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Debt, Pensions and Fiscal Policy under Dynamic Inefficiency and Demographic Change

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Abstract

New evidence suggests that the criterion for dynamic efficiency is not verified for any advanced economy and that over-accumulation of capital may in fact be an issue. This thesis uses a 2-period overlapping generations model with public debt and an endogenous interest rate to study how dynamic inefficiency should optimally be dealt with. The analyses reveal that a pay-as-you-go pension scheme and public debt mitigate dynamic inefficiency. A specific policy mix is required to guide a dynamically inefficient economy characterised by a high debt level towards the welfare maximising Golden Rule growth path. It is demonstrated that this policy mix must comprise of an increase in pay-as-you-go pension contributions and a reduction in the governmental primary budget deficit. In face of the ongoing increase in the dependency ratio, economies suffering from over-accumulation of capital achieve higher welfare levels enforcing a defined benefit pay-as-you-go pension scheme rather than a defined contribution pay-as-you-go pension scheme.

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1 Introduction

Demographic change and dynamic efficiency are two important macroeconomic issues. Demographic change has become a highly debated topic. Advancements in medical and related sciences have led to an increase in longevity. At the same time fertility rates are falling significantly. This evolution will continue and inevitably result in a lower population growth rate and an aging population. Hence, there is a necessity for a holistic approach to foresee the economic effects and to resolve them in an appropriate manner.

Dynamic efficiency in turn has only played a minor role in recent debate. It is often assumed that raising investment is always good for the economy because it leads to higher levels of output in the long run. But a higher level of output does not necessarily increase consumption as more capital also requires more investment to maintain the same capital-output ratio. Therefore, a certain optimal level of capital exists that maximises consumption and welfare. This level is called the Golden Rule level of capital. If an economy accumulates more than the Golden Rule level of capital, consumption must decrease to support the reproduction of capital. In such a case an economy is over-accumulating capital and dynamically inefficient. In a seminal paper, Diamond (1965) showed that in his overlapping generations (OLG) model the Golden Rule level of capital is reached whenever the growth rate of the economy n equals the interest rate r. Over-accumulation occurs if the interest rate r is smaller than the growth rate of the economy n.[1]

Whether an economy is dynamically inefficient is an empirical question. Abel et al. (1989) make use of the fact that in the OLG model established by Diamond (1965) the growth rate of capital equals the growth rate of the economy n in a steady-state. Their test of dynamic efficiency therefore compares the difference between capital income and investments. Using data up to 1985 the results of their test suggests that the United States as well as six other developed countries are dynamically efficient. However, using updated data on mixed income and land rents Geerolf (2013) finds that the criterion for dynamic efficiency is not verified for any advanced economy and that Japan and South Korea unambiguously suffer from capital over-accumulation. Furthermore, by simply comparing the world's interest rates to the growth rate of the global

[1] See more on the Golden Rule level of capital in Ramsey (1928), Phelps (1961) and Cass (1965). Note that in the model by Diamond (1965) the growth rate of the population equals the growth rate of the economy. This is the case, because the efficiency of workers is assumed to remain constant over time.

real gross domestic product (GDP), the world economy appears to be dynamically inefficient (IMF, 2016).[2] This new evidence for dynamic inefficiency and the ongoing demographic change has caught my attention and motivated me to study these important topics further.

The overarching theme of this thesis is to examine the role of debt, pay-as-you-go pension schemes, and fiscal policy under dynamic inefficiency and demographic change. More specifically, it aims to identify optimal fiscal policies for an economy suffering from dynamic inefficiency and threatened by demographic change. The framework used is an OLG model that builds up on work established by Diamond (1965) but extends it with advancements to reflect new findings. In particular, the basic Diamond OLG model assumes a constant debt per capita level and does not model government consumption explicitly. Consequently, the model is one-dimensional with the capital stock being the only endogenous variable. Unlike the Diamond framework, the model used in this thesis incorporates government consumption explicitly, leading to a two-dimensional approach, in which debt and capital are endogenous. This allows the government to run constant primary deficits and, at the same time, makes it possible to derive optimal policy adjustments.

From here the road map of the thesis is as follows. Chapter 2 introduces the OLG model of this thesis. Assuming a Cobb-Douglas production function and a logarithmic utility function concrete steady-states are established for specific values of primary budget deficits run by the government. The focus then lays on the role of debt and fiscal policy in defining the stability and efficiency of the steady-states. Chapter 3 builds up on the previous chapter by extending the model by a pay-as-you-go pension scheme. By modelling pay-as-you-go pension contributions proportional to wages, explicit steady-states and corresponding stabilities can be identified. The efficiency of these steady-states in the model with a pay-as-you-go pension scheme are then compared to the efficiency of the steady-states in the model without such a scheme. Subsequently, it is examined whether a sustainable policy mix, which results in welfare gains, exists. In Chapter 4 demographic change is simulated by an increase in the dependency ratio to study its effects on the economy. A comparison between a defined contribution and a defined benefit pay-as-you-go scheme is made. In addition, optimal fiscal policy adjustment is outlined following an expenditure shock occurring simultaneously with an increase in the dependency ratio. Finally, Chapter 5 concludes and offers suggestions for future research.

[2]See Table A15 from the World Economic Outlook 2016 (IMF, 2016) to compare the world's real GDP growth rate to the world interest rate.

2 The OLG Model

2.1 Introduction

This chapter introduces the overlapping generations (OLG) model used throughout this thesis. The basic structure of the OLG model is the one formulated by Diamond (1965) based on earlier insights of Samuelson (1958). Some extensions established by De La Croix and Michel (2002) and Farmer and Schelnast (2013) are then added to advance the model. The framework allows modelling a large and closed economy with physical capital accumulation over time. This proves to be an ideal tool to study intertemporal aspects of resource allocation, fiscal policy and welfare considerations.

The model is developed in discrete time. People in the model have a finite life and live for two periods. Therefore, at each point in time two generations exist - a generation that is working and receives employment income, and a generation of retirees that are living on savings. There are no bequest motives but an infinite number of generations with finite life. This has two important implications. First, not considering the well-being of future generations when making decisions may lead to inefficient allocation of resources in equilibrium. Pareto optimality is not necessarily given and may be improved by transferring resources between cohorts. Second, the Ricardian Equivalence breaks down in an economy with no bequest motives. This Equivalence was put in a theoretical framework by Barro (1974) and states that the government cannot affect consumption decisions by fiscal policy as forward looking agents internalize the governmental budget constraint. Consequently, not allowing for bequest motives has the advantage that fiscal policy and public debt can directly affect the welfare of the population.

In the basic model by Diamond (1965) debt per working individual is kept constant, implying a certain fixed tax schedule. Debt per worker being constant over time the only endogenous variable in the model is capital. Government consumption is not explicitly modelled, which leaves the government with tax rate as the only fiscal tool. The extension by Farmer and Schelnast (2013) models public consumption proportional to national income. Controlling public consumption expenditure gives the government an additional fiscal tool. This allows the government to run an unbalanced budget in equilibrium and makes the model two-dimensional with both debt and capital being endogenous.

The main focus of the chapter is to establish long run equilibria or so-called steady-states, in which the two endogenous variables are constant over time. It is shown that the primary budget deficit of the government is the main determinant for the existence and the stability of steady-states. A graphical illustration of the model is provided and the role of public debt in defining the efficiency of the steady-states is examined.

2.2 Model Specification

As already mentioned, the model is set up in discrete time. The initial condition of the first period $t = 0$ reflects the history of the economy. From there the economy develops over time up to period $t = \infty$. Given the two-period lifetime of individuals the length of one period is understood to be approximately 30 years. There are three types of goods in every period: capital, labour, and a physical good produced using the first two goods. The physical good existing in each period t can either be consumed or saved.

2.2.1 Population

In each period t there are representative households of a young cohort that are working as well as of an old cohort that is retired. Each young individual is endowed with one unit of labour to be supplied inelastically to firms. The young cohort receives employment income w_t, which is taxed by an income tax rate τ^w with $\tau^w \in [0, 1)$. The after-tax income can then be used for consumption c_t or savings s_t. Thus, the young individuals' budget constraint can be written as:

$$c_t = w_t(1 - \tau^w) - s_t \tag{2.1}$$

Members of the old cohort have stopped working and their consumption d_{t+1} equals their amount of savings and any interest accrued. Life of all individuals is finite and ends after two periods. Generations are not connected by bequest motives, which means individuals do not explicitly consider future generations. This implies that the older generation consumes all of its savings before death. It is assumed that individuals have perfect foresight, so that there is no uncertainty about future endowment and the interest rate r_{t+1}. The old individuals' budget constraint therefore yields:

$$d_{t+1} = s_t(1 + r_{t+1}) \tag{2.2}$$

Combining (2.1) with (2.2) leads to the individuals' budget constraint over the whole lifetime, where the present value of consumption has to equal the disposable income:

$$c_t + \frac{d_{t+1}}{1 + r_{t+1}} = w_t(1 - \tau^w) \tag{2.3}$$

The indiviudals' preferences are defined over their consumption bundle (c_t, d_{t+1}). The welfare is represented by a life cycle utility function:

$$U(c_t, d_{t+1}) = u(c_t) + \beta u(d_{t+1}), \tag{2.4}$$

where β is the psychological discount factor and it is assumed that $\beta \in (0, 1]$. The greater β the higher the individuals' preference to postpone consumption to the second period of their lifetime. The utility function u is assumed to be twice continuously differentiable, strictly increasing and concave:

Assumption 1 *For all $c > 0$, one has $u'(c) > 0$, $u''(c) < 0$, and* $\lim_{c \to 0} u(c) = +\infty$

Assuming a utility function of this fashion implies that there is no satiation but decreasing marginal utilities. Infinite marginal utility of zero consumption leads to an individuals' consumption choice which is always positive when maximising life-cycle utility given that the disposable income is positive.

The population of the young and the old cohort are defined as N_t and N_{t-1}, respectively. The total population at time $t > 1$ is therefore simply the sum of N_t and N_{t_1}. At time $t = 0$ there are in addition to the young generation N_0 also the old generation N_{-1} with $N_{-1} > 0$. All members of the generation $t = -1$ own the fraction s_{-1} of the capital stock K_0 in $t = 0$ such that $s_{-1} = K_0/N_{-1}$. Both the number of indiviuals of each generation as well as the total population N_t and N_{t-1} grow at a constant rate g^N such that $G^N \equiv 1 + g^N$. This implies the following relation:

$$N_t = G^N N_{t-1} = (1 + g^N)N_{t-1}$$

Since $g^N \in (-1, \infty)$ the modelled economy may also shrink at a rate between -1 and 0.

2.2.2 Savings Decision

Individuals in the model maximise life-cycle utility. This maximisation problem includes the choice of consumption today and consumption tomorrow. To get an explicit expression of the saving function logarithmic utilities are assumed in this thesis such that $u(c) \equiv log(c)$. Logarithmic utilities fulfill all properties stated about the utility function in Assumption 1. The household's maximisation problem can then be written as:

$$\max_{c_t, d_{t+1}} U = log(c_t^1) + \beta log(d_{t+1})$$

subject to

$$c_t + \frac{d_{t+1}}{1 + r_{t+1}} = w_t(1 - \tau^w)$$

$$c_t > 0, d_{t+1} > 0$$

Solving this maximisation problem leads to the following first order condition:

$$\frac{d_{t+1}}{c_t} = \beta(1 + r_{t+1}) \tag{2.5}$$

From the first order condition one can infer that consumption growth increases the more patient (high β) individuals are and the higher the rate of return. Combining the present value of lifetime consumption (2.3) with the first order condition (2.5) leads to optimal first period consumption c_t:

$$c_t = \frac{1}{1 + \beta}(1 - \tau^w)w_t \tag{2.6}$$

As $\frac{1}{(1+\beta)} > 0$, Equation (2.6) reveals that assuming logarithmic utilities consumption goods are normal goods. Consumption demand increases in disposable income. Combining optimal first period consumption (2.6) with first period budget constraint of the household (2.1) yields optimal savings s_t:

$$s_t = \frac{\beta}{1 + \beta}(1 - \tau^w)w_t \tag{2.7}$$

In general, an increase in the interest rate has two counteracting effects on the savings decision called income effect and substitution effect. The substitution effect denotes the fact that an increase in the interest rate makes first period consumption c_t more expensive. Individuals therefore tend to postpone consumption to the second period, which raises savings s_t. The income effect reflects the fact that a higher interest rate increases relative income. As first period consumption is a normal good, the income effect leads to an increase in first period consumption c_t and decreases savings s_t. Looking at Equation (2.7), it becomes clear that optimal savings are independent of the interest rate. Thus, logarithmic utilities represent a special case where income effect and substitution effect offset each other. The expression $\beta/(1+\beta)$ defines what fraction of disposable income is saved for old age consumption d_{t+1}. The higher the patience factor β the greater the share of disposable income that is saved.

2.2.3 Technology and Firms

The technology available to the firms can be denoted by a macroeconomic production function:

$$Y_t = F(K_t, A_t),$$

where Y_t denotes the gross national product (GDP) in period t. K_t stands for the capital stock in the beginning of period t and A_t denotes the number of productivity-weighted workers of the economy. A_t is defined as the product of labour force L_t and efficiency of the labour force a_t such that $A_t = a_t L_t$. The GDP depends positively on both factors K_t and A_t. A growing labour force has the same effect on the GDP as growing technological progress (an increasing a_t). Both increase A_t and therefore the GDP. The efficiency of the labour force a_t grows with a growth rate g^a such that $G^a \equiv 1 + g^a > 1$ and $a_{t+1} = G^a a_t$ with $a_0 > 0$. Labour market clearing requires that the following condition holds: $N_t = L_t, \forall t$. Consequently, the labour force L_t grows at the same rate as the population N_t, which is g^N such that $G^N \equiv 1 + g^N > 1$ and $L_{t+1} = G^N L_t$ with $L_0 > 0$. Combining the population growth factor G^N with the technological growth factor G^a yields: $G^N G^a \equiv (1+n)$, where n is the natural growth rate.

The production is assumed to be homogenous of degree 1 and therefore exhibits constant returns to scale: $F(\lambda K_t, \lambda A_t) = \lambda F(K_t, A_t)$ for all $\lambda > 0$. Consequently, the producing sector can be modelled as a single firm that uses the whole capital stock K_t of the economy for production or as a multi-firm environment, where more than one firm produce goods using fractions of K_t. The homogenity of degree 1 also allows us to express the production function of one variable $k_t = K_t/A_t$:

$$Y_t = F(K_t, A_t) = A_t F\left(\frac{K_t}{A_t}, 1\right) = A_t f(k_t),$$

where $f(k_t)$ denotes the production function in its intensive form. $f(k_t)$ is defined on the set of strictly positive numbers $\mathbb{R}_{++}$ and assumed to be twice differentiable. Furthermore the following assumptions about $f(k_t)$ are made:

Assumption 2 *For all* $k_t > 0$

$$\begin{aligned} f(k_t) &> 0 \\ f'(k_t) &> 0 \\ f''(k_t) &< 0 \\ f(0) &= 0 \\ \lim_{k_t \to 0} f'(k_t) &= +\infty \\ \lim_{k_t \to \infty} f'(k_t) &< 1 \end{aligned}$$

The conditions $f'(k_t) > 0$ and $f''(k_t) < 0$ define the concavity of the function that leads to decreasing marginal returns to k_t. $f(0) = 0$ ensures that there can only be production with capital.

It has been shown that GDP Y_t is defined by the production function, which itself depends on capital K_t and labour L_t. It is therefore interesting to know where supply and demand of these two factors come from. The supply of capital is provided by households and lays in their investment decision. The investment decision in turn is determined by savings and public debt. In the model, households save by first buying all issued government bonds B_{t+1} before they invest in firms' capital K_{t+1}. Aggregating savings over all individuals yields the following expression for capital investment I_t that defines the supply of capital:

$$N_t s_t = I_t = K_{t+1} - (1-\delta)K_t - B_{t+1} \tag{2.8}$$

The demand for capital K_t and labour L_t is set by the firms that use it for production. The supply for labour is again offered by households. Summing up, a perfectly competitive firm sector produces by hiring capital K_t from currently old age individuals and by hiring labour L_t from young individuals. Thus, firms' maximise profits by choosing the level of labour L_t and capital K_t:

$$\max_{L_t K_t} \pi = F(K_t, A_t) - w_t L_t - (r_t + \delta) K_t$$

with $(r_t + \delta)$ being the rental rate of capital z_t, which is just the sum of the real interest rate and the depreciation rate. The no-arbitrage condition implies that the real interest rate r_t, representing the return on savings, has to equal the rental rate of capital z_t minus the deprecation rate δ, representing the return on investment in physical capital. This leads us to the two first order conditions of the firms maximisation problem:

$$F_L^{'}(K_t, A_t) = w_t \tag{2.9}$$

$$F_K^{'}(K_t, A_t) = z_t = (r_t + \delta) \tag{2.10}$$

The first order conditions show that the cost of labour w_t equals the marginal product of labour $F_L^{'}(K_t, A_t)$ and that the rental cost of capital z_t equals the marginal product of capital $F_K^{'}(K_t, A_t)$. Now, two things have to be decided to establish explicit motion dynamics of capital and debt. First, the depreciation rate δ is set to 1. This means that over one period the whole capital stock has to be replaced, which is a realistic assumption given that one period can be understood as 30 years. Hence, a depreciation rate of 1 indicates a full depreciation of the capital stock over one period. Second, a specific production function that fulfills all the properties mentioned above has to be assumed. This function is the Cobb-Douglas production function which take on the following form:

$$Y_t = F(K_t, A_t) = A_t f(k_t) = A_t^{1-\alpha} K_t^{\alpha} = A_t (k_t)^{\alpha},$$

where $k_t \equiv K_t / a_t L_t$ and coefficient α denotes the elasticity of capital. Also, note that $\alpha \in (0, 1)$. Assuming a Cobb-Douglas production function and a depreciation rate of 1, Equations (2.9) and (2.10) can be rewritten as:

$$F_L^{'}(K_t, L_t) = f(k_t) - k_t f^{'}(k_t) = w_t = (1 - \alpha) a_t k_t^{\alpha} \tag{2.11}$$

$$F_K^{'}(K_t, L_t) = f^{'}(k_t) = (1 + r_t) = \alpha k_t^{\alpha-1} \tag{2.12}$$

2.2.4 Government

In this section, the government sector and its two main tasks are introduced. The first task is to consume and the second task is to fund this consumption. Government consumption is unproductive and termed Q_t. In addition, it is assumed that these government expenditures do not appear in the households' utility function. Thus, the households' utility is independent of government expenditures. The government finances its consumption Q_t levying taxes proportional to the wage by setting the tax rate τ^w that is between 0 and 1. The model also allows for a case in which the tax revenues are not sufficient to fund government expenditure. In order to close the positive gap between government expenditure and tax revenues, the government issues bonds B_t. The households invest in these bonds and require interest in return. The governments' budget constraint therefore is defined as follows:

$$(1 + r_t)B_t + Q_t = B_{t+1} + \tau^w N_t w_t \tag{2.13}$$

The left hand side of the budget constraint represents government spending, consisting of redemption of bonds issued in period t, interest rate payments and consumption. This is financed by the right hand side comprising of tax revenues and interest rate. Dividing Equation (2.13) by A_t and rearranging it leads to the following the budget constraint in per efficient capita values, indicated by lower case letters:

$$(1 + n)b_{t+1} = (1 + r_t)b_t + q_t - \tau^w \frac{w_t}{a_t} \tag{2.14}$$

In their book Farmer and Schelnast (2013) mention that empirical evidence shows a stable relationship between public expenditure and GDP. Based on their argumentation, a simplified functional form for public expenditures per efficient capita is used, where public expenditures per efficient capita are a proportion of GDP per efficient capita y_t such that:

$$q_t = \Gamma y_t = \Gamma k_t^{\alpha}, \tag{2.15}$$

where Γ denotes public expenditure per (additional) unit of GDP. The tax income per capita yields: $\tau^w w_t$. Using Equation (2.11) this tax income per capita can be rewritten as: $\tau^w(1-\alpha)a_t k_t^\alpha$. As a result, it is easy to see that, $\tau^w(1-\alpha)$ denominates the income tax revenue per (additional) unit of GDP in Equation (2.14). Hence, the primary deficit ratio θ can be written as:

$$\theta = \Gamma - \tau^w(1-\alpha) \tag{2.16}$$

Using this argumentation, the government's budget constraint per efficient capita in Equation (2.14) can be rewritten as follows:

$$(1+n)b_{t+1} - (1+r_t)b_t = \theta y_t = \theta k_t^\alpha \tag{2.17}$$

Thus, the government controls two fiscal tools: the tax rate τ^w and public expenditure per unit of GDP Γ. Later in this thesis it is shown how the choice of these tools and the implied primary budget defitic ratio θ affect long run equilibria.

2.3 Market Equilibrium

An intertemporal equilibrium is a sequence of endogenous variables that attains equilibrium in all time periods t. In each time period, all consumers must maximise utility, the representative firm must maximise profits, and the capital market, the goods market, and the labour market must be in equilibrium. The market equilibrium can be established by aggregating over all individuals and firms in the economy. The key variables in a market equilibrium are the physical capital stock K_t and public debt B_t, which are the endogenous variables in the model. As mentioned before, the capital demand is set by the firms. The supply of capital is set by the individuals and is determined by the result of their savings decision and by the level of public debt. Note that before households are able to invest in capital they have to buy all issued government bonds. Aggregating savings over all individuals and recalling Equation (2.8) and that full depreciation is assumed, the capital market can be written as:

$$N_t s_t = K_{t+1} + B_{t+1} \tag{2.18}$$

Having defined capital market the next step is to establish the goods market. This requires defining aggregate consumption at time t as $C_t^T = N_t c_t + N_{t-1} d_t$. As aggregate consumption is defined, the goods market can be written as:

$$Y_t = C_t^T + Q_t + K_{t+1} \tag{2.19}$$

The last market to be in equilibrium is the labour market. Each young individual is endowed with one unit of labour to be supplied inelastically to firms, which leads to the following labour market clearing condition:

$$N_t = L_t, \forall t \tag{2.20}$$

With all consumers maximising utility and all firms maximising profit based on Walras' law, the capital market, the goods market and the labour market must be in equilibrium. Walras' law states that the sum of nominal excess demands on all three markets is equal to zero for all feasible prices. Thus, if two of the three markets are in equilibrium the third has to be balanced as well, according to Walras' law.

2.4 The Golden Rule

One property of this OLG model with no bequest motives is that its outcome by the market mechanism may not be Pareto optimal. In fact, the economy may over-accumulate capital and suffer from production inefficiency. In order to understand the concept of Pareto-efficiency in the model setting the Golden Rule capital intensity has to be introduced. The Golden Rule capital intensity k_{GR} is the capital intensity that results in the highest sustainable consumption per unit of efficient capita and in welfare maximisation. Hence, solving for the Golden Rule capital intensity requires maximising total consumption C_t^T. This is achieved by starting with the goods market clearing condition:

$$Y_t = C_t^T + Q_t + K_{t+1} \tag{2.21}$$

Defining the goods market per efficient capita and solving for total consumption per efficient capita yields:

$$c_t^T = y_t - q_t - k_{t+1}(1+n) \tag{2.22}$$

The next steps are to replace government consumption per efficienct capita q_t by a fixed share Γ of GDP per efficient capita y_t and to guide the economy into a steady-state, in which time subscripts can be ignored. The steady-state consumption maximisation problem now reads as follows:

$$\max_k c_t^T = (1-\Gamma)f(k) - k(1+n) \tag{2.23}$$

Using Equation (2.12) the first order condition to the maximisation problem can then be written as:

$$\begin{aligned} (1-\Gamma)f'(k) &= (1+n) \\ (1-\Gamma)(1+r) &= (1+n) \end{aligned} \tag{2.24}$$

The capital intensity that solves Equation (2.24) is the Golden Rule capital intensity k_{GR}. It is obvious that this condition depends on the expenditure ratio Γ. As previously mentioned, modelling government consumption explicitly is an extension made to the OLG model formulated by Diamond (1965) suggested by Farmer and Schelnast (2013). In Diamond's model, Γ equals 0 and the Golden Rule simplifies to the following well-known condition: $r = n$. Thus, in Diamond's OLG model welfare is maximised whenever the natural growth rate of the economy n equals the interest rate r. However, in the model used here unproductive government is modelled and assumed neither to enter the household utility nor the firms maximisation problem. Thus, a higher expenditure ratio Γ comes with a lower Golden Rule capital intensity. To see this, one can solve for the Golden Rule capital intensity k_{GR} using Equations (2.24) and (2.12):

$$k_{GR} = \left[\frac{\alpha(1-\Gamma)}{(1+n)}\right]^{\frac{1}{1-\alpha}} \tag{2.25}$$

2.5 Pareto-Efficiency

Since the Golden Rule capital stock per efficient capita k_{GR} has been derived, Pareto-efficiency can now be analysed. Pareto-efficiency in the given setting means that there is no technically feasible path for the economy that makes one cohort better off without making another one worse off. Note that in the case of $k > k_{GR}$ the equlibrium capital stock per efficient capita exceeds that Golden Rule level, which means that $(1 - \Gamma)(1 + r) < (1 + n)$. The opposite is the case if $k < k_{GR}$.

If $k > k_{GR}$, the economy has overcummulated capital along its growth path. The deriviation of the Golden Rule condition has shown that the consumption level in such a situation is not maximised. Thus, reducing the existing capital stock until $k = k_{GR}$ increases the feasible consumption level of households of the current and all future periods. The welfare of the current generation increases as the additional consumption comes without any costs. Future generations' welfare rises as placing the economy on the Golden Rule path maximises their consumption. An economy with over-accumulated capital is called dynamically inefficient, as the above mentioned reduction of capital would lead to a Pareto improvement of the current and all future generations.

If $k < k_{GR}$, the economy has accumulated insufficient capital along its growth path. A Pareto improvement as in a dynamically inefficient economy cannot be found since raising the capital stock to the Golden Rule level would increase consumption in future periods but requires additional savings made by at least some generations. Additional savings require giving up consumption, which lowers welfare. Consequently, a Pareto improvement that improves the welfare of all generations is not possible in this case. This is why economies with $k < k_{GR}$ are called dynamically efficient. As the economies may be dynamically efficient the Golden Rule steady-state is an interesting benchmark in theory but not a policy advice. It ignores dynamic adjustment processes and potential welfare losses for certain generations on the way to a Golden Rule capital intensity. Whether an economy's equilibrium turns out to be dynamically efficient or inefficient depends on a combination of preferences and the technology of the private sector, which is unlikely to happen.

2.6 Medium Run

The goal is now to find the medium run laws of motion of the two endogenous variables capital and debt measured per efficient capita. This allows to analyse how the economy evolves over time. To establish the laws of motion, the capital market in Equation (2.18) has to be combined with the government budget constraint of Equation (2.13). As it is worked with variables per efficient capita, both the capital market as well as the government budget contraint have to be divided by the number of productivity-weighted workers A_t. In the case of the capital market, this yields:

$$\frac{s_t}{a_t} = k_{t+1}(1+n) + b_{t+1}(1+n) \tag{2.26}$$

Dividing the government budget constraint by productivity-weighted workers A_t yields:

$$(1+n)b_{t+1} = (1+r_t)b_t + q_t - \tau^w \frac{w_t}{a_t} \tag{2.27}$$

Both the capital market and the government budget constraint depend on debt per efficient capita b_{t+1}. This fact can be exploited to combine the two:

$$\frac{s_t}{a_t} = k_{t+1}(1+n) + b_t(1+r_t) + q_t - \tau^w w_t \tag{2.28}$$

Equation (2.28) reveals that today's savings need to fund tomorrow's capital stock, today's debt plus interest payments and today's primary budget deficit. The next step is to get to an equation that relates the capital intensity of today to past accumulation and fixed parameters only. To get there insert optimal savings from Equation (2.7), the wage from Equation (2.11), the rental rate of capital from Equation (2.12), the primary deficit ratio from Equation (2.16) and assume that $\theta_t = \theta_{t+1} = \theta$, $\Gamma_t = \Gamma_{t+1} = \Gamma$ and $\tau_t = \tau_{t+1} = \tau, \forall t$:

$$\varphi(1-\tau^w)k_t^\alpha = k_{t+1}(1+n) + b_t \alpha k_t^{\alpha-1} + \theta k_t^\alpha,$$

where $\varphi \equiv \frac{\beta}{1+\beta}(1-\alpha)$ is a constant and the share of total disposable income that is saved. For the purpose of fiscal adjustment policies, it is important to establish the effects of the two policy tools (τ^w,Γ) on the one-period development of debt and capital. To analyse these effects, the laws of motion of debt and capital intensity finally need to be derived explicitly. For simplicity and analytical purposes, define the the debt-capital ratio as $\eta_t = b_t/k_t$ such that the motion equation of the capital intensity can be written as follows:

$$k_{t+1}(1+n) = \varphi(1-\tau^w)k_t^\alpha - \eta_t \alpha k_t^\alpha - \theta k_t^\alpha \tag{2.29}$$

For the second motion equation that defines the law of motion of the debt-capital ratio, rewrite Eqution (2.14) to get:

$$\eta_{t+1}(1+n)k_{t+1} = \alpha k_t^{\alpha}\eta_t + \theta k_t^{\alpha} \tag{2.30}$$

Formally, equations (2.29) and (2.30) represent a system of two non-linear first order equations in the variables (k_t,η_t). If the capital intensity and the debt-capital ratio is given for a certain period t, then the future dynamics of both variables depend on the two laws of motion and are therefore known. This allows for an analysis of how medium run levels of capital intensity and debt are affected by changes in the two policy tools, namely the tax rate τ^w and the public expenditures per additional unit of GDP Γ. The capital intensity and debt per efficient capita of the subsequent period read as follows:

$$k_1(1+n) = \varphi(1-\tau^w)k_0^{\alpha} - \eta_0\alpha k_0^{\alpha} - \theta k_0^{\alpha} \tag{2.31}$$

$$b_1(1+n) = \alpha k_0^{\alpha}\eta_0 + \theta k_0^{\alpha} \tag{2.32}$$

Without further analysis, Equation (2.32) reveals that, *ceteris paribus*, the debt issued in the next period decreases when the primary deficit ratio decreases. Two cases are considered to analyse the effect of a change in the primary deficit ratio on the next period capital intensity. On one hand, it is obvious that an increase in public expenditure today leads to a lower capital intensity tomorrow because a change in public expenditure does not disturb the decision making of either the firms nor the households. On the other hand, when the government decides to raise the tax rate, not only is the primary deficit ratio lowered but is also savings, as one can infer from Equation (2.7). Hence, to examine the overall effect of an increase in the tax rate one has to consider the saving effect and the deficit effect. To illustrate this, k_0^{α} is factored out in Equation (2.31):

$$k_{t+1}(1+n) = \left[\frac{\beta}{1+\beta}(1-\alpha)(1-\tau^w) - \eta_t\alpha - \Gamma + \tau(1-\alpha)\right]k_t^{\alpha} \tag{2.33}$$

Knowing that per definition $\beta \in (0,1]$ it is easy to see that an increase in the tax rate has a positive effect on the future capital intensity as the following conditions always holds true:

$$\begin{aligned}(1-\alpha) &> \frac{\beta}{1+\beta}(1-\alpha)\\ 1 &> \frac{\beta}{1+\beta}\end{aligned} \tag{2.34}$$

The higher tax rate reduces savings but at the same time increases tax revenues (assuming a constant tax base) which leads to a lower future debt stock per capita and crowding-in of private capital. It can be concluded that the mentioned crowding-in effect dominates the negative saving effect such that capital per efficient capita increases in response to a higher tax rate.

Having discussed the effects of the two policy tools on the medium run capital accumulation and government debt level, the focus is turned towards medium run growth. Assuming a constant natural growth rate n, the medium run GDP growth at time t is only determined by the change in capital intensity and defined as follows:

$$g_t^Y = \frac{Y_{t+1}}{Y_t} - 1 = \frac{A_{t+1}^{1-\alpha}K_{t+1}^{\alpha}}{A_t^{1-\alpha}K_t^{\alpha}} - 1 = \frac{A_{t+1}(K_{t+1}/A_{t+1})^{\alpha}}{A_t(K_t/A_t)^{\alpha}} - 1 = (1+n)\left(\frac{k_{t+1}}{k_t}\right)^{\alpha} - 1 \tag{2.35}$$

It can be inferred that for a given k_t the growth rate solely depends on k_{t+1}. Thus the effects of the policy tools on medium run GDP growth are analogous to their effects on k_{t+1}. It can be concluded that an increase in public expenditures Γ lowers medium run growth rates while an increased tax rate τ^w results in higher medium run growth.

2.7 Long Run

2.7.1 Growth Equilibria

Having analysed at medium run dynamics this section establishes long-run growth equilibria of the economy, in which the capital intensity and the debt-capital ratio are constant over time. Subsequently, the impact of the two policy tools on these state states is analysed. While the policy tools certainly affect steady-state levels of the capital intensity, it can be proven that long-run GDP growth is independent of the policy tools and government debt. From Equation (2.35), it was learned that the growth rate depends on the endogenous variables k_{t+1} and k_t.

If the economy is placed on a steady-state growth path, the capital intensity is constant over time. This means $\Delta k_t = k_{t+1} - k_t = 0$ and $k_{t+1} = k_t = k, \forall t$. Thus, (2.35) can be written as:

$$\begin{aligned} g_t^Y &= \frac{Y_{t+1}}{Y_t} - 1 = \frac{A_{t+1}^{1-\alpha} K_{t+1}^{\alpha}}{A_t^{1-\alpha} K_t^{\alpha}} - 1 = \frac{A_{t+1}(K_{t+1}/A_{t+1})^{\alpha}}{A_t(K_t/A_t)^{\alpha}} - 1 \\ &= (1+n)\left(\frac{k_{t+1}}{k_t}\right)^{\alpha} - 1 = (1+n)\left(\frac{k}{k}\right)^{\alpha} - 1 = (1+n) - 1 = n \end{aligned} \tag{2.36}$$

It can be inferred that the growth rate of an economy on its steady-state path simply equals the natural growth rate n. As this natural growth rate is exogenous in the model, it cannot be affected by any type of governmental action since the government debt level does not impact the long-run economic growth.

After proving neutrality of debt level on the long-run GDP growth, the next step is to analyse the existance of long run growth equilibria. To do so, recall the two Equations (2.29) and (2.30) representing the laws of motion. Combining the fact that the capital intensity in a given steady-state is constant over time, where $\Delta k_t = k_{t+1} - k_t = 0$, with law of motion from Equation (2.29) gives:

$$\Delta k_t = k_{t+1} - k_t = \frac{\varphi(1-\tau^w)k_0^{\alpha} - \eta_0 \alpha k_0^{\alpha} - \theta k_0^{\alpha}}{(1+n)} - k_t = 0$$

$$\Leftrightarrow k^{1-\alpha}(1+n) = \varphi(1-\tau^w) - \alpha\eta - \theta$$

$$\Leftrightarrow \eta = \frac{\varphi(1-\tau^w) - \theta}{\alpha} - \frac{k^{1-\alpha}(1+n)}{\alpha} = \frac{\varphi(1-\tau^w) - \theta}{\alpha} - \frac{(1+n)}{f'(k)} \tag{2.37}$$

Equation (2.37) gives an explicit relation between k and η. The relation is named kk-phaseline and comprises all combinations of (k,η) for which the capital intensity is constant over time. The second term on the right hand side with its negative sign in front indicates the negative slope in a (k,η)-diagram. Combining $\Delta\eta_t = \eta_{t+1} - \eta_t = 0$ with Equation (2.30) a second feasible equlibria can be established:

$$\Delta\eta_t = \eta_{t+1} - \eta_t = \frac{\alpha k_t^{\alpha}\eta_t + \theta k_t^{\alpha}}{(1+n)k_{t+1}} - \eta_t = \frac{\alpha\eta + \theta}{(1+n)k^{1-\alpha}} - \eta = 0$$

$$\Leftrightarrow (1+n)k^{1-\alpha} = \alpha + \frac{\theta}{\eta} = \frac{\varphi(1-\tau_t-\sigma^P) - \alpha\eta - \theta}{(1+\omega\sigma^P)}$$

$$\Leftrightarrow \eta = \frac{\theta}{(1+n)k^{1-\alpha} - \alpha} = \frac{\theta k^{\alpha-1}}{(1+n) - f'(k)} \tag{2.38}$$

The relation (2.38) analogous to (2.37) represents a explicit relation between k and η. It is termed the *bb*-phaseline and shows all combinations of (k,η) for which the debt per efficient capita remains constant over time. The duality of two explicit relationships between k and η can now be exploited to solve for closed form solutions of the debt per efficient capita. Define $\epsilon \equiv \varphi(1-\tau^w) - \theta - \alpha$ and use Equations (2.38) and (2.37) together with simple algebraic operations to derive the second degree polynomial equation:

$$\alpha + \frac{\theta}{\eta} = \varphi(1-\tau^w) - \alpha\eta - \theta$$

$$\Leftrightarrow \alpha\eta^2 - \epsilon\eta + \theta = 0 \tag{2.39}$$

This second degree polynomial equation has the following two closed form solutions:

$$\eta^1 = \frac{\epsilon + \sqrt{\epsilon^2 - 4\alpha\theta}}{2\alpha} \qquad \eta^2 = \frac{\epsilon - \sqrt{\epsilon^2 - 4\alpha\theta}}{2\alpha} \tag{2.40}$$

The solutions η^1 and η^2 are termed the $\eta\eta$-phaselines and represent all combinations of (k,η) for which η does not change over time. As the solutions for the debt per efficient capita is independent of k, the two $\eta\eta$-phaselines are horizontal and parallel to the abscissa in a (k,η)-diagram. In fact, η^1 and η^2 solely depend on model parameters and the two policy tools. Therefore, η^1 and η^2 give solutions for the steady-state debt level per capital for a given combination of the tax rate τ^w and the public expenditure per additional unit of GDP.

2.7.2 Graphical Illustration and Efficiency

The aim of this subchapter is to graphically illustrate the model and its equilibria, and to examine their efficiency. The graphical exposition is achieved by choosing values for all exogenous parameters in the model. First, in order to make a reasonable calibration recall that one time period is set to 30 years. For the discount factor β a value of 0.8 is chosen such that it reflects yearly discounting with a factor of 0.993. The growth rates of the population g^N and of the technology g^a are both chosen to be 0.34 such the natural growth rate results in $n = 0.8$, matching a yearly GDP growth rate of 2 percent. The tax rate τ^w is set to 0.25 and the share

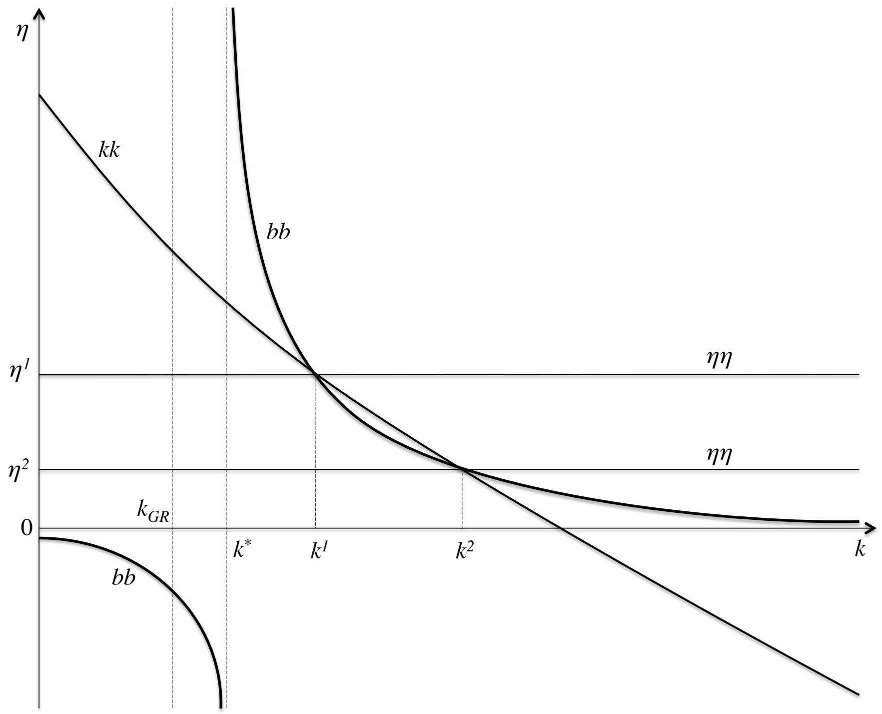

Figure 1: Steady-states with a primary budget deficit

of capital in value added α to 0.15. Finally, the following three different values for the primary budget ratio θ are considered: $\theta = 0.01$, $\theta = 0$ and $\theta = -0.01$. These three cases represent a government with a primary budget deficit, a balanced primary budget and a primary budget surplus, respectively. Together with the tax rate τ^w of 0.25 these deficit ratios imply the following corresponding expenditure ratios: $\Gamma = 0.2225$, $\Gamma = 0.2125$ and $\Gamma = 0.2025$. The three different primary deficit scenarios are depicted in Figures 1, 2 and 3.

The plots are structured so that capital intensity k is on the abscissa and the debt-capital ratio is on the ordinate. Recalling that the bb-phaseline and kk-phaseline defined by Equations (2.38) and (2.37) are monotonic functions in k that comprise all combinations of (k,η) for which the debt per efficient capita and the capital intensity stay constant over time, it is clear that steady-states exist where these two phaselines cross each other. In addition to the bb-phaseline and the kk-phaseline the two horizontal $\eta\eta$-phaseline representing the two roots of the second order polynomial η^1 and η^2 are included. Furthermore, they cross the bb-phaseline and the kk-phaseline in the equilibria. Both the Golden Rule capital stock k_{GR} and the capital stock k^* that solves $f'(k) = 1 + n$ are also presented in the graphs.

The main conclusions from the graphical illustration are:

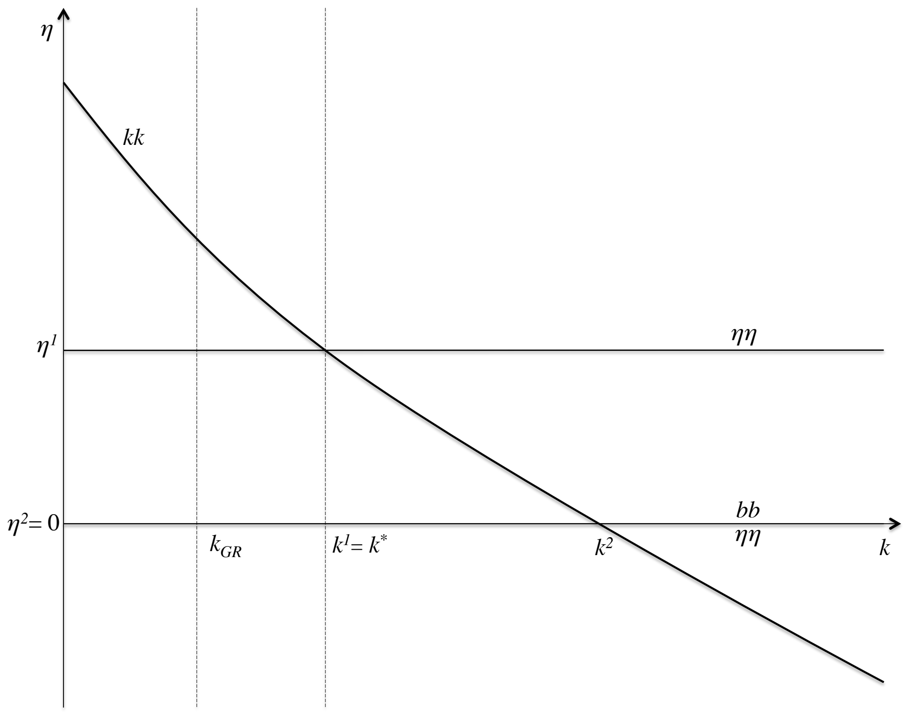

Figure 2: Steady-states with a balanced primary budget

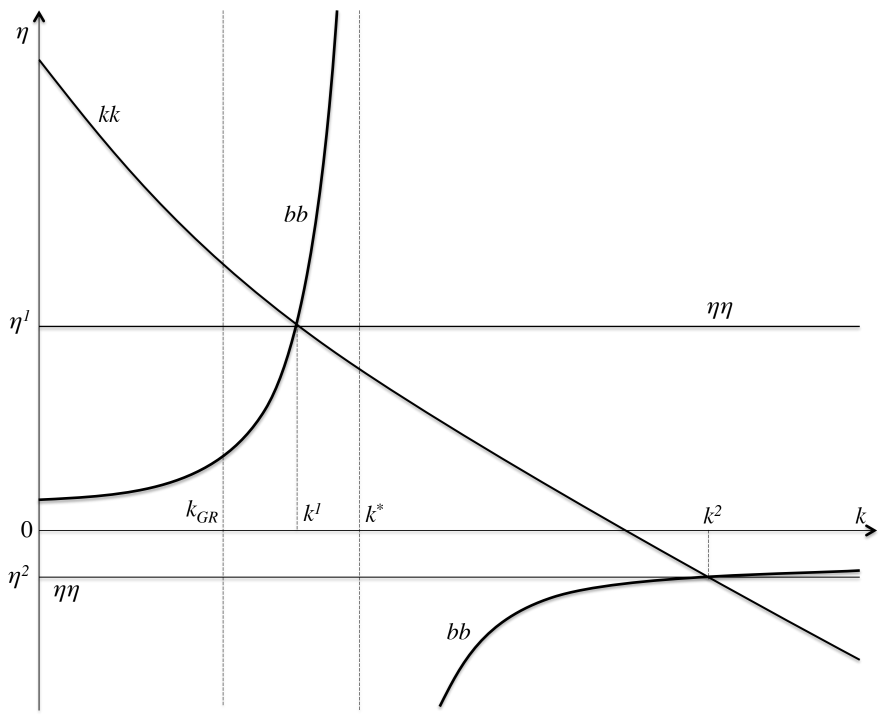

Figure 3: Steady-states with a primary budget surplus

- Primary budget deficit ($\theta = 0.01$)
  - In the case of a primary budget deficit, depicted in Figure 1, the first finding is that both long run equilibria (k^1,η^1) and (k^2,η^2) are dynamically inefficient and come with a positive government debt level. The inefficiency draws from the fact that k^1 and k^2 are greater than the Golden Rule capital intensity k_{GR}. Thus, if over-accumulation of capital occurs together with a primary budget deficit only positive debt level equilibria are possible. A closer examination of Equation (2.40) reveals that the square root has to be positive in order for the debt-capital ratio to have a solution in the realm of the real numbers. In order to have a solution in the realm of the real numbers, the condition $\Phi^2 > 4\alpha\theta$ has to hold true. For the chosen calibration of the model, this means that the primary budget deficit θ has to be smaller than 0.0215. It can hence be concluded that a long run primary budget deficit, which is greater than 2.15%, is impossible to implement.
- Balanced primary budget deficit ($\theta = 0$)
  - The case of a balanced primary budget with $\theta = 0$ is illustrated in Figure 2. It reveals that given a balanced primary budget there will always be a solution with $\eta = 0$. This solution is defined by Equation (2.38), which represents the *bb*-phaseline. Considering the equation, it is apparent that if $\theta = 0$, there will be a solution where $\eta = 0$. This steady-state (k^2,η^2) thus occurs where the *kk*-phaseline crosses the abscissa. While Equation (2.38) is not defined for the k that solves $f'(k) = 1 + n$, Equation (2.37) can be rewritten as: $\eta = \epsilon/\alpha$. Given the chosen parameter set a positive debt-capital ratio results. Note that again both steady-states are located to the right of the Golden Rule capital intensity and are therefore dynamically inefficient.
- Primary budget surplus ($\theta = -0.01$)
  - Figure 3 plots the situation in which the government is running a primary budget surplus. One can infer from the graph that also in the case of a primary budget deficit both steady-states (k^1,η^1) and (k^2,η^2) are dynamically inefficient. The important difference now is that the second steady-state comprises a negative debt level. This negative debt level implies that the government is lending funds to the private firms.

In this section, the long run equilibria of a model with public debt has been discussed for the three different cases of a primary deficit, a primary surplus and a balanced primary budget. One of the important findings is that in any primary budget deficit scenario the steady-state capital intensity is greater than the Golden Rule capital intensity. Therefore, neither of the equilibria is efficient. However, the crowding-out of capital by public debt make the high debt-capital ratio equilbria (k^1,η^1) less inefficient than the low debt-capital ratio equilibria (k^2,η^2). Another important conclusion is that there is a negative relationship between the primary budget deficit θ and the debt-capital ratio η^1 of the low capital intensity steady-state (k^1,η^1). The higher the primary budget deficit the lower the sustainable debt-capital ratio η^1. The observed inefficiency of the long run equilibria calls for policy interventions that places the economy closer to the Golden Rule capital intensity to increase welfare.

2.8 Steady-State Stability Analysis

2.8.1 Analytical Approach

In the previous chapters the long run steady-states have been defined for different cases of governmental primary budget deficits. The next step in this chapter is to mathematically examine the stability properties of the different steady-states based on Farmer and Schelnast (2013). The stability porperties of a steady-state are of special interest with regards to policy interventions or shocks to one of the model parameters as this may push the economy out of a steady-state in both the medium and long run.

In order to establish the steady-state stability condition, the equilibrium dynamics near the steady-state are approximated linearly using the Jacobian matrix of the first partial derivatives of the two Equations (2.29) and (2.30), that represent the laws of motion. The linear approximation of both laws of motion around the steady-state values k and η is performed using a first-order Taylor approximation:

$$k_{t+1} = k + \frac{\partial k_{t+1}}{\partial k_t}(k_t - k) + \frac{\partial k_{t+1}}{\partial \eta_t}(\eta_t - \eta)$$

$$\eta_{t+1} = k + \frac{\partial \eta_{t+1}}{\partial k_t}(k_t - k) + \frac{\partial \eta_{t+1}}{\partial \eta_t}(\eta_t - \eta)$$

This system of equations can be written in a more compact form using matrix and vector notation:

$$\begin{bmatrix} k_{t+1} \\ \eta_{t+1} \end{bmatrix} = \mathbb{J} \begin{bmatrix} k_t \\ \eta_t \end{bmatrix} + \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \mathbb{J} \right] \begin{bmatrix} k \\ \eta \end{bmatrix}, \tag{2.41}$$

where $\mathbb{J}$ denotes the Jacobian matrix that is just first partial derivatives and is defined as follows:

$$\mathbb{J} = \begin{bmatrix} \dfrac{\partial k_{t+1}}{\partial k_t} & \dfrac{\partial k_{t+1}}{\partial \eta_t} \\ \dfrac{\partial \eta_{t+1}}{\partial k_t} & \dfrac{\partial \eta_{t+1}}{\partial \eta_t} \end{bmatrix}$$

The linearised difference equation system from Equation (2.41) has a general solution that is defined as follows:

$$k_t = k + \kappa_1 e_1^k (\lambda_1)^t + \kappa_2 e_2^k (\lambda_2)^t \tag{2.42}$$

$$\eta_t = \eta + \kappa_1 e_1^\eta (\lambda_1)^t + \kappa_2 e_2^\eta (\lambda_2)^t, \tag{2.43}$$

where κ_1 and κ_2 are constants. The eigenvectors are defined as $e_i = \left(e_i^k, e_i^\eta\right)^T$ with $i = 1, 2$ and the eigenvalues of the Jacobian matrix $\mathbb{J}$ of the dynamic system Equations (2.29) and (2.30) are defined as λ_i with $i = 1, 2$. To understand what eigenvector and eigenvalues are, its properties are discussed briefly now. Eigenvectors and eigenvalues of the Jacobian matrix $\mathbb{J}$ are related in a way such that they solve the following equation: $\mathbb{J}e = \mathbb{J}\lambda$. This equation can be rewritten as $[\mathbb{J} - \lambda I]e = 0$ with I being the $2x2$ identity matrix. Assuming that the eigenvector is a non-zero vector the system has as a solution only if the matrix $[\mathbb{J} - \lambda I]$ is singular, which means that its determinant has to be zero. Mathematically it can be written as: $|\mathbb{J} - \lambda I| = 0$. In the current case, the solution of this system is a polynomial of 2 degrees indicating that the Jacobian matrix $\mathbb{J}$ of Equations (2.29) and (2.30) has two eigenvalues (λ_1, λ_2) and two eigenvectors $(e_1^k, e_1^\eta)^T$ and $(e_2^k, e_2^\eta)^T$.

The general solution of the difference equation system defined by Equations (2.42) and (2.43) reveals that the system's stability depends on the absolute values of eigenvalues (λ_1, λ_2) of the Jacobian matrix:

- If $|\lambda_1| \leq |\lambda_2| < 1$, the dynamic system is asymptotically stable. This means that the economy always converge towards the steady-state in finite time.

- If $|\lambda_1| < 1 \leq |\lambda_2|$, the dynamic system is saddle-path stable. Hence, only points on a stable branch guide the economy towards a steady-state.
- If $1 < |\lambda_1| \leq |\lambda_2|$, the dynamic system is unstable. The dynamic system is explosive and will never reach a steady-state.

To derive stability conditions, it is therefore necessary to solve for the eigenvalues. In order to do that, one has to calculate the partial derivatives of the Jacobian matrix at any steady-state first. The results of this calculations are shown here while the full derivations can be found in Appendix A.

$$\frac{\partial k_{t+1}}{\partial k_t} = \alpha \tag{2.44}$$

$$\frac{\partial k_{t+1}}{\partial \eta_t} = \frac{-\alpha k^{\alpha}}{(1+n)} \tag{2.45}$$

$$\frac{\partial \eta_{t+1}}{\partial k_t} = 0 \tag{2.46}$$

$$\frac{\partial \eta_{t+1}}{\partial \eta_t} = \frac{\alpha(1+\eta)}{k^{1-\alpha}(1+n)} \tag{2.47}$$

It is now possible to solve for the eigenvalues of the Jacobian matrix by using partial derivatives at the steady-state derived above and by setting the determinant of the matrix $[\mathbb{J} - \lambda I]$ equal to zero. This yields:

$$det\ [\mathbb{J} - \lambda I] = \begin{vmatrix} \frac{\partial k_{t+1}}{\partial k_t} - \lambda_1 & \frac{\partial k_{t+1}}{\partial \eta_t} \\ \frac{\partial \eta_{t+1}}{\partial k_t} & \frac{\partial \eta_{t+1}}{\partial \eta_t} - \lambda_2 \end{vmatrix} = \begin{vmatrix} \alpha - \lambda_1 & \frac{-\alpha k^{\alpha}}{(1+n)} \\ 0 & \frac{\alpha(1+\eta)}{k^{1-\alpha}(1+n)} - \lambda_2 \end{vmatrix}$$

$$= (\alpha - \lambda_1)\left(\frac{\alpha(1+\eta)}{k^{1-\alpha}(1+n)} - \lambda_2\right) = 0$$

This demonstrates that the eigenvalues are of the following values:

$$\lambda_1 = \alpha \qquad \lambda_2 = \frac{\alpha(1+\eta)}{k^{1-\alpha}(1+n)} \tag{2.48}$$

The first eigenvalue λ_1 is smaller than unity, as for the elasticity of capital α it is assumed that $0 < \alpha < 1$. The second eigenvalue λ_2 in turn may be smaller or greater than unity. As the first value is below unity it can already be concluded that there can either be saddle-path or a asymptotically stable steady-state, while an explosive, unstable steady-state can be ruled out. In what follows, the steady-state stability properties are analysed for the three different scenarios of a government with a primary budget deficit, with a balanced primary budget, and finally of a government with a primary budget surplus. The set of parameters remain as previously defined.

- Primary budget deficit ($\theta = 0.01$)

    - Previous analysis showed that in case of a government running a primary budget deficit there are two dynamically inefficient steady-state (k^1,η^1) and (k^2,η^2). Compared to (k^2,η^2) the steady-state (k^1,η^1) comes with a low capital intensity k^1 and a high positive debt-capital ratio η^1, while the higher capital intensity steady-state (k^2,η^2) comes with a high capital intensity k^2 and a small positive debt-capital ratio η^1. The high debt-capital ratio η^1 in the case of the lower capital intensity steady-state leads to $\lambda_2 > 1$ such that the steady-state (k^1,η^1) becomes saddle-path stable. In turn, the low but positive debt-capital ratio η^2 leads to a $\lambda_2 < 1$ such that the steady-state (k^2,η^2) becomes asymptotically stable. The dynamics of the debt-capital ratio and the capital intensity in the case of a primary budget deficit are depicted in Figure 4.

- Balanced primary budget ($\theta = 0$)

    - The case of a government running a balanced primary budget was illustrated in Figure 2. It demonstrates the two dynamically inefficient steady-states (k^1,η^1) and (k^2,η^2). The steady-state (k^1,η^1) has a relatively low capital intensity but a high debt-capital ratio. Due to the high debt-capital ratio, λ_2 is greater than 1, causing the convergence dynamics of steady-state (k^1,η^1) to be defined by a saddle-path. The second equilibrium (k^2,η^2) has a debt-capital ratio of 0 and a relatively high capital intensity such that $\lambda_2 < 1$. This indicates a steady-state of asymptotic stability. A

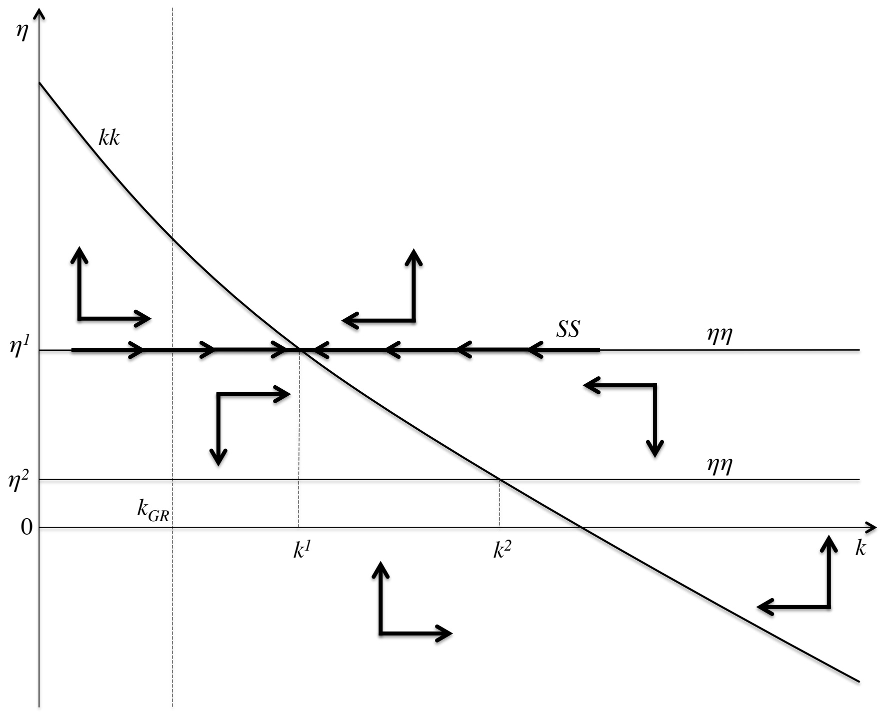

Figure 4: Dynamics in the case of a primary budget deficit

graphical illustration of the dynamics in a situation with a balanced primary budget can be found in Figure 5.

- Primary budget surplus ($\theta = -0.01$)

  - Figure 3 plotted the two dynamically inefficient steady-states in a scenario of a government with a primary budget surplus. The graph reveals a high debt-capital ratio η^1 that causes λ_2 to be greater than unity. Hence, the high debt steady-state (k^1,η^1) is saddle-path stable. In the case of the high capital intensity steady-state (k^2,η^2) the debt-capital ratio η^2 is negative, causing the nominater in Eqaution (2.48) to become small and λ_2 to be less than 1. The steady-state (k^2,η^2) convergence dynamics are therefore characterised by asymptotic stability. The dynamics are provided in Figure 6.

The analysis of the steady-states reveals that whenever there are two steady-states the steady-state with the higher debt-capital ratio (k^1,η^1) is saddle-path stable and the steady-state with the low debt-capital ratio (k^2,η^2) is asymptotically stable.

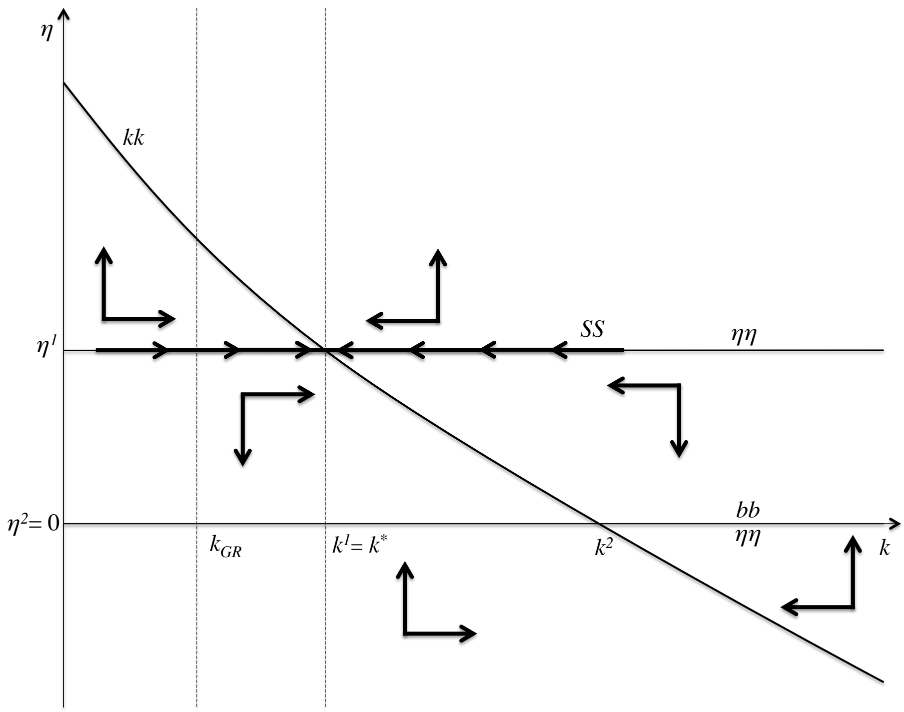

Figure 5: Dynamics in the case of a balanced primary budget

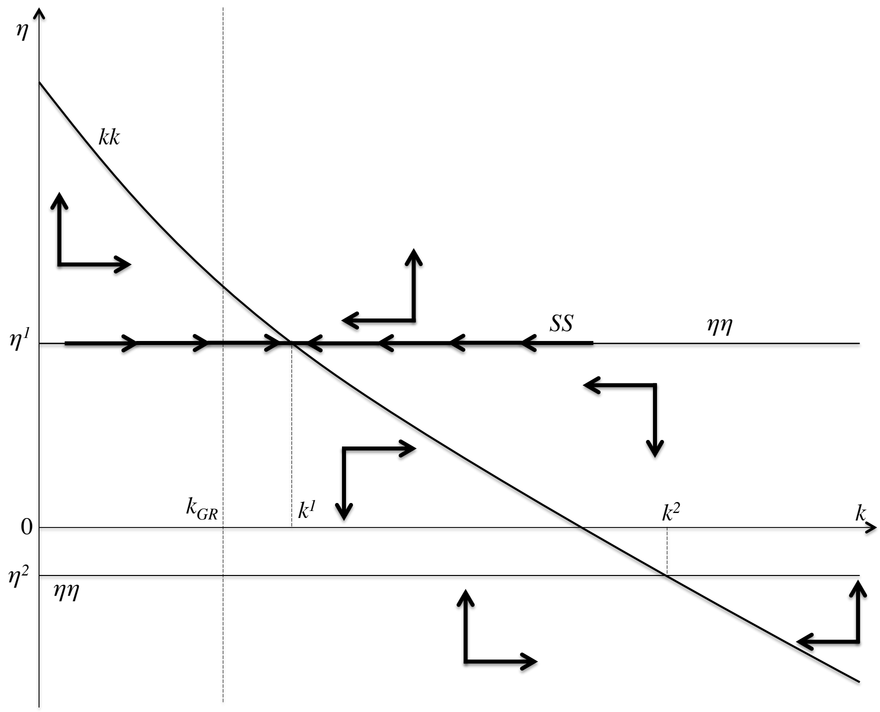

Figure 6: Dynamics in the case of a primary budget surplus

2.8.2 Intuitive Approach

This section aims to offer economic intuition for the depicted dynamics in Figures 4 to 6 and to explain the conditions under which the economy moves to one or the other steady-state. As previously mentioned, the kk-phaseline and the $\eta\eta$-phaseline display all combinations of k_t and η_t for which the capital intensity k_t and the debt-capital ratio η_t stay constant over time. Thus, if the economy is not placed on one of these lines, both endogenous variables either grow or shrink over time. The dynamics of k_t and η_t are determined by their position with respect to the kk-phaseline and the $\eta\eta$-phaseline. Being placed to the left of the kk-phaseline defines the movement of k_t and being placed above or below one of the $\eta\eta$-phaselines determines the movement of η_t. It is therefore necessary to distinguish between different areas in the phase diagram. The dynamics of the capital intensity and the debt-capital ratio in each area in the phase diagrams are represented by different pairs of arrows. The arrows parallel to the abscissa indicate the movement of the capital intensity while the arrows parallel to the ordinate describe the movement of the debt-capital ratio.

Figure 5 is now used to explain the economic rationale behind the dynamics of the capital intensity and debt-capital ratio of the model in general. The first thing to notice is that the capital intensity increases in all areas to the left of the kk-phaseline, indicated by right pointing arrows. In turn, the capital intensity decreases in all areas to the right of the kk-phaseline, indicated by right pointing arrows. The economic rationale behind these dynamics is that the capital intensity increases to the left of the kk-phaseline, since savings out of net wage income are larger than the intensity-sustaining private investment demand and the demand for savings in order to cover interest on public debt and to sustain the prevailing public debt to private capital ratio. The capital intensity decreases to the right of the kk-phaseline, since savings out of net wage income are lower than the intensity-sustaining private investment demand and the demand for savings in order to cover interest on public debt and to sustain the prevailing debt-capital ratio.

In all three different primary budget deficit scenarios the low capital intensity steady-state (k^1,η^1) is characterised by saddle-path stability. This saddle-path is termed SS in the graph and gives all combinations of k_t and η_t for which the economy moves towards the long-term growth equilibrium. Thus, if the economy initially is placed on this stable arm, then the interactions of market participants guide the economy towards the low capital intensity steady-state (k^1,η^1). All pairs of arrows in the neighbourhood of this steady-state point away from the low capital intensity steady-state, which makes the equilibrium unstable whenever the economy is not placed on the unique saddle-path.

For example, if the economy is placed to the left of the steady-state and slightly above the saddle-path, the economy is in a region characterised by a low capital intensity. The firms' demand for capital consequently increases over time, indicated by the right pointing arrow. Due to the low capital intensity, interest rates are very high leading to large interest payment on the existing public debt, which itself is already relatively high. The government is therefore forced to issue more bonds resulting in a growing debt-capital ratio indicated by the upwards pointing arrow. These dynamics guide the economy towards the kk-phaseline that will be crossed eventually. The high public debt in this new region is at a very high level too, leading to huge interest payments on outstanding bonds that can only be repaid by issuing even more bonds, culminating in an increasing debt burden. This results in households investing more of their savings in government bonds B_t and less in real capital K_t. This crowding-out of private capital leads to a decline in the capital intensity, as illustrated by the left pointing arrow.

What happens if in turn the economy jumps to a region sightly below the saddle-path? As the debt burden and interest rate payments are lower than on the stable arm, the debt-capital ratio decreases, indicated by the downwards pointing arrows below the saddle-path. The economy will then move towards the high capital intensity steady-state (k^2,η^2). This high capital intensity steady-state (k^2,η^2) is asymptotically stable as all arrows in its neighbourhood point towards the steady-state. Thus, as long as a shock to the high capital intensity steady-state (k^2,η^2) places the economy on a point below the saddle-path SS, the economy will return to the same steady-state in finite time.

3 Public Pensions

3.1 Introduction

The previous chapter introduced an OLG model with public debt and a government controlling the tax rate τ^w and the expendiure ratio Γ as its fiscal tools. The long run equilibria established under different govermental primary budget deficit scenarios turned out to be dynamically inefficient. Individuals over-accumulate capital in equilibrium, which undermines their consumption possibilities and welfare. This calls for a welfare-enhancing governmental policy instrument. Public pensions may be such a policy tool. It is therefore the main purpose of this chapter to examine whether public pensions and more precisely a pay-as-you-go pension scheme can serve as a policy tool increasing welfare.

The OLG presented in the former chapter proves to be an ideal tool to analyse implications of pay-as-you-go pension schemes. Thus, this chapter builds up on exactly the same model but extends it by a pay-as-you.go pension scheme. The government then not only controls its tax rate and expenditures but also a public pension scheme. The chapter starts by describing the different designs of a pays-as-you-go pension scheme and how they disturb the savings behaviour of households. Subsequently, it is analysed how pay-as-you-go pension contributions affect medium run dynamics. Furthermore, the long run equilibria are established and it is illustrated how pay-as-you-go pension schemes affect the efficiency of these steady-states. Finally, it is demonstrated how a mix of the fiscal tools and the pay-as-you-go contribution rate can be used to increase welfare.

3.2 Pay-as-you-go Pension Schemes

The public pension scheme in the developed world is normally designed as a pay-as-you-go pension scheme and often part of the social security system of a country. Otto von Bismarck the chancellor of the German Empire introduced the world's first public pension scheme in Germany in the 1880s. Thereafter, other developed countries followed his example by providing pensions to their citizens as well. Unfortunately, there are still parts in Africa, Asia and Latin America where pension systems have not been established yet (Blake, 2007). The pay-as-you-go pension scheme is an unfunded scheme where young people who are working contribute and retired people periodically receive pensions. This means that pensions paid out to the retired people in period t are the contributions made by the working generation. The balanced budget condition of the pay-as-you-go pension system therefore is:

$$\begin{aligned} \beta_t^P N_{t-1} &= \sigma_t^P N_t \\ \beta_t^P D_t &= \sigma_t^P \\ \beta_t^P &= \sigma_t^P (1 + g_t^N), \end{aligned} \tag{3.1}$$

where β_t^P is the pay-as-you-go payment each retired person receives at time t, σ_t^P is the pay-as-you-go contribution made by each working age person and D_t is the dependency ratio at time t. The dependency ratio is defined as the ratio of the number of retirees N_{t-1} to the number of working age individuals N_t.

A pay-as-you-go pension scheme can be designed in different ways, but there are two main classes of pay-as-you-go schemes: defined benefit schemes and defined contribution schemes. Within these classes, two different types are presented here. The first type uses lump sum payments. For a defined benefit scheme, this means that retirees are guarateed fixed lump sum benefits β^P by dynamic adjustments of σ_t^P. This has important implications. Equation (3.1) reveals that if benefits β_t^P are fixed, the young generation is bearing all risks. If, for example, the population growth rate g_t^N decreases and the dependency ratio therefore raises the contribution, σ_t^P has to increase to guarantee the fixed benefits of the retirees, leaving the young with the risk of a change in demographics. Another risk the young face in a defined benefit scheme with fix lump sum benefits is a reduction in wage income. If wages drop, the young generation has to spend a higher percentage of their wage income to keep up with the defined lump sum benefits. There are also defined contribution schemes that define the contribution as a fixed lump sum payment σ_t^P. In such a scheme benefits β_t are endogenously adjusted. This implies that the old generation is bearing the risk of demographic change. The benefits per retiree drop following a negative shock to the population growth. However, the young generation is still left with the risk of a change in wage.

The second type of defined benefit and contributions schemes does not fix benefits and contributions as a lump sum payment but make them proportional to wages. If pay-as-you-go pension contributions proportionally depend on wages, σ_t^P can be replaced by $\sigma_t^P = \tau_t^P w_t$ in Equation (3.1):

$$\begin{aligned} \beta_t^P N_{t-1} &= \tau_t^P w_t N_t \\ \beta_t^P &= \tau_t^P w_t (1 + g_t^N), \end{aligned} \tag{3.2}$$

where τ_t^P is the pension contribution rate. It defines a fraction of the wage income w_t between 0 and 1 that has to be paid into the pay-as-you-go pension scheme. This contribution rate τ_t^P differs depending on which class - defined benefit scheme or defined contribution scheme - is supposed to be modelled:

$$\tau_t^P \equiv \begin{cases} \tau_t^{P_C} & \text{if a defined contribution scheme is modelled,} \\ \dfrac{\tau_t^{P_B}}{1 + g_t^N} & \text{if a defined benefit scheme is modelled,} \end{cases} \tag{3.3}$$

where $\tau_t^{P_C} \in [0, 1)$ is a fraction of wage income w_t that has to be contributed to a defined contribution scheme and $\tau_t^{P_B}/(1+g_t^N)$ is a fraction of wage income w_t that has to be paid into a defined benefit scheme. As the fraction of wage to be contributed to the pension scheme is always assumed to be between 0 and 1, the following condition for $\tau_t^{P_B}$ holds true: $0 \leq \tau_t^{P_B} \leq 1 + g_t^N$. Having defined τ_t^P for a defined benefit scheme and a defined contribution scheme in Equation (3.3), the balanced budget condition of a defined contribution scheme depending proportionally on wages reads as follows:

$$\begin{aligned} \beta_t^P N_{t-1} &= \tau_t^{P_C} w_t N_t \\ \beta_t^P &= \tau_t^{P_C} w_t (1 + g_t^N) \end{aligned} \tag{3.4}$$

Observing Equation (3.4), it is obvious to see that contributions are not a fixed lump sum payment anymore but just a fixed proportion of wage income. Hence, the old generation faces the risk of both a change in demographics and a change in wage income of the young. Recalling Equation (3.3), the balanced budget condition of a defined benefit scheme can be written as:

$$\begin{aligned} \beta_t^P N_{t-1} &= \frac{\tau_t^{P_B}}{1 + g_t^N} w_t N_t \\ \beta_t^P &= \tau_t^{P_B} w_t \end{aligned} \tag{3.5}$$

As the young cohort contributes by paying $\sigma_t^P = (\tau_t^{P_B} w_t)/(1+g_t^N)$, it faces the risk of demographic change. If the population growth rate g_t^N drops, every individual of the younger cohort has to contribute a higher payment σ_t^P to the pay-as-you-go scheme. While the young cohort bears the risk of demographic change, the old cohort bears the risk of a change in wages. Equation (3.5) illustrates that the benefits retirees receive depend on the level of working income of the young cohort. Therefore, if wages drop, the absolute value young individuals have to contribute decreases together with the benefits for retirees.

3.3 Pensions and Savings Behaviour

Introducing a public pay-as-you-go pension scheme alters both the first and second period budget constraints of households. To study the effects of the pension scheme on the savings behaviour, one has to maximise the households' utility subjet to new budget constraints. In an economy with a pay-as-you-go pensions scheme, the budget contraints of young individuals (2.1) changes to the following:

$$c_t = w_t(1-\tau^w) - s_t - \sigma_t^P \tag{3.6}$$

While individuals are free to choose between consumption and savings they are forced to pay taxes and to contribute σ_t^P to the pay-as-you-go pension scheme. The old individual's budget constraint therefore yields:

$$d_{t+1} = \sigma_{t+1}^P(1+g_{t+1}^N) + s_t(1+r_{t+1}) \tag{3.7}$$

Equation (3.7) illustrates that the return on pay-as-you-go contributions equals the growth rate of the population at time $t+1$. Inserting Equation (3.7) into Equation (3.6) results in the following lifetime budget constraint:

$$\begin{aligned} c_t + \frac{d_{t+1}}{1+r_{t+1}} &= w_t(1-\tau^w) - \sigma_t^P + \sigma_{t+1}^P \frac{1+g_{t+1}^n}{1+r_{t+1}} \\ &= w_t(1-\tau^w) - \frac{\beta_t^P}{1+g_t^n} + \frac{\beta_{t+1}^P}{1+r_{t+1}} \end{aligned} \tag{3.8}$$

Assuming pay-as-you-go pension scheme with defined lump sum contributions such that $\sigma_t^P = \sigma_{t+1}^P = \sigma^P, \forall t$ Equation (3.8) reduces to:

$$c_t + \frac{d_{t+1}}{1 + r_{t+1}} = w_t(1 - \tau^w) + \sigma^P \frac{g_{t+1}^N - r_{t+1}}{1 + r_{t+1}} \tag{3.9}$$

Equation (3.9) shows that the pay-as-you-go pension scheme affects the present value whenever the growth rate of the population g_{t+1}^N is not equal to the interest rate r_{t+1}. If $g_{t+1}^N > r_{t+1}$, the present value of lifetime consumption is higher with a pay-as-you-go pension scheme than without and vice versa. Having established the budget constraints in a model with pensions, it is now possible to formulate the households maximisation problem. The only change to Chapter 3 is that logarithmic utility is now maximised subject to the new present value of the lifetime budget constraint:

$$\max_{c_t, d_{t+1}} U = log(c_t) + \beta log(d_{t+1})$$

subject to

$$c_t + \frac{d_{t+1}}{1 + r_{t+1}} = w_t(1 - \tau^w) - \sigma_t^P + \sigma_{t+1}^P \frac{1 + g_{t+1}^N}{1 + r_{t+1}}$$

$$c_t > 0, d_{t+1} > 0$$

Following the same procedure as in previous chapters one can first solve this maximisation problem for optimal first period consumption c_t and then insert this optimal consumption choice into the first period budget constraint defined by Equation (3.6) to get to optimal savings:

$$s_t = \frac{\beta}{1 + \beta}[w_t(1 - \tau^w) - \sigma_t^P] - \sigma_{t+1}^P \frac{1 + g_{t+1}^N}{(1 + \beta)(1 + r_{t+1})} \tag{3.10}$$

In order to study the effect of a pay-as-you-go scheme, it makes senses to again assume a defined contribution scheme with contributions being constant over time such that $\sigma_t^P = \sigma_{t+1}^P = \sigma^P$ for all t. In a defined contribution scheme, the optimal savings decision can then be simplified as:

$$s_t = \frac{\beta}{1 + \beta} w_t(1 - \tau^w) - \sigma^P \left[1 + \frac{g_{t+1}^N - r_{t+1}}{(1 + \beta)(1 + r_{t+1})}\right] \tag{3.11}$$

Taking the partial derivative of optimal savings with respect to pay-as-you-go contributions σ^P yields:

$$\frac{\partial s_t^*}{\partial \sigma^P} = -\left[1 + \frac{g_{t+1}^N - r_{t+1}}{(1+\beta)(1+r_{t+1})}\right] < 0 \tag{3.12}$$

The first thing to be concluded is that the partial derivative of optimal savings with respect to pay-as-you-go contributions σ^P is negative. A unit change in pay-as-you-go contributions σ^P lowers savings by exactly one unit if the growth rate of the population equals the interest rate. If the growth rate of the population is greater than the interest rate, a unit change in pay-as-you-go contributions σ^P lowers optimal savings by more than one. If instead the growth rate of the population is smaller than the interest rate a unit change in pay-as-you-go contributions σ^P lowers optimal savings by less than one. The fact that a pay-as-you-go pension scheme lowers savings raises hopes that pay-as-you-go pension schemes can help to mitigate the problem of over-accumulation of capital.

3.4 Market Equilibrium

The introduction of a pay-as-you-go pension scheme distorts savings but does not change the market equilibria of the model. The goods, the capital, and the labour market stay the same and clear based on Walras' law:

$$N_t = L_t, \forall t$$

$$N_t s_t^T = K_{t+1} + B_{t+1} \tag{3.13}$$

$$Y_t = C_t^T + Q_t + K_{t+1}$$

The firms' maximisation problem is unaffected by the pension scheme such that the marginal product of labour defining the wage and the marginal product of capital defining the capital rental rate also remain the same in equilibrium:

$$F_L^{'}(K_t, L_t) = w_t = (1-\alpha)a_t k_t^{\alpha} \tag{3.14}$$

$$F_K^{'}(K_t, L_t) = 1 + r_t = \alpha k_t^{\alpha-1} \tag{3.15}$$

3.5 Medium Run

Knowing the households' optimal savings behaviour in a model with a pay-as-you-go pension scheme makes it possible to study how the economy evolves over time. This then allows to see how the two endogenous variables capital and debt react to changes in the pension contribution rate τ^P. The fundamental equation of motion again lays in the capital market from Equation (3.13) and dividing it by A_t yields:

$$\frac{s_t}{a_t} = k_{t+1}(1+n) + b_{t+1}(1+n) \tag{3.16}$$

Optimal savings differ depending on the design and extent of the pay-as-you-go scheme. But before distinguishing between a defined contribution and a defined benefit scheme the medium run dynamics of the general case of a pay-as-you-go pension scheme is derived. In this general case, the contributions are modelled such that: $\sigma_t^P = \tau^P w_t$. Assuming this general design of a pay-as-you-go pension scheme, the opitmal savings from Equation (3.10) can be rewritten as follows:

$$s_t = \frac{\beta}{1+\beta}(1-\tau_t^w-\tau^P)w_t - \tau^P w_{t+1}\frac{1+g^N}{(1+\beta)(1+r_{t+1})} \tag{3.17}$$

Now the government budget constraint from Equation (2.27) and optimal savings from Equation (3.17) have to be inserted into the capital market from Equation (3.16):

$$\frac{1}{a_t}\left[\frac{\beta}{1+\beta}[w_t(1-\tau_t^w-\tau^P)] - w_{t+1}\tau^P\frac{1+g^N}{(1+\beta)(1+r_{t+1})}\right] = k_{t+1}(1+n) + b_t(1+r_t) + q_t - \tau_t^w\frac{w_t}{a_t} \tag{3.18}$$

Next, a constant primary deficit ratio is assumed, implying that $\theta_t = \theta_{t+1} = \theta$, $\Gamma_t = \Gamma_{t+1} = \Gamma$, and $\tau_t^w = \tau_{t+1}^w = \tau^w, \forall t$ hold. Using the expression of the wage and capital rental rate from Equations (3.14), (3.15), and the definition of the primary budget deficit from Equation (2.16), the capital market from Equation (3.18) can be rewritten as:

$$\varphi(1-\tau^w-\tau^P)k_t^\alpha - \omega\tau^P(1+n)k_{t+1} = k_{t+1}(1+n) + b_t\alpha k_t^{\alpha-1} + \theta k_t^\alpha,$$

where $\varphi = \frac{\beta}{1+\beta}(1-\alpha)$ and $\omega = \frac{(1-\alpha)}{\alpha(1+\beta)}$ are constants. The next step is to establish the effects of the policy tools on the one-period development of debt and capital. In addition to the tax rate τ^w and public expenditure ratio Γ there is now also a third policy tool being the pension contribution rate τ^P. The government determines the magnitude of its policy tool τ^P by choosing a certain value for τ^{P_C} in a defined contribution scheme and a certain value τ^{P_B} in a defined benefit scheme. To analyse the effects of the policy tools one has to specify the laws of motion of debt and capital intensity. Defining the debt-capital ratio as $\eta_t = b_t/k_t$ the law of motion of the capital intensity can be written as follows:

$$k_{t+1}(1+n)(1+\omega\tau^P) = \varphi(1-\tau^w-\tau^P)k_t^\alpha - \eta_t\alpha k_t^\alpha - \theta k_t^\alpha \tag{3.19}$$

The law of motion of the debt-capital ratio remains the same compared to Chapter 2 and comes from the government budget contraint in Equation (2.14), which does not directly depend on the pension contribution rate:

$$\eta_{t+1}(1+n)k_{t+1} = \alpha k_t^\alpha \eta_t + \theta k_t^\alpha \tag{3.20}$$

The two equations (3.19) and (3.20) represent a system of two non-linear first order equations in the variables (k_t,η_t) in a model with a pay-as-you-go pension scheme. If the capital intensity and the debt-capital ratio is given for a certain period t, the future dynamics of both variables depend on the two laws of motion and are therefore known. This allows for an analysis of how medium run levels of capital intensity and debt are affected by changes in the new policy tool: the pension contribution rate τ^P. The next period levels of capital intensity and debt now depend not only on Γ,τ^w but also on τ^P:

$$k_1 = \frac{1}{(1+n)(1+\omega\tau^P)}\left[\varphi(1-\tau^w-\tau^P)k_0^\alpha - \eta_0\alpha k_0^\alpha - \theta k_0^\alpha\right] \tag{3.21}$$

$$b_1 = \frac{1}{(1+n)}\left[\alpha k_0^\alpha \eta_0 + \theta k_0^\alpha\right] \tag{3.22}$$

In a model with public pensions, the same arguments as in Chapter 2 can be used to conclude that an increase in the tax rate τ^w reduces the next period capital and debt per efficient capita. It is also apparent that a higher expenditure ratio Γ still lowers the next period capital intensity and increases the next period debt per efficient capita. The pension contribution rate τ^P does not affect the next period debt level but affects the next period capital intensity. Taking the derivative of the future capital intensity with respect to τ^P yields:

$$\frac{\partial k_{t+1}}{\partial \tau^P} = \left[\frac{-1}{1+\omega\tau^P}\right]\frac{\beta}{1+\beta}(1-\alpha)k_t^\alpha < 0 \tag{3.23}$$

This indicates that an increase in the pay-as-you-go contributions lowers the future capital intensity in the medium run. Intuitively, higher pay-as-you-go contributions reduce savings and therefore capital accumulation. Recalling Equation (2.35), it is known that GDP growth g_t^Y reads as follows:

$$g_t^Y = \frac{Y_{t+1}}{Y_t} - 1 = \frac{A_{t+1}^{1-\alpha}K_{t+1}^\alpha}{A_t^{1-\alpha}K_t^\alpha} - 1 = \frac{A_{t+1}(K_{t+1}/A_{t+1})^\alpha}{A_t(K_t/A_t)^\alpha} - 1 = (1+n)\left(\frac{k_{t+1}}{k_t}\right)^\alpha - 1 \tag{3.24}$$

It was inferred that for a given k_t the growth rate solely depends on k_{t+1}. Thus, by increasing $\tau_t^{P_C}$ in a defined contribution scheme or $\tau_t^{P_B}$ in a defined benefit scheme that government lowers medium run GDP growth.

3.6 Long Run

After looking at medium run dynamics, the attention is now turned towards long-run dynamics of an economy with public pensions. Equation (2.36) has shown that long run growth is simply equal to the natural growth rate n. Hence, changes in the pension contribution rate τ^P cannot affect long run growth rates. However, the following section outlines how the pension scheme alters the long run equilibria, in which the capital intensity and the debt-capital ratio are constant over time.

3.6.1 Growth Equilibria

The long run equilibria of an economy with a pay-as-you-go pension scheme can be established using the two laws of motion from Equations (3.19) and (3.20). In the long run, it is known that the capital intensity and the debt-capital ratio are constant over time. This means that $\Delta k_t = k_{t+1} - k_t = 0$ and $k_{t+1} = k_t = k, \forall t$. Thus, the first law of motion from Equation (3.19) in a model with a pay-as-you-go pension scheme can be written as:

$$\Delta k_t = k_{t+1} - k_t = \frac{\varphi(1-\tau^w-\tau^P)k_t^\alpha - \eta_t\alpha k_t^\alpha - \theta k_t^\alpha}{(1+n)(1+\omega\tau^P)} - k_t = 0$$

$$\Leftrightarrow k^{1-\alpha}\big[(1+n)(1+\omega\tau^P)\big] = \varphi(1-\tau^w-\tau^P) - \alpha\eta - \theta$$

$$\Leftrightarrow \eta = \frac{\varphi(1-\tau^w-\tau^P)-\theta}{\alpha} - \frac{k^{1-\alpha}\big[(1+n)(1+\omega\tau^P)\big]}{\alpha} = \frac{\varphi(1-\tau^w-\tau^P)-\theta}{\alpha} - \frac{(1+n)(1+\omega\tau^P)}{f'(k)} \tag{3.25}$$

Equation (3.25) gives an explicit relation between k and η and is again termed the kk-phaseline. It represents all combinations of (k,η) for which the capital intensity is constant over time. The second term on the right hand side, that contains k and is negative, indicates the descending slope in a (k,η)-diagram. Combining $\Delta\eta_t = \eta_{t+1} - \eta_t = 0$ with our second law of motion from Equation (3.20), a second feasible equlibrium condition can be established:

$$\Delta\eta_t = \eta_{t+1} - \eta_t = \frac{\alpha k_t^\alpha \eta_t + \theta k_t^\alpha}{(1+n)k_{t+1}} - \eta_t = \frac{\alpha\eta + \theta}{(1+n)k^{1-\alpha}} - \eta = 0$$

$$\Leftrightarrow (1+n)k^{1-\alpha} = \alpha + \frac{\theta}{\eta} = \frac{\varphi(1-\tau^w-\tau^P) - \alpha\eta - \theta}{(1+\omega\tau^P)}$$

$$\Leftrightarrow \eta = \frac{\theta}{(1+n)k^{1-\alpha} - \alpha} = \frac{\theta k^{\alpha-1}}{(1+n) - f'(k)} \tag{3.26}$$

Equation (3.26) analogous to Equation (3.25), shows an explicit relation between k and η. It is termed bb-phaseline and comprises all combinations of (k,η) for which the debt per efficient capita remains constant over time. The fact of having two explicit relationships between k and η can now be exploited to solve for closed form solutions for the debt per efficient capita. Define $\psi \equiv [\varphi(1-\tau^w-\tau^P)-\theta]/(1+\omega\tau^P) - \alpha$ and use Equations (3.26) and (3.25) together with simple algebraic operations to derive the second degree polynomial equation:

$$\alpha + \frac{\theta}{\eta} = \frac{\varphi(1-\tau^w-\tau^P) - \alpha\eta - \theta}{(1+\omega\tau^P)}$$

$$\Leftrightarrow \frac{\alpha}{(1+\omega\tau^P)}\eta^2 - \psi\eta + \theta = 0 \tag{3.27}$$

This second degree polynomial equation has the following two roots:

$$\eta^1 = \frac{\psi + \sqrt{\psi^2 - (4\alpha\theta)/(1+\omega\tau^P)}}{(2\alpha)/(1+\omega\tau^P)} \qquad \eta^2 = \frac{\psi - \sqrt{\psi^2 - (4\alpha\theta)/(1+\omega\tau^P)}}{(2\alpha)/(1+\omega\tau^P)} \tag{3.28}$$

The solutions η^1 and η^2 are termed $\eta\eta$-phaselines and show all combinations of (k,η) for which η does not change over time. As the solutions for the debt per efficient capita is independent of

k, the two $\eta\eta$-phaselines are horizontal and parallel to the abscissa in a (k,η)-diagram. In fact, η^1 and η^2 solely depend on model parameters and the three policy tools. Therefore, η^1 and η^2 give solutions for the steady-state debt level per capital for a given combination of the tax rate τ^w, the expenditure ratio Γ, and the share of income that has to be paid into the pay-as-you-go scheme τ^P.

3.6.2 Graphical Illustration and Efficiency

The analysis in Chapter 2 has shown that the steady-state equilibria in a model without pensions are inefficient because the equilibrium capital intensities have always proven to be greater than the Golden Rule capital intensity. This was the case for all three different governmental primary budget deficit scenarios considered. The goal of this section, therefore, is not only to graphically illustrate the model but also to show how pay-as-you-go contribution improve or exacerbate efficiency of long run equilibria. Equations (3.12) and (3.23) revealed that savings and the medium run capital intensity decrease whenever the scope of the pay-as-you-go scheme is extended. This has raised hope that a pay-as-you-go pension scheme can place an economy on a more efficient growth path. To analyse this, one has to understand the effect of a change in pay-as-you-go contributions τ^P on the long run equilibria. The long run equilibiria are defined by the kk-phaseline from Equation (3.25) and the bb-phaseline from Equation (3.26). The latter equation reveals that the bb-phaseline depends on the capital intensity and the exogenous parameters n,α and θ only. It is therefore independent of the pension contribution rate τ^P. This is not the case for the kk-phaseline. Equation (3.25) reveals that the kk-phaseline depends on pay-as-you-go contributions. In fact, it is obvious that the higher the contribution rate τ^P the lower the kk-phaseline's intercept and slope. Considering this and that the bb-phaseline is independent of τ^P, a certain value of the pension contribution rate τ^P must exist that places the economy on the Golden Rule growth path. To find the pension contribution rate that places the low capital intensity long run equilibrium (k^1,η^1) of the economy on the Golden rule growth path τ^P_{GR}, one has to equalize Equations (3.25) and (3.26), replace the capital intensity k by the Golden Rule capital intensity from Eqaution (2.25) and subsequently solve for τ^P:

$$\tau^P_{GR} = \frac{1}{\varphi/\alpha + (1-\Gamma)\omega}\left[\frac{\varphi(1-\tau^w) - \theta - \alpha(1-\Gamma)}{\alpha} - \frac{\theta}{\alpha(1-\Gamma) - \alpha}\right] \tag{3.29}$$

In the following, a graphical exposition of the model with a pay-as-you-go pension scheme is presented for an economy with a government running either a primary budget deficit, a balanced primary budget, or a primary budget surplus. The parameter values chosen for this illustration remain the same as in Chapter 2. The graphs show the previous long run equilibria from an economy without a pension scheme, the long run equilibria of an economy with a moderate pay-as-you-go pension scheme as well as of an economy with an extensive pay-as-you-go pension scheme that places the low capital intensity long run equilibrium (k^1,η^1) on the Golden Rule path.

- Primary budget deficit ($\theta = 0.01$)

  - Firstly, the effects of a pay-as-you-go pension scheme are discussed for an economy with a government running a primary budget deficit. A graphical illustration of such an economy is provided by Figure 7. Previous analyses have shown that two inefficient long run equilibria with a positive debt-capital ratio exist in the same economy but without public pensions (see Figure 1). These two old, long run equilibria are termed A and B in Figure 7. They can now be compared to the long run equilibria with a pay-as-you-go pension scheme in place. As mentioned previously, the higher the pension contribution rate τ^P the lower the intercept and slope of the kk-phaseline. Therefore, increasing the τ^P starting from zero makes the old equilibrium A more inefficient, while the old equilibrium B becomes more efficient. This continues until the kk-phaseline becomes tangent to the bb-phaseline such that there is only one unique long run equilibrium ($k^1 = k^2$,$\eta^1 = \eta^2$) termed C. For the parameter values chosen this happens for a $\tau^P = 0.047$. If one would keep increasing τ^P, there would not be an equilibrium anymore. The reason for it lays in the closed form solutions from Eqaution (3.28). In order to get solutions in the realm of the real numbers $\psi^2 > (4\alpha\theta)/(1 + \omega\tau^P)$ has to hold true. *Ceteris paribus*, this condition only holds for a τ^P that is slighty lower than 0.047. Economic intuition for the breakdown of an equilibrium in the realm of the real numbers is that in such cases households do not save enough anymore to fund the government's primary deficit and the interest payments in the long run. For very high values of τ^P, however, $\psi^2 > (4\alpha\theta)/(1+\omega\tau^P)$ holds again. In such cases, the kk-phaseline crosses the bb-phaseline in areas which imply a negative debt-capital ratio. The negative debt-capital ratio means that the government has aquired assets and receives interest payments from the private sector that the government funds. Receiving interest rate payments makes it possible to run a primary budget deficit in the long run. The Golden Rule equilibrium is one with a negative debt capital ratio and is achieved by a $\tau^P_{GR} = 0.27$. In Figure 7, this equilibrium is termed E and the corresponding kk-phaseline is termed kk_{GR}.

■ Balanced primary budget ($\theta = 0$)

□ Here, the effects of a pay-as-you-go pension scheme are discussed for an economy with a government running a balanced primary budget. To get a good understanding of the effects, Figure 8 shows the graphical exposition of the model. The long run equilibria termed A and B in the graphs are the already known equilibria that exist in an economy wihout a public pension scheme. Their corresponding kk-phaseline is dashed and termend kk_0. To better understand the effects of pay-as-you-go pension scheme, note that for a $\theta = 0$ the roots of the second degree polynomial equation read as follows:

$$\eta^1 = \frac{\psi + \sqrt{\psi^2}}{(2\alpha)/(1+\omega\tau^P)} \qquad \eta^2 = \frac{\psi - \sqrt{\psi^2}}{(2\alpha)/(1+\omega\tau^P)}$$

This is the case because with $\theta = 0$ the second term of the square root in Equation (3.28) becomes zero and the square root reduces to $\sqrt{\psi^2}$ with $\psi \equiv \varphi(1-\tau^w-\tau^P)/(1+\omega\tau^P)-\alpha$. Hence, ψ is positive as long as: $\tau^P < \varphi(1-\tau^w)-(\alpha/\alpha\omega+\varphi)$. If τ^P is greater than this expression, ψ is negative. If ψ is positive, η^2 equals zero while η^1 is positive. If τ^P is so great that ψ is negative, η^1 becomes zero and η^2 negative. It can then be shown that whenever η^1 or η^2 are not zero, their corresponding capital intensity equals k^* that solves $f'(k) = 1+n$. The long run equilibria C and D in Figure 8 occur if a moderate pay-as-you-go pension scheme with $\tau^P = 0.05 < \varphi(1-\tau^w)-(\alpha/\alpha\omega+\varphi)$ is chosen. Their corresponding kk-phaseline is simply termed kk. The new high debt-capital ratio equilibrium (k^1,η^1) named C comes with a lower debt-capital ratio but keeps its capital intensity at k^*, because ψ is positive for the value chosen for τ^P. A pension contribution rate of 0.05 in this case does not help to increase efficiency. In turn, the low debt-capital long run equilibrium

(k^1,η^1) becomes more efficient by jumping from B to D while keeping a debt level of 0. For $\tau^P > \varphi(1-\tau^w) - (\alpha/\alpha\omega + \varphi)$, the first root η^1 is zero and its corresponding equilibrium capital intensity is smaller than k^*. The pay-as-you-go contribution rate τ^P_{GR} is greater than $\varphi(1-\tau^w) - (\alpha/\alpha\omega + \varphi)$. In Figure 8, the Golden Rule long run equilibrium is termed E and the Golden Rule kk-phaseline is termed kk_{GR}. While in case of a government running a primary budget deficit the Golden Rule path required a governement holding positive assets, the Golden Rule path in the case of a balanced primary budget can be achieved with a debt-capital ratio of zero.

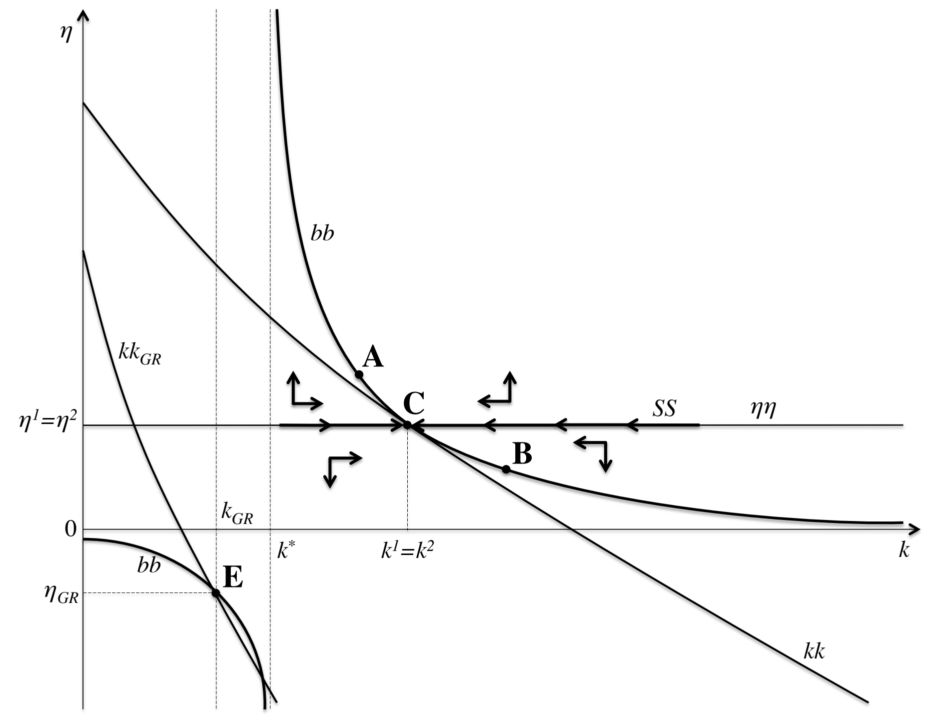

Figure 7: Steady-states with a primary budget deficit and a pay-as-you-go pension scheme

■ Primary budget surplus ($\theta = -0.01$)

□ Lastly, the case of a government running a primary budget surplus is discussed. In this scenario pay-as-you-go pension schemes prove to be very effective in improving efficiency. Figure 9 illustrates the model without a pension scheme and the model with a pay-as-you-go go pension scheme that puts the high debt-capital long run equilibrium (k^1,η^1) on the Golden Rule growth path. The long run equilibria that exist in a model without pensions are termed A and B and their corresponding kk-phaseline is termed kk_0. Knowing that a higher pension contribution rate τ^P pushes

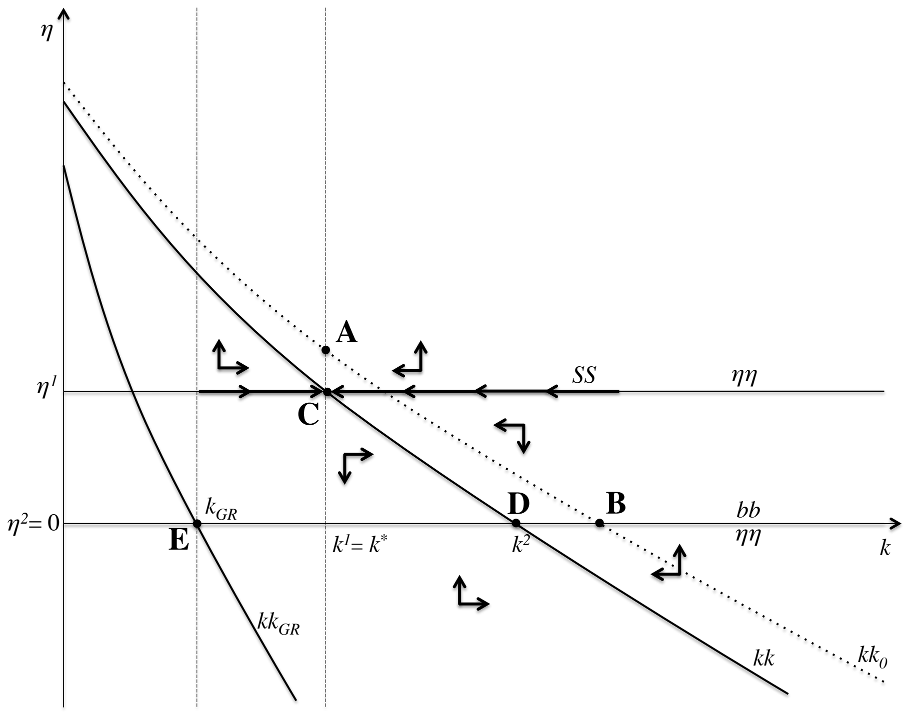

Figure 8: Steady-states with a balanced primary budget and a pay-as-you-go pension scheme

the intercept of the kk-phaseline down while the slope becomes steeper, it is obvious that the pension scheme is very effective in increasing efficiency. To place the high debt-capital long run equilibrium (k^1,η^1) termed C on the Golden Rule growth path, the pension contribution rate is set to $\tau^P = \tau^P_{GR} = 0.16$. This long run equilibrium, in contrast to the cases of a primary budget deficit and a balanced primary budget, comes with a positive debt-capital ratio. With $\tau^P = \tau^P_{GR}$ there also is a low debt-capital ratio long run equilibrium (k^2,η^2) termed D. It is much more efficient than the corresponding equilibrium B in a model without pensions and comes with a positive debt-capital ratio.

The graphical illustration reveals that generally there is a negative relationship between the pay-as-you-go contribution rate τ^P and the equilibrium capital intensities (k^1,k^2) and debt-capital ratios (η^1,η^2). The economic intuition behind this relationship is that a higher pension contribution rate τ^P lowers aggregate savings which are invested in capital and bonds. As a result, the equilibrium capital intensities decrease and only a lower debt-capital ratio can be sustained.

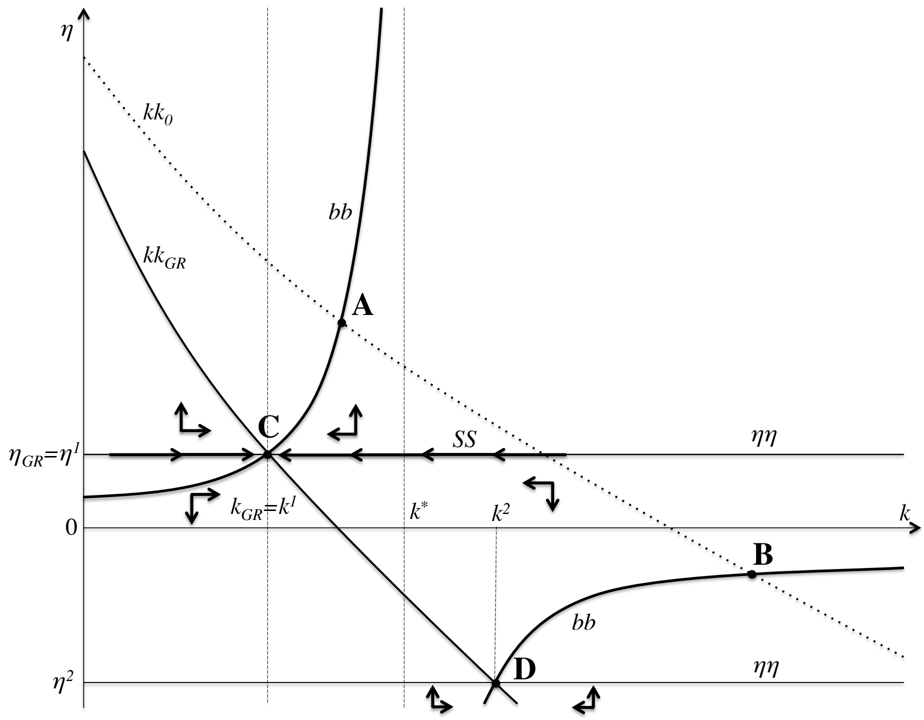

Figure 9: Steady-states with a primary budget surplus and a pay-as-you-go pension scheme

3.7 Steady-State Stability Analysis

This section comprises two parts. In the first part, the general steady-state conditions of long run equlibria in an economy are analysed. In the second part specific dynamics at the high debt-capital ratio steady-states (k^1,η^1) are outlined. These dynamics are of special interest when it comes to fiscal adjustment policy.

3.7.1 General Steady-State Stability Analysis

This section evalutes the general steady-state conditions of long run equilibria in economies with a pay-as-you-go pension scheme. The approach to examine the stability conditions is the same as the one used in Chapter 2, that described an economy without a pension scheme. The same steps are therefore followed: In order to establish the steady-state stability conditions, the equilibrium dynamics near the steady-state are approximated linearly using the Jacobian matrix of the first partial derivatives of the two Equations (3.19) and (3.20). The linear approximation of both laws of motion is performed using a first-order Taylor approximation:

$$k_{t+1} = k + \frac{\partial k_{t+1}}{\partial k_t}(k_t - k) + \frac{\partial k_{t+1}}{\partial \eta_t}(\eta_t - \eta)$$

$$\eta_{t+1} = k + \frac{\partial \eta_{t+1}}{\partial k_t}(k_t - k) + \frac{\partial \eta_{t+1}}{\partial \eta_t}(\eta_t - \eta)$$

The compact form of this system of equations using matrix and vector notation reads as follows:

$$\begin{bmatrix} k_{t+1} \\ \eta_{t+1} \end{bmatrix} = \mathbb{J} \begin{bmatrix} k_t \\ \eta_t \end{bmatrix} + \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \mathbb{J} \right] \begin{bmatrix} k \\ \eta \end{bmatrix}, \tag{3.30}$$

where $\mathbb{J}$ denotes the Jacobian matrix that is defined by the first partial derivatives and reads as follows:

$$\mathbb{J} = \begin{bmatrix} \frac{\partial k_{t+1}}{\partial k_t} & \frac{\partial k_{t+1}}{\partial \eta_t} \\ \frac{\partial \eta_{t+1}}{\partial k_t} & \frac{\partial \eta_{t+1}}{\partial \eta_t} \end{bmatrix}$$

The linearised difference equation system in Equation (3.30) has a general solution:

$$k_t = k + \kappa_1 e_1^k (\lambda_1)^t + \kappa_2 e_2^k (\lambda_2)^t \tag{3.31}$$

$$\eta_t = \eta + \kappa_1 e_1^\eta (\lambda_1)^t + \kappa_2 e_2^\eta (\lambda_2)^t, \tag{3.32}$$

where κ_1 and κ_2 are constants. The eigenvectors are defined as $e_i = \left(e_i^k, e_i^\eta\right)^T$ with $i = 1, 2$. The eigenvalues of the Jacobian matrix $\mathbb{J}$ of the dynamic system equations from Equations (3.19) and (3.20) are defined as λ_i with $i = 1, 2$. The general solution of the system of difference from Equations (3.31) and (3.32) has the same form as the equivalent system in Chapter 2. The system's stability therefore depends on the absolute values of the eigenvalues (λ_1, λ_2) of the Jacobian matrix. Hence, it can be recalled that an equilibrium is asymptotically stable if $|\lambda_1| \leq |\lambda_2| < 1$, saddle-path stable if $|\lambda_1| < 1 \leq |\lambda_2|$, and explosive if $1 < |\lambda_1| \leq |\lambda_2|$. The specific values of the eigenvalues are now needed to derive the stability conditions. In order to evaluate them, the partial derivatives of the Jacobian matrix at any steady-state must be calculated. The results of this calculations are shown here while the full derivations can be found in Appendix A:

$$\frac{\partial k_{t+1}}{\partial k_t} = \alpha \tag{3.33}$$

$$\frac{\partial k_{t+1}}{\partial \eta_t} = \frac{-\alpha k^{\alpha}}{(1+\omega\tau^P)(1+n)} \tag{3.34}$$

$$\frac{\partial \eta_{t+1}}{\partial k_t} = 0 \tag{3.35}$$

$$\frac{\partial \eta_{t+1}}{\partial \eta_t} = \frac{\alpha(1+\omega\tau^P+\eta)}{k^{1-\alpha}(1+n)(1+\omega\tau^P)} \tag{3.36}$$

The eigenvalues of the Jacobian matrix can now be calculated using the partial derivatives at the steady-state derived above and by setting the determinant of the matrix $[\mathbb{J} - \lambda I]$ equal to zero. This yields:

$$det\ [\mathbb{J} - \lambda I] = \begin{vmatrix} \frac{\partial k_{t+1}}{\partial k_t} - \lambda_1 & \frac{\partial k_{t+1}}{\partial \eta_t} \\ \frac{\partial \eta_{t+1}}{\partial k_t} & \frac{\partial \eta_{t+1}}{\partial \eta_t} - \lambda_2 \end{vmatrix} = \begin{vmatrix} \alpha - \lambda_1 & \frac{-\alpha k^{\alpha}}{(1+\omega\tau^P)(1+n)} \\ 0 & \frac{\alpha(1+\omega\tau^P+\eta)}{k^{1-\alpha}(1+n)(1+\omega\tau^P)} - \lambda_2 \end{vmatrix}$$

$$= (\alpha - \lambda_1)\left(\frac{\alpha(1+\omega\tau^P+\eta)}{k^{1-\alpha}(1+n)(1+\omega\tau^P)} - \lambda_2\right) = 0$$

The eigenvalues in the case of an economy with a pay-as-you-go pension scheme hence take on the following form:

$$\lambda_1 = \alpha \qquad \lambda_2 = \frac{\alpha(1+\omega\tau^P+\eta)}{k^{1-\alpha}(1+n)(1+\omega\tau^P)} \tag{3.37}$$

Equivalent to an economy without pension schemes the first eigenvalue λ_1 is smaller than unity because the elasticity of capital α is assumed to be between 0 and 1. The second eigenvalue λ_2 in turn may be smaller or greater than unity. Long run equilibria that are explosive can therefore be ruled out. In what follows, using the analytical approach the steady-state stability properties are analysed for the three different scenarios of a government running a primary budget deficit, a balanced primary budget and a primary budget surplus. To understand the dynamics intuitively, one can recall Chapter 2, as the intuition behind the dynamics stays the same.

- Primary budget deficit ($\theta = 0.01$)

  - Figure 7 plots the case of a primary budget deficit and pay-as-you-go pension scheme with contributions that lead to a unique long run equilibrium termed C. It can be shown that in this case λ_2 equals 1, indicating a saddle-path long run equilibrium. For pay-as-you-go contribution rates that are too small to create a unique equilibrium, two equilibiria exist. In such a case it can be shown that the low capital intensity but high debt-capital ratio long run eqilibrium (k^1,η^1) is always saddle-path stable while the high capital intensity but low debt-capital ratio long run eqilibrium (k^2,η^2) is asymptotically stable.

- Balanced primary budget ($\theta = 0$)

  - The case of a government running a balanced primary budget and a pay-as-you-go pension scheme is illustrated in Figure 8. The steady-state (k^1,η^1) termed C has a relatively low capital intensity leading to a $\lambda_2 > 1$ and to saddle-path stability. The second equilibrium (k^2,η^2) termed D comes with a negative debt-capital ratio and a relatively high capital intensity such that $\lambda_2 < 1$, making the equilibrium asymptotically stable.

- Primary budget surplus ($\theta = -0.01$)

  - Figure 9 plots the steady-states in a scenario of a government with a primary budget surplus and a pay-as-you-go pension scheme placing the first steady-state on the Golden Rule growth path. The first steady-state (k^1,η^1) termed C is saddle-path stable as the relatively high debt-capital ratio together with the relatively low capital intensity causes λ_2 to be greater than unity. In the case of the high capital intensity steady-state (k^2,η^2) termed D, the debt-capital ratio η^2 is negative causing the the nominator in Eqaution (3.37) to become small and λ_2 to be less than unity. The steady-state's (k^2,η^2) convergence dynamics are therefore characterised by asymptotic stability.

As in the model without pensions from Chapter 2, the stability analysis shows that whenever there are two steady-states, the steady-state with the higher debt-capital ratio (k^1,η^1) is saddle-path stable and the steady-state with the lower debt-capital ratio (k^2,η^2) is asymptotically stable. Additionally, it has been demonstrated that when there is a unique tangent steady-state, this steady-state is characterised by saddle-path stability.

3.7.2 Specific Dynamics at the High Debt Steady-State

To derive optimal fiscal adjustments to changes or shocks to certain parameters, more specific knowledge about the dynamics in the neighbourhood of the high debt steady-state (k^1,η^1) is required. The linearised difference equation system of the laws of motion was presented by Equations (3.31) and (3.32). Using this linearised difference equation system, a specific linear approximation of the dynamics that respects the initial saddle-path SS is performed based on Farmer and Schelnast (2013). The first eigenvector $\left(e_1^k, e_2^\eta\right)^T$ can be used to solve the following matrix equation:

$$\mathbb{J}\begin{bmatrix} e_1^k \\ e_1^\eta \end{bmatrix} = \lambda_1 \begin{bmatrix} e_1^k \\ e_1^\eta \end{bmatrix} \tag{3.38}$$

From the steady-state stability analysis, it is known that $\lambda_1 = \alpha$. The partial derivatives from Equations (3.33) to (3.36) can be inserted into the Jacobian matrix $\mathbb{J}$ to solve for the eigenvectors:

$$\begin{bmatrix} e_1^k \\ e_2^\eta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.39}$$

Knowing the value of the first eigenvector, it is possible to establish specific solutions to the linearised difference equation system in Equations (3.31) and (3.32). To respect the initial saddle-path SS, explosive dynamics have to be prevented by setting κ_2 equal to zero. Together with Equation (3.39), the specific linearly approximated solutions of the dynamics of the debt-capital ratio η_t and the capital intensity k^t read as follows:

$$\eta_t = \eta^1, \tag{3.40}$$

$$k_t = k^1 + \kappa_1(\alpha)^t, \tag{3.41}$$

where η^1 is the first root of the second degree polynomial in Equation (3.27). It determines the debt-capital ratio of the low capital intensity steady-state (k^1,η^1) that coincides with the saddle-path SS. Setting t equal to zero yields: $\kappa_1 = k_0 - k^1$. Summing up, the specific dynamics at the high debt steady-state can be approximated by the following laws of motion of η_t and k_t:

$$\eta_t = \eta^1, \qquad t = 0, 1, 0, ..., \tag{3.42}$$

$$k_t = k^1 + (k_0 - k^1)\alpha^t \qquad t = 0, 1, 0, ..., \tag{3.43}$$

The purpose now is to find a specific mix of the three policy tools that ensures that the economy is maintained on the stable arm. Note first that Equations (3.42) and (3.43) are over-identified if b_0, k_0 and therefore η_0 are historically given. The determinacy is restored by making parameters in η^1 endogenous. For obvious reasons, the chosen parameters are the three policy tools (Γ,τ^w,τ^{P_C}). To prevent the equilibrium dynamics from a breakdown and to keep the economy on the stable arm, the government is not free to choose any values for its policy tools. In fact, the policy tool mix (Γ,τ^P,τ^w) must satisfy the condition that the first root of the second degree polynomial in Equation (3.27) equals η_0 at any point in time:

$$\eta^1 = \eta_0 = \frac{\dfrac{\varphi(1-\tau^w-\tau^P)-\theta}{(1+\omega\tau^P)} - \alpha + \sqrt{\left[\dfrac{\varphi(1-\tau^w-\tau^P)-\theta)}{(1+\omega\tau^P)} - \alpha\right]^2 - (4\alpha\theta)/(1+\omega\tau^P)}}{\dfrac{2\alpha}{(1+\omega\tau^P)}} \tag{3.44}$$

As long as this Equation holds true the government manages to maintain the economy on the stable arm at a debt-capital ratio equal to η_0. If the government fails to respect the condition by its policy tool mix, the economy either jumps above the saddle-path, where it collapses due to an explosive debt-capital ratio, or jumps below the saddle-path, where the economy converges to the low debt long run equilibrium (k^2,η^2) that is very dynamically inefficient. Thus, there is one Equation (3.44) to determine Γ,τ^w and τ^P. This shows that a change to any of the three policy tools (Γ,τ^{P_C},τ^w) has to be answered by an appropriate adjustment to the remaining two tools to restore fiscal sustainability.

3.8 Efficiency Improvement Policy

It has been shown that the long run equilibria in the economy with a pay-as-you-go pension scheme are more efficient than in the same economy without a public pension scheme. The interesting question that arises now is whether an economy that is in an inefficient long run

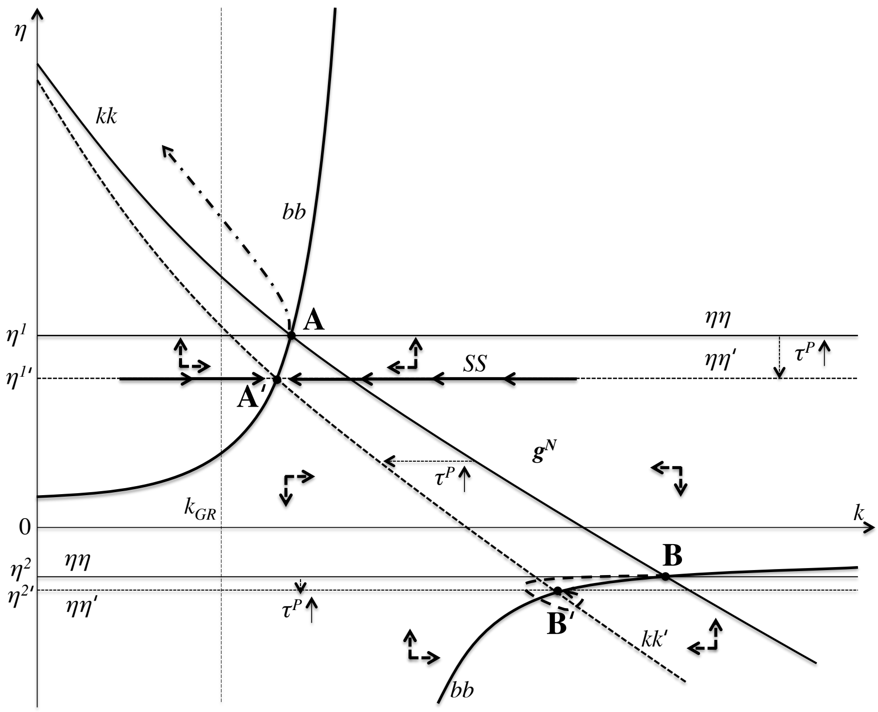

Figure 10: Effects of an increase in the pension contribution rate τ^P

equilibrium can simply increase the scope of the pay-as-you-go pension scheme by raising the pension contribution rates τ^P to increase welfare. Whether this is possible and how efficiency can be improved is illustrated using the example of an economy with a government running a primary budget surplus.

3.8.1 Medium and Long Run Dynamics

To understand the evolvement of the economy over time following an increase in the pension contribution rate τ^P, the location of the pre- and post-increase long run equilibria as well as the dynamics of the economy at the time of the shock need to be known. As the results of the following analysis hold for both a defined benefit and a defined contribution scheme, the analysis is demonstrated for the general case. All information required to locate the pre- and post-adjustment long run equilibria of the economy has already been provided.

It is now assumed that the government initially runs a primary budget surplus with $\theta = -0.01$ and that a small-sized pay-as-you-go pension scheme is in place with a contribution rate $\tau^P = 0.03$. The resulting long run equilibria are illustrated in Figure 10. The initial high debt-capital ratio equilibrium (k^1, η^1) is termed A, while the low debt-capital ratio equilibrium

(k^2,η^2) is termed B. Previous analysis has shown that an increase in τ^P does not affect the bb-phaseline but lowers the intercept of the kk-phaseline from Equation (3.25) and makes its slope steeper. The kk-phaseline induced by a higher τ^P is termed kk' in Figure 10. The post-increase long run equilibria termed A' and B' unsurprisingly come with a lower capital intensity and a lower debt-capital ratio. The initial steady-states A and B are put in an off-steady-state position. Therefore, one has to recall the medium run dynamics that are defined by the two laws of motion from Equations (3.21) and (3.22). The increase of τ^P to $\tau^{P'}$ is assumed to take place during period $t = 1$. Therefore, the evolution of the capital intensity k_t and debt per efficient capita b_t has to be evaluated at $t = 2$. Knowing that the economy started in a long run equilibrium, k_1 equals k_0 such that the capital intensity and debt per efficient capita read as follows at time $t = 2$:

$$k_2 = \frac{1}{(1+n)(1+\omega\tau^{P'})}\left[\varphi(1-\tau^w-\tau^{P'})k_0^\alpha - \eta_0\alpha k_0^\alpha - \theta k_0^\alpha\right] \tag{3.45}$$

$$b_2 = \frac{1}{(1+n)}\left[\alpha k_0^\alpha \eta_0 + \theta k_0^\alpha\right] \tag{3.46}$$

Equation (3.45) reveals that a change in the pension contribution rate has a one period lagged effect on the capital intensity k_t and an indirect two period lagged effect on the debt per efficient capita b_t. It can be concluded that at time $t = 1$ only the long run equilibria change from A to A' and from B to B' but the economy remains in either A or B.

The attention is firstly focused on the dynamics at the low debt-capital ratio long run equilibrium (k^2,η^2) that is termed B in the graph. At time $t = 1$, the dynamics of the economy placed in B are determined by the dynamics of the newly induced long run equilibrium $(k^{2'},\eta^{2'})$ termed B'. Thinking back to the steady-state stability analysis, it is known that the low debt-capital ratio steady-state $(k^{2'},\eta^{2'})$ is asymptotically stable. The economy will therefore move from B to B' in finite time. The transition leads to a lower debt-capital ratio and a lower capital intensity and hence to an efficiency improvement. The dynamics at the high debt-capital ratio equilibrium (k^1,η^1) look different compared to the ones at the low debt-capital ratio equilibrium $(k^{2'},\eta^{2'})$ just analysed. At the time of the shock $t = 1$, the economy stays at its initial long run equilibrium (k^1,η^1) named A in Figure 10. However, as mentioned before, the medium run dynamics are then defined by the newly induced long run equilibrium $(k^{1'},\eta^{1'})$ named A'. This

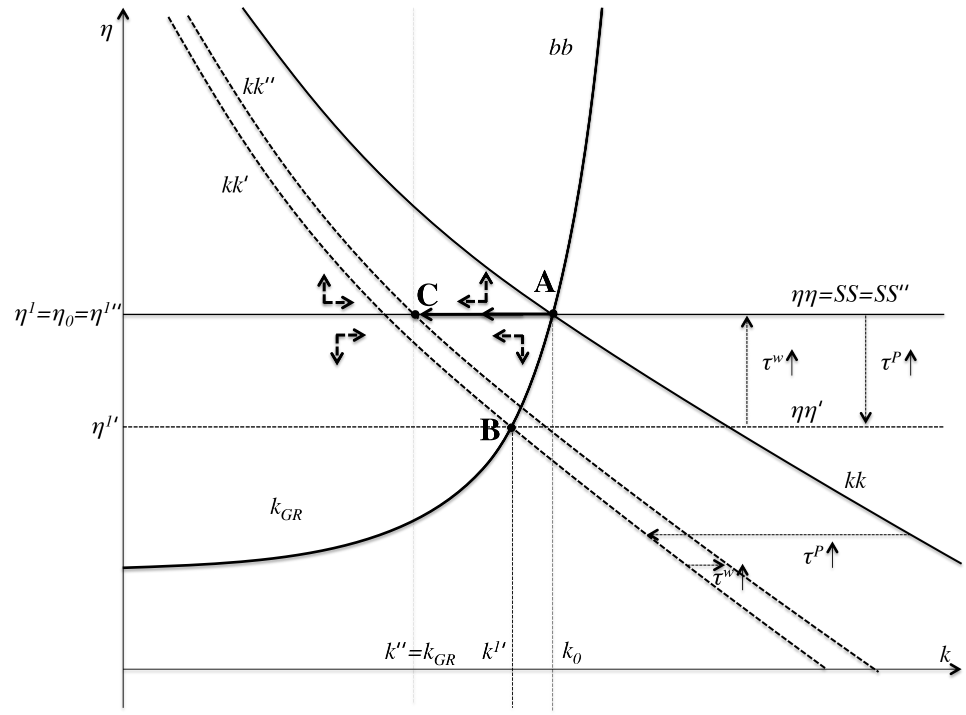

Figure 11: Dynamics following a sustainable increase in τ^P and τ^w

long run equilibrium is saddle-path stable and puts the economy placed in A on an explosive debt path on which the economy will collapse in finite time. In the graph, this is illustrated by the dashed arrow pointing away from point A to the top left. In order to keep the economy starting in a high debt-capital equilibrium on a sustainable path, the government needs to use fiscal adjustment policy.

3.8.2 Policy Mix

Increasing the pension contribution rate τ^P at a high debt-capital ratio equilibrium (k^1,η^1) has proven to put the economy on an unsustainable path. To prevent these explosive dynamics, the government has to choose a policy mix (Γ,τ^w,τ^{P_C}) that keeps the economy on the intitial saddle-path at a level of $\eta^1 = \eta_0$, which was defined by Equation (3.44). The equation reveals that whenever the government raises the pay-as-you-go contribution rate by $\Delta\tau^P > 0$, it has to lower the primary deficit ratio θ by either increasing the tax rate τ^w or by decreasing the expenditure ratio Γ in order to restore a sustainble path. Here it is assumed that the government chooses to keep the expenditure ratio constant and to immediately adjust the tax rate by $\Delta\tau^w$ given a change to τ^P. Unfortunately, a closed form solution for $\Delta\tau^w$ given a certain $\Delta\tau^P$ has

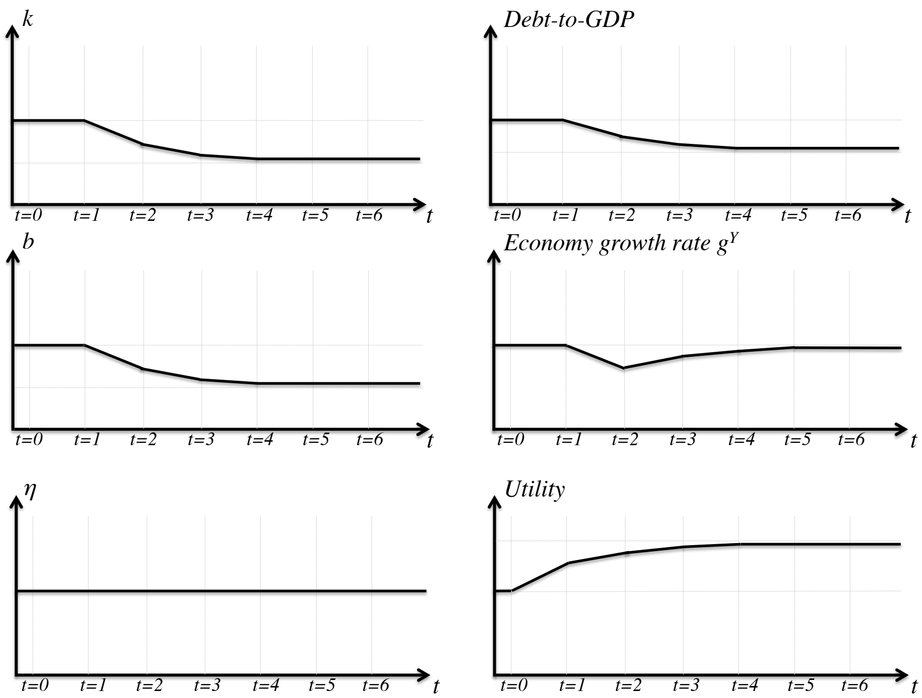

Figure 12: Time-evolution of key variables following a sustainable increaes in τ^P and τ^w

not been found such that one has to numerically examine the optimal adjustment of the tax rate given an increase in the pension contribution rate. Hence, note that the economy considered is one with a government running a primary budget surplus ($\theta = -0.01$) and a small-sized pay-as-you-go pension scheme is in place ($\tau^P = 0.03$). In this case the optimal policy mix given a constant Γ requires an increase in the pension contribution rate by $\Delta\tau^P = 0.0395$ together with a rise in the tax rate by $\Delta\tau^w = 0.01932$. Choosing this policy mix, the government places the post-adjustment long run equilibrium ($k^{1''}$,$\eta^{1''}$) on the Golden Rule growth path.

To understand how the phaselines change, recall that the higher the contribution rate τ^P the lower the kk-phaseline's intercept and slope. The higher tax rate only pushes the intercept slightly up because $\varphi < (1-\alpha)$ but does not affect the slope. The results are summarized in Figure 11. The economy is initially placed in the high debt-capital ratio equilibrium ($k^1 = k_0$,$\eta^1 = \eta_0$) termed A in the graph. An increase in τ^P shifts the intercept and the slope of the kk-phaseline termed kk' and therefore the $\eta\eta$-phaseline termed $\eta\eta^{'}$ down. This induces the long run equilibrium ($k^{1'}$,$\eta^{1'}$) named B. Since the rise in τ^P is immediately met by an appropriate increase in τ^w, the $\eta\eta$-phaseline shifts up again to a level of η_0 while the kk-phaseline slightly raises its intercept. This results in the final post-adjustments long run equilibrium ($k^{1''}$,$\eta^{1''}$)

termed C. This equilibrium comes with the initial debt-capital ratio η_0 and the Golden Rule capital intensity $k^{1''} = k_{GR}$.

The time-evolution of key economic variables is depicted in Figure 12. During the transition from A to C, the capital intensity k_t and debt per efficient capita b_t decrease proportionally causing the debt-capital ratio η to stay constant at any point in time. However, the debt-to-GDP ratio $\widetilde{b_t}$, which is defined as $\widetilde{b_t} \equiv B_t/Y_t = b_t/(k_t)^{1-\alpha}$, decreases. The growth rate of the economy g^Y defined in Equation (3.24) that only equals the natural growth rate n in equilibrium declines following the policy adjustments before it raises and converges to the level of the natural grwoth rate again. This evolvement of the growth rate of the economy g^Y is solely caused by the decrease in the capital intensity. The first generation affected is the one born in period $t = 1$. Their wage is unaffected as the adjustment only has a one period lagged effect. However, the interest rate they receive on their savings increases due to the lower capital intensity at time $t = 2$. The result is a rise in utility. The utility of future generations then converges to a higher level as the economy moves closer to the Golden Rule capital intensity.[3]

The analysis has been conducted for an economy with a government running a primary budget surplus. It can be shown that also in the case of a government running either a primiry budget deficit or a balanced primary budget for a given Γ, an increase in τ^P at the high debt-capital ratio equilibrium (k^1,η^1) has to be answered by a rise in the tax rate τ^w to keep public debt on a sustainable path and to place the economy on the Golden Rule path. Extending the scope of the pay-as-you-go scheme at the low debt-capital ratio long run equilibrium (k^1,η^1) would lead to a convergence to the newly induced equilibrium without further policy action.

[3]Note that utility stated in this thesis is always measured relatively. It is the difference between the utility that would have been realised without the shock and the actually realised utility following the shock. In general, utility follows an upwards trend because the growth rate of technology g^a is assumed to be greater than zero.

4 Demographic Change

4.1 Introduction

Most countries in the world face demographic changes and are confronted with its challenges. There are two main reasons: the first reason is that fertility rates are declining.[4] As a result the addition of new workers is reduced. The second reason is the increased life exptectancy causing people to live longer on average.[5] For a given retirement age, these two phenomenona lead to a higher dependency ratio. Table 1 presents the past and the forecasted increase in the dependency ratio. It can be inferred that in some of the countries the dependency ratio is almost going to double between 2015 and 2075. This has important implications that policy makers should be aware of in order to respond with appropriate adjustments. This chapter aims to illustrate some of the consequences of demographic change and how they should be dealt with.

First, a permanent increase in the dependency ratio is simulated for a government able to keep its expenditure ratio fixed. In a second step, it is assumed that the rise in the dependency ratio occurs simultaneously with a positive expenditure shock. This is motivated by the fact that a higher share of elderly people that live longer on average challenges the fiscal position of the government. OECD (2013) expects the combined public health and long-term care expenditure in OECD countries to rise from currently 6% to 9.5% of GDP in 2060. As a response to this expenditure shock optimal fiscal adjustment policy is derived.

4.2 Demographic Shock under Defined Contributions

Here, the effects of a permanent drop in the population growth rate g^N that leads to an increase in the dependency ratio are analysed. It is assumed that the economy is intially located in the high debt-capital ratio long run equilibrium (k^1,η^1). In addition, a pay-as-you-go pension scheme with defined contribution rates is in place. Then the evolvement of the economy over time is examined following a negative shock to g^N. To understand the dynamics over time, the

[4] See Table 2 in Appendix C

[5] See the related rise of the life expectancy at birth depicted in Table 3 in Appendix C

pre- and post-shock long run equilibria, the effect at the time of the shock and the dynamics around the post-shock steady-states must be known.

TABLE 1: Dependency Ratio

in %	Year						
	1950	**1975**	**2000**	**2015**	**2025**	**2050**	**2075**
Belgium	18.1	25.2	28.3	30.6	37.1	51.0	54.0
Denmark	15.6	23.7	24.2	33.0	37.7	45.3	53.4
Finland	11.9	18.1	24.8	35.0	44.0	48.8	54.7
France	19.5	24.5	27.3	33.3	40.9	52.3	55.8
Germany	16.2	26.5	26.5	34.8	41.4	59.2	63.1
Greece	12.4	20.9	26.7	33.0	39.2	73.4	75.2
Italy	14.3	21.6	29.2	37.8	45.6	72.4	67.0
Netherlands	13.9	19.3	21.9	30.2	39.0	53.0	59.7
Norway	16.0	24.9	25.9	27.4	32.5	43.1	51.2
Portugal	13.0	19.6	26.8	34.6	42.4	73.2	77.6
Spain	12.8	19.0	26.9	30.6	38.6	77.5	70.4
Sweden	16.8	26.3	29.5	33.8	38.2	45.5	51.6
Switzerland	15.8	21.5	24.9	29.0	35.4	54.6	58.1
United Kingdom	17.9	25.5	27.0	31.0	35.9	48.0	53.0
United States	14.2	19.7	20.9	24.6	32.9	40.3	49.3
OECD	13.9	19.5	22.5	27.9	35.2	53.2	58.6

Source: The Organisation for Economic Co-operation and Development (OECD)

4.2.1 Effect on Long Run Equilibria

The long run equilibria exist where the kk-,$\eta\eta$- and the bb-phaselines cross eachother. Thus, one must look at how these phaselines are affected by a decrease in g^N to understand the changes to the long run equilibrium. In a defined contribution scheme, the kk-phaseline from Equation (3.25) and the bb-phaseline from Equation (3.26) read as follows:

$$\eta = \frac{\varphi(1-\tau^w-\tau^{P_C})-\theta}{\alpha} - \frac{k^{1-\alpha}\big[(1+g^N)(1+g^a)(1+\omega\tau^{P_C})\big]}{\alpha} \tag{4.1}$$

$$\eta = \frac{\theta}{(1+g^N)(1+g^a)k^{1-\alpha}-\alpha} \tag{4.2}$$

The intercept of the kk-phaseline in Equation (4.1) remains the same but the negative slope becomes less steep following a drop in the population growth rate g^N. The first thing to notice about the bb-phaseline in Equation (4.2) is that it yields $\eta = 0$, which is independent of g^N, for a government running a balanced primary budget ($\theta = 0$). As explained in previous chapters,

k^* is the capital intensity that solves $f'(k) = 1 + n$. The bb-phaseline is not defined for $k = k^*$ as its denominator becomes 0. Furthermore, it has been seen that in the case of a primary budget deficit the bb-phaseline converges to negative infinity approaching k^* from the left-hand side and to positive infinity approaching k^* from the right-hand side. In the case of a primary budget surplus the bb-phaseline converges to positive infinity approaching k^* from the left-hand side and negative infinity approaching k^* from the right-hand side. As previously mentioned, k^* is the capital intensity that leads to a denominator of 0. Looking at the denominator one can infer that the lower g^N the higher k^*. It can be concluded that a decrease in g^N raises k^*, which defines the point where the bb-phaseline flips from negative to positive infinity and vice versa. The less steep slope of the kk-phaseline together with the change of the bb-phaseline increases the equilibrium capital intensity long run equilibrium (k^1,η^1). The final phaseline to look at is the high debt-capital $\eta\eta$-phaseline from Equation (3.28) that in a defined contribution scheme can be written as:

$$\eta^1 = \frac{\psi + \sqrt{\psi^2 - (4\alpha\theta)/(1 + \omega\tau^{P_C})}}{(2\alpha)/(1 + \omega\tau^{P_C})}, \tag{4.3}$$

where ψ under a defined contribution schemes is defined as $\psi \equiv [\varphi(1 - \tau^w - \tau^{P_C}) - \theta]/(1 + \omega\tau^{P_C}) - \alpha$. It becomes instantly clear that the $\eta\eta$-phaseline does not depend on g^N such that the long run equilibrium debt-capital ratio remains the same following a demographic change. The long run capital intensity, in turn, increases. This holds for all three cases of a government running either a primary budget deficit, a primary budget surplus, or a balanced primary budget. Figure 13 illustrates pre-shock long run equilibrium (k^1,η^1) termed A and the post-shock long run equilibria ($k^{1\prime}$,$\eta^{1\prime}$) termed B for an economy with a government running a balanced primary budget. The parameter values chosen to depict this graphical illustration are the same as in Chapter 2 and complemented with the addition of pay-as-you-go contributions $\tau^{P_C} = 0.04$. Looking at the graph, one can observe that $\eta\eta$-phaselines are unaffected by the change in g^N, while the kk-phaseline's slope becomes less steep.

4.2.2 Dynamics in the Medium and Long Run

Knowing the location of the pre- and post-shock long run equilibria the steady-state stability conditions and the effect at the time of the shock can be evaluated to then derive the dynamics over time. The steady-state stability analysis in Chapter 3 revealed that the high debt-capital

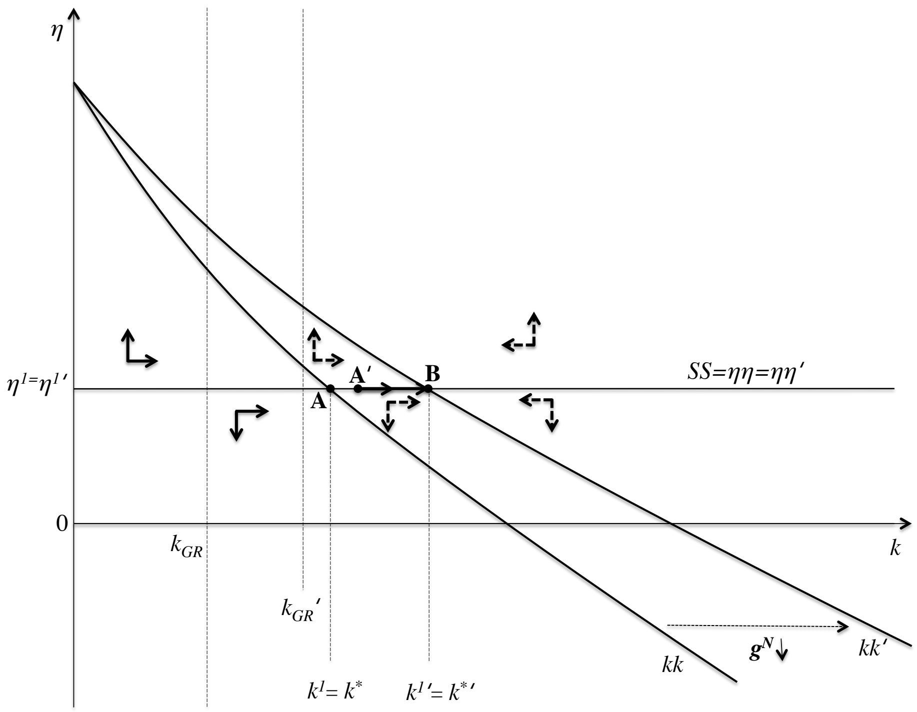

Figure 13: Demographic change under a defined contribution scheme

ratio steady-states are saddle-path stable. This holds for the long run equilibria before and after the demographic shock.

The effect at the time of the shock can be evaluated recalling Equations (3.21) and (3.22), which determine the evolution of k_t and b_t over time. In the case of a defined contributions they read as follows:

$$k_t = \frac{1}{(1+g_t^N)(1+g^a)(1+\omega\tau^{P_C})}\Big[\varphi(1-\tau^w-\tau^{P_C})-\eta_{t-1}\alpha-\theta\Big]k_{t-1}^{\alpha} \tag{4.4}$$

$$b_t = \frac{1}{(1+g_t^N)(1+g^a)}\Big[\alpha\eta_{t-1}+\theta\Big]k_{t-1}^{\alpha} \tag{4.5}$$

If it is assumed that the negative shock to the population growth rate g^N occurs at time t, it becomes instantly clear that the capital intensity k_t and the debt per efficient capita b_t increase as long as the expressions in squared brackets is positive. For the chosen parameter values and the high debt-capital ratio equilibria considered in this thesis, the expression in squared brackets is, in fact, positive. Thus, both the capital intensity k_t and the debt per efficient capita b_t increase at the time of the shock. To understand what happens with the debt-capital ratio

η_t divide Equation (4.5) by Equation (4.4), which yields:

$$\eta_t = \frac{(\alpha\eta_{t-1} + \theta)(1 + \omega\tau^{P_C})}{\varphi(1 - \tau^w - \tau^{P_C}) - \eta_{t-1}\alpha - \theta} \quad (4.6)$$

Equation (4.6) reveals that η_t only depends on constants and the debt-capital ratio from the period $t-1$. Thus, the debt-capital ratio η_t remains at its initial equilibrium value at the time of the shock. To sum up the effects at the time of the shock, the economy jumps to a point with a higher capital intensity and higher debt per efficient capita but with the same debt-capital ratio. In Appendix B it is demonstrated that this new starting point results in a lower capital intensity than the capital intensity of the post-shock long run equilibrium. This means that the economy does not jump to the post-shock long run equilibrium straight away.

Before and after the demographic shock, the high debt-capital ratio steady-state (k^1,η^1) is saddle-path stable. Earlier analyses have shown that this saddle-path coincides with the high debt-capital ratio $\eta\eta$-phaseline, which remains unchanged following a drop in the population growth rate. Hence, after the jump from the initial long run equilibrium (k^1,η^1) to a point on the $\eta\eta$-phaseline with a higher capital intensity the economy gradually moves along the saddle-path increasing its capital intensity until it arrives at the post-shock high debt-capital ratio long run equilibrium ($k^{1\prime}$,$\eta^{1\prime}$). During the whole transition from the pre- to the post-shock equilibrium the debt-capital ratio is kept constant. Figure 13 illustrates this transition. The economy starts in the pre-shock long run equilibrium (k^1,η^1) termed A before it jumps to point A' and then finally moves along the saddle-path to the post-shock long run equilibrium ($k^{1\prime}$,$\eta^{1\prime}$) termed B.

To further illustrate the dynamics, Figure 14 plots the time-evolution of key variables. Note that the shock is assumed to take place at $t = 1$ and that government is running a balanced primary budget. The plots nicely show that the capital intensity k and debt per efficient capita b leap up at $t = 1$ and then increase in the same magnitude causing η to remain constant over time. The intuition behind the increasing capital intensity is that the permanent drop in g^N makes labour force scarce and increases wages. The firms shift from human capital to real capital such that the capital intensity rises and the interest rate decreases. The higher wage and the lower interest rate allow financing a higher capital intensity while keeping the debt-capital constant. The debt-to-GDP ratio $\widetilde{b_t}$ instead increases following a negative shock to g^N. The first generation affected by the shock is the one born in $t = 0$. These households experience a

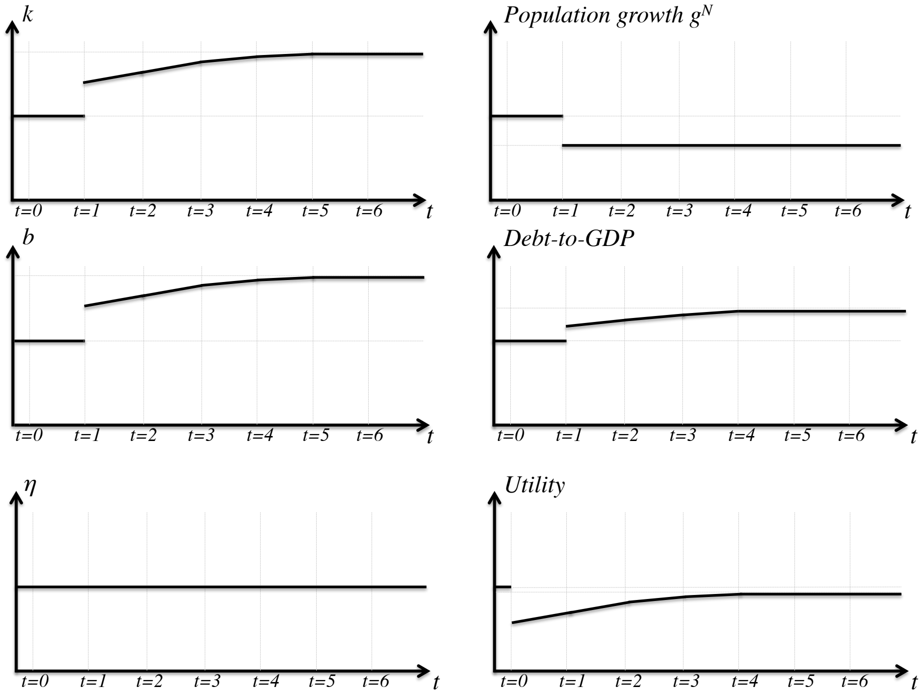

Figure 14: Time-evolution of key variables under a defined contribution scheme following a demographic shock

drop in utility as they receive a lower return on savings and pension contributions but a wage defined by pre-shock capital accumulation. The utility of future generations slightly rises and converges to a level below the pre-shock level. This indicates that the negative effect on utility of a reduced interest rate dominates the positive effect of a higher wage.

4.3 Demographic Shock under Defined Benefits

Now the effects of a permanent increase in the dependency ratio are analysed for an economy with a defined benefit pay-as-you-go pension scheme. This is done by focusing on the high debt-capital ratio long run equilibrium (k^1,η^1). Thus, it is assumed that the economy is initially placed in this low capital intensity but high debt-capital ratio long run equilibrium (k^1,η^1). To examine the evolvement of the economy over time, the pre- and post-shock long run equilibria, the effect at the time of the shock, and the dynamics around the post-shock steady-states are derived.

4.3.1 Effect on Long Run Equilibria

Again, one has to look at how the kk-,$\eta\eta$- and the bb-phaselines are affected by a decrease in g^N to understand the changes to the long run equilibrium. In a defined benefit scheme the kk-phaseline from Equation (3.25) and the bb-phaseline from Equation (3.26) read as follows:

$$\eta = \frac{\varphi\left(1-\tau^w-\dfrac{\tau^{P_B}}{1+g^N}\right)-\theta}{\alpha} - \frac{k^{1-\alpha}\left[(1+g^N)(1+g^a)\big(1+\omega\dfrac{\tau^{P_B}}{1+g^N})\right]}{\alpha} \tag{4.7}$$

$$\eta = \frac{\theta}{(1+g^N)(1+g^a)k^{1-\alpha}-\alpha} \tag{4.8}$$

Examining the intercept of the kk-phaseline in Equation (4.7), it is straight forward that a decrease in the population growth rate slightly lowers the intercept. Taking the derivative of the slope with respect to g^N yields: $-k^{1-\alpha}(1+g^a)/\alpha < 0$. A decline in g^N therefore leads to an increase of the slope of the kk-phaseline and makes it less steep. The effect of demographic change on the bb-phaseline under a defined benefit pay-as-you-go scheme is the exact same as in a defined contribution scheme. Therefore, the bb-phaseline just equals zero if a balanced primary budget is assumed. For $\theta \neq 0$ a decrease in g^N raises the capital intensity k^* for which the phaseline flips from negative to positive infinity. Very similar to the case of a defined contribution scheme the less steep slope of the kk-phaseline increases the equilibrium capital intensity of the long run equilibrium (k^1,η^1) regardless of the type of primary deficit the government runs. The last phaseline to look at is the high debt-capital ratio $\eta\eta$-phaseline from Equation (3.28). In the case of defined benefit scheme ψ is defined as $\psi \equiv [\varphi(1-\tau^w-\frac{\tau^{P_B}}{1+g^N})-\theta]/(1+\omega\frac{\tau^{P_B}}{1+g^N})-\alpha$ and the $\eta\eta$-phaseline can be written as:

$$\eta^1 = \frac{\psi+\sqrt{\psi^2-(4\alpha\theta)/(1+\omega\frac{\tau^{P_B}}{1+g^N})}}{(2\alpha)/(1+\omega\frac{\tau^{P_B}}{1+g^N})} \tag{4.9}$$

Unlike under defined contribution schemes, the debt-capital ratio of the long run equilibrium depends on the growth rate of the population. To understand how it reacts to a change in the growth rate of the population, one has to take the partial derivative of η^1 with respect to g^N. Using computational power, the partial derivative can be derived and analysed for various sets of parameter values. The analysis shows that η^1 rises in g^N given the parameter values and the high debt-capital ratio equilibria considered in this thesis. Hence, if g^N drops, η^1 declines as well.

To sum up, following a reduction in g^N the low capital intensity steady-state comes with a higher capital intensity $k^{1\prime}$ and a lower debt-capital ratio $\eta^{1\prime}$. The increase in the capital intensity mainly results from the less steep slope of the kk-phaseline. The intuition behind it is again that labour force gets scarce due to a lower g^N, which increases wages. The firms then shift from human capital to real capital and the capital intensity increases. While in the case of a defined contribution scheme it was possible to finance the higher capital intensity by increased wages while keeping the debt-capital ratio η^1 constant, this is not possible in the case of a defined benefit scheme anymore. In the case of a defined benefit scheme, the wage increases but so does the share of wages $\tau^P = \tau_t^{P_B}/(1+g_t^N)$ that has to be paid into the pay-as-you-go pensions scheme. There is therefore less room to finance a high public debt level and a higher capital intensity. Consequently, the equilibrium debt-capital ratio η^1 decreases. The effects of a drop in the population growth rate on the long run equilibrium (k^1,η^1) for an economy with a government running a balanced primary budget and a initial pay-as-you-go contribution rate of $\tau^P = \tau^{P_B}/(1+g^N) = 0.04$ are depicted in Figure 15. Note that the pre-shock long run equilibrium (k^1,η^1) is termed A and the post-shock long run equilibrium ($k^{2\prime}$,$\eta^{2\prime}$) is termed B.

4.3.2 Dynamics in the Medium and Long Run

Having defined where the pre- and post-shock long run equilibria are located, the evolvement of the economy over time starting in its initial equilibrium can be examined. Knowing the location of the pre-shock and post-shock long run equilibria, the evolvement of the economy is now defined by the effect at the time of the shock and the dynamics around the post-shock long run equilibria. To verify the effects of a negative shock to g^N, recall Equations (3.21) and (3.22) that determine the evolution of k_t and b_t over time. Under a defined benefit scheme these laws of motion can be written as:

$$k_t = \frac{1}{(1+g_t^N)(1+g^a)(1+\omega\frac{\tau^{P_B}}{1+g_{t-1}^N})}\left[\varphi(1-\tau^w-\frac{\tau^{P_B}}{1+g_{t-1}^N})-\eta_{t-1}\alpha-\theta\right]k_{t-1}^{\alpha} \tag{4.10}$$

$$b_t = \frac{1}{(1+g_t^N)(1+g^a)}\left[\alpha\eta_{t-1}+\theta\right]k_{t-1}^{\alpha} \tag{4.11}$$

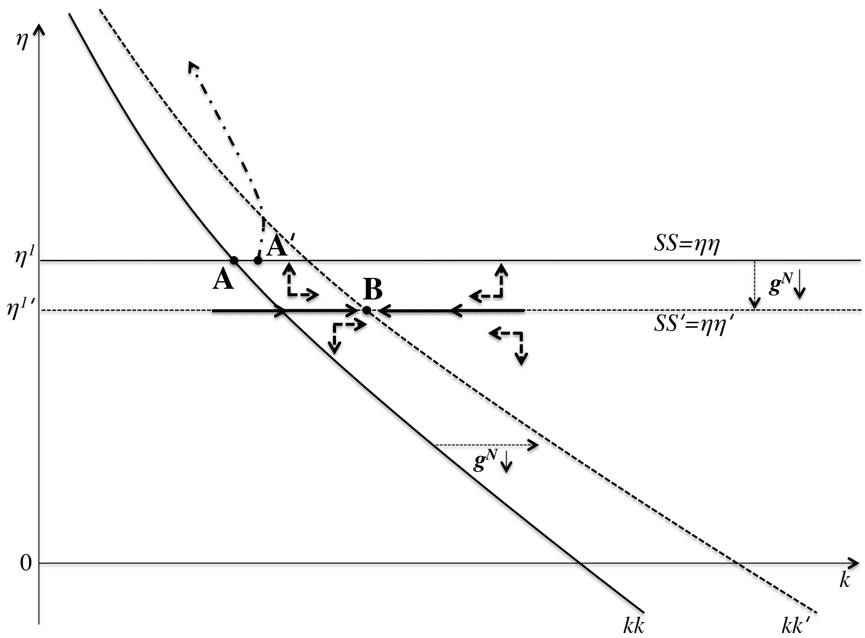

Figure 15: Dynamics after demographic change under a defined benefit scheme

The two Equations (4.10) and (4.11) demonstrate that a shock to the population growth rate g_t^N at time $t = 1$ increases both the capital intensity k_t and debt per efficient capita b_t. To understand how the debt-capital ratio η_t reacts to demographic change, Equation (4.11) needs to be divided by Equation (4.10). This yields:

$$\eta_t = \frac{(\alpha\eta_{t-1} + \theta)(1 + \omega\frac{\tau^{P_B}}{1+g_{t-1}^N})}{\varphi(1 - \tau^w - \frac{\tau^{P_B}}{1+g_{t-1}^N}) - \eta_{t-1}\alpha - \theta} \tag{4.12}$$

Equation (4.12) reveals that the debt-capital ratio η_t only depends on either constant parameters and or parameters from the previous period. Consequently, at the time of the shock $t = 1$ the debt capital ratio η_t remains constant and the capital intensity increases. This horizontal leap is illustrated in Figure 15 by the jump from the initial high debt-capital ratio equilibrium (k^1,η^1) termed A to point A'. The post-shock high debt-capital ratio long run steady-state ($k^{1\prime}$,$\eta^{1\prime}$) is saddle-path stable but comes with a lower debt-capital ratio $\eta^{1\prime}$. Since this capital intensity $\eta^{1\prime}$ now defines the post-shock $\eta\eta$-phaseline as well as the post-shock saddle-path, the economy in point A' is placed in an off-steady state position. The economy is on an unsustainable debt path, which is indicated by dashed arrows leading away from point A' in the graph.

For a better understanding of the dynamics, Figure 16 depicts the time-evolution of key variables of the economy. At the time of the shock at $t = 1$, the capital intensity k, debt per efficient capita b, and the debt-capital ratio η increase instantly due to the lower population growth. As discussed this puts the economy on an explosive debt path such that debt per efficient capita b and the debt-capital ratio η increase until the economy collapses. After the time of the shock, the capital intensity k increases further for one more period. Afterwards, the economy crosses the kk-phaseline and the whole economy starts to shrink to zero. The generation born in $t = 0$ experiences a decline in utility, as their savings pay lower interest, but receives a wage defined by pre-shock capital accumulation. The utility then starts to rise because the capital intensity starts to decrease, placing the economy closer to the Golden Rule capital intensity until at some point the capital intensity shrinks to zero. People then realise no utility as they die. This is illustrated by the sharp drop in utility after period $t = 6$.

The main lesson learnt is that at a high debt-capital equilibrium (k^1,η^1) with $\eta^1 > 0$ an increase in the dependency ratio, *ceteris paribus*, puts public debt and the economy on an unsustainable path. This happens even if the government manages to keep the expenditure ratio constant. To stay on a sustainable path the government would need to reduce the primary budget deficit by increasing taxes or lowering expenditures to prevent the debt from exploding.

4.4 Fiscal Adjustment Policy

As mentioned in the introduction to this chapter, government expenditures are likely to rise following an increase in the dependency ratio. In the model framework, this is simulated by a permanent negative shock to the population growth rate g^N that occurs simultaneously with a permanent positive shock to the governmental expenditure ratio Γ. As a reaction to these shocks, a vertical fiscal adjustment, the optimal fiscal policy in situations where a government is restricted from freely choosing its fiscal tools (Γ,τ^w,τ^P), is presented. Assuming that the government is not able to just scale back the expenditures and keeps τ^P fixed, the optimal fiscal response must be conducted by adjusting the tax rate τ^w. The simulation considered aims to match the projection of the OECD countries. Recall that one period in the model can be understood as 30 years. Starting in 2018 the next period in the model starts in 2048. As mentioned before, the OECD (2013) expects the combined public health and long-term care

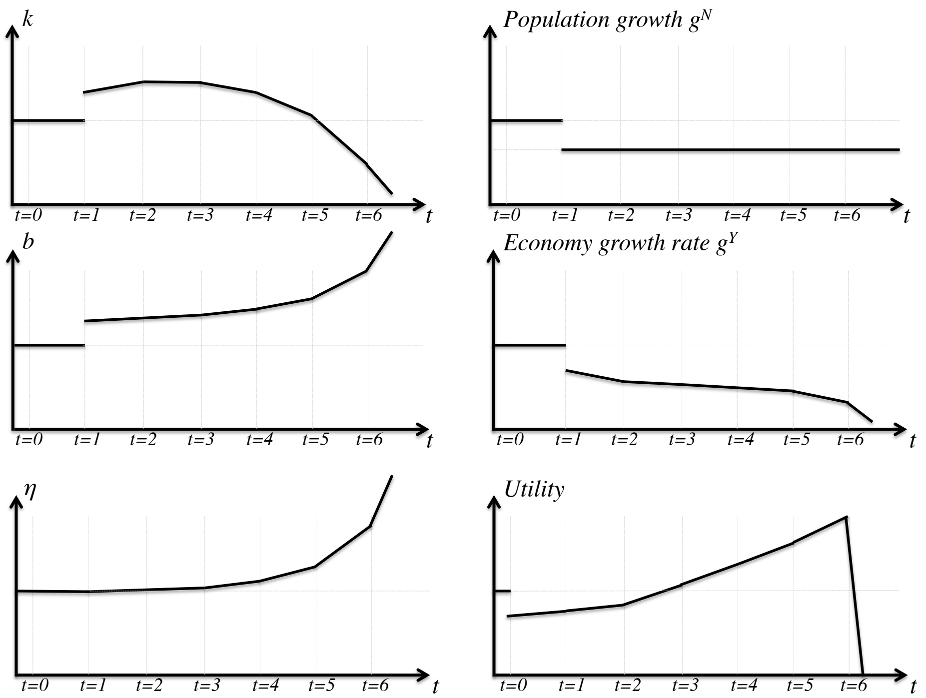

Figure 16: Time-evolution of key variables under a defined benefit scheme following a demographic shock

expenditure in OECD countries to rise from currently 6% to 9.5% of GDP in 2060. Assuming a linear trend, the combined public health and long-term care expenditure increase to 8.5% of GDP in year 2048. Thus, the shock to the expenditure ratio is set to $\Delta\Gamma = 0.025$. The dependency ratio is assumed to rise from 0.286 to 0.518, approximately matching the figures for the OECD countries in Table 1. The fiscal adjustment is illustrated for a pay-as-you-go scheme under defined contribution and under defined benefit. Initially, the pension contribution rate τ^P is set to 0.08 for both pension scheme designs. The government runs a balanced primary budget with a tax rate τ^w equal to 0.35. All other parameter values stay the same as in previous chapters. Finally, the economy is initially placed in the dynamically inefficient high debt steady-state (k^1,η^1) and the government runs a balanced primary budget.

4.4.1 Expenditure Shock under a Defined Contribution Scheme

4.4.1.1 Effect on Long Run Equilibrium

An exogenous positive shock to the expenditure ratio Γ leads to a higher value $\Gamma' \equiv \Gamma + \Delta\Gamma$. If the government does not adjust the tax rate τ^w, the shock induces a primary budget deficit ratio θ that rises to $\theta^{'} = \Gamma^{'} - \tau^{w}(1-\alpha)$. To see how this higher $\theta^{'}$ changes the high debt-capital ratio η^{1}, which is independent of g^{N} under a defined contribution scheme, one must recall Equation (3.28). It becomes instantly clear that a higher primary deficit ratio lowers the pre-shock long run debt-capital ratio η_{0} to the post-shock long run debt-capital ratio $\eta^{1^{'}}$. The negative relationship between the primary deficit ratio and the long run debt-capital ratio makes intuitively sense. A higher primary deficit reduces the capability of paying interest expenses and hence the sustainable long run debt must be at a lower level.

Having learnt that the equilibrium long run debt-capital ratio decreases following the exogenous shocks, the attention is now turned towards the post-shock long run capital intensity $k^{1^{'}}$. This capital intensity is defined by the kk-phaseline from Equation (4.1). A higher primary deficit ratio reduces the intercept of the kk-phaseline and the lower growth rate of the population g^{N} makes the slope less steep. The two effects are counteractive to the capital intensity. The reduced intercept induces a lower equilibrium capital intensity, while the increased slope causes a higher equilibrium capital intensity. Together with the reduced post-shock equilibrium debt-capital ratio $\eta^{1^{'}}$ the effect of the flatter slope outweighs the effect of the lower intercept. Consequently, the post-shocks equilibrium capital intensity $k^{1^{'}}$ is higher than the initial capital intensity k_{0}. The post-shock kk-phaseline is termed $kk^{'}$ in the graph. The findings of this paragraph are summarized in Figure 17.

4.4.1.2 Fiscal Adjustment and Dynamics

To illustrate that fiscal adjustment is necessary to keep the economy on a sustainable path the motion dynamics are first derived for the case of no fiscal response. The laws of motion of the capital and the debt per efficient capita are defined by Equations (4.4) and (4.5). The expenditure shock and the demographic shock are assumed to take place at the beginning of period $t = 1$. Thus, it is important to notice that the primary budget deficit does not have an instantaneous effect on the economy but an effect that is lagged by one period. At the time of the shock in period $t = 1$, only the change in the population growth rate g^{N} has an immediate impact on the capital intensity and debt per efficient capita. In the previous section on demographic change under a defined contribution scheme it was shown that both the capital and the debt per efficient capita increase following a negative shock to g^{N}. Both variables increase proportionally, keeping the debt-capital ratio constant. The capital intensity and debt

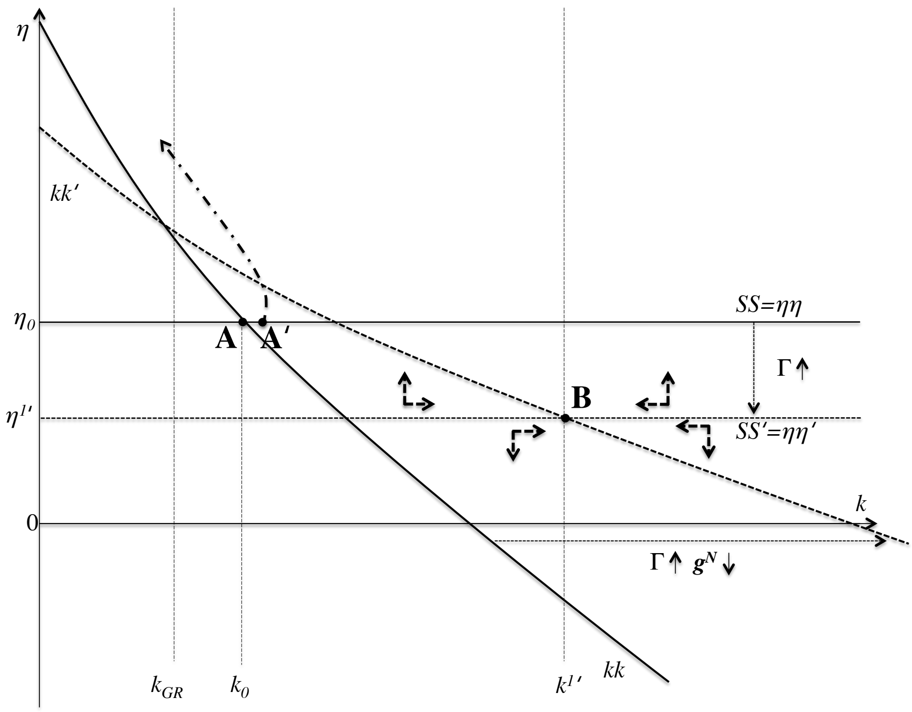

Figure 17: Effects of an expenditure and a demographic shock without fiscal response under a defined contribution scheme

per efficient capita at the time of the shock k_1 and b_1 are therefore greater than the capital intensity and debt per efficient capita of the pre-shock long run equilibrium k_0 and b_0. The debt-capital ratio, however, remains at its pre-shock long run equilibrium level such that $\eta_0 = \eta_1$ holds. The capital intensity shifts up at time $t = 1$ but the economy stays at the pre-shock stable arm, which is defined by the $\eta\eta$-phaseline at a level of η_0. Figure 17 shows this by the leap from the pre-shock steady-state A to A' at the time of the shock in period $t = 1$. Since the two shocks immediately imply the new long run steady-state B, point A' is placed in an off steady-state position. Without further action from the government during period $t = 1$, the economy takes on an explosive debt path, in which the economy shrinks up to economic collapse.

Assuming that τ^{P_C} is kept fixed, the optimal fiscal response to the two contemporaneous shocks requires an adjustment of the tax rate τ^w during period $t = 1$ in a way that keeps the economy on the pre-shock saddle-path SS. In order to respect the pre-shock saddle-path SS, the debt-capital ratio η^1 has to remain at its initial level η_0, which is defined by Equation (3.44). Given the permanent shock to the expenditure ratio $\Delta\Gamma = 0.025$ the tax rate has to be

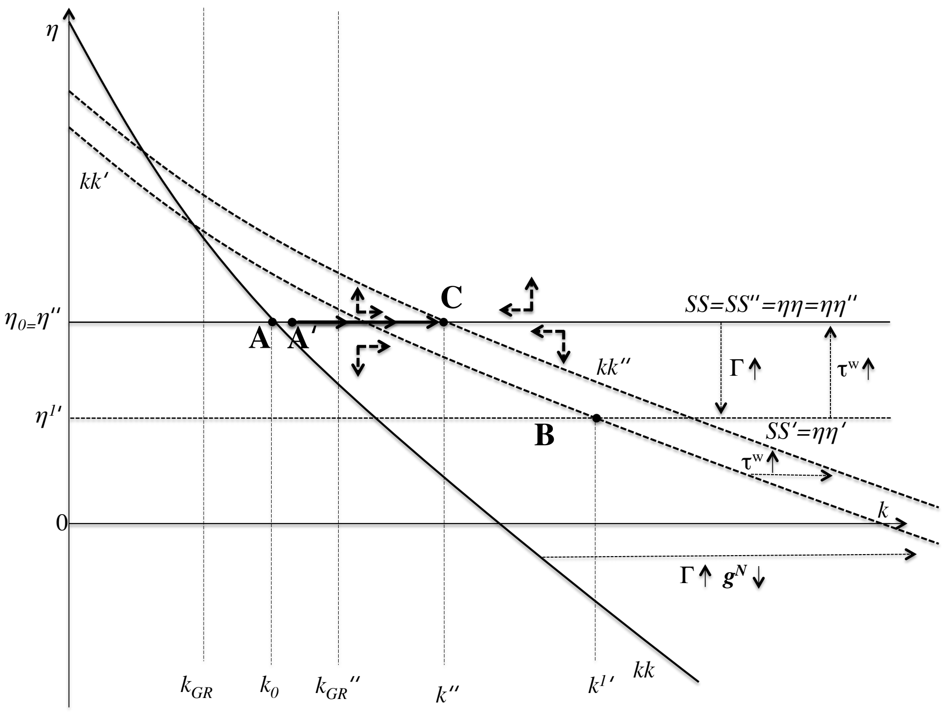

Figure 18: Optimal fiscal adjustment policy under a defined contribution scheme

permanently increased by $\Delta\tau^w = 0.0312$ to maintain public debt on the sustainable path.[6]

Knowing that the debt-capital ratio stays constant after the optimal fiscal adjustment the motion dynamics of the capital intensity and debt per efficient capita still have to be discussed. The next step is therefore to verify where the new long run equilibrium ($k^{2''}$,$\eta^{2''}$) is located after the shocks and after the optimal fiscal response. This can be done by examining the effect the optimal fiscal tax response has on the kk-phaseline from Equation (4.1). It is not difficult to see that the slope is unaffected by the optimal tax response. The post-shock and post-tax-response kk-phaseline termed $kk^{''}$ therefore has the same slope as the kk-phaseline termed $kk^{'}$, which occurs if the government takes no action after the shocks, but a flatter slope than the initial pre-shock kk-phaseline termed kk. This induces a post-tax-response long run equilibrium ($k^{2''}$,$\eta^{2''}$) with the capital intensity $k^{2''}$ that is higher than the initial capital intensity k_0 and a debt-capital ratio $\eta^{2''}$ that is equal to the pre-shock debt-capital ratio η_0.

[6]Note that $\Delta\tau^w = 0.0312$ must be found numerically given the assumed parameters

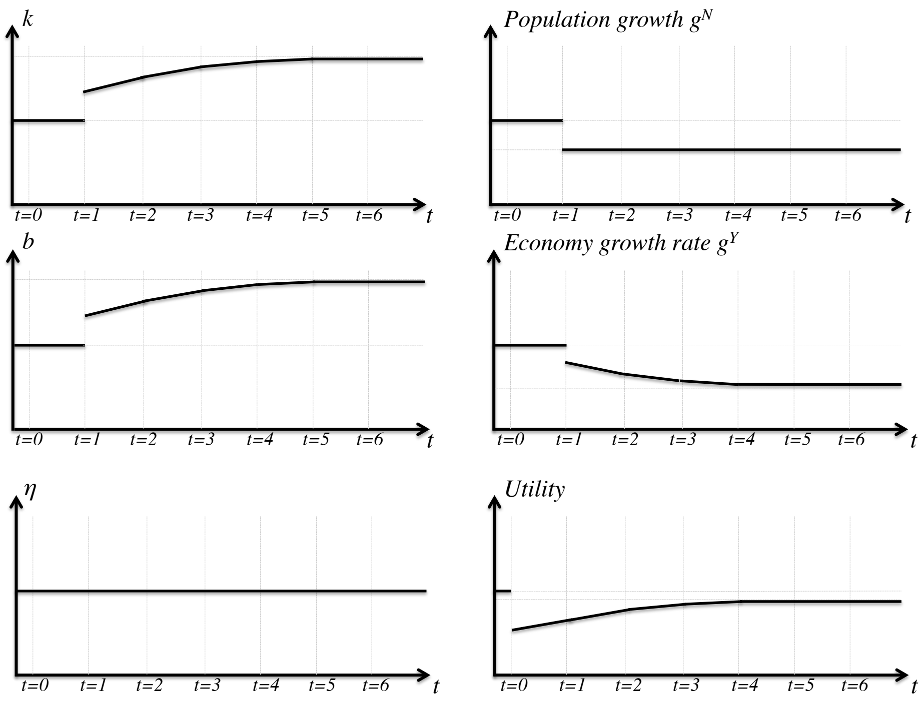

Figure 19: Time-evolution of key variables following an optimal fiscal adjustment under a defined contribution scheme

Figure 19 plots the time-evolution of key variables following that scenario. The graphs illustrate that the capital intensity k and debt per efficient capita b jump up at $t = 1$ due to the permanent change in population growth rate. Subsequently, the two variables converge to the level of the new long run equilibrium ($k^{2''}$,$\eta^{2''}$). The debt-capital ratio remains constant at the level of η_0. The intuition behind the raising capital intensity once more is that the permanent drop in g^N makes labour force scarce and increases wages. Labour force gets relatively more expensive than capital and firms shift from human capital to real capital such that the capital intensity increases and the interest rate decreases. The generation born in $t = 0$ is the first one affected by the shocks. The capital intensity that jumps up at time $t = 1$ defines the interest rate that the generation born in $t = 0$ receives on their savings. The increased capital intensity lowers this interest rate such that the households are worse off. The utility of future generations then rises and converges to a level below the pre-shock level because except for the generation born in $t = 0$ they earn higher wages. The fact that the utility level is below the one that would be reached without any shocks indicates that the negative effect that the lower interest rate has on utility dominates the positive effect of a higher wage.

4.4.2 Expenditure Shock under a Defined Benefit Scheme

4.4.2.1 Effect on Long Run Equilibrium

Like under a defined contribution scheme, a permanent positive shock to the expenditure ratio Γ leads to a higher value $\Gamma' \equiv \Gamma + \Delta\Gamma$. If the government does not adjust the tax rate τ^w the shock induces a primary budget deficit ratio θ that raises to $\theta' = \Gamma' - \tau^w(1-\alpha)$. This higher θ' lowers the high debt-capital ratio η^1 defined in Equation (4.9). The same equation reveals that under a defined benefit scheme η^1 also depends on the growth rate of the population g^N, which is assumed to decrease at time $t = 1$. As discussed earlier, this drop in g^N further reduces η^1 to the post-shock long run equilibrium debt-capital ratio $\eta^{1'}$.

To define the post-shock long run equilibrium capital intensity $k^{1'}$, recall the kk-phaseline from Equation (4.7). The higher primary deficit ratio Γ' and lower population growth rate reduce the intercept of the kk-phaseline. The slope of the kk-phaseline increases. However, it does not rise as much as under a defined contribution scheme. Thus, the slope under a defined benefit scheme is steeper than under a defined contribution scheme. Again, the two effects are counteractive to the capital intensity. The reduced intercept induces a lower equilibrium capital intensity, while the increased slope causes the equilibrium capital intensity to rise. It is demonstrated in Figure 20 that the reduced post-shock equilibrium debt-capital ratio $\eta^{1'}$ and the effect of the flatter slope outweigh the effect of the lower intercept. This leads to a post-shock equilibrium capital intensity $k^{1'}$ that is higher than the initial capital intensity k_0.

4.4.2.2 Fiscal Adjustment and Dynamics

The next step is to illustrate that fiscal adjustment must be used to prevent the economy from a deviation of the sustainable path. The motion dynamics of the capital intensity and debt per efficient capita under a defined benefit scheme are defined by laws of motion from Equations (4.10) and (4.11). At the time of the shocks in period $t = 1$, only the drop in g^N affects the two endogenous variables. The altered deficit ratio Γ' only has a one-period lagged effect. Thus, capital intensity k_t and debt per efficient capita b_t increase proportionally, keeping the debt-capital ratio η_0 defined in Equation (4.12) constant. Figure 20 depicts this by a jump from the initial long run equilibrium (k_0,η_0) termed A to the point A'. At time $t = 1$, the shocks immediately induce the post-shock equilibrium ($k^{1'}$,$\eta^{1'}$) termed B. This equilibrium is

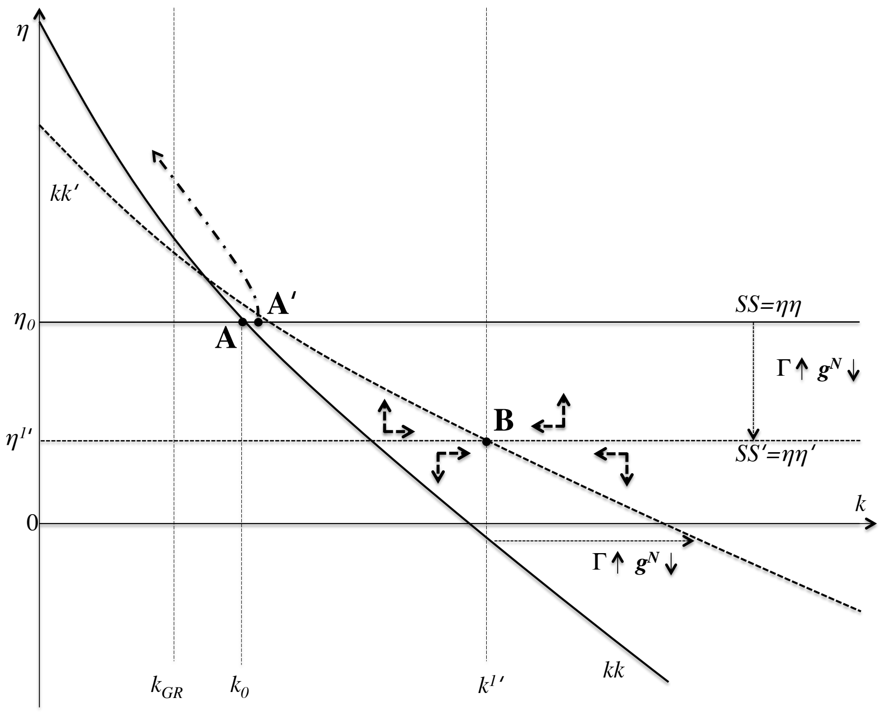

Figure 20: Effects of an expenditure and demographic shock without fiscal response under a defined benefit scheme

saddle-path stable and puts the economy in A' in an off steady-state position. If the government fails to react in a situation like this, the economy's debt level would explode, indicated by the dashed arrows pointing away from A'. Thus, assuming that τ^{P_B} is kept fixed the optimal fiscal response to the two contemporaneous shocks to g^N and Γ requires an adjustment of the tax rate τ^w during period $t = 1$. The change to the tax rate $\Delta\tau^w$ must ensure that the debt-capital ratio η^1 is maintained at its pre-shock level η_0 defined by Equation (3.44). In the scenario considered, the tax rate must be increased by $\Delta\tau^w = 0.3863$ to adhere to the condition in Equation (3.44) and to respect the initial saddle-path SS.[7] This increase in the tax rate ensures keeping the economy on the stable arm. Since in the case of a defined benefit scheme not only the shock to Γ but also the reduced g^N pushed the $\eta\eta$-phaseline down, the tax rate must be increased more than under a defined contribution scheme.

Now, the post-tax-adjustment long run equilibrium ($k^{2''}$,$\eta^{2''}$) is located. Examining the kk-phaseline in Equation (4.7), one can conclude that the slope is unaffected by the change in tax rate, while the intercept is increased. Figure 21 plots this change to the kk-phaseline termed $kk^{'}$, which occurs if the government takes no action after the shocks, by a parallel shift.

[7]Note again that $\Delta\tau^w = 0.3863$ must be found numerically given the assumed parameters.

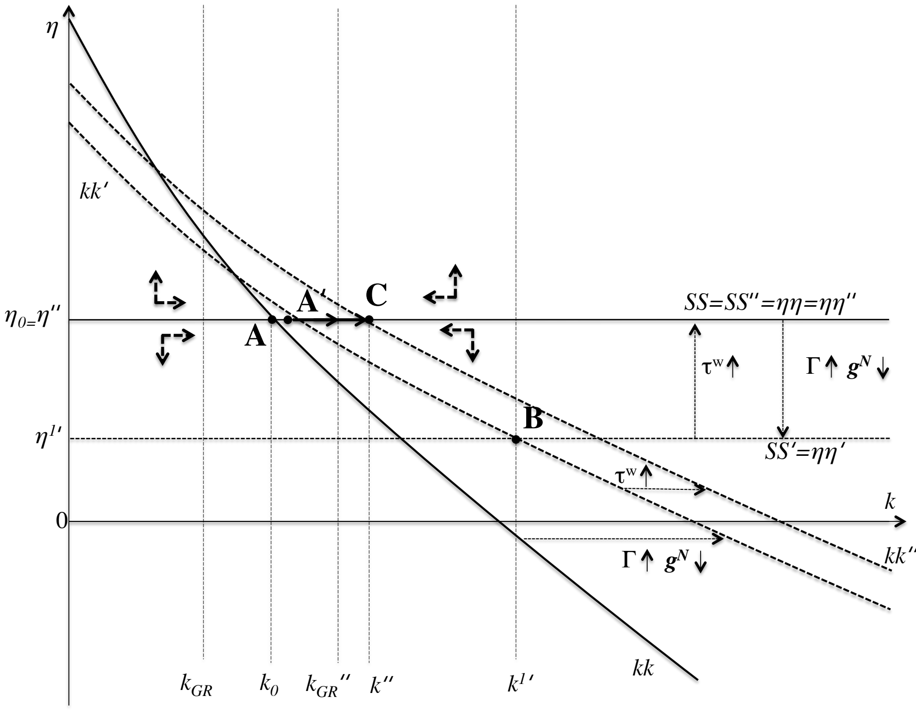

Figure 21: Optimal fiscal adjustment policy under a defined benefit scheme

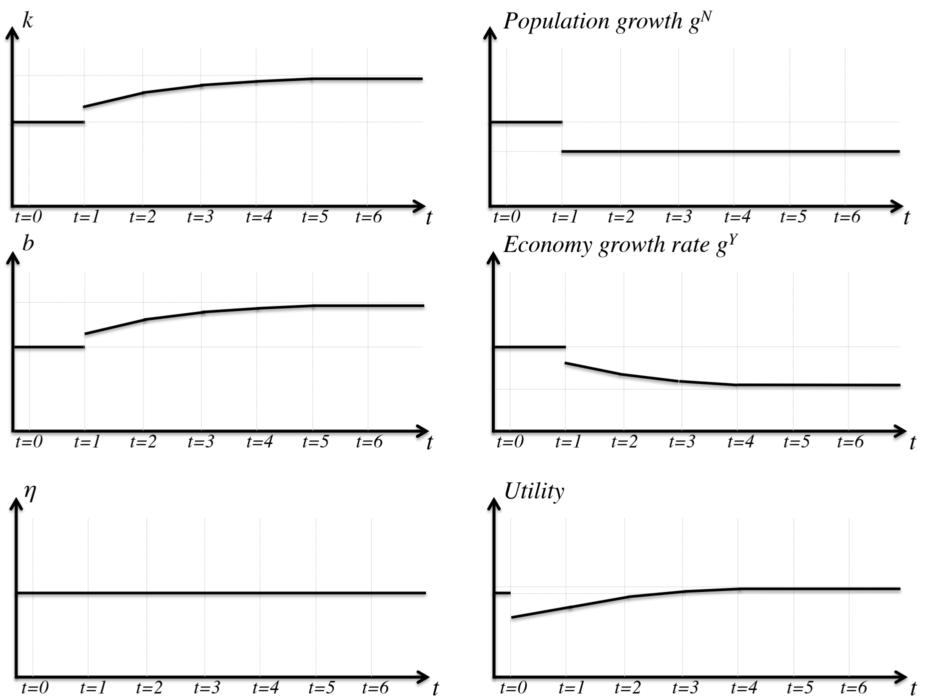

Figure 22: Time-evolution of key variables following an opitmal fiscal adjustment under a defined benefit scheme

This leads to the new and final kk-phaseline termed kk''. The kk-phaseline kk'' induces the post-tax-response long run equilibrium ($k^{2''}$,$\eta^{2''}$) termed C with capital intensity $k^{2''}$ that is higher than the initial capital intensity k_0 and a debt-capital ratio $\eta^{2''}$ that is equal to the pre-shock debt-capital ratio η_0. However, compared to the case of a defined contribution scheme the difference between k_0 and $k^{2''}$ is smaller because, as mentioned before, the kk-phaseline kk'' under a defined benefit scheme is steeper than under a defined contribution scheme. Intuitively, the negative shock to the population growth rate g^N only increases the pension contribution rate τ^P under a defined benefit scheme. This extension expends the pay-as-you-go scheme and reduces savings. Consequently, the new long run equilibrium capital intensity $k^{2''}$ under a defined benefit scheme is smaller and more efficient than the equivalent capital intensity under a defined contibutions scheme.

The time-evolution of key variables following the permanent shocks and the fiscal adjustment is depicted in Figure 22. The graphs confirm that the capital intensity k and debt per efficient capita b jump up at $t = 1$ due to the permanent change in the population growth rate. Thereafter, firms shift from scarce labour force to capital, inducing a proportional increase in the capital intensity and debt per efficient capita maintaining a debt-capital ratio at a level of η_0. The generation born in $t = 0$ is the first one affected by the shocks and fiscal adjustment. While having a wage determined by pre-shock accumulation, interest rate on savings, households of this generation receive, is reduced. This negative effect on utility is only insufficiently offset by the risk sharing mechanism of a defined benefit scheme pay-as-you-go pension scheme. The utility of future generations rises and converges to a level above the pre-shock level. This is a important difference to the case of a defined contribution scheme, in which the new long run utility is below the pre-shock level. The rise in long run utility under a defined benefit scheme is not surprising since, as mentioned previously, the pension contribution rate τ^P increases causing savings and capital accumulation to be reduced. Given a fiscal adjustment that keeps the economy on a sustainable path this leads to a post-shocks long run equilibrium ($k^{2''}$,$\eta^{2''}$) that is less inefficient that the pre-shock long run equilibrium (k^2,η^2).

The main conclusion is that a defined benefit pay-as-you-go pension scheme can improve efficiency following a negative demographic shock and a positive expenditure shock in the given scenario. This holds true under the condition that optimal fiscal adjustment is performed as response to the shocks.

5 Conclusion

The analysis presented in this thesis investigates the role of public debt, pay-as-you-go pension schemes, and fiscal policy in situations characterised by over-accumulation of capital and changing demographics. It is based on an extension of the OLG model formulated by Diamond (1965), which displays an appropriate framework to analyse inter-temporal welfare considerations and optimal policy adjustments.

The thesis begins by presenting the baseline two-period OLG model used throughout the analysis. It models a closed and large economy with an endogenous interest rate and an endogenous public debt stock. Individuals have a finite life and no bequest motives. In this set-up, equilibria may not be efficient even though a Golden Rule level of capital, which maximises consumption and welfare, exists. Fiscal adjustments by the government can therefore directly augment or worsen the well-being of the population. The government controls two fiscal tools for its policy interventions: the tax rate and the expenditure rate. They allow the government to run a constant unbalanced primary budget deficit in a steady-state. Assuming a Cobb-Douglas production function and a logarithmic utility function, explicit long run equilibria are established for a given primary budget deficit ratio. The graphical illustration of the model reveals that the long run equilibria are dynamically inefficient since households over-accumulate capital. It also shows that long run equilibria with a high debt level tend to have a lower equilibrium capital stock and thus tend to be more efficient. This is called the crowding-out effect of public debt. Public debt therefore proves to be a tool to mitigate the problem of dynamic inefficiency. However, being placed in a dynamically inefficient long run equilibrium an additional policy tool is needed to reach the Golden Rule growth path.

Considering the necessity of an extra policy tool to maximise welfare, the baseline OLG model is extended by a pay-as-you-go pension scheme. This scheme models contribution rates proportionally to wage income. It is distinguished from a defined contribution scheme, in which the contribution rate is fixed, and a defined benefit scheme, in which the contribution rate depends negatively on the growth rate of the population. It is demonstrated that the pay-as-you-go pension scheme disturbs the individuals' savings decision. The higher the pay-as-you-go contribution rate the lower the individual and aggregate savings that are invested in firms' capital and government bonds. This mechanism leads to a negative relationship between the pay-as-you-go contribution rate and the equilibrium capital and debt levels. As a result, pay-as-you-go contribution rates arise that place long run equilibria on the Golden Rule growth path. However, starting at a dynamically inefficient long run equilibrium, the Golden Rule level of capital can never be reached by simply increasing the pay-as-you-go contribution rate. It is shown that at high debt level long run equilibria, every rise in the contribution rate puts the economy on an explosive debt path. The economic intuition behind this finding is that a higher contribution rate lowers savings such that the initial debt level cannot be sustained. The government is therefore required to choose a specific mix of its three policy tools when aiming to guide the economy towards the Golden Rule growth path. The condition the government has to adhere to is derived in this thesis. It reveals that the optimal policy mix must include an increase in the pay-as-you-go contribution rate together with a reduction of the primary deficit ratio. Conducting this policy intervention a Pareto improvement, in which every generation is better off, is achieved. In addition, it is demonstrated that the debt-to-GDP ratio is decreasing during the transition from the dynamically inefficient long run equilibrium to the Golden Rule equilibrium. This may be of special interest for policy makers that are forced to reduce the debt-to-GDP ratio by imposed regulations.

In the last part, this thesis outlines the effects of demographic change on an economy that is initially placed in a dynamically inefficient equilibrium with a positive debt level. First, a simulation of a permanent increase in the dependency ratio shows that a defined benefit pay-as-you-go pension scheme can never be sustained under demographic change. Every increase in the population growth rate must be countered by a rise in the tax rate or a reduction in government expenditures to prevent a deviation from the stable arm. This holds true even under the assumption that an aging population does not cause additional governmental spending. Given the same assumption, a defined contribution pay-as-you-go pension scheme, in contrast, has proven to be sustainable under an increasing dependency ratio.

As an aging population is expected to raise public health and long-term care expenditures, a second simulation considers a permanent positive shock to the dependency ratio and a permanent positive shock to the expenditure ratio occurring simultaneously. Assuming that the government cannot simply scale back the expenditures and does not want to adjust the pay-as-you-go scheme, the tax rate has to be increased to maintain the economy's stable debt path. The analysis demonstrates that the tax rate increase that respects this sustainable path is higher under a defined benefit scheme than under a defined contribution. Thus, defined benefit schemes require a higher fiscal discipline. The permanent drop in population growth rate makes labour force scarce and increases, which prompts firms to shift from human capital to real capital. Thus, the capital intensity increases after the shocks and the rise in taxes. However, under a defined benefit scheme the capital intensity does not rise as much as under a defined contribution scheme. The economic rationale behind this is that the higher dependency ratio increases the contribution rate of the defined benefit scheme, which lowers savings and capital accumulation. In a state of dynamic inefficiency this lower capital intensity of the defined benefit scheme implies higher consumption and utility levels. In conclusion, dynamically inefficient economies facing an increasing dependency ratio achieve higher long run welfare levels when applying a defined benefit pays-as-you-go scheme rather than a defined contribution scheme.

In the model presented in this thesis long run growth per capita is solely determined by the efficiency of labour force. Since the efficiency of labour force is exogenous to the model, growth can not be influenced by public policy. Based on the "New Growth Theory" a research and development sector could be added hereby including technological progress as an endogenous factor. In addition, the model could be extended with a fully funded occupational pension scheme. In such a framework one could then study how a fully funded and a pay-as-you-go scheme affect efficiency, the fiscal position of the government and economic growth.

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Appendices

A: Calculation of the Jacobian matrix and its Eigenvalues

Model without Pensions

In the following the Jacobian matrix is derived for the case of an economy without a pension scheme. In order to derive the partial derivatives of the Jacobian matrix, recall the medium run dynamic equations (2.29) and (2.30):

$$k_{t+1}(1+n) = \varphi(1-\tau^w)k_t^\alpha - \eta_t \alpha k_t^\alpha - \theta k_t^\alpha \tag{A.1}$$

For the law of motion of the debt-capital ratio rewrite Equation (2.14) to get:

$$\eta_{t+1}(1+n)k_{t+1} = \alpha k_t^\alpha \eta_t + \theta k_t^\alpha \tag{A.2}$$

Using simple algebra, these laws of motion can be written as:

$$k_{t+1} = \frac{1}{(1+n)}\big[\varphi(1-\tau^w) - \alpha\eta_t - \theta\big]k_t^\alpha \tag{A.3}$$

$$\eta_{t+1} = \frac{1}{(1+n)k_{t+1}}\big[\alpha\eta_t + \theta\big]k_t^\alpha \tag{A.4}$$

Now exploit the fact that in a steady-state the $\Delta k_t = k_{t+1} - k_t = 0$ has to hold true and that one can ignore time subscripts:

$$\Delta k_t = k_{t+1} - k_t = \frac{\varphi(1-\tau^w)k_t^\alpha - \eta_t \alpha k_t^\alpha - \theta k_t^\alpha}{(1+n)} - k_t = 0$$

$$\Leftrightarrow k^{1-\alpha}(1+n) = \varphi(1-\tau^w-\tau^P) - \alpha\eta - \theta \tag{A.5}$$

Next, exploit the fact that in a steady-state $\Delta\eta_t = \eta_{t+1} - \eta_t = 0$ has to hold true and that one again can ignore time subscripts:

$$\Delta\eta_t = \eta_{t+1} - \eta_t = \frac{\alpha k_t^\alpha \eta_t + \theta k_t^\alpha}{(1+n)k_{t+1}} - \eta_t = \frac{\alpha\eta + \theta}{(1+n)k^{1-\alpha}} - \eta = 0$$

$$\Leftrightarrow k^{1-\alpha}(1+n) = \alpha + \frac{\theta}{\eta} = \varphi(1-\tau^w) - \alpha\eta - \theta \tag{A.6}$$

The next step is to calculate the partial derivatives of the Jacobian matrix:

$$\mathbb{J} = \begin{bmatrix} \frac{\partial k_{t+1}}{\partial k_t} & \frac{\partial \eta_{t+1}}{\partial \eta_t} \\ \frac{\partial k_{t+1}}{\partial \eta_t} & \frac{\partial \eta_{t+1}}{\partial k_t} \end{bmatrix}$$

Let's start with the first partial derivative $\frac{\partial k_{t+1}}{\partial k_t}$. In order to derive it, one has to first take the derivative of k_{t+1} in Equation (A.3) with respect to k_t and then insert Equation (A.5) that holds at the steady-state:

$$\frac{\partial k_{t+1}}{\partial k_t} = \frac{\alpha}{(1+n)(1+\omega\tau^P)}\left[\varphi(1-\tau^w-\tau^P) - \alpha\eta_t - \theta\right]k_t^{\alpha-1} = \alpha \tag{A.7}$$

Let's continue with the parital derivative $\frac{\partial k_{t+1}}{\partial \eta_t}$. In order to derive it, one has to first take the derivative of k_{t+1} in Equation (A.3) with respect to η_t and then evaluate it at the steady-state:

$$\frac{\partial k_{t+1}}{\partial \eta_t} = \frac{-\alpha k_t^\alpha}{(1+n)} = \frac{-\alpha k^\alpha}{(1+n)} \tag{A.8}$$

Having derived the partial derivatives of k_{t+1} with respect to both k_t and having η_t and evaluted it at the steady-state, the attention is now turn towards the partial derivatives of η_{t+1}. Inserting Equation (A.3) into Equation (A.4) yields:

$$\eta_{t+1} = \frac{\alpha\eta_t + \theta}{\varphi(1-\tau^w-) - \alpha\eta_t - \theta} \tag{A.9}$$

It immediately becomes clear that (A.9) does not depend on k_t and hence the partial derivative of η_{t+1} with respect to k_t reads as follows:

$$\frac{\partial \eta_{t+1}}{\partial k_t} = 0 \tag{A.10}$$

In order to calculte the partial derivative of η_{t+1} with respect to η_t evaluated at the steady-state, one has to take the derivative of η_{t+1} in Equation (A.9) with respect to $\eta_t$and then insert the steady-state conditions from Equations (A.5) and (A.5):

$$\begin{aligned}
\frac{\partial \eta_{t+1}}{\partial \eta_t} &= \frac{\alpha[\varphi(1-\tau^w) - \alpha\eta_t - \theta] - (\alpha\eta_t + \theta)(-\alpha)}{[\varphi(1-\tau^w) - \alpha\eta_t - \theta]^2} \\
&= \frac{\alpha[\varphi(1-\tau^w)]}{[\varphi(1-\tau^w) - \alpha\eta_t - \theta]^2} \\
&= \frac{\alpha[k^{1-\alpha}(1+n) + \alpha\eta + \theta]}{[k^{1-\alpha}(1+n)]^2} \\
&= \frac{\alpha}{k^{1-\alpha}(1+n)} + \frac{\alpha[\alpha\eta + \theta]}{[k^{1-\alpha}(1+n)]^2} \\
&= \frac{\alpha}{k^{1-\alpha}(1+n)} + \frac{\alpha[\eta k^{1-\alpha}(1+n)]}{[k^{1-\alpha}(1+n)]^2} \\
&= \frac{\alpha}{k^{1-\alpha}(1+n)} + \frac{\alpha\eta}{k^{1-\alpha}(1+n)} \\
&= \frac{\alpha(1+\eta)}{k^{1-\alpha}(1+n)} \\
&= \frac{\alpha(1+\eta)\eta}{(\alpha\eta + \theta)}
\end{aligned} \tag{A.11}$$

From here, it is now possible to solve for the eigenvalues of the Jacobian matrix by using partial derivatives at the steady-state derived above and by setting the determinant of the matrix $[\mathbb{J} - \lambda I]$ equal to zero. This yields:

$$det\ [\mathbb{J} - \lambda I] = \begin{vmatrix} \frac{\partial k_{t+1}}{\partial k_t} - \lambda_1 & \frac{\partial k_{t+1}}{\partial \eta_t} \\ \frac{\partial \eta_{t+1}}{\partial k_t} & \frac{\partial \eta_{t+1}}{\partial \eta_t} - \lambda_2 \end{vmatrix} = \begin{vmatrix} \alpha - \lambda_1 & \frac{-\alpha k^\alpha}{(1+n)} \\ 0 & \frac{\alpha(1+\eta)}{k^{1-\alpha}(1+n)} - \lambda_2 \end{vmatrix}$$

$$= (\alpha - \lambda_1)\left(\frac{\alpha(1+\eta)}{k^{1-\alpha}(1+n)} - \lambda_2\right) = 0$$

It is now easy to see that the eigenvalues are of the following values:

$$\lambda_1 = \alpha \qquad \lambda_2 = \frac{\alpha(1+\eta)}{k^{1-\alpha}(1+n)} \tag{A.12}$$

Model with a Pay-as-you-go Pension Scheme

In the following, the Jacobian matrix is derived for the case of an economy with a pay-as-you-go pension scheme. In order to derive the partial derivatives of the Jacobian matrix, recall the medium run dynamic Equations (3.19) and (3.20):

$$k_{t+1}(1+n)(1+\omega\tau^P) = \varphi(1-\tau^w-\tau^P)k_t^\alpha - \eta_t\alpha k_t^\alpha - \theta k_t^\alpha$$

$$\eta_{t+1}(1+n)k_{t+1} = \alpha k_t^\alpha \eta_t + \theta k_t^\alpha$$

Using simple algebra these laws of motion can be written as:

$$k_{t+1} = \frac{1}{(1+n)(1+\omega\tau^P)}\big[\varphi(1-\tau^w-\tau^P) - \alpha\eta_t - \theta\big]k_t^\alpha \tag{A.13}$$

$$\eta_{t+1} = \frac{1}{(1+n)k_{t+1}}\big[\alpha\eta_t + \theta\big]k_t^\alpha \tag{A.14}$$

Now exploit the fact that in a steady-state the equation $\Delta k_t = k_{t+1} - k_t = 0$ has to hold true and that one can ignore time subscripts:

$$\Delta k_t = k_{t+1} - k_t = \frac{\varphi(1-\tau^w-\tau^P)k_t^\alpha - \eta_t\alpha k_t^\alpha - \theta k_t^\alpha}{(1+n)(1+\omega\tau^P)} - k_t = 0$$

$$\Leftrightarrow k^{1-\alpha}\big[(1+n)(1+\omega\tau^P)\big] = \varphi(1-\tau^w-\tau^P) - \alpha\eta - \theta \tag{A.15}$$

Now exploit the fact that in a steady-state $\Delta\eta_t = \eta_{t+1} - \eta_t = 0$ has to hold true and that one again can ignore time subscripts:

$$\Delta\eta_t = \eta_{t+1} - \eta_t = \frac{\alpha k_t^\alpha \eta_t + \theta k_t^\alpha}{(1+n)k_{t+1}} - \eta_t = \frac{\alpha\eta + \theta}{(1+n)k^{1-\alpha}} - \eta = 0$$

$$\Leftrightarrow k^{1-\alpha}(1+\omega\tau^P)(1+n) = (1+\omega\tau^P)\left[\alpha + \frac{\theta}{\eta}\right] = \varphi(1-\tau^w-\tau^P) - \alpha\eta - \theta \tag{A.16}$$

The next step is to calculate the partial derivatives of the Jacobian matrix:

$$\mathbb{J} = \begin{bmatrix} \dfrac{\partial k_{t+1}}{\partial k_t} & \dfrac{\partial \eta_{t+1}}{\partial \eta_t} \\ \dfrac{\partial k_{t+1}}{\partial \eta_t} & \dfrac{\partial \eta_{t+1}}{\partial k_t} \end{bmatrix}$$

Let's start with the first partial derivative $\dfrac{\partial k_{t+1}}{\partial k_t}$. In order to derive it, one has to first take the derivative of k_{t+1} in Equation (A.13) with respect to k_t and then to insert Equation (A.15) that hold at the steady-state:

$$\frac{\partial k_{t+1}}{\partial k_t} = \frac{\alpha}{(1+n)(1+\omega\tau^P)}\left[\varphi(1-\tau^w-\tau^P)-\alpha\eta_t-\theta\right]k_t^{\alpha-1} = \alpha \tag{A.17}$$

Let's continue with parital derivative $\dfrac{\partial k_{t+1}}{\partial \eta_t}$. In order to derive it one has to first take the derivative of k_{t+1} in Equation (A.13) with respect to η_t and then evaluate it at the steady-state:

$$\frac{\partial k_{t+1}}{\partial \eta_t} = \frac{-\alpha k_t^{\alpha}}{(1+\omega\tau^P)(1+n)} = \frac{-\alpha k^{\alpha}}{(1+\omega\tau^P)(1+n)} \tag{A.18}$$

Having derived the partial derivatives of k_{t+1} with respect to both k_t and η_t and having evaluted it at the steady-state, the attention is again turned towards the partial derivatives of η_{t+1}. Inserting Equation (A.13) into Equation (A.14) yields:

$$\eta_{t+1} = \frac{(1+\omega\tau^P)(\alpha\eta_t+\theta)}{\varphi(1-\tau^w-\tau^P)-\alpha\eta_t-\theta} \tag{A.19}$$

It immediately becomes clear that Equation (A.19) does not depend on k_t and hence the partial derivative of η_{t+1} with respect to k_t reads as follows:

$$\frac{\partial \eta_{t+1}}{\partial k_t} = 0 \tag{A.20}$$

In order to calculate the partial derivative of η_{t+1} with respect to η_t evaluated at the steady-state one has to take the derivative of η_{t+1} in Equation (A.19) with respect to η_t and then to insert the steady-state conditions from Equations (A.15) and (A.16):

$$
\begin{aligned}
\frac{\partial \eta_{t+1}}{\partial \eta_t} &= \frac{\alpha(1+\omega\tau^P)[\varphi(1-\tau^w-\tau^P)-\alpha\eta_t-\theta]-(1+\omega\tau^P)(\alpha\eta_t+\theta)(-\alpha)}{[\varphi(1-\tau^w-\tau^P)-\alpha\eta_t-\theta]^2} \\
&= \frac{\alpha(1+\omega\tau^P)[\varphi(1-\tau^w-\tau^P)]}{[\varphi(1-\tau^w-\tau^P)-\alpha\eta_t-\theta]^2} \\
&= \frac{\alpha(1+\omega\tau^P)[k^{1-\alpha}(1+n)(1+\omega\tau^P)+\alpha\eta+\theta]}{[k^{1-\alpha}(1+n)(1+\omega\tau^P)]^2} \\
&= \frac{\alpha(1+\omega\tau^P)}{k^{1-\alpha}(1+n)(1+\omega\tau^P)}+\frac{\alpha(1+\omega\tau^P)[\alpha\eta+\theta]}{[k^{1-\alpha}(1+n)(1+\omega\tau^P)]^2} \\
&= \frac{\alpha(1+\omega\tau^P)}{k^{1-\alpha}(1+n)(1+\omega\tau^P)}+\frac{\alpha(1+\omega\tau^P)[\eta k^{1-\alpha}(1+n)]}{[k^{1-\alpha}(1+n)(1+\omega\tau^P)]^2} \\
&= \frac{\alpha(1+\omega\tau^P)}{k^{1-\alpha}(1+n)(1+\omega\tau^P)}+\frac{\alpha\eta}{k^{1-\alpha}(1+n)(1+\omega\tau^P)} \\
&= \frac{\alpha(1+\omega\tau^P+\eta)}{k^{1-\alpha}(1+n)(1+\omega\tau^P)} \\
&= \frac{\alpha(1+\omega\tau^P+\eta)\eta}{(\alpha\eta+\theta)(1+\omega\tau^P)}
\end{aligned}
\tag{A.21}
$$

It is now possible to solve for the eigenvalues of the Jacobian matrix by using partial derivatives at the steady-state derived above and by setting the determinant of the matrix $[\mathbb{J}-\lambda I]$ equal to zero. This yields:

$$
det\ [\mathbb{J}-\lambda I] = \begin{vmatrix} \frac{\partial k_{t+1}}{\partial k_t}-\lambda_1 & \frac{\partial k_{t+1}}{\partial \eta_t} \\ \frac{\partial \eta_{t+1}}{\partial k_t} & \frac{\partial \eta_{t+1}}{\partial \eta_t}-\lambda_2 \end{vmatrix} = \begin{vmatrix} \alpha-\lambda_1 & \frac{-\alpha k^{\alpha}}{(1+\omega\tau^P)(1+n)} \\ 0 & \frac{\alpha(1+\omega\tau^P+\eta)}{k^{1-\alpha}(1+n)(1+\omega\tau^P)}-\lambda_2 \end{vmatrix}
$$

$$
= (\alpha-\lambda_1)\left(\frac{\alpha(1+\omega\tau^P+\eta)}{k^{1-\alpha}(1+n)(1+\omega\tau^P)}-\lambda_2\right) = 0
$$

It is now easy to see that that the eigenvalues are of the following values:

$$
\lambda_1 = \alpha \qquad \lambda_2 = \frac{\alpha(1+\omega\tau^P+\eta)}{k^{1-\alpha}(1+n)(1+\omega\tau^P)} \tag{A.22}
$$

B: Comparison of Capital Intensities

In Subchapter 4.2 the dynamics of an economy with a defined contribution pay-as-you-go pension scheme are analysed following a negative shock to the population growth rate g_t^N at time t. It has already been demonstrated that the capital intensity of the pre-shock equilibrium is smaller than the capital intensity k_t at the time t of the shock. Here, it is shown that capital intensity at time t of the shock is smaller than the post-shock long run equilibrium capital intensity. Let's define the capital intensity at time t of the shock k_t, the pre-shock capital intensity simply as k, and the post-shock long run capital intensity as k'. Furthermore, the pre-shock population growth rate is termed g^N and the post-shock population growth rate is termed $g^{N'}$ with $g^N > g^{N'}$. Note that the deb-capital ratio η_t remains constant before and after the shock and is hence just termed η. In the following, it is shown that k_t is smaller than k':

$$k' > k_t$$

$$\left[\frac{\varphi(1-\tau^w-\tau^P)-\alpha\eta-\theta}{(1+g^{N'})(1+g^a)(1+\omega\tau^P)}\right]^{\frac{1}{1-\alpha}} > \frac{\varphi(1-\tau^w-\tau^P)-\alpha\eta-\theta}{(1+g^{N'})(1+g^a)(1+\omega\tau^P)}k^\alpha$$

$$\left[\frac{\varphi(1-\tau^w-\tau^P)-\alpha\eta-\theta}{(1+g^{N'})(1+g^a)(1+\omega\tau^P)}\right]^{\frac{\alpha}{1-\alpha}} > k^\alpha$$

$$\left[\frac{\varphi(1-\tau^w-\tau^P)-\alpha\eta-\theta}{(1+g^{N'})(1+g^a)(1+\omega\tau^P)}\right]^{\frac{\alpha}{1-\alpha}} > \left[\frac{\varphi(1-\tau^w-\tau^P)-\alpha\eta-\theta}{(1+g^{N})(1+g^a)(1+\omega\tau^P)}\right]^{\frac{\alpha}{1-\alpha}}$$

$$g^N > g^{N'}$$

As $g^N > g^{N'}$ was initially assumed, it can be concluded that the post-shock long run capital intensity k' is greater than the capital intensity at time t of the shock k_t.

C: Tables

TABLE 2: Fertility rates

	Year			
	1960	**1980**	**2000**	**2015**
Belgium	2.5	1.7	1.6	1.7
Denmark	2.5	1.5	1.8	1.7
Finland	2.7	1.6	1.7	1.7
France	2.7	1.9	1.9	1.9
Germany	2.4	1.6	1.4	1.5
Greece	2.2	2.2	1.3	1.3
Italy	2.4	1.7	1.3	1.4
Netherlands	3.1	1.6	1.7	1.7
Norway	2.9	1.7	1.9	1.7
Portugal	3.1	2.2	1.6	1.3
Spain	2.9	2.2	1.2	1.3
Sweden	2.2	1.7	1.5	1.9
Switzerland	2.4	1.6	1.5	1.5
United Kingdom	2.7	1.9	1.6	1.8
United States	3.7	1.8	2.1	1.8

Source: OECD

TABLE 3: Life expectancy at birth

	Year			
	1960	**1980**	**2000**	**2015**
Belgium	69.8	73.3	77.8	81.1
Denmark	72.4	74.3	76.9	80.8
Finland	69.0	73.6	77.7	81.6
France	70.3	74.3	79.2	82.4
Germany	69.1	72.9	78.2	80.7
Greece	-	75.3	78.6	81.1
Italy	-	74	79.9	82.6
Netherlands	73.5	75.9	78.2	81.6
Norway	73.8	75.9	78.8	82.4
Portugal	63.9	71.4	76.9	81.2
Spain	69.8	75.4	79.3	83.0
Sweden	73.1	75.9	79.7	82.3
Switzerland	71.4	75.7	79.9	83.0
United Kingdom	70.8	73.2	77.9	81.0
United States	69.9	73.7	76.7	78.8

Source: OECD