Master thesis

A stochastic DCF model with a recursive calculation of the value of an enterprise

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Executive summary

The present thesis focuses on the discounted cash flow model. The discounted cash flow model is also known under the name DCF model. The DCF model is used for valuation of an enterprise. As the name implies the discounted cash flow model determines the value of the enterprise by discounting a future cash flow to a present value. However, the discounting rate used for the discounting is in itself a function of the enterprise value, i.e. the enterprise value in the DCF model is a result of a function that depend on the result of that same function. In the literature this is known as a recurrence problem (Larkin, 2011) or a circularity problem (Mohanty, 2003).

For a DCF model in discrete time the recurrence problem has been solved with iteration algorithms without major difficulties. However, for a stochastic DCF model in continuous time the recurrence problem seems to be more difficult to solve. It is therefore asked how it is possible to account for the recursive nature of the enterprise value in a stochastic DCF model.

The recurrence problem is solved by proposing an iterative DCF model. The iterative DCF model takes its starting point with the Monte Carlo method. The Monte Carlo method is suitable for numerically generating a cash flow as a function of time. The cash flow is then discounted backwards step by step until present time. This stepwise discounting allows for an isolation of the enterprise value such that there is no recursive or circular relationship between the enterprise value and the discounting rate.

With the proposed iterative DCF model it becomes possible to see how the recursively calculated enterprise value compare to a more simple DCF model. For a low leveraged enterprise there is no difference between the iterative DCF model and a simple DCF model. Thus, for the low leveraged enterprise it would seem ok to use a simple DCF model when determining a value of an enterprise.

However, for a high leveraged enterprise it surprisingly turns out that the error on the enterprise value by using a simple DCF model is quite large. Thus, for the high leveraged enterprise an iterative DCF model should be used or it should at least be borne in mind that there is a large error involved in using a simple DCF model.

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List of frequent used symbols

Symbol	Variable	Translation to Danish	Symbol
EV	Enterprise value	Virksomhedsværdi	EV
MV	Market value of equity	Markedsværdi af egenkapital	MV
B	Net interest-bearing debt	Netto rentebærende gæld	$NRBG$
EK	Equity	Egenkapital	EK
IC	Invested capital	Investeret kapital	IC
TR	Revenue	Omsætning	Oms
$EBIT$	Earnings before interest and tax	Resultat af primær drift	$EBIT$
$NOPAT$	Net operating profit after tax	Driftsresultat før skat	$NOPAT$
CF	Cash flow	Pengestrøm	CF
g	growth	vækst	g
Π	EBIT margin	Overskudsgrad	OG
Λ	Asset turnover	Aktivernes omsætningshastighed	AOH
σ	Revenue volatility	Standard afvigelse af omsætning	σ
var	Variance	Varians	var
L_B	Book leverage	Indre gearing	L_B
L_M	Market leverage	Markeds gearing	L_M
τ	Tax rate	Skattesats	τ
y	Yield on debt	Netto lånerente	NLR
$wacc$	Weighted average cost of capital	Vægtede kapitalomkostninger	$wacc$
E[]	Expected value	Forventet værdi	E[]
T	Time	Tid	t

1. Introduction

One of the more popular models for valuation of a business enterprise is the Discounted Cash Flow model otherwise known as the DCF model, (Plenborg, 2000).

In the present thesis focus will be directed to the DCF model for modeling the value of an enterprise.

Some of the reasons to the popularity of the DCF model is that it is relatively intuitive and also reasonably straightforward to implement in praxis. It is furthermore independent of accounting principles, (Plenborg, 2000).

The DCF model is a relatively simple model in that the value of the enterprise is modeled as a function of the present value of the cash flow that the enterprise can generate in the future to the security holders of the enterprise, (Brealey & Meyers, 2003). The present value is determined by applying a discounting factor to the cash flow.

Thus, the DCF model is a relatively simple model where in principle only two factors is needed as inputs in order to model the enterprise value. The first factor being the cash flow that the enterprise can generate in the future to the security holders of the enterprise. The second factor being the discounting factor used to discount the cash flow to a present value.

Despite of this simplicity of the DCF model it does have some disadvantages just like any other theoretical model has a number of disadvantages.

1.1. Purpose of the present project

In view of the popularity of the DCF model it is desirable to see if at least one of the disadvantages of the DCF model can be overcome.

The purpose of the present project is therefore to propose an improvement to the DCF model in order to overcome at least one of the disadvantages of the model such that the measure of the model can be made more accurate and precise.

The proposal can be used to shed light on the limitations of the DCF model in certain situations and also expand the scope of the DCF model. Thus, the purpose is not only to propose an improvement to the DCF model, but also to gain knowledge about the limitations of the DCF model.

Specifically the present project will be looking at a stochastic version of the DCF model. The reason for this is that the stochastic DCF model not only models the expected enterprise value it also models the volatility of the enterprise value. The stochastic DCF model therefore gives an idea of how precise the measure of the enterprise value in the DCF model is.

1.2. Problem statement

The DCF model has is a model that in principle only requires two factors as inputs to the model. These two factors are the cash flow that the enterprise can generate in the future to the security holders of the enterprise and the discounting factor used to discount the cash flow to a present value.

The recursive nature of the enterprise value

A disadvantage of the DCF model that the present project will focus on in order to improve the model is the fact that the discounting factor is in itself a function of the enterprise value.

In other words the DCF model reveal a recursive nature of the enterprise value, because the enterprise value is in itself a function of the enterprise value. This means that the DCF model is a model where the output of the model is also an input to the model.

Thus, when the discounting factor is applied to the cash flow that the enterprise can generate in the future to the security holders of the enterprise, the resulting enterprise value is only an approximation to the enterprise value that should have been the output of the DCF model.

Models that have such a recursive relationship between the input parameters and output parameters are often referred to as *recursive models*.

A stochastic DCF model

Some configurations of the DCF model does allow to account for the recursive nature of the enterprise value without making the DCF model too complex. For example (Larkin, 2011) and (Mohanty, 2003) configured the DCF model to a deterministic setting and thereby managed to provide for the recursive nature of the enterprise value in the DCF model.

However, a deterministic DCF model overvalues the enterprise and does not provide for a measure of the precision of the enterprise value. In order to avoid this overvaluation and also provide for a measure of the precision of the enterprise value a stochastic DCF model will be looked at in the present thesis. The term stochastic refers to the fact that one or more of the input parameters are non deterministic, i.e. one or more of the input parameters are random variables.

If nothing is done to overcome the problem with the recursive nature of the enterprise value in the stochastic DCF model some of the advantages by having a stochastic DCF model in the first place are lost.

It is therefore desirable to see if it is possible to turn to a stochastic DCF model and at the same time account for the recursive nature of the enterprise value in the DCF model.

1.2.1. Questions

The problem with the DCF model discussed above leads to the following main question:

> **How is it possible to account for the recursive nature of the enterprise value in a stochastic DCF model?**

In this setting the recursive nature refers to the fact that for the DCF model the enterprise value is in itself a function of the enterprise value. The term stochastic refers to the fact that one or more of the input parameters are non deterministic.

The answer to the above main question will lead to a proposal to an improvement of the DCF model in a stochastic setting. In the proposal the enterprise value will be determined recursively.

The proposed improvement to the DCF model in the stochastic setting will be referred to as an *iterative DCF model*.

With the proposed improvement to the DCF model it becomes possible to address the following sub questions:

1) Can the proposed improvement to the DCF model provide for an analytical solution of the enterprise value?

2) How does the enterprise value of the proposed improvement to the DCF model compare to the enterprise value of a simple DCF model?

The above sub questions come natural when a new model or an improvement to a model is proposed.

1.3. Motivation for the questions

In general the DCF model is one of the more popular models that are used for valuation of business corporations and enterprises. Any improvement to one of its disadvantages is therefore desirable.

Until now it has only been possible to account for the recursive nature of the enterprise value in a deterministic DCF model, (Larkin, 2011) and (Mohanty, 2003). However, a deterministic DCF model does not measure the precision of the enterprise value. So if a measure of the precision of the enterprise value is needed it is only for a limited number of enterprises that a stochastic DCF model can be used.

Thus, it is specifically desirable to account for the recursive nature of the enterprise value in a stochastic DCF model, because this will increase the number of enterprises that the stochastic DCF model can be applied to.

Why is an analytical solution desirable?

It is desirable to provide for an analytical solution of the enterprise value in the proposed improvement to the DCF model, because this will give a high usability.

In addition a sensitivity analysis of the proposed model will be more rigid if an analytical solution can be provided for.

In this setting an analytical solution means a closed form expression. With a closed form expression the enterprise value can be expressly written using elementary arithmetic operations and well known functions.

Why is a comparison of the enterprise value desirable?

It is desirable to compare the enterprise value in the proposed improvement to the DCF model with the enterprise value in a simple DCF model, because that will tell how different the enterprise value is in the two models.

If there is a little difference in enterprise value the simple DCF model is adequate, but if there is a large difference in enterprise value the simple DCF model is not adequate.

In this way knowledge will also be gained on the limitations of a simple DCF model. This is important, because it should be avoided to use a simple DCF model in cases where it will give an inaccurate measure of the enterprise value.

A *simple DCF model* is simple in the way that it is a DCF model where the recursive nature of the enterprise value is not accounted for. In such implementations of the DCF model a qualified guess is made on the enterprise value. This guess goes into the DCF model in order to determine an enterprise value. The output of such an implementation of the DCF model will only be an approximation to the real enterprise value.

The enterprise value in the proposed improvement to the DCF model will be compared with the enterprise value in a simple DCF model for two test cases. The first test case being a low leveraged enterprise and the second test case being a high leverage enterprise.

The enterprise value in the proposed improvement to the DCF model will also be compared to the observed enterprise value for both the low leveraged enterprise and the high leveraged enterprise. The result of such a comparison is nice to know. However, a comparison with only two enterprise forms too small a statistical basis to determine a correlation between the model and any market observations.

Both the accuracy and precision of the enterprise value is desirable to compare between models.

The *accuracy* is a measure of how close the enterprise value of a DCF model comes to a “true enterprise value”. The term “true enterprise value” in this setting merely means the output that should have been the output of the DCF model if it had been calculated consistently.

The *precision* is a measure of how wide the enterprise value in a stochastic DCF model distributes itself around an average enterprise value. The more the enterprise value deviates from the average enterprise value the more volatile is the enterprise value.

1.4. Method

In order to answer the main question of the problem statement and the associated sub questions the present project is divided into two parts. The first part is a theoretical part leading to a proposal to an improvement to the DCF model in a stochastic setting. The second part shows the results of the proposal to an improvement to the DCF model applied to two different test cases. The first test case being a low leveraged enterprise and the second test case being a high leveraged enterprise.

An overview of the first and second part of the present project is illustrated below in Figure 1.

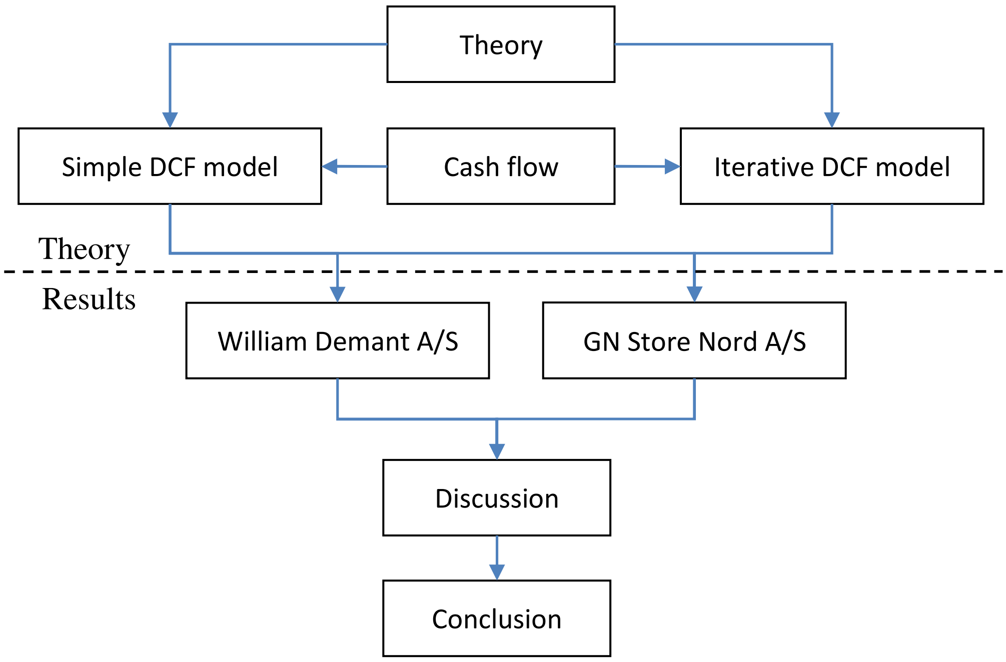

Figure 1. Structure of the present project.

In the following a summary of the sections of the present project will be given.

<u>Theory section</u>

In the theory section the DCF model will be introduced and the different configuration and implementation choices of the DCF model is presented.

The configuration of a simple DCF model is introduced and the configuration of the proposed improvement to the DCF model is also introduced. The proposed improvement to the DCF model will be introduced as an "iterative" DCF model.

Cash flow section

In the cash flow section an analytical expression for a cash flow to the security holders of the enterprise will be determined.

The cash flow is going to be structured such that it will be a function of the initial state of the enterprise, the tax rate and key financial ratios that can be found in the financial statement of an enterprise.

That cash flow will go into both the simple DCF model and the proposed improvement to the DCF model.

Section on a simple DCF model

In the section on a simple DCF model an expression for the enterprise value in the simple DCF model will be determined. This expression can be written analytically as a closed form solution.

An expression for the precision of the enterprise value in the simple DCF model will also be determined. This can also be done analytically as a closed form solution.

The above establishes the simple DCF model as a point of reference for the proposed improvement to the DCF model.

Section on an iterative DCF model

The section on an iterative DCF model will establish a proposal to an improvement to the DCF model in the stochastic setting. The proposed improvement will be referred to as an iterative DCF model.

The proposed improvement to the DCF model will take its starting point in a Monte Carlo simulation of a cash flow. Thus, the proposed improvement to the DCF model will be a numerical model.

An approximation to the enterprise value in the iterative DCF model will also be suggested. In the approximation the enterprise value can be expressed analytically as a closed form solution.

Results section

In the results section the proposed improvement to the DCF model in the stochastic setting will be applied to two test cases.

The first test case will be the enterprise GN Store Nord A/S and the second test will be the enterprise William Demant A/S. Both enterprises are listed in the NASDAQ OMX Copenhagen 20 index.

The two enterprises are chosen as test cases, because they operate on the same market, but have different leverage. GN Store Nord A/S is an enterprise with a historically relatively low book leverage and William Demant A/S is an enterprise with a historically high book leverage.

Despite of the high statistical uncertainty when using only two test cases, they do illustrate the performance of the proposed improvement to the DCF model and give interesting results.

The sensitivity of the enterprise value in the proposed improvement to the DCF model with respect to the leverage will then be determined by illustrating the enterprise value in the proposed improvement to the DCF model as a function of leverage.

1.5. Target group

The target group of this project is anyone interested in improvements to the DCF model and especially anyone who is interested in how large an error there is involved in calculating the enterprise value without accounting for the recursive nature of the enterprise value

The DCF model is a quantitative model and is as such mathematical in nature. Mathematics will therefore be involved in the present project. It can be difficult to set the level of mathematics. Too much mathematics will make anything tiresome and too little mathematics will make it difficult to keep track of what is what. However, the level of the mathematics in the present project will require some knowledge of basic equation solving, differentials and integration.

1.6. Delimitation

Since the purpose of the present project is on an improvement of the DCF model the focus will be on exactly that. For this reason the following limitations are made

1) The budget will be kept as simple as possible. This includes the following:

- The number of entries in the budget will be kept to a minimum.
- There will be no transient taking the budget from a short term period to a long term period.

2) Options will not be included.

3) The discounting of the cash flow will be under the physical measure. Thus, a switch to the risk neutral measure will not be made.

4) A longer assessment of the future of the enterprise will not be made. This has the following consequences:

- The choice of budget drivers will primarily be assumed to not deviate too much from the historic values.

- The choice of the required returns will not be very robust.

5) The stochastic process for the cash flow will be kept as simple as possible. Thus, potential macroeconomic chocks will not have any effect on the valuation.

6) The market value of the net interest-bearing debt will be assumed to equal the book value.

2. Theory

The theory section will start with a short introduction to the DCF model. The different configuration and implementation choices of the DCF model is then presented.

2.1. A short introduction to the DCF model

In the DCF model the value of the enterprise is modeled as a function of the present value of the cash flow that the enterprise can generate in the future to the security holders of the enterprise, (Brealey & Meyers, 2003). Mathematically this can be written as:

$$EV = \mathrm{PV}(CF) \tag{1.1}$$

where EV is the enterprise value, CF is the cash flow and the function PV denotes present value.

The *enterprise value* is the value that reflects the value of the whole business. Thus, for a leveraged enterprise the enterprise value is the value of the equity and the value of any liabilities and obligations, (Pratt, Reilly, & Schweihs, 2000).

The present value of the cash flow that the enterprise can generate in the future to the security holders of the enterprise is determined by discounting with a discounting rate reflecting the required return on the cash flow to the security holders. Thus, the enterprise value is a function of the cash flow and the discounting rate $EV = f(CF, wacc)$, where $wacc$ is the required return on the cash flow to the security holders, (Brealey & Meyers, 2003). The required return is also referred to as the weighted average cost of capital.

In order for the present value to converge to a finite value it is assumed that there is a limited expansion of the enterprise. Thus, the enterprise should not grow so much that its value explodes. In a DCF model this means that the cash flow should not grow more than the discounting.

It is also assumed that all surplus liquidity generates a net present value of zero, i.e. all surplus liquidity is invested at a rate of return that corresponds to the required rate of return (Plenborg, 2000). In cases where this is not fulfilled the enterprise value in the DCF model will either overestimate or underestimate the true value of the enterprise. For example, if the surplus liquidity is invested to a rate of return that is less than the required rate of return, the DCF model will overestimate the value of the enterprise as compared to the result of the DDM model.

2.2. Configurations of the DCF model

The above formulation of the DCF model in equation (1.1) makes the DCF model appear relatively simple. However, the simplicity of the DCF model depends very much on how it is configured. The DCF model is controlled by choosing the setting of the different characteristics of the DCF model. Two of the more dominant characteristics of the DCF model that can be set are the time domain and the capital structure.

Table 1 illustrates the characteristics together with their different settings.

Characteristic	Setting	
Time domain	Discrete time	Continuous time
Capital structure	Intrinsic	Extrinsic

Table 1. Configurations options for a DCF model.

As can be seen from Table 1 there is two settings for each characteristic. The time domain can be in discrete time or in continuous time and the capital structure can be intrinsic or extrinsic. An extrinsic capital structure accounts for the recursive nature of the enterprise value in a DCF model, but an intrinsic capital structure does not. This will be explained later.

The two characteristics can be combined in four different ways resulting on four different configurations of the DCF model. Figure 2 illustrates a flow chart that can be used for choosing one of these four configurations of the DCF model.

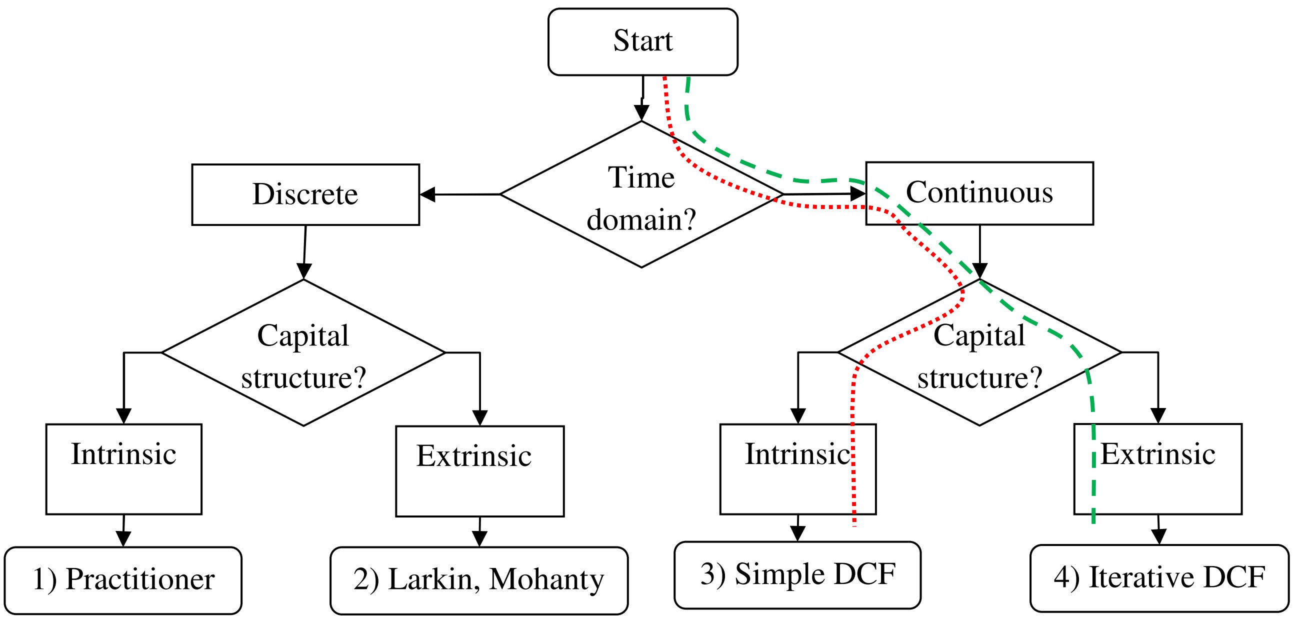

Figure 2. Flow chart for configuring the DCF model.

The flow chart of Figure 2 has four paths. Each path corresponds to one of the four configurations of the DCF model.

The first path is the configuration that is usually seen in praxis: A practitioner configures the DCF model in discrete time and do not account for the recursive nature of the enterprise value. This will overvalue the enterprise, because of the choice of time domain and because of the choice of capital structure.

The second path is the configuration of (Larkin, 2011) and (Mohanty, 2003). They configured the DCF model in discrete time, but accounts for the recursive nature of the enterprise value. This will overvalue the enterprise, because of the choice of time domain

The third path is illustrated with a dotted red line and the forth path is illustrated with a dashed green line. These two configurations of the DCF model will be introduced in the following.

2.2.1. An iterative DCF model

The DCF model configured according to the dashed green path of Figure 2 is termed an iterative DCF model. Thus, the iterative DCF model is configured with a combination consisting of an extrinsic capital structure combined with a continuous time domain.

The reason for terming such a configuration an iterative DCF model is that it is possible to determine the enterprise value of such a DCF model by iteration. This terminology corresponds to the terminology of (Larkin, 2011) and (Mohanty, 2003) who have both looked at a non stochastic DCF model with an extrinsic capital structure.

In the following the effects of such a configuration will be explained.

<u>An extrinsic capital structure</u>

The iterative DCF model is configured with an extrinsic capital structure. The term *extrinsic capital structure* is used to reflect that it is the market value of the equity and the market value of the debt that is used in the DCF model.

In a DCF model the market value of the equity and the market value of the debt should be used to determine the capital structure, because the cash flow should be discounted with a discounting rate reflecting the expectations of the market. A DCF model configured with an extrinsic capital structure is therefore a closer representation of an enterprise than a DCF model configured with an intrinsic capital structure.

The leverage of an enterprise where the capital structure is represented with the market value of the equity and the market value of the debt is the ratio between the market value of the debt and the market value of the equity. This will be called the *market leverage* in the following. Mathematically the market leverage is written as $L_M = B/MV$ where L_M is the market leverage, B is the market value of the debt and MV is the market value of the equity.

Alternatively the market leverage can be expressed as the ratio between the market value of the debt and the enterprise value. Mathematically this is written as $L_M = B/EV$ where EV is the enterprise value. For the present project this last definition of the market leverage will be used.

It will be assumed that the book value of the debt equals the market value of the debt, i.e. $D = B$.

Continuous time domain

The iterative DCF model is configured with a *continuous time domain*. This means that the variables in the iterative DCF model such as the cash flow and the discounting factor is known for all real numbers $t \in \mathbb{R}$.

A DCF model should be configured with a continuous time domain, because an enterprise in real life operates in continuous time. A DCF model configured with an continuous time domain is therefore a closer representation of an enterprise than a DCF model configured with a discrete time domain.

Furthermore, the continuous time domain allows for a convenient way of making the DCF model stochastic.

2.2.2. A simple DCF model

The DCF model configured according to the dotted red path of Figure 2 is termed a *simple DCF model*. Thus, the simple DCF model is configured with a combination consisting of an intrinsic capital structure combined with a continuous time domain and a cut off budget.

As for the iterative DCF model the simple DCF model is configured with a continuous time domain and a cut off budget. However, for the capital structure selection the simple DCF model is configured with an intrinsic capital structure. The effect of this capital structure selection will be explained in the following.

Intrinsic capital structure

The iterative DCF model is configured with an extrinsic capital structure. The term *intrinsic capital structure* is used to reflect that it is the book value of the equity and the book value of the debt that is used in the DCF model.

With an intrinsic capital structure the leverage of the enterprise is the ratio between the book value of the debt and the book value of the equity. This is also called the *book leverage*. The book leverage is written as $L_B = D/EK$ where L_B is the book leverage, D is the book value of the debt and EK is the book value of the equity.

Alternatively the book leverage is the ratio between the book value of the debt and the invested capital. Mathematically this is written as $L_B = D/IC$, where IC is the invested capital. For the present project this last definition of the book leverage will be used.

2.2.3. The difference between a simple DCF model and an iterative DCF model

The configuration of the iterative DCF model and the simple DCF model has been discussed above. Table 2 shows the differences and similarities between the iterative DCF model and the simple DCF model with respect to the three characteristics of a DCF model. The cash flow is also addressed in Table 2.

	Simple DCF model	Iterative DCF model
Cash flow	The same in both models	
Time domain	Continuous time	
Capital structure	Intrinsic	Extrinsic

Table 2. Similarities between the simple DCF model and the iterative DCF model.

In Table 2 it can be seen that the cash flow that goes into the iterative DCF model and the simple DCF model is the same.

The time domain is also the same in both models, i.e. the continuous time domain is selected in both the iterative DCF model and the simple DCF model.

Thus, the only difference between the simple DCF model and the iterative DCF model is that the iterative DCF model has the market leverage as an input and the simple DCF model has the book leverage as an input.

<u>Problems with the configurations that have been considered so far in the literature</u>

Research has been performed on DCF models configured with an extrinsic capital structure, (Larkin, 2011) and (Mohanty, 2003). However, in this research the time domain was chosen to be a *discrete time domain*. Such a choice of time domain will result in the enterprise being overvalued. Thus, the advantage of a higher accuracy is lost, because of the choice of time domain.

In view of this there is motivation for configuring the DCF model according to the dashed green line of the flowchart of Figure 2. Such a configuration of the DCF model combines an extrinsic capital structure with a continuous time domain This is closer to how an enterprise is configured in real life and it will therefore overcome the disadvantage of overvaluation mentioned above.

3. Cash flow of an enterprise

The first input to the DCF model is a future cash flow to the security holders. The present section will therefore focus on establishing an expression for the future cash flow that can go into the DCF model.

A *structural form model* will be used to generate the cash flow. This means that the cash flow is generated as a function of endogenous variables. These endogenous variables are the budget drivers and initial state of the enterprise. The cash flow can thereby be expressed by economic parameters representing the initial state and the future development of the enterprise.

3.1. The enterprise budget

In order to keep things as simple as possible a simple form of budget will be chosen for generating the cash flow. The enterprise budget has been inspired by Thomas Plenborg with a few modifications.

Table 3 shows the cash flow statement for the chosen budget. The cash flow statement shows the effects of income and cost together with changes in the balance and reflects how the liquidity of the enterprise has changed through a period of time. The period of time in Table 3 is a calendar year. The income statement, balance sheet and budget alignment are shown in Appendix on page 81.

	Entry	Drives
+	Net operating profit after tax $(NOPAT)$	$\Pi \cdot TR(1-\tau)$
+	Depreciations and amortizations (DA)	DA
-	Net working capital (ΔNWC)	$\Delta GM + \Delta DR - \Delta CR$
=	Cash flow from operations (CFD)	
-	Cash flow to investments (CFI)	$\Delta PPE + DA$
=	Cash flow (CF)	$NOPAT - \Delta IC$

Table 3. Cash flow statement.

In Table 3 it can be seen that the cash flow to the security holders is the net operating profit after tax subtracted with the change in invested capital (IC). In the continuous time domain the cash flow to the security holders is written as:

$$CF(t) = NOPAT - \frac{\mathrm{d}\,IC}{\mathrm{d}\,t} \tag{2.1}$$

The *invested capital* is the sum of the inventory of goods and materials (GM), the debtors (DR) and investments in property, plant and equipment (PPE) minus the creditors (CR). The four entries collected under the invested capital are assumed to be driven by the asset turnover (Λ) on the revenue (TR) such that the invested capital can be written as $IC = TR/\Lambda$. The asset turnover is defined as $\Lambda^{-1} \equiv v_{GM} + v_{DR} + v_{PPE} - v_{CR}$ with v_{GM} being the driver for the debtors, v_{DR} being the driver for the debtors, v_{PPE} being the driver for the property, plant and equipment and v_{CR} being the driver for the creditors.

The *net operating profit after tax* is the margin on earnings before interest and tax (Π) multiplied with the revenue, the tax rate being (τ).

3.2. The revenue of the enterprise

Both the net operating profit after tax and the invested capital are functions of the revenue. The revenue is defined in continuous time as a *Geometric Brownian motion*. For such a process the relative growth of the revenue in time (t) will consist of two terms:

$$\frac{1}{TR}\frac{\mathrm{d}TR}{\mathrm{d}t} = g + \sigma\frac{\mathrm{d}W_t}{\mathrm{d}t} \tag{2.2}$$

The first term is an average deterministic growth g. The second term is a stochastic term. where σ is the volatility of the relative growth of the revenue and W_t is a *Wiener process*. The Wiener process has stationary independent increments that are normally distributed, (Hull, 2000).

The solution to the above stochastic differential equation can be found with the help of Itos' lemma. Thus, the solution to equation (2.2) is:

$$TR(t) = TR_0 e^{(g-\sigma^2/2)t} e^{\sigma W_t} \tag{2.3}$$

where TR_0 is the initial state of the revenue. Since the revenue is a stochastic process it is uncertain how the revenue will evolve through the budget period. This is illustrated in Figure 3 where two examples of a path of the revenue is shown together with the expected evolution of the revenue $(\mathrm{E}[TR])$.

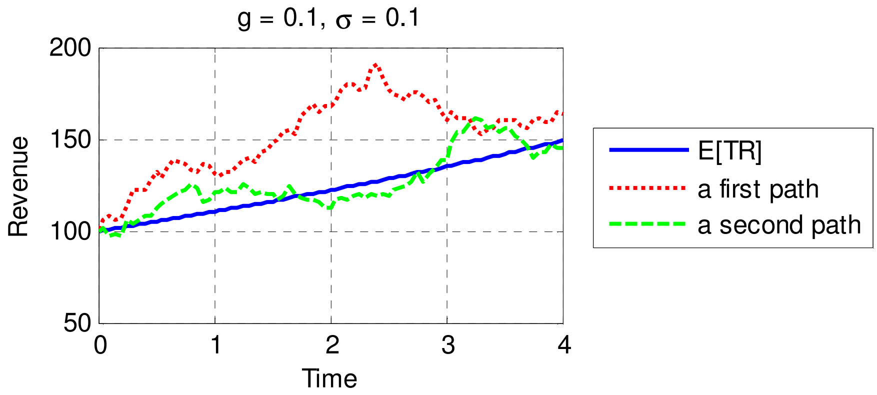

Figure 3. Paths of the revenue.

The expected evolution of the revenue shown as in Figure 3 is the average of all the possible paths. Since the revenue is a Geometric Brownian motion the expected evolution of the revenue is the exponential function: $\mathrm{E}[TR] = TR_0 e^{g \cdot t}$.

3.3. A structural form model of the cash flow

The revenue of equation (2.3) is now inserted into the cash flow to the security holders. This results in the following expression:

$$CF(t) = TR_0 e^{(g-\sigma^2/2)t} e^{\sigma W_t} \left(\Pi(1-\tau) - \frac{1}{\Lambda}\left(g + \sigma \frac{\mathrm{d}W_t}{\mathrm{d}t} \right) \right) \tag{2.4}$$

The above cash flow is going to be used in both the simple DCF model and the iterative DCF model. The cash flow is a function of the margin on earnings before interest and tax, the asset turnover, the tax rate and the process for the revenue.

Cash flow of an enterprise

4. The enterprise value in a simple DCF model

The purpose of the present project is to answer how it is possible to account for the recursive nature of the enterprise value in a stochastic DCF model. Thus, an iterative DCF model will be proposed. As a point of reference for the proposed model a simple DCF model will be considered. Such a simple DCF model is often seen expressed as numbers in an Excel worksheet. However, in order to have a more manageable expression for the simple DCF model an analytical expression will be shown in the following.

The enterprise value in a DCF model has been presented as the present value of the future cash flow to the security holders $EV = \mathrm{PV}(CF)$. When all of the future cash flow is accumulated the enterprise value becomes:

$$EV = \int_0^T \mathrm{PV}\left(CF(t)\right)\mathrm{d}t + c \tag{3.1}$$

Since the simple DCF model is configured with a continuous time domain the accumulation of the present value of the cash flow is achieved by integration from present time to a time T. The constant of integration has been expressed as c.

The present value of the cash flow is determined by discounting it back to present time with a *discounting factor* d_t. In a DCF model the rate of discounting is the *weighted average cost of capital* $(wacc)$. The weighted average cost of capital equals the relative change of the enterprise value in time, (Brealey & Meyers, 2003):

$$\frac{1}{EV}\frac{\mathrm{d}EV}{\mathrm{d}t} = wacc \tag{3.2}$$

The above differential equation can be solved by integration which results in the discounting factor being $d_t = e^{-wacc \cdot t}$. A discounting of a cash flow at time t to a present value is therefore determined as the discounting factor multiplied onto the cash flow: $\mathrm{PV}\left(CF(t)\right) = e^{-wacc \cdot t} CF(t)$.

Inserting the discounting factor and the cash flow in the expression for the enterprise value results in the following enterprise value:

$$EV = TR_0 \int_0^T e^{-\left(r+\sigma^2/2\right)t} e^{\sigma W_t}\left(\Pi(1-\tau) - \frac{1}{\Lambda}\left(g + \sigma\frac{\mathrm{d}W_t}{\mathrm{d}t}\right)\right)\mathrm{d}t + c$$

where the translated *discounting rate* is $r = wacc - g$.

4.1. The expected enterprise value in the simple DCF model

The enterprise value is the value that is to be expected from all of the possible cash flows in the future. Thus, the expected enterprise value is the double integration:

$$\mathrm{E}\left[EV\right]=\mathrm{E}\left[TR_0\int_0^T e^{-\left(r+\sigma^2/2\right)t}e^{\sigma W_t}\left(\Pi\left(1-\tau\right)-\frac{1}{\Lambda}\left(g+\sigma\frac{\mathrm{d}W_t}{\mathrm{d}t}\right)\right)\mathrm{d}t\right]+c \tag{3.3}$$

where $\mathrm{E}[\cdot]$ denotes the *expected value*. The expectation can be moved inside the integration. Furthermore, the expectation of the Wiener function is zero. This means the expected enterprise value can be reduced to:

$$\mathrm{E}\left[EV\right]=TR_0\left(\Pi\left(1-\tau\right)-\frac{g}{\Lambda}\right)\int_0^T \mathrm{E}\left[e^{-\left(r+\sigma^2/2\right)t}e^{\sigma W_t}\right]\mathrm{d}t+c$$

The first exponential in the kernel of the integral is deterministic. It is therefore only necessary to consider the expectations of the exponential of the Wiener function. This expectation can be written as $\mathrm{E}\left[e^{\sigma W_t}\right]=e^{\sigma^2/2}$, (Hull, 2000). The enterprise value is therefore expected to be:

$$EV_s=TR_0\left(\Pi\left(1-\tau\right)-g/\Lambda\right)\frac{1-e^{-rT}}{r}+IC-TR_0/\Lambda \tag{3.4}$$

where the integration also has been carried out and the constant of integration written as $c=IC-TR_0/\Lambda$. The expectation sign has been omitted and the subscript s used to denote the simple DCF model. The discounting rate is $r=wacc-g$.

If the enterprise is a going concern the expected enterprise value can be written as the limit:

$$\lim_{t\to\infty}\left(EV_s\right)=\frac{EVA_0}{r}+IC \tag{3.5}$$

where the constant $EVA_0=TR_0\left(\Pi\left(1-\tau\right)-wacc/\Lambda\right)$ represents an economic value added. From this simple DCF model a number of observations can be made. One observation is that the enterprise value is greater than the invested capital when the economic value added is positive. This happens when the return on the invested capital after tax $\left(ROIC_\tau\right)$ is greater than the weighted average cost of capital, $ROIC_\tau>wacc$, where $ROIC_\tau=\Lambda\cdot\Pi\left(1-\tau\right)$.

4.2. The volatility of the enterprise value in the simple DCF model

The precision of the simple DCF model can be measured with the *volatility* of the enterprise value. The volatility of the enterprise value is the square root of the *variance* of the enterprise value. The variance is defined as the difference between the expected enterprise value squared and the expected square of the enterprise value:

$$\mathrm{var}(EV) = \left(\mathrm{E}[EV]\right)^2 - \mathrm{E}\left[EV^2\right] \tag{3.6}$$

The first term in the variance has been determined above. For a going concern it can be written as $\left(\mathrm{E}[EV]\right)^2 = \left(EVA_0/r + IC\right)^2$ where $EVA_0 = TR_0\left(\Pi(1-\tau) - wacc/\Lambda\right)$. The second term is more difficult to handle. However, it can be shown that it can be written as (Hull, 2000):

$$\mathrm{E}\left[EV^2\right] = \frac{EVA_0^2}{r\left(r+0.5\sigma^2\right)} + 2\frac{EVA_0 IC}{r} + IC^2 \tag{3.7}$$

Putting it all together means that the volatility of the enterprise value in the simple DCF model can be written as:

$$\sigma_{EV} = EVA_0\sqrt{\frac{1}{r^2} - \frac{1}{r\left(r+0.5\sigma^2\right)}} \tag{3.8}$$

5. The enterprise value in an iterative DCF model

In a simple DCF model the future cash flow to the security holders is simply discounted to a present value with the weighted average cost of capital as a discounting rate. However, this is in reality only an approximation to the correct enterprise value, because the weighted average cost of capital is a function of the enterprise value. Thus, the value of the enterprise in a DCF model is of a recursive nature, i.e. the enterprise value is in itself a function of the enterprise value.

In this section it will be analyzed how it is possible to account for the recursive nature of the enterprise value in the DCF model. A DCF model accounting for this recursive nature is called an *iterative DCF model* in the following.

5.1. Decomposing the weighted average cost of capital

The weighted average cost of capital goes into the simple DCF model as a single parameter. However, the weighted average cost of capital is actually a function of a number of parameters. As the name implies the weighted average cost of capital is an average of the required returns to the security holders.

For an enterprise with a capital structure consisting of equity and debt alone, the weighted average cost of capital is a weighted average of the required return to the equity holders and the required return to the debt holders after tax, (Miles & Ezzell, 1980):

$$wacc = \omega_{EK} k_e + \omega_B y_\tau \tag{4.1}$$

where k_e is the required return to the equity and y_τ is the required return on the net interest-bearing debt after tax, i.e. $y_\tau = y(1-\tau)$. The two weights in the weighted average cost of capital are:

$$\begin{aligned} \omega_{EK} &= \frac{EV - B}{EV} \\ \omega_B &= \frac{B}{EV} \end{aligned} \tag{4.2}$$

where EV is the value of the enterprise and B is the value of the debt. The first weight is the value of the equity relative to the value of the enterprise. The second weight is the value of the debt relative to the value of the enterprise. Since there is only equity and debt the two weights sum up to one: $\omega_{EK} + \omega_B = 1$.

The weights represent the market value of the security that have been issued. This is why the enterprise value goes into the weights.

The *weighted average cost of capital* can be written more simple as:

$$wacc = k_e - s \cdot L_M \tag{4.3}$$

where s is the *spread* between the required return on the equity and the required return on the debt after tax $s = k_e - y(1-\tau)$ and L_M is the market leverage.

The market leverage is the ratio between the net interest-bearing debt and the enterprise value $L_M = B/EV$.

When the leverage is defined as the ratio between the outstanding debt and the enterprise value the leverage is between zero and one $L_M \in [0;1[$. A leverage of zero means that there is no outstanding debt. In such an extreme the weighted average cost of capital corresponds to the required return on the equity: $wacc_{L=0} = k_e$.

5.2. Predicting the enterprise value in an iterative DCF model

In the simple DCF model the market leverage is approximated with the book leverage. However, in the iterative DCF model the weighted average cost of capital is decomposed to a function of the required return to the equity, the spread and the market leverage.

So what consequence will it have for the enterprise value when the assumption that the market leverage is equal to the book leverage is relaxed?

The elasticity of the enterprise value with respect to the weighted average cost of capital is negative, because the enterprise value will increase when the weighted average cost of capital decreases while it will decrease when the weighted average cost of capital increases.

Since the sign of the elasticity of the enterprise value with respect to the weighted average cost of capital is know it is possible to predict how the enterprise value will change from the simple DCF model to the iterative DCF model. This can be done just from considering potential changes in the discounting rate.

The elasticity of the weighted average cost of capital with respect to the leverage is negative, because the weighted average cost of capital will decrease when the leverage increases while it will increase when the leverage decreases.

This is under the assumption that the spread is greater than zero, i.e. the required return to the equity is greater than the yield on the debt after tax: $k_e > y_\tau$.

Thus, the elasticity of the enterprise value with respect to leverage is positive, because the enterprise value will increase when the leverage increases while it will decrease when the leverage decreases.

In a case where the book leverage is greater than the market leverage, i.e. $L_B > L_M$, the enterprise value will increase when approximating the market leverage with the book leverage. This is because the leverage goes from a low value to a higher value. Thus, the simple DCF model will overvalue the enterprise when compared to the iterative DCF model.

The book leverage relative to the market leverage corresponds to the *book-to-market ratio* k/i:

$$\frac{L_B}{L_M} = \frac{EV}{IC} = k/i \tag{4.4}$$

Thus, when the book-to-market ratio is greater than one the simple DCF model will overvalue the enterprise value. Said in other words when the enterprise value is greater than the invested capital the simple DCF model overvalues the enterprise.

5.3. Iterating the enterprise value

The decomposed form of the weighted average cost of capital is now used as discounting rate. This gives the following recursive expression for the enterprise value in a DCF model:

$$\boxed{EV = \int_0^\infty CF(t)\exp\left(-\left(k_e - s \cdot B(t)/EV(t)\right)t\right)\mathrm{d}t} \tag{4.5}$$

From the above integral equation it is clear that the enterprise value appears on both sides in the equality and that the enterprise value is in itself a function of the enterprise value, i.e. a recursive function. Thus, the enterprise value cannot be found simply by putting parameters into the right side of the equality and integrating. Instead the equation has to be solved for the enterprise value.

There is a number of ways of solving an integral equation. Below is summarized some of the methods for which an integral equation can be solved:

Method 1) Solve the equation for the enterprise value analytically.

Method 2) Guess at an enterprise value and iterate until the equation balances.

Method 3) Linearise the integral equation by setting up a set of linear equations.

Method 4) Simulate the cash flow and iterate backwards.

Ad method 1. The most attractive method to find the enterprise value would be to solve the equation for the enterprise value analytically. However, the equation for the enterprise value is an integral equation. Such an equation is very difficult to solve analytically. In the continuous time setting it is in fact not possible to analytically isolate the enterprise value. Thus, an analytical solution cannot be pursued.

Ad method 2. A less attractive method to find the enterprise value would be to simply guess repeatedly at a random enterprise value until the correct enterprise value is found. The correct enterprise value is found when the left side of the equation for the enterprise value equals the right side of the equation for the enterprise value, i.e. the equation balances. However, it is very difficult to guess. A numerical routine therefore has to be implemented. Such a numerical routine is not very attractive to implement.

Ad method 3. It is possible to structure the guessing method such that the guess on the enterprise value is a qualified guess. This makes the guessing method more attractive. The guessing method can be structured by writing the enterprise value as a sum of known functions with a weight associated to each function. From this a linear system of equations can be set up with the weights as unknowns. However, this approach is not easy to implement in a tool like Excel.

Ad method 4. Far into the future the present value of a cash flow only contributes with very little to the overall enterprise value. Thus, the cash flow can be simulated numerically up to a time where the cash flow only contributes with very little. The cash flow farthest into the future can then be discounted back to present time one step at a time. Such an approach will be pursued in the following.

5.3.1. Monte Carlo simulation of the cash flow

The method used to solve for the enterprise value numerically will essentially comprise two steps which are summarized in the following:

Step 1) A cash flow is generated numerically in continuous time.

Step 2) The cash flow is discounted analytically step by step.

Ad step 1. The first step is a bit counterintuitive because a continuous cash flow cannot be generated numerically. This is because a computer works in discrete time and is simply not able to generate anything continuous. However, a computer with large enough processing power can generate an approximation to a continuous cash flow.

Ad step 2. The second step is an analytically step where the future cash flow is discounted stepwise backwards by starting with the cash flow which is the farthest into the future. In this step it is exploited that the present value of cash flow far into the future only contributes with very little to the overall enterprise value.

Thus, the first thing to do in order to find a correct value of the enterprise is to generate the approximation to the continuous cash flow. For this it is assumed that a differential $\mathrm{d}\,x$ can be approximated as $\mathrm{d}\,x \approx x_n - x_{n-1}$, where the subscript n represents the *sampling number*. The sampling number is a natural number $n \in \mathbb{N}^+$.

With such an approximation of the differential the cash flow can be simulated numerically in discrete time according to the following expression:

$$CF_n = NOPAT_n - \frac{IC_n - IC_{n-1}}{\Delta t} \quad \text{for} \quad n = 1,2,3.... \tag{4.6}$$

The time differential is denoted as Δt and represents the time period between two sampling events: $\mathrm{d}t \approx \Delta t = t_n - t_{n-1}$.

When the sampling is one year apart the cash flow will correspond to the cash flow in discrete time, i.e. a situation where a budget period is one year and the entries in the budget is sampled once a year. Thus, in order for the numeric approximation to be an approximation to a continuous time setting the sampling needs to take place less than a year apart. The smaller the time period between sampling events the closer the numeric approximation will be to a continuous time setting.

Both the net operating profit after tax and the invested capital is in the simple budget a function of the revenue. The sampling of these three variables is therefore needed. Table 4 shows the sampling of the net operating profit after tax, the revenue and the invested capital.

Variable	**Evolution**			
Sampling	–	1	$\cdots$	n
Revenue (TR)	TR_0	$TR_0\left(1+g\Delta t+\sigma\cdot\varepsilon\sqrt{\Delta t}\right)$	$\cdots$	$TR_{n-1}\left(1+g\Delta t+\sigma\cdot\varepsilon\sqrt{\Delta t}\right)$
NOPAT	–	$TR_1\Pi(1-\tau)$	$\cdots$	$TR_n\Pi(1-\tau)$
Invested capital (IC)	IC_0	$\frac{TR_1}{\Lambda}$	$\cdots$	$\frac{TR_n}{\Lambda}$

Table 4. Sampling of the variables in the cash flow.

The Wiener process has been approximated as $\mathrm{d}W_t \approx \varepsilon\sqrt{\Delta t}$, (Hull, 2000), where ε is the inverse of the standard normal distribution, i.e. a normal distribution with mean 0 and volatility 1.

The invested capital in the simple budget experiences a jump when there is a difference in asset turnover between the initial state and the first sampling, i.e. the initial state of the invested capital is IC_0 while it is defined from the asset turnover at the first sampling.

The weighted average cost of capital goes into the DCF model as a second input. Thus, the weighted average cost of capital also needs to be sampled. This is done according to:

$$wacc_n = k_e - s\cdot L_n \tag{4.7}$$

The leverage is the ratio between the net interest-bearing debt and the enterprise value: $L_n = B_n / EV_n$. The net interest-bearing debt is defined from the book leverage L_B and Table 5 shows the sampling of the net interest-bearing debt.

Variable	Evolution			
Sampling	–	1	$\cdots$	n
Net interest-bearing debt (B)	B_0	$L_B \cdot IC_1$	$\cdots$	$L_B \cdot IC_n$

Table 5. Sampling of the net interest-bearing debt.

The net interest-bearing debt also experiences a jump when there is a difference in book leverage between the initial state and the first sampling, i.e. the initial state of the net interest-bearing debt is B_0, but at the first sampling it is defined from the book leverage.

5.3.2. Discounting the cash flow

The cash flow is now going to be discounted back to present time in order to arrive at an enterprise value.

The enterprise is assumed to be a going concern. This means that the cash flow continues until infinity.

Using a terminal value to discount the infinite cash flow

It is very difficult to handle a process that continuous until infinity. However, when there is a limited expansion of the enterprise the present value of a cash flow will decrease exponentially with time. A limited expansion of the enterprise means that the cash flow to the security holders does not grow with more than the required return on the enterprise. Far into the future the present value of a cash flow will therefore approach zero:

$$\lim_{t \to \infty} \left(\mathrm{PV}\left(CF\left(t\right) \right) \right) = 0 \tag{4.8}$$

Thus, the longer into the future the cash flow goes the less is the contribution to the overall enterprise value. This means that the infinite cash flow can be handled by collecting the cash flow in the far future under a terminal value.

The term "far future" refers to the fact that a budget is often divided up into a period defining a near future and a period defining a far future[*]. The near future can be said to be a transition period taking the budget from present time to a far future.

The number of budget years in each of the periods can vary from enterprise to enterprise and the budget drivers used in the two periods can be very different. For example, the

[*] For the present project a transition period taking the budget from present time to a far future has been delimited. Thus, there no distinction between the near future and the far future will be made.

growth rate for the enterprise in the far future is often chosen to be smaller than the expected macro economical growth rate – otherwise the enterprise would become larger than the economy as a whole.

The terminal value is a technical method that uses mathematical manipulations to collect the cash flow of the far future. It is often referred to as Gordons growth formula. However, in general it is not all cash flows that are suited to be collected as one term.

To keep things general the cash flow will therefore be discounted one time period at a time. This will give the same result as using a terminal value, but the method is more general.

Discounting the infinite cash flow one time period at a time

In order to discount the infinite cash flow one period at a time a last cash flow is defined. This means that the enterprise value becomes:

$$EV = \int_0^T CF(t) \exp\left[-\left(k_e - s \cdot B(t)/EV(t)\right)t\right] \mathrm{d}t \tag{4.9}$$

The cash flow at time T is thus assumed to be the last cash flow. In that respect it can be called the terminal cash flow. The terminal cash flow at time T is sampled with sampling number n.

The integration should of course expand over a time period such that the terminal cash flow contributes very little to the overall enterprise value. A threshold χ can be defined for which this happens.

For example when a cash flow in the future at the time T contributes with less than $\chi = 1\ \%$ to the enterprise value, the contribution of cash flows following that is neglected. Such a threshold is also known as a stopping criteria. This is useful for stopping a numerical routine such that it does not continue indefinitely.

Discounting the terminal cash flow

The discounting of the last cash flow is done one time period back. Such discounting is expressed mathematically as:

$$EV_{n-1} = CF_n \exp\left(-wacc_{n-1}\Delta t\right) \tag{4.10}$$

Inserting the decomposed form of the weighted average cost of capital gives:

$$EV_{n-1} = CF_n \exp\left(-\left(k_e - s \cdot B_{n-1}/EV_{n-1}\right)\Delta t\right) \tag{4.11}$$

The above expression for the cash flow discounted back one time period comprises an enterprise value in two places. Thus, in order to arrive at a correct enterprise value the equation needs to be solved for the enterprise value, i.e. the enterprise value needs to be isolated such that it appears by itself on a side of the equality. This isolation can be performed with the *Lambert function* W_L :

$$EV_{n-1} = \frac{s \cdot B_{n-1}\Delta t}{W_L\left(s \cdot B_{n-1}\Delta t \exp\left(k_e \Delta t\right) / CF_n\right)} \tag{4.12}$$

The enterprise value now only appears one place. Thus, by taking one sample of the cash flow at a time it becomes possible to isolate the enterprise value at each step of the discounting.

The Lambert function is defined as the function that satisfies the equation $x = W_L\left(x\right)e^{W_L(x)}$. For the sampling number n the root is $x_n = s \cdot B_{n-1}\Delta t \exp\left(k_e \Delta t\right) / CF_n$. Alternatively the root can be written as: $x_n = s \cdot L_B \Delta t \exp\left(k_e \Delta t\right) \big/ \left(ROIC_\tau \left(1 + g\Delta t\right) - g\right)$.

The spread multiplied on the book leverage and the time step makes the root a small number. This means that the Lambert function can be approximated with the *Padé approximant*:

$$W_L\left(x\right) \approx \ln\left(1+x\right) \frac{1 + \frac{123}{40}x + \frac{21}{10}x^2}{1 + \frac{143}{40}x + \frac{713}{240}x^2} \tag{4.13}$$

The approximation with the Padé approximant is valid for roots being less than the natural number $e \approx 2.7$.

With the Padé approximant the Lambert function can be implemented in an Excel worksheet without the use of a macro.

<u>The simple DCF model as an approximation to the iterative DCF model</u>

As the leverage approaches zero the root x_n also approaches zero and the Lambert function can be approximated with the Taylor series truncated at first order $W_L\left(x\right) \approx x$. In this case the enterprise value becomes:

$$\lim_{L_B \to 0}\left(EV_{n-1}\right) = CF_n \exp\left(-k_e \Delta t\right) \tag{4.14}$$

This is the same result as for the simple DCF model. Thus, for enterprises with a small leverage the simple DCF model can be used as an approximation.

<u>Discounting to present time</u>

Until now the discounting has only gone one time step back. Thus, the enterprise values of equation (4.12) needs to be discounted further back. Discounting back one more time period results in the enterprise value:

$$EV_{n-2} = \left(CF_{n-1} + EV_{n-1}\right)\exp\left(-wacc_{n-2}\Delta t\right) \tag{4.15}$$

It can be seen that it is now not only the cash flow that goes into the enterprise value. The enterprise value of the previously discounted cash flow also goes into the discounting. This is how the last cash flow is discounted all the way back. Inserting the decomposed form of the weighted average cost of capital gives:

$$EV_{n-2} = \left(CF_{n-1} + EV_{n-1}\right)\exp\left(-\left(k_e - s\cdot B_{n-2}/EV_{n-2}\right)\Delta t\right) \tag{4.16}$$

Again the enterprise value can be isolated with the help of the Lambert function:

$$EV_{n-2} = \frac{s\cdot B_{n-2}\Delta t}{W_L\left(s\cdot B_{n-2}\Delta t\exp\left(k\Delta t\right)/\left(CF_{n-1}+EV_{n-1}\right)\right)} \tag{4.17}$$

The above expression for the enterprise value at sampling number $n-2$ is very similar to the enterprise value one sampling number higher. Only the enterprise value of the higher sampling number is added as mentioned.

The discounting continuous one time period at a time all the way back to present time. Thus, the enterprise value at present time becomes:

$$\boxed{EV_0 = \frac{s\cdot B_0\Delta t}{W_L\left(s\cdot B_0\Delta t\exp\left(k_e\Delta t\right)/\left(CF_1+EV_1\right)\right)}\Delta t} \tag{4.18}$$

The extra time differential at the end of the expression for the enterprise value at present time accounts for the numeric integration.

The above method of discounting the last cash flow back in time can be illustrated with the time line shown in Figure 4.

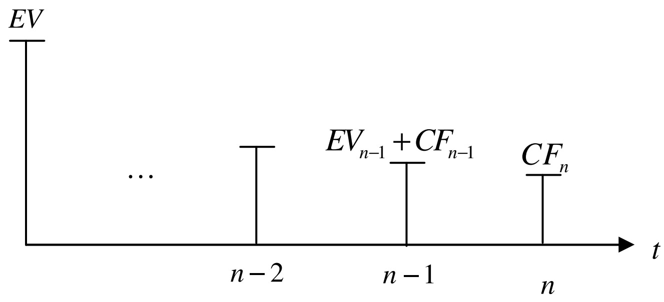

Figure 4. Timeline for discounting a cash flow.

In Figure 4 the abscissa shows the points in time and the ordinate shows the enterprise value at the time points of sampling. It can be seen that at the latest time point only the cash flow goes into the valuation. That cash flow is discounted back to the previous time point and together with the cash flow at the previous time point it is discounted back once more. This continues all the way back to present time.

In principle an analytical expression for the enterprise value at present time has been found. However, such an analytical expression for the enterprise value at present time will become very complicated. This is because all the sampled cash flows goes into the enterprise value recursively:

$$EV = f\left(CF_1, f\left(CF_2, f\left(CF_3, f\left(\cdots\right)\right)\right)\right) \tag{4.19}$$

The expression for the enterprise value at present time would therefore be very large. Only for enterprises operating over a small time scale a clear expression for the enterprise value could be achieved.

5.4. An approximation to the iterative DCF model

In equation (4.14) it was seen that for small leveraged enterprises the simple DCF model is an approximation to the iterative DCF model. However, this approximation is only valid when the leverage is small. It is therefore desirable to see if it possible to determine an approximation that is valid for both high leveraged enterprises and low leveraged enterprises.

To determine a better approximation to the iterative DCF model a time invariant discounting rate is forced into the DCF model. This means that the integral equation can be analytically solved.

The discounting rate is the weighted average cost of capital. The weighted average cost of capital is expressed as: $wacc(t) = k_e - s \cdot B(t)/EV(t)$. In a steady state condition both the enterprise value and the net interest-bearing debt will grow exponentially as $EV(t) = EVe^{gt}$ and $B(t) = L_B ICe^{gt}$, where L_B is the book leverage. This time dependent behavior of the enterprise value and the net interest-bearing debt actually means that the market leverage becomes independent of time:

$$L_M = \frac{B(t)}{EV(t)} = \frac{L_B IC}{EV} \tag{4.20}$$

<u>The enterprise value</u>

Inserting the time invariant discounting rate into the integral equation for the enterprise value greatly simplifies things:

$$EV = \int_0^\infty CF(t) \exp\left[-\left(k_e - s \cdot L_B IC/EV\right)t\right] \mathrm{d}t + c \tag{4.21}$$

where the integration constant is $c = IC - TR_0/\Lambda$. Thus, the integral can be taken the same way as for the simple DCF model. The integral thereby disappears and an algebraic equation is left:

The enterprise value in an iterative DCF model

$$EV = \frac{TR}{k_e - s \cdot L_B IC/EV - g}\left(\Pi(1-\tau) - \frac{k_e - s \cdot L_B IC/EV}{\Lambda}\right) + IC$$

The above enterprise value corresponds with the enterprise value that can be arrived at by starting from the EVA model. Thus, there is consistency between the DCF model and the EVA model.

It is possible to rearrange the enterprise value such that it can be written as a second order polynomial equation for the enterprise value:

$$r_k EV^2 - \left(s \cdot L_B IC + r_k EV_0 + CF_0\right) EV + s \cdot EV_0 L_B IC = 0 \tag{4.22}$$

where $r_k = k_e - g$, $CF_0 = TR\left[\Pi(1-\tau) - g/\Lambda\right]$ and $EV_0 = IC - TR/\Lambda$ have been introduced. A second order polynomial equation always has an analytical solution. For the above second order polynomial equation the analytical solution is:

$$EV_a = \frac{s \cdot L_B IC + r_k EV_0 + CF_0 + \sqrt{\left(s \cdot L_B IC + r_k EV_0 + CF_0\right)^2 - 4 r_k s \cdot EV_0 L_B IC}}{2 r_k} \tag{4.23}$$

Where the subscript a has been used to denote the enterprise value in the approximation to the iterative DCF model. Thus, an expression for the enterprise value in an iterative DCF model have been arrived at. It is under the assumption that the weighted average cost of capital is time invariant. The enterprise value in such an approximation is a square root function of the book leverage.

The square root function is characterized by an intersection with the ordinate axis and a slope. The slope is influenced by all of the input parameters. The intersection is found at a leverage of zero. This gives an enterprise value of $EV = CF_0/r_k + EV_0$ which corresponds to the result of the simple DCF model, when there is no leverage.

The volatility of the enterprise value in the approximation to the iterative DCF model

The volatility of the enterprise value in the approximation to the iterative DCF model is the square root of the variance. The variance is defined as:

$$\operatorname{var}(EV_a) = \left(\mathrm{E}[EV_a]\right)^2 - \mathrm{E}\left[EV_a^2\right] \tag{4.24}$$

The first term in the variance was determined above. However, the second term cannot be expressed as a closed form solution. As an approximation to the above the volatility of the enterprise value from the simple DCF model will be used as a measure for the volatility of the enterprise value in the iterative DCF model.

That expression is a function of the weighted average cost of capital:

$$\sigma_{EV} = EVA_0 \sqrt{\frac{1}{r^2} - \frac{1}{r\left(r + 0.5\sigma^2\right)}} \tag{4.25}$$

where $r = wacc_a - g$, $EVA_0 = TR\left(\Pi\left(1-\tau\right) - wacc_a/\Lambda\right)$ and $wacc_a = k_e - s \cdot L_M$

6. Results

In the previous section a model that accounts for the recursive nature of the enterprise value in a stochastic DCF model was proposed. With this an answer to the main question of the problem statement was given.

In the present section the proposed model will be used to determine the enterprise value of a low leveraged enterprise and a high leveraged enterprise.

6.1. Enterprise value of a low leveraged enterprise

6.1.1. Introduction to GN Store Nord A/S

As a low leveraged enterprise the business GN Store Nord A/S is chosen as a test case. GN Store Nord A/S produces hearing aids, head sets and audiometric equipment for the health care industry, (GN Store Nord A/S, 2012).

Figure 5 shows the performance of GN Store Nord A/S on the stock exchange since the beginning of the millennium. The historic income statement and balance sheet of GN Store Nord A/S is shown in Appendix on page 82.

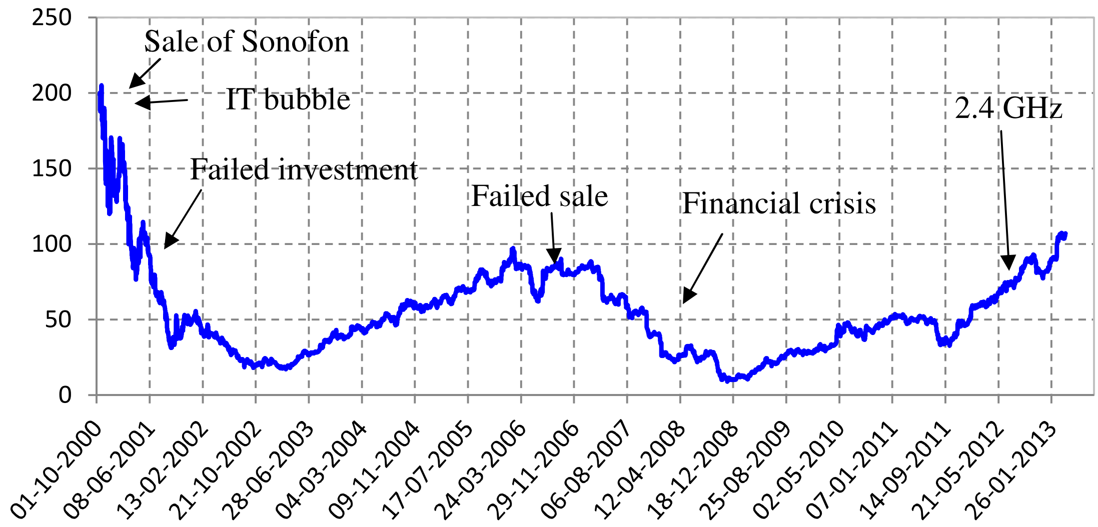

Figure 5. Historic share price of GN Store Nord A/S.

In Figure 5 it can be seen that the price of a GN Store Nord A/S share started with a dramatic fall from a price of 200 DKK in the year 2000 and has gone through two dips. The share price fell to its lowest below 10 DKK in the second dip in 2008. Today the share price is on an uphill slope and has passed 100 DKK in the beginning of 2013.

The top at 200 DKK in year 2000 was reached with the sale of the mobile communications company Sonofon. The fall from the top was due to bad investments, (GN Store Nord A/S, 2002). The first dip lasted from the year 2000 and all the way to the year 2006. In 2006 part of GN Store Nord A/S was supposed to be sold to a competitor. However, that sale was prohibited by the Bundeskartellamt in Germany and GN Store Nord A/S went into the second dip, (GN Store Nord A/S, 2007).

The second dip culminated with the financial crisis in the end of 2007-2008, (GN Store Nord A/S, 2008). To move out of the second dip GN Store Nord A/S decided to take its own path away from the competitors and launch products with a new wireless technology called 2.4 GHz, (GN Store Nord A/S, 2012).

Leverage of GN Store Nord A/S

GN Store Nord A/S is chosen as a test case, because it is an enterprise with a historically low book leverage at least when compared to an enterprise such as William Demant A/S. The historic book leverage of GN Store Nord A/S is shown in Figure 6 together with the historic yields. The period is from the year 2004 to the year 2012. The book leverage and yield are plotted in pairs as a phase diagram.

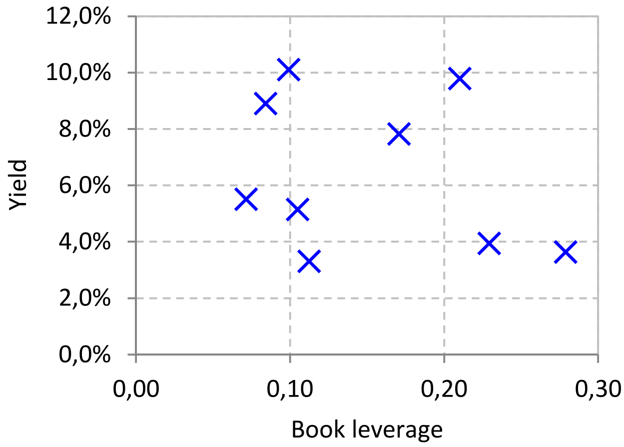

Figure 6. Historic book leverage of GN Store Nord A/S

The *book leverage* is defined as the average net interest-bearing debt of an income period and the average invested capital of the income period:

$$\text{book leverage} = \frac{\langle \text{net interest-bearing debt} \rangle}{\langle \text{invested capital} \rangle} \tag{5.1}$$

In a short hand notation the book leverage is written as $L_B = \langle B \rangle / \langle IC \rangle$, where L_B is the book leverage, B is the net interest-bearing debt and IC is the invested capital.

The average book leverage of an income period is taken as the average between the book leverage of the opening balance of the income period and the book leverage of the closing balance of the income period. The same principle will be applied to the invested capital. For GN Store Nord A/S an income period is a calendar year.

The book leverage as defined above will take on a value between zero and one as long as the book value of the equity is positive. This is because the invested capital that goes into the denominator of the book leverage is the sum of the book value of the equity and the net interest-bearing debt. When the equity is positive the invested capital is always greater than the net interest-bearing debt resulting in a fraction between 0 and 1.

In Figure 6 it can be seen that the historic book leverage of GN Store Nord A/S has been in the interval around 0.1 and 0.2.

The *yield* is the return to the debt holders. It is defined as the ratio between the interest payments in the income period and the average net interest-bearing debt in the income period:

$$yield = \frac{interest\ payments}{\langle net\ interest\text{-}bearing\ debt \rangle} \tag{5.2}$$

With the short hand notation the yield is written as $y = I/\langle B \rangle$, where y is the yields and I is the interest payment.

In Figure 6 it can be seen that the historic yield of GN Store Nord A/S has been in the interval between a 3 % and 10 %.

6.1.2. Values of the input parameters for the valuation of GN Store Nord A/S

The input parameters to the DCF model for the valuation of GN Store Nord A/S are now to be determined. The input parameters to both the simple DCF model and the iterative DCF model comprise the variables of the initial state, the budget drivers and the returns to the security holders.

However, it is very difficult to predict the future. This is also not the purpose of the present project. Instead the strategic analysis used in the following to arrive at a best guess for the input parameters will be kept as simple as possible. In addition nice numbers will be used as inputs in order to keep the comparison between the low leveraged enterprise and a high leveraged enterprise as clean as possible.

The initial state of GN Store Nord A/S

The *initial state* is characterized by the value of the invested capital at the time of valuation (IC), the value of the net interest-bearing debt at that time (B) and the value of the revenue that has been generated in the income period up to that time (TR).

The valuation of GN Store Nord A/S is as of the date of the publication of the latest annual report. The latest annual report of GN Store Nord A/S is the report of the income period 2012. This report was published February 21st 2013. Thus, the valuation is as of that day and the initial state of the enterprise of that day is needed.

For the revenue GN Store Nord A/S reported a value of just above $TR = 6.000$ DKK million in the income period of 2012.

It is possible that this value of the revenue has changed up to the publication of the annual report. However, since there only is a little more than a month separating the end of 2012 and the time of valuation it will be assumed that the initial state of the revenue has not changed much. For convenience a revenue of $TR = 6.000$ DKK millions is therefore used in the initial state of GN Store Nord A/S.

The *net interest-bearing debt* is defined as the liabilities that require interest payments. Thus, the net interest-bearing debt is determined by subtracting the non operating assets and the non interest-bearing debt from the total liabilities:

$$\left(net\ interest\ bearing\ debt\right) = \left(liabilities\right) - \left(non\ operating\ assets\right) - \left(non\ interest\ bearing\ debt\right) \quad (5.3)$$

In the annual report of the income period 2012 GN Store Nord A/S reported a closing balance with a net interest-bearing debt of $B = 230$ DKK millions. This value can also have changed up to the time of valuation. However, since there only is a little more than a month separating the end of 2012 and the time of valuation it will be assumed that the initial state of the net interest-bearing debt has not changed much. For convenience a net interest-bearing debt of $B = 300$ DKK millions is therefore used in the initial state of GN Store Nord A/S.

The *invested capital* is defined as the interest bearing capital invested in the operations. Thus, the invested capital is determined by subtracting non operating assets and non interest-bearing debt from the total assets:

$$\left(invested\ capital\right) = \left(total\ assets\right) - \left(non\ operating\ assets\right) - \left(non\ interest\text{-}bearing\ debt\right) \quad (5.4)$$

The invested capital can also be written as the sum of the equity and the net interest-bearing debt, i.e. $IC = EK + B$, where IC is the invested capital and EK is the equity. Using this means that the invested capital of GN Store Nord A/S in the closing balance of the income year of 2012 was just about $IC = 6.000$ DKK millions as reported implicitly in the annual report of 2012.

Again there is only a little more than a month between the end of 2012 and the time of valuation. Thus, a value of $IC = 6.000$ DKK millions for the initial state of the invested capital of GN Store Nord A/S will be used.

Returns to the security holders and the tax rate of GN Store Nord A/S

The returns to the security holders are the yields on the net interest-bearing debt (y) and the required return to the equity (k_e). The corporate tax rate (τ) is also considered in the following.

In Figure 6 the historic yield was shown. It took on values more or less in the interval between a yield of 3 % and a yield of 10 %. So what will the yields of GN Store Nord A/S be in the future?

There is a number of methods of calculating the yield, see for example Mertons structural model on default and yields, (Merton, 1973). Many of these methods are, however, complex. The historic values of the yield will therefore be relied upon instead. Thus, it will be assumed that the future yield of GN Store Nord A/S will be $y = 5\ \%$.

The *required return to the equity* is a measure of the future expectations of the risk of the enterprise. As for the yield there is a number of methods of calculating the required return to the equity, see for example the CAPM model, (Sharpe, 1964). Sydbank A/S, however, estimates a required return to the equity of $k = 10\ \%$ in the future, (Imsgard, 2013). This value will be used for the valuation of GN Store Nord A/S.

The *tax rate* is determined by the Corporation Tax Act (Selskabsskatteloven). At present time it is 25 % as determined by the Corporation Tax Act § 17, stk. 1. It is difficult to determine how the tax rate will move in the future. For convenience it will therefore be assumed that the future tax rate of GN Store Nord A/S will be $\tau = 25\ \%$.

Budget drivers of GN Store Nord A/S

The budget drivers are the EBIT margin (Π), the asset turnover (Λ), the book leverage (L_B) as well as the revenue growth (g) and the revenue volatility (σ).

The *EBIT margin* is defined as the ratio between the earnings before interest and tax in the income period and the revenue in the income period:

$$EBIT\ margin = \frac{EBIT}{revenue} \tag{5.5}$$

With the short hand notation from prior sections the EBIT margin is written as $\Pi = EBIT/TR$, where Π is the EBIT margin.

The *asset turnover* is defined as the ratio between the revenue in the income period and the average invested capital:

$$asset\ turnover = \frac{revenue}{\langle invested\ capital \rangle} \tag{5.6}$$

where the angle brackets denote the average of the value of the invested capital. With the short hand notation from prior sections the asset turnover is written as $\Lambda = TR/\langle IC \rangle$, where Λ is the asset turnover, TR is the revenue and IC is the invested capital.

The realized asset turnover and the EBIT margin for GN Store Nord A/S from the year 2002 until the year 2012 are shown together in the below phase diagram of Figure 7.

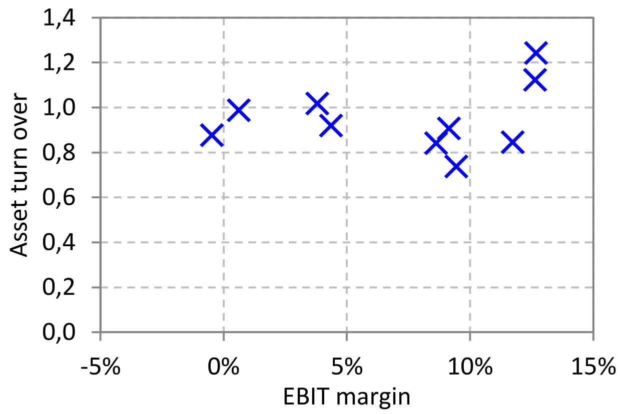

Figure 7. Historic EBIT margin and asset turn over of GN Store Nord A/S.

In Figure 7 it can be seen that the realized asset turnover of GN in the period 2002 to 2012 has been approximate constant around 1 with only minor deviations. The EBIT margin however has experienced fluctuations between a value of 0 and up to a value of 15 %.

So what will the EBIT margin and the asset turnover be in the future? CFO of GN Store Nord A/S Anders Boyer beliefs that with the current technology and the focus on keeping cost down a goal of an EBIT margin of 20 % in the future can be reached. It will thus be assumed that the EBIT margin of GN Store Nord A/S in the future will be $\Pi = 20\ \%$.

There is no indications from the management of GN Store Nord A/S that focus is on bringing the asset turnover up. Instead focus is on the high end products where the margin is highest. It will therefore be assumed that the asset turnover in the future will be as it has been in the past. An asset turnover of $\Lambda = 1$ for GN Store Nord A/S will therefore be used.

For the book leverage is was seen in Figure 6 that its past value have been in the interval between around 0.1 and 0.2. It will be assumed that GN Store Nord A/S will continue to have a low leverage in the future. Thus, a book leverage of $L_B = 0.1$ for GN Store Nord A/S will be used.

The growth is assumed to follow a Geometric Brownian motion such that the relative growth can be written as the sum $g + \sigma \mathrm{d}W_t/\mathrm{d}t$, where g is an average deterministic growth, σ is the volatility and W_t is a Wiener process.

The frequency density of the historic relative growth per quarter of GN Store Nord A/S is shown in Figure 8.

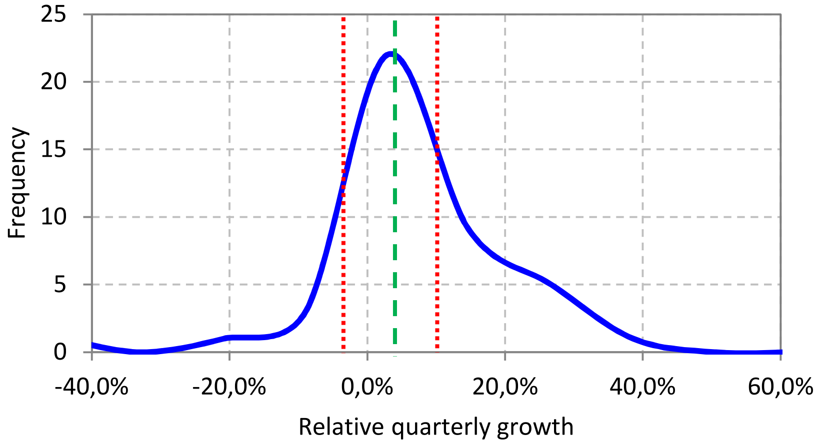

Figure 8. Frequency density for the relative revenue growth of GN Store Nord A/S.

In the frequency density diagram of Figure 8 the observed relative growth of the revenue of GN Store Nord A/S per quarter is shown along the abscissa while the frequency of the observations is shown along the ordinate. The frequency tells how many instances there has been of an observation.

The average relative growth is shown with the dashed green line. It is approximate at 2 % per quarter. Thus, the yearly average relative growth has historically been four times as much. However, it will be assumed that the revenue of GN Store Nord A/S will grow at around $g = 5\ \%$ in the future.

The volatility is shown as the distance between the two dotted red lines. It is approximate 10 %. Thus, the yearly volatility has historically been approximate $\sigma = 20\ \%$. However, it will be assumed that the volatility of the revenue of GN Store Nord A/S will be $\sigma = 10\ \%$ in the future.

Table 6 summarizes the chosen input parameters to both the simple DCF model as well as the iterative DCF model.

Initial state		**Drivers**		**Returns**	
Revenue (TR)	6000	EBIT margin (Π)	20 %	Equity (k_e)	10 %
Invested capital (IC)	6000	Asset turnover (Λ)	1	Yield (y)	5 %
Net interest-bearing debt (B)	300	Leverage (L_B)	0.1	Tax rate (τ)	25 %
		Growth (g)	5 %		
		Volatility (σ)	10 %		

Table 6. Input parameters for the valuation of GN Store Nord A/S.

6.1.3. The enterprise value of GN Store Nord A/S

With the input parameters determined above the enterprise value of GN Store Nord A/S as of February 21st 2013 can now be modeled in both the simple DCF model and the iterative DCF model.

The enterprise value of GN Store Nord A/S in the simple DCF model

To begin with the enterprise value of GN Store Nord A/S in the simple DCF model will be determined.

In the simple DCF model the expected enterprise value as a function of the cut off year of the budget was shown to have the following analytical expression:

$$EV_s = TR\left(\Pi\left(1-\tau\right)-g/\Lambda\right)\frac{1-e^{-rT}}{r}+IC-TR/\Lambda \tag{5.7}$$

where the subscript s denotes the simple DCF model and $r = wacc - g$. The input parameters can be inserted directly into the above analytical expression for the enterprise value.

The simple DCF model is distinguished in that an intrinsic capital structure is chosen. In the simple DCF model the weighted average cost of capital is therefore written as:

$$wacc_s = k_e - s \cdot L_B \tag{5.8}$$

The net interest bearing debt does not go into the enterprise value in the simple DCF model. This is because the book leverage that goes into the weighted average cost is taken as the future book leverage of the enterprise and not as the ratio between the net interest-bearing debt of the initial state of the enterprise and the invested capital of the initial state of the enterprise.

When the simple DCF model is used to determine the enterprise value the input parameters that have been chosen should of course be kept with and not changed because the resulting enterprise value seems too far away from what can be observed. If an input parameter is changed to fit an observed enterprise value the simple DCF model is no longer a model for the enterprise value but instead a model used for determining an input parameter implicitly. Such an input parameter could for example be the weighted average cost of capital.

For example if the above weighted average cost of capital in the simple DCF model has a value that is close to the growth of the enterprise it would result in a very large value of the enterprise value. In such a case it is tempting to change the value of the weighted average cost of capital, but then it is no longer the enterprise value that is determined, instead it is the weighted average cost of capital that is determined implicitly.

Only when the weighted average cost of capital is changed such that the output would be consistent with the input is it fair to go back and change the weighted average cost of capital. But then it would be an iterative DCF model and the weighted average cost of capital should be changed such that the output indeed is consistent with the input of the DCF model.

In Figure 9 the enterprise value of GN Store Nord A/S as modeled by the simple DCF model is shown as a function of when the budget is cut off. The solid blue line is the enterprise value as a function of the budget cut off year in the simple DCF model determined according to equation (5.7). The black circles shows the enterprise value as a function of the budget cut off year in the simple DCF model determined numerically with a Monte Carlo method. The dashed green line is the enterprise value in the simple DCF model when there is an infinite number of years in the budget. The dotted red line is the enterprise value in an EVA model.

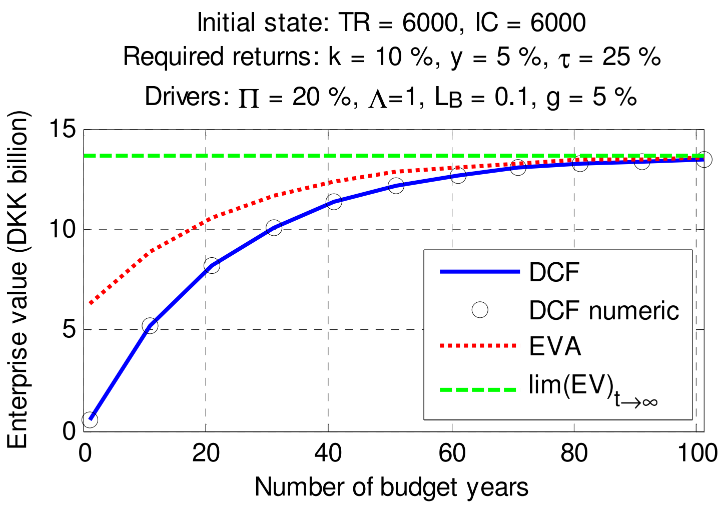

Figure 9. The enterprise value of GN Store Nord A/S in the simple DCF model.

It can be seen from Figure 9 that the enterprise value in the simple DCF model begins to saturate more and more the higher the number of budget years there is added to the budget. It saturates to the same value as the enterprise value in an EVA model saturates at. Furthermore, the enterprise value determined numerically in the simple DCF model with a Monte Carlo method results in the same values as the closed form solution.

Since it is assumed that GN Store Nord A/S is a going concern the budget should only be cut off after an infinite number of budget years. For the simple DCF model it is in fact possible to include an infinite number of budget years in the budget without cutting the budget off. Doing this results in an enterprise value of GN Store Nord A/S of:

$$\lim_{t\to\infty}(EV_s) = \frac{TR}{wacc-g}\left(\Pi(1-\tau) - \frac{wacc}{\Lambda}\right) + IC = 13.7 \text{ DKK billion} \tag{5.9}$$

where the weighted average cost of capital is $wacc_s = 9.38\ \%$.

The volatility of the enterprise value in the simple DCF model can be expressed with the following analytical expression:

$$\sigma_{EV} = EVA_0 \sqrt{\frac{1}{r^2} - \frac{1}{r\left(r + 0.5\sigma^2\right)}} \tag{5.10}$$

where $r = wacc - g$ and $EVA_0 = TR\left(\Pi\left(1-\tau\right) - wacc/\Lambda\right)$. When the input parameters are inserted into the above expression a volatility of $\sigma_{EV} = 2.47$ DKK billion is attained.

<u>The market value of GN Store Nord A/S</u>

The market value of the equity $\left(MV\right)$ of GN Store Nord A/S was at the end of 2012 observed to be just below $MV = 14$ DKK billion. The net interest-bearing debt adds little to this so the market value of the enterprise was in the area of $EV_M = 14$ DKK billion at the end of 2012.

The observed market value of the enterprise is very close to the enterprise value modeled by the simple DCF model. However, care should be taken when doing a direct comparison between the market value of the enterprise and the enterprise value modeled by the simple DCF model. This is because the simple DCF model is only a representation of the enterprise value it models. Thus, a number of factors are left out and it will therefore only be a coincidence if the enterprise value for a single test case in the simple DCF model comes close to the market value of the enterprise.

<u>The enterprise value of GN Store Nord A/S in the iterative DCF model</u>

The enterprise value in the iterative DCF model is now going to be determined. In the iterative DCF model the enterprise value is at a given time t modeled as:

$$EV_t = \frac{s \cdot B_t \Delta t}{W_L\left(s \cdot B_t \Delta t \exp\left(k_e \Delta t\right) / \left(CF_{t+1} + EV_{t+1}\right)\right)} \tag{5.11}$$

where the time t corresponds to the sampling numbers n that was introduced in the derivation of the iterative DCF model.

There is no analytical expression for the enterprise value discounted all the way back to present time. Thus, the input parameters cannot simply be inserted into a formula as with the simple DCF model. Instead the enterprise value needs to be calculated iteratively by applying the above expression one step at a time until the cash flow to the security holders has been discounted all the way back to present time. As in the simple DCF model it is assumed that GN Store Nord A/S is a going concern for an infinite number of years. Thus, an infinite number of budget years should be included in the budget.

However, since the iterative DCF model is a numerical model it is not possible to include an infinite number of budget years in the budget. Instead of including an infinite number of budget years in the budget the enterprise value in the iterative DCF model can be determined as a function of the number of budget years in the budget as was done with the simple DCF model.

Figure 10 shows the enterprise value of GN Store Nord A/S as modeled by the iterative DCF model as a function of when the budget is cut off. The value of the enterprise in the iterative DCF model is illustrated with a solid blue line while the value of the enterprise in an iterative EVA model is illustrated with a dotted red line. In Appendix on page 84 is given an example of the implementation of the numerical method of the iterative DCF model in Excel and in Appendix on page 85 is given an example of the implementation of the numerical method of the iterative DCF model in Matlab. In Excel the columns of the worksheet is used for the expected evolution of the variables in the iterative DCF model and in Matlab a For loop is used to generated the evolution of the variables in the iterative DCF model. The enterprise value is then found in the end as the expectation of all the paths of the variables. The EVA model is also shown in Appendix on page Appendix F: An iterative EVA model.

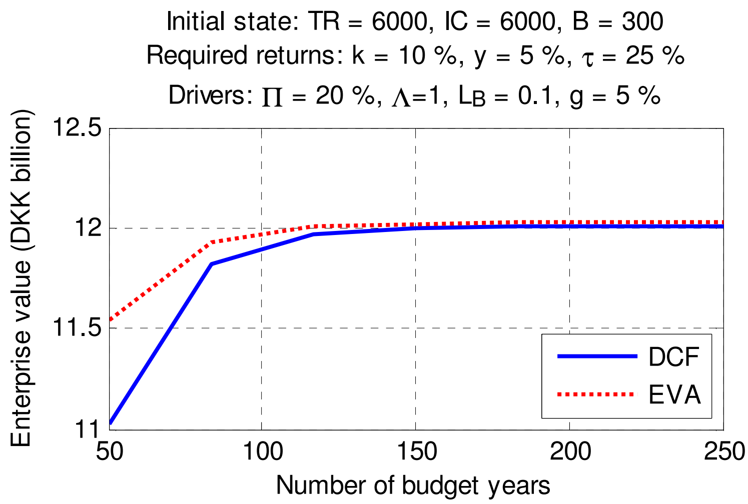

Figure 10. The enterprise value of GN Store Nord A/S as a function of budget years.

It can be seen from Figure 10 that the enterprise value in the iterative DCF model begins to saturate the higher the number of budget years there is added to the budget.

In Figure 10 there has been included up to 250 budget years. For that many budget years the enterprise value of GN Store Nord A/S in the iterative DCF model is about $EV_i = 12.1$ DKK billion, where the subscript i denotes the iterative DCF model.

In Figure 10 the enterprise value in the iterative DCF model is shown together with the enterprise value in an iterative EVA model. It can be seen that the enterprise value in the iterative EVA model also begins to saturate more and more the higher the number of budget years there is added to the budget. In fact the enterprise value in the iterative EVA model saturates at the same value as the iterative DCF model saturates at. This is a first indication that the iterative DCF model is consistent with an iterative EVA model as it should be.

The above enterprise value of Figure 10 have been calculated numerically by sampling a continuous cash flow. However, it is important that the sampling is high enough in order to capture that the cash flow is continuous.

In order to ensure this the enterprise value of GN Store Nord A/S as modeled by the iterative DCF model can be plotted as a function of the number of samples per year. This will show how high a sampling per year is necessary in order to achieve an enterprise value that converges.

Figure 11 shows the enterprise value of GN Store Nord A/S as modeled by the iterative DCF model as a function of the number of samples per year. The value of the enterprise in the iterative DCF model is illustrated with a solid blue line while the value of the enterprise in an iterative EVA model is illustrated with a dotted red line. The dashed green line shows the enterprise value when there is one year between the samples. This would correspond to a discrete time setting.

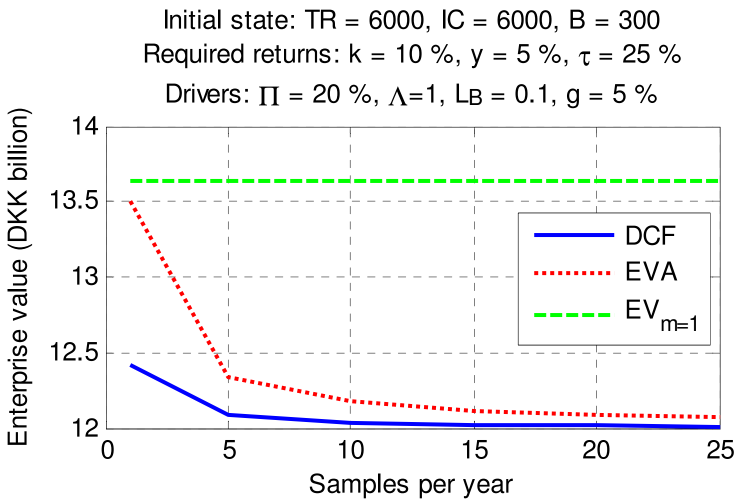

Figure 11. The enterprise value of GN Store Nord A/S as a function of sampling.

It can be seen from Figure 11 that the enterprise value in the iterative DCF model begins to saturate more and more the higher the sampling is.

When there is only one sample per year the enterprise value of GN Store Nord A/S can be written with the analytical expression from discrete time:

$$EV_{m=1} = TR\frac{1+g}{k_e - g}\left[\Pi(1-\tau) - \frac{1}{\Lambda}\frac{k_e}{1+k_e} + \frac{IC}{TR}\frac{s \cdot L_B}{1+k_e}\right] + \frac{IC + s \cdot B}{1+k_e} = 13.6 \text{ DKK billion} \quad (5.12)$$

where the subscript $m = 1$ denotes a sampling of the cash flow of one times per year, i.e. a discrete time domain. The derivation of the above expression is shown in Appendix on page 87.

When there are 25 samples per year the enterprise value of GN Store Nord A/S in the iterative DCF model has saturated to the value of $EV_{m=25} = 12.1$ DKK billion.

Thus, it is clear that sampling only once per year results in an overvaluation of the enterprise. This is one of the reasons to why it is interesting to look at a stochastic DCF model such that this overvaluation can be removed.

In Figure 11 the enterprise value in the iterative DCF model is shown together with the enterprise value in an iterative EVA model. It can be seen that the enterprise value in the iterative EVA model also begins to saturate more and more the higher the number of budget years there is added to the budget.

In fact the enterprise value in the iterative EVA model saturates at the same value as the iterative DCF model saturates at. This is a second indication that the iterative DCF model is consistent with an iterative EVA model as it should be.

Thus, in order to determine the value of an enterprise in the iterative DCF model the enterprise value needs to convergence both with respect to the number of samples per year and the number of budget years in the budget. When the enterprise value converges with respect to these two numerical parameters the iterative DCF model is seen to be consistent with the iterative EVA model.

With the enterprise value determined above the market leverage can now be determined as $L_M = B/EV_i = 0.025$ and the weighted average cost of capital in the iterative DCF model can finally be determined as $wacc_i = k_e - s \cdot L_M = 9.84\ \%$.

The volatility of the enterprise value in the iterative DCF model is a function of the volatility of the revenue. Numerically the volatility of the enterprise value in the iterative DCF model can be determined as:

$$\sigma_{EV} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(EV_i - \mathrm{E}\left[EV\right]\right)^2} \tag{5.13}$$

Thus, a large number (N) of enterprise values needs to be generated. This can for example be done with the Data Table function of Excel. The Data Table generates a matrix with a number of rows and columns. Each cell in the matrix comprises the enterprise value for a random path of the cash flow.

With $N = 10.000$ paths the volatility of the enterprise value becomes $\sigma_{EV} \approx 2$ DKK billion.

<u>The enterprise value of GN Store Nord A/S in an approximation to the iterative DCF model</u>

As an approximation to an iterative DCF model the following enterprise value was determined:

$$EV_a = \frac{s \cdot L_B IC + r_k EV_0 + CF_0 + \sqrt{\left(s \cdot L_B IC + r_k EV_0 + CF_0\right)^2 - 4 r_k s \cdot EV_0 L_B IC}}{2 r_k} \tag{5.14}$$

where $r_k = k_e - g$, $CF_0 = TR\left[\Pi(1-\tau) - g/\Lambda\right]$ and $EV_0 = IC - TR/\Lambda$.

In order to see how close the above analytical expression comes to the numerically calculated enterprise value the input parameters will be put into the expression. This will also make it possible to see if the approximation can come closer to the numerically calculated enterprise than the simple DCF model does.

When the input parameters are inserted into the analytical expression, an enterprise value of $EV_a = 12.7$ DKK billion for GN Store Nord A/S is attained. This is a little above the enterprise value of 12 DKK billion determined in the iterative DCF model and a little below the enterprise value of 13 DKK billion determined in the simple DCF model.

With the enterprise value determined above the market leverage can now be determined as $L_M = B/EV_i = 0.024$ and the weighted average cost of capital in the approximation to the iterative DCF model can be determined as $wacc_a = k_e - s \cdot L_M = 9.85\ \%$.

The volatility of the enterprise value in the approximation to the iterative DCF model is approximated with:

$$\sigma_{EV} = EVA_0 \sqrt{\frac{1}{r^2} - \frac{1}{r\left(r + 0.5\sigma^2\right)}} \qquad (5.15)$$

where $r = wacc_i - g$, $EVA_0 = TR\left(\Pi\left(1-\tau\right) - wacc_i/\Lambda\right)$ and $wacc_i = k_e - s \cdot L_M$. With the above weighted average cost of capital the volatility of the enterprise value of GN Store Nord A/S in the approximation to the iterative DCF model becomes $\sigma_{EV} = 1.94$ DKK billion.

<u>Comparison between the simple DCF model and the iterative DCF model</u>

A comparison between the enterprise value of GN Store Nord A/S in the simple DCF model and the enterprise value of GN Store Nord A/S in the iterative DCF model can now be performed.

Figure 12 shows the probability density function for the enterprise value in both the simple DCF model and the iterative DCF model. The solid blue line is the enterprise value in the iterative DCF model while the dotted red line is enterprise value in the simple DCF model.

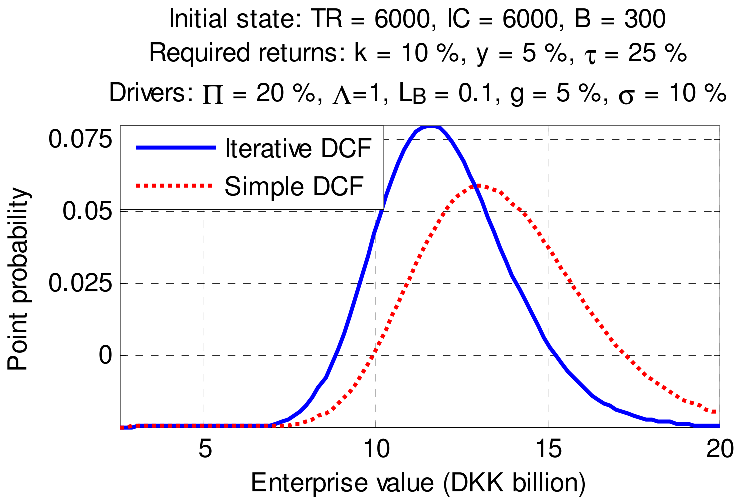

Figure 12. Probability density function of the enterprise value of GN Store Nord A/S.

In Figure 12 it can be seen that the probability density function for the simple DCF model and the iterative DCF model are not that far from each other.

The expected enterprise value in the simple DCF model was found above to be $EV_s = 13.7$ DKK billion while the expected enterprise value in the iterative DCF model was found above to be $EV_i = 12.1$ DKK billion. Thus, the probability density function for the enterprise of the simple DCF model is around 1 DKK billion to the right of the probability function for the enterprise value in the iterative DCF model. This is also evident in Figure 12.

The volatility of the enterprise value in the simple DCF model was found above to be $\sigma_{EV_s} = 2.47$ DKK billion while the volatility of the enterprise value in the iterative DCF model was found above to be $\sigma_{EV_i} = 1.95$ DKK billion. Thus, the width of the probability function for the enterprise value in the simple DCF model is seen to be almost identical to the width of the probability function for the enterprise value in the iterative DCF model.

Table 7 summarizes the main results for the valuation of GN Store Nord A/S in the simple DCF model, the iterative DCF model and the approximation to the iterative DCF model. Both the enterprise value, the weighted average cost of capital and the volatility of the enterprise value are summarized. The market value of the enterprise is also shown.

Model	**Enterprise value** (DKK billion)	**wacc**	σ_{EV} (DKK billion)
Simple DCF model	13.7	9.38 %	2.47
Iterative DCF model	12.1	9.84 %	≈ 2
Approximation to the iterative DCF model	12.7	9.85 %	1.94
Market value of the enterprise	≈14	–	–

Table 7. Comparison of DCF models for valuation of a low leveraged enterprise.

It was predicted that for an enterprise with a book-to-market ratio greater than one, the weighted average cost of capital in the simple DCF model would be smaller than the weighted average cost of capital in the iterative DCF model. Since the weighted average cost of capital is smaller in the simple DCF model than it should be, the simple DCF model overvalues the enterprise. For GN Store Nord A/S the simple DCF model overvalues the enterprise with 1 DKK billion.

The relative difference between the enterprise value of GN Store Nord A/S in the simple DCF model and the enterprise value of GN Store Nord A/S in the iterative DCF model is about 8 %. Even though 1 DKK billion could be said to be a large amount a relative difference of 8 % is not the much. It is therefore questionable if it is worth the trouble to use an iterative DCF model when an enterprise with a low leverage is to be valuated.

6.2. Enterprise value of a high leveraged enterprise

The relative difference between the enterprise value in the simple DCF model and the enterprise value in the iterative DCF model is now to be determined for a relatively high leveraged enterprise.

6.2.1. Introduction to William Demant A/S

As a high leveraged enterprise the business William Demant A/S is chosen as a test case. William Demant A/S also produces hearing aids, head sets and audiometric equipment for the health care industry, (William Demant A/S, 2012).

Figure 13 shows the performance of William Demant A/S on the stock exchange since the beginning of the millennium. The historic income statement and balance sheet of William Demant A/S is shown in Appendix on page 83.

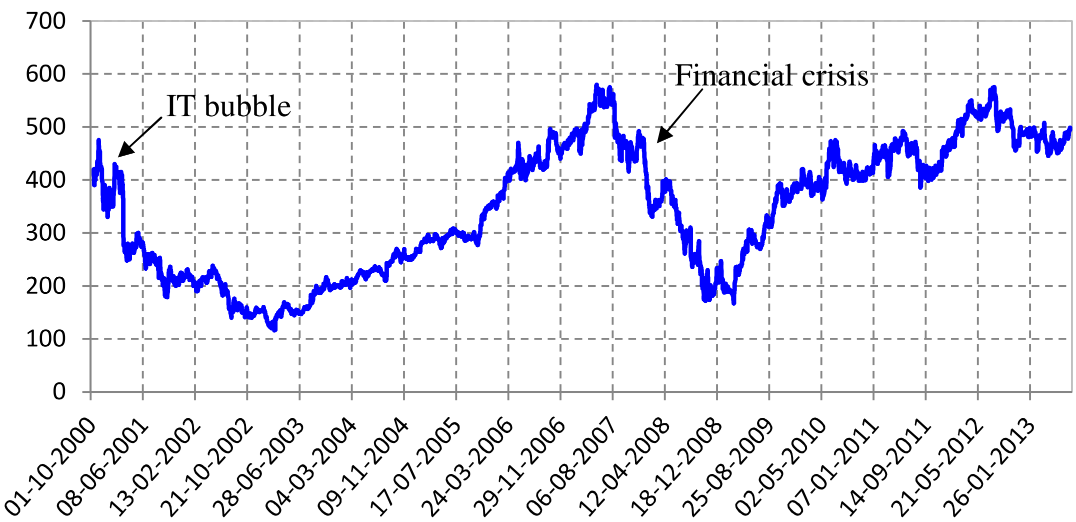

Figure 13. Historic share price of William Demant A/S.

William Demant A/S has not had as dramatic a history since the millennium as GN Store Nord A/S. However, in Figure 13 it can be seen that the share price of William Demant

A/S did go down after the IT bubble as well as after the financial crisis. Thus, Williams Demant A/S has experienced two dips very similar to what GN Store Nord A/S has experienced since the beginning of the millennium.

Even though William Demant A/S has not experienced the same shock to a failed investment and to a failed sales process the share price of William Demant A/S seems to be on a slower climb than the share price of GN Store Nord A/S. A likely explanation to this could be that William Demant A/S is pursuing a more old fashioned technology than GN Store Nord A/S, (William Demant A/S, 2012).

Leverage of William Demant A/S

William Demant A/S is chosen as a test case, because it is an enterprise with a historically high book leverage at least when compared to an enterprise such as GN Store Nord A/S.

The historic book leverage of William Demant A/S is shown in Figure 14 together with the historic yields. The period is from the year 2004 to the year 2012. The book leverage and yield are plotted in pairs as a phase diagram. For comparison the average book leverage and average yield of GN Store Nord A/S has been illustrated as the red square.

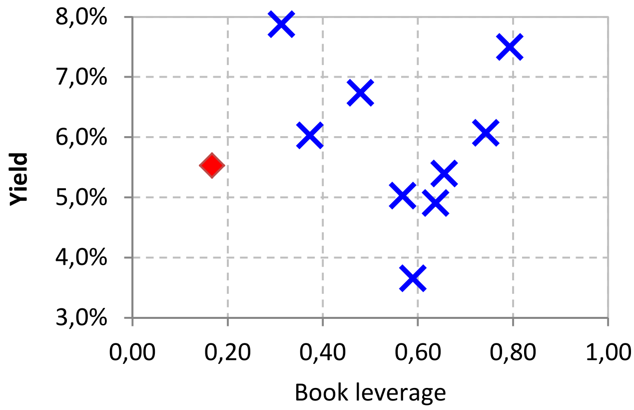

Figure 14. Historic book leverage of William Demant A/S

It can be seen that the book leverage of William Demant A/S is found more or less in the interval between a book leverage of 0.3 and a book leverage of 0.8. Many of the observations are placed around a book leverage of 0.6. The yield is found more or less in the interval between a yield of 4 % and a yield of 8 %.

6.2.2. Values of the input parameters for the valuation of William Demant A/S

The input parameters to both the simple DCF model and the iterative DCF model for William Demant A/S are now going to be determined.

The initial state of William Demant A/S

The valuation of William Demant A/S is as of the date of the publication of the latest annual report. The latest annual report of William Demant A/S is the report of the income period 2012. This report was published February 26th 2013. Thus, the valuation is as of that day and the initial state of the enterprise of that day is needed.

In the annual report of the income period 2012 William Demant A/S reported a closing balance with a net interest-bearing debt of 1.804 DKK millions. There are two months separating the end of 2012 and the time of valuation of William Demant A/S. It will therefore be assumed that the initial state of the net interest-bearing debt has not changed much. For convenience a value of $B = 1.800$ DKK millions for the net interest-bearing debt of William Demant A/S will therefore be used.

For the revenue William Demant A/S reported a value of just above 8.500 DKK million in the income period of 2012. Again there is little time up to the date of valuation. It will therefore be assumed that the initial state of the revenue has not changed much. Thus, a value of $TR = 8.500$ DKK millions for the initial state of the revenue of William Demant A/S will be used.

William Demant A/S does not explicitly report the value of the invested capital. However, it has been seem that the invested capital can be derived from the equity and the net interest bearing debt. Using this means that the invested capital in the closing balance of the income year of 2012 was just below $IC = 6.000$ DKK millions. This value will be used for the initial state of the invested capital of William Demant A/S.

Returns to the security holders and tax rate of William Demant A/S

In Figure 14 the historic yield of William Demant A/S was shown. It took on values more or less in the interval between a yield of 4 % and a yield of 8 %. But what will the yields of William Demant A/S be in the future? Since there is no simple way to answer this question it will be assumed that the future yield will correspond to the observed historic yield. Thus, a yield of $y = 5\ \%$ will be used for William Demant A/S.

The enterprise of William Demant A/S is very similar to that of GN Store Nord A/S. It will therefore be assumed that the risk on the equity of William Demant A/S is similar to the risk on the equity of GN Store Nord A/S. The same value for the required return to the equity will therefore be used for both enterprises. This means that the required return to the equity of William Demant A/S will be $k_e = 10\ \%$ in the future.

The tax rate for William Demant A/S will also be assumed to be the same as for GN Store Nord /AS. Thus, a tax rate of $\tau = 25\ \%$ for William Demant A/S will be used.

<u>Budget drivers of William Demant A/S</u>

The realized asset turnover and the EBIT margin for William Demant A/S from the year 2004 until the year 2012 are shown together in the below phase diagram of Figure 15. For comparison the average EBIT margin and average asset turnover of GN Store Nord A/S has been illustrated as the red square.

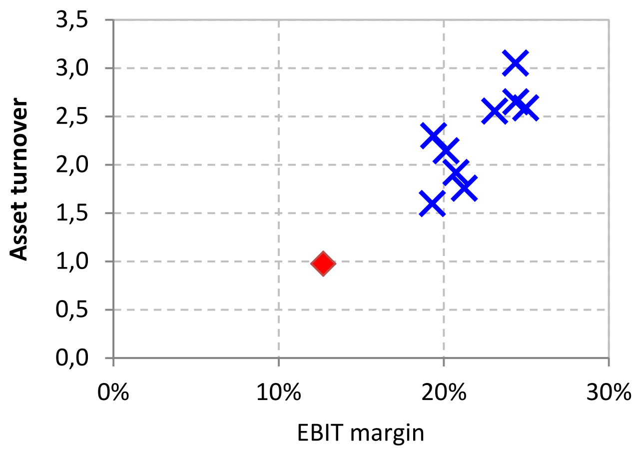

Figure 15. Historic EBIT margin and asset turnover of William Demant A/S.

In Figure 15 it can be seen that the realized asset turnover of William Demant A/S in the period 2004 to 2012 has been between 1.5 and 3. The EBIT margin has been close to 20 %. It will be assumed that the future asset turnover and EBIT margin of William Demant A/S not will be different than the historic values. Thus, for the asset turnover a value of $\Lambda = 2$ will be used for William Demant A/S and for the EBIT margin a value of $\Pi = 20\ \%$ will be used for William Demant A/S.

For the book leverage is was seen in Figure 14 that many of the observed values of the historic book leverage of William Demant A/S was around 0.6. It will therefore be assumed that William Demant A/S will continue to have such a relative high leverage in the future. Thus, a book leverage of $L_B = 0.6$ for William Demant A/S will be used.

When it comes to the growth of the revenue of William Demant A/S it is assumed that it follows a Geometric Brownian motion. The average deterministic growth and volatility of the relative growth of the revenue of William Demant A/S is illustrated in Figure 16 where a frequency density of the historic semiannual relative growth of William Demant A/S is shown.

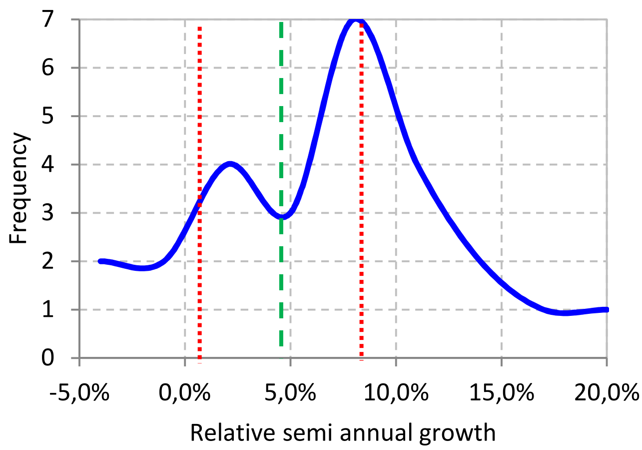

Figure 16. Frequency density for the relative revenue growth of William Demant A/S.

In the frequency density diagram of Figure 16 the observed relative growth is shown along the abscissa while the frequency of the observations is shown along the ordinate. The frequency tells how many instances there has been of the observation.

The average relative growth is shown with the dashed green line. It is approximate at 5 % semiannual. The yearly average relative growth is double that. Thus, the average relative growth of William Demant A/S has historically been 10 % per year. In the following it will be assumed that William Demant A/S will grow at a rate of $g = 5\ \%$ per year in the future. This will correspond to the assumed growth rate of the revenue of GN Store Nord A/S.

The volatility is shown as the distance between the two dotted red lines. It is approximate 8 % semiannual. To determine the yearly volatility the semiannual volatility has to be multiplied with the square root of two. Thus, the volatility of the revenue of William Demant A/S has historically been around 12 % per year. In the following it will be assumed that the revenue of William Demant A/S will have a volatility of $\sigma = 10\ \%$ per year in the future. This will correspond to the assumed volatility of the revenue of GN Store Nord A/S.

Table 8 summarizes the chosen input parameters to both the simple DCF model as well as the iterative DCF model.

Initial state		**Drivers**		**Returns**	
Revenue (TR)	8.500	EBIT margin (Π)	20 %	Equity (k)	10 %
Invested capital (IC)	6.000	Asset turnover (Λ)	2	Yield (y)	5 %
Net interest-bearing debt (B)	1.800	Leverage (L_B)	0.6	Tax rate (τ)	25 %
		Growth (g)	5 %		
		Volatility (σ)	10 %		

Table 8. Input parameters for the valuation of William Demant A/S.

When the input parameters for the valuation of William Demant A/S are compared to the input parameters for the valuation of GN Store Nord A/S it can be seen that when it comes to the debt and the leverage the two enterprises are different from each other. The initial state of the revenue and the asset turnover are also different between the two enterprises. However, the rest of the parameters are the same for both enterprises.

6.2.3. The enterprise value of William Demant A/S

With the input parameters determined above the enterprise value of William Demant A/S as of February 26th 2013 can now be modeled in both the simple DCF model and the iterative DCF model.

<u>The enterprise value of William Demant A/S in the simple DCF model</u>

To begin with the enterprise value of William Demant A/S in the simple DCF model will be determined.

The weighted average cost of capital for William Demant A/S in the simple DCF model is $wacc_s = k_e - s \cdot L_B = 6.3\ \%$. This is lower than the weighted average cost of capital of GN Store Nord A/S, simply because the book leverage of William Demant A/S is higher than the book leverage of GN Store Nord A/S. It is also relative close to the growth of $g = 5\ \%$. Other things aside a relative large enterprise value is therefore to be expected in the simple DCF model.

When the input parameters of William Demant A/S is inserted into the simple DCF model the enterprise value becomes:

$$EV_s = \frac{TR}{wacc - g}\left(\Pi\left(1-\tau\right) - \frac{wacc}{\Lambda}\right) + IC = 86\ \text{ DKK billion} \tag{5.16}$$

This is a relatively large enterprise value compared to GN Store Nord A/S despite the fact that the two enterprise are very similar. Only the asset turnover and the leverage are different between the two enterprises.

The volatility of the enterprise value of William Demant A/S in the simple DCF model becomes:

$$\sigma_{EV} = EVA_0\sqrt{\frac{1}{r^2} - \frac{1}{r\left(r + 0.5\sigma^2\right)}} = 43\ \text{ DKK billion} \tag{5.17}$$

where $r = wacc_s - g$, $EVA_0 = TR\left(\Pi\left(1-\tau\right) - wacc_s/\Lambda\right)$ and $wacc_s = k_e - s \cdot L_B$.

The market value of William Demant A/S

The market value of the equity was at the end of 2012 observed to be $MV = 27.4$ DKK billion. With a net interest-bearing debt of $B = 1.800$ DKK billion the market value of the enterprise of William Demant A/S at the end of 2012 was approximately $EV_M = 29$ DKK billion.

The observed enterprise value of 29 DKK billion is therefore not very close to the enterprise value of 86 DKK billion modeled in the simple DCF model.

The enterprise value of William Demant A/S in the iterative DCF model

The enterprise value in the iterative DCF model is now going to be determined. In the iterative DCF model the enterprise value is at a given time t modeled as:

$$EV_t = \frac{s \cdot B_t \Delta t}{W_L\left(s \cdot B_t \Delta t \exp\left(k_e \Delta t\right) / \left(CF_{t+1} + EV_{t+1}\right)\right)} \tag{5.18}$$

As in the simple DCF model it is assumed that William Demant A/S is a going concern for an infinite number of years. Since it is very difficult to include an infinite number of budget years in the budget the enterprise value will be determined as a function of the number of budget years.

Thus, Figure 17 shows the enterprise value of William Demant A/S as modeled by the iterative DCF model as a function of when the budget is cut off. The value of the enterprise in the iterative DCF model is illustrated with a solid blue line while the value of the enterprise in an iterative EVA model is illustrated with a dotted red line.

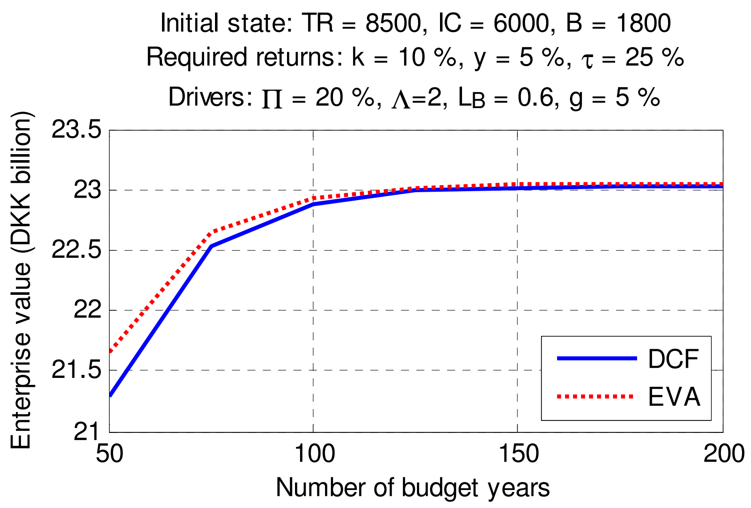

Figure 17. The enterprise value of William Demant A/S as a function of budget years.

It can be seen from Figure 17 that the enterprise value of William Demant A/S in the iterative DCF model begins to saturate more and more the higher the number of budget years there is added to the budget.

In Figure 17 there has been included up to 200 budget years. For that many budget years the enterprise value of William Demant A/S in the iterative DCF model is about $EV_i = 23$ DKK billion, where the subscript i denotes the iterative DCF model.

In Figure 17 the enterprise value in the iterative DCF model is shown together with the enterprise value in an iterative EVA model. The enterprise value in the iterative EVA model saturates at the same value as the iterative DCF model saturates at.

To ensure that the time steps are small enough to represent a continuous time cash flow the enterprise value of William Demant A/S as a function of the number of samples per year should be determined.

Thus, Figure 18 shows the enterprise value of William Demant A/S as modeled by the iterative DCF model as a function of the number of samples per year. The value of the enterprise in the iterative DCF model is illustrated with a solid blue line while the value of the enterprise in an iterative EVA model is illustrated with a dotted red line. The dashed green line shows the enterprise value when there is one year between the samples. This would correspond to a discrete time setting.

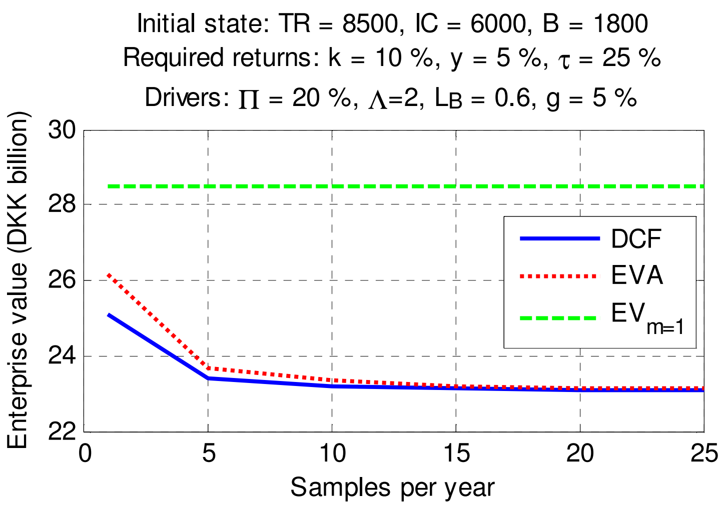

Figure 18. The enterprise value of William Demant A/S as a function of sampling.

It can be seen from Figure 18 that the enterprise value in the iterative DCF model begins to saturate more and more the higher the sampling is.

When there are 25 samples per year the enterprise value of William Demant A/S in the iterative DCF model has saturated to the value of $EV_{m=25} = 23$ DKK billion.

The result from discrete time is illustrated with the dashed green line. Using the analytical expression from discrete time the enterprise value of William Demant A/S can be written as:

$$EV_{m=1} = TR\frac{1+g}{k_e - g}\left[\Pi(1-\tau) - \frac{1}{\Lambda}\frac{k_e}{1+k_e} + \frac{IC}{TR}\frac{s \cdot L_B}{1+k_e}\right] + \frac{IC + s \cdot B}{1+k_e} = 28.5 \text{ DKK billion} \quad (5.19)$$

where the subscript $m = 1$ denotes a sampling of the cash flow of one times per year, i.e. a discrete time domain.

In Figure 18 the enterprise value in the iterative DCF model is shown together with the enterprise value in an iterative EVA model. The enterprise value in the iterative EVA model saturates at the same value as the iterative DCF model saturates at.

With the enterprise value determined above the market leverage can now be determined as $L_M = B/EV_i = 0.078$ and the weighted average cost of capital in the iterative DCF model can finally be determined as $wacc_i = k_e - s \cdot L_M = 9.51\ \%$.

Finally, the volatility of the enterprise value of William Demant A/S in the iterative DCF model becomes $\sigma_{EV} \approx 7$ DKK billion calculated numerically with $N = 10.000$ paths.

<u>The enterprise value of William Demant A/S in an approximation to the iterative DCF model</u>

As an approximation to an iterative DCF model the following enterprise value was determined:

$$EV_a = \frac{s \cdot L_B IC + r_k EV_0 + CF_0 + \sqrt{\left(s \cdot L_B IC + r_k EV_0 + CF_0\right)^2 - 4 r_k s \cdot EV_0 L_B IC}}{2 r_k} = 27 \text{ DKK billion}$$

where $r_k = k - g$, $CF_0 = TR\left[\Pi(1-\tau) - g/\Lambda\right]$ and $EV_0 = IC - TR/\Lambda$.

With the enterprise value determined above the market leverage can now be determined as $L_M = B/EV_i = 0.066$ and the weighted average cost of capital in the approximation to the iterative DCF model can be determined as $wacc_a = k_e - s \cdot L_M = 9.59\ \%$. With this value of the weighted average cost of capital the volatility of the enterprise value of William Demant A/S in the approximation to the iterative DCF model becomes:

$$\sigma_{EV_a} \approx EVA_0 \sqrt{\frac{1}{r^2} - \frac{1}{r\left(r + 0.5\sigma^2\right)}} = 5.9 \text{ DKK billion} \tag{5.20}$$

where $r = wacc_a - g$, $EVA_0 = TR\left(\Pi\,(1-\tau) - wacc_a/\Lambda\right)$ and $wacc_a = k_e - s \cdot L_M$.

<u>Comparison between the simple DCF model and the iterative DCF model</u>

A comparison between the enterprise value of William Demant A/S in the simple DCF model and the enterprise value of William Demant A/S in the iterative DCF model can now be performed. Figure 19 shows the probability density function for the enterprise value in both the simple DCF model and the iterative DCF model. The solid blue line is the enterprise value in the iterative DCF model while the dotted red line is enterprise value in the simple DCF model.

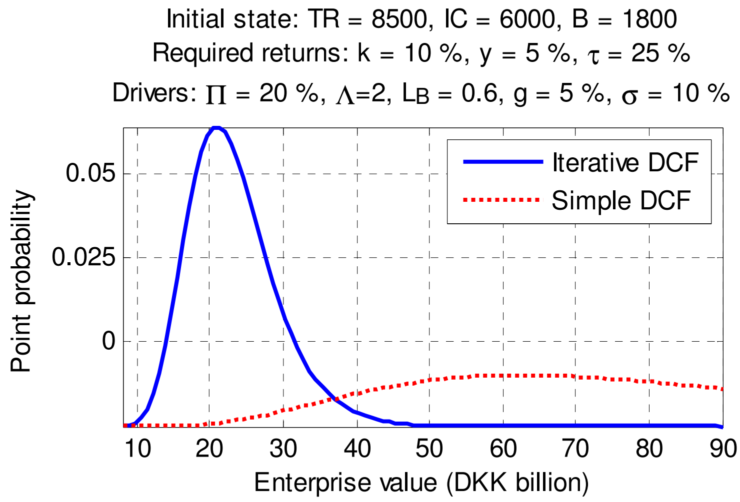

Figure 19. Probability density function of the enterprise value of William Demant A/S.

In Figure 19 it can be seen that the probability density function for the simple DCF model and the iterative DCF model are very different.

The expected enterprise value in the simple DCF model was found to be $EV_s = 86$ DKK billion while the expected enterprise value in the iterative DCF model was found to be $EV_i = 23$ DKK billion.

The volatility of the enterprise value in the simple DCF model was found above to be $\sigma_{EV_s} = 43$ DKK billion while the volatility of the enterprise value in the iterative DCF model was found above to be $\sigma_{EV_i} \approx 7$ DKK billion.

This means that the probability density function for the enterprise value of William Demant A/S in the simple DCF model will be wide and to the right in Figure 19 while the probability density function for the enterprise value of William Demant A/S in the iterative DCF model will be thin and to the left in Figure 19.

Table 9 summarizes the main results for the valuation of William Demant A/S in the simple DCF model, the iterative DCF model and the approximation to the iterative DCF model. Both the enterprise value, the weighted average cost of capital and the volatility of the enterprise value are summarized. The market value of the enterprise is also shown.

Model	**Enterprise value** (DKK billion)	**wacc**	σ_{EV} (DKK billion)
Simple DCF model	86	6.25 %	43.2
Iterative DCF model	23	9.51 %	≈ 7
Approximation to the iterative DCF model	27	9.59 %	5.9
Market value of the enterprise	≈29	–	–

Table 9. Comparison of DCF models for valuation of a high leveraged enterprise.

For the low leveraged enterprise of GN Store Nord A/S there was not much difference between the simple DCF model and the iterative DCF model. However, for the high leveraged enterprise of William Demant A/S there is a large difference between the simple DCF model and the iterative DCF model.

The weighted average cost of capital in the simple DCF model is more than three percent points lower than the weighted average cost of capital in the iterative DCF model. Other things aside this means that the cash flow to the security holders will not be discounted as hard in the simple DCF model as in the iterative DCF model. This results in a enterprise value that is much larger in the simple DCF model than in the iterative DCF model.

The absolute difference between the enterprise value of William Demant A/S in the simple DCF model and the enterprise value of William Demant A/S in the iterative DCF model is more than 60 DKK billion.

The relative difference between the enterprise value of William Demant A/S in the simple DCF model and the enterprise value of William Demant A/S in the iterative DCF model is 270 %. This means that the simple DCF model overvalues William Demant with 270 %.

For GN Store Nord A/S it was questionable if it was worth the effort to use an iterative DCF model, because the difference between the simple DCF model and the iterative DCF model was not that big. However, for William Demant A/S there will be a big difference between the output of a simple DCF model and an iterative DCF model. Thus, for a valuation of William Demant A/S there is something to gain by using an iterative DCF model instead of a simple DCF model.

6.3. The enterprise value as a function of leverage

Both the simple DCF model and the iterative DCF model was used above to determine the enterprise value of GN Store Nord A/S and the enterprise value of William Demant A/S. The reason for looking at two test cases was to see what happens in the simple DCF model and the iterative DCF model for an enterprise has a low leverage and for an enterprise with a high leverage.

It was seen that there was not much difference between the result of the simple DCF model and the result of the iterative DCF model when the two models were applied to GN Store Nord A/S. This is because GN Store Nord A/S operates with a relative small leverage. However, for William Demant A/S there was a big difference between the result of the simple DCF model and the result of the iterative DCF model. This is because William Demant A/S operates with a relative high leverage. Thus, the difference between the enterprise value in the simple DCF model and the enterprise value in the iterative DCF model will depend on the leverage of the enterprise.

In Figure 20 the enterprise value as a function of leverage is illustrated for the simple DCF model, the iterative DCF model and the approximation to the iterative DCF model. The parameters describing the enterprise are those for GN Store Nord A/S.

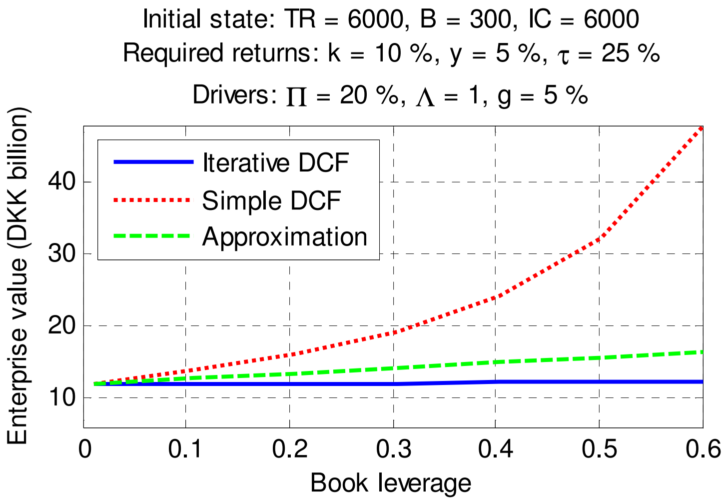

Figure 20. The enterprise value as a function of leverage.

In Figure 20 it can be seen that for a very small leverage the enterprise value in the simple DCF model and the iterative DCF model will be the same. Thus, for an enterprise with a capital structure consisting of equity alone there will be no difference between the enterprise value determined in the simple DCF model and the enterprise value determined in the iterative DCF model. As the leverage approaches zero the enterprise value can therefore be written as:

$$\lim_{L_B \to 0}(EV) = \frac{TR}{k-g}\left(\Pi(1-\tau) - \frac{k}{\Lambda}\right) + IC \tag{5.21}$$

When the leverage starts to go up the enterprise value will also start to go up. This is the same in both the simple DCF model and the iterative DCF model. The enterprise value is therefore an increasing function of the leverage. It is not as easy to see in the iterative DCF model as in the simple DCF model, because the elasticity of the enterprise value in the simple DCF model is much higher than the elasticity of the enterprise value in the iterative DCF model. The presence of the simple DCF model in Figure 20 therefore gives a large interval on the ordinate axis making the behavior of the iterative DCF model more difficult to see.

The enterprise value is maximized in both the simple DCF model and the iterative DCF model when the leverage is as high as possible. This is a corner solution and reflects the assumption that the yield is independent on debt.

However, there is a difference between how much the enterprise value increases with leverage in the simple DCF model and how much the enterprise value increases with leverage in the iterative DCF model.

<u>The behavior of the simple DCF model with respect to leverage</u>

In the simple DCF model the enterprise value almost explodes as the leverage goes up. The explosion in the enterprise value in the simple DCF model can be contributed to the fact that when the leverage goes up the weighted average cost of capital goes down and approached the growth of the revenue. The difference between the weighted average cost of capital and the growth of the revenue will therefore be closer to zero thereby dividing the future cash flow with a small number. This causes the enterprise value to explode.

<u>The behavior of the iterative DCF model with respect to leverage</u>

In the iterative DCF model the enterprise value does not show the same kind of exploding behavior with leverage. The enterprise value in the iterative DCF model is an increasing function of leverage, but it does not increase as much as is seen in the simple DCF model. Thus, the enterprise value does not increase as much with leverage as would be indicated by the simple DCF model.

Thus, the difference between the enterprise value in the simple DCF model and the enterprise value in the iterative DCF model will be an increasing function of leverage. The higher the leverage is the higher the difference between the two models will be.

<u>The behavior of the approximation to the iterative DCF model with respect to leverage</u>

The analytical expression is merely an approximation to the iterative DCF model. However, in Figure 20 it can be seen that the analytical expression does not move too far away from the iterative DCF model when the leverage increase.

The difference between implementing the iterative DCF model numerically and simply using the analytical expression is therefore not that big. At least the difference is smaller than the difference between the simple DCF model and the iterative DCF model. The approximation can be expressed as a closed form solution and is therefore much more simple to implement than the Monte Carlo method used to implement the iterative DCF model.

In Figure 20 the asset turnover and the invested capital are by coincidence chosen such that the enterprise value is a linear function of leverage in the approximation to the iterative DCF model:

$$EV = \frac{CF_0}{k_e - g} + \frac{k_e - y(1-\tau)}{k_e - g} IC \cdot L_B \tag{5.22}$$

where $CF_0 = TR\left[\Pi(1-\tau) - g/\Lambda\right]$. To arrive at such a linear function the relation between the asset turnover and the invested capital is $\Lambda = TR/IC$.

6.3.1. An alternative approximation to the iterative DCF model

Figure 20 shows that the enterprise value in the iterative DCF model stays almost at a constant level as a function of leverage. This is because the elasticity of the enterprise value in the iterative DCF model with respect to the leverage is relatively low.

At small leverages the level of the enterprise value in the iterative DCF model is the same as in the simple DCF model. This means that the enterprise value at small leverages can expressed as:

$$EV_a = \frac{TR}{k-g}\left(\Pi(1-\tau) - \frac{k}{\Lambda}\right) + IC \qquad (5.23)$$

Since the enterprise value in the iterative DCF model stays almost at a constant level the above expression also approximates the enterprise value in the iterative DCF model at high leverages.

A disadvantage of such an approximation is that the yield does not go into the enterprise value.

6.4. The influence of the input parameters

In order to go into further details on the influence of input parameters the following approximation to the enterprise value in the iterative DCF model will be used:

$$EV_a = \frac{s \cdot L_B IC + r_k EV_0 + CF_0 + \sqrt{\left(s \cdot L_B IC + r_k EV_0 + CF_0\right)^2 - 4 r_k s \cdot EV_0 L_B IC}}{2 r_k}$$

where $r_k = k - g$, $CF_0 = TR\left[\Pi(1-\tau) - g/\Lambda\right]$ and $EV_0 = IC - TR/\Lambda$. The approximation can be expressed as a closed form solution and is therefore attractive to work when the influence of the input parameters are to be investigated. It is a square root function of the book leverage. Such a square root function is defined by a non linear slope and an intersection with the ordinate axis. The intersection with the ordinate axis is at a leverage of zero. In such a case the enterprise has a capital structure consisting of equity entirely.

The sensitivity of intersection with the ordinate axis and the slope of the square root function will now be examined. Two of the more interesting input parameters are the yield and the equity return. These are both weighted as a function of the capital structure and in the end determine the value of the weighted average cost of capital.

Figure 21 shows the enterprise value in the approximation to the iterative DCF model. The figure to the left shows the enterprise value as a function of leverage for three different values of the yield while the figure to the right shows the enterprise value as a function of leverage for three different values of the equity return.

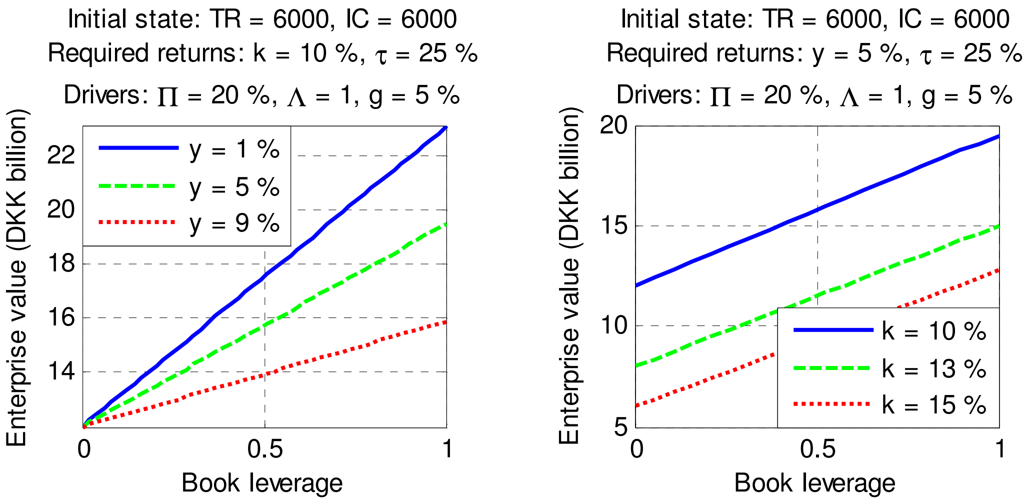

Figure 21. The enterprise value as a function of leverage.

In Figure 21 it can be seen that the yield does not influence the intersection of the function for the enterprise value. This is because the enterprise value is not a function of the yield when the book leverage is zero. The slope of the square root function, however, goes down when the yield increases.

The equity return influence the intersection in such a way that the intersection goes down when the equity return increases. Furthermore, the slope of the square root function goes down when the equity return increases..

The influence of the remaining input parameters on the intersection and the slope is shown in Table 10. A green up arrow illustrates that the intersection or the slope goes up when the input parameter increases and a red down arrow illustrates that the intersection or the slope goes down when the input parameter increases.

Parameter	Intersection	Slope
Revenue (TR)	↑	↑
Invested capital (IC)	↑	↑
EBIT margin (Π)	↑	↑
Asset turnover (Λ)	↑	↓
Growth (g)	↑	↑
Equity return (k_e)	↓	↓
Yield (y)	–	↓
Tax rate (τ)	↓	↑

Table 10. Influence of the input parameters.

Both the revenue and the invested capital has a positive effect on the intersection and the slope. Thus, when either the revenue or the invested capital increases both the intersection and slope will increase.

An increase in the EBIT margin will also lead to an increase in both the intersection and slope. Finally, this is also seen with the growth. An increase in the growth will also lead to an increase in both the intersection and slope.

An increase in the asset turnover will only lead to an increase of the intersection. This means that the intersection increases when the asset turnover increases, but the slope decreases when the asset turnover increases.

An increase in the tax rate will lead to a decrease of the intersection and an increase of the slope. This means that the intersection decreases when the asset turnover increases and the slope increases when the asset turnover increases.

7. Discussion

A DCF model establishes an integral equation for the value of an enterprise. This integral equation have been solved recursively with the Monte Carlo method. The Monte Carlo method is suitable for implementation in a Excel worksheet and it resembles how a more simple DCF model would be implemented in Excel. During the derivation and testing of the iterative DCF model a number of observations have been made. These observations include the following:

> Observation 1) Convergence of the enterprise value in the iterative DCF model with respect to time.
>
> Observation 2) Convergence of the enterprise value in the iterative DCF model with respect to sampling.
>
> Observation 3) Convergence of the enterprise value in the iterative DCF model as the leverage approaches zero.
>
> Observation 4) Convergence of the enterprise value in the iterative DCF model with respect to the EVA model.
>
> Observation 5) Converges of the Monte Carlo method with respect to the closed form solution.

Ad observation 1. To arrive at a solution for the integral equation it was assumed that the contribution of future cash flows to the enterprise value becomes less and less the further into the future the cash flow goes. This means that the enterprise value in the iterative DCF model should converge to a finite value with time. In Figure 10 and Figure 17 this convergence was seen to be fulfilled.

Ad observation 2. The iterative DCF model is a continuous time model. However, it is implemented non-continuously. The enterprise value in the iterative DCF model should therefore converge to a finite value with the sampling. In Figure 11 and Figure 18 this convergence was seen to be fulfilled.

Ad observation 3. The difference between the iterative DCF model and the simple DCF model is a function of leverage. Thus, for a leverage approaching 0 the enterprise value of the iterative DCF model should approach the enterprise value of the simple DCF model. In equation (4.14) this checked out to be fulfilled.

Ad observation 4. The iterative DCF model should give the same result for the enterprise value as other income based models for valuation. When the iterative DCF model is compared to for example an EVA model the same enterprise value should therefore be found. In Figure 10, Figure 11, Figure 17 and Figure 18 it was seen that the enterprise value in the iterative DCF model converged to the same enterprise value as in an EVA model.

Ad observation 5. There is no closed form solution for the enterprise value in the iterative DCF model. Instead the Monte Carlo method was chosen as a numerical tool for solving the integral equation numerical and determining the enterprise value in the iterative DCF model. It is therefore difficult to know how good the chosen numerical method is. However, in the simple DCF model it was possible to express the enterprise value as a closed form solution and in Figure 9 it was seen that Monte Carlo method returned the same enterprise value as the closed form solution.

None of the above observations sets off any alarms with respect to the robustness of the chosen numerical method. Thus, the Monte Carlo method seems to be a robust method, which can be used to determine the enterprise value in the iterative DCF model.

7.1. Perspectives

In the following it will be discussed what additional aspects of the iterative DCF model in a stochastic setting could be interesting to investigate. Three perspectives are:

> Perspective 1) Solve the integral equation of the DCF model with one or more of the other methods.
>
> Perspective 2) Do a regression analysis on the iterative DCF model.
>
> Perspective 3) Use the approximation to the enterprise value in the iterative DCF model as a terminal value.

Ad perspective 1) The Monte Carlo method was used to solve the integral equation for the enterprise value numerically. However, there is at least three other methods that can be used to solve the integral equation. It could be interesting to solve the integral equation with at least one of the other methods for several reasons. A first reason is that it could serve as a verification of the Monte Carlo method. A second reason is that it could be determined if one of the other methods were more rigid.

Ad perspective 2) The iterative DCF model was only tested on two cases. It would be interesting to apply the iterative DCF model to a larger number of cases and do a regression analysis, because that would tell if the iterative DCF model can explain more of the observed enterprise values than a simple DCF model can.

Ad perspective 3) The approximation to the enterprise value in the iterative DCF model gives an analytical expression for the enterprise value. It can therefore be used as a terminal value and thereby reduce the number of budget years. This would be an advantage. It is therefore interesting to see effects of such a terminal value.

8. Conclusion

The recursive nature of the enterprise value in a stochastic DCF model (iterative DCF model) has been accounted for by using a Monte Carlo method to generate a cash flow and discount that cash flow stepwise backwards with the help of the Lambert function. This gives a numerical result for the enterprise value in the iterative DCF model.

However, a stochastic DCF model with a recursive calculation of the enterprise value cannot provide for an analytical solution of the enterprise value. It was only possible to provide for an analytical solution that was an approximation to the enterprise value. In this approximation it is assumed that the discounting rate is time invariant. The approximation comes close to the numerical result that can be determined with the Monte Carlo method.

For the low leveraged enterprise GN Store Nord A/S the enterprise value determined in the iterative DCF model was only a little smaller than the enterprise value determined in the simple DCF model.

For the high leveraged enterprise William Demant A/S the enterprise value determined in the iterative DCF model was a lot smaller than the enterprise value determined in the simple DCF model.

In fact the error between the enterprise value in the iterative DCF model and the enterprise value in the simple DCF model grows exponentially with the leverage. The greater the leverage the greater is the error between the enterprise value in the iterative DCF model and the simple DCF model and the greater. This means that the greater the leverage is the more will the simple DCF model overvalue the enterprise.

Thus, for enterprises with a low leverage a simple DCF model can be used to model the enterprise value. But when the leverage is high the iterative DCF model should be used or it should at least be borne in mind that the simple DCF model will overvalue the enterprise quite a bit.

For the low leveraged enterprise GN Store Nord A/S the volatility of the enterprise value determined in the iterative DCF model was only a little smaller than the volatility of the enterprise value determined in the simple DCF model.

For the high leveraged enterprise William Demant A/S the volatility of the enterprise value determined in the iterative DCF model was a lot smaller than the volatility of the enterprise value determined in the simple DCF model.

Thus, the iterative DCF model results in an expected value of the enterprise that is smaller, but more precise than the enterprise value of the simple DCF model.

<u>What to take away</u>

For a low leveraged enterprise it is not necessary to account for the recursive nature of the enterprise value in the DCF model. Instead a simple DCF model provides for a good approximation.

Conclusion

For a high leveraged enterprise it is necessary to account for the recursive nature of the enterprise value in the DCF model. This can be done by using an iterative DCF model or an approximation to the iterative DCF model. As an alternative it should at least be borne in mind that the enterprise is being overvalued if the simple DCF model is used for the valuation.

It is also important to keep in mind that increasing the leverage of an enterprise in the hope of increasing the enterprise value does not lead to as large an increase in enterprise value as would be suggested by the simple DCF model.

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Appendix A: Budget

Entry	Drivers
+ Revenue (TR)	g, σ
- Variable cost (VC)	
- Fixed cost (FC)	
- Depreciations and amortizations (DA)	
= Earnings before interest and tax $(EBIT)$	$\Pi \cdot TR$
- Interest payment (I)	$y \cdot B$
= Earnings before tax (EBT)	$\Pi \cdot TR - y \cdot B$
- Tax payment (T)	$(\Pi \cdot TR - y \cdot B)\tau$
= Earnings (E)	$(\Pi \cdot TR - y \cdot B)(1-\tau)$

Income statement.

Entry	Drivers
+ Equity (EK_0)	$IC_0 - B_0$
+ Debt (B_0)	
= Invested capital (IC_0)	

Opening balance sheet (liabilities).

Entry	Drivers
+ Equity (EK)	$EK_0 + E - CF_{EK} = TR/\Lambda - L_B IC$
+ Debt (B)	$L_B IC$
= Invested capital (IC)	TR/Λ

Closing balance sheet (liabilities).

Entry	Drivers
+ Net operating profit after tax $(NOPAT)$	$\Pi \cdot TR(1-\tau)$
+ Depreciations and amortizations (DA)	DA
- Net working capital (ΔNWC)	$\Delta GM + \Delta DR - \Delta CR$
= Cash flow from operations (CFD)	
- Cash flow to investments (CFI)	$\Delta PPE + DA$
= Cash flow (CF)	$NOPAT - \Delta IC$
- Interest payment (I)	$y \cdot B$
+ Tax shield (V_τ)	$y \cdot B \cdot \tau$
+ Change in obligations (ΔB)	$L \cdot IC - B_0$
= Cash flow to the equity (CF_{EK})	$CF - y \cdot B(1-\tau) + \Delta B$

Cash flow statement.

Appendix B: GN Store Nord A/S financial statement 2003-2012

GN A/S (mDKK)	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
TR	4.742	5.548	6.644	6.766	5.981	5.624	4.729	5.145	5.564	6.251
EBIT	86	532	823	257	230	65	8	2.595	1.284	616
I	-45	37	21	60	66	117	71	33	28	69
E	250	508	850	348	-67	-56	-70	1.855	865	321
B	784	245	720	1.387	1.516	1.592	1.029	960	1.269	230
<B>		515	483	1.054	1.452	1.554	1.311	995	1.115	750
EK	4.473	4.580	5.349	4.900	4.482	5.361	5.349	5.254	4.653	5.680
<EK>		4.527	4.965	5.125	4.691	4.922	5.355	5.302	4.954	5.167
IC	5.257	4.825	6.069	6.287	5.998	6.953	6.378	6.214	5.922	5.910
<IC>		5.041	5.447	6.178	6.143	6.476	6.666	6.296	6.068	5.916
MV	8.094	12.672	18.305	17.571	8.121	2.080	5.616	10.336	9.634	13.980
EV	8.878	12.917	19.025	18.958	9.637	3.672	6.645	11.296	10.903	14.210
<EV>		10.897	15.971	18.991	14.297	6.655	5.159	8.971	11.100	12.557
L_B		10%	9%	17%	24%	24%	20%	16%	18%	13%
Π	2%	10%	12%	4%	4%	1%	0%	50%	23%	10%
Λ		1,1	1,2	1,1	1,0	0,9	0,7	0,8	0,9	1,1
y		7,2%	4,4%	5,7%	4,5%	7,5%	5,4%	3,3%	2,5%	9,2%

Appendix C: William Demant A/S financial statement 2003-2012

WD A/S (mDKK)	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
TR	3676	4120	4522	5085	5488	5374	5701	6892	8041	8555
EBIT	855	1003	1102	1270	1267	1042	1149	1430	1709	1653
I	28,2	38,6	36,8	61,4	96,9	139	94	116	103	132
E	618,3	716,9	790,6	900,7	894,5	682	795	988	1199	1151
B	633	902	1.111	1.392	1.800	1.908	1.575	1.869	1.548	1.804
<B>		767	1.006	1.251	1.596	1.854	1.742	1.722	1.709	1.676
EK	522	646	757	671	435	532	1.302	2.443	3.304	4.059
<EK>		584	701	714	553	483	917	1.873	2.874	3.682
IC	1.155	1.547	1.867	2.063	2.234	2.440	2.877	4.312	4.852	5.863
<IC>		1.351	1.707	1.965	2.149	2.337	2.659	3.595	4.582	5.358
MV	13.710	16.989	22.315	28.274	28.063	12.718	22.894	24.173	27.397	27.419
EV	14.343	17.891	23.426	29.666	29.863	14.626	24.469	26.042	28.945	29.223
<EV>		16.117	20.658	26.546	29.764	22.244	19.548	25.256	27.494	29.084
L_B		0,57	0,59	0,64	0,74	0,79	0,66	0,48	0,37	0,31
Π	23%	24%	24%	25%	23%	19%	20%	21%	21%	19%
Λ		3,0	2,6	2,6	2,6	2,3	2,1	1,9	1,8	1,6
y		5,0%	3,7%	4,9%	6,1%	7,5%	5,4%	6,7%	6,0%	7,9%

Appendix D: Excel worksheet

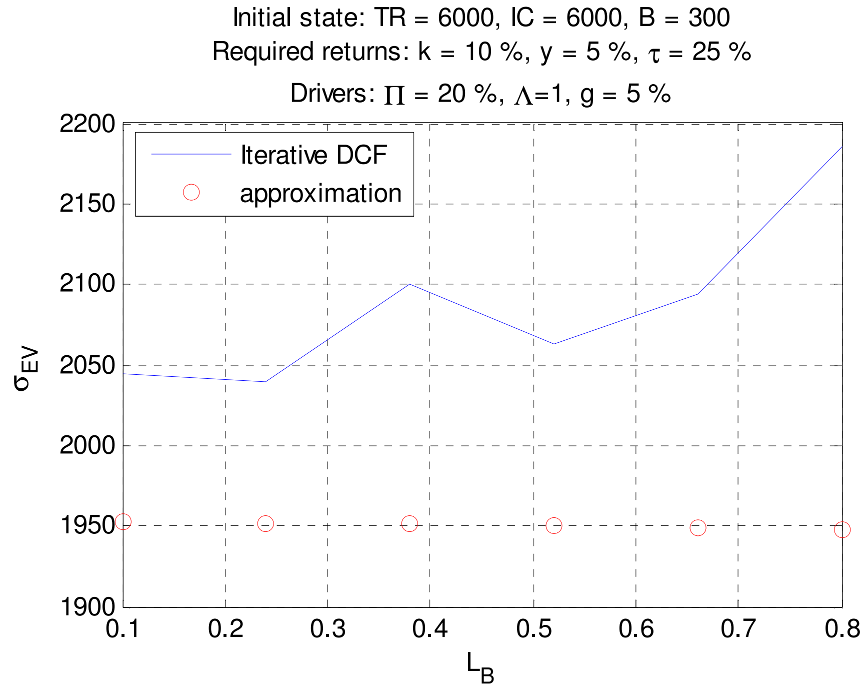
t	TR	IC	CF	B	x	Pade	EVtemp	Check	EV
0,0	6000	6000		6000	0,00022	0,00022	142.229	142.229	**11.852**
0,1	6.025	6.025	604	603	0,00002	0,00002	142.784	142.784	
0,2	6.050	6.050	606	605	0,00002	0,00002	143.369	143.369	
0,3	6.075	6.075	609	608	0,00002	0,00002	143.957	143.957	
0,3	6.101	6.101	611	610	0,00002	0,00002	144.547	144.547	
0,4	6.126	6.126	614	613	0,00002	0,00002	145.140	145.140	
0,5	6.152	6.152	616	615	0,00002	0,00002	145.735	145.735	
0,6	6.177	6.177	619	618	0,00002	0,00002	146.332	146.332	
0,7	6.203	6.203	622	620	0,00002	0,00002	146.932	146.932	
0,8	6.229	6.229	624	623	0,00002	0,00002	147.534	147.534	
0,8	6.255	6.255	627	625	0,00002	0,00002	148.138	148.138	
0,9	6.281	6.281	629	628	0,00002	0,00002	148.745	148.745	
1,0	6.307	6.307	632	631	0,00002	0,00002	149.355	149.355	
1,1	6.333	6.333	635	633	0,00002	0,00002	149.967	149.967	
1,2	6.360	6.360	637	636	0,00002	0,00002	150.581	150.581	
1,3	6.386	6.386	640	639	0,00002	0,00002	151.198	151.198	
1,3	6.413	6.413	643	641	0,00002	0,00002	151.817	151.817	
1,4	6.439	6.439	645	644	0,00002	0,00002	152.439	152.439	
1,5	6.466	6.466	648	647	0,00002	0,00002	153.063	153.063	
1,6	6.493	6.493	651	649	0,00002	0,00002	153.690	153.690	
1,7	6.520	6.520	653	652	0,00002	0,00002	154.319	154.319	
1,8	6.547	6.547	656	655	0,00002	0,00002	154.951	154.951	
1,8	6.575	6.575	659	657	0,00002	0,00002	155.585	155.585	
1,9	6.602	6.602	662	660	0,00002	0,00002	156.222	156.222	
2,0	6.630	6.630	664	663	0,00002	0,00002	156.862	156.862	
2,1	6.657	6.657	667	666	0,00002	0,00002	157.504	157.504	
2,2	6.685	6.685	670	669	0,00002	0,00002	158.149	158.149	
2,3	6.713	6.713	673	671	0,00002	0,00002	158.796	158.796	
2,3	6.741	6.741	675	674	0,00002	0,00002	159.446	159.446	
2,4	6.769	6.769	678	677	0,00002	0,00002	160.098	160.098	
2,5	6.797	6.797	681	680	0,00002	0,00002	160.753	160.753	
2,6	6.825	6.825	684	683	0,00002	0,00002	161.411	161.411	
2,7	6.854	6.854	687	685	0,00002	0,00002	162.071	162.071	
2,8	6.882	6.882	690	688	0,00002	0,00002	162.734	162.734	
2,8	6.911	6.911	693	691	0,00002	0,00002	163.400	163.400	
2,9	6.940	6.940	695	694	0,00002	0,00002	164.068	164.068	
3,0	6.969	6.969	698	697	0,00002	0,00002	164.739	164.739	
3,1	6.998	6.998	701	700	0,00002	0,00002	165.413	165.413	
3,2	7.027	7.027	704	703	0,00002	0,00002	166.089	166.089	
3,3	7.056	7.056	707	706	0,00002	0,00002	166.768	166.768	
3,3	7.086	7.086	710	709	0,00002	0,00002	167.450	167.450	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Initial state	
TR	6.000
IC	6.000
B	6.000
Drivers	
Π	20%
Λ	1
g	5%
L_B	0,1
Returns	
k	10%
y	5%
τ	25%
s	6,3%
Time	
m	12
Δt	0,083

Appendix E: Matlab script

```
% INITIAL STATEE
TRstart = 6000; ICstart = 6000; Bstart = 300;

% REQUIRED RETURNS
k = 0.1; tau = 0.2; y = 0.05; s = k-y*(1-tau);

% DRIVERS
OG = 0.2; AOH = 2; g = 0.05; L = 0.1; sigma = 0.1;

% SAMPLING AND TIME STEP
m = 12; T = 200; dt = 1/m; t = 0:dt:dt*T*Nt;

% NUMBER OF PATHS
M = 10000;

% CASH FLOW
TR = TRstart*ones(M,1); IC = ICstart*ones(M,1);
 B =  Bstart*ones(M,1);

for n = 1:T*m
    TR(:,n+1) = TR(:,n) + g*TR(:,n)*dt + ...
                sigma*TR(:,n).*normrnd(0,sqrt(dt),M,1);
    IC(:,n+1) = TR(:,n+1)/AOH;
    CF(:,n+1) = TR(:,n+1)*OG*(1-tau) - (IC(:,n+1)-IC(:,n))/dt;
     B(:,n+1) = IC(:,n+1)*L;
end

% DISCOUNTING
N = length(t)-1;
EV(:,N) = s*B(:,N)*dt./lambertw(s*B(:,N)*dt*exp(k*dt)./CF(:,end));

for n = 1:N-1
    EV(:,N-n) = s*B(:,N-n)*dt./lambertw(s*B(1,N-n)*dt*exp(k*dt)...
                ./(EV(:,N-n+1)+CF(:,N-n+1)));
end

% ENTERPRISE VALUE
mean(EV*dt)
```

Appendix F: Volatility of the enterprise value in the iterative DCF model



Appendix E: The enterprise value in an iterative DCF model in discrete time

In the following a closed form expression for the enterprise value in an iterative DCF model in discrete time will be derived. The enterprise value will correspond to the value expressed numerically in an Excel spreadsheet in (Larkin, 2011) and (Mohanty, 2003).

In discrete time the enterprise value is the sum of the future cash flow to the security holders:

$$EV = \sum_{n=1}^{\infty} d_n \cdot CF_n \tag{1.1}$$

where d_n is the discounting factor and CF_n is the cash flow. In discrete time the discounting factor is $d_n = (1 + wacc_n)^{-n}$, where $wacc$ is the weighted average cost of capital. When the cash flow has a constant relative growth the sum is a geometric series which can be expressed as a closed form solution:

$$EV = \frac{CF_1 + EV_1}{1 + wacc_0} \tag{1.2}$$

where EV_1 is called the terminal value. The weighted average cost of capital in the terminal value is $wacc_0 = k_e - s \cdot B / EV$. Inserting this into the terminal value gives:

$$EV = \frac{CF_1 + EV_1}{1 + k_e - s \cdot B / EV}, \tag{1.3}$$

The above equation can be directly solved for the enterprise value resulting in:

$$EV = \frac{CF_1 + EV_1 + s \cdot B}{1 + k_e} \tag{1.4}$$

The terminal value is:

$$EV_1 = \frac{CF_2}{wacc_1 - g} \tag{1.5}$$

The weighted average cost of capital in the terminal value is $wacc_1 = k_e - s \cdot B_1 / EV_1$, where $B_1 = L_B TR(1 + g) / \Lambda$. Inserting this into the terminal value gives:

$$EV_1 = \frac{CF_2}{k_e - s \cdot B_1 / EV_1 - g} \tag{1.6}$$

Solving the equation for the terminal value results in:

$$EV_1 = \frac{CF_2 + s \cdot B_1}{k_e - g} \tag{1.7}$$

Inserting the terminal value into the enterprise value gives:

$$EV = \frac{CF_1 + \dfrac{CF_2 + s \cdot B_1}{k_e - g} + s \cdot B}{1 + k_e} \tag{1.8}$$

The two cash flows can be written as:

$$\begin{aligned} CF_1 &= TR(1+g)\left[\Pi(1-\tau) - \frac{1}{\Lambda}\right] + IC \\ CF_2 &= TR(1+g)^2\left[\Pi(1-\tau) - \frac{1}{\Lambda}\frac{g}{1+g}\right] \end{aligned} \tag{1.9}$$

In terms of the budget drivers the enterprise value is:

$$EV = TR\frac{1+g}{k_e - g}\left[\Pi(1-\tau) - \frac{1}{\Lambda}\frac{k_e}{1+k_e} + \frac{IC}{TR}\frac{s \cdot L_B}{1+k_e}\right] + \frac{IC + s \cdot B}{1+k_e} \tag{1.10}$$

Appendix F: An iterative EVA model

In the following the enterprise value in the Economic Value Added model will be determined. This is to see if the iterative DCF model is consistent with the Economic Value Added model. The Economic Value Added model is also referred to as the *EVA model*.

In the EVA model the enterprise value is modeled as a function of the future economic value added discounted to a present day value:

$$EV = \int_0^T e^{-wacc \cdot t} EVA(t)\,\mathrm{d}t + IC_0 \tag{1.11}$$

The *economic value added* is defined as the net operating profit after tax $(NOPAT)$ subtracted with the *capital charge*. The capital charge is the required return $(wacc)$ on the invested capital (IC), (Stewart, 1990):

$$EVA = NOPAT - wacc \cdot IC \tag{1.12}$$

An assumption of the EVA model is that clean surplus accounting is used in the budgeting. This means that all income and cost goes into the income statement and not put directly on the balance sheet. When doing this the EVA model becomes independent of the accounting principles used in the budget, (Rådgivningsudvalget, 2002).

When the weighted average cost of capital is inserted into the discounting factor in the integral, the EVA model expresses the enterprise value as a recursive function:

$$EV = \int_0^T e^{-\left(k_e - s \cdot B / EV(t)\right)t} EVA(t)\,\mathrm{d}t + IC_0 \tag{1.13}$$

This was the same thing that happened in the DCF model. So the problem of solving an integral equation is also present in the EVA model. There exist a number of approaches to solve such an integral equation. In the following the same solution approach as in the DCF model is going to be used.

The economic value added in continuous time needs to be sampled in discrete time. Since there are no differentials involved it is relatively simple to sample the economic value added. Thus, the continuous evolution of the revenue can in discrete time be approximated with the following relation:

$$EVA_n = NOPAT_n - wacc_{n-1} IC_{n-1} \quad \text{for} \quad n = 1, 2, 3.... \tag{1.14}$$

It can be seen that the economic value added at sampling number n comprises the capital charge at the previous sampling, i.e. the capital charge for going into a period appears in the economic value added.

Inserting the expression for the net operating profit after tax and the weighted average cost of capital into the economic value added gives:

$$EVA_n = TR_n \Pi(1-\tau) - (k_e - s \cdot B_{n-1}/EV_{n-1}) IC_{n-1} \quad \text{for} \quad n = 1,2,3.... \tag{1.15}$$

Since there are no options involved the revenue evolves according to $TR_n = TR_{n-1}(1+g\Delta t)$ in the deterministic case. As before the invested capital evolves from the initial state IC_0 into $IC_n = TR_n/\Lambda$ for $n > 0$. Finally, the net interest-bearing debt evolves from the initial state B_0 into $B_n = L_B IC_n$.

The numerically generated economic value added now has to be discounted back to a present value in order to arrive at an enterprise value. Under the going concern assumption the cash flow evolves indefinitely into the future. Thus, it also has to be assumed that the expansion of the enterprise is not larger than the discounting. Far into the future the present value of the economic value added will therefore approach zero $\lim_{t\to\infty}(\mathrm{PV}(EVA(t))) = 0$.

Thus, the longer into the future the economic value added goes the less is the contribution to the enterprise value. The same stopping criteria as in the DCF model can therefore be used, i.e. when an economic value added in the future contributes with less than 1 % to the enterprise value the contribution of economic value added following that is neglected.

In the following the stopping criteria is assumed to be reached at sampling number n. The economic value added at sampling number n will thus be assumed to be the last economic value added. This last economic value added is then discounted back one time period. The enterprise value for that last economic value becomes:

$$EV_{n-1} = \left(NOPAT_n - (k_e - s \cdot B_{n-1}/EV_{n-1}) IC_{n-1}\right) \exp\left(-(k_e - s \cdot B_{n-1}/EV_{n-1}) \Delta t\right) \tag{1.16}$$

In the above equation for the enterprise value the enterprise value actually appears in three places. In order to arrive at a correct enterprise value the equation of course needs to be solved for the enterprise value, i.e. the enterprise value needs to be isolated such that it appears by itself on a side of the equality.

The equation for the enterprise value in the EVA model is just a bit more complex than the equation for the enterprise value in the DCF model. This means that the equation cannot be solved with the Lambert function. Instead the equation has to be solved directly or it can be simplified. This simplification results in the following expression for the enterprise value:

$$EV_{n-1} = \frac{s \cdot B_{n-1} \Delta t}{k_e \Delta t + \chi_n} \tag{1.17}$$

where the root χ_n satisfies the equation $\Gamma_n(\chi_n) = 0$.

The root function Γ being given by:

$$\Gamma_n(\chi_n) = e^{\chi_n}(\chi_n + k_e\Delta t)(NOPAT\Delta t + \chi_n IC_{n-1}) - s \cdot B_{n-1}\Delta t^2 \tag{1.18}$$

If Excel is used to implement the EVA model there is two ways to solve for the enterprise value numerically:

Root method 1) The Excel solver feature can be used.

Root method 2) A macro implementing a root finding method can be used.

The Excel solver feature is not especially suitable for determining the root, because the root method has to be implemented at each step backwards in time. Instead the second root method is going to be used. In the second root method a macro is called each time the root has to be determined.

When the enterprise value corresponding to the last economic value added has been found the discounting continues one time period further back. Thus, discounting back one time period from the second last time point results in the enterprise value:

$$EV_{n-2} = \left(NOPAT_{n-1} - \left(k_e - s \cdot B_{n-2}/EV_{n-2}\right) IC_{n-2} + EV_{n-1}\right)\exp\left(-\left(k_e - s \cdot B_{n-2}/EV_{n-2}\right)\Delta t\right) \tag{1.19}$$

The equation is very similar to the one above for the last time point. Only the enterprise value of the higher sampling number is added.

Again the root function can be used to simplify an expression for the enterprise value:

$$EV_{n-2} = \frac{s \cdot B_{n-2}\Delta t}{k_e\Delta t + \chi_{n-1}} \tag{1.20}$$

where χ_{n-1} satisfies the root function:

$$\Gamma_{n-1}(\chi_{n-1}) = e^{\chi_{n-1}}(\chi_{n-1} + k_e\Delta t)\left((NOPAT_{n-1} + EV_{n-1})\Delta t + \chi_{n-1} IC_{n-1}\right) - s \cdot B_{n-1}\Delta t^2 \tag{1.21}$$

The procedure continues all the way back to present time where the enterprise value can be expressed as:

$$\boxed{EV_0 = \frac{s \cdot B_0\Delta t}{k_e\Delta t + \chi_1}\Delta t + IC_0} \tag{1.22}$$

At this last discounting the extra time differential is used to account for the numeric integration and the initial state of the invested capital is added.