 715 data points between the January 8 2013 and 22 April 2015. From Figure 11.1 we clearly see several expected connections. First, within each day the CDS Spread is higher for larger yield. This is a result of the survival probability curve being smaller for larger future time points. Such a connection maps the reality realistically. Second, we observe that the CDS spread with different yields move together. Again, this is a result of the probability curve, and is therefore also a connection we would expect.

Using the equations from above we can obtain a daily volatility of the CDS Spread. We use the past 225 days to calculate the daily volatility, and we use the CDS Spread with 5Y yield because the option for which we will later calculate the volatility will be on a 5Y CDS contract. Figure 11.2 shows the calculated CDS spread volatilities for the remaining 715-225=490 observations. Using the past 225 observations leads to a very smooth volatility "curve" due to auto regression. However, from [Garman and Klass, 1980] we know that the discussion about the number of lags to consider is a balance between good estimates and robustness.

Figure 11.4 shows the CDS volatility curve using only 100 past observations. In this case we observe more fluctuations than in Figure 11.2. Figure 11.3 shows the CDS volatility curve using only 10 past days and in this case it becomes clear that the estimates are not that robust. Again, we refer to [Garman and Klass, 1980] for further discussion of the number of lags. From now on we will choose to use the past 100 daily observations.

Next step will be calculating the implied volatility of the CDS option for the same time period as with the the CDS volatility. In Chapter 10 we discussed how to use the Bloomberg CDSO option

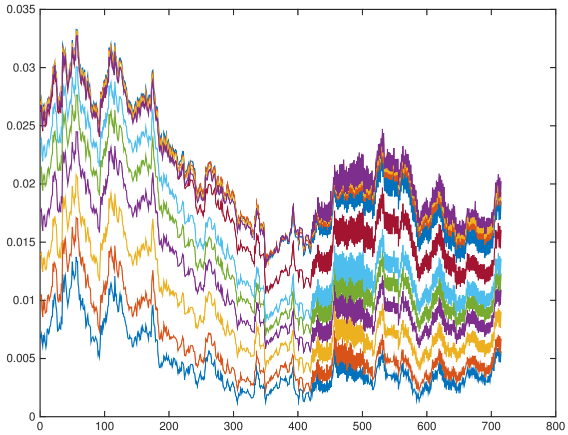

Figure 11.1: CDS Spread with different yield, Italy, 08/01/2013-22/04/2015

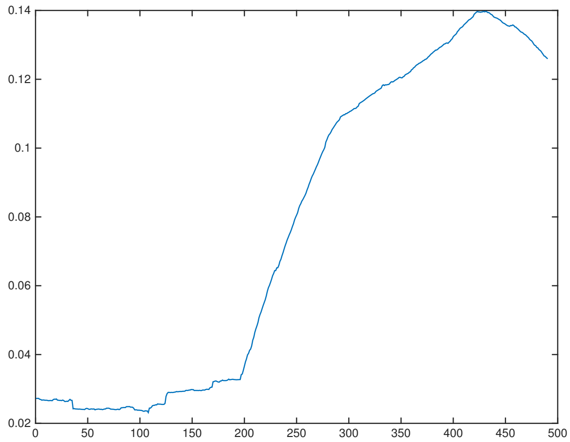

Figure 11.2: Daily CDS Spread volatility, Italy, 11/12/2013-22/04/2015, 225 past obs

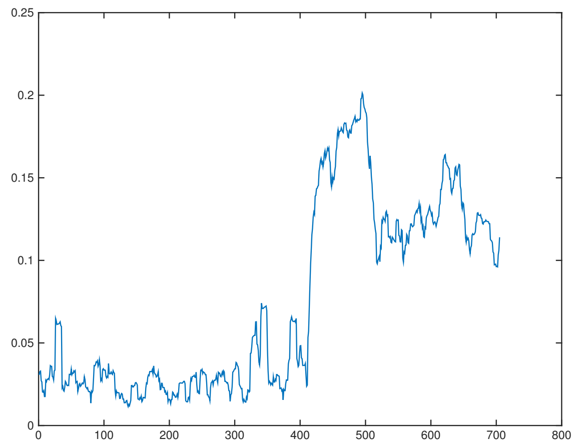

Figure 11.3: Daily CDS Spread volatility, Italy, 11/12/2013-22/04/2015, 10 past obs

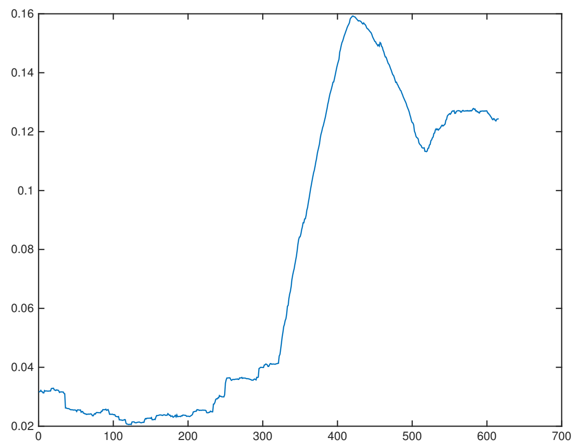

Figure 11.4: Daily CDS Spread volatility, Italy, 11/12/2013-22/04/2015, 100 past obs

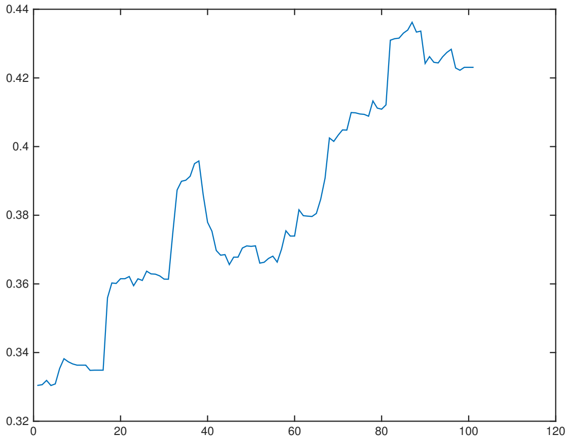

Figure 11.5: Implied volatility of a 5Y CDS option, Italy, 11/12/2013-02/05/2014

calculator and in this section we will use the same method. Due to the very time consuming method when getting the CDSO option we will use only 101 time points.

Figure 11.5 shows the implied volatility of the CDS option from 11 December 2013 to 2 May 2014.

In Figure 11.6 we show the two different volatility curves plotted in the same graph but using different axes. Note that we only plot the CDS Spread volatilities that correspond to the same date of the collected implied volatility. The figure clearly shows a relationship between the implied volatility of the CDS option and the volatility of the CDS spread. This relationship is confirmed by a correlation coefficient of 0.896. In the end of this chapter we will discuss the meaning of this correlation between the two volatilities.

Example: Germany

In this subsection we briefly perform calculations similar to the ones performed in the above example using the Federal Republic of Germany as reference entity.

Figure 11.7 shows the CDS spreads with reference entity the Federal Republic of Germany and the same yields as with the previous example. In this example we will only plot the implied volatility and the CDS spread volatility. The results can be viewed in Figure 11.8. Again, we notice a correlation between the two types of volatility, and with a positive correlation of 0.532 it is clear that the movements in the CDS Spread are mirrored in the movements in the implied volatility. We will end this chapter with a discussion of why this is the case and what we can deduce from such correlation.

Results of Comparison

The relationship between the implied volatility of an option and the volatility of the underlying asset is discussed in [Shu and Zhang, 2003] and [Christensen and Prabhala, 1998]. They both suggest that the implied volatility which is extracted from the option price might be a good indicator of the future realized volatility of the underlying asset. However, both articles mentions that this relationship will only exists if the option pricing model is correct. Additionally, the relationship is more likely to hold if

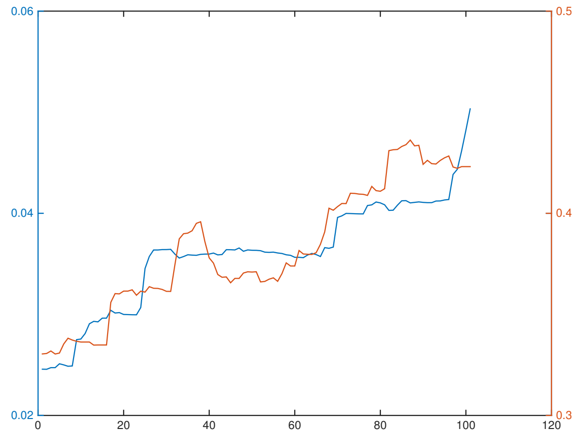

Figure 11.6: Implied Volatility + CDS Spread volatility, Italy, 11/12/2013-02/05/2014

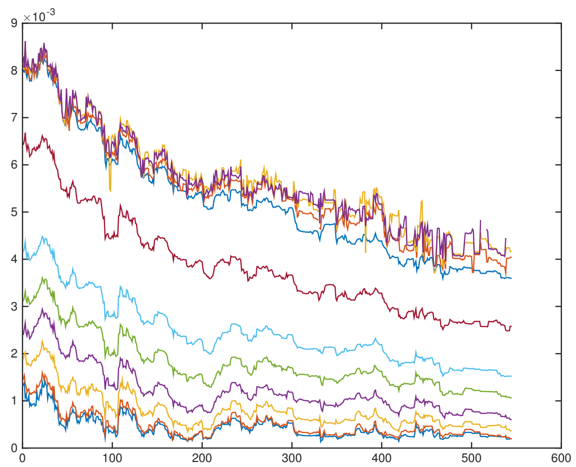

Figure 11.7: CDS Spread with different yield, Germany, 08/01/2013-22/04/2015

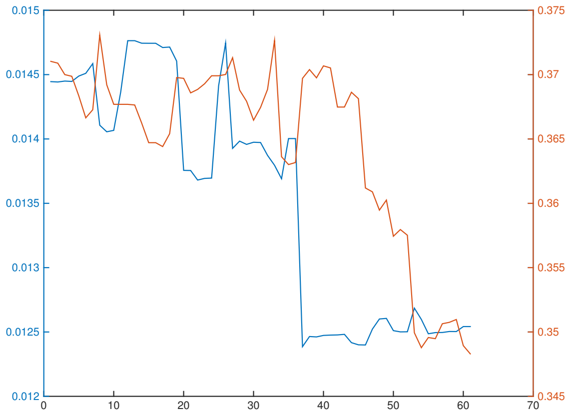

Figure 11.8: Implied Volatility + CDS Spread volatility, Germany, 11/12/2013-06/03/2015

the market is efficient. Lets for now assume that the model we used in Part II presented in [Brigo, 2005] and the model Bloomberg uses in their CDSO calculator are both correct. Then the last statement about market efficiency draws our attention. [Shu and Zhang, 2003] argues that since option traders are generally institutional traders and therefore posses more information than the average financial market trader, it is expected that implied volatility is better in forecasting future volatility than historical volatility. In Figure 11.9 we plot the volatility of the CDS spread against the implied volatility with a 30 days lag. We find that the correlation between the implied volatility and the CDS volatility is 0.7971. Furthermore, when running a linear regression of the CDS spread volatility on the implied volatility, we find a coefficient of 0.3638 with tstat=13.134 hence significant.

The above findings are exactly what we would have expected given that implied volatility implicit reflects the underlying asset. We will not make an unshakable conclusion and we will not claim that these findings imply that the CDS option is *only* traded for the purpose of hedging risk exposure. However, the fact that the correlation between the implied volatility from the market price of the CDS Option calculated in CDSO Bloomberg and the volatility from [Markit, b] is close to 1 suggest that the market for CDS option is in fact efficient as suggested by [Shu and Zhang, 2003]. Had the correlation between those two kinds of volatility been 0 or even negative, it would have suggested that the market was inefficient or reflecting something else than the underlying asset. I feel obligated to mention that other studies e.g. in [Canina and Figlewski, 1993] find no relationship between implied volatility and realized volatility. However, [Christensen and Prabhala, 1998] claim that such conclusions are the result of measurement errors. Therefore, we conclude that the outcome of the empirical studies in this chapter serve only as an indicator that the market for CDS option might be very efficient and actually reflects some of the movements in the underlying CDS contract.

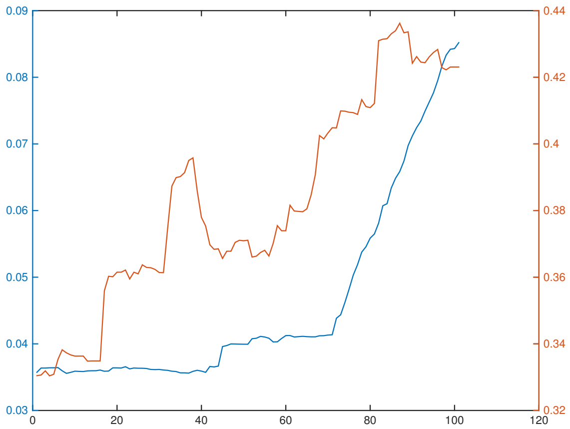

Figure 11.9: Implied Volatility lag 25 days + CDS Spread volatility, Italy

Chapter 12

Profiting from Trading on CDS Option Market

In the previous chapter we saw a coherence of the CDS market movements in the CDS option market. This was an indicator that the option market was in fact reflecting the underlying asset, hence the option market can be used for hedging risk exposure. In this chapter we will look at the option market from at different point of view. From [Alloway] we know that almost all traded CDS options have very short term maturities, which might indicate that the market is mainly speculative. We will discuss this claim and then assuming that we were trading on the CDS option market for profiting purpose we will look at the outcome and the performance. Again, it is worth noticing that the findings in the previous chapter and in this chapter do not lead to any conclusion that is set in stone. This is mainly due to the lack of data which we will discuss in Chapter 14.

12.1 Short Term Maturity for CDS option

We know very little about the market for CDS option but several articles among them [Alloway] indicate that the traded CDS option has a very short maturity of three to six months. Let us for moment discuss the value of the short-term option on a CDS contract to the buyer of the option. If an investor buys a CDS call option with a maturity of 6 month, he buys the right to enter into a CDS contract in 6 month. If the investor truly buys the CDS option for the protection then the 6 month maturity suggest that he is worried about the underlying reference entity defaulting 6 month from now. However, he cannot be worried that the underlying reference entity will default before the 6 month given that most traded CDS options are knock-out options. A knock-out option means that if the reference entity defaults before the maturity of the option the protection will stop (See the Appendix in Chapter 15 for further explanation).

Despite the lack of data and literature about this paradox of investors being worried in 6 month but not before, it is fair to assume that short-term maturity options might be traded for different reasons. The following section will assume that the short-term maturity options are traded for speculative purposes. We will then look at the trading strategy from an investor's point of view and calculate the profit of this strategy over a period of time. We will see that in our dataset it is not possible to profit from buying CDS call option. However, the dataset that we have is too small to conclude that this is always the case.

12.2 Performance of trading CDS options

12.2.1 The strategy

Let us say that we are working at a huge hedge fund wishing to enter the CDS option market to earn a profit. The CDS option market is a rather complicated market to trade in given that the CDS contract works like a forward contract, hence there is zero cost related to entering into the contract. This means that if the trader chooses to exercise the option at the time of the option expiry, he cannot close the position by selling a CDS contract at a low price and earn the difference. This can be illustrated as in Figure 12.1. At time 0 the trader buys the option by paying the option premium illustrated by the negative black line. Buying this option he obtains the right to enter into a CDS contract at a pre-specified premium K. At time T_a the option expires as the trader can then choose whether or not he will exercise the option. In Figure 12.1 the black lines between T_a and T_b illustrate the pre specified premium K. At time T_a the price of the CDS spread has evolved and the difference between the market price of the premium and the pre-specified premium can be illustrated by the red lines. In this case the trader will earn the differences (the red lines) until the end of the CDS contract. If the CDS contract was bought by paying an upfront price the trader would have closed the position at time T_a when he chooses to exercise. Instead, we need to discount back the difference in payments to obtain the profit from trading the CDS contract

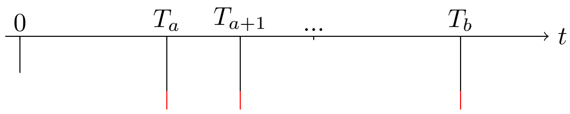

Figure 12.1: Earning af profit on a CDS option

Next we will walk through a numerical example trading a CDS option on the Federal Republic of Germany. Consider the following trading strategy:

- On 20 December 2013: Buy a Call option on a 5Y CDS contract with expiry data on 20 June 2014 and Strike=20bps. Pay the option price 0.0066934.
- On 20 June 2014 the 5Y CDS spread was 21.12648bps hence one would exercise the option. The profit will then be the difference between the pre specified spread and the 5Y spread equal to 1.12648 bps.

One would earn the above profit every payment day (every quarter) for the next 5 years. If we take these future profits, discount them back until 20 June 2019 we can calculate the return of the investment (ROI).

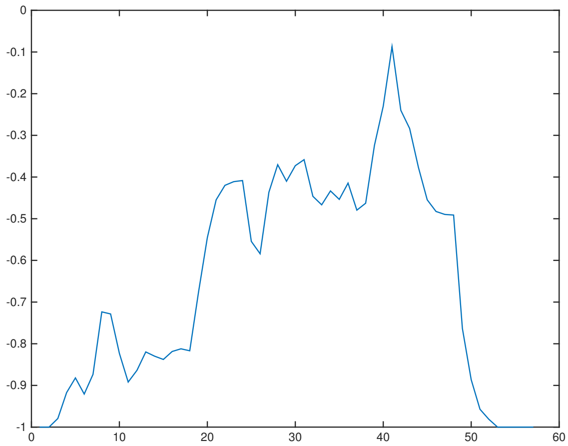

Figure 12.2: Return of buying one 5Y Call option with maturity 6 month later for 57 consecutive days, Germany

$$
\begin{aligned}
ROI =& \frac{(PV_{profit} - Investment)}{Investment} && (12.1)\\
=& \frac{(0.0012648 - 0.0066934)}{0.0066934} && (12.2)\\
=& -0.811 && (12.3)
\end{aligned}
$$

The above calculation implies that the return of investment in this case was -81%. However, before we can conclude that trading CDS options would be a horrible investment strategy, we would have to do two things. First, we will have to calculate the return of investment for several trading days. Second, we will have to compare the performance of this strategy to simply investing in the market portfolio. Those two things are exactly what we will do in the next subsection by introducing the Capital Asset Pricing Model (CAPM).

12.2.2 CAPM

Consider the above-mentioned trading strategy. For multiple days we now wish to calculate the return of buying one call option, exercising the option at maturity 6 months later, thereby earning a profit for 5 years. If the Call option is out-of-the-money at expiration day the return is -100%. Figure 12.2 shows the return of trading by this method on CDS option with the reference of Germany for 57 consecutive days. Even though the investor exercises the option most days earning a profit is not enough to cover the pre-paid option premium. In this short period the trading strategy is performing very badly. However, the reader should notice that the data-collection is too small for any conclusion to be made.

Let us now compare the above returns to the return of the market in the same time period. To do this we will use the famous capital asset pricing model (CAPM) developed by Treynor, Jack, Lintner, John, Mossin, Jan and Sharpe, William F. independently (see Sharpe [1964]). The model builds upon the Mean-Variance model by [Markowitz, 1968]. The CAPM states that the expected excess return on

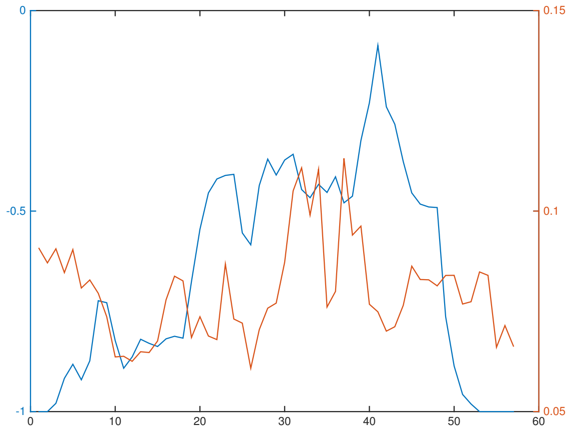

Figure 12.3: Return of the trading CDS option with reference Germany + Return on SP500

a risky asset is given by the product of systematic risk and the excess market return. The systematic risk is denoted β and expresses the non-diversifiable risk. The CAPM is therefore given by

$$\mathbb{E}[R_i] - R_f = \beta(\mathbb{E}[R_m] - R_f) \tag{12.4}$$

where $\beta = Cov(R_i, R_m)/Var(R_m)$ Figure 12.3 shows the return of the option-strategy and the return of the market, the market being the SP500 index. Even though the return of the option strategy is negative we see that the movements are positively correlated to the movements in the market. When using the CAPM model we find a beta of 3.72, but the tstat is 1.31¡2 hence insignificant. In a moment we will discuss why the CAPM produces insignificant results in our setting.

The previous calculations and research was made under the assumption that CDS options was mainly traded for the purpose of profiting. The speculative trading impression was a result of CDS options having mostly short-term maturity as mentioned in [Alloway]. The above research suggests that trading CDS option is a non-profit trading strategy. However, due to the lack of data we cannot finally conclude that this is the case. Therefore, we will - in the rest of this chapter - assume that trading the CDS option is for other reasons than speculating, and we will examine if the CDS option on Germany serves as a good hedge for specific risk exposure. In Figure 12.3 we already saw that the movements in the market (SP500) were correlated to the movements in trading the CDS options on Germany. This means that the CDS option on Germany is not well suited for hedging risk exposure to the market. A natural next step would be to test whether the CDS option on Germany can hedge risk exposure to the German market represented in by the DAX-index. Remember that a portfolio with $\beta = 0$ is market neutral and therefore independent of the movements in the market. Therefore, what we are looking for is a negative beta to ensure that a combined portfolio with both market portfolio and the CDS option on Germany will be market neutral. Figure 12.4 shows the return on the option-strategy and the return on the DAX-index. Given that the correlation between the DAX-index and SP500 is 0.991 in the specific time period that we are looking at, it is not surprising that the beta relating the DAX index and the option-strategy is also positive around 3. Once again we notice that the beta is insignificant and therefore we cannot conclude the positive relationship without using a larger dataset.

Next step would be to perform the same calculation but with Italy as the reference entity. The

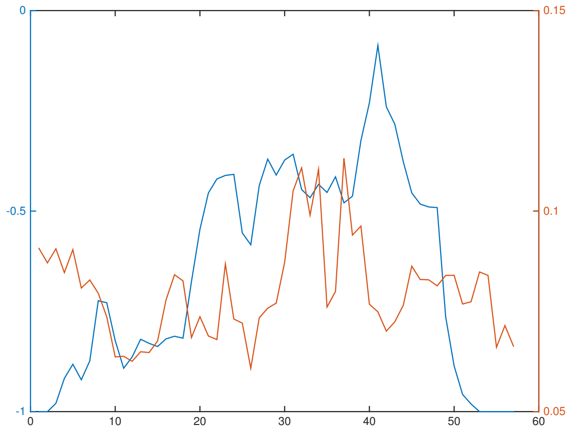

Figure 12.4: Return of the trading CDS option with reference Germany + Return on DAX

results of such research would have been even more interesting given that we have 100 observations and Italy has a lower credit rating. However, when looking at our data we observe that for all of our 100 observations the options are never in-the-money resulting in a profit of -100%. This is due to the major decrease in the fair CDS spread in the specific time period on which we posses data as illustrated in Figure 12.5.

A natural final step would be to perform the above calculation on a much larger dataset and on multiple countries, indices and even single companies. However, as we will return to in Chapter 14 the research of this thesis is lacking information and data on relevant variables. E.g. we observe missing data points about smaller countries with lower credit rating when collecting data from [Markit, b].

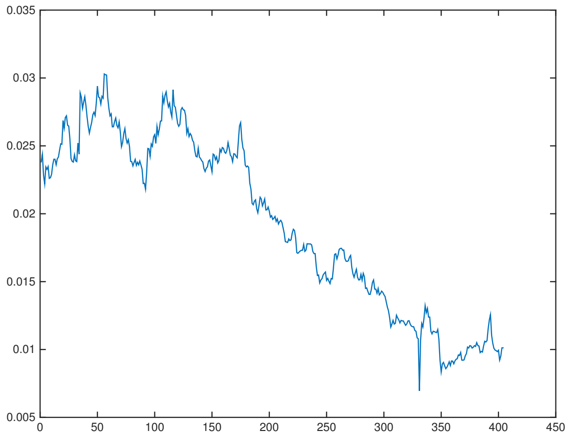

Figure 12.5: Calculated fair spread of a 5Y Call option, Italy

Chapter 13

Concluding Remarks

The purpose of this thesis was to investigate whether the relatively new derivative the Credit Default Swaption makes a difference – in terms of value or risk - to investors, financial institutions, companies or even to private individuals.

Firstly, I described the CDS option and the different components of the CDS option. I discussed the underlying Credit Default Swap, the market for the CDS and the market for the CDS option. Furthermore, I described how financial institutions, investors and large companies might benefit from this new product.

Secondly, I investigated how to price the CDS option. Based on [Brigo and Morini, 2005] I arrived at an extension of the Black-Scholes formula and thereby I succeeded in pricing the CDS option. I discussed the critique of the Black-Scholes formula given that the critique is also relevant for our pricing model. I then presented the most common extensions of the Black-Scholes formula.

Thirdly, I presented an empirical analysis of the Credit Default Swaption. By successfully calibrating the survival probability of a reference entity I was able to calculate the option price of a CDS. Afterwards, I discussed my results and compared the results to the research of others.

Finally, I analyzed data on implied volatility of the CDS options with Germany and Italy as reference entity – two countries with different credit ratings. I find that the CDS options are often short-term-maturity options indicating that investors trade the CDS option for the short-term profit and not for the long-term reduction in risk exposure. However, the profit from trading the CDS option in the time frame in which I have data is negative. I also find that the implied volatility of the CDS option significantly forecasts the volatility of the CDS spread no matter the credit rating of the reference entity. This suggests that the CDS option reflects the underlying asset thereby indicating that the CDS options might be traded not only for profiting but also for reducing different risk exposures.

During the entire thesis my research has been limited by the lack of data on traded CDS options. Therefore I cannot conclude that the CDS options are traded only for the purpose of profiting or only for the purpose of hedging risk exposure. However, we would also expect the incitements for trading the CDS options to be mixed. Therefore, I encourage the reader to further analyze the market for CDS option when the financial institutions publish more data.

This thesis can be used not only for researching the CDS option but also as a guideline on how to analyze new derivatives.

Chapter 14

Further research

I end this thesis with a discussion of additional research that I would have wished to investigate. A huge drawback in this thesis when describing the CDS option is the lack of data. Therefore, I will discuss the research that we could have done if we have had data on traded CDS options.

For now assume that we have unlimited data on the CDS options.

Firstly, when we have data on the trading amount of the CDS option we can confirm that the foundation of this thesis is justifiable. Additionally, having data on the amount of which the CDS options are traded will give us more answers to why the CDS options are being used. If the trading amount of the CDS option is comparable to the trading amount of the underlying reference entity it would indicate that investors try to hedge their risk exposure in the reference entity. This is also what we discussed in Chapter 4.

Secondly, if we have data on price of the CDS option in the market we are able to calculate the true implied volatility and not just the volatility found in the Bloomberg CDSO Calculator. This implies that we can verify the research made in Chapter 12. This includes further tests of the relationship between the implied volatility of the CDS option and the historical volatility of the CDS contract.

Thirdly, I would perform the profit-analysis from Chapter 12 for multiple reference entities and larger dataset. By this analysis we would be able to conclude whether or trading the CDS options yield any profit and if the profit differs depending on the credit rating of the reference entity.

Finally, I would use the additional data to search for investment assets that are negatively correlated with the CDS Option. This would suggest a new hedging possibility. Next, I would compare the trading amount of the investment asset to the trading amount of the CDS Option. I would then be able to conclude if the CDS option serves as a good hedging instrument.

I end this chapter with a note on how to extend the theoretical pricing formula. A natural next step would be investigating the data on the CDS contract to see how it fit with the assumptions of the Black-Scholes model. Knowing the behaviour of the different variables in the CDS contract will indicate how to improve the pricing formula.

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Chapter 15

Appendix

15.1 Legal Terms regarding the CDS Contract

In this section I present some important legal terms regarding the CDS contract. In the thesis I have made some assumptions for simplicity. These assumption will also be presented in this section.

Quaterly Role

In 2003 the market for CDS contract was regulated. Before 2003 the CDS contracts had maturity for a total number of years. However, according to [O'Kane, 2011] rolling contracts were introduced. This means that there exists four standard maturity dates - 20 March, 20 June, 20 September and 20 December. Then a 5Y CDS contract will have maturity the first standard date after the five years. In this thesis I have assumed that this was familiar to the reader hence all treated CDS contracts have maturity dates on the standard dates. The Bloomberg calculator CDSO does also take the standard dates into account.

Credit Event

There exists a number of different credit event triggering a "default" but in different ways. Throughout this thesis I have only used data on "Complete Restructuring of Debt" and I refer to [O'Kane, 2011] for an analysis of the different credit events.

Knockout vs Non-Knockout options

The different between a knockout option and a non-knockout option is their reaction to a default before the option maturity. If a credit event occur before the option maturity the knockout option stops immediately and you cannot enter the CDS contract. When buying a non-knockout option this is possible. The most traded option is the knockout option hence I have only used that type of option during the entire thesis.

15.2 Treasury Yield vs. Swap Rates

In this thesis we have used the US Treasury Yield Data while one may argue that we should have chosen to use swap rates. [Feldhütter and Lando, 2008] test and show that the swap rate is a better proxy for the risk-less rate than the treasury yield for all maturities. The focus of this thesis is not how to best proxy the risk-less rate and we have only chosen the treasury yield for simplicity and data availability.

15.3 Bloomberg CDSO calculator

The Bloomberg CDSO calculator is tool to help price these the Credit Default Swaption. In the Bloomberg CDSO calculator it is possible to calculate the CDS option on different trading days and with different maturities. In this thesis I have used the previously mentioned CDS spread data from Markit together with the Bloomberg CDSO calculator. Figure 15.1 is a screen-shot from the Bloomberg CDSO calculator illustrating the method behind the calculated CDS option prices.

15.4 Additional Results

In this section I discuss additional results of CDS option prices and implied volatility. In Table 15.1 I have illustrated the differences in the CDS option price when using Bloomberg CDSO compared to when I calculate the price. I use the implied volatility of the Bloomberg CDSO calculator.

STRIKE=0,0170	CDS spread (Bloomberg)	Bloomberg implied vol	OptionPrice (Bloomberg)
	0,0177882	0,37536	0,0064005
	CDS spread (Thesis)	Bloomberg implied vol	OptionPrice (Thesis)
	0,017662859401105	0,37536	0,006216517855587

Table 15.1: Comparing the prices calculated from this thesis to the prices of Bloomberg CDSO

Table 15.2 shows CDS option prices for different strikes both with the Bloomberg CDSO calculator and the method used in my empirical implementation.

15.5 Guide to Data Appendix

Code

In the Data Appendix the first part will be the code from the empirical implementation. I have tried to make the coding as readable as possible.

Bloomberg CDSO

When calculating the implied volatility of the CDS Option I used the Bloomberg CDSO calculator. I collected the screen-shots with the calculated implied volatility that I have used. If the reader wish to go through the 200 screen-shots, making sure that I did not make any mistakes, I can easily forward them.

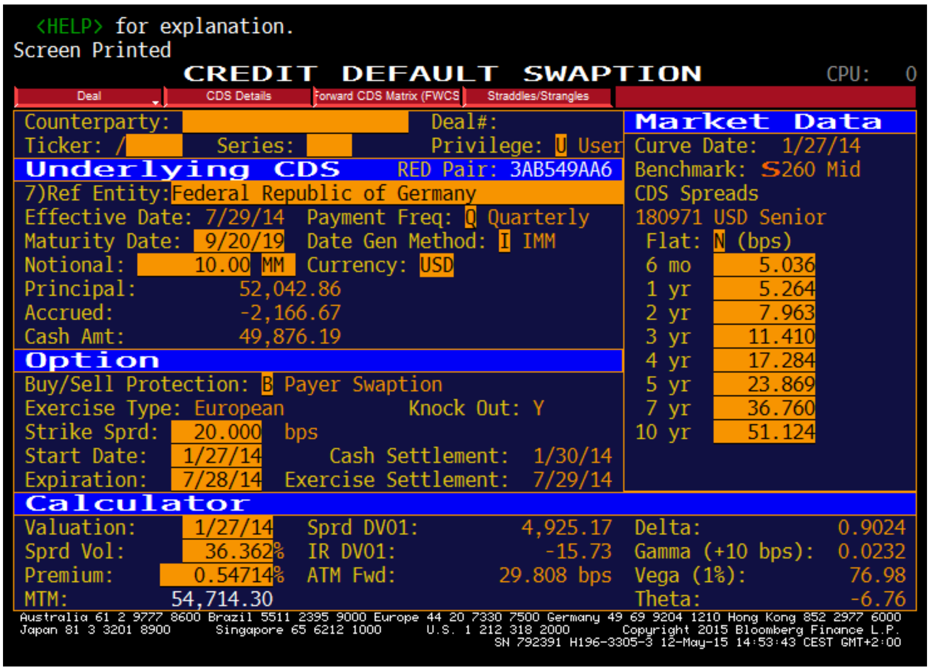

Figure 15.1: Example of calculating CDSO option prices

Strikes	Option Premium (Bloomberg)	Option Premium (Thesis)
100	0,037727 0	0,037791341
140	0,0184444	0,018205341
150	0,0138983	0,013635266
160	0,0097999	0,009558477
170	0,0064005	0,006216518
180	0,0038492	0,003733995
190	0,002128	0,002070469
200	0,0010835	0,001062611
210	0,0005012	0,000507101
220	0,0002234	0,000226305
230	0,0000915	0,0000950
250	0,0000129	0,0000143
300	0	6,32E-08

Table 15.2: Comparing the Option Premium find by Bloomberg CDSO to the Option Premium find in this thesis, different strikes

The Data Appendix

Code from Matlab

Master File

```
%Great Master Instruction

%run MasterYield to calculate yield curve, change to
    right yield document
ZeroCoupon='Italy190313.xlsx';
A=xlsread(ZeroCoupon);
R_save=nan(1,404);
%MasterYield
%for i=123:123

par=A(1,9:14);
     time = 1:1:30;
     par=par';
     r_new_trial=nan(1,601);
     t_y= 0:0.05:30;
     timepoint=1;
    %yield = A(timepoint,1:30);
     %r = yield;

    for j=1:601
          Ti=t_y(j);
          r_new_trial(j)=PriceZeroFixedTrue(Ti,par);
    end

d=0.4;
%yieldcurve1=plot(t_y,r_new_trial)%,time,yield,'*'); % nr
     jeg vil plotte en alene


% plot with changes
```

```
Y=par;
%PlotYield
%yieldcurve2=plot(t_y,yield_p,time,yield,'*'); % nr jeg
    vil plotte en alene
%print -depsc Yield190313


%rund MasterSurvival to calculate survival curve, change
    to right Spread
%document
Ti=1;
C=MasterSurvival(Ti,A,Y);

PlotSurvival
q1=calc';
figure; hold on
a1 = plot(t_1,q1);
xlim([0 t_1(end)])
M1 = '19032013';
legend(a1, M1);
xlabel('Years')
ylabel('Probability')
print -depsc SurvItaly190313



%plot calculated spreads against observed
SpreadFitCheck

%Run CalculateR to calculate the corresponding coupon

R=CalculateR(1,A,Y,C);
R_save(1)=R;
```

Extract the yield curve

```
%Price at time t of a risk-less zero-coupon bond with
    maturity s
function [price] = PriceZeroFixedT(s,par)

% set date for your fixed t

e_1_fix=exp(-s/par(5));
e_2_fix=exp(-s/par(6));

price= par(1)+par(2)*(1-e_1_fix)./(s./par(5))+par(3)*((1-
```

```
    e_1_fix)./(s/par(5))-e_1_fix)+...
     par(4)*((1-e_2_fix)./(s/par(6))-e_2_fix);
end

function rate=YieldCurv(time,Y)

beta_0=Y(1);
beta_1=Y(2);
beta_2=Y(3);
beta_3=Y(4);
tau_1=Y(5);
tau_2=Y(6);

if time<1

e_1=exp(-1/tau_1);
e_2=exp(-1/tau_2);

rate_1=beta_0+beta_1*(1-e_1)./(1/tau_1)+beta_2*((1-e_1)
    ./(1/tau_1)-e_1)+...
     beta_3*((1-e_2)./(1/tau_2)-e_2);

rate=rate_1*time;

else

e_1=exp(-time/tau_1);
e_2=exp(-time/tau_2);

rate=beta_0+beta_1*(1-e_1)./(time/tau_1)+beta_2*((1-e_1)
    ./(time/tau_1)-e_1)+...
     beta_3*((1-e_2)./(time/tau_2)-e_2);
end
```

Calibrating the Survival Curve

```
function C = MasterSurvival(tid,A,Y)
%%
%Master Survival

% Extract S_obs from specific t, different yields

%Sobs='ItaSpread0502014.xlsx';
%Mat=xlsread(Sobs);
S_obs=A(tid,1:8);
```

```
%survival prob
    GetSurvivalFirstDocument
    PlotSurvival
    C=par;
end

function [minimum] = BestSurvival( t, T,Yieldpar,Calpar,d
    ,S_obs, freq)

par = Calpar;

S_calc = nan(1,length(T));

for i=1:length(T)
Ti  = T(i);
ProtectionLeg(i) = Prot(t,Ti, Yieldpar,par,d,freq);
PremiumLeg(i)= Prem(t,Ti, Yieldpar,par,freq);

S_calc(i)=ProtectionLeg(i)./PremiumLeg(i);
end

minimum =sum( ((S_calc-S_obs)./S_obs).^2);

end

function [survival] = SurvivalProb(t,t_pay,par)
% skal bruge t og t_pay

% t=0 kun her i starten

alpha_1=par(1);
alpha_2=par(2);
alpha_3=par(3);
alpha_4=par(4);
alpha_5=par(5);

e_1_1=exp(-alpha_1*(t_pay-t));
e_2_1=exp(-alpha_3*(t_pay-t)^2);
e_3_1=exp(-alpha_5*(t_pay-t)^3);

survival = 1./(1+alpha_2+alpha_4)*(e_1_1+alpha_2*e_2_1+
    alpha_4*e_3_1);

end
```

```
% Create matching timepoints

C=nan(1,5);
T = [0.5,1,2,3,4,5,7,10];
freq =0.25 ; % * ones(length(T))
t=0;

%%
d= 0.4;

para_0=[0.5,0.5,0.5,0.5,0.5];
my_optim = optimoptions(@fmincon,'Display','off', '
    TolFun',...
10^-20,'TolX',10^-20,'MaxFunEvals',10000,'
    DiffMinChange',10^-20);
problem=createOptimProblem('fmincon','objective',@(x)
    ...
BestSurvival(t,T,Y,x,d,S_obs,freq) ,'x0',para_0,'
    options',my_optim);
gs=GlobalSearch;
[par]=run(gs,problem);

%C=par;
```

The Protection Leg

```
function [protection] = Prot(t,T, Yieldpar,Calpar,d,freq)

summation= 0;
t_pay =t+freq:freq:T;
for j = 1:length(t_pay)

    if j==1
        t_pay_snit = t_pay(j)/2;
        tj_minus = t;
    else
        t_pay_snit=(t_pay(j) + t_pay(j-1)) / 2 ;
        tj_minus = t_pay(j-1);
    end

    B = PriceZeroFixedTrue(t_pay_snit,Yieldpar) ; %
        changed
    Qtj_minus = SurvivalProb(t,tj_minus,Calpar);
    Qtj = SurvivalProb(t,t_pay(j),Calpar) ;
    ep=exp(-B*t_pay_snit);
```

```
    summation = summation+ep*(Qtj_minus-Qtj); %changed *
        t_pay_snit
end

modi_2=mod(T-t,0.25);
T_snit=(T+(T-modi_2))/2;
P_modi=PriceZeroFixedTrue(T_snit,Yieldpar); %changed
Q_modi_minus=SurvivalProb(t,(T-modi_2),Calpar);
Q_modi=SurvivalProb(t,T,Calpar);
k=exp(-P_modi*T_snit)*(Q_modi_minus-Q_modi);
summation_l=summation+k; %changed *(T+(T-modi_2))/2

protection = (1-d)*summation_l;


end
```

The Premium Leg

```
function [premium] = Prem(t,T, Yieldpar,Calpar,freq)
summation1= 0 ;
summation2= 0 ;

t_pay =t+freq:freq:T;
for j = 1:length(t_pay)
    if j==1
        t_pay_snit = t_pay(j)/2;
        t_pay_inc = t_pay(j)/2;
        tj_minus = t ;
    else
        t_pay_snit=(t_pay(j) + t_pay(j-1)) / 2 ;
        t_pay_inc = (t_pay(j)-t_pay(j-1))/2 ;
        tj_minus = t_pay(j-1) ;
    end

    B = PriceZeroFixedTrue(t_pay_snit,Yieldpar); %changed
    Qtj_minus = SurvivalProb(t,tj_minus,Calpar);
    Qtj = SurvivalProb(t,t_pay(j),Calpar) ;
    su=exp(-B*t_pay_snit)*t_pay_inc*(Qtj_minus-Qtj);
    summation1 = summation1+su; %changed *t_pay_snit
end
modi_2=mod(T-t,0.25);
T_snit=(T+(T-modi_2))/2;
T_inc=(T-(T-modi_2))/2;
B_2 = PriceZeroFixedTrue(T_snit,Yieldpar); %changed
Q_1 = SurvivalProb(t,(T-modi_2),Calpar);
```

```
Q_2   =   SurvivalProb(t,T,Calpar);
k=exp(-B_2*T_snit)*(Q_1-Q_2)*T_inc;

summation1_r=summation1+k;%changed *(T+(T-modi_2))/2

 for j = 1:length(t_pay)
      if j==1
         t_pay_inc = t_pay(j);
      else
          t_pay_inc = t_pay(j) - t_pay(j-1);
      end
      su=0;
      B = PriceZeroFixedTrue(t_pay(j),Yieldpar) ; %changed
      Qtj = SurvivalProb(t,t_pay(j),Calpar);
      trial=exp(-B*t_pay(j));
    su=trial*t_pay_inc*Qtj ;
    summation2 = summation2+su;%changed*t_pay(j)
 end
T_inc=(T-(T-modi_2));
B_2 = PriceZeroFixedTrue(T,Yieldpar); %changed
k=exp(-B_2*T)*T_inc*SurvivalProb(t,T,Calpar);
summation2_r=summation2+k;%changed *T
premium = summation1_r + summation2_r;
end
```

Calculating CDS option

```
%calculate d_1 manually

strike = 0.0250;
sigma= 0.37536; %0.382775342891781; %0.393446866701957;
d_1 = (log(R_trial2/strike)+sigma^2*a/2)./(sigma*sqrt(a))
    ;
d_2 = d_1-sigma*sqrt(a);

parent_trial=R_trial2*normcdf(d_1)-strike*normcdf(d_2);

Call_prem=parent_trial*premium;

Volatility = blsimpv(R_trial2, strike, 0, b, 0.0064005);


sigma=0.37536;
R_bloom=0.0177882;
d_1 = (log(R_bloom/strike)+sigma^2*a/2)./(sigma*sqrt(a));
d_2 = d_1-sigma*sqrt(a);
```

```
parent_trial=R_bloom*normcdf(d_1)-strike*normcdf(d_2);

Call_bloomberg=parent_trial*premium;

Volatility_bloom = blsimpv(R_bloom, strike, 0, b,
    0.0038492);

T_E=0.11781;
t=0;
S_pay=0.0182767;
K_ex=0.014;
sigma=0.4;
Risky=4.8354;

divide=S_pay/K_ex;
d_1_pay=(log(divide)+sigma^2*(T_E-t)/2)./(sigma*sqrt(T_E-
    t));
d_2_pay=d_1_pay-sigma*sqrt(T_E-t);

Black_pay=S_pay*normcdf(d_1_pay)-K_ex*normcdf(d_2_pay);

option_payer=Black_pay*Risky;
```

Comparing to White-article

```
%Extract yield curve
     ZeroCoupon= 'YieldOpen1.xlsx';
     A=xlsread(ZeroCoupon);

     %%
     %yield curve
   timepoint=2;
   % Y=nan(1,5);

% Uses FitSpotRate + GetSpecificSpotRates finding Nelson
    siegel parameters
% for fixed timepoint
min=30/360;

     time = [min,2*min,3*min,6*min
         ,1,2,3,4,5,6,7,8,9,10,12,15,20,25,30];
     t_y= 0:0.05:30;
     yield = A(timepoint,1:19);
     r = yield;
```

```
[r_new,par] = GetSpecificSpotRates(r,t_y,time);
 %run med denne dato, specielt data, gemmer i figure
  Y=par;
  yieldcurve1=plot(t_y,r_new,time,yield,'*');
  print -depsc OpenGammaYield

  %%
%Master Survival

% Extract S_obs from specific t, different yields

Sobs='SurvOpen.xlsx';
Mat=xlsread(Sobs);
S_obs=Mat(1,1:8);

%survival prob
    GetSurvivalFirstDocument
    PlotSurvival

   %Plot survival
q1=calc';
figure; hold on
a1 = plot(t_1,q1); M1 = '020514';
legend(a1, M1);
print -depsc OpenGammaSurv

%%
% Spread calc
Spread_plot=nan(8,1);

for j=1:8
     Si=T(1,j);
     Spread_plot(j)=Prot(0,Si, Y,C,d,freq)/Prem(0,Si, Y,C,
         freq);
end
T_c=T';
SpreadCurve=plot(T_c,Spread_plot,T_c,S_obs,'*');
print -depsc OpenGammaSpread

%Si=S_obs(1,i)

%%
%CALCULATE R
LGD=1-d;
summation_R=0;
a=0.11781; %days between 05/02 og 20/03 trading, 32 eller
```

```
     almindeling = 43?
b=1918/365.25; %Days between 20/03-14 og 20/06-19 trading
     .. eller alm = 1918,
for i=a+0.25:0.25:b-0.000000001
     A=YieldCurv(i,Y);
     su=exp(-A*i)*(SurvCurv(i-0.25,C)-SurvCurv(i,C));
     summation_R = summation_R+su;
end
A_b=YieldCurv(b,Y);
modi=mod(b-a,0.25);

Upper_R=LGD*summation_R+LGD*exp(-A_b*b)*(SurvCurv(b-modi,
    C)-SurvCurv(b,C));

summation_R2=0;
for i=a+0.25:0.25:b-0.00000001
     A2=YieldCurv(i,Y);
     su=exp(-A2*i)*0.25*SurvCurv(i,C);
     summation_R2 = summation_R2+su;
end

Lower_R=summation_R2+modi*exp(-YieldCurv(b,Y)*b)*SurvCurv
    (b,C);


R_True=Upper_R/Lower_R;
```

Calculating CDS volatility

```
%% Calculate daily vol

Spreads= 'ItalySpread.xlsx';
SP=xlsread(Spreads);
t_spread1=1:715;
plot(t_spread1,SP);
print -depsc CDSSpread_ITA;

Spread6M=nan(715,5);
for i=1:715
     Spread6M(i,1)=SP(i,6);
end
for i=1:715
     if i==1
          Spread6M(i,2)=0;
     else
          s6i=Spread6M(i,1);
```

```
        s6i1=Spread6M(i-1,1);
        divs6=s6i/s6i1;
    Spread6M(i,2)=log(divs6);
    end
end
AB=nan(715,1);
for i=1:715
AB(i)=Spread6M(i,2);
end
ABC=nan(715,1);
for i=1:715
        ABC=cumsum(AB);
end
for i=1:715
    if i<100 %changed
        Spread6M(i,3)=0;
    elseif i==100 %changed
        Spread6M(i,3)=AB(i);
    else
        Spread6M(i,3)=AB(i)-AB(i-100); %changed
    end
end
for i=1:715
    Spread6M(i,4)=Spread6M(i,3)/100; %changed
end
ABCD=nan(1,100); %changed
ABCDE=nan(1,615); %changed
for j=1:615 %changed
        for i=j:99+j %changed
            Spread6M(i,5)=(Spread6M(i,2)-Spread6M(100,4))
                ^2; %changed
        end
        for i=0:99 %changed
        ABCD(i+1)=Spread6M(i+j,5);
        end
    tests=sum(ABCD);
    kvad=sqrt(1/99*tests); %changed
    ABCDE(j)=kvad;
end
t_ABCDE=1:615; %changed
plot(t_ABCDE,ABCDE);
print -depsc CDSvol_ITA10;
CDSvol=nan(1,101);
for i=1:101
  CDSvol(i)=ABCDE(i+250);
end
```

```
t_CDS=1:101;
plot(t_CDS,CDSvol);
test2= 'DataITA1.xlsx';
CDSOVol=xlsread(test2);
CDSOvol1=nan(1,101);
for i=1:101
CDSOvol1(i)=CDSOVol(i,1);
end
plot(t_CDS,CDSOvol1);
print -depsc CDSOvol_ITA100;
[ax,p1,p2] = plotyy(t_CDS,CDSvol,t_CDS,CDSOvol1);
print -depsc CDSvol_CDSOvol_ITA;
corrtest = corrcoef(CDSvol,CDSOvol1);

X0 = CDSOvol1;          % Initial predictor set (matrix)
%predNames0 = series(1:4); % Initial predictor set names
%T0 = size(X0,1);            % Sample size
Y0 = CDSvol;              % Response data
%respName0 = series{5};     % Response data name
M0 = fitlm(X0,Y0);
```

Calculating beta

```
Yields='YieldToGerm.xlsx';
B_trial=xlsread(Yields);
Spreads='CDSToOption.xlsx';
C_trial=xlsread(Spreads);
Return_disc=nan(1,57);
RiskF='RiskFree1.xlsx';
R_f=xlsread(RiskF);
OptionData='DataGe.xlsx';
z=xlsread(OptionData);
SP='SP500.xlsx';
SP500=xlsread(SP);

for j=1:57
    par=B_trial(j,1:6);
    summation=0;
price=C_trial(j,6);
if price>0.0020
    Return=price-0.0020;
    for i=1:0.25:5
        B=PriceZeroFixedTrue(i,par);
        su=Return*exp(-B*i);
        summation=su+summation;
```

```
        end

        Return_disc(j)=summation;

 else
        Return_disc(j)=0;
 end
 end

 Rf=nan(1,57);
 R_afk=nan(1,57);
 for i=1:57
        Option=z(i,2);
 R_afk(i)=(Return_disc(i)-Option)/Option;
 end

 for j=1:57
        par=B_trial(j,1:6);
        Rf(j)=PriceZeroFixedTrue(0.5,par);
 end

Rm=nan(1,57);
 for i=1:57
        market=SP500(i,1);
        market2=SP500(i,2);
Rm(i)=(market2-market)/market;
 end

 beta_t=cov(R_afk,Rm)/var(Rm);


 Data=nan(2,57);
 for i=1:57
        %Data(i,1)=R_afk(i);
        Data(1,i)=Rf(i);
        Data(2,i)=(Rm(i)-Rf(i));

 end

%beta_trial=nan(1,57);
%for i=1:57
%      beta_trial(i)=R_afk(i)/Rm(i);
%end

X0 = Data';          % Initial predictor set (matrix)
%predNames0 = series(1:4); % Initial predictor set names
```

```
%T0 = size(X0,1);               % Sample size
Y0 = R_afk';                % Response data
%respName0 = series{5};      % Response data name
M0 = fitlm(X0,Y0);

[ax,p1,p2] = plotyy(t_afk,R_afk,t_afk,Rm);
print -depsc R_afk_Rm_SP500;
print -depsc R_afk_Rm_DAX;
%RSP=Rm;
%RDAX=Rm;
trytry=plot(t_afk,R_afk);
print -depsc R_afk;

corr(R_SP,RDAX);
```