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Vulnerable Derivatives and Good Deal Bounds. A Structural Model

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Abstract

We price vulnerable derivatives - i.e. derivatives where the counterparty may default. These are basically the derivatives traded on the OTC markets. Default is modeled in a structural framework. The technique employed for pricing is Good Deal Bounds. The method imposes a new restriction in the arbitrage free model by setting upper bounds on the Sharpe ratios of the assets. The potential prices which are eliminated represent unreasonably good deals. The constraint on the Sharpe ratio translates into a constraint on the stochastic discount factor. Thus, tight pricing bounds can be obtained. We provide a link between the objective probability measure and the range of potential risk neutral measures which has an intuitive economic meaning. We also provide tight pricing bounds for European calls and show how to extend the call formula to pricing other financial products in a consistent way. Finally, we numerically analyze the behavior of the good deal pricing bounds.

Key words: Incomplete markets, Good deal bounds, Vulnerable options.

1 Introduction

Vulnerable derivatives are derivatives that bear counterparty risk – in other words, the writer of the option may not deliver the underlying. The main reason for having counterparty risk is the fact that these options are traded over-the-counter (OTC). If traded on an organized exchange, the counterparty risk associated with the option disappears due to the presence of the market maker. Our main application is the pricing of equity linked derivatives, mainly options, since they represent a major class of derivatives where one party bears the counterparty risk and there are many possible variations of the option payoff. According to BIS, the OTC equity-linked option gross market value in the first half of 2010 is USD 22.18 bln[1]. Moreover, there has been an increase in the volume of equity linked derivatives in the last few years. Thus, it is necessary to have consistent pricing of vulnerable options.

In the previous literature, vulnerable options were priced using structural models for default, i.e. a model for credit risk that takes into account the value of the assets of the option writer (counterparty) in order to define default. The main ingredients for such a framework are the dynamics of the stock and the dynamics of the assets of the counterparty. The papers were assuming market completeness - i.e that both the underlying stock and the assets of the counterparty are traded assets. Such papers include Johnson and Stulz (1987), Jarrow and Turnbull (1995), Hull and White (1995) and Klein (1996).

One of the main limitations of the above mentioned framework is the assumption that the assets of the counterparty, or the default "trigger", are liquidly traded on the market. It is a strong assumption, which allows us to obtain a unique price for the vulnerable option. If both the stock and the assets of the counterparty are traded on the market, we have a complete market model and,

[1]Statistics available through the Bank for International Settlements, at http://bis.org/statistics/derstats.htm

hence, a unique price.

However, if the assets of the counterparty are not liquidly traded, we are not in a complete market setup and hence, we are not entitled to use the formula derived in the previous section. One of the consequences of having an incomplete market setup is the fact that we no longer have a unique equivalent martingale measure (EMM) and consequently, we do not a unique price. One could simply calculate the bounds of the prices generated by the interval of all possible risk-neutral measures. These bounds are known as the no-arbitrage bounds. However, they are too large to be of any practical use.

Another alternative would be to pick one of the possible equivalent martingale measures, according to some criterion chosen by the researcher/implementer of the model. The literature adopting this path is vast. For further reference to different strands of literature dealing with this approach see Schweizer (2001), Henderson and Hobson (2008) and Barrieu and Karoui (2005). However, there is no clear-cut way of choosing between different criteria and some of them are somewhat ad-hoc, in the sense that they do not have any clear economic interpretation.

In contrast to this, Cochrane and Saa-Requejo (2000) proposed the method of good deal bounds. The good deal approach aims at obtaining an interval of "reasonable" prices in incomplete markets, rather than concentrating on obtaining a unique price. Since the no-arbitrage bounds are too large to be used, Cochrane and Saa-Requejo (2000) propose to not only rule out arbitrage opportunities, but also trade opportunities which are too favorable to be observed on a real market. These unrealistically-favorable deals are considered "too good to be true", hence the name of "good deal bounds". One possible measure for the "goodness" of a deal is its Sharpe Ratio and thus, trades/portfolios which have a Sharpe Ratio (SR) above a certain threshold are eliminated. The SR is chosen as a measure for the “goodness of the deal” because of its intuitive meaning, but also due to a large empirical literature which can tell us the range of the Sharpe Ratios observed on the market. Thus, the bound on the SR will not be arbitrary. The procedure reduces the set of possible prices for the claims traded. Thus, the good-deal bounds methodology leads to a much tighter interval of possible prices than the bounds obtained by no-arbitrage.

The next step in developing a theory for “good deal bounds” was taken by Björk and Slinko (2006). They proposed a new framework for solving the optimization problem defined by Cochrane and Saa-Requejo (2000) while at the same time allowing for more complex dynamics for the underlying assets, such as jump-diffusion processes, to be taken into account.

The first to notice the complete market inconsistency in the case of vulnerable options were Hung and Liu (2005). They priced the vulnerable options using the structural model set up by Klein (1996) and using “good deal bounds” as defined by Cochrane and Saa-Requejo (2000).

In contrast, we use the good deal bounds framework proposed by Björk and Slinko (2006) which allows for a higher degree of tractability and hence, we can deal with the more general problem of pricing a derivative claim on equity, rather than just options. Besides pricing European vulnerable options as an application of the good deal bounds with counterparty risk, we also show how results obtained for complete markets, non-vulnerable options pricing, can be extended in the incomplete market case, for the pricing of vulnerable options. Thus, we show how the same techniques can be used to infer the pricing bounds for the exchange options and the barrier options from the pricing bounds of the European calls. Then, we presents a few numerical results and conclude.

2 Setup

In this paper, we model default in the classical Merton framework, while dropping the assumption of market completeness.

Let $(\Omega, \mathcal{F}, P, \mathbf{F})$ be a filtered probability space. On this space, we have $\tilde{W}$, a two-dimensional P-Wiener process:

$$\tilde{W} = \begin{pmatrix} \tilde{W}^1 \\ \tilde{W}^2 \end{pmatrix},$$

with $\tilde{W}^1$ and $\tilde{W}^2$ being independent scalar P-Wiener processes. The filtration $\mathcal{F}_t$ is is generated by $\tilde{W}$. Our market has a risk free asset, the bank account, denoted by B_t, and a liquidly traded risky asset, S_t. The derivative claims contracted are over the counter and written on S_t. Our counterparty's assets Y_t are not traded, but we know their dynamics. The dynamics of the traded and non-traded assets under the objective probability measure P are:

$$dY_t = \mu_t Y_t dt + Y_t \bar{\sigma}_t d\tilde{W}_t, \tag{1}$$

$$dS_t = \alpha_t S_t dt + S_t \bar{\gamma}_t d\tilde{W}_t, \tag{2}$$

$$dB_t = B_t r dt. \tag{3}$$

The parameter r represents the constant risk-free interest rate. The coefficients μ_t and α_t are scalar deterministic functions of time and $\bar{\sigma}_t$ and $\bar{\gamma}_t$ are positive deterministic functions of time specified as follows:

$$\bar{\gamma}_t = (\gamma_t, \quad 0),$$

$$\bar{\sigma}_t = (\sigma_t \rho, \quad \sigma_t \sqrt{1-\rho^2}).$$

Remark 2.1 *One can extend the dynamics of the stock and the assets of the* *counterparty to a jump-diffusion setup. However, in that case we would have a market with one traded asset and four sources of risk. Hence, it would be difficult to know if the size of the good deal bounds interval is driven by the presence of the counterparty risk or by the presence of jumps in the traded asset.*

On the OTC market, we are trading a European derivative $\mathcal{X}$ with the payoff $\Phi(S_T)$. If there were no counterparty risk, its price at time t would be the conditional expected value of the discounted payoff $\Phi(S_T)$, where the expectation is taken under the risk-neutral measure. With counterparty risk, we need to specify how default occurs and the value recovered in case of default. In our case, default occurs if the value of the assets of the counterparty at T falls below the claims written against the counterparty, denoted by D. If default occurs, the payoff of claim $\mathcal{X}$ becomes $\mathcal{R}$, the recovery payoff which is given by:

$$\mathcal{R} = (1-\beta)\frac{Y_T}{D}\Phi(S_T).$$

The logic behind the above formula is straightforward. One gets a proportional part of the value of the claim, corresponding to how much the assets of the counter-party have fallen below the value of the claim. However, there are some deadweight costs associated with the bankruptcy procedure, which are captured by the β parameter. For these reasons, β needs to belong to $[0,1]$.This recovery specification is very close to the specification for recovery of treasury.

Thus, we can write the payoff of a vulnerable claim as:

$$\Phi^V(S_T, Y_T) = \Phi(S_T) I\left\{Y_T \geq D\right\} + \mathcal{R} I\left\{Y_T < D\right\}, \tag{4}$$

where D is the total value of the claims against the counter-party. We notice that, in general, the vulnerable version of a contract function $\Phi(x)$, denoted by

$\Phi^V(x, y)$ is given by:

$$\Phi^V(x, y) = \Phi(x)\left\{I\{y \geq D\} + \frac{(1-\beta)y}{D} I\{y < D\}\right\}. \tag{5}$$

We denote

$$G(y) = I\{y \geq D\} + \frac{(1-\beta)y}{D} I\{y < D\} \tag{6}$$

2.1 The Equivalent Martingale Measures

Since we are in an incomplete market set-up, we do not have a unique equivalent martingale measure (EMM), but a whole class of EMM. For any potential EMM $Q \sim P$ we define the corresponding likelihood process L by:

$$L_t = \frac{dQ}{dP} \quad \text{on } \mathcal{F}_t. \tag{7}$$

Since $\mathcal{F}_t = \mathcal{F}_t^{\tilde{W}}$, L_t must have dynamics of the form:

$$dL_t = L_t \varphi_t' d\tilde{W}_t, \tag{8}$$

$$L_0 = 1, \tag{9}$$

where $\varphi_t = (\varphi_t^1, \quad \varphi_t^2)'$ is adapted to $\mathbf{F}$. Thus, the dynamics of the two assets under the potential martingale measure Q are:

$$\begin{aligned} dY_t &= (\mu_t + \bar{\sigma}_t\varphi_t)Y_t dt + Y_t\bar{\sigma}_t dW_t, \\ dS_t &= (\alpha_t + \bar{\gamma}_t\varphi_t)S_t dt + S_t\bar{\gamma}_t dW_t, \\ dB_t &= B_t r dt, \end{aligned}$$

where W_t is a Q-Wiener process. Since S_t is a traded asset, its drift must equal the risk-free interest rate under an equivalent martingale measure. Thus, for Q to be a martingale measure, φ has to satisfy the **martingale condition**:

$$r = \alpha_t + \bar{\gamma}_t \varphi_t \tag{10}$$

The martingale condition does not determine a unique Girsanov kernel φ_t, but only the first term of the φ_t. Thus, we do not have a unique equivalent martingale measure, but we obtain a class of martingale measures. They are defined as the class of measures obtained by (7)- (9) and satisfying the martingale condition (10).

3 The good deal bound problem

As previously mentioned, the "good deal bound" valuation framework rests on the idea of placing constraints on the Sharpe ratio of the claim to be priced. The problem becomes that of finding the highest and the lowest arbitrage free price processes, subject to a constraint on the maximum Sharpe Ratio (SR). However, if we want to be consistent, we should look for a framework allowing us to place an upper bound on the SR not only of the derivative under consideration, but also of all **portfolios** that can be formed on the market consisting of the underlying assets, the derivative claim and the money account. It then turns out that binding the Sharpe Ratio of all possible portfolios is equivalent to using the Hansen-Jagannathan bounds.

An extended version of the Hansen Jaganathan bounds is derived and proven in Björk and Slinko (2006). This inequality provides the bounds for the Sharpe ratio of the assets on the market, as well as for all derivatives and self financing portfolios formed on the market, and reads as follows:

$$|SR_t|^2 \leq \|\lambda_t\|^2.$$

Here we denote by λ_t the market price of risk and by SR_t the Sharpe ratio on a particular asset derivative or self financing portfolio on the market; $\| \bullet \|$ stands for the Euclidean norm.

As we can see, the Sharpe ratio is bounded by the norm of the price of risk on the market. Standard theory gives us the relationship between the Girsanov kernel, φ_t, and the market price of risk:

$$\varphi_t = -\lambda_t.$$

Thus, our pricing problem can be reformulated as follows: we are trying to find the highest and the lowest arbitrage free pricing processes, subject to an upper bound on the norm of the market price for risk or, equivalently, a bound on the Girsanov kernel φ_t for every t. Dealing with the market price of risk translates into dealing with the Girsanov kernel of the equivalent martingale measures.

Following the above reasoning, we can now define the good deal bounds. In the definition below and the rest of the paper, for a random variable Y, the notation $E_t^Q[Y]$ stands for the conditional expected value of Y, taken at time t and under the risk measure Q.

Definition 3.1 *The **upper good deal bound** price process for a vulnerable option is defined as the optimal value process for the following optimal control* *problem:*

$$\max_{\varphi} \quad E_t^Q[e^{-r(T-t)}(\Phi(S_T)I\{Y_T \geq D\} + \mathcal{R}I\{Y_T \leq D\})],$$

$$dY_t = (\mu_t + \bar{\sigma}_t\varphi_t)Y_t dt + Y_t\bar{\sigma}_t dW_t,$$

$$dS_t = rS_t dt + S_t\bar{\gamma}_t dW_t, \tag{11}$$

$$\alpha_t + \bar{\gamma}_t\varphi_t = r, \tag{12}$$

$$\|\varphi_t\|^2 \leq C^2. \tag{13}$$

The ***lower good deal bound*** *process is the optimal value process for a similar optimal control problem, with the only difference that we minimize the expression, subject to the same constraints.*
We denote the optimal value process by $V(t, S_t, Y_t)$, where V is the ***optimal value function***.

Before proceeding, let us comment on the structure of the optimization problem. The objective function is the arbitrage-free price for the payoff function, where the expectation is computed under the risk neutral measure generated by φ. Since we must select this measure from a continuum of eligible EMM, we maximize with respect to the Girsanov kernel φ.

The optimization is subject to the dynamics of the assets on the market, under the appropriate probability measure.

The constraints (11)-(12) are the usual constraints on the drift of the traded assets on the market that establish the probability measure as a risk neutral measure.

If all elements of φ could be identified from these constraints, we would be in a complete market setup and would be able to find a unique price. Since the number of traded assets is smaller than the number of risk sources, we cannot price all risk factors and need the last inequality in order to tighten the no arbitrage price bounds. We will refer to the inequality (13) as the good deal bounds condition.

4 Solving the HJB

The optimization problem stated above is a standard stochastic optimal control problem and it will be solved with the aid of the Hamilton Jacobi Bellman equation. We restrict ourselves to the case when the market price of risk depends only on the stock and the assets of the counterparty; thus, we have $\varphi_t = \varphi(t, S_t, Y_t)$. According to the general theory of dynamic programing, the optimal value function satisfies the Hamilton Jacobi Bellman equation, where $\mathcal{A}$ denotes the infinitesimal operator for (S, Y):

$$\begin{aligned}&\frac{\partial V}{\partial t}(t,s,y) + \sup_{\varphi} \mathcal{A}V(t,s,y) - rV(t,s,y) = 0,\\&V(T,s,y) = \Phi^V(s,y),\end{aligned}$$

where $\Phi^V(s,y)$ is defined by (4). The infinitesimal operator is given by:

$$\begin{aligned}\mathcal{A}V &= \frac{\partial V}{\partial s} rs + \frac{\partial V}{\partial y}(\mu_t + \bar{\sigma}_t \varphi_t) y \\&+ \frac{1}{2}\frac{\partial^2 V}{\partial s^2} s^2 \bar{\gamma}_t \bar{\gamma}_t' + \frac{1}{2}\frac{\partial^2 V}{\partial y^2} y^2 \bar{\sigma}_t \bar{\sigma}_t' + \frac{\partial^2 V}{\partial s \partial y} sy \bar{\gamma}_t \bar{\sigma}_t'.\end{aligned}$$

The first step in solving the PDE is to solve the embedded static maximization problem for each t, s, y. In our case, for fixed t, s, y, the static problem takes the form:

$$\max_{\varphi} \quad \frac{\partial V}{\partial y} \sigma \varphi y, \tag{14}$$

$$\alpha + \bar{\gamma}\varphi = r, \tag{15}$$

$$\|\varphi\|^2 \leq C^2. \tag{16}$$

We notice that the above problem is, in fact, a linear optimization problem and therefore, the solution will be a boundary solution. Thus, both constraints are binding. Since the Girsanov kernel φ is a (2,1) matrix, by solving the system of equations:

$$\alpha + \bar{\gamma}\varphi = r,$$
$$\|\varphi\|^2 = C^2,$$

we obtain:

$$\hat{\varphi}(t,s,y)' = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad \pm\sqrt{C^2 - (\tfrac{r-\alpha_t}{\gamma_t})^2} \,\right). \tag{17}$$

Thus, we have two candidates for the optimal φ and it remains to be determined which is the optimal one. Since our objective function is linear in φ:

$$\frac{\partial V}{\partial y}\sigma\varphi y,$$

and σ and y are positive by assumption, we need to investigate the sign of $\frac{\partial V}{\partial y}$ to decide which of the possible Girsanov kernels we choose.

Lemma 4.1 *Let the good deal bound price process be defined as in Definition 3.1. If φ does not depend on s and y, we have*

$$\frac{\partial V}{\partial y} \geq 0. \tag{18}$$

Proof. First, we will show that the payoff function is non-decreasing in y. Then, we will prove that this implies that the associated pricing function is non-decreasing in y and hence, so is the optimal value function.

One can easily see that the payoff function $\Phi^V(s,y)$ is non-decreasing in y if we go back to equation (5). If $y \geq D$, we have

$$\Phi^V(x,y) = \Phi(x)$$

and if $y < D$, $\Phi^V(s,y)$ is linear in y with a positive coefficient, from the assumptions on β and D and the definition of $\Phi(x)$.

Let $\Pi^Q(t,s,y)$ be a pricing function, i.e.

$$\Pi^Q(t,s,y) = E^Q[e^{-r(T-t)}\Phi^V[S_T, Y_T]|S_t = s, Y_t = y], \tag{19}$$

where Q is some admissible EMM.

We now want to prove that if the payoff function $\Phi^V(s,y)$ is increasing in y and the Girsanov kernel is a deterministic function of time

$$\varphi(t,s,y) = \varphi(t),$$

also the pricing function $\Pi^Q(t,s,y)$ is increasing in the variable y. We solve the SDE giving the dynamics of Y_t under Q and obtain the following formula for Y_T, given $Y_t = y$:

$$Y_T = y \exp\left(\int_t^T \left[\mu_s + \bar{\sigma}_s\varphi_s - \frac{1}{2}\sigma_s^2\right] ds + \int_t^T \bar{\sigma}_s dW_s\right).$$

Thus, for a given φ which does not depend on s and y, we can write $Y_T = yZ$, where Z is a lognormal variable that does not depend on y. It can easily be seen that if $\Phi(s,y)$ is increasing in the second variable, then also the pricing function $\Pi^Q(t,s,y)$ is increasing in the variable y.

We know that $V = \Pi^Q$ when Q is generated by $\hat{\varphi}$. Since we see from (17) that $\hat{\varphi}$ does not depend on s and y, we conclude that $\Pi^Q(t,s,y)$ and thus V is nondecreasing in y. ∎

Proposition 4.1 *The Girsanov kernel corresponding to the upper good deal bound EMM is*

$$\hat{\varphi}'_{max} = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad \sqrt{C^2 - (\tfrac{r-\alpha_t}{\gamma_t})^2} \right),$$

The Girsanov kernel for the lower good deal bound EMM is given by

$$\hat{\varphi}'_{min} = \left(-\tfrac{\alpha_t - r}{\gamma_t}, \quad -\sqrt{C^2 - (\tfrac{r-\alpha_t}{\gamma_t})^2} \right),$$

5 Extending the formula

5.1 Linearly homogeneous payoffs and exchange options

In this section, we consider derivatives written on several assets with linearly homogeneous payoffs - i.e. derivatives whose non-vulnerable payoffs $\Phi(s^1, s^2)$ have the property:

$$\Phi(\lambda s^1, \lambda s^2) = \lambda \Phi(s^1, s^2), \quad \forall \lambda \geq 0.$$

The most common example of a claim with a linearly homogeneous payoff would be the exchange option with the payoff $\mathcal{X} = \max[S_T^1 - S_T^2, 0]$. A well-known result in mathematical finance relates the non-vulnerable pricing problem of Φ to the simpler problem of pricing ψ, defined by the contract function:

$$\psi(z) = \Phi(z, 1). \tag{20}$$

We would like to see if it is possible to find such a relation between vulnerable versions of the contracts defined above, as well as what the simplified pricing problem would look like for good deal bounds.

Let us consider a market formed by a risk free asset, the bank account, denoted by B_t, and two liquidly traded risky asset, S_t^1 and S_t^2. The derivative claims contracted are over the counter and written on both assets. As previously, the assets of our counterparty Y_t are not traded, but we know their dynamics. The dynamics of the traded and non-traded assets under the objective probability measure P are:

$$\begin{aligned} dY_t &= \mu_t Y_t dt + Y_t \bar{\sigma}_t d\tilde{W}_t, & (21) \\ dS_t^1 &= \alpha_t^1 S_t^1 dt + S_t^1 \bar{\gamma}_t^1 d\tilde{W}_t, & (22) \\ dS_t^2 &= \alpha_t^2 S_t^2 dt + S_t^2 \bar{\gamma}_t^2 d\tilde{W}_t, & (23) \\ dB_t &= B_t r dt. \end{aligned}$$

As previously, we denote the vulnerable version of the contract function $\Phi(S_t^1, S_t^2)$ by $\Phi^V(S_t^1, S_t^2, Y_t)$ and the vulnerable version of the contract function $\psi(S_t)$ by $\psi^V(S_t, Y_t)$. We remember that, in general, the vulnerable version of a contract function $F(x)$, denoted by $F^V(x, y)$ is given by:

$$F^V(x, y) = F(x)G(y),$$

with $G(y)$ as given by equation (6). By applying the risk neutral valuation formula to the claim $\mathcal{Y}$ with payoff $\Phi^V(S_t^1, S_t^2, Y_t)$, we obtain the following expression for the price of the claim, $\Pi(t, \mathcal{Y})$:

$$\Pi(t, \mathcal{Y}) = E_t^Q \left[e^{(-r(T-t))} \Phi^V(S_T^1, S_T^2, Y_T) \right] = S_t^2 E_t^2 \left[\Phi \left(\frac{S_T^1}{S_T^2}, 1 \right) G(Y_T) \right],$$

where $E^2[\bullet]$ is the expectation operator taken under the equivalent martingale measure Q^2 where S^2 is the numeraire. We denote $\frac{S_t^1}{S_t^2}$ by Z_t. Under Q^2, Z is a martingale, and has a zero rate of return. We note that in order to obtain a similar result as in the non-vulnerable claims case, we would need Y_t to also be a Q^2-martingale. However, since we are not in a complete market set-up and Y_t is not a traded asset, we have to take a different route.

Should Y_t be a traded asset, we could define it as $Y_t = \tilde{Y}_t e^{c_t t}$ where $c_t = r + \bar{\gamma}_t^2 \bar{\sigma}_t'$. Notice that $\tilde{Y}_t$ is a martingale under Q^2. A few easy computations show the price of the vulnerable claim $\mathcal{Y}$ at time t:

$$
\begin{aligned}
\Pi(t, \mathcal{Y}) &= S_t^2 E_t^2 \left[\Phi(Z_T, 1) I \left\{ \tilde{Y}_T \geq De^{-cT} \right\} \right] \\
&+ S_t^2 E^2 \left[\frac{(1-\beta)\tilde{Y}_T}{De^{-cT}} \Phi(Z_T, 1) I \left\{ \tilde{Y}_T < De^{-cT} \right\} \right] \\
&= S_t^2 E^2 \left[\psi(Z_T, \tilde{Y}_T) \right],
\end{aligned}
$$

where the default barrier for the vulnerable claim $\mathcal{Y} = \psi(Z_T, \tilde{Y}_T)$ is De^{-cT}.

In our case, Y_t is not a traded asset and we do not have a unique martingale measure, so we can use good deal bounds to obtain tighter pricing bounds. We remember that the good deal bounds are defined as follows:

Definition 5.1 *The* ***upper good deal bound*** *price process for a vulnerable exchange option is defined as the optimal value process for the following optimal control problem:*

$$
\begin{aligned}
\max_{\varphi} \quad & E^Q[e^{-r(T-t)}\mathcal{X}], \\
& \alpha_1 + \bar{\gamma}_1 \varphi_t = r, \\
& \alpha_2 + \bar{\gamma}_2 \varphi_t = r, \\
& \|\varphi_t\|^2 \leq C^2,
\end{aligned}
$$

where Y_t, S_t^1 and S_t^2 have dynamics as given by equation (21)-(23). The ***lower good deal bound*** *is the optimal value process for a similar optimal control* *problem, except that we minimize instead of maximize subject to the same constraints.*

We will show how to obtain this equivalent good deal bounds problem which allows a direct transfer from the pricing problem of a vulnerable claim with linearly homogeneous payoff $\Phi(S_t^1, S_t^2)$ to the pricing problem of a vulnerable claim on only one asset, which is a more simple problem. We will do this by obtaining equivalent expressions to the objective function and the constraints under the new measure Q^2 and involving a Girsanov kernel corresponding to the change of measure $P \to Q^2$, denoted by ψ.
We will present how we have obtained the equivalent problem:

- We apply a standard change of measure to the objective function of the upper good deal bound problem and obtain: $E^Q[e^{-rT}\mathcal{X}] = S_0^2 E^2[\mathcal{Z}]$ where

$$\mathcal{Z} = \psi(Z_t)G(Y_t)$$

  and $Z_T = \frac{S_T^1}{S_T^2}$ and $G(Y_t)$ is as defined in equation (5); $E^2(\bullet)$ denotes the expectations operator under Q^2.

  We denote by ϕ the Girsanov kernel corresponding to the change of measure $P \to Q^2$.

- We can easily derive the dynamics of $\frac{S_t^1}{S_t^2}$ and Y_t under Q^2 using the Girsanov transformation and the fact that $\frac{S_t^1}{S_t^2}$ should be a martingale under the new measure.

$$dY_t = (\mu + \bar{\sigma}\psi_t)Y_t dt + Y_t \bar{\sigma} dW_t^2 \tag{24}$$

$$d\left(\frac{S_t^1}{S_t^2}\right) = \frac{S_t^1}{S_t^2}(\bar{\gamma}_1 - \bar{\gamma}_2)dW_t^2 \tag{25}$$

  where W_t^2 is Q^2-Wiener.

- The next step is to derive the **martingale conditions** corresponding to Q^2. These are derived by setting $\frac{S_t^1}{S_t^2}$ and $\frac{B_t}{S_t^2}$ as martingales under Q^2, as required by the definition of the new measure. We obtain:

$$r - \alpha_2 = \bar{\gamma}_2 \psi_t - \gamma_2^2,$$
$$\alpha_1 - \alpha_2 = \gamma_1 \gamma_2 \rho_{12} - \gamma_2^2 - (\bar{\gamma}_1 - \bar{\gamma}_2)\psi_t,$$

- The next step in our equivalence problem is to take the good deal bound condition for the transformation $P \to Q$:

$$\|\varphi_t\|^2 \leq C^2,$$

  and find an equivalent good deal bound condition for the transformation $P \to Q^2$. We define the following transformations:

  - $P \to Q$, defined by $L = \frac{dQ}{dP}$ on $\mathcal{F}_T$ with $dL = L\varphi' d\tilde{W}$;
  - $P \to Q^2$, defined by $L^2 = \frac{dQ^2}{dP}$ on $\mathcal{F}_T$ with $dL^2 = L^2 \phi' d\tilde{W}$;
  - $Q \to Q^2$, defined by $L^{1,2} = \frac{dQ^2}{dQ}$ on $\mathcal{F}_T$ with $dL^{1,2} = L^{1,2}\bar{\gamma}_2 dW$

  We notice that

$$\frac{\frac{dQ^2}{dP}}{\frac{dQ}{dP}} = \frac{dQ^2}{dQ}.$$

  The above equation together with the dynamics of the three Radon-Nikodym derivatives yield the following relation between φ and ϕ:

$$\varphi = \phi - \bar{\gamma}_2'.$$

Thus, the good deal bounds constraint becomes:

$$\|\phi - \bar{\gamma}_2'\|^2 \leq C^2.$$

Thus, we have reduced the problem of pricing a vulnerable claim written on two assets to the problem of pricing a vulnerable claim written on one asset. We can summarise the result as follows:

Proposition 5.1 *The upper good deal bound price process defined in Definition 5.1 is also the optimal value process for the optimal control problem given below:*

$$\begin{aligned}
\max_{\phi} \quad & S_t^2 E^2[\mathcal{Z}], \\
& dY_t = (\mu + 2\bar{\sigma}\phi - \bar{\sigma}\bar{\gamma}_2) Y_t dt + Y_t \bar{\sigma} dW_t^2, \\
& d\left(\frac{S_t^1}{S_t^2}\right) = \frac{S_t^1}{S_t^2}(\bar{\gamma}_1 - \bar{\gamma}_2) dW_t^2, \\
& r - \alpha_2 = \bar{\gamma}_2 \psi_t - \gamma_2^2, \\
& \alpha_1 - \alpha_2 = \gamma_1 \gamma_2 \rho_{12} - \gamma_2^2 - (\bar{\gamma}_1 - \bar{\gamma}_2)\psi_t, \\
& \|\phi - \bar{\gamma}_2'\|^2 \leq C^2.
\end{aligned}$$

The lower good deal bound is the optimal value process for a similar optimal control problem, where we minimize subject to the same constraints as above.

By a reasoning very similar to that in the previous section, we calculate the upper good deal bound Girsanov kernel.

6 Small numerical example - the vulnerable option

In this section, we will implement the good deal bounds and obtain the price of a vulnerable option when the stock is liquidly traded. Thus, the payoff of our non-vulnerable claim is $\Phi(S_T) = \max[S_T - K, 0]$ where K is the strike price. We need to compute the price for a vulnerable European call, hence our payoff is:

$$\Phi^V(S_T, Y_T) = \max[S_T - K, 0]\left\{I\{Y_T \geq D\} + \frac{(1-\beta)Y_T}{D} I\{Y_T < D\}\right\}.$$

The above payoff of a vulnerable option was first priced in complete markets by Klein (1996). It was later priced in incomplete markets by Hung and Liu (2005) using "good deal bounds" as defined by Cochrane and Saa-Requejo (2000). We note that using the good deal bounds as defined by Björk and Slinko (2006), the upper and the lower GDB price can be derived in a manner similar to that of Klein (1996). The difference between the formulae is due to the fact that the drift under the risk neutral measure of the assets of the counterparty Y_t is no longer given by the risk-free rate, as in the complete market case, but by the Girsanov transformations corresponding to the Girsanov kernels identified in proposition 4.1.

Proposition 6.1 (Vulnerable Options) *The upper good deal bound price of a vulnerable option is given by:*

$$\begin{aligned}\Pi(t) &= S_t\mathcal{N}[a_1, b_1, \rho_2] - e^{-r(T-t)}K\mathcal{N}[a_2, b_2, \rho_2] \\ &+ \frac{1-\beta}{D} S_t Y_t \exp\left\{\int_t^T [\mu_s + \bar{\sigma}_s\hat{\varphi}_s + \sigma_s\gamma_s\rho]\, ds\right\}\mathcal{N}[-a_3; b_3; -\rho_2] \\ &- e^{-r(T-t)}\frac{K(1-\beta)}{D} Y_t \exp\left\{\int_t^T (\mu_s + \bar{\sigma}_s\hat{\varphi}_s) ds\right\}\mathcal{N}(a_4, b_4, -\rho_2),\end{aligned}$$

where

$$a_1 = \frac{\ln\frac{S_t}{K} + \int_t^T \left\{r + \frac{1}{2}\gamma_s^2\right\} ds}{\sqrt{\int_t^T \gamma_s^2 ds}},$$

$$
\begin{aligned}
b_1 &= \frac{\ln \frac{Y_t}{D} + \int_t^T \left[\mu_s + \bar{\sigma}_s \hat{\varphi}_s + \sigma_s \gamma_s \rho - \frac{1}{2}\sigma_s^2\right] ds}{\sqrt{\int_t^T \sigma_s^2 ds}}, \\
a_2 &= \frac{\ln \frac{S_t}{K} + r(T-t) - \frac{1}{2}\int_t^T \gamma_s^2 ds}{\sqrt{\int_t^T \gamma_s^2 ds}}, \\
b_2 &= \frac{\ln \frac{Y_t}{D} + \int_t^T \left[\mu_s + \bar{\sigma}_s \hat{\varphi}_s - \frac{1}{2}\sigma_s^2\right] ds}{\sqrt{\int_t^T \sigma_s^2 ds}}, \\
a_3 &= \frac{\ln \frac{S_t}{K} + \int_t^T \left\{r + \frac{1}{2}\gamma_s^2 + \sigma_s \gamma_s \rho\right\} ds}{\sqrt{\int_t^T \gamma_s^2 ds}}, \\
b_3 &= \frac{\log \frac{D}{Y_t} - \int_t^T \left\{\mu_s + \bar{\sigma}_s \hat{\varphi}_s + \gamma_s \sigma_s \rho + \frac{1}{2}\sigma_s^2\right\} ds}{\sqrt{\int_t^T \|\bar{\sigma}_s\|^2 ds}}, \\
a_4 &= \frac{\ln \frac{S_t}{K} + \int_t^T \left[r + \gamma_s \sigma_s \rho - \frac{1}{2}\gamma_s^2\right] ds}{\sqrt{\int_t^T \gamma_s^2 ds}}, \\
b_4 &= \frac{\ln \frac{D}{Y_t} + \int_t^T \left[\mu_s + \bar{\sigma}_s \hat{\varphi}_s + \frac{1}{2}\sigma_s^2\right] ds}{\sqrt{\int_t^T \sigma_s^2 ds}}, \\
\rho_2 &= \frac{\rho \int_t^T \sigma_s \gamma_s ds}{\sqrt{\int_t^T \sigma_s^2 ds}\sqrt{\int_t^T \gamma_s^2 ds}}, \\
\hat{\varphi}_t &= \left(-\frac{\alpha_t - r}{\gamma_t}, \quad \sqrt{C^2 - (\frac{r-\alpha_t}{\gamma_t})^2}\,\right)'.
\end{aligned}
$$

The lower good deal bound price is given by a similar pricing formula, with the only exception that

$$
\hat{\varphi}_t = \left(-\frac{\alpha_t - r}{\gamma_t}, \quad -\sqrt{C^2 - (\frac{r-\alpha_t}{\gamma_t})^2}\,\right)'.
$$

Proof. The results can be derived in a manner similar to that of Klein (1996) and are therefore omitted. Detailed computations are presented in Murgoci

(2008) ∎

The figure 1 plots the upper and lower GDB prices for a one-year European call option with strike 100 and good deal bound constant C=1.5, against the initial stock price. As would be expected, the size of the good deal bound price interval depends heavily on the moneyness of the option. The parameters used are the following: the risk free rate$r$ is 4%; the expected return and the volatility of the stock are given by $\alpha = 0.1$ and $\sigma = 0.45$; the assets of the counterparty have a drift $\mu = 0.1$ and volatility $\gamma = 0.2$; the assets of the counterparty and the stock have an instant correlation equal to 0.3; the dead weight loss β is 0.3. The value of the claims against the counterparty is given by $D = 100$. Since the good deal bound price interval depends on the distance to default of the counterparty, the upper graph presents results obtained when assuming that initially, the assets of the counterparty are 104; hence, the counterparty is very close to default. The lower graph assumes the assets of the counterparty to be 120, and the counterparty is far from default.

Besides the GDB prices, the figure also presents the prices obtained from the Black-Scholes formula and the formula for a vulnerable option by Klein (1996). We notice that although the upper good deal bound price and the Black Scholes price are numerically different, the difference is so small that it cannot be distinguished in the graph. Moreover, the complete market price is always very close to the upper good deal bound price and the Black-Scholes price. The figure supports the intuition that the discount due to counterparty risk will be higher when the counterparty is close to default.

Remark 6.1 (Barrier Options) *The formula for a vulnerable European call to price vulnerable barrier options can be used in a manner similar to the non vulnerable case derived in Björk (2004). Let Ψ denote a claim on the traded stock price. We denote by Ψ_{LO} the down and out version of the vulnerable*

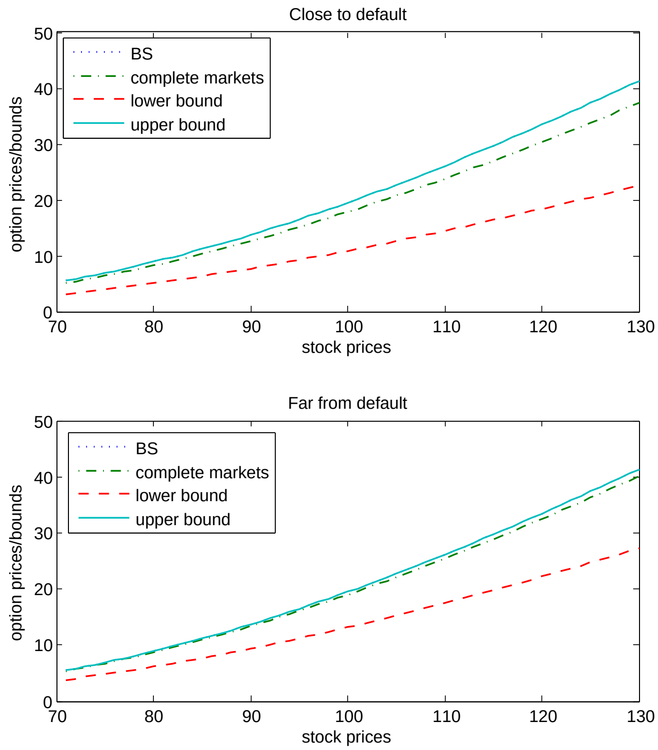

Figure 1: Upper and Lower GDB prices for a European Call

The figure presents the upper and lower GDB prices plotted against the initial stock price for a one-year European call option with strike 100 and good deal bound constant C=1.5. The figure also presents the price obtained using the Black Scholes formula and the formula for a vulnerable option obtained by Klein (1996). The upper graph presents results obtained when assuming that initially, the assets of the counterparty are 104, compared to the claims $D = 100$. The lower graph assumes the claims against the counterparty to still be 100 but the assets of the counterparty are 120.

claim - i.e. a claim which pays Ψ *as long as the underlying of the claim is above a certain level* L *and* 0 *otherwise. An easy example is the down and out option, with the payoff:*

$$\Psi_{LO}(S_T) = \begin{cases} \max[S_T - K, 0], & \textit{if } S_t > L, \textit{for all } t \leq T \\ 0, & \textit{otherwise} \end{cases}$$

We denote by Ψ_{LO}^V *the vulnerable version of a down-and-out claim. Using the same reasoning as in Björk (2004), one can prove the formula:*

$$\Pi(0, \Psi_{LO}^V) = e^{-rT} \left\{ E_{0,s,y}^Q \left[\Psi_L^V(S_T, Y_T) \right] - \left(\frac{L}{s} \right)^{\frac{2\tilde{r}}{\gamma^2}} E_{0,\frac{L^2}{s},y\left(\frac{L}{s}\right)^{2\rho}}^Q \left[\Psi_L^V(S_T, Y_T) \right] \right\},$$

where $\tilde{r} = r - \frac{1}{2}\sigma^2$ *and* $E_{0,s,y}^Q[\bullet]$ *denotes the conditional expected value taken at time* $t = 0$*, given that* $S_0 = s$ *and* $Y_0 = y$*.*

Since only the no-arbitrage assumption (the existence of a martingale measure) and not the market completeness (the unicity of this measure) is used in the proof, it must hold also in incomplete markets as long as we have picked a risk neutral measure Q *according to some criteria. Choosing the Girsanov kernels as in proposition 4.1, we obtain the upper and the lower good deal bound prices for the down-and-out barrier option.*

7 Stability of the GDB prices

In this section, we investigate how sensitive is the good deal bounds formula to the choice of the good deal bound constant C. As previously mentioned, this constant is chosen by the implementer depending on his/her experience and past market performance. The figure 2 presents results for the at-the-money European call computed with the same parameters as in the previous question. As before, we present results for both the case when the counterparty is close to default and when the counterparty is far from default. Since the upper GDB price is already very close to the Black-Scholes price, it is not sensitive to the choice of C. The lower GDB price proves to be sensitive to the choice of C, losing four monetary units for an increase of one in the GDB constant. An interesting note is the fact that the lower GDB price deterioration due to an increase in C is similar for both cases. We also note that values in the Sharpe ratio observed on the financial markets are significantly lower than 1.5, with the Sharpe ratio of S&P 500 around 0.4.

8 Conclusion

We price vulnerable derivative claims - i.e. options where the counterparty may default. These are basically options traded on the OTC markets. Default is modeled in a structural framework.

We price the claims in the more realistic, incomplete market pricing problem. The technique employed for pricing is Good Deal Bounds. The method imposes a new restriction in the arbitrage free model by setting upper bounds on the Sharpe ratios of the assets. The potential prices which are eliminated represent unreasonably good deals, as defined by Cochrane and Saa-Requejo (2000) and Björk and Slinko (2006). The constraint on the Sharpe ratio translates into a constraint on the stochastic discount factor. Thus, tighter pricing bounds can be obtained. We provide a link between the objective probability measure and the range of potential risk neutral measures which has an intuitive economic meaning. We also provide tighter pricing bounds for European calls and show how to extend the call formula to pricing other financial products in a consistent way. Specific examples for exchange options and barrier options are computed.

Finally, we analyze numerically the behavior of the good deal pricing bounds

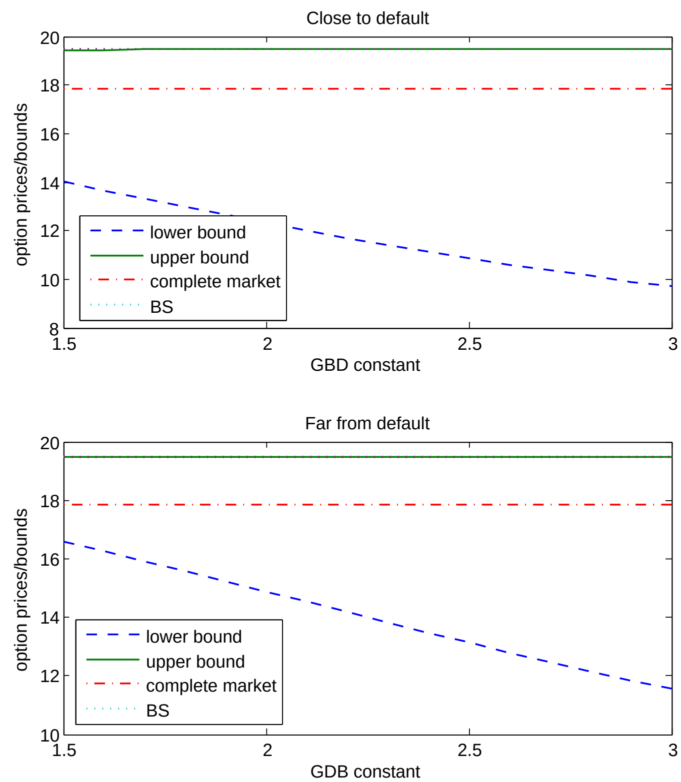

Figure 2: The upper and lower GDB prices when we vary the GDB constraint
The figure presents the upper and lower GDB prices for at-the-money one-year European call option with strike 100. The good deal bound constant C varies between 1.5 and 3. The figure also presents the price obtained using the Black Scholes formula and the formula for a vulnerable option obtained by Klein (1996). The upper graph presents results obtained when assuming that initially, the assets of the counterparty are 104, compared to the claims $D = 100$. The lower graph assumes the claims against the counterparty to still be 100 but the assets of the counterparty are 120.

interval and analyze what impact the good-deal bound constraint has on the size of the good deal price interval. We show that the lower good deal bound is sensitive to the choice of GDB constant. However, the values for which we have applied the good deal bounds are extremely conservative and significantly higher than what is usually observed on the market. In order to obtain tighter bounds, it is important to have good econometric studies regarding the Sharpe Ratio of the investment opportunities existing on the market. The good deal bounds are model dependent and it would be interesting to compare the impact of different models of credit risk on the good deal bound interval. Jaschke and Küchler (2001) show that the lower good deal bound is a coherent risk measure. From this point of view, it would be interesting to compare how sensitive the lower good deal bound is to modeling choices in the context of counterparty risk. Good deal hedging is another interesting direction to extend current work on good deal bounds.

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