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The Case of Soybeans

Richter, Martin; Sørensen, Carsten

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Martin Richter & Carsten Sørensen

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Martin Richter

Finansbanken A/S

Kalkbrænderihavnen 7

DK-2100 Copenhagen

Denmark

Carsten Sørensen

Department of Finance

Copenhagen Business School

Solbjerg Plads 3

DK-2000 Frederiksberg

Denmark

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Abstract

This paper sets up and estimates a continuous-time stochastic volatility model using panel data of soybean futures and options in an integrated time-series study. The model of commodity price dynamics is within the class of affine asset pricing models, and option prices are determined using a standard inversion of characteristic functions approach. Our modeling acknowledges that commodities exhibit seasonality patterns in both spot price level and volatility. The estimation method is based on a state space formulation of the model and a quasi maximum likelihood approach. Estimation results are obtained based on weekly observations of soybean futures prices and options prices from the Chicago Board of Trade in the period October 1984 to March 1999. The empirical results support the conceptual ideas in the theory of storage, but not the view that convenience yields behave like timing options.

1 Introduction

Seasonality is known to be one of the empirical characteristics that make commodities strikingly different from stocks, bonds, and other conventional financial assets (see e.g. the discussion in Routledge, Seppi and Spatt (2000)), and is especially important for agricultural commodities with a seasonal harvesting pattern. This paper sets up a continuous-time stochastic volatility model of commodity prices with a seasonality feature in both the commodity spot price and the spot price volatility, and parameters of the model are estimated using panel data of soybean futures prices and soybean option prices from the Chicago Board of Trade (CBOT) over the period 1984-1999. The estimation approach is analogous to the approach in Schwartz (1997), who sets up and estimates models of commodity futures prices using the Kalman-filter and a state-space approach in a context without seasonality and stochastic volatility.[1] The analysis in this paper is thus based on a state space formulation of the problem which results in an integrated time series study where it is consistently taken into account how model parameters affect both the cross sectional characteristics of derivatives prices and the time series characteristics of commodity prices. Moreover, as another innovative feature compared to e.g. Schwartz (1997), the integrated time series study is based on the simultaneously use of both futures prices as well as option prices across different exercise prices and maturities.

The commodity price behavior is modeled in continuous-time as part of a system of stochastic differential equations within the class of asset pricing models where the drift and diffusion coefficients are affine functions of the relevant basic state-variables. In our framework there are three basic state-variables: the commodity spot price, the convenience yield, and the volatility of the spot price. Affine models are tractable for asset pricing purposes and, in our case, for the valuation of commodity futures contracts and options written on commodity futures. Due to the tractability of affine models, they are widely used for modeling of term structures of interest rates and pricing options on equity and foreign exchange rates.[2] In particular, it

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[1]See also Schwartz and Smith (2000) and Sørensen (2002) for empirical analyzes of commodity price models following this approach.

[2]See e.g. Duffie and Kan (1996) for a general specification of affine term structure models, Heston (1993) and Bakshi, Cao and Chen (1997) for affine models of equity returns, and Backus, Foresi and Telmer (1996) and Bates (1996) for affine models of foreign exchange rates. Duffie, Pan and Singleton (2000) provide a discussion of other applications as well as a general treatment of the affine class of models, with the possibility of jumps included.

is in general possible to determine relevant conditional characteristic functions within affine models which in turn facilitates the evaluation of option prices through an inversion approach, building on the ideas suggested by Heston (1993) for a specific stochastic volatility model of currency and equity returns.

The modeling of commodity spot price behavior in this paper differs from models of interest rates, equity returns, and foreign exchange by the fact that the involved stochastic differential equations are inhomogeneous in time since the drift and diffusion coefficients are functions of calendar time, due to the seasonality feature, and by the inclusion of a convenience yield. On the other hand, we build on the same traditional no-arbitrage approach to the pricing of commodity derivatives as used in analyzing the term structure of interest, equity derivatives, and foreign exchange derivative, and also used widely for analyzing commodity contracts and commodity hedging in related papers.[3] It may be noted that the evaluation of futures and option pricing expressions within affine models can in principle be reduced to the problem of solving ordinary differential equations (as in e.g. Heston (1993)). However, due to the seasonality feature the relevant ordinary differential equations in this paper do not have known analytical solutions, and the specific implementation of futures and option pricing expressions used in our estimation procedure relies heavily on numerically methods.

The estimation approach is based on quasi maximum likelihood and the panel data estimation approach suggested originally by Chen and Scott (1993) in the context of models of the term structure of interest rates. In particular, at each sample date we observe a panel of futures prices and option prices which are related to the basic state-variables through a non-linear measurement equation. The measurement equation is based on the pricing results that we establish for the specific affine model of commodity price behavior with respect to relevant futures and option prices. Following the ideas in Chen and Scott (1993), we assume that all but three contracts are observed with measurement error. The latent value of the basic state-variables can be exactly filtered out at each sample date by inversion based on the three contract prices which are observed without error, and the likelihood of the full sample can then be described based on the likelihood of the filtered paths of the state-variables. In the

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[3]See e.g. Black (1976), Brennan and Schwartz (1985), McDonald and Siegel (1986), Gibson and Schwartz (1990,1991), Jamshidian and Fein (1990), Bjerksund (1991), Brennan (1991), Cortazar and Schwartz (1994), Dixit and Pindyck (1994), Schwartz (1997), Hilliard and Reis (1998,1999), Miltersen and Schwartz (1998), and Schwartz and Smith (2000) for advances on no-arbitrage models with a focus on the valuation of commodity contingent claims, such as real options, and hedging.

evaluation of the likelihood of the filtered state-variables we apply a quasi maximum likelihood approach where the relevant densities are substituted by Gaussian densities with appropriate first and second order moments.

The estimation approach is similar to the estimation approach used in related papers on interest rates and equity returns in affine asset pricing models, and the specific estimation approach could potentially be modified and extended in various ways.[4] For example, one could in principle relax the assumption that three specific contract prices are observed without measurement error by using an extended Kalman filter approach, or more advanced filtering algorithms, and it is possible to approximate the relevant transition densities using for example refinements as in Singleton (2000). Furthermore, it is possible to extend the model and approach to models including jumps in the commodity price dynamics (daily price limits imposed at CBOT, though, complicate this issue). However, it may be noted that the choice of model and estimation approach is motivated and restricted in part by numerical considerations, since the model already in its current form has several parameters to be estimated and the estimation is computationally demanding with respect to both the evaluation of involved derivative prices and numerical evaluation and optimization of the log-likelihood criteria function.

The estimated commodity price model describes the simultaneous dynamics of the commodity spot price, the convenience yield, and the volatility of the commodity spot price. When discussing our specific estimation results, based on soybean futures and option contracts from CBOT, we are especially interested in how the results relate to the basic ideas of the *theory of storage* by Kaldor (1939), Working (1948, 1949), Brennan (1958), and Telser (1958). The theory of storage explains the differences between the spot and futures prices with different contract horizons in terms of advantages of physical inventory compared to advantages of ownership of a futures contract for future delivery of the commodity. Physical inventory is associated with cost of carry such as storage costs and the interest forgone by investing in storage. On the other hand, physical inventory gives rise to a convenience yield from being able to profit from temporary price increases due to temporary shortages of the particular commodity or from being able to maintain a production process despite abrupt shortages of

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[4]See, e.g. Pearson and Sun (1994) and Dai and Singleton (2000) for estimation of models of the term structure of interest rates, Duffie and Singleton (1997) for estimation of a model of swap yields, Duffie and Singleton (1999) and Duffee (1999) for models of credit risky bonds, and Bates (2000), Chernov and Ghyssel (2000) and Pan (2002) for estimation of equity models using options data. This list of related papers concerned with estimation of affine asset pricing models is very partial, and this is a fast growing research area.

the commodity used as input.[5] The theory of storage predicts a negative relationship between supply/inventories and convenience yields. Thus, when inventories are full and supply is plenty the convenience yield from marginal storage is low, and vice versa when inventories are close to being empty.[6] Furthermore, since holding inventory allows for flexibility in the timing of the use of a given commodity, the convenience yield may be viewed as a *timing option*. Recently, Routledge, Seppi and Spatt (2000) has thus provided a formal model supporting this idea building on the apparatus of Wright and Williams (1989), Williams and Wright (1991), Deaton and Laroque (1992, 1996), Chambers and Bailey (1996), among others, who formalize the theory of storage in competitive rational expectations models.

Our empirical results are to some degree in accordance with the theory of storage and are qualitatively consistent across different time periods. The seasonal components in both the soybean convenience yields and volatilities are thus at the highest just before harvesting when inventories are low and supply scarce. The spot price is estimated to be positively correlated with the convenience yield, so this is also a period where spot prices are relatively high. While this supports the ideas in the theory of storage, our results on the other hand do not support the view that the convenience yield behaves like a timing option. In particular, our estimation results suggest that convenience yields and volatilities are slightly negatively correlated contrary to how options depend on volatility.

Furthermore, our empirical results demonstrate that the soybean commodity spot price is positively correlated with volatility. This is opposed to the so-called *leverage effect* in equity markets where the correlation between stocks and volatility is usually estimated negative. This estimation result is, however, consistent with the observation that the implied volatility "smile" or "smirk" in soybean options has opposite slope of the post-1987 crash "smirk" observed for equity options.

The paper is organized as follows. In section 2, the model and estimation approach as well as the numerical implementation is described in detail. In section 3 the data and estimation results are presented and discussed. A final section concludes.

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[5]When we refer to the convenience yield in the following and in the formal continuous-time model, we are in fact referring to the *net convenience yield*, i.e. the convenience yield net of storage costs, unless otherwise specified.

[6]Since inventories/supply of agricultural commodities vary with the season, a seasonality pattern in convenience yields is, therefore, well in accordance with the prediction of the theory of storage, as discussed by, e.g. Fama and French (1987).

2 The model and estimation approach

Let $W = (W_1, W_2, W_3)$ be a three-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. $F = \{\mathcal{F}_t : t \geq 0\}$ denotes the standard filtration of W and, formally, $(\Omega, \mathcal{F}, F, \mathcal{P})$ is the basic model for uncertainty and information arrival in the following. We will assume that W has a constant "instantaneous" correlation matrix Σ with entries ρ_{ij} and ones in the diagonal.

2.1 Commodity price dynamics

The model of commodity price behavior incorporates important realistic features such as stochastic convenience yields and stochastic volatility as well as a seasonal feature. That these features are important and realistic in the specific context is documented and discussed more extensively in the subsequent data description. Also, the modeling captures well-known phenomena of commodity prices such as the mean-reverting feature of convenience yields which has been pointed out by, e.g. Gibson and Schwartz (1990) and Bessembinder, Coughenour, Seguin and Smoller (1995). In fact, in the special case without stochastic volatility and seasonality, the model suggested below coincides with the model proposed by Gibson and Schwartz (1990).

Let P_t denote the spot commodity price at time t. The convenience yield and the seasonal adjusted spot price volatility state-variable at time t are denoted δ_t and v_t, respectively. The dynamics of the three-dimensional process (P, δ, v) are described by the following stochastic differential equation system,

$$dP_t = P_t \left[(r - \delta_t + \lambda_P e^{\nu(t)} v_t) \, dt + e^{\nu(t)} \sqrt{v_t} \, dW_{1,t} \right] \tag{1}$$

$$d\delta_t = (\alpha(t) - \beta \delta_t + \lambda_\delta \sigma_\delta e^{\nu(t)} v_t) \, dt + \sigma_\delta e^{\nu(t)} \sqrt{v_t} \, dW_{2,t} \tag{2}$$

$$dv_t = (\theta - \kappa v_t + \lambda_v \sigma_v v_t) \, dt + \sigma_v \sqrt{v_t} \, dW_{3,t} \tag{3}$$

where β, κ, θ, λ_P, λ_δ, λ_v, σ_δ, and σ_v, as well as the correlation parameters of W, are constant parameters that we will estimate in the subsequent sections. $\alpha(t)$ and $\nu(t)$ are deterministic functions that aim at describing seasonal patterns in convenience yields and volatilities; the functional forms of $\alpha(t)$ and $\nu(t)$ are described and discussed below.

The parameters λ_P, λ_δ, and λ_v describe the risk premia on commodity price uncertainty, convenience yield uncertainty, and volatility uncertainty. We will assume markets are complete and the existence of an equivalent martingale measure, Q, so that prices on contingent claims

can be uniquely derived by evaluating the relevant expectations of the discounted payoff under Q. Using a standard terminology, we will also refer to Q as the risk-neutral probability measure in the following. Under the risk-neutral probability measure, risk premia are zero and the dynamics of (P, δ, v) are described by,

$$dP_t = P_t \left[(r - \delta_t) \, dt + e^{\nu(t)} \sqrt{v_t} \, dW_{1,t}^Q \right] \tag{4}$$

$$d\delta_t = (\alpha(t) - \beta \delta_t) \, dt + e^{\nu(t)} \sigma_\delta \sqrt{v_t} \, dW_{2,t}^Q \tag{5}$$

$$dv_t = (\theta - \kappa v_t) \, dt + \sigma_v \sqrt{v_t} \, dW_{3,t}^Q. \tag{6}$$

From equations (1) and (4) it follows that the convenience yield, δ_t, can be interpreted as the return shortfall on an inventory position of the specific commodity.[7] Basically, in order to break-even on the marginal inventory position, there must be a gain, or *convenience yield*, from holding inventory to exactly offset the return shortfall.

The parameters β and κ determine the degree of reversion to the deterministic seasonal pattern in the convenience yield and the long-run volatility level θ, respectively. Likewise, σ_δ and σ_v are parameters that, together with the level of v_t, determine the volatility of the convenience yield and the volatility of the stochastic volatility of the commodity price. The parameter r is the short default-free interest rate which is assumed constant (but we allow to vary over time in our empirical analysis).[8]

The deterministic seasonal functions $\alpha(t)$ and $\nu(t)$ are defined as,

$$\alpha(t) = \alpha_0 + \sum_{k=1}^{K^\alpha} \left(\alpha_k \cos(2\pi k t) + \alpha_k^* \sin(2\pi k t) \right) \tag{7}$$

and

$$\nu(t) = \sum_{k=1}^{K^\nu} \left(\nu_k \cos(2\pi k t) + \nu_k^* \sin(2\pi k t) \right) \tag{8}$$

where K^α and K^ν determine the number of terms in the sums and α_0, α_k, α_k^*, $k = 1, \ldots, K^\alpha$, and ν_k, ν_k^*, $k = 1, \ldots, K^\nu$ are constant parameters to be estimated. The specific functional form of the deterministic seasonal components were originally suggested by Hannan, Terrell,

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[7]In fact, the convenience yield may be defined as the return shortfall on holding the commodity; see, e.g. MacDonald and Siegel (1984).

[8]Our pricing results will carry through if interest rates are stochastic but distributed independently of the soybean dynamics. In this case, discounting is done by using the appropriate zero coupon bonds, as in our empirical approach.

and Tuckwell (1970) as an alternative to standard dummy variable methods in econometric modeling of seasonality. Due to the continuous time feature of our analysis this is a flexible and natural choice of modeling the seasonal aspects of commodity price behavior which is also applied in Sørensen (2002).

It may be noted that the specific modeling of commodity behavior that incorporates stochastic volatility and seasonality, as in the above stochastic differential equation systems, is chosen mainly in order to keep the model within the so-called affine asset pricing class where drift terms and squared diffusion terms are affine functions of the state-variables. However, the specific choice is also driven in part by the desire to keep the model sufficiently parsimonious to allow for numerical estimation; for example, the seasonality pattern in the volatility of the spot price and the volatility of the convenience yield is chosen to be identical for this reason.

2.2 Futures prices

Following Cox, Ingersoll and Ross (1981), the futures price at time t on a futures contract for delivery at time $T \geq t$ can be obtained by taking the relevant expectations of the future spot price, $F_t(T) = \mathrm{E}_t^Q[P_T]$. Equivalently, by using the Feynman-Kac formula, $F_t(T) = F(P_t, \delta_t, v_t, t; T)$ where $F(\cdot)$ can be found as the solution to a partial differential equation on the form

$$
\begin{aligned}
&\tfrac{1}{2}e^{2\nu(t)}vP^2\frac{\partial^2 F}{\partial P^2} + \tfrac{1}{2}\sigma_\delta^2 e^{2\nu(t)}v\frac{\partial^2 F}{\partial \delta^2} + \tfrac{1}{2}\sigma_v^2 v\frac{\partial^2 F}{\partial v^2} + \rho_{12}\sigma_\delta e^{2\nu(t)}vP\frac{\partial^2 F}{\partial P\partial \delta} + \rho_{13}\sigma_v e^{\nu(t)}vP\frac{\partial^2 F}{\partial P\partial v} \\
&+ \rho_{23}\sigma_\delta\sigma_v e^{\nu(t)}v\frac{\partial^2 F}{\partial \delta\partial v} + (r-\delta)P\frac{\partial F}{\partial P} + (\alpha(t)-\beta\delta)\frac{\partial F}{\partial \delta} + (\theta-\kappa v)\frac{\partial F}{\partial v} + \frac{\partial F}{\partial t} = 0
\end{aligned}
\tag{9}
$$

with terminal condition $F(P, \delta, v, T; T) = P$.

For a given and fixed expiration date T, the solution to the partial differential equation in (9) is given by

$$
F(P, \delta, v, t; T) = Pe^{A(t;T) + B(t;T)v + D(t;T)\delta}
\tag{10}
$$

where

$$
D(t;T) = -\frac{1}{\beta}\left(1 - e^{-\beta(T-t)}\right)
\tag{11}
$$

and $A(t;T)$ and $B(t;T)$ are the solutions to the ordinary differential equations

$$
\begin{aligned}
&\tfrac{1}{2}\sigma_\delta^2 e^{2\nu(t)}\left(D(t;T)\right)^2 + \rho_{12}\sigma_\delta e^{2\nu(t)}D(t;T) + \tfrac{1}{2}\sigma_v^2\left(B(t;T)\right)^2 \\
&+ (\rho_{13}\sigma_v e^{\nu(t)} + \rho_{23}\sigma_\delta\sigma_v e^{\nu(t)}D(t;T) - \kappa)B(t;T) + B'(t;T) = 0, \; B(T;T) = 0
\end{aligned}
\tag{12}
$$

and

$$r + \alpha(t)D(t;T) + \theta B(t;T) + A'(t;T) = 0, \ A(T;T) = 0. \tag{13}$$

Solutions to the ordinary differential equations in (11) and (13) are not available in closed form. Solutions are, however, easily obtained numerically with very high precision and speed using for example Runge-Kutta methods; this is the approach followed in our empirical analysis as described in the subsequent section on the numerical implementation.

2.3 Option prices

We will consider options written on commodity futures. By using Ito's lemma on the futures price, as described in equation (10), it can be seen that the dynamics of the futures price under the equivalent martingale measure, Q, are given by

$$dF_t(T) \ = \ F_t \sqrt{v_t} \left[e^{\nu(t)} \, dW^Q_{1,t} + \sigma_v B(t;T) \, dW^Q_{3,t} + \sigma_\delta e^{\nu(t)} D(t;T) \, dW^Q_{2,t} \right]. \tag{14}$$

Let C_t be the price of a European call option written on the futures price of a futures contract expiring at time T. The option has strike price K and matures at time τ; $t \leq \tau \leq T$. The call price can be determined by taking the relevant expectations,

$$C_t = \mathrm{E}^Q_t \left[e^{-r(\tau-t)} \max[0, F_\tau(T) - K] \right]. \tag{15}$$

From the Feynman-Kac's formula, and the dynamics of F_t and v_t in equations (14) and (6), it follows that the option price can be represented as $C_t = C(F_t, v_t, t)$ where the function $C(\cdot)$ is the solution to the partial differential equation[9]

$$\tfrac{1}{2}\sigma_F^2(t;T)vF^2\tfrac{\partial^2 C}{\partial F^2} + \tfrac{1}{2}\sigma_v^2 v\tfrac{\partial^2 C}{\partial v^2} + \sigma_{Fv}(t;T)vF\tfrac{\partial^2 C}{\partial F \partial v} + (\theta - \kappa v)\tfrac{\partial C}{\partial v} + \tfrac{\partial C}{\partial t} - rC \ = \ 0 \tag{16}$$

where

$$\begin{aligned}
\sigma_F^2(t;T) \ = \ & e^{2\nu(t)} + \sigma_v^2 \left(B(t;T) \right)^2 + \sigma_\delta^2 e^{2\nu(t)} \left(D(t;T) \right)^2 + 2\rho_{13}\sigma_v e^{\nu(t)} B(t;T) \\
& + 2\rho_{12}\sigma_\delta e^{2\nu(t)} D(t;T) + 2\rho_{23}\sigma_v\sigma_\delta e^{\nu(t)} B(t;T)D(t;T)
\end{aligned}$$

and

$$\sigma_{Fv}(t;T) = \rho_{13}\sigma_v e^{\nu(t)} + \sigma_v^2 B(t;T) + \rho_{23}\sigma_v\sigma_\delta e^{\nu(t)} D(t;T)$$

---

[9]Note that the observations about futures price dynamics and option pricing in equations (14) and (15) serve to show that for the option pricing in this paper, the underlying futures price F is a "sufficient statistic" for the state-variables P and δ. The option prices will also satisfy a partial differential equation which is analog to (9), as all other contingent claims will.

and with terminal condition $C(F, v, \tau) = \max[0, F - K]$. $\sigma_F(t; T)$ describes the volatility on the underlying futures price whereas $\sigma_{Fv}(t; T)$ describes the "instantaneous" rate of covariance between the futures price and the stochastic volatility process.

This is basically the Heston (1993) model but with $\sigma_F^2(t; T)$ and $\sigma_{Fv}(t; T)$ being functions of time instead of constants (and the underlying being a futures price instead of a spot price). Hence, using the arguments and approach in Heston (1993), pp. 330-331 and his Appendix, one obtains the following option pricing formula on the form of Black (1976),

$$C(F, v, t) = e^{-r(\tau - t)} [FP_1 - KP_2] \tag{17}$$

where (see equation (18) in Heston (1993))

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left[\frac{e^{-i\phi \ln[K]} f_j(x, v, t; \phi)}{i\phi} \right] d\phi , \quad j = 1, 2 \tag{18}$$

and where $x = \ln F$ and $f_j(\cdot; \phi)$, $j = 1, 2$, denote characteristic functions that are obtained as solutions to partial differential equations with terminal condition $e^{i\phi x}$. Since coefficients in the particular partial differential equations are affine and the terminal conditions are exponential-affine, the solutions for f_j, $j = 1, 2$, are exponential-affine and on the form

$$f_j(x, v, t; \phi) = e^{a_j(t) + b_j(t)v + i\phi x} , \quad j = 1, 2 \tag{19}$$

where $a_j(t)$ and $b_j(t)$ are solutions to the ordinary differential equations

$$b_j'(t) = \left(\tfrac{1}{2}\phi^2 - u_j i\phi \right) \sigma_F^2(t; T) + (k_j(t) - i\phi\sigma_{Fv}(t; T)) b_j(t) - \tfrac{1}{2}\sigma_v^2 (b_j(t))^2 , \quad b_j(\tau) = 0 \tag{20}$$

and

$$a_j'(t) = -\theta b_j(t) , \quad a_j(\tau) = 0 \tag{21}$$

with $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $k_1(t) = \kappa - \sigma_{Fv}(t; T)$, and $k_2(t) = \kappa$.

Again, the numerical implementation of the above option pricing formula in (17) requires several numerical considerations which are provided in a subsequent section on the numerical implementation of the estimation approach.

2.4   The estimation approach

The estimation approach is based on formulating the model in state space form so that the model is represented by a *measurement* equation, and a *transition* equation. In our case the state space model is non-linear and the specific approach relies on a general state space

estimation approach suggested originally by Chen and Scott (1993) for estimation of models of the term structure of interest rates.

The data are in the following observed at equidistant time points, t_n, $n = 0, \ldots, N$, and $\Delta = t_{n+1} - t_n$. The transition equation describes the dynamics of the basic unobserved state-variables. In our case, there are three state-variables which can be summarized by the three-dimensional state-vector $X_n = (p_{t_n}, \delta_{t_n}, v_{t_n})$ where $p_t = \log P_t$. The transition equation is given by the discrete-time solution to the stochastic differential system in (1), (2), and (3) between the observation time points.

The measurement equation relates the state-variables to the observed (log-) futures prices and option prices.[10] Following the approach in Chen and Scott (1993), we will assume that three prices on soybean contingent claims are observed without measurement error. In particular, this allows one to filter out the three unobserved state-variables by solving three equations with three unknowns at all observation time points. Let the vector Y_n denote the panel data observation of (log-) futures prices and options prices at time t_n, $n = 0, \ldots, N$. We have $Y_n' = (Y_{(1),n}', Y_{(2),n}')$ where $Y_{(1),n}$ is the three-dimensional vector of contract prices observed without measurement error while $Y_{(2),n}$ is the m-dimensional vector of contract prices that are observed with measurement error. Formally, the measurement equation is given by

$$
\begin{pmatrix} Y_{(1),n} \\ Y_{(2),n} \end{pmatrix} = \begin{pmatrix} f_{(1),n}(X_n) \\ f_{(2),n}(X_n) \end{pmatrix} + \begin{pmatrix} \underline{0} \\ \epsilon_n \end{pmatrix}
\tag{22}
$$

where $\underline{0}$ is a three-dimensional vector of zeros and ϵ_n, $n = 0, 1, \ldots, N$, are serially uncorrelated measurement error terms that are assumed identical m-dimensional normally distributed with mean zero and variance-covariance matrix Γ; $f_{(1),n}$ and $f_{(2),n}$ are a three-dimensional valued function and an m-dimensional valued function, respectively, that return the relevant theoretical value of the logarithm to the futures prices and options prices, as implied by the futures and options pricing expressions in (10) and (17).

In our implementation, we assume that a short maturity at-the-money call option, the corresponding futures price, and the futures price on a long maturity futures contract are

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[10]In the concrete estimations, we have chosen to impose measurement noise on the logarithm to the futures price which facilitates comparison with, e.g. Schwartz (1997), Schwartz and Smith (2000), and Sørensen (2002). This implies that the measurement noise enters proportionally to the futures price. On the other hand, we have chosen to simply add measurement noise to the observed option prices without transformation since, e.g. a logarithmic transformation will inappropriately tend to allow more noise on in-the-money options than on at-the-money options and out-of-the-money options.

observed without measurement error; i.e. these are the contract prices that enter $Y_{(1),n}$ at the different sample dates. ($Y_{(2),n}$ has dimension $m = 10$ in our implementation since six (log-) futures prices and four option prices are assumed observed with measurement noise.) Having observed a total of three futures and option prices without measurement noise, it is possible to filter out the three-dimensional state-vector X_n by simply inverting the relationship between $Y_{(1),n}$ and X_n, as given by the pricing expressions in (10) and (17) that are the entries in $f_{(1),n}$. Specifically, we have the relationship

$$X_n = f_{(1),n}^{-1}(Y_{(1),n}). \tag{23}$$

Whenever we refer to the value of X_n in the following, it is assumed that X_n is observed through the filtering equation (23).

Let ψ denote the set of parameters entering the description of state-variable dynamics in (1), (2), and (3) and in the pricing results in (10) and (17). We are interested in estimating the parameters in ψ as well as the variance-covariance matrix Γ of the measurement error terms in the measurement equation (22). Following Chen and Scott (1993), the log-likelihood function has the following form:

$$
\begin{aligned}
l(Y_0, Y_1, \ldots, Y_N; \psi, \Gamma) \;=\; & \sum_{n=1}^{N} \left[\log p(X_n | X_{n-1}) - \log \left| \frac{\partial f_{(1),n}}{\partial X} \right| \right. \\
& \left. - \frac{m}{2} \log(2\pi) - \frac{1}{2} \log |\Gamma| - \frac{1}{2} \epsilon_n' \Gamma^{-1} \epsilon_n \right].
\end{aligned}
\tag{24}
$$

The first term on the right-hand side of (24) is the log-likelihood value of the implied state-variables as filtered through the relationship in (23), and $p(X_n | X_{n-1})$, $n = 1, \ldots, N$, denote the relevant conditional densities of the Markovian discrete-time solution to the stochastic differential equation system (1), (2), and (3) which describes the transition equation.[11] The second term on the right-hand side of (24) is the Jacobian determinant of the transformation between the observed contract prices in $Y_{(1),n}$ and X_n. The last terms on the right-hand side of (24) are the likelihood of the normally distributed measurement errors.

The estimation results obtained in the following analysis is obtained by numerical maximization of the log-likelihood function in (24). However, since the conditional densities $p(X_n | X_{n-1})$, $n = 1, \ldots, N$, are not known in closed-form analytical form, we have chosen to rely on a quasi maximum likelihood approach where these conditional densities are substituted by densities from the three-dimensional normal distribution; the appropriate first and

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[11]Note that the involved stochastic differential equation system is inhomogeneous due to the seasonality feature of the model, though, the time-dependence of the relevant conditional densities is suppressed in (24).

second order moments follow from Lemma 1, as stated and proved in the appendix. Likewise, in our calculation of standard errors on parameter estimates, we follow the general quasi maximum likelihood approach suggested originally by White (1982); see also, e.g. Hamilton (1994), pp. 126,145.

The concrete implementation and optimization involves many numerical considerations and, therefore, we have allocated the following subsection to this issue. However, at this point it can be noted that the variance-covariance matrix of the measurement errors, Γ, can be concentrated out of the log-likelihood function. From the first order conditions of the maximization of (24), it is thus seen that the estimator of Γ (for a given set of parameters ψ) has the form

$$\hat{\Gamma} = \frac{1}{N} \sum_{n=1}^{N} \epsilon_n \epsilon_n' = \frac{1}{N} \sum_{n=1}^{N} \left(Y_{(2),n} - f_{(2),n}(X_n) \right) \left(Y_{(2),n} - f_{(2),n}(X_n) \right)' \qquad (25)$$

and, $N\hat{\Gamma} \sim \mathtt{W}_m(N,\Gamma)$ where $\mathtt{W}_m(N,\Gamma)$ is the m-dimensional Wishart distribution with N degrees of freedom. Substituting the expression for Γ in (25) into the log-likelihood function (24) simplifies the numerical optimization problem since we only have to determine the maximum likelihood parameter values of ψ with a numerical optimization routine. The numerical optimization is done in GAUSS using the method of Broyden, Fletcher, Goldfarb and Shanno (BFGS). We use the standard GAUSS options in the numerical BFGS optimization algorithm.

2.5   Numerical implementation

In our implementation of the above estimation approach we observe data at a weekly frequency and at a total of 753 sample dates. At each sample date we have a panel-observation consisting of eight (log-) futures prices and five option prices. The evaluation of the log-likelihood function, hence, involves the evaluation of a large number of theoretical futures and options prices using the formulas stated in (10) and (17). In this section we will briefly describe how these pricing expressions are implemented and how the log-likelihood function is calculated in a relatively fast but still accurate way. Using the implementation described in the following, the evaluation of a single value of the log-likelihood function in GAUSS takes about one minute on a Pentium III 500 MHz computer while the evaluation of a single option price on average is done in about one hundredth of a second. A single iteration of the BFGS algorithm takes about one hour, and the parameter estimates presented in a subsequent section are obtained by using a total of about three weeks of computer time.

The evaluation of a futures price as in (10) requires solving the system of ordinary differential equations in (12) and (13). The numerical solution to (12) and (13) is implemented using the classical Runge-Kutta method of fourth order accuracy combined with Richardson extrapolation. The discrete time steps used in the classical Runge-Kutta method are always chosen in order not to exceed five days or, equivalently, $(5/365.25=)$ 0.01369 years. In the Richardson extrapolation this solution is combined with the Runge-Kutta solution with half as many time steps by using the relevant Richardson extrapolation formula for numerical methods of fourth order accuracy.

The evaluation of an option price as in (17) requires the numerical evaluation of integrals of the form in (18). The numerical integration is implemented by the Gauss-Laguerre quadrature formula using twenty points in the evaluation of the integrand function. Each evaluation of such an integral thus requires solving numerically the system of ordinary differential equations in (20) and (21) for twenty different values of the running variable ϕ as well as simultaneously solving the ordinary differential equation for $B(\cdot)$ in (12) since the coefficients $\sigma_F^2(\cdot)$ and $\sigma_{Fv}(\cdot)$ in (20) are functions of $B(\cdot)$. This system of forty one $(= 20 + 20 + 1)$ first order differential equations is solved simultaneously using the classical Runge-Kutta method combined with Richardson extrapolation following exactly the same procedure as described for futures prices above.[12]

Evaluation of the log-likelihood function in (24) requires inverting the relationship between the unobserved state-vector X_n and the observed prices in $Y_{(1),n}$, as described conceptually by the filtering equation (23). In our implementation, we assume that a short maturity at-the-money call option, the corresponding futures price, and the futures price on a long maturity futures contract are observed without measurement error. The inversion is carried out by first solving for the implied value of the volatility state-variable, v_{t_n}, in the call option price by solving one equation with one unknown by the Newton-Raphson method (in this step we use that the underlying futures price is observed without noise). In solving for the implied

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[12]We checked the numerical precision of this procedure for option prices by considering the Heston (1993) model for a variety of times to maturity, option moneyness, and parameter values. Option prices were compared to prices obtained by the Heston (1993) closed-form expression (using forty points in the evaluation of the involved integral). Running through different grids of parameter values, the worst case deviation was of order 10^{-6} for values of the underlying and option prices compatible with observed prices; as a comparison, the option prices in the data used in the estimation are quoted with only three decimal points.

volatility state-variable by the Newton-Raphson method, we use that

$$\frac{\partial C}{\partial v} = e^{-r(\tau-t)} \left[F \frac{\partial P_1}{\partial v} - K \frac{\partial P_2}{\partial v} \right] \tag{26}$$

and

$$\frac{\partial P_j}{\partial v} = \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left[b_j(t) \frac{e^{-i\phi \ln[K]} f_j(x,v,t;\phi)}{i\phi} \right] d\phi \ , \ \ j = 1, 2 \tag{27}$$

which follows by differentiating the call option expression in (17) and (18) for a fixed value of the underlying futures price. Evaluating the relevant derivatives in this step thus require evaluation of integrals of the same form as in the evaluation of options prices, and the same procedure is applied in this case.[13] Having solved for the implied value of the volatility state-variable, we use the exponential affine structure of the futures prices (i.e. the log-futures prices are affine functions of the state-variables) to solve for the values of the two remaining state-variables p_{t_n} and δ_{t_n} by simply solving two *linear* equations with two unknowns. Furthermore, the Jacobian determinant of the transformation between $Y_{(1),n}$ and X_n is calculated as a by-product of the numerical calculations in this inversion approach since we have that

$$\left| \frac{\partial f_{(1),n}}{\partial X} \right| = \frac{\partial C}{\partial v} \left(D(t_n; T_s) - D(t_n; T_l) \right) \tag{28}$$

where T_s and T_l refer to the relevant maturities of the involved short and long futures contract and $D(\cdot)$ is defined in (11).

Finally, the relevant conditional means and conditional variances that enter the normal density functions in the quasi maximum likelihood approach are determined by solving numerically the integrals in (30) and (31), as stated in Lemma 1 in the appendix. Since data are observed at a weekly frequency we have $\Delta = 1/52 = 0.01923$ years. The numerical integration is in this case done by using the trapezoidal rule and using the second-order Runge-Kutta method (Heun's method) with one time step to solve the differential equation in (32) and obtain the value of Φ in the right-end-point of the integrals. Many of the entries in (30) and (31) can be calculated in explicit analytical form, but not all, and the suggested approximation is very accurate compared to for example a left-end-point approximation of the integrals. The later approximation corresponds to the case where the relevant normal densities in the quasi maximum likelihood approach are obtained by an Euler approximation of the continuous-time

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[13]It can be noted that the relevant system of forty one first order differential equations must only be solved once at each sample date in order to determine the relevant derivatives in the different Newton-Raphson steps as well as the calculation of all option prices with the same maturity.

differential equation system in (1), (2), and (3). In our estimations we originally obtained starting parameter values using such an Euler approximation, but with the specific data the estimates and the likelihood of observations are not ignorably altered by going to a more accurate approximation of the relevant moments in the conditional densities.[14]

3 Empirical results

In this section we will describe the soybean futures and options data and, subsequently, present the formal estimation results based on the model and estimation approach presented above.

3.1 Data

The data consists of weekly observations of soybean futures prices and options on soybean futures from CBOT in the period from the beginning of soybean options trading on CBOT in October 1984 and to March 1999. The futures prices and option prices involved are settlement prices at CBOT for Wednesdays (or Tuesday if Wednesday is unavailable). In fact, settlement prices are available on a daily basis but the weekly sampling frequency is chosen mainly to reduce problems due to microstructure issues such as daily price limits imposed by CBOT.

The soybean futures and options on soybean futures at CBOT are relatively liquid contracts. For example, in 1991 (at the middle of our sampling period) soybean futures contracts at CBOT are ranked number ten worldwide most heavily traded futures contract based on daily volume in Ritchken (1996, pp. 22-23) and is one of the few futures contracts on physicals ranked in top fifteen (only crude oil futures contracts at NYMEX and corn futures contracts at CBOT are also ranked in top fifteen).[15]

The Futures data

The CBOT soybean futures have seven expiration months in the sampling period: January, March, May, July, August, September, and November. At any particular date, more than seven soybean futures contracts may be traded since for example futures with expiration in

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[14]In particular, the likelihood is significantly affected for values of the volatility state-variable being very close to zero.

[15]The industrial use of soybeans is based on "crushing" the soybean into soybean oil (used in cooking oil, margarine, mayonnaise etc.) and soybean meal (a protein supplement used in livestock and poultry feeds), and the CBOT also offers futures and options contracts on these principal soybean by-products.

July this year and next year may be traded simultaneously. The maximum number of futures contracts that are open at all the dates in our sample is eight. In order to have the same number of futures observations at each sample date, we have thus chosen only to include the eight futures contracts with the shortest time to expiration at any sample date.

In the estimation approach we need the times to maturity for the different futures contracts at the different sample dates. The contractual terms at CBOT specify that physical delivery (in the form of an inventory receipt) may occur at any business day throughout the expiration month and, hence, that the last delivery day is the last business day of the month. However, the *last trading day* of the futures contract is the seventh business day preceding the last business day of the delivery month. Since the last trading day is the last day that a contract can in principle be closed at CBOT without physical delivery, we have chosen this day as the expiration date when constructing the times to maturity for the involved futures contracts in the estimations presented in the next section.

Table 1 provides summary statistics with respect to the involved soybean futures contracts.

[INSERT TABLE 1 ABOUT HERE]

The table states the average number of days to maturity and mean and standard deviation of all the soybean futures prices in the dataset. In addition, the tables provide summary statistics for the futures contracts categorized into expirations months as well as the time to maturity of the contracts. In the following, the terminology "1. Maturity" is used as notation for the futures contract that has the shortest time to maturity at a given sample date; the "2. Maturity" represents the futures contract with the second shortest time to maturity, and so on.

The table suggests at least three features of the soybean futures prices that are captured by the model in section 2. (i) Futures prices display a seasonal pattern since the soybean futures price is on average high in July prior to the US harvesting and low in November after the US harvesting. (ii) The variability of futures prices also seem to have a seasonal pattern since the standard deviations for the futures prices, when grouped into expiration months, display the same time pattern as for the level of futures prices. And, (iii) the variability of long futures prices seems lower than that of short futures prices. The lower variation of futures prices with a long time to maturity compared to futures prices with a short time to maturity is suggested by the different standard deviations for the futures prices grouped into their time to maturity in Table 1. The tabulated standard deviations are thus monotonically decreasing for increasing

16

maturities. This pattern is consistent with the so-called "Samuelson hypothesis" that predicts that long futures prices are less volatile than futures prices on contracts with short time to maturity.

The time series aspects of the soybean futures data are illustrated in Figure 1.

[INSERT FIGURE 1 ABOUT HERE]

In the figure we have graphed the time-series for the futures contracts with the shortest time to maturity and the futures contracts with the longest time to maturity over the sample period. A visual inspection of the figure suggests that the time series of the futures price on the shortest maturity contract is more volatile than the futures price on the longest maturity contract. In particular, if one enter a horizontal line at the overall futures price mean at 627.49 cents per bushel (cf. Table 1) in the figure, the deviations from this overall mean is more significant for the short maturity futures price over the sample period. Again, this is in line with the "Samuelson hypothesis" and a feature of the soybean futures prices which is captured in the formal modeling in section 2 by including a mean-reverting convenience yield; cf. the discussion in Bessembinder, Coughenour, Seguin and Smoller (1996).

The options data

The involved CBOT soybean option contracts are options written on soybean futures prices. The relevant futures contracts expire in: January, March, May, July, August, September, and November. The option contract expires around one month prior to the expiration of the underlying futures contract. Specifically, according to the CBOT contract specification, the *last trading day* is the Friday which precedes by at least five business days the last business day of the month preceding the option month. The *expiration date* is the Saturday following the last trading day.

Both call options and put options are traded. Also, at any particular sample date a variety of options contracts, which differ with respect to the underlying futures and with respect to the strike prices, are quoted. However, in our sample we have only included five call option contracts at each sample date. The specific call option contracts are selected in order to represent very liquid contracts. The sample thus includes three call option contracts that have the shortest possible time to maturity (though, we ignore very short contracts so that all the considered options have at least 15 days to maturity). The three call options with a short maturity are an at-the-money (ATM) option, an out-of-the-money (OTM) option, and

17

an in-the-money (ITM) option. The ATM option is chosen as the call option that has a strike price as close as possible to the current underlying futures price. The OTM call option is the quoted call option with a strike price just higher than the ATM call option and, likewise, the ITM call option is the quoted call option with a strike price just lower than the ATM call option. Furthermore, our sample includes two longer term ATM options that are written on the next maturing futures prices.

The CBOT options on soybean futures are American-style. Since our pricing formulas are relevant only for European-style options, the option data are adjusted for the early exercise premium. The early exercise premia is estimated using a standard Black-Scholes framework and applying a trinomial lattice model combined with Richardson extrapolation, following Broadie and Detemple (1996).[16] The relevant interest rates are obtained by interpolation of 3 months, 6 months, and 1 year to maturity T-bills available from the Federal Reserve H.15 release. The rates are quoted on a banker's discount basis and converted appropriately; see Jarrow (1996), chapter 2. The American-style CBOT option prices *minus* the estimated early exercise premia are the European-style input data used in the empirical analysis.

Table 2 provides summary statistics on the involved input call option prices.

[INSERT TABLE 2 ABOUT HERE]

Instead of exhibiting summary statistics of the raw option prices, we have chosen to represent the option data by implied volatilities in Table 2. The relevant implied volatilities of the involved options on soybean futures are backed out by using the Black (1976) formula. Three features of the option data can be pointed out: (i) The implied volatilities have a clear seasonal pattern, (ii) the implied volatilities for the ATM options are slightly increasing as a function of time to maturity, and (iii) the volatilities on the short maturity contract suggest a "smile" or "smirk" pattern. Note, however, that the "smirk" is downward sloping as a function of the strike price in contrast to the upward sloping "smirk" in equity index options observed after the October 1987 market crash; see, e.g. Rubinstein (1994).

In Figure 2 the time series properties of the implied volatilities for the shortest ATM option are displayed.

[INSERT FIGURE 2 ABOUT HERE]

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[16]The results in Broadie and Detemple (1996) suggest that this method is efficient with respect to accuracy, and we are grateful to Mark Broadie for supplying the relevant program code while one of the authors was visiting Columbia University.

As suggested by Figure 2, the implied volatilities are quite variable over time. This, combined with the clear seasonal pattern in implied volatilities displayed in Table 2, is our basic motivation for the formal modeling of the stochastic volatility in section 2.

3.2 Estimation results

In this section we will present and discuss the estimation results based on the above data set and the estimation approach described in section 2.4. We start by discussing the full-sample estimates but, subsequently, we also present estimation results where the data is split into two sub-samples in order to investigate the parameter stability of the model over time.

The parameter estimates obtained from the state space estimation approach and the full sample of observations are tabulated in Table 3.

[INSERT TABLE 3 ABOUT HERE]

In the following discussion of the full-sample parameter estimates in Table 3, we will start by focusing on the structural parameters of the soybean spot price volatility process and the convenience yield process parameters and, subsequently, we discuss the implications of the estimated measurement noise structure.

Stochastic volatility and seasonality

The volatility of the implied soybean spot price is determined by the process v_t and the seasonal function $\nu(t)$. In particular, as inferred from the spot price dynamics in (1), or the risk-neutral analog in (4), $e^{2\nu(t)}v_t$ is the rate of variance on the soybean spot price and, hence, $e^{\nu(t)}\sqrt{v_t}$ is the spot price volatility. Since the seasonal function $\nu(t)$ captures any regular and systematic seasonal pattern in the spot price volatility, $\sqrt{v_t}$ can be interpreted as the seasonally adjusted spot price volatility. Under the risk-neutral measure, the long-run level for v_t is estimated to be $0.0679 \ (= \theta/\kappa)$ which implies a value of the long-run seasonal adjusted volatility of about 26.1% $(= \sqrt{0.0679})$. Furthermore, the volatility process exhibits a strong degree of mean-reversion since the parameter estimate of κ is significantly higher than zero.[17] The parameter estimate of $\kappa \ (= 2.2708)$ implies that the so-called half-life of shocks to the seasonal adjusted variance rate v_t is 0.305 years $(= \log 2/\kappa)$, under the risk-neutral probability measure. Basically, this

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[17]Throughout the discussion we will refer to an estimate being significantly differently from a given value if the given value is not within the estimate plus/minus two standard deviations (the usually applied asymptotic 95% confidence interval). The 95% confidence interval for κ is thus $[1.7400, 2.8016]$.

means that it takes about three to four months before a given shock in volatility (e.g. due to changes in harvest expectations or temporary changes in demand for soybeans) is expected to have levelled off to half of its immediate effect. Under the true probability measure, however, the degree of mean-reversion is even stronger and estimated to be 5.0754 ($= \kappa - \lambda_v \sigma_v$) which corresponds to a half-life of shocks of 0.137 years, or about one and a half months. Moreover, under the true probability measure the long-run level of the seasonal adjusted volatility, as reflected in the estimates in Table 3, is 17.4% ($= \sqrt{\theta/5.0754}$). These long-run volatility levels seem well in accordance with the overall average Black-Scholes implied volatility of 19.99%, as tabulated in the summary statistics in Table 2.

The seasonal variation in volatilities is captured by the seasonal function $\nu(t)$ defined in (8). This function is a sum of self-repeating trigonometric functional terms. In our estimation we have on beforehand limited the number of terms to two in order to keep the model sufficiently parsimonious for numerical optimization to be possible and, hence, in our implementation this function is described by a total of four parameters: ν_1, ν_1^*, ν_2, and ν_2^*.[18] All the seasonal parameters are estimated to be significantly different from zero. The seasonal pattern in volatilities that the parameter estimates reflect is illustrated in Figure 3 where the seasonal function $\nu(t)$ is plotted.

[INSERT FIGURE 3 ABOUT HERE]

The figure illustrates that the seasonal function has two local maxima and two local minima (though, this feature is much more clear for the convenience yield seasonal function $\alpha(t)$, as to be discussed below). The global maximum is achieved in late July about two months prior to the beginning of the US harvest (which occur from mid to late September and through October).[19] The flowering and pod filling of the plant, which is of crucial importance for the final seed yield and soybean production occur around this time, and this is thus a period where adverse weather conditions may significantly affect next year's US supply of soybeans. At this time of the year the volatility in soybeans is estimated to be 44% ($= e^{0.365} - 1$)

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[18] We have estimated the model including additional terms in the seasonal functions $\nu(t)$ and $\alpha(t)$ (the analog for the convenience yield process) but keeping all other parameters fixed at the estimated values in Table 3. This did not affect the estimated seasonal patterns substantially. Also, it may be noted that in a similar analysis with futures price observations and seasonality in convenience yields, Sørensen (2002) limits the relevant number of terms involved in the seasonal function to two based on a formal criteria, the Akaike Information Criteria.

[19] The global maximum of the seasonal function $\nu(t)$ is reached on July 21 where the function value is 0.365 while the global minimum is reached on March 23 where the function value is -0.249.

20

above "normal," where normal refers to the case when $\nu(t) = 0$. On the other hand, the soybean volatility is low in March prior to the US planting (which occur in May and June) and at its lowest the volatility is 22% ($= 1 - e^{-0.249}$) below normal. As discussed above, the seasonal adjusted volatility reverts over time to a long-run level of around 17.4%. The estimated seasonal pattern, however, suggests that the mean-reverting level is closer to 25.1% ($= 1.44 \times 17.4\%$) in July and 13.6% ($= 0.78 \times 17.4\%$) in March. These mean-reverting levels are slightly lower but of the same magnitude as the average volatilities tabulated in Table 2. Note in addition that in Table 2 the minimum average implied volatility is obtained for the options written on the May futures while the maximum average implied volatility is obtained for the option written on the September futures. But since these option contracts expire one month prior in April and August, respectively, and since the average times to maturity are 78.49 days and 59.04 days, these option contracts are precisely the contracts that cover the relevant periods through late March and late July, respectively.

Convenience yields and theory of storage

The parameters estimated for the convenience yield process in (2), (5), and (7) also reveal a clear seasonal pattern. In particular, the convenience yield process reverts to a level that depends on the season, as reflected in the seasonal function $\alpha(t)$. The degree of tendency towards this "normal" seasonal level is described by the parameter β. Note that while the convenience yield drift term has this simple interpretation under the risk neutral measure in (5), the dynamics are more complicated under the true probability measure since the convenience yield risk premium is assumed proportional to the time-varying volatility (including the seasonality feature of volatility). However, since the convenience yield risk premium parameter λ_δ is estimated close to (and not significantly different from) zero, we will continue the discussion as if $\lambda_\delta = 0$ or, equivalently, as if the risk-neutral dynamics and the true dynamics coincide. The estimate of β is 0.8145 which imply a half-life of unexpected shocks to the convenience yield process of about 0.851 years ($= \log 2/\beta$). Hence, the mean-reverting tendency is seemingly less strong than the similar feature in the volatility process, as discussed above. The seasonal function $\alpha(t)$ is described by a total of five parameters: α_0, α_1, α_1^*, α_2, and α_2^*. These are all estimated to be significantly different from zero. The parameter α_0 is a constant and without a seasonality feature the long-run level for the convenience yield process would thus be 0.0751 ($= \alpha_0/\beta$); we interpret this as the "normal" level of the convenience yield when adjusted for

seasonality.

The seasonal pattern of the mean-reverting level of the convenience yield is also illustrated in Figure 3. The seasonal variation in $\alpha(t)$ is very similar to the seasonal variation in $\nu(t)$, though amplitudes are different. However, it is much more clear that $\alpha(t)$ has two local maxima and two local minima.[20] Again, the global maximum is achieved in July about two months prior to the beginning of the US harvesting while the global minimum is achieved at the end of the US harvesting. While the US soybean production accounts for 50-60% of world soybean production other key producers are Argentina and, especially, Brazil which are both located in the Southern Hemisphere. Together these countries account for 20-30% of world soybean production. The Brazilian soybean harvesting takes place in March and April and the local maximum in Figure 3 is reached about two months earlier, while the local minimum is reached during the ending of the Brazilian soybean harvesting. The seasonal pattern in convenience yields is thus related to the supply of soybeans. When supplies are low convenience yields are high, and vice versa. This is consistent with the theory of storage which predicts a negative relationship between inventories and convenience yields. The basic idea is that the holding of physical inventory of soybeans gives rise to a convenience yield from being able to profit from temporary soybean supply shortages or keep a production process running *especially* when inventories are low. Furthermore, the parameter estimates suggest that the soybean spot price and the convenience yield are positively correlated with a correlation coefficient of 0.3979. Following the above kind of reasoning, this is also consistent with the observation that when supplies and inventories are scarce the equilibrium soybean spot price and the convenience yield are simultaneously high; and vice versa when supplies and inventories are plenty.

On the other hand, supporters of the theory of storage often view the convenience yield as an *option* to profit from temporary shortages of the particular commodity. This would suggest a positive correlation between the convenience yield and the volatility; however, this view is not supported by the parameter estimate of ρ_{23} which is not significantly different from zero and even estimated negatively.

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[20]The global maximum of the seasonal function $\alpha(t)$ is reached on July 15 where the function value is 0.746, while the global minimum is reached on October 30 where the function value is -0.723. A local maximum is reached on February 7, and a local minimum is reached on April 16.

Inverse leverage effects

The correlation coefficient between the soybean spot price and the soybean volatility is estimated to be 0.4078. This reflects that the volatility tends to be high when soybean spot prices are high and in which case supplies/inventories tend to be scarce and the arrival of new supply or demand information may have significant effect on prices. While this positive correlation thus seems natural and an intuitively appealing feature for commodities like soybeans, this is different from equity markets where spot prices and volatilities are usually estimated to be negatively correlated; in equity markets this phenomenon is often referred to as the *leverage* effect. Moreover, this feature is consistent with the observation that the volatility "smirk" in soybean option prices has opposite slope of that implied from equity options after October 1987, as described in our data description. Note also that the estimated positive correlation between the spot price and the volatility seems intuitively to support the observation that OTM call option should trade at relatively higher implied volatilities. Thus, in order to have a positive payoff on the call option, the underlying spot price must increase and, since the volatility in this case also tends to increase, this will strengthen the upward potential of the particular option (while keeping downside risk limited).

Risk premia and investment returns

As seen from Table 3, in all cases the risk premia parameters are estimated with large standard errors. Only the risk premium on volatility risk is significantly different from zero, and the implications with respect to the long-run level for the seasonal adjusted volatility under the risk-neutral measure, as reflected in market prices, and the true probability measure was discussed earlier on. While no strong inference can be drawn from the other risk premia estimates, it seems relevant to consider briefly the implications of the parameter point estimates for the implied excess returns on e.g. soybean futures positions. We will consider an investor who exposes himself to soybean price risk by going long in futures contracts and at the same time places an amount equal to the futures price in a safe bank account. Since the drift rate of the futures price is zero under the risk-neutral probability measure, the excess return on this investment policy is described by the drift rate of the futures price under the true probability measure. Using Ito's lemma on the futures price expression in equation (10), and by comparison with the risk-neutral dynamics of the futures price in (14), it is thus seen that

the expected rate of excess return on this investment position is

$$E_t\left[\frac{dF_t(T)}{F_t(T)}\right]\bigg/dt = \left(\lambda_P e^{\nu(t)} + \lambda_\delta \sigma_\delta e^{\nu(t)} D(t;T) + \lambda_v \sigma_v B(t;T)\right) v_t \qquad (29)$$

where $D(t;T)$ and $B(t,T)$ are described in (11) and (12). Assume that we are looking at the futures position at May 17 in a given year; this date is chosen as a point in time t where the seasonal component in volatility is zero (i.e. $\nu(t) = 0$). Also, assume that the spot price volatility is $\sqrt{v_t} = 0.20$ (i.e. $v_t = 0.04$). If the involved futures contract is very short (i.e. $t = T$), we have that $D(t;t) = B(t;t) = 0$ and that the position is only exposed to spot commodity price risk. Inserting the parameter point estimates in Table 3 into (29), the expected excess return in this case is negative and equal to -2.51%. Longer term futures prices are negatively related to the convenience yield (see equations (10) and (11)). Hence, since the risk premium on convenience yield risk is estimated negative, longer term futures contracts will have a higher, and possibly positive, overall risk premium when using the point estimates in Table 3. Also, as seen by numerical inspection, futures prices at the point estimates are negatively related to volatility. Since the risk premium on volatility risk is estimated negative, this also implies that longer term futures contracts will have a higher, and possibly positive, overall risk premium. However, when solving (11) and (12) for $D(t;T)$ and $B(t,T)$, and by insertion in (29), we find that the overall expected excess return is -1.95% if the involved futures contract has time to maturity equal to six months. Likewise, the overall expected excess return is -1.77% if the involved futures contract has time to maturity equal to one year.[21] In all the cases considered above, the risk premium on the long futures investment position is negative. Furthermore, since the futures price is expected to drift downwards this is also a case where the futures price is above the expected future spot price, and thus a situation usually referred to as "contango."[22] On the other hand, the only risk premia parameter which is significantly different from zero is λ_v and if we set $\lambda_P = \lambda_\delta = 0$, normal backwardation would be the situation. It is thus concluded that no strong inference can be drawn from the risk premia point estimates with respect to the quantitative implications as well as the qualitative implications for expected soybean investment returns.

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[21]By solving (11) and (12) numerically, we in the particular cases find that $B(t;t+0.50) = $ -0.03151, $D(t;t+0.50) = -0.41071$, $B(t;t+1.00) = $ -0.03475, and $D(t;t+1.00) = $ -0.68403.

[22]see, e.g. Hull (2000, p. 74) for this definition of contango versus normal backwardation.

Structure of measurement error covariance matrix

The measurement noise term added in the state space specification of the model is estimated to have a covariance matrix as implied by the stated correlation matrix of measurement errors in Table 3. Thus, using the one-to-one relationship between the tabulated correlation matrix (with the relevant estimated standard deviation in the diagonal), it is straightforward to obtain the relevant covariance matrix, and vice versa. Moreover, applicable standard errors on the covariance matrix can be obtained by using the result stated in section 2.4 that $N\hat{\Gamma}$ is Wishart distributed. In particular, having N=753 observation dates the standard errors on the standard deviation estimates in the diagonal can be obtained as 0.0364 ($= 1/\sqrt{753}$) times the true standard deviation parameter (which, as usual, may by approximated by the point estimate). Since we are only interested in the overall structure of the measurement errors, we have only tabulated the point estimates on the covariance matrix in Table 3, and, in addition, we have chosen to represent the covariance matrix in the form of the implied correlation matrix in order to focus on a normalized measure of how the model errors co-vary which potentially could give some indications and directions for how to improve the formal underlying futures and option pricing model.

The measurement noise error vector has dimension ten, as described in the data description. The first four entries in the vector are related to option prices in the following order: a short maturity in-the-money call option, a short maturity out-of-the-money call option, a medium maturity at-the-money call option, and a long maturity at-the-money call option. In general, the longer the maturity the less precise is the formal model in providing at-the-mark option prices, as can be inferred from the higher standard deviations on the measurement error terms for long maturity contracts. Also, it is evident that the measurement errors on the medium maturity and long maturity call options are positively correlated so that if the medium maturity call option is underpriced by the theoretical option pricing model then the long maturity call option contract will also tend to be underpriced; the point estimate of the relevant correlation coefficient is 0.7310. It is likewise seen that the measurement errors on the short in-the-money call option and out-of-the-money call option are negatively correlated. This could be consistent with, and thus indicative of, an even steeper empirical volatility surface than produced by the formal option pricing model with parameters as in Table 3 since a relatively flat model volatility curve would tend to produce undervalued in-the-money options whenever out-of-the money options are overvalued, and vice versa.

The last six entries in the measurement noise errors correspond to relative errors on futures prices with different times to maturity. Number five entry relates to the shortest time to maturity futures contract observed with measurement error, number six relates to the second shortest time to maturity futures contract observed with measurement error, and so on. In the estimation approach we have assumed that futures prices on a short maturity contract and the longest available futures contract are observed without measurement error. Hence, it is not surprising that short and long contracts are seemingly fitted better by the model than medium maturity contracts, as reflected in the higher standard deviations on the imposed measurement errors for medium maturity contracts. Note that the standard deviations on the individual error terms on futures prices are estimated of the same numerical magnitude as in Schwartz (1997), Schwartz and Smith (2000), and Sørensen (2002) who use only commodity futures data and Kalman filtering estimation approaches. For example, Sørensen (2002) provides results for soybean futures where the measurement errors are assumed uncorrelated and with identical standard deviation which he estimates to be 0.187. As apparent from the correlation matrix in Table 3, especially measurement errors for contracts that have maturities close to each other are highly positively correlated while the correlations decrease systematically for contracts having increasingly dispersed maturities.

Finally, the estimated correlations indicate slight co-variation between option prices and futures prices measurement errors. In particular, the measurement errors related to short maturity options have higher correlations with the measurement errors related to short maturity futures while the medium/long maturity option measurement errors are more correlated with measurement errors on futures prices of similar maturities. The correlations are in general positive, however, no clear pattern is evident since the measurement error related to short maturity in-the-money call options is negatively correlated with the measurement error on futures prices of similar maturity while the short maturity out-of-the-money call option measurement error at the same time is positively correlated with the same futures price measurement error.

Structural changes

In order to provide insights into whether the basic dynamic commodity price and contingent claims pricing model is robust over time, we have estimated the model for two equally long non-overlapping sub-periods. The relevant sub-samples thus cover the periods: October 1984 to November 1991 and November 1991 to March 1999. This investigation is especially mo-

tivated by qualitative considerations such as: Does seasonal patterns change systematically over time? And, are correlation structures qualitatively robust over time? The following analysis intentionally focuses on these conceptual issues rather than on quantitative econometric considerations with respect to diagnostic checking of the estimated model.

Seasonal patterns may vary over time due to especially production and storage innovations as well as changes in government policies with respect to intervention in grain markets. For example, Frechette (1997) analyzes the "flattering" of the soybean futures profile up through the 1970s where the Brazilian soybean production increased rapidly and changed the seasonal pattern in world soybean production. However, such dramatic changes in world soybean production patterns have not occurred in the time periods relevant for our investigation. Also, although US government policies prevailing in the 1970s and 1980s provided a floor under grain prices, soybeans had a relatively low support level and, thus, also no dramatic changes in government intervention policies with respect to soybeans seem to have occurred in the relevant estimation periods. In sum, it is not obvious beforehand what kind of parameter instability one would expect to find due to changes in soybean markets conditions when splitting the data into sub-samples.

The sub-period estimation results are tabulated in Table 4 and Table 5, respectively.

[INSERT TABLE 4 AND TABLE 5 ABOUT HERE]

We will be brief in our discussion of the differences between the estimates in the two sub-periods and will mainly point out and focus on differences in the persistence of shocks, seasonal patterns, and correlations of relevance to the above full-sample discussion of the relation to the theory of storage and the "inverse leverage effect."

It may first be noted that volatility shocks and convenience yield shocks seemingly are more persistent in the first sub-period than the second sub-period, as reflected in the estimates of the mean-reversion parameters κ and β. Thus, the estimates of κ indicate that, under the risk-neutral measure, the half-lives of shocks to volatility equal 0.940 years in the first sub-period and 0.160 years in the second sub-period (while the full-sample analog is 0.305 years, as discussed earlier). While this is true under the risk-neutral measure, as reflected in options and futures prices, the relevant mean-reversion parameter under the true probability measure is given by $\kappa - \lambda_v \sigma_v$. Hence, under the true probability measure the persistence of shocks to the volatility is much more similar across the two sub-periods, and relevant half-lives are 0.141 years and 0.093 years, respectively, as opposed to 0.137 years for the full-sample estimation

results.

In order to see more clearly the estimated seasonality patterns in volatilities across the two sub-periods, we have graphed the relevant seasonal components in Figure 4.

[INSERT FIGURE 4 ABOUT HERE]

The figure shows the estimated seasonal function $\nu(t)$ for the full-sample as well as the estimated seasonal function for the first sub-period, $\nu_1(t)$, and the estimated seasonal function for the second sub-period, $\nu_2(t)$. The estimated seasonal functions seem very similar with amplitudes of similar magnitude and with minima and maxima attained at about the same time of the year. The amplitude is slightly higher in the second sub-period and, if not simply due to randomness, this could be due to the above-mentioned price floors maintained by the US government programs up through the 1980s.

I Figure 5 we have similarly illustrated the seasonal patterns of the mean-reverting level of the convenience yield as estimated in the first sub-period, $\alpha_1(t)$, the second sub-period, $\alpha_2(t)$, as well as for the full sample, $\alpha(t)$.

[INSERT FIGURE 5 ABOUT HERE]

While the estimated functions attain minima and maxima at the same times of the year, the amplitudes seem very different. However, though the amplitude for the first sub-period estimation is much larger, it may be noted that the mean-reversion tendency is at the same time much lower than for the second sub-period, as reflected in the β estimates. Hence, the overall seasonal effect on the level of the convenience yield does not differ very substantially across the two sub-periods.

Finally, it is seen from Table 4 and Table 5 that the estimated correlation coefficients (ρ_{12}, ρ_{13}, and ρ_{23}) are very similar across the two sub-periods. Thus, for example, the conclusion that convenience yields do not behave like timing options is robust over time since in both sub-periods the correlation coefficient, ρ_{23}, between the convenience yield and the volatility state-variable is close to zero and, in fact, consistently estimated slightly negative. Also, the "inverse leverage effect" is robust over time since the correlation coefficient, ρ_{13}, between the spot price and the volatility state-variable is consistently estimated positively and of the same magnitude across the sub-periods.

28

4   Conclusions

In the paper we have set up a stochastic volatility model of commodity price behavior and estimated the model parameters using panel data of soybean futures prices and soybean option prices. The empirical analysis was based on an integrated time-series estimation approach which makes use of the information in both the time series characteristics of the involved soybean product prices as well as the information inherent in futures price structures and option price structures across different exercise prices and maturities.

Besides estimating model parameters, which among other things facilitate implementation of hedging strategies and related applications of the specific model, we find a clear seasonal pattern in both volatilities and convenience yields. Also, our results are to some extent consistent with the theory of storage in the sense that the estimations suggest that convenience yields tend to be low when inventories/supply is high, and vice versa. However, our results do not support the view that convenience yields are like timing options since convenience yields are not positively correlated with volatility. Furthermore, we find a positive correlation between spot prices and volatility, as opposed to the so-called leverage effect in equity markets. This, however, is consistent with a different sloping "smirk" in the implied volatility curve for soybean options.

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Appendix

Lemma 1 *Let $p_t = \log P_t$. The conditional means and conditional variances of the process $X_t = (p_t, \delta_t, v_t)$, described by the stochastic differential equation system (1), (2), and (3), are given by*

$$\mathrm{E}_t[X_{t+\Delta}] = \Phi(t+\Delta)\left(X_t + \int_t^{t+\Delta} \Phi^{-1}(u)l(u)\,du\right) \tag{30}$$

$$\mathrm{Var}_t[X_{t+\Delta}] = \Phi(t+\Delta)\left(\int_t^{t+\Delta} \mathrm{E}_t\left[v_u\right]\Phi^{-1}(u)\Sigma(u)\Phi^{-1}(u)'\,du\right)\Phi(t+\Delta)' \tag{31}$$

where

$$l(u) = \begin{pmatrix} r \\ \alpha(u) \\ \theta \end{pmatrix}, \quad \Sigma(u) = \begin{pmatrix} e^{\nu(u)} & \rho_{12}\sigma_\delta e^{\nu(u)} & \rho_{13}\sigma_v e^{\frac{1}{2}\nu(u)} \\ \rho_{12}\sigma_\delta e^{\nu(u)} & \sigma_\delta^2 e^{\nu(u)} & \rho_{23}\sigma_v\sigma_\delta e^{\frac{1}{2}\nu(u)} \\ \rho_{13}\sigma_v e^{\frac{1}{2}\nu(u)} & \rho_{23}\sigma_v\sigma_\delta e^{\frac{1}{2}\nu(u)} & \sigma_v^2 \end{pmatrix}$$

and where Φ is the unique solution to the matrix differential equation

$$\frac{d\Phi(u)}{du} = L(u)\Phi(u), \quad \Phi(t) = I \tag{32}$$

with

$$L(u) = \begin{pmatrix} 0 & -1 & -\frac{1}{2} + \lambda_P e^{\frac{1}{2}\nu(u)} \\ 0 & -\beta & \lambda_\delta\sigma_\delta e^{\frac{1}{2}\nu(u)} \\ 0 & 0 & -\kappa + \lambda_v\sigma_v \end{pmatrix}.$$

In particular, the conditional expectations entering the right hand side integral of (31) are described by (30) and given by $\mathrm{E}_t[v_u] = v_t e^{-\kappa(u-t)} + \frac{\theta}{\kappa}(1 - e^{-\kappa(u-t)})$.

Proof: The dynamics of $X_t = (p_t, \delta_t, v_t)$ can be written in matrix form as

$$dX_t = (L(t)X_t + l(t))\,dt + \sqrt{v_t}\,\sigma(t)\,dW_t \tag{33}$$

where $\sigma(t) = \mathrm{diag}[e^{\frac{1}{2}\nu(t)}, \sigma_\delta e^{\frac{1}{2}\nu(t)}, \sigma_v]$ is a diagonal matrix with the three arguments in the diagonal and zeros off the diagonal. By using Ito's lemma, and the definition of Φ in (32), it is seen that X can be characterized as the solution to ($s \geq t$)

$$X_s = \Phi(s)\left(X_t + \int_t^s \Phi^{-1}(u)l(u)\,du + \int_t^s \sqrt{v_u}\,\Phi^{-1}(u)\sigma(u)\,dW_u\right). \tag{34}$$

Note that $\int_t^s \sqrt{v_u}\,\Phi^{-1}(u)\sigma(u)\,dW_u$, $s \geq t$, is a martingale starting at zero at time t. Hence, by taking conditional expectations in (34) (and setting $s = t + \Delta$), one obtains (30). Furthermore, since the two first terms on the right hand side of (34) are $\mathcal{F}_t$-measurable, the result in (31) follows by an application of e.g. Karatzas and Shreve (1991), Proposition 2.17, p. 144. ∎

Table 1: Summary statistics for soybean futures, 1984-1999.

		Days to maturity	Settlement prices	
Contracts	Number of observations	Mean	Mean	Std. deviation
All	6024	209.53	627.49	86.08
Grouped into time to maturity:				
1. Maturity	753	28.60	622.86	97.13
2. Maturity	753	81.75	624.82	95.24
3. Maturity	753	132.52	626.84	92.24
4. Maturity	753	183.57	627.88	87.92
5. Maturity	753	234.00	628.75	83.81
6. Maturity	753	284.36	628.99	80.09
7. Maturity	753	337.77	629.18	76.38
8. Maturity	753	393.63	630.59	72.68
Grouped into expiration months:				
January	893	215.46	622.51	81.96
March	877	210.84	628.02	85.24
May	872	212.17	634.03	87.13
July	883	214.26	638.37	91.18
August	812	197.62	632.63	90.16
September	808	197.84	621.02	84.49
November	879	216.55	615.77	80.09

Table 2: Summary statistics for soybean call options, 1984-1999.

| Contracts | Number of observations | Days to maturity | Implied volatilities | |
		Mean	Mean	Std. deviation
All	3765	71.95	19.99	6.60
Grouped into time to maturity and moneyness:				
1. Maturity – ITM*	753	40.50	18.88	6.53
1. Maturity – ATM*	753	40.50	19.61	6.99
1. Maturity – OTM*	753	40.50	21.58	7.19
2. Maturity – ATM*	753	93.71	19.81	6.38
3. Maturity – ATM*	753	144.69	20.06	5.51
Grouped into expiration months:				
January	548	66.65	18.13	4.72
March	636	77.32	17.13	3.72
May	645	78.49	16.71	4.17
July	631	81.50	19.40	4.84
August	431	71.85	25.17	8.40
September	374	59.04	26.28	8.76
November	500	60.17	21.47	5.74

* The average call prices are: 30.58 for 1. Maturity – ITM, 15.68 for 1. Maturity – ATM, 8.31 for 1. Maturity – OTM, 25.25 for 2. Maturity – ATM, and 31.37 for 3. Maturity – ATM.

Table 3: Parameter estimates for soybean contracts, Full sample: Oct. 1984 – Mar. 1999.

Volatility parameters:	κ: 2.2708 (0.2654)	θ: 0.1542 (0.0085)	σ_v: 0.7414 (0.0299)	ν_1: -0.4096 (0.0387)	ν_1^*: -0.2626 (0.0483)	ν_2: 0.2250 (0.0229)	ν_2^*: 0.1293 (0.0223)
Convenience yield parameters:	β: 0.8145 (0.2100)	σ_δ: 0.9933 (0.1345)	α_0: 0.0612 (0.0086)	α_1: -0.2683 (0.0276)	α_1^*: 0.1915 (0.0130)	α_2: 0.2338 (0.0255)	α_2^*: 0.2689 (0.0600)
Correlations and risk premia parameters:	ρ_{12}: 0.3979 (0.0110)	ρ_{13}: 0.4078 (0.0461)	ρ_{23}: -0.0784 (0.0772)	λ_P: -0.6274 (1.3001)	λ_δ: -0.1282 (1.2286)	λ_v: -3.7829 (1.5610)	

Correlation matrix of measurement errors (standard deviation estimates in diagonal):

$$
\begin{pmatrix}
0.9606 & & & & & & & & & \\
-0.3075 & 0.8040 & & & & & & & & \\
0.0489 & 0.2150 & 2.4068 & & & & & & & \\
0.0164 & 0.1373 & 0.7310 & 3.4679 & & & & & & \\
-0.2480 & 0.1719 & 0.1118 & 0.1053 & 0.0180 & & & & & \\
-0.1871 & 0.1153 & 0.1435 & 0.1549 & 0.8947 & 0.0265 & & & & \\
-0.1236 & 0.0850 & 0.2187 & 0.2398 & 0.7175 & 0.9137 & 0.0279 & & & \\
-0.0638 & 0.0566 & 0.2922 & 0.3084 & 0.5302 & 0.7388 & 0.9238 & 0.0254 & & \\
-0.0688 & 0.0755 & 0.2306 & 0.2765 & 0.3838 & 0.5751 & 0.7466 & 0.8870 & 0.0193 & \\
-0.1199 & 0.0808 & 0.1573 & 0.2042 & 0.2053 & 0.3663 & 0.4892 & 0.6014 & 0.8191 & 0.0116
\end{pmatrix}
$$

Log-likelihood function value: **8468.59**

Notes: Heteroscedasticity-consistent standard errors from White (1980) in parentheses.

Table 4: Parameter estimates for soybean contracts, 1. half: Oct. 1984 – Nov. 1991.

Volatility parameters:	κ:	0.7373 (0.5038)	θ:	0.1103 (0.0110)	σ_v:	0.7626 (0.0282)	ν_1:	-0.3825 (0.0371)	ν_1^*:	-0.1448 (0.0423)	ν_2:	0.2461 (0.0332)	ν_2^*:	0.0861 (0.0186)
Convenience yield parameters:	β:	0.3186 (0.1921)	σ_δ:	0.7385 (0.0887)	α_0:	0.0872 (0.0090)	α_1:	-0.2103 (0.0336)	α_1^*:	0.1571 (0.0139)	α_2:	0.2300 (0.0335)	α_2^*:	0.2454 (0.0516)
Correlations and risk premia parameters:	ρ_{12}:	0.3549 (0.0150)	ρ_{13}:	0.4989 (0.1052)	ρ_{23}:	-0.0638 (0.0836)	λ_P:	-3.0441 (2.0377)	λ_δ:	-1.9042 (1.6564)	λ_v:	-5.4792 (1.8105)		

Correlation matrix of measurement errors (standard deviation estimates in diagonal):

$$
\begin{pmatrix}
1.0093 \\
-0.1240 & 0.8056 \\
0.1506 & 0.2121 & 2.6630 \\
0.0842 & 0.1804 & 0.6474 & 3.6959 \\
-0.2235 & 0.1148 & 0.1393 & 0.0651 & 0.0142 \\
-0.1720 & 0.0662 & 0.0958 & 0.0409 & 0.9347 & 0.0219 \\
-0.1396 & 0.0662 & 0.1087 & 0.0256 & 0.8492 & 0.9599 & 0.0251 \\
-0.1216 & 0.0656 & 0.1563 & 0.0505 & 0.7250 & 0.8476 & 0.9381 & 0.0244 \\
-0.1333 & 0.0901 & 0.0660 & 0.0475 & 0.5411 & 0.6404 & 0.7345 & 0.8644 & 0.0206 \\
-0.1787 & 0.0821 & -0.0131 & 0.0266 & 0.3112 & 0.3578 & 0.4195 & 0.5321 & 0.8027 & 0.0129
\end{pmatrix}
$$

Log-likelihood function value: 4205.60

Notes: Heteroscedasticity-consistent standard errors from White (1980) in parentheses.

Table 5: Parameter estimates for soybean contracts, 2. half: Nov. 1991 – Mar. 1999.

Volatility parameters:	κ: 4.3447 (0.3286)	θ: 0.2572 (0.0144)	σ_v: 0.8736 (0.0351)	ν_1: -0.4185 (0.0190)	ν_1^*: -0.2587 (0.0268)	ν_2: 0.2193 (0.0108) $\quad \nu_2^*$: 0.1421 (0.0103)
Convenience yield parameters:	β: 1.1901 (0.0794)	σ_δ: 1.1621 (0.1401)	α_0: 0.0328 (0.0047)	α_1: -0.1875 (0.0068)	α_1^*: 0.1777 (0.0073)	α_2: 0.1489 (0.0039) $\quad \alpha_2^*$: 0.1879 (0.0035)
Correlations and risk premia parameters:	ρ_{12}: 0.3972 (0.0115)	ρ_{13}: 0.5085 (0.0361)	ρ_{23}: -0.0409 (0.1197)	λ_P: -0.7322 (2.1073)	λ_δ: -1.8687 (1.9719)	λ_v: -3.5911 (2.8661)

Correlation matrix of measurement errors (standard deviation estimates in diagonal):

$$
\begin{pmatrix}
0.9013 & & & & & & & & & \\
-0.4926 & 0.7990 & & & & & & & & \\
-0.1575 & 0.3329 & 1.9643 & & & & & & & \\
-0.1543 & 0.2437 & 0.7839 & 2.5335 & & & & & & \\
-0.2784 & 0.1928 & 0.0113 & 0.0066 & 0.0213 & & & & & \\
-0.2068 & 0.1145 & 0.0766 & 0.0743 & 0.8746 & 0.0303 & & & & \\
-0.1114 & 0.0506 & 0.1487 & 0.1832 & 0.6274 & 0.8831 & 0.0297 & & & \\
-0.0207 & 0.0071 & 0.2015 & 0.2410 & 0.3607 & 0.6371 & 0.8998 & 0.0254 & & \\
-0.0263 & 0.0403 & 0.1731 & 0.1870 & 0.1877 & 0.4560 & 0.6987 & 0.8859 & 0.0178 & \\
-0.0891 & 0.0948 & 0.1543 & 0.1252 & 0.0098 & 0.2526 & 0.4272 & 0.5733 & 0.8113 & 0.0107
\end{pmatrix}
$$

Log-likelihood function value: 4642.37

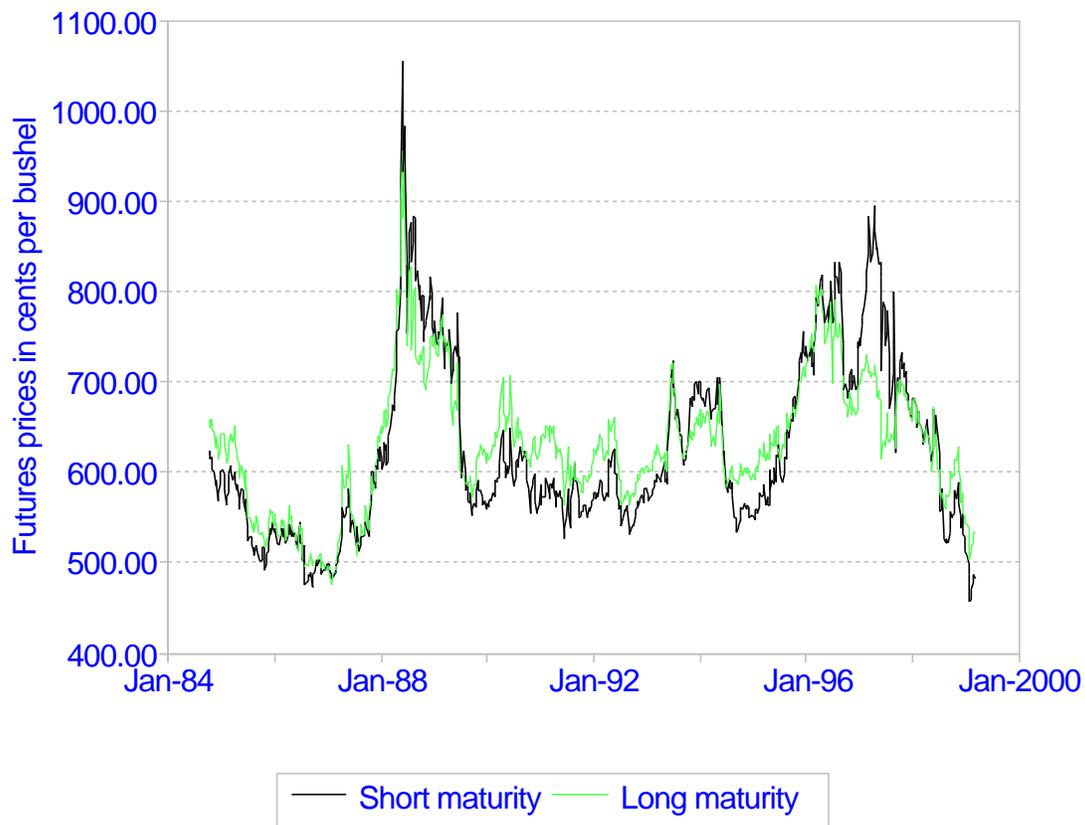*Notes*: Heteroscedasticity-consistent standard errors from White (1980) in parentheses.

Figure 1: **Time series of soybean futures prices.** The figure displays the futures prices on the contracts with the shortest (1. Maturity) time to maturity and the longest (8. Maturity) time to maturity over the full sample period from October 1984 to March 1999.
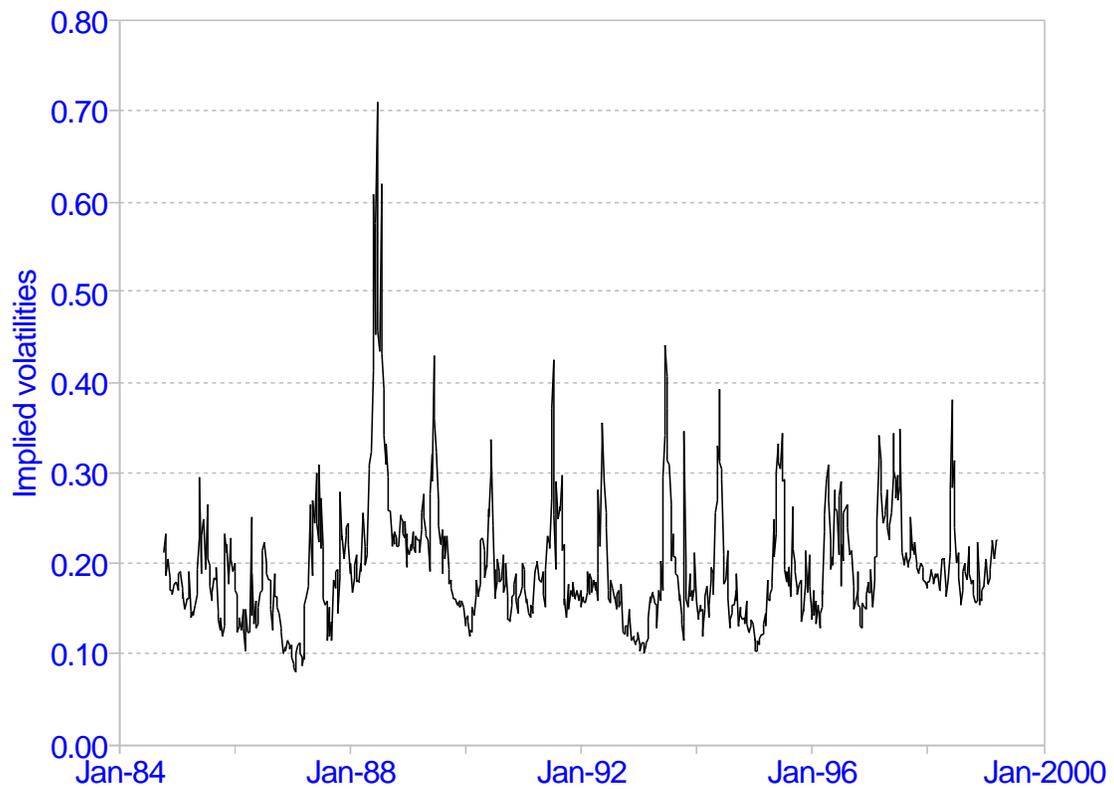
Figure 2: **Time series of implied volatilities on soybean call options.** The figure displays the implied volatilities on the at-the-money call option contracts with the shortest (1. Maturity) time to maturity over the full sample period from October 1984 to March 1999.
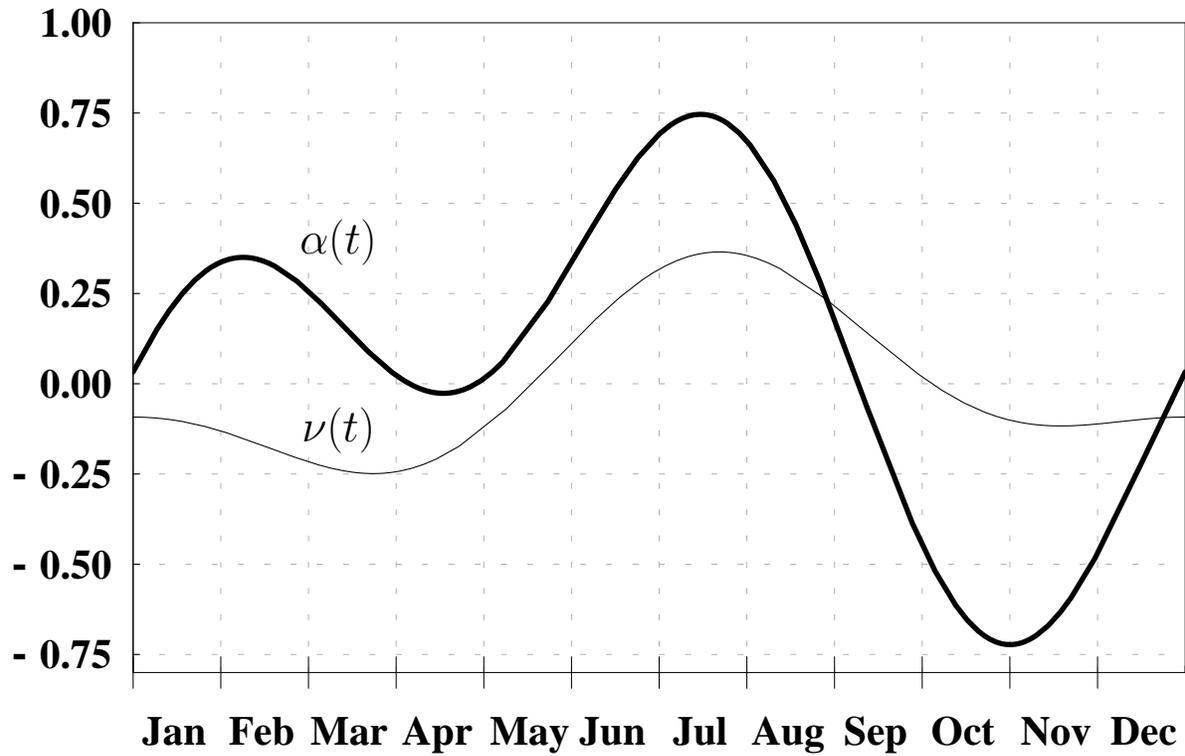
Figure 3: **The seasonal functions** $\alpha(t)$ **and** $\nu(t)$**.** The figure displays the target seasonal level of convenience yields, $\alpha(t)$, and the seasonal component in the soybean price volatility, $\nu(t)$.
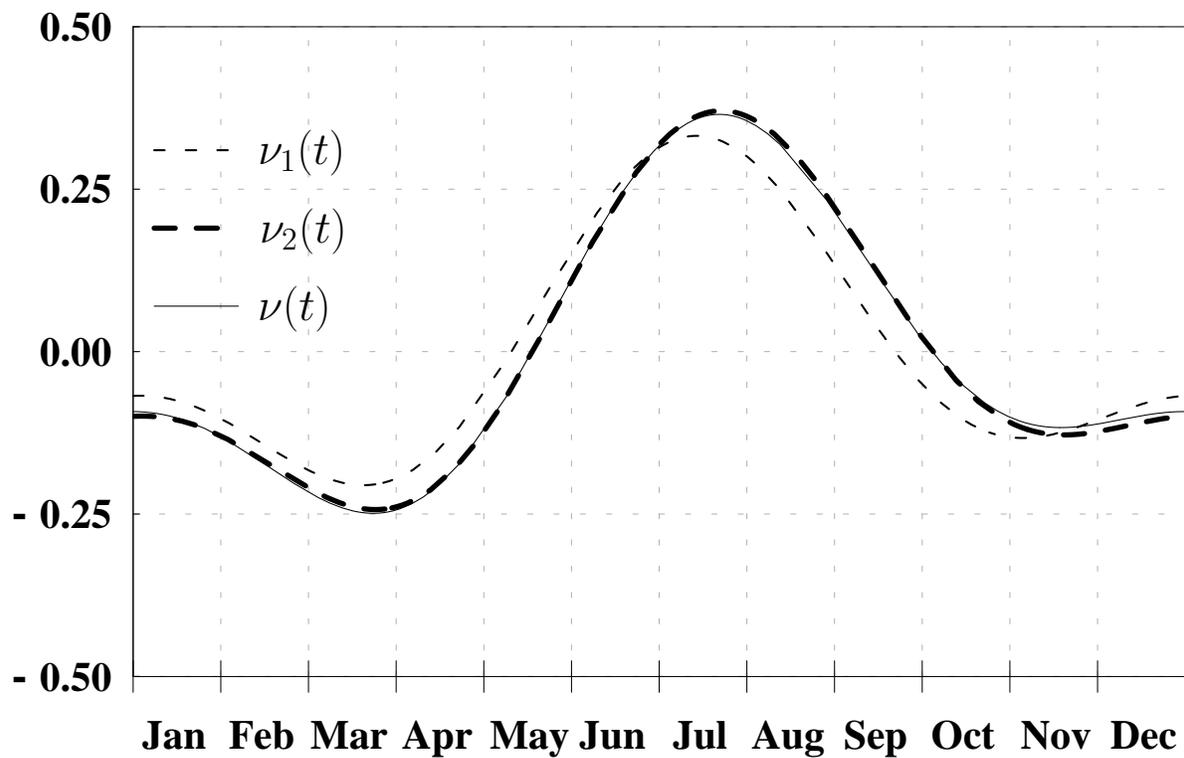
Figure 4: **The three volatility seasonal functions** $\nu(t)$**,** $\nu_1(t)$**, and** $\nu_2(t)$**.** The figure displays the seasonal component in the soybean price volatility for the full sample, $\nu(t)$, the first sub-period, $\nu_1(t)$, and the second sub-period, $\nu_2(t)$
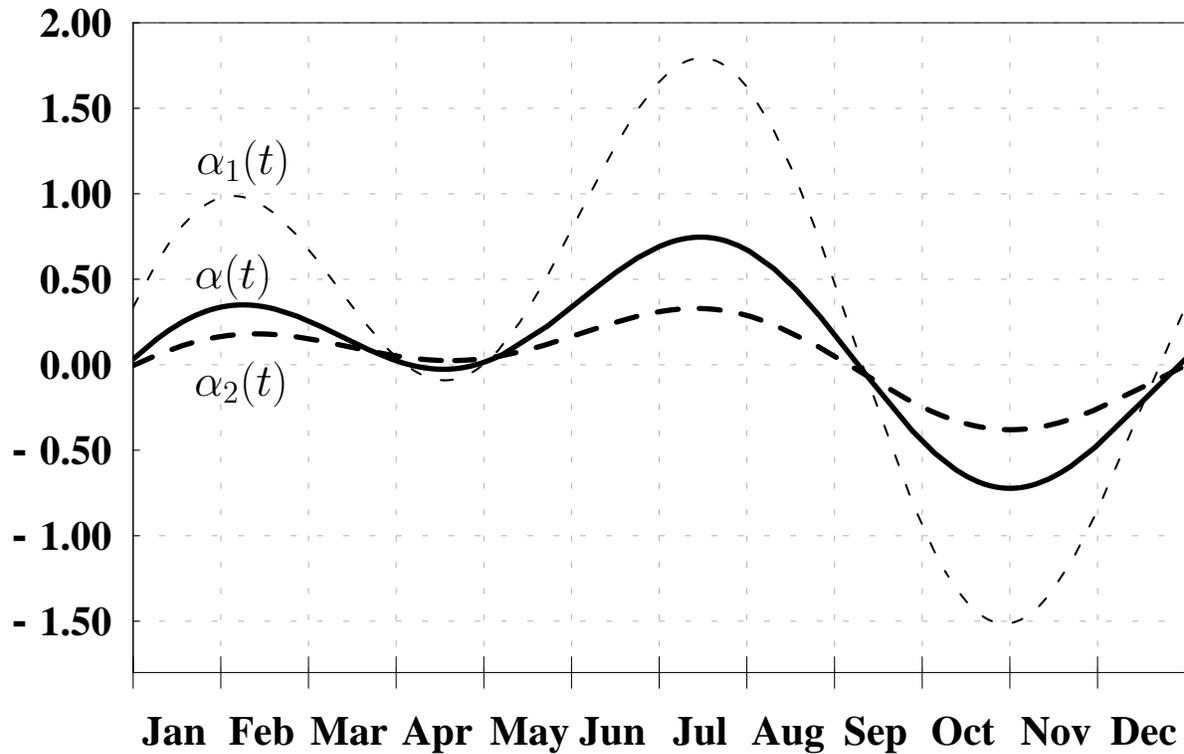
Figure 5: **The three convenience yield seasonal functions** $\alpha(t)$**,** $\alpha_1(t)$**, and** $\alpha_2(t)$**.** The figure displays the relative seasonal level of convenience yields for the full sample, $\alpha(t)$, the first sub-period, $\alpha_1(t)$, and the second sub-period, $\alpha_2(t)$