ALM Models in a Danish Perspective

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Abstract

Because Danish pension funds used to guaranteed a minimum investment return of up to 4.5 percent, the current market conditions of low interest rates and increases in longevity poses a challenge through proper risk management to honor future liabilities. Given the limited literature in the specific area of with-profit schemes, papers investigating asset-liability management models relevant to public pensions and defined benefit schemes have encouraged the study of with-profit schemes in a Danish perspective. With the introduction of Solvency II, to increase the capital requirements and reduce the probability of default, the pension funds are expected to accumulate a larger amount of capital to keep the funding ratio above one. While asset-liability management models can be used to stochastically model the future shocks of the economy, the objective of it can very significantly, with the purpose of this paper being a maximization of collective bonus potential through comparing four different portfolio theories. One of the difficulties of estimating an asset-liability management model, to achieve a correct assessment of the future shocks to the economy, through the use of stochastic models such as Vasicek and Geometric Brownian Motion, is the sensitivity to the selected time horizon of available data. To understand the influencing factors and the level of sensitivity to the asset-liability management model, a practical working model within the Danish pension sector is constructed by applying a set of realistic assumption.

The output shed lights on how different portfolio theories maximize the collective bonus potential, while assessing how it is influenced by shocking or adjusting of the variables of the model. Through a comparison between the four portfolio theories proposed, the Sortino ratio is discovered to outperform the tangency portfolio in the constructed default model, while the minimum variance portfolio outperform the risk parity in providing the lowest variance. By reducing the level of the risk-based constraint imposed by Solvency II, the probability of default increases from 0 to 5-10 percent. To reduce the exposure, the implications of a of risk factors are discussed, suggesting a focus driven by shortfall risk and heding instrument. In conclusion, the asset-liability management model proves to be a superior risk management tool, for the continued use in the Danish pension market.

Contents

1 Introduction

The Danish pension system has been ranked the best in the world since 2012 (Melbourne Mercer Global Pension Index, 2017; J. G. Andersen, 2016), but is still faced with a number of challenges. The most common schemes in the Danish pension system are the with-profit *(gennemsnitsrenteprodukt)* and unit-link *(markedssrenteprodukt)*. A prominent feature of with-profit contracts, in the Danish pension system, is the use of minimum guaranteed investment returns *(tegningsgrundlag)*. The guarantee is set upon initiation of the contract and is fixed until the termination of the contract or death of the policyholders. Throughout the 1980s and 1990s, the guaranteed minimum investment returns were fixed at up to five percent per year (Finanstilsynet, 2016), which poses a problem given the current market conditions. As interest rates have started to decline the pension funds are put under pressure to meet the guaranteed investment returns. With-profit contracts are no longer being offered, but the majority of the accumulated reserves are still made up of with-profit contracts and will continue to do so in the coming decades (Steffensen & Ramlau-Hansen, 2017).

Another challenge is the rise in expected lifetime of the individuals. In with-profit contracts, the benefits are usually a whole life annuity, that runs until the death of the policyholder. The difficulties of estimating the expected lifetime of individuals and interest rates are some of the problems faced by the pension fund. This proves the need for proper risk management tools and the need for an asset-liability management model.

The modeling of future shocks is not a new phenomenon (Guo, 1996). Much literature of asset-liability management models exists, but it is often focused on defined benefit schemes or public pensions. The objective of the asset-liability management model can vary greatly and the model could be used to maximize the value of the shareholders while keeping the probability of default at a minimum (Duarte, Valladão, & Veiga, 2017), maximize the excess return per unit of risk (Platanakis & Sutcliffe, 2017), etc. Within the pension sector, there is an increasing focus on risk management. With the introduction of Solvency II and the prudent-person principle, the fund must, to a greater extent, understand their risk exposure. A study investigates the effect of different regulatory systems (Blome, Fachinger, Franzen, Scheuenstuhl, & Yermo, 2007), and concludes that quantitative restrictions affect the funding cost of the pension fund. Another study investigates the interest rate's effect on public pensions (Chen & Matkin, 2017), and find that it would take seven years to rebuild the funding ratio to the previous level if the interest rate were reduced by one percentage point, ceteris paribus.

The guaranteed minimum investment returns, provided by with-profit contracts, are very risky to the pension funds. It is important to understand how the future might behave, and what the pension funds can do to mitigate their exposure. The interest in the topic is encouraged by the limited literature of asset-liability management models in with-profit schemes, from a Danish perspective.

2 Research Question & Limitations

2.1 Research Question

Pension funds are facing restrictions regarding their investment policy and how they conduct business. They have obligations to their policyholders, as they must be able to honor the future liabilities. The majority of the liabilities in Danish pension funds are tied to with-profit contracts, where the policyholders are guaranteed a minimum investment return. While it is important for the pension funds to secure a return on the assets under management, they are forced to stay in line with the strict regulations set by Solvency II and the Danish financial supervisory authorities. High returns are correlated with risky investment, but pension funds are required to manage their risk with care; they have to follow the prudent-person principle, hence they are unable to achieve a high return simply by partaking in risky ventures. A large proportion of the asset value has to be invested in safe assets, like bonds, to ensure future liabilities are met with safety. Assets and liabilities are affected by various shocks in the economy, which might stress the funding ratio and increase the probability of default. The pension fund should use an asset-liability management model to account for future stochastic shocks and to get a better understanding of how these affect them. It enables the pension fund to develop a more robust strategy and improve the overall risk management.

This provides an interesting case and drives the investigation of how risk management is applied in Danish pension funds. An asset-liability management model will be constructed to understand the importance of such a model and how it is utilized. It is applied to Danish pension funds to provide realistic scenarios of the future that stays within the required regulations. The economic consumption of the retired population is contingent on pension funds providing retirement benefits, hence the risk management of such an entity is important to avoid default scenarios. This lead to the following research question:

> **How does a Danish pension fund maximize the collective bonus potential through the utilization of an asset-liability management model?**

To answer the research question various topics are described and discussed. Initially, the state of the Danish pension market is described, to understand the setting for which the asset-liability management model is applied. The assets incorporated in the model and their impact is explained, while it is shown how the liabilities of a pension fund is calculated and what implications they might have. The assumptions of the asset-liability management model are discussed following the construction of the model. Furthermore, an analysis of the outcome of the model is conducted, with a subsequent assessment of the sensitivity of the model, and explore how adjustments affect the result. Additionally, various risk parameters influencing the decision making will be discussed, to get an understanding of what the pension funds' options are to mitigate the exposure. Lastly, the discoveries and discussions are combined to suggest how the asset-liability management model can be improved to support further studies.

The purpose of the paper is to understand how Danish pension funds can maximize the expected bonus potential to the policyholders, with stochastic market changes affecting the value of assets and liabilities. It is relevant to any financial entity with needs to meet future financial expectations, to understand how asset-liability management models work, and the benefits such a modeling of the future provide. The approach to answering the research question is a combination of both theoretical and practical work. The practical construction of a simplified working model combined with a theoretical discussion of sensitivity factors and risk parameters provide a foundation to discuss the implications of applying such a model in Danish pension funds to improve the risk management. Increased regulatory supervision puts more emphasis on pension funds to account for shocks in the asset-liability ratio, ensuring the capital soundness comply with regulations and the financial supervisory authorities.

2.2 Limitations

The resources available are limited and combined with the scope of the paper, it provides a series of limitations. To answer the research question, it is important to define a series of concepts, definitions and the extent of the paper. The purpose of constructing an asset-liability management (ALM) model is not to provide a sophisticated model identical to the models used in practice. To investigate the research question, such an extensive model is unnecessary. Construction of an ALM model is difficult and faces the pension fund with a series of implications. The purpose is to construct the model in a simplified setting based on realistic assumptions and limited capabilities. It is a tool to understand the importance of using it in practice, and what benefits it provides in the current Danish market.

The paper investigates the ALM model with respect to Danish pension funds, hence everything is related to Danish pension funds unless explicitly stated otherwise. There are different kind of pension funds; public, occupational and private. The discussion of public pension plans is excluded, given it is controlled by the government and the risks of the public pension fund are fewer compared to private and occupational pension funds. The private and non-industry specific pensions[1] funds have to compete for policyholders, which is the reasoning for excluding industry-specific pension funds, as they are secured a stream of new policyholders, which reduces the costs of marketing and competition. The assessment of a maximization of collective bonus potential is only applied to open pension funds. There are differences in some of the open pension funds, but they will be assumed identical and no distinction is considered. Danish pension funds must comply with Solvency II, but only the pillar I of the Solvency II regulations are relevant to the investigation of the asset-liability management model. Pillar I is the quantitative requirements, which is measurable and directly influence the risk management of the pension fund. The paper will not focus on the minimum capital requirement, because of its similarities with solvency capital requirements and function mostly as an intervention process for regulatory

[1] Industry specific pension funds are limited to individuals in a specific industry.

purposes.

It is worth noting the difference between pension scheme and pension fund. The analysis will be conducted at the fund level, focusing on the investments, risk management and managing of liabilities. A pension scheme is the customer level of the fund and refers to the general contracts. Unless explicit, the point of view is from the management of the pension funds. Neither the tax implications of the pension fund or policyholders are relevant to the research question, as the goal is to maximize the collective bonus potential. Within the paper collective bonus potential and expected bonus potential are used interchangeably.

The paper will use a large amount of mathematics some of which is actuarial calculations, however, the focus of the paper is not the actuarial component, but how the calculations affect the risk management decisions. For simplification, the calculations of the individualized bonus potential is approximated as a dividend. When comparing the accumulated bonus potential, it has not been adjusted for time-value. The accumulation of bonus potential does not discount each cash-flow, thus it is assumed an identical amount of bonus potential at time 0 is worth the same as at time 40. The paper is not a mathematical discussion of stochastic models, instead, it applies the mathematics to answer the research question. All of the returns of assets or investments are assumed to be log-returns. Financial instruments, derivatives, and hedging instruments are used interchangeably. The paper does not consider transaction costs, due to not being relevant in answering the research question.

The definition of liabilities used in this paper is identical to the reserves of the policyholders. Liabilities include the value of the contracts of the policyholders and the capital requirements of Solvency II. When referring to the owners of the pension funds they are assumed to be different from policyholders. The paper makes a distinction between property and private equity, where private equity is assumed to be an alternative investment while property is not. In practice, both types of investments are usually considered alternative investments, given the nature of them compared to traditional investments in bonds and public equity. Covered bonds are used as an asset in the construction of the asset-liability management model. As the covered bonds are related to the Danish real estate market, it is assumed to be identical to a mortgage-backed security.

The purpose of the paper promotes a discussion of the risks affecting Danish pension funds. Risk is a broad term, thus some clarification of what is meant by risk is needed. Risk is how the pension fund is affected by changes in market conditions, and how it affects the expected bonus potential. It is divided into various risk parameters influencing the overall exposure and decision making of the pension fund. Danish pension funds are subject to many different risk parameters, some of which is fundamental to discuss in this analysis. The risk parameters that will be thoroughly discussed are interest rate risk, longevity risk, funding risk, regulatory risk, hedging risk, contract risk and market risk, as well as an analysis of the sensitivity of constructing an asset-liability management model. These all contribute to the discussion and analysis of the research question. Other risk parameters have been omitted due to lack of relevance to the analysis. The omitted risk parameters include inflation, credit risk, expense risk, operational risk, accounting risk, etc. Inflation is excluded from the asset-liability management model, as it is not assumed to contribute to the assessment of the model. It is an important factor in pension savings, as it affects the purchasing power of the policyholders' benefit, but not assumed relevant to answer the research question. It could be argued that interest rate risk should be under market risk, the reason it has been separated as an individual risk parameter is due to its prominent influence. Interest rate risk is the most influential risk parameters a pension fund faces, hence the reason it is being treated as a standalone risk parameter.

3 Pension Schemes and the Overall Market

The Danish pension system has its origin back in 1891, where an old age support was introduced. It was a tax-financed means-tested[2] support for all individuals (J. G. Andersen, 2016). Denmark was the second country in the world, after Germany, to introduce a pension system. Throughout the years the pension system has undergone significant changes. The last of these fundamental changes took place in the late 1980s, where a more decentralized solution became a reality, with the inclusion of occupational pensions in most collective agreements (Green-Pedersen, 2003). In the following section, the Danish pension sector will be discussed, what type of pensions are available, what products exist, and how it is regulated.

3.1 Danish Pension Schemes

Like in most OECD countries, the Danish pension system is built around a complex multi-pillar system. The goal of dividing the pension schemes into pillars is to separate the objectives of pension schemes (World Bank, 1994). The public pensions offer a basic coverage and the goal is to reduce poverty in retirement. The intentions of the occupational schemes are to ensure a high level of income replacement, while the private pensions schemes cover the voluntary savings plans and insurances. In the following section the public, occupational, and private pension schemes will be discussed.

Pillar I: Public Pension Schemes The first pillar is administrated by the public sector, where two different schemes exist; the social pension *(folkepension)* and the labor market supplementary pension scheme (ATP). The social pension is an unfunded pay-as-you-go[3] scheme, financed through tax revenues. The social pension is further divided into a basic pension and a yearly pension supplement *(ældrecheck)* (Møller & Nielsen, 2015). The pension consists of a flat-rate pension to all individuals during retirement, unless they have a work-related income above a certain amount. The social pension also includes a means-tested supplement *(pensionstillæg)* for low-income individuals. The yearly pension supplement is another means-tested benefit, based on the individual's total wealth (Møller & Nielsen, 2015).

Within the last decade, the social pension has been subject to significant changes. The changing demographics, with a growing elderly population, is preasuring the system (Finansministeriet, 2011). In order to improve the government's finances in the coming decades and to maintain our current welfare system, the retirement age have increased from 65 up to 68 years, based on birth year. The size of the benefits has changed as

[2]Means-tested is a method to determine whether an individual qualifies for financial support, based on the individual's income and is a way to prevent individuals from falling into or remaining in poverty (Oxford Reference, 2018b)

[3]A pay-as-you-go scheme is financed through tax revenue opposed to a funded scheme, where the individual's previous contributions and investment returns pay for the future benefits (Oxford Reference, 2018a)

well, in favor of the low-income individuals, so that this group will receive higher benefits (Finansministeriet, 2011). The second scheme in the public pension is ATP. It entails some social security features and is, despite the nature of the scheme, classified as a first pillar pension by most Danish pension experts (J. G. Andersen, 2016). ATP is a fully funded mandatory and supplementary pension scheme for all wage earners. The contributions are a fixed, based on the individuals working hours, and partly covered by the employer. The number of policyholders within the scheme has increased steadily over the decades and ATP now covers most of the population (C. Andersen & Skjodt, 2008).

Pillar II: Occupational Pensions The occupational schemes are aimed at income replacement to prevent poverty among elders. These schemes have become more widespread since the late 1980s and early 1990s, due to a result of the collective bargains *(overenskomstforhandlinger)* and tax incentives by the government. The occupational pensions cover an overwhelming majority of the workforce and the contributions are steadily increasing, due to expanding coverage and gradually rising contribution rates (C. Andersen & Skjodt, 2008). The collective agreements determine the minimum contributions within the private sector, the contributions rates are ranging between 12 and 18 percent. The employer pays two-third of the contributions while the employee pays the remaining one-third (J. G. Andersen, 2011). Since these schemes have been established through collective agreements, there is a wide variation in terms of what the pension schemes offer. Most of them are sector-wide and thus benefit from economy of scale and offer a wide range of additional insurance products (C. Andersen & Skjodt, 2008).

The civil servant schemes *(tjenestemandspension)* is also a part of the occupational pillar. It is characterized by a very advantageous pension schemes, where the individual is guaranteed a certain benefits based on the ending salary, during retirement. This type of pension scheme is slowly replaced by ordinary employment contracts, due to its very high cost for the employer (M. B. Andersen & Kristiansen, 2009).

Pillar III: Private Pensions The last pillar covers the voluntary private pensions schemes. These schemes appeal to the individuals not covered by an occupational scheme or those that want to supplement their occupentional pension savings (C. Andersen & Skjodt, 2008). The private pensions are eligible to the same tax advantages as the occupational pensions and contributions are tax-deductible up to a certain limit. The contributions are rather low, compared to the occupational schemes. Since the beginning of the 1990s, there has been a shift away from the private pension, as most individuals are now covered by occupational schemes.

3.2 Product Definitions

The pension schemes in the Danish multi-pillar system have some very fundamental differences, when it comes to benefits, premiums, and whom bears the risk. Pension schemes are generally divided into two categories, in between exists a number of hybrid products that combines elements from both ends of the spectrum.

Defined Benefit The defined benefit scheme, in its pure form, entitles the individual to a specific payment stream in the future. The distinguishing feature is that the benefits are usually determined by the individual's salary, age or employment history. The risk is transferred to the sponsor and the individual bears no risk. Low investment returns, low mortality rates, etc. have no effect on their future benefits (Kemp & Patel, 2012). The policyholders have little to no influence over the investments and for the pension fund to lower their risk, the contributions are usually invested more conservatively (Kemp & Patel, 2012). However, the individuals must rely on the pension funds continued solvency and their ability to pay the guaranteed benefits in the future.

The only pure defined benefit scheme in Denmark is the civil servant pension since the benefits are dependent upon the individuals ending salary (Kristiansen, 2005). The public pension carries some elements of the defined benefit structure, but the benefits themselves, are not based upon the individual's salary, but on a flat-rate pension and means-tested supplement.

Defined Contribution In the defined contribution schemes, contribution are usually defined as a fixed amount or a percentage of the salary. The contributions are individual and the policyholder's contributions are, to some extent, invested according to the individual's own risk preferences (C. Andersen & Skjodt, 2008). The benefits are dependent on the terminal value of the contributions and accumulated returns. The risk is borne by the individual, rather than the pension fund, and the policyholder carries a significant downside risk. If they suffer from poor investment returns the individual will end up with lower benefits during his or her retirement, and vice versa.

Hybrid products Most pension schemes are rarely pure defined benefit or defined contribution, but a hybrid of the two. The Danish pension schemes borrow some of the risk and contribution characteristics from the defined contribution schemes but often include a minimum guarantee on investment returns or more investment flexibility.

One of the most common hybrids is the with-profit scheme, which have two distinguishing features; the use of minimum guarantee on investment returns and the use of the contribution principle *(kontributionsprincippet)*. The investment risk is borne by the pension fund, but the individual is still dependent upon the continued solvency of the pension fund (C. Andersen & Skjodt, 2008). The individuals receive a yearly guaranteed return on their accumulated contributions, a predefined average rate, despite the actual market returns. In years with high returns the pension fund accumulates capital, so that in years with negative returns, the fund can draw upon the accumulated capital to pay the guaranteed return. The accumulated capital is the collective bonus potential *(kollektivt bonuspotentiale)*, and redistributed fairly among the policyholders, according to the contribution principles (Erhvervs- og Vækstministeriet, 2015). The mathematical concepts behind this will be introduced in section 6.2.

Another common scheme in the Danish pension system is the unit-link product. It closely resembles the defined contribution scheme, but the individual policyholder decides how the contributions should be invested. They are able to pick specific stocks, bond or mutual funds, which requires a lot of financial insight. The high complexity of the unit-link scheme and recent turmoil on the stock market have resulted in a less than anticipated interest in the product (Iversen, 2013).

The most widespread version of the unit-link scheme, is a more simplified version where the individual picks the desired level of risk. The risk is borne solely by the investor, however, a guarantee can be included at a cost. Until the late 1990s almost all occupational and private pensions were based upon the with-product schems (Iversen, 2013). The unit-link product was introduced as an alternative to the with-product scheme and is the majority of new contracts, The transition towards unit-link is slow and with-profit products still make up the majority of accumulated pension savings and will continue to do so for the decades to come (Steffensen & Ramlau-Hansen, 2017). The unit-link scheme was introduced after heavy criticism of with-profit, both from an academic and political perspective. The decreasing interest rate was putting an increasing pressure on the pension funds and their ability to meet the guaranteed return (C. Andersen & Skjodt, 2008).

3.3 Overall Market

In Denmark, there are 29 pension funds covering the second and third pillars. PFA Pension, Danica Pension, Nordea Liv & Pension and PensionDenmark have a total market share of just above 50 percent[4] (Munck, 2017). The majority of pension funds are industry specific funds, which are often mandatory for individuals working within specific sectors, and other individuals are excluded from signing a contract with these pension funds. This influences the market behavior and lowers the competition for the remaining pension funds since they do not have to compete for customers.

For non-industry specific companies, the competition is high. The pension funds primarily compete on the costs (for the policyholder), since this is the most transparent parameters. The costs can be divided into three categories; the administrative costs, investment costs and the cost of the various additional insurances.

Within the last decades, the insurance costs have been lowered significanlty, and most of the pension funds are losing money on these products. In the 2016 PFA pension had a loss of DKK 490 million on health and accident insurances, and in 2015 this number amounted to DKK 655 million[5] (PFA Pension, 2017b). Many pension funds are in similar situations, though not at the same scale as PFA Pension. The administrative costs are at

[4] Measured in terms of the contribution rates.

[5] Compared to a revenue of DKK 1.954 million and a profit of DKK 136 million in 2016 (PFA Pension, 2017b).

a low level, leaving the investment costs as the only real profit for most of these companies. Within recent years most pension funds have had to raise their costs, to cover some of their losses (Svendsen, 2016).

Pension products are also offered by banks, but they only offer contracts within the third pillar, and the contributions are relatively modest (C. Andersen & Skjodt, 2008). The contracts are primarily signed as an addition or as an alternative to the individuals not covered by an occupational pension scheme. The banks play a significant role and account for nearly 60 percent of the total contributions in the pillar (C. Andersen & Skjodt, 2008). The products are very similar, to those offered by pension funds, and the policyholders benefit from the same tax advantages. However, there are some rather significant differences between the products offered by banks and pension funds. Pension funds must include an insurance product in their contracts, i.e. a risk element. Banks are only allowed to offer savings products with phased withdrawals or lump sum payments, identical to annuity benefit *(livsrente)* and pure endowment *(kapitalpension)*, and upon death, the accumulated savings are paid out to the heirs. A pension policy offered by a pension fund is technically a bet on your life; the policyholder pays the premiums until they retire and the longer they live the more benefits they will receive. If the policyholder dies before retirement, they will only a term insurance, depending on the contract. The pension fund will keep the accumulated return and the probability of death is accounted for in the contract, this concept is explained in detail in section 6.1. Most individuals have a pension contracts due to being risk-adverse since this a better alternative to the uncertainty of having no income during their retirement years.

3.4 Regulation

The Danish pension system is subject to different laws, regulations, and guidelines created by the government and the Danish Financial Supervisory Authority (FSA) and the local implementations of the regulations created by the European Commission. On January 1, 2016 the Solvency II regime was introduced as a replacement of the Solvency I regime from 1979 (European Commission, 2002). The regime was created with The European Insurance and Occupational Pensions Authority (EIOPA) and was introduced in order to develop a single market in insurance services in Europe. The development of Solvency II began in 2002 and was postponed multiple times until its approval in 2014 (European Commission, 2015). It is important to note that the Solvency II regulation mostly applies to with-profit schemes since unit-link customers carry most of the risk themselves.

The goal of Solvency II is to establish a more financially robust insurance industry and ensure that these companies will be able to survive though economic downturns (European Commission, 2007). This is done through a framework that establishes new rules for how these companies should cover their risk exposures, with the use of economic risk-based constraints. It is a comprehensive set of rules that will help the creation of a more harmonized system throughout the entire European Union and promote more competition, transparency and consumer protection (European Commission, 2007).

Despite the implementation in 2016, pension funds are given a lot of leeway and some of the risk measures and valuation methods are not fully implemented until 2032 (European Commission, 2007). In Denmark, the implementation has been much more rapid and on January 1, 2016, the Solvency II regime was fully implemented. FSA wanted to be a rolemodel and in Denmark the new quantitative requirements and guidelines have been gradually implemented throughout the last decade (Bork, 2017).

The prudent-person principle was introduced alongside Solvency II. The idea behind the principle is to ensure that pension funds invest prudently and it requires the pension funds to invest the assets in the best interest of policyholders' (Finanstilsynet, 2014). Pension funds are only allowed to invest in assets and instruments whose risk can be properly identified, measured, managed and where they fully understand the risk and the complexity of the investment.

Pillar I: Quantitative Requirements The Solvency II regime is also built around a three-pillar system. It draws inspiration from the Basel III requirements, that is the equivalent regulation within the banking sector. The first pillar covers the new and more comprehensive quantitative requirements. The risk measurements are specific to the individual pension fund's risk profile, allowing for a more optimal allocation of capital. The aim is to quantify the risk that the pension funds are exposed to and ensure that they have sufficient capital to withstand severe shocks in the economy. The Solvency Capital Requirement (SCR) and the Minimum Capital Requirements (MCR) are introduced as a supervisory ladder, in order for the authorities to take the necessary interventions if the capital reserve is too low (Douglas, Noss, & Vause, 2017). It is based upon a Value-at-Risk calculation on a 99.5% confidence level over a 1-year time horizon.

The SCR is a stress test imposed to reflect a shock in interest rates, mortality rates, equity prices, etc. Pension funds can use their own complete or partial internal models, but it has to be validated by the supervisory authorities. The MCR is a more prudent threshold, at 25-40% of the level of SCR, representing the minimum amount of capital the pension funds must hold (European Commission, 2007). If the capital falls below the SCR, the supervisors are required to take action, with the aim of restoring the capital reserves. If the capital reserves continue to decline, the level of intervention will be increased. If the capital reserve, despite the interventions, falls below the MCR, the liabilities will either be transferred to another pension fund or the pension fund will be liquidated (European Commission, 2007).

Solvency II also requires the pension funds to calculate and set up technical provisions, corresponding to the current value of the policyholder's reserves and a risk margin. The technical provision is the value of the reserve if they were to transfer the amount to a different pension fund (EIOPA, 2012b). The values of the balance sheets have had significant changes in terms of the valuation of both the assets and liabilities. The items on the balance sheet are no longer valued based on book values but on market-values, in order to better reflect the risk that arises from particular balance sheet items. Assets should be valued at the amount for which they can be exchanged between different investors, while the liabilities should be valued at the amount for which they could be transferred to another pension fund.

Pillar II: Governance & Risk Management The second pillar requires pension funds to implement solid and efficient risk management and governance processes. All pension funds must implement their 'own risk and solvency assessment' which is both an internal assessment process and a supervisory tool for the authorities to ensure prudent management practices. Pension funds have to comply with the 'supervisory review process', that ensures the pension funds are well managed and meet the risk management standards imposed by Solvency II (European Commission, 2007).

Pillar III: Disclosure & Transparency Requirements The last pillar focus on the reporting to the supervisory authorities and disclosure to the public, in terms of 'the solvency and financial condition report' and 'the regular supervisory report' (PwC, 2013). The pillar aims at ensuring a common reporting standard and better risk monitoring across the entire European Union. The reports will tighten the market discipline and help the supervisory authorities perform their reviews, and get a better aggregate view of the industry.

4 Literature Review

4.1 Risk Parameters

Risk parameters is a broad category, where the pension fund must carefully consider which parameters it wishes to mitigate and which it desires to keep. Some of the risk parameters the pension fund is able to adjust through investment policy or other decisions, and other parameters they have no way of influencing. Risk management is used to make decisions to ensure the proper risk is mitigated. Risk management enables the pension fund to acquire a deep understanding of risk factors and how they influence the fund. Numerous risk parameters exist, but only the most important relating to the ALM model will be considered.

Interest rate risk One of the most prominent risks is the interest rate risk, as it has a strong impact on liabilities, as a short-term fall in the interest rate would force the liabilities upward and increase the requirements of the fund, and vice versa. The interest rate changes the scale of the liabilities and any adjustments affect the funding ratio through a change in liabilities (Chen & Matkin, 2017). How the market behaves is unknown, however, the interest rate is a risk that is easier to hedge against through financial instruments. Given the impact, it has on liabilities it is usually hedged by the fund (Danske Bank Group, 2016). The pension fund must account for the influence the interest rate has on the asset prices, as the price of bonds are directly affected by changes in interest rate (Collie, 2012).

Longevity risk Longevity is a difficult parameter to estimate, but it proves to be very influential to the pension fund's liabilities (Danske Bank Group, 2016). The risk of an individual living longer than anticipated is based on the historical improvements in life expectancy, which has been increasing for very long time periods (European Comission, 2017). Longevity risk can be very costly to pension funds as they be required to provide benefits for a longer time. It is important for the pension fund to adequately forecast the expected lifetime, as failing to do so results in more payouts than expected. Two factors contribute to the forecast error; longevity shock and computational risk (Fong, Piggott, & Sherris, 2015). Longevity shock is the risk of an unexpected, immediate and permanent increase in longevity. Computational risk is the risk of forecasting longevity incorrectly, by not factoring in the expectation of future health improvements; medical, technical or lifestyle changes (Fong et al., 2015).

Funding risk Having larget liabilities than assets is known as funding risk, which is a problem with increasing liabilities or decreasing assets. In such a situation the liabilities are said to be underfunded, which brings potential costs; low liquidity, increase in cost of capital, disrupt or prevent mergers or acquisitions and/or lead to lawsuits (Mangiero, 2012). The pension fund is required to have sufficient assets to match the value of their liabilities and being unable to do so triggers the authorities to intervene (European Commission, 2007). The pension fund's level of funding affects the asset allocation and risk aversion. In the case of a very poor funding ratio, the fund is likely to allocate a larger fraction of capital to bonds and safe assets, whereas funds with solid funding have more freedom to invest in riskier assets (Rauh, 2008). A low funding ratio also brings a drawback of poor credit ratings and potential financial distress.

Regulatory risk Regulatory risk is the risk of new regulations the fund must comply with, which is something they are unable to influence. The purpose of regulations is to provide security to the policyholders (Blome et al., 2007) and ensure the pension funds manage the capital safely. Regulations are often either prudent-person rules or quantitative portfolio restrictions (Davis, 2001). Solvency II is a regulatory risk, which has forced pension funds to retain more capital in reserve, thus putting pressure on the solvency of the funds.

Hedging risk Hedging is not affected by outside factors but instead caused by the fund itself. In risk management, it is not just about reducing the overall risk exposure. One must be consciously aware of which risks are worth having and which are not. It is costly to reduce risks, as a risk premium is paid to another party willing to take on the risk. When mitigating risk you reduce the risk of loss at the expense of risk of profit. It is essential that the pension fund understands the risk factors associated, and thus selectively take action towards which risks to keep and which to mitigate. The pension fund can use financial instruments to mitigate undesired risk factors. It is usually done via derivatives, transferring the risk from one party to another, which imposes a hedging risk. Mitigating risk is about know-how; some risk is worth keeping if the fund has sufficient knowledge about how to handle it (Mangiero, 2012). Using hedging instruments can reduce the volatility of the liabilities, but it can also bring the risk of over-hedging. If a pension fund mitigates too much risk, they become subject to over-hedging. In the case of over-hedging they would be subject to a high cost, as mitigating risk it is usually done by paying another party a risk premium[6] to take over the risk.

Contract risk The contracts a pension fund provides are associated with contract risk. The fund needs to be aware of potential risks associated with its contracts, based on what kind of guarantees it provides. A wide variety of contracts exist, each with different contract risk embedded. In defined benefit schemes the risk is borne by the fund, whereas the risk associated with defined contribution is borne by the policyholder (Kemp & Patel, 2012). Newer contracts are hybrids containing elements from both ends of the spectrum.

Market risk The potential change in value due to market movements, is called the market risk. It is likely to affect both assets and liabilities with fluctuations in market

[6]It depends on the type of derivative being used; options have premiums, forwards do not.

conditions (Danske Bank Group, 2016). It is important to note that market risk is not strictly considered the risk of a drop in value of financial assets, but instead the effect it has on the asset-liability matching issue (Kemp & Patel, 2012). Market risk is a combination of different market movements affecting the fund; some of which is shortfall risk, bond risk, and equity risk.

Shortfall risk covers the risk of earning a return on an investment portfolio, which is less than the risk-free interest rate over some time period (Bodie, 1991). Pension funds have the luxury of very long time horizons, which means they are not as vulnerable to short-term fluctuations as other investors might be. This contributes to a large discussion about time diversification; whether or not the risk of investing in stocks subside as the time horizon increases (The Vanguard Group, 2008). In a fully funded defined benefit scheme it is less beneficial to invest in risky assets, as the upside is shared by the policyholder and the fund, whereas the downside is carried solely by the fund (Bodie, 1991).

Bond risk is the risk of bond prices dropping, which is correlated to the interest rate. Equity risk is the risk of equity prices dropping, which is based on future prospects as seen by market participants.

4.2 Portfolio Theory

Modern portfolio theory is a mathematical framework for how risk-averse investors should select a portfolio of assets in order to maximize the expected return, for a given level of risk. One of the most important properties is that the asset's risk and return should be examined by how it contributes to a portfolio's overall risk. By investing in different sectors, industries, and countries the pension fund is able to maximize its return while lowering the overall risk. The covariance must be considered since a high amount of assets in itself does not lower the risk, if the assets are highly correlated (Markowitz, 1952). The risk is defined as the variance of the asset returns and the portfolio framework is also known as mean-variance analysis.

Mean-Variance Analysis In portfolio theory, the investors always face a trade-off between the level of risk and the expected return. The main assumptions of the mean-variance analysis are that; the investors choose among different portfolios over fixed future time periods, and investors would prefer as high expected return and lowest variance as possible (Munk, 2016). A portfolio is mean-variance efficient if it has the minimum variance among all portfolios for a given expected return. From these portfolios, the efficient-frontier of the risky assets can be constructed.

The investors are able to invest in N different assets, $\boldsymbol{\mu}$ is a vector of expected returns and Σ is the covariance matrix of the returns. The weight of the individual assets, i, is denoted π_i and a portfolio vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)$ must satisfy (Munk, 2016)

$$\boldsymbol{\pi}\mathbf{1} = \pi_1 + \pi_2 + \ldots + \pi_N = 1,$$

meaning that the weight of all assets must sum to one[7]. The expected return, variance and standard deviation is then given by (Munk, 2016):

$$\mu(\boldsymbol{\pi}) = \boldsymbol{\pi}\boldsymbol{\mu} = \sum_{i=1}^{N} \pi_i \mu_i, \tag{4.1}$$

$$\sigma^2(\boldsymbol{\pi}) = \boldsymbol{\pi}\Sigma\boldsymbol{\pi} = \sum_{i=1}^{N}\sum_{j=1}^{N} \pi_i \pi_j \Sigma_{ij}, \tag{4.2}$$

$$\sigma(\boldsymbol{\pi}) = \sqrt{\boldsymbol{\pi}\Sigma\boldsymbol{\pi}} = \left(\sum_{i=1}^{N}\sum_{j=1}^{N} \pi_i \pi_j \Sigma_{ij}\right)^{1/2}. \tag{4.3}$$

The covariance matrix Σ is non-singular and the inverse is denoted by Σ^{-1}. For the calculation of the portfolios. the following auxiliary constants are introduced (Munk, 2016):

$$A = \boldsymbol{\mu}\ \Sigma^{-1}\boldsymbol{\mu}, \tag{4.4}$$

$$B = \mathbf{1}\ \Sigma^{-1}\boldsymbol{\mu}, \tag{4.5}$$

$$C = \mathbf{1}\ \Sigma^{-1}\ \mathbf{1}, \tag{4.6}$$

$$D = AC - B^2. \tag{4.7}$$

The efficient portfolios are determined by solving a quadratic minimization or maximization problem for each portfolio. These problems are solved by using the Lagrange optimization technique that finds the local maxima and minima of a function under the given equality constraints (Sydsæter, Hammond, & Strøm, 2012).

Minimum-Variance Portfolio The portfolio with the lowest possible variance is the minimum-variance portfolio, also known as the global minimum-variance portfolio. The portfolio is determined by solving the minimization problem (Merton, 1972)

$$\begin{aligned} &\min_{\boldsymbol{\pi}} \boldsymbol{\pi}\Sigma\boldsymbol{\pi}, \\ &s.t.\ \boldsymbol{\pi}\mathbf{1} = 1. \end{aligned} \tag{4.8}$$

There is no constraint on the expected return, but the portfolio on the efficient-frontier with the lowest variance is the minimum-variance portfolio. The same conclusion can be reached using the following equation (Munk, 2016);

[7] When applying linear algebra one of the matrices have to be transposed in order to multiply them. The transposed matrix is flipped diagonally, which can be denoted $\boldsymbol{\pi}\Sigma\boldsymbol{\pi} = \boldsymbol{\pi}\Sigma\boldsymbol{\pi}^\top$, but it is implied in the coming equations.

$$\boldsymbol{\pi}_{min} = \frac{1}{C}\Sigma^{-1}\ \mathbf{1}. \tag{4.9}$$

The mean, variance, and standard deviation can then be calculated using the auxiliary constants calculated in equation (4.4) - (4.7):

$$\mu_{min} = \frac{B}{C}, \tag{4.10}$$

$$\sigma^2_{min} = \frac{1}{C}, \tag{4.11}$$

$$\sigma_{min} = \frac{1}{\sqrt{C}}. \tag{4.12}$$

The assets with the largest expected weight are expected to be the assets with the lowest standard deviation. The correlation have a significant role in terms of the diversification effect. The portfolio might include an asset with a high standard deviation if it has a low or negative correlation with the remaining assets.

Tangency Portfolio The minimum-variance portfolio is optimized under the assumption that only risky assets exist. For the determination of the tangency portfolio the notion of the risk-free asset is introduced. The risk-free asset has a variance of zero and is uncorrelated with the rest of the assets. The investor can combine any portfolio of a risky asset with an investment in the risk-free asset to obtain an optimal portfolio. The slope of the tangent is the Sharpe ratio given by (Sharpe, 1994)

$$SR = \frac{\bar{\mu} - r_f}{\sigma_(\bar{\mu})}, \tag{4.13}$$

where $\bar{\mu}$ is the expected return of the minimum-variance portfolio from equation (4.10), r_f is risk-free rate, and $\sigma_(\bar{\mu})$ is the standard deviation of the portfolio. The Sharpe ratio is defined as the expected return earned in excess of the risk-free rate per unit of risk, thus isolating the performance related to the risk-taking investments. The tangency portfolio is maximization of the risk/return ratio. The tangency portfolio is found by solving the following maximization problem:

$$\begin{gathered}\max_{\boldsymbol{\pi}} SR,\\ s.t.\ \boldsymbol{\pi}\mathbf{1} = 1.\end{gathered} \tag{4.14}$$

If the risk-free rate is higher the expected return, the tangency portfolio would be downward-sloping and the optimization would require a short position in the tangency portfolio. This would instead lead to a minimization of the Sharpe ratio. No matter the size of the risk-free rate the optimization is achieved by solving $SR' = 0$, which leads to the equations (Munk, 2016):

$$\boldsymbol{\pi}_{tan} = \frac{1}{B - C\ r_f}\Sigma^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}), \tag{4.15}$$

$$\mu_{tan} = \frac{A - B\ r_f}{B - C\ r_f}, \tag{4.16}$$

$$\sigma_{tan}^2 = \frac{A - 2B\ r_f + C\ r_f^2}{(B - C\ r_f)^2}, \tag{4.17}$$

$$\sigma_{tan} = \frac{\sqrt{A - 2B\ r_f + C\ r_f^2}}{|B - C\ r_f|}. \tag{4.18}$$

Since the tangency portfolio maximizes the Sharpe ratio, the assets with the highest weights are expected to be the assets with the largest Sharpe ratios. However, the correlation once again have a significant influence over the optimization, and an asset with a low Sharpe ratio could have a high weight if its correlation with the other assets is low or negative.

Sortino Ratio The Sharpe ratio has become widely used for calculating risk-adjusted returns but it does have its limitations. If the assets are not normally distributed, have a high degree of kurtosis or negative skewness, the Sharpe ratio is not able to correctly assess the risk. Since the ratio does not distinguish between upside and downside volatility, very high returns would lower the overall Sharpe ratio, despite an increase in the expected portfolio return.

An alternative risk measure is the Sortino ratio, which puts more emphasis on the downside risk in the investment decision. It is a modified Sharpe ratio where the standard deviation has been replaced by the downside deviation and only returns falling below a desired target return are considered risky. The Sortino ratio is defined as (Rollinger & Hoffman, 2013)

$$ST = \frac{R - T}{TDD}. \tag{4.19}$$

The expected return is defined as R and the target ratio for the investment strategy is defined as T. The target downside deviation, TDD, is given by

$$TDD = \sqrt{\frac{1}{N}\sum_{i=1}^{N} (Min(x_i - T,\ 0))^2}, \tag{4.20}$$

where x_i is the portfolio's return and T is the target ratio. The measure is the deviation of the realized returns that underperforms the target return and all returns above the target are zero. The optimal Sortino portfolio is found by solving the following optimization problem (Rollinger & Hoffman, 2013):

$$
\begin{aligned}
&\max_{\boldsymbol{\pi}} ST, \\
&s.t.\ \boldsymbol{\pi}\mathbf{1} = 1.
\end{aligned} \tag{4.21}
$$

This leads to the following equations for the calculation of the mean, variance and standard deviation for the Sortino portfolio:

$$\boldsymbol{\mu}_s = \boldsymbol{\pi}\boldsymbol{\mu}, \tag{4.22}$$

$$\boldsymbol{\sigma}_s^2 = \boldsymbol{\pi}\Sigma\boldsymbol{\pi}, \tag{4.23}$$

$$\boldsymbol{\sigma}_s = \sqrt{\boldsymbol{\pi}\Sigma\boldsymbol{\pi}}. \tag{4.24}$$

The Sortino ratio accounts for the frequency and magnitude of below-target returns, which will result in a portfolio with an emphasis downside risk. The correlation plays a significant role and the portfolio could include an asset with a high downside risk if it has a low or negative correlation with the remaining assets.

Risk Parity The risk parity method has a much higher focus on the risk allocation and how it is allocated between the assets. It is a risk diversification strategy, where the portfolio is constructed such that each asset will contribute equally to the overall risk of the portfolio.

Risk aversion or investment restrictions on leverage, along with a return requirement, could force an investor to increase the risk to reach a high return. Traditional 60/40 asset allocation portfolios tend to have a high allocation to high-risk assets and returns are dependent on strong equity markets. Stocks often account for more than 80% and 90% of the total portfolio risk (Qian, 2012), which means that these portfolios are not risk diversified. In the risk parity approach the overall risk is defined as (Maillard, Roncalli, & Teiletche, 2008):

$$\sigma(\pi) = \sqrt{\boldsymbol{\pi}\Sigma\boldsymbol{\pi}}, \tag{4.25}$$

and the risk contribution from each individual assets, i, is defined as

$$\sigma_i(\pi) = \frac{\pi_i(\Sigma\boldsymbol{\pi})_i}{\sqrt{\boldsymbol{\pi}\Sigma\boldsymbol{\pi}}}. \tag{4.26}$$

The following is true for all asset volatilities:

$$\sigma(\pi) = \sum_{i=1}^{N} \sigma_i(\pi), \tag{4.27}$$

and

$$\sigma_i(\pi) = \frac{\sigma(\pi)}{N} \tag{4.28}$$

must be true for the volatility of the individual assets. The optimal risk parity portfolio is found by solving the following optimization problem (Maillard et al., 2008):

$$\begin{aligned} &\max_{\boldsymbol{\pi}} \ \boldsymbol{\pi\mu}, \\ &s.t. \ \boldsymbol{\pi 1} = 1, \\ &s.t. \ \sigma_i(\pi) = \sigma_j(\pi) \ for \ all \ i, j. \end{aligned} \tag{4.29}$$

The risk parity weights are determined by maximizing the return, under the constraint that the total portfolio weights must sum to one and the marginal risk contribution from each asset must be the same. This leads to the following equations:

$$\boldsymbol{\pi}_{rp} = \frac{\Sigma\pi}{\sqrt{\boldsymbol{\pi}\Sigma\boldsymbol{\pi}}}, \tag{4.30}$$

$$\boldsymbol{\mu}_{rp} = \boldsymbol{\pi\mu}, \tag{4.31}$$

$$\boldsymbol{\sigma}_{rp}^2 = \boldsymbol{\pi}\Sigma\boldsymbol{\pi}, \tag{4.32}$$

$$\boldsymbol{\sigma}_{rp} = \sqrt{\boldsymbol{\pi}\Sigma\boldsymbol{\pi}}. \tag{4.33}$$

The risk parity portfolio could be used in order to more efficiently allocate risk while delivering stable returns to the pension fund (Lee, 2014).

4.3 Stochastic Models

Any variable that changes over time, in an uncertain way, is said to follow a stochastic process. Stochastic models are very popular within finance and are widely used to price financial instruments and assets. They are used in order to obtain numerical results in applications where closed-form analytical solutions are not available and in order to create complex ALM models.

The future is uncertain and in order to manage the complex relationship between the asset and liabilities of a pension fund, the possible future outcomes have to be modeled. By including random variation in one or more inputs over time and incorporating the properties of the asset classes in the stochastic models, a wide range of outcomes, reflecting the scale and complexity of the risk, are produced. From the output of a stochastic model statistical distribution can be determined (EIOPA, 2012a). Stochastic models are a critical part of an ALM analysis and helps the pension fund understand and manage risk.

Real-World vs Risk-Neutral Measure An ALM model in itself have a wide range of applications and in order to ensure consistent valuation of the assets and liabilities, the distinction between the risk-neutral and the real-world measures is necessary. It is very important to ensure consistency in the use of different models and that the dynamics are captured. This would otherwise require an alteration of the output of the model, or lead to incorrect results and inconsistency between asset and liability modeling. None of the methods are better than the other, it all depends on the purpose of the ALM model (Institute of Actuaries of Australia, 2007).

Real-World The intuition behind the real-world measure is very simple and more transparent than the risk-neutral measure. The broad idea is to capture market dynamics, risk and returns in the way that the pension funds are affected. The models account for the investor's risk aversion and the expected risk premium, which yields realistic cash-flows and their associated probabilities (Society of Actuaries, 2016). The idea is to give a realistic dynamic of market prices and to estimate the probability of extreme events to plan and manage the risk of the pension funds. The models can be implemented on a wide range of application and include features of the financial markets which make the model very easy to understand, though they are significantly harder to calibrate (Institute of Actuaries of Australia, 2007).

Risk-Neutral The concept behind the risk-neutral measure, also known as market-consistent valuations, is less intuitive but is still very important. In the risk-neutral world, all investors are assumed to be risk-neutral in order to achieve market consistency in the pricing of the assets and liabilities (Jakhria, Mirzai, & Muller, 2013). The probability of future states and cash-flows are adjusted to the risk-neutral probabilities[8] and the expected returns are then equal to the risk-free rate (Boukfaoui, 2013). The risk-neutral theory provides a mathematical framework for pricing of assets and liabilities in an arbitrage-free framework and is widely used in finance, which models like Black–Scholes is built upon.

The real-world and risk-neutral measurements will often be applied together. Measuring risk always involve returns and how these evolve over time. When closed-form solutions exist for asset prices, the prices can be simulated off the real-world dynamics and the risk-neutral measure is not required (Institute of Actuaries of Australia, 2007). The risk-neutral measure is used to price the liabilities and assets, that do not have a closed-form solution, in order to combine risk management and ensure consistency in the pricing of assets and liabilities.

Markov Property Before getting into the actual models, some background knowledge about the stochastic models and their properties are needed. A very important stochastic process is the Markov process, which refers to a memoryless property. The fundamental

[8] The risk-neutral probability is also known as state-contingent price, state price or martingale price (Berk & DeMarzo, 2014).

Markov property is defined as a sequence $X_0, X_1, \ldots, X_{n-1}$ of random variables, which are a Markov chain if (Rønn-Nielsen & Hansen, 2014),

$$X_{n+1} \perp (X_0, X_1, \ldots, X_{n-1}) | X_n \quad for\ n = 1, 2, \ldots$$

meaning that the one-step conditional probability distribution of the future value depends only upon the current value and not the sequence of events that precede it. The immediate future, represented by X_{n+1}, is independent of the entire past given the present (Rønn-Nielsen & Hansen, 2014).

Stock prices are often assumed to follow a Markov process, where the only relevant information is the current stock prices. The predictions are uncertain and the Markov property implies that the probability distribution of the stock price, at any future time, does not depend on the past prices, but only the current stock price. The Markov property is also consistent with the weak form of market efficiency, which states that the present stock price only contains past information and that the stock follows a random walk (Fama, 1998). The Markov property simplifies a lot of financial problems and provides a solution to the complex problems, that could not otherwise have been solved.

Wiener Processes A stochastic Markov process with a mean of zero and variance of one is known as a Wiener process. The Wiener process is used to simulate random walks and is also referred to as a Brownian motion. The Wiener process plays an immense role in the stochastic models and is an efficient tool within finance to price derivatives and other financial instruments. The process is too simple to create a realistic model of the market and price movements, but the properties of the model are very important and the following stochastic models all include elements of the Wiener process. A Wiener process for a variable W has the following properties (J. C. Hull, 2012),

Property I: The change ΔW during a small period of time Δt is

$$\Delta w = \epsilon \sqrt{\Delta t}, \tag{4.34}$$

where ϵ has a standardized normal distribution $\phi(0, 1)$.

Property II: The values of ΔW for any two different short time intervals, Δt, are independent.

The Wiener process can be extended to the Generalized Wiener process, which includes a drift term creating a non-stationary process. The basic Wiener process had a drift rate of zero, meaning that the expected value of x at any future time would be equal to its current value. For variable x the process can be defined as (J. C. Hull, 2012)

$$dx = a\ dt + b\ dW, \tag{4.35}$$

where a and b are constants. The $b\ dW$ term can be regarded as noise, while the $a\ dt$ term, the drift, is the increase in value of x over time period t. For the small time interval Δt, the change Δx in the value of x is given by

$$\Delta x = a\Delta t + b\ \epsilon\sqrt{\Delta t}. \tag{4.36}$$

Just like in the Wiener process, ϵ has a standard normal distribution with mean 0 and variance 1, while Δx has a normal distribution with mean $a\ \Delta t$ and variance of $b^2\Delta t$

Vasicek A term structure model is needed to describe the evolution of the interest rate and price the assets and liabilities. The interest rate, r, at time t, is referred to as the short rate since it applies to an infinitesimally short period of time t (J. C. Hull, 2012). The term structure models are constructed so that they capture the behavior of the short-term interest rate and gives an approximation of how the interest rate might evolve in the future.

In the short rate one-factor Vasicek model the dynamics for the risk-neutral measure of r is used to price the zero-coupon bonds and enables the discounting of cash-flows in the ALM model to time t. The Vasicek model is given by (Vasicek, 1977)

$$dr = a(b-r)dt + \sigma\ d\epsilon, \tag{4.37}$$

and a discrete version of the model is defined as

$$\Delta r = a(b-r)\Delta t + \sigma\epsilon\sqrt{\Delta t}, \tag{4.38}$$

where a, b and σ are constants and ϵ is the Wiener process. The Vasicek model follows an Ornstein–Uhlenbeck stochastic process and is a stationary distribution, which resembles the Markov process and Wiener process (Vasicek, 1977). The instantaneous drift rate $a(b-r)$ pulls the process towards the model's long-term mean b, while the mean-reversion speed is defined as a (Praz, 2017).

In the Vasicek model, the price of a zero-coupon, $P(t,\ T,\ r(t))$, a bond that pays one at time t is defined as (J. C. Hull, 2012),

$$P(t,\ T,\ r(t)) = e^{A(t,\ T)-B(t,\ T)r(t)}. \tag{4.39}$$

The short rate r at time t is calculated with the following equations:

$$B(t,\ T) = \frac{1-e^{-a(T-t)}}{a}, \tag{4.40}$$

$$A(t,\ T) = \frac{(B(t,\ T)-T+t)(a^2b-\sigma^2/2)}{a^2} - \frac{\sigma^2\ B(t,\ T)^2}{4a}. \tag{4.41}$$

The term structure at time t is given by

$$R(t,\ T,\ r(t)) = -\frac{1}{T-t}A(t,\ T) + \frac{1}{T-t}B(t,\ T)r(t), \tag{4.42}$$

and is a function of $r(t)$ which can be determined once the variables a, b, and σ have been calibrated. The shape of the term structure is independent of $r(t)$ but is dependent on t and can change over time (J. C. Hull, 2012). The Vasicek model is an example of an equilibrium model, where today's term structure of interest rates is an output of the model (J. C. Hull, 2012). However, short rate models can also be defined as a no-equilibrium model where the term structure and the interest rate is instead an input. Equilibrium models only give an approximate fit to the real term structure, where the no-arbitrage models are more consistent with today's term structure, making the models more popular among traders. However, the advantage of the Vasicek model is that the model parameters can be calibrated using historical data, which is not the case for no-arbitrage models (J. C. Hull, 2012).

Geometric Brownian Motions In order to simulate the price of a stock, assumptions about the stock and its properties must be made. The stock prices are assumed to be lognormally distributed stock prices are assumed to be stochastic; given the prices today, we don't know the stock price tomorrow. The stock prices are assumed to changes continuously and increase over time (Benninga, 2014).

The stocks are assumed to follow a random walk which is assumed to be a Markov process, though you could argue that the stock prices, over a very short time horizon, are Martingale. A Martingale process is defined as (Cuthbertson & Nitzsche, 2005)

$$E(X_{n+1}|X_1, ..., X_n) = X_n, \tag{4.43}$$

and is, like a Markov process, a random walk. The conditional expected value of X_{n+1}, given all the past observations, is equal to the most recent observation. If the weak-form of the efficient market hypothesis is assumed to be true, it would be assumed that the stock prices follow a Martingale process. In a perfect market, this would be true since the future stock prices cannot be predicted (Cuthbertson & Nitzsche, 2005). In reality, the stock prices follow neither a Martingale nor Markov process, but the Markov process provides better results when simulating the stock prices since only the current prices would be sufficient for the probability distribution of the future prices.

The price path of stocks is simulated using a stochastic model, known as a Geometric Brownian Motion, which follows the above assumptions. It is a model built upon a Generalized Wiener process with the assumption that over a short time period, Δt, the expected return, μ, is constant and that volatility, σ, is not dependent on the stock price. These assumptions leave us with the following continuous model (J. C. Hull, 2012):

$$\frac{dS}{S} = \mu\ dt + \sigma\ dz. \tag{4.44}$$

A discrete version of the model can be expressed by

$$\frac{\Delta S}{S} = \mu\ \Delta t + \sigma\epsilon\sqrt{\Delta t}, \tag{4.45}$$

where μ and σ are constants and the Wiener process expressed by ϵ. In the discrete time version, the dt and dz notations have been replaced with Δt, in order to express the change in the stock price, S, over the small time interval Δt. The notation is used to indicate the movement from small changes to the limit; as the time interval goes towards zero, $\Delta t \to 0$, it will become a continuous process (J. C. Hull, 2012).

A Monte Carlo simulation[9] is used to sample the random outcomes from a standard normal distribution with a mean of zero and a variance of one, $\phi(0, 1)$. Multiple simulations of the price path will be calculated since the stochastic process yields a single possible future stock price. By repeating the process a complete probability distribution of the stock will be obtained.

In practice, it is more accurate to simulate $ln\ S$ instead of S, so Itô's lemma[10] is applied to the above equations and we are left with the following expressions for the process (J. C. Hull, 2012),

$$d\ ln\ S = \left(\hat{\mu} - \frac{\sigma^2}{2}\right) dt + \sigma\ d\epsilon. \tag{4.46}$$

The equation used to express the price path is

$$S(t + \Delta t) = S(t)\ exp\left[\left(\hat{\mu} - \frac{\sigma^2}{2}\right)\Delta t + \sigma\epsilon\sqrt{\Delta t}\ \right], \tag{4.47}$$

and it follows

$$S(T) = S(0)\ exp\left[\left(\hat{\mu} - \frac{\sigma^2}{2}\right)T + \sigma\epsilon\sqrt{\Delta t}\ \right], \tag{4.48}$$

where $\hat{\mu}$ is the estimated mean, σ is the volatility and both of the variables are constant. The above model indicates that the change in $ln\ S$ between time 0 and time T is distributed with mean $(\mu - \sigma^2/2)T$ and variance $\sigma^2 T$:

$$ln\ S_T \sim \phi\left[ln\ S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma^2 T\ \right]. \tag{4.49}$$

The result shows that S_T is normally distributed and the variable has a lognormal distribution since the natural logarithm of the variable is normally distributed, with the above parameters (J. C. Hull, 2012).

Cholesky Decomposition When the asset paths are simulated using the stochastic models, it is important to account for the correlation between the different assets, in order to generate a scenario that captures more realistic market dynamics. A method for generating correlated errors involves the use of Cholesky's decomposition. The method

[9] A Monte Carlo simulation is a procedure for sampling random variables in order to obtain numerical results (Raahauge, 2016). Within finance, the method is used to price derivatives and other financial instruments without closed-form solutions.

[10] Itô's lemma is an important mathematical proof for the stochastic process, that enables the function of a variable to be calculated from the stochastic process for the variable itself (J. C. Hull, 2012).

has some very desirable properties, it is a very simple method that can be applied to any sampling distribution with a symmetric and positive definite correlation matrix and the martingale distribution remain intact (Iman & Conover, 1982).

If a square matrix **C** is symmetric and positive definite, the matrix can be decomposed. The symmetry means that $a_{ij} = a_{ji}$ for $i, j, ..., N$, while positive definite means that

$$\mathbf{vCv} > 0$$

is true for all vectors **v**. The Cholesky decomposition constructs the lower triangular matrix denoted **L**, whose transpose serves as the upper triangular part, meaning that

$$\mathbf{LL} = \mathbf{C}. \tag{4.50}$$

Writing out the equation in components, the following analogs of equations are obtained:

$$L_{ii} = \left(a_{ii} - \sum_{k=1}^{i-1} L_{ik}^2 \right)^{1/2}, \tag{4.51}$$

$$L_{ji} = \frac{1}{L_{ii}} \left(a_{ij} - \sum_{k=1}^{i-1} L_{ik} L_{jk} \right) \quad j = i+1, i+2, ..., N. \tag{4.52}$$

By applying the equations in the order $i = 1, 2, ...N$, where only the components a_{ij} for $j \geq i$ are referenced, the Cholesky decomposition matrix is obtained (Press, Teukolsky, Vetterling, & Flannery, 1992).

In order to obtain the correlated errors, ϵ_i, independent errors from a univariate standardized normal distribution, x_i, are generated. Let **C** denote the correlation matrix where **L** is the Cholesky decomposition of the matrix. The number of input variables is denoted K and the sample size is denoted N. Let **X** denote the **N**×**K** matrix whose columns contain the uncorrelated randomly generated numbers, x_i. By multiplying the matrices the correlated errors are obtained:

$$\mathbf{E} = \mathbf{XL}. \tag{4.53}$$

Multiplying the matrices will results in K vectors with a multivariate distribution according to the correlation matrix (Iman & Conover, 1982), the correlated random numbers that remains are used in the stochastic models above, to ensure that the models capture the dynamics of the actual markets.

4.4 Mortality model

The value of a pension fund liabilities is influenced by the mortality intensity and the expected lifetime of an individual. The properties and definitions of the distribution of lifetime are necessary before the actual mortality model is introduced. The stochastic variable T denotes the uncertain lifetime of an individual and P specifies the probability of the individual dying before age t. The distribution function of the individual's lifetime can then be defined as (Norberg, 2002),

$$F(t) = P(T \leq t) \tag{4.54}$$

and the survival function can be defined as

$$\overline{F}(t) = 1 - F(t) = P(T > t), \tag{4.55}$$

which is the probability of the individual surviving age t. The distribution function can also be expressed in terms of the density by the integral (Steffensen & Ramlau-Hansen, 2017)

$$F(t) = \int_0^t f(s)ds, \tag{4.56}$$

which is an important property used in the calculations of the liabilities of a pension fund. When considering a probability of an individual, we are interested in the probability of surviving or dying, within a given time interval. If we know that an individual has survived age x, it will affect the probability that the policyholder will also survive age $x + t$. This gives us the conditional distribution function (Norberg, 2002)

$$P(T \leq x + t \mid T > x) = \frac{F(x+t) - F(x)}{\overline{F}(x)}, \tag{4.57}$$

and the survival function is defined as

$$P(T > x + t \mid T > x) = \frac{\overline{F}(x+t)}{\overline{F}(x)}. \tag{4.58}$$

The mortality intensity defines the distribution of T in terms of the conditional probability that determines whether an individual dies 'within the small time interval' of Δt. The probability is defined as μ, and by rearranging the conditional functions (4.57) and (4.58), the probability of dying can be described in terms of the mortality intensity

$$F(t) = e^{-\int_0^t \mu(s)ds}\mu(t), \tag{4.59}$$

and the survival function

$$\overline{F}(t) = e^{-\int_0^t \mu(s)ds}. \tag{4.60}$$

The mortality intensity is defined as a distribution of T and specified by μ, thus calculations of the liabilities can be based on these stochastic variables. There are various models of mortality intensity and it is often a trade-off between simple models and advanced models, with a very high number of variables, that describes the reality more accurately. One of the oldest and still a very popular mortality model is the Lee-Carter model (Steffensen & Ramlau-Hansen, 2017);

$$\mu(x,t) = e^{\beta_1(x)+\beta_2(x)k(t)}. \tag{4.61}$$

The model includes an age-dependent mortality, β_1, and improvements across calendar years, $\beta_2(x)k(t)$, and a random-walk with a Brownian motion, $k(t)$. The FSA has introduced a simple version of a Lee-Carter model that all pension funds must use when calculating the liabilities. The pension funds are allowed to use their own internal models, but they have to be approved by FSA. The standard equation is given by

$$\mu(x,t) = \mu(x,t_0)(1 - R(x))^{(t-t_0)}. \tag{4.62}$$

The model is deterministic, $\mu(x,t_0)$ is a benchmark for the observed mortality and $R(x)$ is the benchmark for the expected lifetime improvements. The benchmarks for the observed mortality is based on data from pension funds and the lifetime improvements are based on forecasts of data from 1987-2016 (Finanstilsynet, 2017a). The benchmarks include data for both men and women, but Danish insurance companies are not allowed to discriminate based on gender and a unisex mortality table is used instead (Steffensen & Ramlau-Hansen, 2017). The FSA benchmarks can be seen in appendix C.

4.5 Asset-Liability Management

The available literature investigating Asset-Liability Management (ALM) models is mainly related to traditional defined benefit schemes (Drijver, Haneveld, & Vlerk, 2000; Platanakis & Sutcliffe, 2017; Schwaiger, Lucas, & Mitra, 2010), where the main objective is a minimization of cost to the sponsoring company. Other studies focus on valuation of public pensions (Ezra, 1980; Chen & Matkin, 2017), and discusses implications of unfunded public pension and the sensitivity of actuarial inputs. Some studies provide an assessment of how regulations affect pension fund's decision making (Blome et al., 2007; Davis, 2001).

The paper will apply the knowledge and findings of prior literature to a Danish pension fund setting, focusing on with-profit contracts, to understand the strengths and weaknesses of ALM models. Pension funds are important and the insolvency must be prevented, as the policyholders are financially dependent of the pension fund. Solvency II and other regulations are implemented to improve the surveillance of funding level while ensuring the funds stay solvent and are able to meet the future minimum guaranteed investment returns (Blome et al., 2007). The majority of pension holders in Denmark have with-profit contracts that provide the policyholder with a guarantee.

The funding ratio is a good measure of the pension fund's financial situation. It is the ratio of assets to liabilities and a high funding ratio means a high amount of assets proportionate to liabilities. When a pension fund is providing guarantees through its contracts, it is essential that they manage their assets such they are able to honor those guarantees in the future. The fund is monitored by the FSA; making sure the funding levels and solvency requirements meet the regulations. To properly manage the assets of the fund with respect to future liabilities, pension funds use an ALM model. This ensures the shocks of the assets match the shocks of the liabilities to secure enough assets in the future to cover the liabilities.

The ALM model provides a better understanding of the possible risk factors influencing the pension fund and helps to manage the assets and liabilities. When utilizing an ALM model multiple scenarios could be considered to account for multiple cases of future behavior, which can be achieved through scenario-based programming[11] (Platanakis & Sutcliffe, 2017). There are a variety of ALM models, but the four main categories are stochastic programming, dynamic programming, portfolio theory, and stochastic simulation (Platanakis & Sutcliffe, 2017), with the two first cases being a subcategory of scenario-based programming. Stochastic programming[12] is a popular technique for solving ALM problems, as it takes probability distributions of uncertain parameters into account, and support the decision making for unknown future events (Drijver et al., 2000). An ALM model can be either a 'one-period static' model or a 'multi-period dynamic' model (Blome et al., 2007). In reality, pension funds are able to make changes depending on how the future evolves, thus the ALM model will be constructed as a multistage stochastic programming model. There is a wide variety of objectives function of ALM models using stochastic programming; some possibilities are to maximize the terminal wealth of the policyholder or to minimize the present value of contributions (Platanakis & Sutcliffe, 2017). The purpose of the ALM model will be a maximization of the total expected bonus potential.

[11] Scenario-based programming is conducting multiple possible scenarios, to understand how the future might behave.

[12] Stochastic programming is an in-depth analysis of possible future states taking stochastic variables into account. The stochastic nature of the variables demands multiple scenarios to understand how the distribution of the future might look.

5 Assets

The accumulated Danish retirement saving make up a significant part of the aggreate finanical wealth, and by the end of 2016 Danish pension funds had more than DKK 2,300 billion under management (Finanstilsynet, 2017b). This makes pension funds some of the largest institutional investors, but their investment strategies differ significantly from other investors. The liabilities of a pension fund represent the value of the future benefits for the policyholders, which span decades into the future. As most long-term assets have a lifespan of 20-30 years, duration matching[13] poses a challenge for pension funds, with the uncertainty from the long investment horizon being reflected in its investment strategy.

The majority of assets consist of bonds and equity, but long-term investments like alternative investments and property are common as well. The alternative investments are often long-term and illiquid, providing the pension funds with higher expected returns. The weight of the individual asset classes varies between pension funds and depends on their risk profile, the policyholders' average age, the current state of the financial markets, etc. The average pension fund's portfolio consists of 60-80 percent bonds, 15-30 percent equity, 5-10 percent property, financial instruments, private equity, and alternative investments (Finanstilsynet, 2017b).

5.1 Bonds

The most abundant asset class in pension funds are bonds, that are characterized by low returns and risk, which makes them a very safe investment. They can be defined as a tradable loan, where the owner of the bond is promised a predefined payment schedule. One of the main advantages of bonds are the long maturities and cash-flow structure, which helps the pension fund more adequately match their assets and liabilities. Many different types and variations of bonds exist, some of which are; government bonds, covered bonds, and high-yield bonds.

Government bonds, or sovereign bonds, are issued by the government where default is highly unlikely, which means that the bonds are often used as a benchmark for a risk-free investment. They are traded in highly liquid markets and have relatively low returns compared to the other assets.

Danish covered bonds are issued by mortgage institutions, persistent demand for mortgage-financing means that the Danish covered bond market is one of the largest in the world (Danske Bank, 2017). The mortgage bonds are callable and unique to the Danish bond market with the maturity usually ranging between 10-30 years. They can be redeemed by the issuer prior to the bond's maturity, which makes it unlikely that the bond price exceeds the face value (Danske Bank, 2017) since the bonds are backed by

[13]Duration is a measurement of the sensitivity of a fixed-income investment to the change in the yield curve. Where duration matching is an immunization strategy, and the duration of the assets and liabilities are matched, so that the change in interest rate will influence the price of both at the same rate (Munk, 2016).

property, which makes them a very secure investment.

High-yield bonds, also known as junk bonds, are bonds issued by corporations with a credit rating below BBB, which means they carry a high risk of default (Berk & DeMarzo, 2014). Investors are compensated by the risk, in terms of a higher yield, compared to investment grade bonds. They are widely held by investors, usually through collateralized debt obligations or a mutual fund, in order to gain a higher expected return.

5.2 Equity

An ownership in a listed company is referred to as equity and entitles the owner to a share of the future cash-flows, and control rights of the company. There are two main sources of return from equity; the distribution of the profit in terms of dividends, and the increase in share price over time. The advantage of equity compared to bonds is the higher expected return. Over the last 100 years, the historical real return on equity has provided a 7.7 percent risk premium over bonds (Cuthbertson & Nitzsche, 2005). Pension funds and other institutional investors devote a part of their assets in equity in order to increase exposure and achieve a higher expected return. The return distribution is negativity skewed and highly volatile with fat tails. Asset pricing models, like the 'capital asset pricing model', suggests that the higher expected return in equity is correlated with an increasr in the variance (Markowitz, 1952).

5.3 Property

Property is a a more long-term investment and most pension funds hold 5-10 percent, which includes real estate, commercial building, shopping malls, etc. Property provides both very stable and high returns (Cambridge Associates, 2017a), and the payment profile of property is similar to that of a bond; the value of the property can be regarded as the face value, with the rent as the yields. Most pension funds own entire investment funds dedicated to the investment in property where the returns are similar to a dividend payment. Property is a very illiquid investment class and rarely traded, which means that its harder to determine the correct value. Bonds and equity are sold in liquid markets where tracking the value of the investment is easy and can be determined based on the daily trading prices. The market value of property is based on estimates of future cash-flows (PFA Pension, 2018), which are both complicated and resourceful. Properties are valued less often, which creates a smoothing effect on the returns and gives a biased impression of the evolution of the prices.

Property tends to have low correlation with the remaining assets and is often used to diversify the pension funds portfolio (PFA Pension, 2018). Property is a favorable investment due to the high returns and long duration, and the pension fund can provide living spaces where their customers are prioritized (PFA Ejendomme, 2018).

5.4 Alternative Investments

Within the last decades alternative investments have become very popular and a crucial part of institutional portfolios. Alternative investment can be defined as those assets that are not part of traditional asset classes like cash, equity, bonds, etc. (World Economic Forum, 2015). The most common alternative investments are private equity, venture capital, hedge funds, infrastructure, and social investments. They are defined by being accessable only to wealthy individuals or institutions, due to the high capital requirements. Alternative investments are very illiquid and if liquidated prematurely a large discount on the selling price is incurred. The investments are long-term and very volatile, but provide very high returns (Cambridge Associates, 2017b). The correlation with traditional assets is low, thus making alternative investments a favorable way to diversify the portfolio.

Private equity as an alternative investment, which is defined as long-term investment in a capital fund, where the investors in return get a stake in unquoted companies with the potential for a high return (British Private Equity & Venture Capital Association, 2012). Private equity funds invest in companies with a high potential for growth, or companies that are poorly managed by providing the needed capital and operational improvements the company needs to grow. Investor's return is gained by working with the company's management team in order to improve the strategic foundation by making intelligent investments and operational improvements. This can lead to an increase in the value of the company, thus producing capital gains for the investors.

5.5 Financial Instruments

Financial instruments cover a broad range of products offered in over-the-counter[14] (OTC) and on exchanges. One of the most simple instruments is forwards which are agreements between two parties to exchange an asset in the future for a certain price. They are usually traded in OTC markets, which makes them very flexible and the investors are able to customize the instruments to suit their needs. The party with the long position agrees to buy the underlying assets, while the investor with the short position agrees to sell the assets at a future date at a predetermined price.

Futures share many similarities with the forwards and the underlying features are the same. The difference is that these instruments are traded on exchanges all with the same standardized features, which improves the liquidity and transparency of these instruments. By using an exchange, certain risks are eliminated as the exchange oversee the transactions, and margin requirements[15] limit the counterparty risk.

Options are another financial instrument and can be traded in both OTC markets and

[14]Over-the-counter markets are decentralized markets where investors are able to trade directly without the supervision of an exchange. The transactions are less transparent, but more specialized products are traded in these markets.

[15]If the balance of a trader's account falls as a result of the changing price of the underlying assets, the investor will be required to deposit money in the account to ensure that they are able to meet their future obligations.

on exchanges. A call option gives the owner the right, but not the obligation, to buy the underlying asset at a certain date for a certain price, while a put option gives the owner the option, but not the oblation, to sell the underlying asset at a future date at a predetermined price. The option provides the right to exercise the option, but the holder can decide not to do so, which distinguishes the option from forwards and futures.

Swaps are an OTC agreement between to investors to exchange cash-flows in the future, based on a notional principal amount.

Financial instruments have a broad range of applications and are defined by either the value in the market or by the value of the underlying asset, index, or interest rate. All of these instruments can be used to hedge the risk of the pension fund to lower overall exposure.

6 Liabilities

The liabilities of a pension fund are subject to change depending on factors such as interest rates and mortality rates (Steffensen & Ramlau-Hansen, 2017) since the liabilities are a sum of many different contracts.

A pension fund has signed numerous contracts at very different time points, all with different agreements, which requires a set of actuaries to calculate the total liability of the fund (Ezra, 1980). The liabilities drives the asset allocation with respect to the specific level of risk undertaken by the fund, as mentioned in section 3.4. They serve as future obligations, which must be met in order for the fund to stay solvent.

The majority of new contracts signed are based on unit-link products, as mentioned in section 3.2. The majority of risk is borne by the policyholder in unit-link products, and the properties of the products are identical to an investment account in a different financial institution. The risk of with-profit contracts is borne by the pension fund and since these account for the majority of accumulated pension savings. To understand the implication of with-profit contracts in risk management, the mathematics behind is discussed in the following section.

6.1 Contracts

The product design of pension contracts can vary tremendously, the common denominator is the requirement of an insurance element[16]. To understand how interest rate and mortality rate affects the liabilities, the mathematics driving a simple pension contract will be illustrated in details. Suppose the contract consists of a deferred whole life annuity *(livrente)*, a term insurance *(livsforsikring)*, a pure endowment *(kapitalpension)* and is paid by a level premium (Steffensen & Ramlau-Hansen, 2017).

The deferred whole life annuity is the primary benefit component of the pension contract. It entitles the policyholder to receive a set of payments from retirement until death. This component allows the contributions to be tax deductible, as they serve to provide the policyholder with an income stream once retired.

It pays out a benefit from time m until time T, if death has not occurred before time m, where m is time of retirement and T is time of death. The whole life benefit is given by b^a, interest rate is r and mortality rate is μ. The value of the deferred whole life annuity at time t is calculated by

$$b^a \int_m^{\infty} e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds, \tag{6.1}$$

where the present value at time t is the expected payout from time m to time ∞. The contract lasts until death occurs, however, as this is unknown the maturity of the contract is set to infinity.

[16]Examples of insurance are disability insurance and term insurance, where the recipients would receive a payment in case of disability or death, respectively.

The term insurance is used to provide heirs with a benefit, a lump sum, in case death of the policyholder occurs before retirement. The term insurance is very common to include in the contract if the policyholder has financially dependent heirs.

It pays out a benefit at time of death, $t = T$, if death occurs before retirement, m. The death benefit is given by b^d, and the value of the insurance at time t is calculated by

$$b^d \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds, \tag{6.2}$$

where the present value at time t is the expected payout between time t and time m, given the likelihood of death, $\mu(s)$, occurring.

Pure endowment is a benefit that provides a large lump sum payment upon retirement. Saving for this benefit does not provide any favorable tax incentives, as was the case of the deferred whole life annuity. It is merely to provide the policyholder with a significant increase of capital; usually, to pay off outstanding debt, travel the world, etc.

It pays a benefit at time m, if death has not occurred before time m. The lump sum benefit is given by b^p, and the value of the pure endowment at time t is calculated by

$$b^p e^{-\int_t^m (r(\tau)+\mu(\tau))d\tau}, \tag{6.3}$$

where the present value at time t is the expected payout at time m.

Level premium is how the policyholder pays for the contract. It can be done either as a single premium, identical to a lump sum, or a level premium, identical to an annuity. It is very common to use a level premium, as very few have the sufficient capital to pay for the contract in one installment. The policyholder pays for the contract from initiation until retirement or death, whichever occurs first.

It pays for the contract from time t until time m or T. The payment is given by π, and the value of the level premium at time t is calculated by

$$\pi \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds, \tag{6.4}$$

where the present value at time t is the expected payments from time t to time m.

The reserve is given by the combination of the individual sections of the contract from equation (6.1) - (6.4). The reserve, $V(t)$, is a measure of the present value of the contract at time t, and is equivalent to how much capital the pension fund must keep in reserve for that specific contract.

The payment coefficients, b^a, b^d, b^p and π, must be set at contract initiation to ensure $V(0) = 0$, from which the reserve will follow a convex function until time m. It will experience a drop due to the pure endowment at time m, from which it will follow a concave function until time n, with n being the end of the contract in case of a temporary annuity instead of whole life, see figure 6.1.

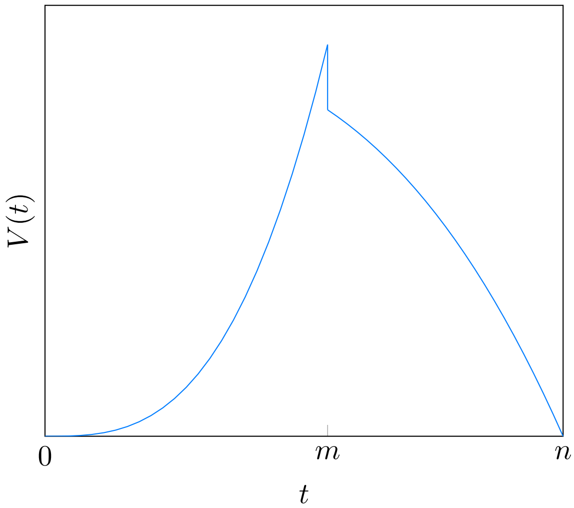

Figure 6.1: Development of the reserve over time.

The reserve is found by adding all the benefits and subtracting the payments,

$$\begin{aligned} V(t) = b^a \int_m^\infty e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds + b^p e^{-\int_t^m (r(\tau)+\mu(\tau))d\tau} \\ + b^d \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds - \pi \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds. \end{aligned} \tag{6.5}$$

The dynamic of $V(t)$ provides an intuitive understanding of how interest rate and mortality rate affects the reserve. The dynamics is the differential equation of $V(t)$ with respect to t, where a thorough illustration of the differentiation can be seen in appendix B. To illustrate the dynamics of the reserve, it is divided into a saving phase and a retirement phase.

Saving Phase The dynamics are used to understand the development of the reserve in each time step. The dynamics of the saving phase where $t < m$ is

$$\frac{d}{dt} V(t) = (r(t) + \mu(t)) V(t) + \pi - b^d \mu(t), \tag{6.6}$$

or

$$\frac{d}{dt} V(t) = r(t)V(t) + \pi - \mu(t) \left(b^d - V(t) \right). \tag{6.7}$$

The dynamics of the saving phase has been rearranged to show the interest gain, $r(t)V(t)$, and the mortality gain, $\mu(t) \left(b^d - V(t) \right)$. The mortality gain is also known as the risk premium (Steffensen & Ramlau-Hansen, 2017). The risk premium in this setting is to be understood as the probability of death, $\mu(t)$, times the sum at risk/net loss, $b^d - V(t)$, in case of death.

Retirement Phase As one retires only the deferred whole life annuity coefficient remains, thus the dynamics of $V(t)$ in the retirement phase where $t > m$ is

$$\frac{d}{dt} V(t) = (r(t) + \mu(t)) V(t) - b^a, \tag{6.8}$$

or

$$\frac{d}{dt}V(t) = r(t)V(t) - b^a + \mu(t)V(t). \tag{6.9}$$

The arrangement in (6.9) is identical to (6.7).

6.2 Technical Reserve

When setting up with-profit contracts the fund must use a set of artificial parameters for r and μ. The use of the artificial parameters is to calculate a conservative value of the reserve, to ensure that the liabilities are overestimated. By overestimating the value of the reserve, the pension fund is able to account for some of the uncertainty in the used parameters (Steffensen & Ramlau-Hansen, 2017). This calculates the technical reserve, where r and μ is replaced by r^* and μ^* and is an indicator of how much the policyholder has saved up in value until that time point. The equation is very identical to (6.5), r and μ are substituted by the artificial parameters;

$$\begin{aligned} V^*(t) = b^a \int_m^\infty e^{-\int_t^s (r^*(\tau)+\mu^*(\tau))d\tau} ds + b^p e^{-\int_t^m (r^*(\tau)+\mu^*(\tau))d\tau} \\ + b^d \int_t^m e^{-\int_t^s (r^*(\tau)+\mu^*(\tau))d\tau} \mu^*(s) ds - \pi \int_t^m e^{-\int_t^s (r^*(\tau)+\mu^*(\tau))d\tau} ds. \end{aligned} \tag{6.10}$$

It is paramount that r^* and μ^* is chosen conservatively, as failing to do so can have severe consequences to the solvency of the pension fund. When choosing the artificial parameters, the idea is to be one the safe side of what is expected to be realized in the future. This ensures the technical reserve is larger than what the future realized liabilities is expected to be, thus the pension fund is expected to set aside more than enough capital to honor the contracts in the future. Intuitively, $r^* \leq r$ for all cases, as a lower discount rate results in a larger present value. For μ^* it is dependent on the contract. If the sum at risk is positive, $b^d \geq V^*(t)$, then $\mu^* > \mu$ and vice versa (Steffensen & Ramlau-Hansen, 2017). It contributes a larger emphasis on the likelihood of death when the death benefit is larger than the technical reserve.

Through the use of the technical set of artificial parameters, surplus, based on realized values, will accumulate and must be individualized through a redistribution of surplus. b^a, b^p, b^d and π is set based on the artificial parameters, thus π is usually set too high[17]. This is consistent with the conservative requirement, in terms of estimating the parameters, to ensure the technical reserve is larger than the expected future reserve. Surplus accumulates over time, also known as the collective bonus potential, once individualized the policyholder will benefit through an increase in b^a[18].

The bonus potential is calculated through a forward differential equation for BP. By using the artificial parameters, b^a, b^p, b^d and π is set so that V^* is equal to 0 at time

[17]It is usually π that is set too high, but in theory, any of the other payment coefficients could instead be set too low.

[18]In theory, the bonus could be paid out through an increase in any of the payment coefficient, or even a reduction in π, however, it is more customary to increase b^a.

0, while another process V^A identical to V^* uses the realized parameters (Steffensen & Ramlau-Hansen, 2017):

$$\frac{d}{dt}V^*(t) = r^*(t)V^*(t) + \pi - \mu^*(t)\left(b^d - v^*(t)\right), \tag{6.11}$$

$$\frac{d}{dt}V^A(t) = r(t)V(t) + \pi - \mu(t)\left(b^d - v^*(t)\right). \tag{6.12}$$

The bonus potential then arises through the difference between the forward differential equations (6.11) and (6.12),

$$BP = V^A - V^*. \tag{6.13}$$

During the 1980s and 1990s the guaranteed technical rate, r^*, was set at 4.5 percent per year (C. Andersen & Skjodt, 2008). This poses a problem in the current market where this is no longer considered a conservative guarantee. The consequences of setting r^* at 4.5 percent is that the technical reserve will be too low compared to the realized reserves, and more capital is required.

How the liabilities are affected by fluctuations in the underlying factors such as r and μ, were illustrated through the reserve and the dynamics thereof. This also proves the earlier description of interest rate and mortality rate as being the two most influential risk parameters to pension funds. By looking at (6.5), it can be seen that a fall in interest rate would increase the reserve, forcing the pension fund to preserve a larger portion of their capital. How a change in mortality rate affects the liabilities is dependent upon the sum at risk. If the sum at risk is positive, then an increase in mortality would increase the reserve, and vice versa. Danish pension funds are under heavy restrictions by the FSA, requiring them to use realistic parameters of r and μ. This provides safety and security to the pension market and ensures the reserves set aside are reasonable and in accordance with the expectation of the future.

As was mentioned in section 3.3, there is a difference between the pension contract offered by banks and pension funds. The dynamics of $V(t)$ clarify this differences. Equation (6.7) is arranged into two parts; a banking part and an insurance part. The banking part is identical to what is received in a bank and is simply the interest gain on an investment, where the risk premium is the additional gain by favoring a pension fund. The same is evident in equation (6.9), where μ is the additional gain, a pension fund provides contrast to a bank.

If an individual is without any financially dependent heirs, then from a rational point of view, the insurance company is a superior saving model compared to the bank. In case of financially dependent heirs, the pension fund would still likely be the best option as the pension could provide a term insurance, while the policyholder still gains from the mortality risk. Pension funds offer flexibility, as the term insurance can be lowered if the heirs are no longer financially dependent on the policyholder. The flexibility to tailor the contract specifically to the policyholder is what makes a pension fund an economically superior option.

7 Asset-Liability Management Model

Financial products are increasing in complexity, interest rates are challenging, and the hunt for satisfactory return is everlasting. All elements are adding to the riskiness of the pension funds. The increasing complexity of products results in difficult measurements of the risk embedded. The challenging interest rates put a strain on the solvency of pension funds while the hunt for high return drives the pension funds to participate in risky ventures. Due to the important nature of pension funds, the regulatory authorities demand the pension funds to have a thorough understanding of their risk exposure. To comply with the FSA and to achieve better results through a deep assessment of the current, as well as the future, exposures, Danish pension funds make use of ALM models to manage their risk (Nordea Liv og Pension, 2016; Danica Pension, 2016; PenSam Liv, 2017).

The importance of ALM models varies depending on the stakeholder. Each stakeholder has a different interest in the function of an ALM model, with the three most important stakeholders being: Regulators, the owners of the pension fund, and the policyholders. The regulators want the pension funds to understand the risk they are exposed to, and to uphold a minimum level of safety in future benefits. The owners wish to achieve a high profit and the policyholders want a high increase in future benefits combined with safety. In an effort to combine the interest of the stakeholders, the objective of an ALM model in general, would be a maximizing of surplus return while minimizing the risk exposure (Guo, 1996). ALM models are an effective tool for any financial entity with a complex balance sheet management. The explicit relevance of ALM models to pension fund is due to the time horizon of the contracts and various risk exposures. The purpose of a pension fund is to provide policyholders with an income stream during retirement, thus the solvency of pension fund is important.

ALM models work by providing a thorough assessment of possible scenarios of the future, and how it affects the balance sheet of the pension fund. The developments of ALM models are very sophisticated, as significant resources have been invested in improving, maintaining and understanding the models (Gibbs & McNamara, 2007). At the early days of ALM modeling, deterministic models were used by financial entities, but as significant technological advancements have been made, more sophisticated stochastic modeling are now developed (Guglielmo, 2010).

Intuitively, the most notable benefit of ALM models is the assisting in risk management. The need for sophisticated ALM models would follow the introduction of complex new financial products, both investment and contract related, and, as ALM models become increasingly sophisticated the possibility of new products emerges (Guglielmo, 2010). With a continued advancement in ALM models, a series of challenges are expected to follow, some being the risk of inaccurate models and the trade-off of complexity vs. practicality (Gibbs & McNamara, 2007). Pension funds are contingent on the results being reliable, which poses the risk of inaccurate models. Effective models are dependent on reliable estimations of the balance sheet (Guo, 1996), however, such estimations are based on historical data as the best guess for the future, but it is no guarantee history will repeat itself. In the development of ALM models, it is a question of how complex the model should be, as there is an inverse connection between complexity and practicality. It is a trade-off between precision, usability and costs. Some additional costs of highly complex models could be increases in training, employees and software (Guo, 1996).

In section 7.1 the assumptions and the considerations are discussed to provide a base for the model. Section 7.2 deals with the settings of the model and the mathematics behind it. Lastly, in section 7.3 the intuition behind the calculations of the model are explained.

7.1 Assumptions

The construction of the ALM model will serve as a tool in answering the research question. It should provide an understanding of the responsiveness of the model, by analyzing the output and investigating the relationship between some of the input variables and the expected bonus potential. The model will include four different portfolio theories, to get an assessment of the implications of different approaches to investments. It will allow for a discussion of how the ALM model assists in managing risks dependent on the investment policy of a pension fund. The four portfolio theories have different attitudes towards risk and the model helps evaluate the investment strategies and the likelihood of default. The strategies will be tested statistically, to determine if the output differ significantly. The model is simplified due to the resources and scope of the paper but serves its purpose of assisting in understanding how Danish pension funds maximize the collective bonus potential under stochastically changing market conditions. The intention of the model is to stay realistic in accordance to the Danish pension market.

The ALM model[19], will run for 40 years, from 2018 to 2057, and each time point is one year. The synthetic pension fund will consist of a single with-profit product, described in section 6.1, identical for all policyholders. The age of the policyholders are assumed to start from 30 since with-profit contracts are no longer offered. The pension fund will originally have 36,960 policyholders, which is equal to one percent of the Danish population aged 30 and older (Statistics Denmark, 2018), where all are treated as unisex policyholders. The fund will have no inflow of new policyholders, as with-profit contracts are no longer being offered by Danish pension funds. The number of policyholders will slowly decrease over timed based on the mortality model from FSA. As the synthetic pension fund only covers with-profit contracts, being able to provide bonus potential to the policyholders is important.

The assets evolve stochastically and the initial amount considered at time $t = 0$, is set to equal a funding ratio of 1.2 based on the calculated liabilities at time 0. The evolution of the asset allocation depends on the stochastic future, and rebalancing will only occur

[19] A download link for the model is provided in appendix A.

when and if any asset class breaches the predetermined bound, as described in section 7.2. The model is used as a risk management tool and to ensure the pension fund's solvency, a penalty in cases of default is imposed. The likelihood for insolvency must not exceed 0.5 percent according to Solvency II, equivalent to one default during a 200-year time period. It will follow the regulations dictating the boundaries of the pension funds and investigate the effect of the capital requirements through stress-testing the SCR calculations. The requirement is explained in section 7.2 and the calculations are shown in 7.3.

Pension funds are not allowed to have long-term borrowings or make use of other financial derivatives for the purpose of levering investments. This poses non-negativity constraints on the assets, and forces the synthetic fund to have a robust investment policy to enable it to stay solvent even in times of trouble.

When considering the solvency of the fund, only two sources of cash inflow are considered; return or cash inflows from assets, and premiums paid by the policyholders. The model will solely look at the fund's ability to stay solvent without the need to inject new capital from owners, and if the asset value fall below the reserves the fund will be insolvent. The return of assets depends on asset allocation and how the market develops while premiums paid by policyholders depend on active members. It is assumed the pension fund does not carry any cash holdings, thus any net cash-flow through assets, benefits, and premiums is treated as a purchase/sell of new assets, and is distributed according to the current asset allocation.The purpose of the ALM model is to maximize the total expected bonus potential and the redistribution of collective bonus potential will be considered a dividend.

7.2 Settings

The model is composed of 1,000 scenarios, with each scenario, φ, consisting of a unique development of each stochastic variable. The assets consist of four classes; bonds, equity, property, and private equity, denoted by $A = \{\alpha^B, \alpha^S, \alpha^P, \alpha^{PE}\}$. The total value of the assets at time t for scenario φ is, $A_t(\varphi)$. For each asset class, the value $A_t^\alpha(\varphi)$ invested at time t under scenario φ is the previously invested value $A_{t-1}^\alpha(\varphi)$ accrued by its return $r_t^\alpha(\varphi)$, plus any purchases and coupon payments from bonds $\beta_t^\alpha(\varphi)$ and minus any sales $\gamma_t^\alpha(\varphi)$. The value of each asset group is calculated as

$$A_t^\alpha(\varphi) = (1 + r_t^\alpha(\varphi))A_{t-1}^\alpha(\varphi) + \beta_t^\alpha(\varphi) - \gamma_t^\alpha(\varphi). \tag{7.1}$$

The liabilities uses the calculations described by equation (6.5). It is calculated for each time point for each scenario, and vary depending on the change in the interest rate determined by the Vasicek model. The real-world and risk-neutral measurements are used to ensure a consistent pricing within the model. The liabilities and bonds are priced under the risk-neural measure since their cash-flows are known but lies in the future. The risky assets are simulated off the real-world dynamics, in order to capture realistic market behavior. As discount rate, the closed-form solution to the zero-coupon bonds from Vasicek is used, which changes the equation to

$$\begin{aligned} V_t(\varphi) = b^a \int_m^\infty \left(e^{-\int_t^s \mu(\tau)d\tau} ds\right) P(t,T,r(t)) + b^p e^{-\int_t^m \mu(\tau))d\tau} P(t,T,r(t)) \\ + b^d \int_t^m \left(e^{-\int_t^s \mu(\tau)d\tau} \mu(s) ds\right) P(t,T,r(t)) - \pi \int_t^m \left(e^{-\int_t^s \mu(\tau)d\tau} ds\right) P(t,T,r(t)). \end{aligned} \tag{7.2}$$

The model uses the non-anticipativity constraint[20] and future developments are unknown by the model. The model assumes that the future time points are unknown to prevent biased decision making. The constraint for $t \in T$ is

$$\left.\begin{matrix} A_\tau(\varphi) \\ V_\tau(\varphi) \end{matrix}\right\} \tau = 0, 1, \ldots, t. \tag{7.3}$$

To ensure no short selling and the realism of the ALM model, a non-negativity constraint of the assets is imposed. All capital is invested and no cash balance exist with the net cash inflows being invested in the current asset weights

$$\begin{gathered} \omega_t^\alpha(\varphi) \geq 0, \qquad \text{where } \ \omega_t^\alpha \in \Omega_t, \\ \text{and } \ \Omega_t(\varphi) = 1. \end{gathered} \tag{7.4}$$

For simplicity, the model does not use any hedging instruments and only considers the aforementioned asset classes. This makes the fund more volatile, as changes in interest rates will have a higher influence than had it been hedged, ceteris paribus.

The investments in property and private equity is linked to a benchmark of an investment fund. In reality most major pension funds are large enough to own entire investment funds, and cash-flow are identical to a dividend, which makes it possible to model the evolution of the value of those investment through Geometric Brownian Motion. For all the asset classes the weight has been set within a predetermined bound and utilizes a classic 60/40 allocation policy of safe/risky assets. Equity, property and private equity are considered risky assets. Starting values are set as:

$$Bonds = 0.6, \ Equity = 0.3, \ Propterty = 0.05 \text{ and } Private\ Equity = 0.05,$$

while the upper and lower limit are set as:

$$0.5 \geq Bonds \geq 0.7,$$

$$0.2 \geq Equity \geq 0.4,$$

$$0.025 \geq Property \geq 0.075,$$

$$0.025 \geq Private\ Equity \geq 0.075.$$

Rebalancing occurs if any of the asset classes exceed the bounds. This constraint protects the pension fund against reckless investment behavior due to limiting the upper bound

[20] The non-anticipativity is an information constraint, meaning information is only known up until the time point t for which the model has reached.

of risky assets, and it protects the policyholders against unreasonable safe investing due to limiting the lower bound of safe assets. Thus, the policyholders are able to achieve a proper bonus potential without the excessive risk of the pension fund defaulting.

The investment in bonds has been constructed as a single portfolio consisting of three different bonds with a variety of different maturities. The different bonds are government bonds, callable bonds and high-yield bonds. The government and callable bonds have maturities of up to 30 years, while the high-yield bond only has maturity of up to 10 years. The bond portfolio consists of 30 government bonds, 30 callable bonds and 10 high-yield bonds, each with a one-year difference in maturity.

To achieve a good estimation of the evolution of the world's stock market, five different indices is considered; S&P 500, OMX C20, Nikkei 225, DAX and FTSE 100. The allocation between these five indices depends on the used portfolio theory. Four different portfolio theories is considered in the ALM model; Minimum Variance Portfolio, Tangency Portfolio, Risk Parity, and Sortino Ratio. The portfolio theories are based on the same scenario outcomes, which will make them comparable. This will provide insights to how different approaches to investment policies might differ for the possible outcomes of collective bonus potential for pension funds.

Solvency Capital Requirements, SCR, is a capital reserve required by Solvency II and is used as a buffer to how much extra capital the fund must preserve. SCR is added to the liabilities, and this amount will be used as the total side of the liabilities, L. The pension fund's equity is considered zero in this case. The collective bonus potential, BP, will then be redistributed once the funding ratio, F, reaches 1.4, half of the assets exceeding the total liabilities are paid out to policyholders. The condition is set as:

$$L_t(\varphi) = V_t(\varphi) + SCR_t(\varphi), \tag{7.5}$$

$$F_t(\varphi) = \frac{A_t(\varphi)}{L_t(\varphi)}, \tag{7.6}$$

$$F_t(\varphi) \geq 1.4 \Rightarrow BP_t(\varphi) = \frac{1}{2}(A_t(\varphi) - L_t(\varphi)). \tag{7.7}$$

If the asset value falls below the liabilities the pension fund is considered to default. When default occurs a penalty is imposed on the pension fund and all acumulated BP, for that scenario, is set equal to zero,

$$A_t(\varphi) < V_t(\varphi) \Rightarrow BP(\varphi) = 0. \tag{7.8}$$

This improves the realism of the model as the collective bonus potential would result in higher future benefits for the policyholders. In case of default, the asset value is assumed to be lost, and the policyholders will have lost their future benefits. This limits potential reckless investment behavior, as $BP = 0$ provides no bonus potential to the policyholder, and it contradicts the prudent-person principle.

The entire ALM model then becomes:

$$
\begin{aligned}
&\max \ BP(\varphi) \\
&\text{subject to:} \\
&\left.\begin{aligned} A_\tau(\varphi) \\ V_\tau(\varphi) \end{aligned}\right\} \tau = 0, 1, \ldots, t, \\
&\omega_t^\alpha(\varphi) \geq 0, \\
&\Omega_t(\varphi) = 1, \\
F_t(\varphi) \geq 1.4 \Rightarrow &BP_t(\varphi) = \frac{1}{2}(A_t(\varphi) - L_t(\varphi)), \\
A_t(\varphi) < V_t(\varphi) \Rightarrow &BP(\varphi) = 0.
\end{aligned} \tag{7.9}
$$

7.3 Calibrations

The contract for all policyholders is identical and consists of a deferred whole life annuity, a term insurance, a pure endowment and a level premium. The payment coefficients are identical and constant across all policyholders and shown in table 7.1. The time of pension has been assumed specific to each age group, as it has been changed by law through time (Udbetaling Danmark, 2018). The distinction between each policyholder's time of pension as well as the amount in each age group at time 0 of the model, can be seen in appendix D.

	b^a	b^d	b^p	π
Payment Coefficient	240,000	1,000,000	300,000	60,000

Table 7.1: Payment coefficients of the contract in yearly figures.

The risky assets are stochastically modeled using GBM, using historical data from the indices to calibrate the mean and standard deviation, which can be seen in table 7.2 - 7.3. For the S&P 500, OMX C20, Nikkei 225, DAX and FTSE 100, the data is taken from October 11, 1996, to December 29, 2017, as this is the longest time period the indices has identical time points, and the data has been cleared for any data points missing for single indices. The parameters are assumed constant over time, such that the simulations do not affect the parameters of the model. Property and private equity data is based on U.S. quarterly data. They are not as liquid as equities and they are only reported quarterly. The data is a benchmark of 988 real estate funds (Cambridge Associates, 2017b) and 1,421 private equity funds (Cambridge Associates, 2017a). The data ranges from 1986 1st quarter to 2017 3rd quarter. The starting price is assumed to be 100 for all risky assets, making it possible to follow the relative change in the development of the prices.

To calibrate the Vasicek model the 3-month treasury rate is used. The data is taken from January 4, 1982, to December 29, 2017, from the U.S. federal reserve. To calibrate a, b and σ of Vasicek, a regression analysis is used. The change in the interest rate is the response variable, and the actual rate is the explanatory variable. Once the regression is

	S&P 500	OMX C20	Nikkei 225	DAX	FTSE 100
μ	0.07054	0.10986	0.00433	0.08260	0.03405
σ	0.20056	0.21256	0.25249	0.25452	0.19875

Table 7.2: Estimates of mean and standard deviation of the stocks.

	Property	Private Equity
μ	0.13432	0.09786
σ	0.09629	0.10663

Table 7.3: Estimates of mean and standard deviation of property and private equity.

performed, the parameters are found by (J. C. Hull, 2017);

$$\begin{aligned} a &= -\text{Intercept} \cdot \text{Trading days}, \\ b &= -\text{Intercept}/\text{Slope}, \\ \sigma &= \text{Standard error} \cdot \sqrt{\text{Trading days}}, \end{aligned} \tag{7.10}$$

and are shown in table 7.4. Once Vasicek has been calibrated, the zero-coupon bonds and the term structure are obtained as explained in section 4.3, and the zero-coupons are used as discount factors. Another approach to using zero-coupons as discount factors would

	a	b	σ	Short Rate
Parameters	0.14585	0.01999	0.01178	0.01390

Table 7.4: Estimates of the parameters of Vasicek.

have been to simulate the interest rates in each time period, also known as nested simulations[21], however, this would be very computationally demanding and is not necessary for the purpose of this ALM model. The zero-coupon prices are priced under risk-neutral measure, and only applied to pricing of bonds and liabilities, as the future cash-flows of liabilities and bonds are known at present time, and do not have closed-form solutions to real-world measure[22] (Society of Actuaries, 2016). The risky assets, however are priced at market price following the real-world measure, as their future behavior is unknown at present time. The usage of both risk-neutral and real-world measure ensures consistency to the risk-management approach of the ALM model.

The bond portfolio is set up as a portfolio of 30 government bonds, 30 callable bonds, and 10 high-yield bonds. The coupon rate for all bonds at $t = 0$ is set equal to the short rate. Each time step one bond of each category expires and a new is added to the

[21]Nested simulations is where you simulate multiple paths for how the variable will evolve, and for every possible path, in each time step, you do another simulation, which continues. It turns into a simulation inside a simulation for every time step.

[22]To calculate the present value of liabilities and bonds using real-world measure, nested simulations would be needed.

portfolio, for each new bond the coupon is calculated to be at par with a face value of 100. This ensures the future bonds be realistic according to the evolution of the short rate. The government bonds and callable bonds are assumed to be Scandinavian and Danish, respectively, thus they are very safe investments. They both have a rating of AAA rated by S&P (Global Credit Portal, 2017; Falch, Sørensen, Holbek, Østergaard, & Andersen, 2017). The high-yield bonds are risky bonds assumed to have an S&P rating of B. The default rates of AAA and B are 0.00 percent and 3.76 percent, respectively (Vazza et al., 2016), see table 7.5. The high-yield bonds are assumed to be identical to a senior subordinated bond yielding a recovery rate of 33.5 percent (Ou, 2011) in the event of a default.

	S&P Rating	Default Rate	Recovery Rate
Government	AAA	0.0000	N/A*
Callable	AAA	0.0000	N/A*
High-yield	B	0.0376	33.50

Table 7.5: The rating, default rate and recovery rate for the bond types. *Not Applicable: With a default rate of zero the recovery rate is redundant.

To calculate the correlation matrix for the entire asset range, the historical data available for the risky assets is used. The bond portfolio is not based on historical data, but is developed as the model simulates future scenarios. For simplicity, the correlation between risky assets and the bond portfolio is assumed 0.25, constant for all pairs, and is shown in table 7.6. In the short-run, the correlation between bonds and equity tend to be negative, due to changes in investors' risk appetite. If investors become more risk-averse, they will transfer a larger portion of their investments from equity to bonds, ceteris paribus. The correlation between 10-year government bond yields and the S&P 500 returns, has been observed to be moving towards positive values as inflation rates have fallen, and was observed to be 0.5 in 2014 (Rankin & Idil, 2014). When calculating the SCR, the correlation between interest rates and risky assets is dependent on whether interest rates are rising, 0.5, or falling, 0, as shown in table 7.8. Hence, the choice for 0.25 in the correlation between safe and risky assets is realistic compared to the observations up until 2014, and match the assumptions of SCR.

The four asset allocation policies are all a subsection of constant-mix strategies; the purpose of the allocation is to be close to 60/40 in safe/risky assets (Perold & Sharpe, 1995). Only when the allocation deviates far enough from the constant-mix, a rebalance occurs. This strategy provides a concave payoff; the fund purchase risky assets when the prices fall, and vice versa. This is equivalent to selling the insurance of risky assets; the holdings of assets are increased as prices fall, thus the investor go against a falling market. In the Sortino ratio policy, a target ratio must be specified, which is realistic to pension funds considering the artificial parameter set at contract initiation, as described in section

	ZCB	S&P 500	OMX	Nikkei 225	DAX	FTSE 100	PO	PE
ZCB	1.0000	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500
S&P 500	0.2500	1.0000	0.4191	0.1837	0.6091	0.5643	0.7651	0.3711
OMX C20	0.2500	0.4191	1.0000	0.3640	0.6380	0.6746	0.6823	0.3522
Nikkei 225	0.2500	0.1837	0.3640	1.0000	0.3266	0.3597	0.6163	0.2784
DAX	0.2500	0.6091	0.6380	0.3266	1.0000	0.8042	0.7264	0.2754
FTSE 100	0.2500	0.5643	0.6746	0.3597	0.8042	1.0000	0.6766	0.2830
Property	0.2500	0.3711	0.3522	0.2784	0.2754	0.2830	1.0000	0.5697
PE	0.2500	0.7651	0.6823	0.6163	0.7264	0.6766	0.5697	1.0000

Table 7.6: Correlation matrix of the entire asset range.

6.2. The artificial parameters are not utilized in the ALM model, however, the target ratio has been set to match a realistic setting of r^*. The artificial parameter, r^*, is the minimum investment return guaranteed to the policyholder by the pension fund, hence an average of this parameter across the policyholders has been chosen as the target ratio, which is approximately 3 percent. An overview of the policyholders and r^* can be seen in appendix D.

The collective bonus potential is calculated according to equation (7.6) and (7.7). For each time step the funding ratio is calculated, and when it exceeds a funding ratio of 1.4, half of the assets exceeding the liabilities are paid out as a cash dividend. The assets are reduced by the same amount. It provides a reliable measure of the performance of the fund, however, it is done differently in practice. The total sum of

$$BP_t(\varphi), \qquad \text{for } t = 0, 1, \ldots, T \tag{7.11}$$

for each φ, is the measurement of the fund's performance and is used to compare the different investment policies.

The calculation of SCR is a stress-test of different parameters to see how it affects the balance of the assets and liabilities, where the worst outcome of an upward and downward shock is used as the basis for the SCR. The goal of SCR is to ensure the pension fund has sufficient capital to stay solvent with a 99.5 percent probability (Steffensen & Ramlau-Hansen, 2017). The basic model for calculating SCR is very complex, but as the contract in this ALM model is as simple as it is, the calculations can be simplified significantly. Figure 7.1 shows the full model of SCR, where the components used for the ALM calculations of SCR is marked by a red square.

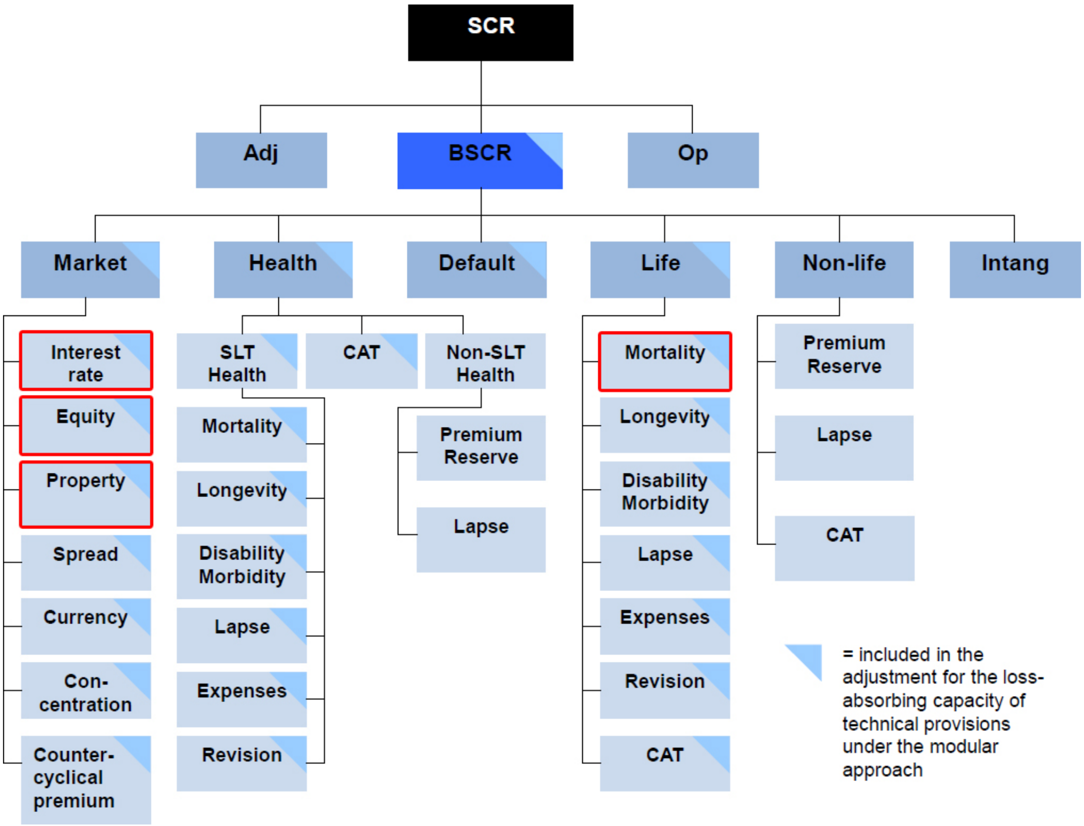

Figure 7.1: Overall structure of the standard SCR formula.

When stressing the interest rate the change is dependent on maturity and the absolute change in interest rate must be at least 1 percentage point (EIOPA, 2012b). To simplify the calculations, the relative change has been kept at a 20 percent change constant for all maturities specific to cases where it yields an absolute change larger than 1 percentage point. According to EIOPA, if the unstressed rate is less than 1 percent, the shocked rate in the downward scenario should be set at 0 percent, however, as the rate might become negative, this limit is not enforced. The stress of equities and properties is 32 percent and 25 percent, respectively (EIOPA, 2012b). Private equity and public equity is assumed to have a correlation of 1 in the SCR calculations. The stress for mortality is an increase of 10 percent and a decrease of 25 percent (EIOPA, 2014). An SCR value is calculated

	Interest rate	Equity	Property	Mortality
Upward shock	20%*	32%	25%	10%
Downward shock	20%*	32%	25%	25%

Table 7.7: The relative shocks for estimating the Solvency Capital Requirement. *Absolute shock change must be at least 1 percentage point.

for each of the individual stress-tests, these values need to be adjusted for correlations (EIOPA, 2012b), to arrive at a single SCR value for the pension fund. The correlations for the market risks are listed in table 7.8. In life risk only mortality is present, thus it requires no correlation adjustment. Finally, the correlation between market risk and life risk is listed in table 7.9.

	Interest rate*	Equity	Property
Interest rate*	1	0/0.5	0/0.5
Equity	0/0.5	1	0.75
Property	0/0.5	0.75	1

Table 7.8: Correlation matrix of the market risk components. *Correlation coefficient is 0 for increasing interest rates and 0.5 for decreasing interest rates.

	Market risk	Life risk
Market risk	1	0.25
Life risk	0.25	1

Table 7.9: Correlation matrix between market and life risk.

8 Analysis

The goal of the ALM model is to maximize the collective bonus potential and get a better understand how various risk parameters effect the pension fund. In the default ALM model, 1,000 scenarios are simulated and each scenario the assets and liabilities are calculated. In order to compare the different portfolio optimization theories all the methods are tested in each scenario; minimum-variance portfolio (MV), tangency portfolio (TP), risk parity (RP), and Sortino ratio (ST). A detailed explanation of the different portfolio methods can be found in section 4.2, and a detailed explanation of the ALM models and it's assumptions can be found in section 7. Other than the default ALM model, a number of test scenarios are conducted. For each test scenario, 100 new scenarios are simulated, to obtain new distributions of the bonus potential. The purpose of the test scenarios is to investigate how the model parameters and variables will affect the pension fund and the expected bonus potential. In total, 3,000 scenarios are simulated, each with approximately 50 million calculations and a run-time of six minutes per simulation.

In the following sections, the output from the ALM model and the various test scenarios will be analyzed. How does the portfolio methods perform and how do changes to the SCR, longevity, payout ratio, etc., affect the distribution of the bonus potential? The analysis will illustratethe sensitivity of the model and how even minor changes will affect a pension fund and the possible bonus potential.

8.1 Default ALM Model

The accumulated with-profit savings, still make up the majority of the total pension savings and will continue to do so in the coming decades. The question is how the pension funds are affected by changes in regulation, longevity, interest rate and how they are able to increase the expected bonus potential for their policyholders while managing the risk. One of the assumptions in the ALM model is that no new with-profit contracts are signed and the amount of policyholders is slowly decaying over time, as seen in figure 8.1 through the reduction of the total assets and liabilities. Over the 40 year time period the assets and liabilities decrease by approximately 80 percent. At the last time period, only 14,772 of the original 36,960 policyholders are alive, and most of the asset value have been liquidated in order to distribute the benefits and the bonus potential. The average asset value of the four portfolio methods are closely linked as seen in figure 8.1. The reason for the similarity in the asset values is explained by the model; as the funding ratio reaches 1.4, half the assets exceeding the liabilities are distributed as a dividend to the policyholders. The average funding ratio, across all simulations, is 1.21 - 1.22 percent, since the asset value is dependent on the value of the liabilities. The portfolio method that provides the highest bonus potential will have a similar funding ratio as the remaining methods, the differences is in the times it takes and how frequent the payout criteria is reached. The simulations apply the law of large numbers, as the mean converges towards the true mean and the difference between the average and the true mean shrinks as the number of trials is increased. Figure 8.2 shows the development of asset value if no bonus is redistributed. In the figure, the funding ratio no longer affects the value of the assets and the difference between the methods becomes clearer; the TP and ST portfolios provide a higher expected return.

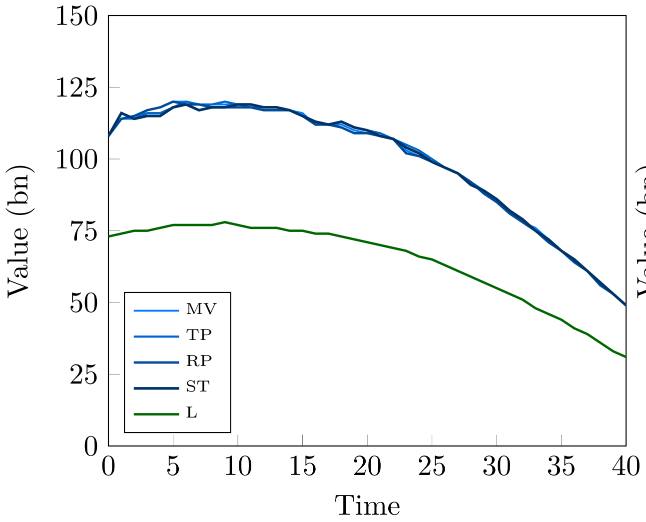

Figure 8.1: Average Assets-Liabilities

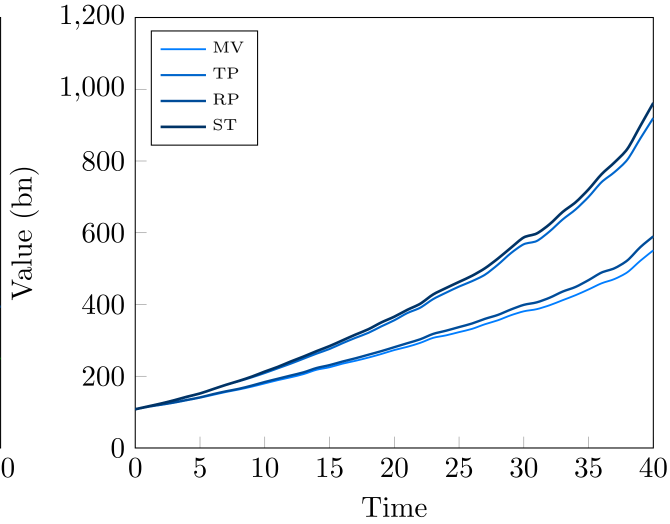

Figure 8.2: Accumulated Asset Values

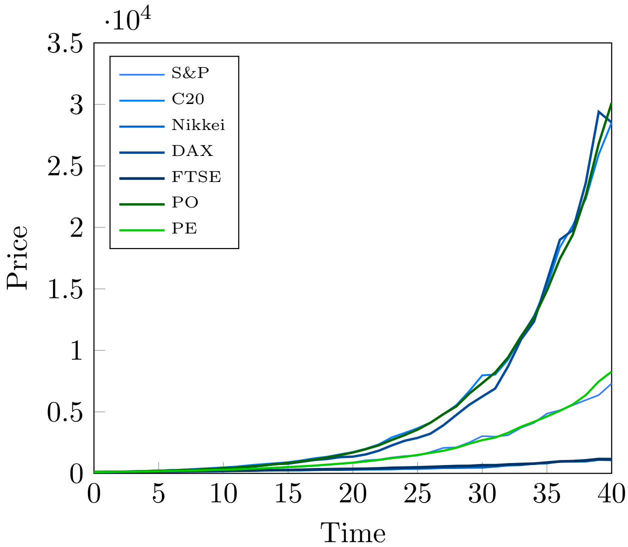

Figure 8.3: Average Asset Prices

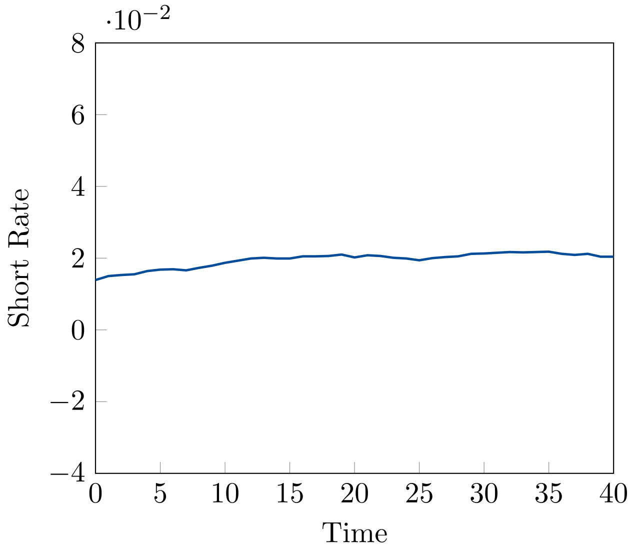

Figure 8.4: Average Short Rate

In section 7.3 the estimation of the asset classes were introduced and the effect of the different means and standard deviations is shown in figure 8.3. OMX C20, DAX, and property are estimated to have the highest expected returns, whereas Nikkei and FTSE 100 are far from the remaining assets. The average short rate is stationary and no significant fluctuation can be observed in figure 8.4.

The stochastic models provides the ALM model stochastic prices and the individual simulations will have significant fluctuations. The output from an individual simulation is shown in figure 8.5 - 8.8, illustrating the value of the asset, liabilities, asset prices, and interest rate fluctuations. The stochastic nature of the values are evident in the simulation and the ST portfolio provides the highest bonus potential of the four portfolio methods. S&P 500, OMX C20, property and PE had the highest expected returns and the short rate increased followed by long periods of negative interest rates. In appendix E, the output from four additional simulations are shown.

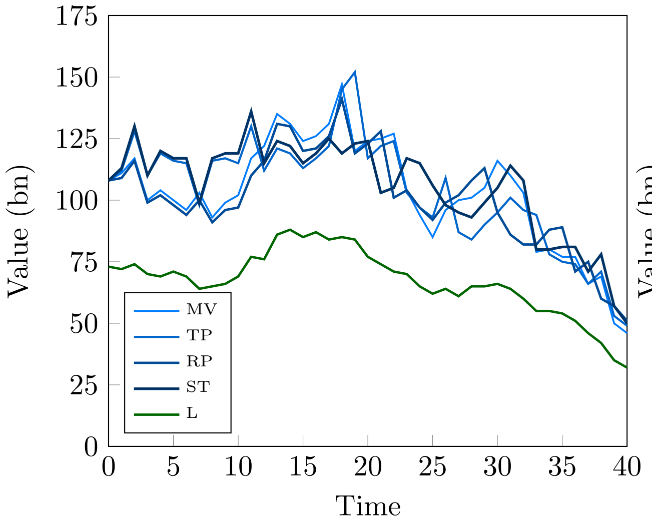

Figure 8.5: Simulation 1 - Assets-Liabities

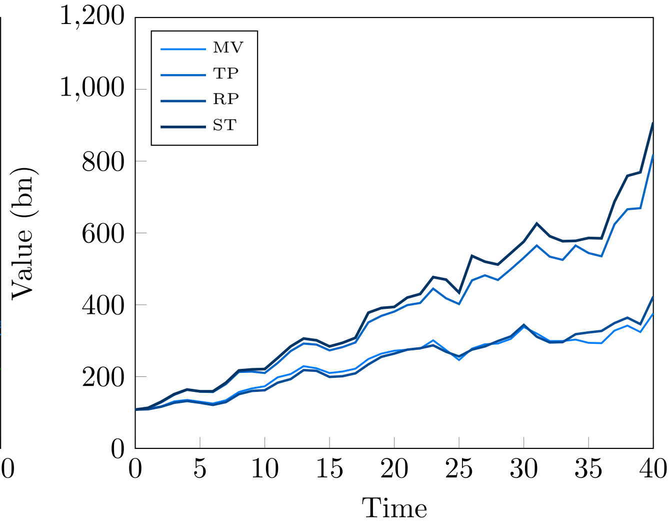

Figure 8.6: Simulation 1 - Acc. Asset Values

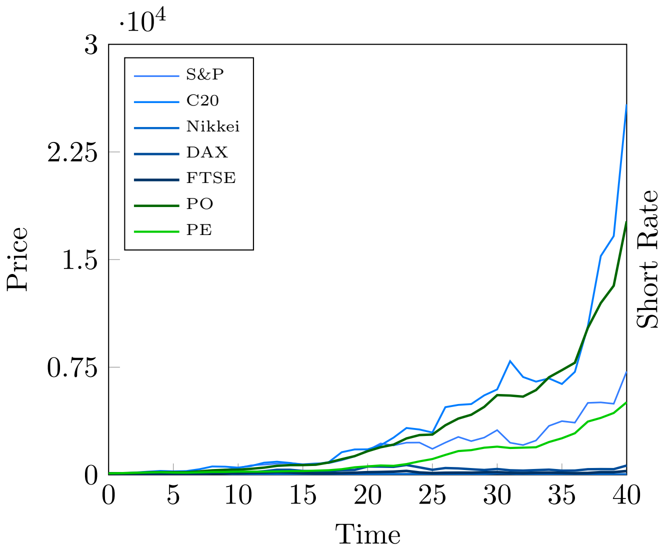

Figure 8.7: Simulation 1 - Asset Prices

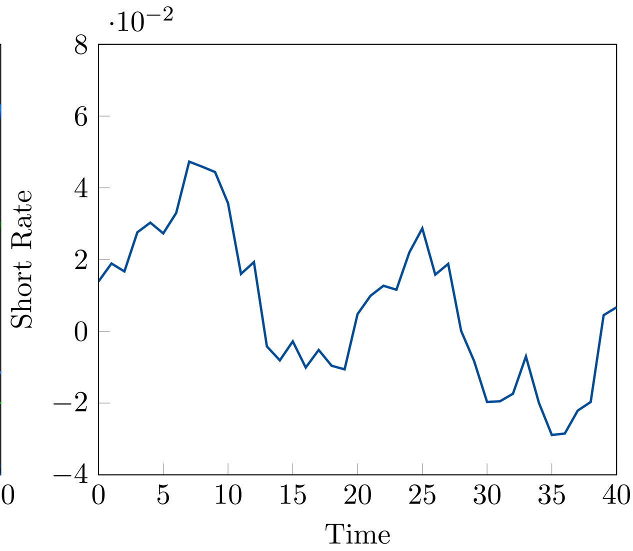

Figure 8.8: Simulation 1 - Short Rate

In table 8.1 the output from the default ALM model is presented. The Sortino ratio provides the highest average bonus potential and the return on the TP is similar, while both MV and RP are underperforming considerably. The ST portfolio has the highest standard deviation, and the default rate for all portfolios is 0 percent. Classical portfolio theory states that risk and return are related and a portfolio with a low variance will have a low expected return. The MV portfolio was expected to provide the lowest return, as it minimizes the variance of the portfolio, as is evident in the above table. The high performance of TP and ST compared to RP is expected, as the two portfolios focus on a maximization of expected return, while the RP diversifies the risk equally across all assets.

(1,000,000)	MV	TP	RP	ST
Mean	227,799	284,627	234,909	290,288
Std Dev	28,711	33,111	29,954	35,171
Default	0%	0%	0%	0%

Table 8.1: Model Output

The expected means seem to be different, but a t-test must be conducted, to determine whether the means are statistically significantly different. The following two-sided hypothesis is used:

$$\begin{gathered} H_0 : \mu_i = \mu_j \quad H_1 : \mu_i \neq \mu_j \\ \forall_i = 1, \ldots, 4 \;\wedge\; \forall_j = 1, \ldots, 4. \end{gathered} \tag{8.1}$$

It is a test for the individual null hypothesis, for all i and j portfolio methods, to verify whether the means are indistinguishable, while the alternative hypothesis is testing if the means are significantly different. It is done under the assumption that the populations are independent and the expected population variances are different. The t-test is calculated by (Stata, 2017)

$$t = \frac{\bar{\mu_i} - \bar{\mu_j}}{\sigma_i^2/n_i + \sigma_j^2/n_j}, \tag{8.2}$$

and the degrees of freedom is given by Satterthwaite's equation,

$$df = \frac{(\sigma_i^2/n_i + \sigma_j^2/n_j)^2}{\dfrac{(\sigma_i^2/n_i)^2}{n_i - 1} + \dfrac{(\sigma_j^2/n_j)^2}{n_j - 1}}, \tag{8.3}$$

where μ_i and μ_j are the sample means, σ_i and σ_j are the standard deviation and n_i and n_j are the number of simulations.

T-Test	MV	TP	RP	ST
MV	-	41.0050	5.4189	43.5242
TP	41.0050	-	35.2125	3.7060
RP	5.4189	35.2125	-	37.9073
ST	43.5242	3.7060	37.9073	-

Table 8.2: T-Tests - ALM Simulations

P-Value	MV	TP	RP	ST
MV	-	0.0000	0.0000	0.0000
TP	0.0000	-	0.0000	0.0002
RP	0.0000	0.0000	-	0.0000
ST	0.0000	0.0002	0.0000	-

Table 8.3: P-Value - ALM Simulations

In table 8.2 and 8.3, the t-tests and p-values for the four portfolio methods are calculated. At a 5 percent significance level, the null hypothesis is rejected for all portfolio theories and the means are therefore significantly different. The very high t-tests indicate just how different the means are.

The stochastic models are based on random samples, thus a sampling distribution of the bonus potentials can be determined. As the amount of simulations is increased the law of large numbers states the sampling distribution will converge to the true distribution of the bonus potential. Knowing the expected mean and standard deviation values provide valuable information, but it provides no insight to the distribution.

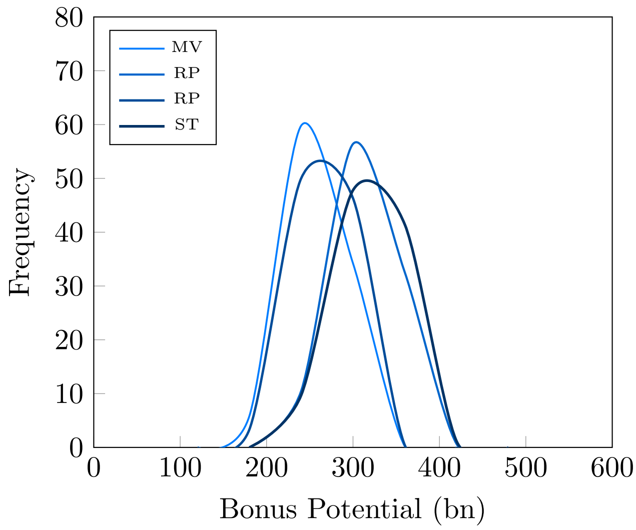

Figure 8.9: Distribution of Bonus Potential

In figure 8.9, the distributions can be seen, and they appears to be normally distributed and centered around their respective means with very small tails. It also shows similarities between the MV and RP portfolios, and the TP and ST portfolios.

	S&P 500	OMX C20	Nikkei 225	DAX	FTSE 100
MV	39.73%	20.07%	23.33%	0.00%	16.87%
TP	22.87%	77.13%	0.00%	0.00%	0.00%
RP	23.44%	20.03%	21.41%	15.47%	19.65%
ST	8.68%	91.32%	0.00%	0.00%	0.00%

Table 8.4: ALM Simulations - Portfolio Weights

The reason for the similarities in the distribution is found in the portfolio method and the way the asset weights are determined. The means and standard deviations, from table 7.2, result in some inconsistencies. The reason why the effect is not stronger is that only 20-40 percent of the assets are invested in equity. The standard deviations, for all 5 indices, are almost identical, but the means differ substantially. This contradicts the CAPM theory, that states that higher risk should be compensated by higher expected return, as the majority of the portfolios end up with weights in equities that underperform.

The portfolios seem to be favoring the S&P 500 and OMX C20, while Nikkei 225 and FTSE 100 perform poorly and used to diversify the portfolios. The DAX index has the second highest expected return, but due to it having the highest standard deviation and positive correlations with the other indices it is rarely included in the portfolios. It is only included in the RP portfolio, as it diversifies to risk equally across all indices. Nikkei 225 has a very low expected return, but it is included in the MV, due to its diversification effects through the low correlation with the remaining indices. The majority of the capital is invested in S&P 500 and OMX C20 since they have a high Sharpe ratio, while FTSE 100 is included due to it having the lowest variance among all indices.

The risk parity portfolio has a higher focus on the risk allocation and is constructed such that each asset in the portfolio will contribute equally to the overall risk of the portfolio. It performs worse than TP and ST, as it focuses on risk diversification by investing in all indices, and the portfolio ends up investing a large portion in assets with low expected return. However, the RP portfolio has been seen to outperform the other theories when it comes to maximizing the Sharpe ratio (Lee, 2014). Through diversification, the portfolio is able to lower the variance, and by levering the investment, it will be able to provide a higher expected return than the other portfolios.

The TP portfolio maximizes the Sharpe ratio, by allocating the entire capital to S&P 500 and OMX C20, as they have the highest risk/return trade-off. The Sortino ratio is an extension of the Sharpe ratio that distinguishes between upside and downside volatility. The ST portfolio allocates a tremendous weight in the OMX C20, which creates an undiversified portfolio. The low diversification of investing highly in a single index is the reason for the higher standard deviation of the TP and ST portfolios.

8.2 Risky Assets

The Solvency II requirements aim at creating a robust financial insurance industry that will ensure that pension funds will be able to survive economic downturns, as the policyholders are dependent on the benefits during their retirement. Solvency II aims at a maximum default probability of less than 0.5 percent (EIOPA, 2014), as this is deemed an acceptable risk level for pension funds. The SCR calculation accounts for the individual pension funds' asset allocation, and pension funds with a higher allocation in risky assets are required to increase their capital reserves. In case the boundary is breached, the financial authorities are able to intervene and ensure that the pension funds have a strategy to improve the financial situation.

Most pension funds tend to hold a large proportion of their assets in bonds, as this is seen as a more secure investment than equity and due to the cash-flow structure provided by bonds. By investing in bonds, the pension fund limits its exposure to interest rate risk, which reduces the capital requirements. The reduction in capital requirement is at the cost of expected return, thus the fund must consider its desired risk profile. How would a 100 percent risky asset portfolio affect the pension fund and the expected bonus potential? How much risk would the pension fund capture and how would this affect their default risk? If the pension fund would be able to significantly increase the bonus potential, but at the cost of a high probability of default, the entire purpose of the pension fund would be obsolete, as the future income of their customers could easily be lost.

(1,000,000)	100%	40%*
Mean	253,130	227,799
Std Dev	175,058	28,711
Default	28%	0%
T-Test	1.4451	-
P-Value	0.1516	-

Table 8.5: Risky Assets - Minimum Variance

(1,000,000)	100%	40%*
Mean	474,639	284,627
Std Dev	200,811	33,111
Default	12%	0%
T-Test	9.4494	-
P-Value	0.0000	-

Table 8.6: Risky Assets - Tangency Portfolio

(1,000,000)	100%	40%*
Mean	259,346	234,909
Std Dev	189,768	29,954
Default	30%	0%
T-Test	1.2861	-
P-Value	0.2014	-

Table 8.7: Risky Assets - Risk Parity

(1,000,000)	100%	40%*
Mean	470,750	290,288
Std Dev	223,178	35,171
Default	15%	0%
T-Test	8.0760	-
P-Value	0.0000	-

Table 8.8: Risky Assets - Sortino Ratio

Table 8.5 - 8.8 display the results of the new simulations, where the old output is the 40 percent risky portfolio marked by the asterisk. The risky portfolio weights consist

of 75 percent equity, 12.5 percent property and 12.5 percent in PE. The 100 percent weight in risky assets have an enormous effect on the expected bonus potential for the TP and ST portfolios, but the default rates rise to unacceptable levels. In the scenarios where the pension fund defaults, the bonus potential will be set to zero, disregarding the accumulated bonus up until that point in time. In practice, rather than paying out the bonus potential as a dividend, it will be individualized and result in higher future benefits for the policyholder. As a pension fund defaults, the majority of the policyholder's reserves are assumed to be lost, and in the ALM model, a default should not have an effect on the distribution.

Among the four portfolios, the TP portfolio now provides the highest bonus potential. When testing if TP and ST provide different expected bonus potentials, we fail to reject the null hypothesis, at a 5 percent significance level, due to a p-value of 0.8552. For the MV and RP portfolios, the difference in the expected bonus potential is low and the added overall risk does not seem to improve the expected return. For all portfolios, we test the following two-sided hypothesis to determine whether the means are significantly different from the default ALM model:

$$
\begin{gathered}
H_0 : \mu(j)_i = \mu(j)_{ALM} \quad H_1 : \mu(j)_i \neq \mu(j)_{ALM} \\
\forall j = 1, \ldots, 4 \ \wedge \ \forall_i = 1, \ldots, N
\end{gathered}
\tag{8.4}
$$

where μ_{ALM} are the means from the original ALM model, j is the portfolio method, μ_i is the average bonus potential from the test scenario and N is the number of test scenarios. The hypothesis for the TP and ST portfolios are rejected, but for the MV and RP portfolios, we fail to reject the hypothesis, and it is not possible to conclude whether these means are different than the ones in the default ALM model.

The variance in the expected bonus potentials increases more than 36 times, which is a reflection of the volatility in the risky assets. The same volatility was present in the original ALM model, but the effect was less significant as only 25-55 percent of the portfolios were invested in risky assets. This has an effect on the default rates, which increase to extreme levels, especially for the MV and RP portfolios, as seen in table 8.5 - 8.8. It would appear that the added risk have a higher effect on the MV and RP portfolios, despite their low increase in the bonus potential. In table 8.4 the portfolio weights were given, and it would seem that the weights could explain why these portfolios perform so poorly. The TP and ST portfolios are primarily invested in S&P 500 and OMX C20 and as these indices perform better on average than the indcies, these portfolios will be able to prevent default, despite their lack of diversification.

If the pension fund increases its risk exposure, it would be subject to an increase in its capital requirements. The SCR is risk-based constraints, limiting excessive risk-taking of the pension fund. As the pension fund increases the amount invested in risky assets to 100 percent, the SCR requirements more than double due to the much higher risk exposure. The purpose of the capital requirements is to protect the pension fund against default scenarios. In table 8.5 - 8.8 the default scenarios range between 12 percent and 30 percent. This proves the increased SCR to not be sufficient to withstand the increase in risk, which might be a result of the simplified model. Even as the SCR does not prevent default scenarios, the number of defaults would likely have been higher if the capital requirements had been static.

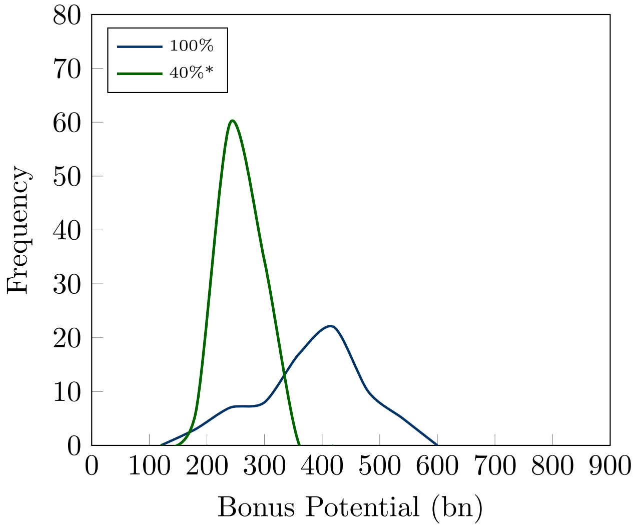

Figure 8.10: Risky Assets - Minimum Variance

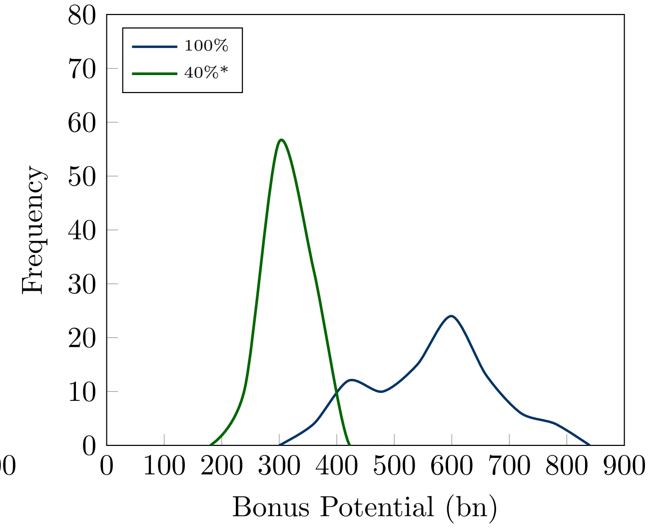

Figure 8.11: Risky Assets - Tangency Portfolio

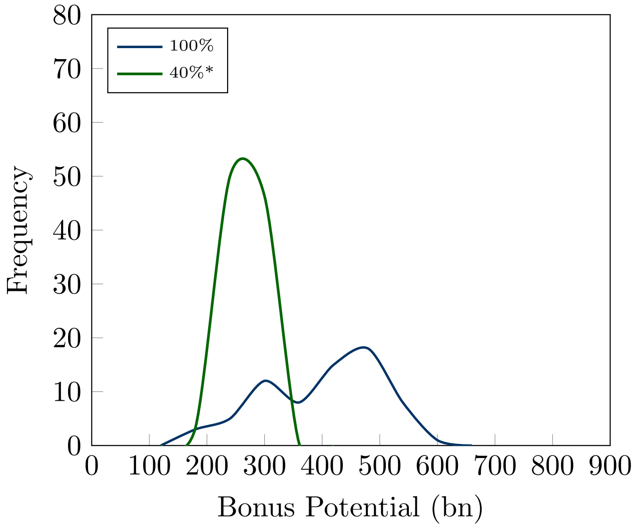

Figure 8.12: Risky Assets - Risk Parity

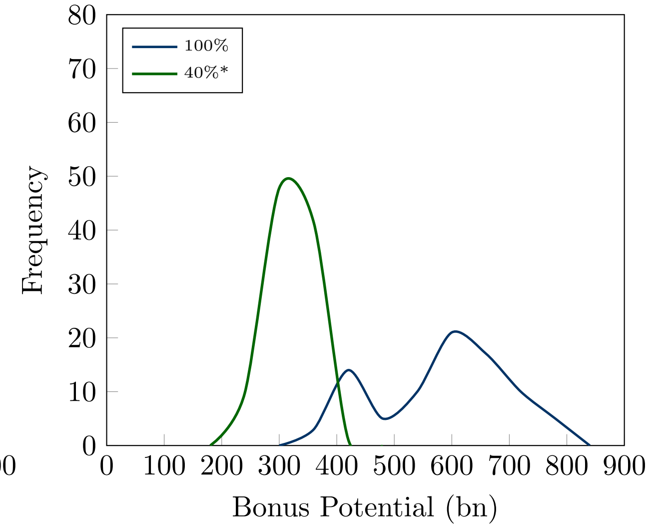

Figure 8.13: Risky Assets - Sortino Ratio

In figure 8.10 - 8.13 the distribution of the bonus potential can be examined and the higher variance is clear. The distribution changes significantly and it would seem that some of the distributions are slightly negatively skewed and with fatter tails. The pension might be able to increase the expected bonus potential, but with 100% risky assets, the default rate would exceed the acceptable probability of default from Solvency II.

8.3 Tolerance Bands

The rate of portfolio rebalancing is a topic often discussed by researchers, in theory, the portfolio should be rebalanced continuously to ensure that the portfolio weights are always equal to the most optimal weights (Munk, 2016). This is not feasible in practice as the transactions costs would reduce the returns drastically, thus it is a trade-off between transaction costs and the lost return from a non-optimal portfolio strategy. Research suggests that discrete trading is not a concern and rebalancing on a monthly, quarterly or yearly basis would result in insignificant losses (Sun, Ayres, Li-Wei, Schouwenaars, & Albota, 2006; Branger, Breuer, & Schlag, 2010). A rebalancing strategy could be based on a monitoring system with an allocation threshold to ensure a high level of risk control for a diversified portfolio (Jaconetti, Kinniry Jr., & Zilbering, 2010). In practice, the frequency of rebalancing fluctuates and by using the above method, most pension funds would obtain a semi-optimal portfolio. They usually use the proceeds of the premiums to buy assets, allowing them to rebalance only when necessary, yielding lower transaction costs.

In order to create a more realistic model, the frequency of the rebalancing is determined through a tolerance band strategy. As the upper or lower limits, from section 7.2, are reached, the entire portfolio will be rebalanced, according to either of the four portfolio methods. This will ensure that a high deviation from the strategies are eliminated and that sub-optimal portfolio weights are minimized. In the following test scenarios, the effect of different tolerance bands will be examined to determine the optimal tolerance bands for the pension fund.

(1,000,000)	5%	10%*	20%
Mean	200,017	227,799	254,228
Std Dev	25,077	28,711	35,338
Default	0%	0%	0%
T-Test	10.417	-	7.2437
P-Value	0.0000	-	0.0000

Table 8.9: Weight Span - Minimum Variance

(1,000,000)	5%	10%*	20%
Mean	250,790	284,627	313,257
Std Dev	30,710	33,111	39,277
Default	0%	0%	0%
T-Test	10.4287	-	7.0433
P-Value	0.0000	-	0.0000

Table 8.10: Weight Span - Tangency Portfolio

(1,000,000)	5%	10%*	20%
Mean	208,573	234,909	263,224
Std Dev	26,490	29,954	36,012
Default	0%	0%	0%
T-Test	9.3614	-	7.604
P-Value	0.0000	-	0.0000

Table 8.11: Weight Span - Risk Parity

(1,000,000)	5%	10%*	20%
Mean	255,344	290,288	320,351
Std Dev	33,126	35,171	41,372
Default	0%	0%	0%
T-Test	10.0002	-	7.0174
P-Value	0.0000	-	0.0000

Table 8.12: Weight Span - Sortino Ratio

The original ALM model had a tolerance band of 10%[23], the output from the default model is marked with the asterisk in table 8.9 - 8.12. The effect is clear, the higher bounds provide the pension fund with the highest possible expected bonus potential. The ST portfolio, with 20% tolerance bands, provides the highest expected bonus potential but it is very similar to the TP. When testing if the expected bonus potential of the two portfolios could be identical, we fail to reject the null hypothesis with a p-value of 0.0882. Identical for all portfolios, we reject the hypothesis. The means are statistically significantly different using the new tolerance bands and the very high t-tests suggest that the differences are immense.

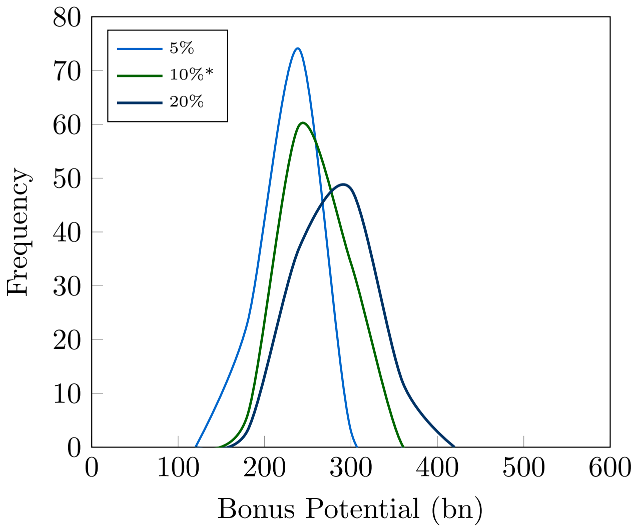

Figure 8.14: Weight Span - Minimum Variance

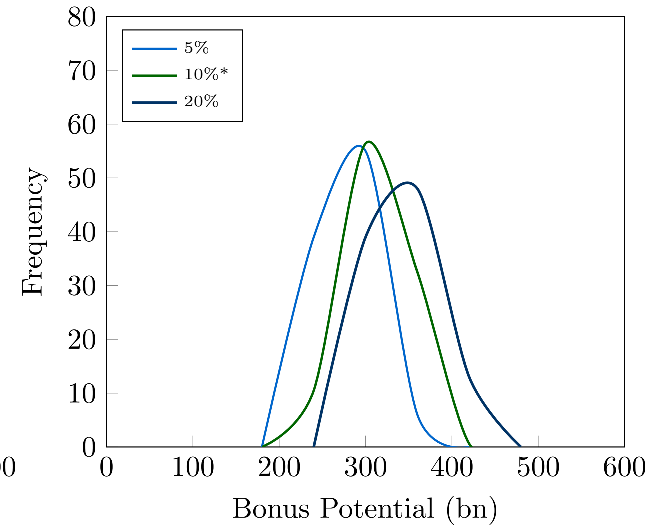

Figure 8.15: Weight Span - Tangency Portfolio

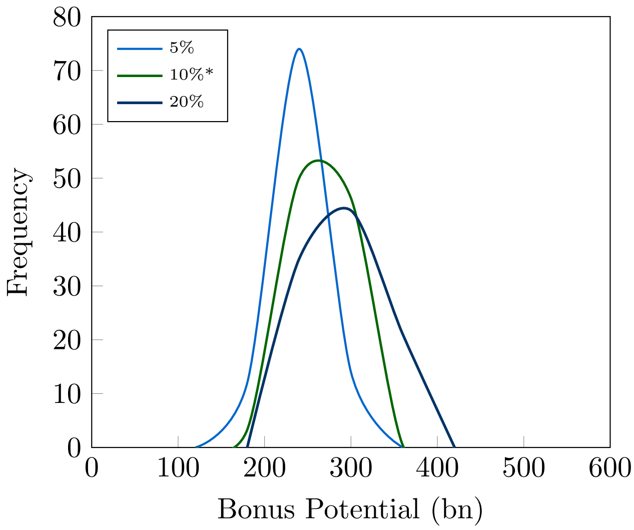

Figure 8.16: Weight Span - Risk Parity

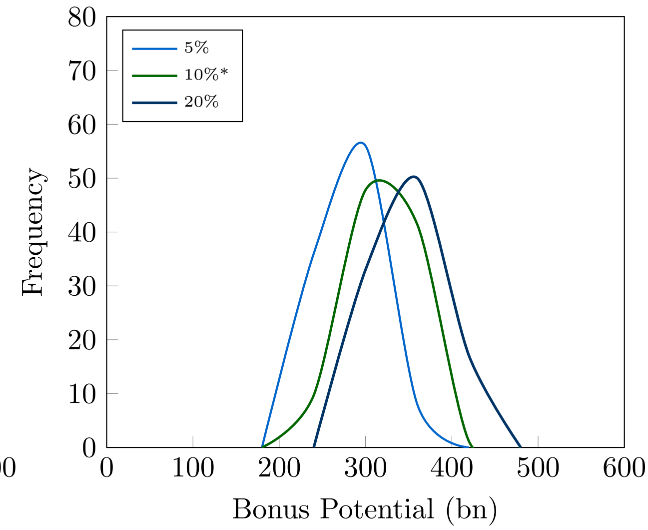

Figure 8.17: Weight Span - Sortino Ratio

[23]The tolerance bands of private equity and property have been scaled to 25%, due to their low weights in the portfolio.

In figure 8.14 - 8.17 the distributions of the expected bonus potentials are shown. The 5% bands reduce the variance of the expected bonus potential of the portfolio methods, and especially for the MV and RP portfolios, as is clearly seen in the figures. For the 20% bands, the kurtosis is slightly lower, which results in distributions being wider with fatter tails. The difference in the variance and the effect it has on the expected bonus potential is due to the weights in the portfolio. The MV and RP portfolios are very diversified, compared to TP and SR, thus high changes in equity prices have little effect on these portfolios. For the TP and SR portfolios, the opposite is the case, as they have high weights in the S&P 500 and OMX C20 indices. Both of these asset classes have a relatively high mean and variance, thus larger fluctuations are more likely, which increase the portfolio variance and result in a distribution with fatter fails.

The change does not seem to have an effect on the defaults, which remains at zero for all portfolios. It would seem that the overall effect of increasing the tolerance bands would be beneficial to policyholders, as it would result in an increase in the expected bonus potential, without increasing the risk of default. An increase in the tolerance band allows for a higher portion of risky assets, which also triggers an increase in SCR, as it increases the risk exposure of the pension fund. This could be the reason the increase in risk does not yield a higher default ratio.

8.4 Solvency Capital Requirements

The Solvency II regime introduced the solvency capital requirement. It is a risk-based constraint that evaluates the risk of the individual pension fund and determines the capital requirements, that ensures the pension fund has sufficient capital to withstand shocks in the market. In practice, the major Danish pension funds hold much more capital than required and it is not unusual to keep 150-300 percent of their SCR in reserves (PFA Pension, 2018; Nordea, 2017; Danica Pension, 2018). It is worth noting, that these numbers are public and could be used to signal the robustness of the pension fund for potential customers or investors, as this is a very transparent key figure.

When introducing regulatory changes, how the pension funds respond is important to consider, as the idea is to promote good risk management. Regulatory changes could create market distortions, cause an increase in prices or give the policyholders a false sense of security. From an economic standpoint regulation of an industry is needed if it can be used to lower market imperfections, information asymmetries, market distortions, etc. Another argument for the regulation within the pension industry is that the public interest theory can be applied to pension funds, which is the idea that the public population needs to be protected by the government (Eling, Schmeiser, & Schmit, 2007). The policyholders depend on the pension funds for their future income, hence defaults must be prevented since the government would otherwise have to provide for these individuals.

In order to test the effect of the capital requirement, the SCR is scaled up and down. The effect on the bonus potential and the default rates will then be analyzed in order to determine whether the capital requirements have the intended effect and if it actually prevents defaults from occurring.

(1,000,000)	-100%	-50%	0%*	+50%	+100%
Mean	162,881	202,514	227,799	244,700	258,161
Std Dev	58,817	43,861	28,711	30,823	37,667
Default	10%	3%	0%	0%	0%
T-Test	10.9081	5.6451	-	5.2598	7.8362
P-Value	0.0000	0.0000	-	0.0000	0.0000

Table 8.13: SCR - Minimum Variance

(1,000,000)	-100%	-50%	0%*	+50%	+100%
Mean	215,684	259,921	284,627	310,596	327,879
Std Dev	56,392	29,710	33,111	35,968	46,579
Default	5%	0%	0%	0%	0%
T-Test	12.0202	7.8429	-	6.9323	9.0597
P-Value	0.0000	0.0000	-	0.0000	0.0000

Table 8.14: SCR - Tangency Portfolio

(1,000,000)	-100%	-50%	0%*	+50%	+100%
Mean	169,089	215,453	234,909	252,182	266,965
Std Dev	61,191	26,115	29,954	32,359	41,275
Default	10%	0%	0%	0%	0%
T-Test	10.6299	7.0036	-	5.123	7.5697
P-Value	0.0000	0.0000	-	0.0000	0.0000

Table 8.15: SCR - Risk Parity

(1,000,000)	-100%	-50%	0%*	+50%	+100%
Mean	214,299	264,745	290,288	315,148	336,145
Std Dev	64,396	31,182	35,171	39,105	49,548
Default	7%	0%	0%	0%	0%
T-Test	11.6281	7.7155	-	6.1147	9.0304
P-Value	0.0000	0.0000	-	0.0000	0.0000

Table 8.16: SCR - Sortino Ratio

The differences in the capital requirements between the portfolio methods will be negligible, as all assets are being stressed by the same percentage. The only difference in the SCR will arise from the different overall weight in equities, but due to the tolerance bands, this will have a minor effect. The effect from the SCR is clear, it has a considerable effect on the default rates, seen in table 8.13 - 8.16, as a reduction in the capital requirement, increases the default rates for all of the portfolios. By completely removing the SCR, the default rate increases to 5-10%. At a 50% reduction, the default rate for the MV portfolio increases to 3%, while the rest remain at 0%, the reason for this is that defaults during the 40 years, is affected by the construction of the portfolio, where a high proportion is invested in the assets that perform poorly.

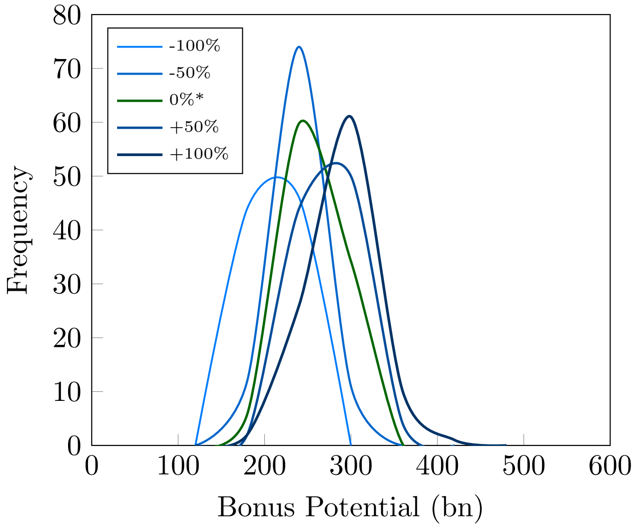

Figure 8.18: SCR - Minimum Variance

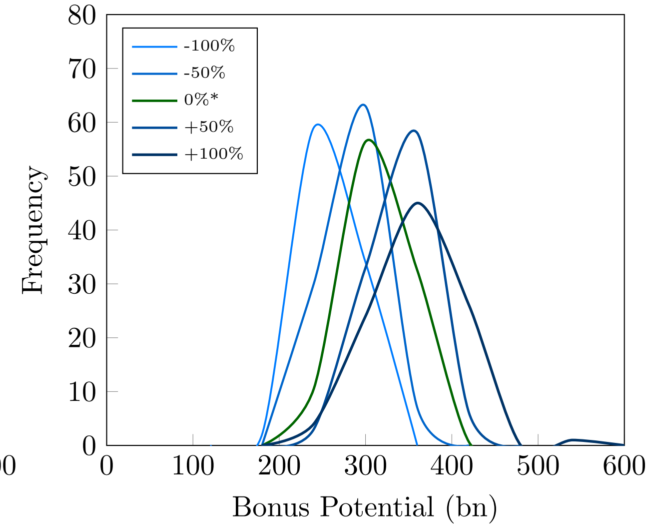

Figure 8.19: SCR - Tangency Portfolio

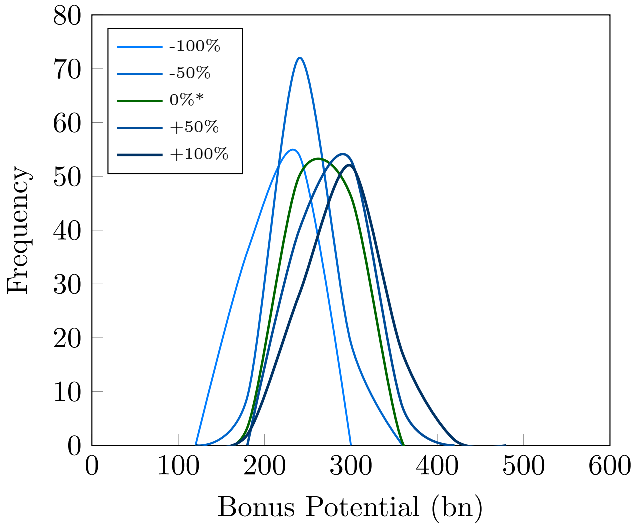

Figure 8.20: SCR - Risk Parity

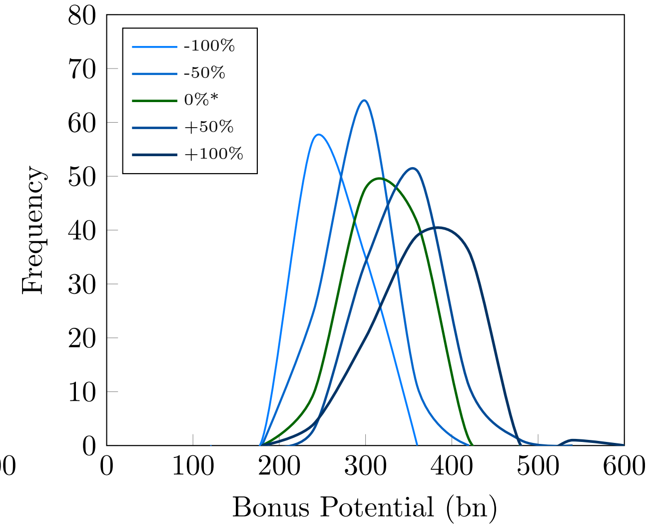

Figure 8.21: SCR - Sortino Ratioy

At a 5 percent significance level, the null hypothesis is rejected for all portfolios. Observing the changes in the expected bonus potential, it is clear that an increase in the SCR will result in a higher bonus potential. The ST portfolio provides the highest expected bonus. Testing the TP and ST portfolio, at a five percent significance level, results in a p-value of 0.0942 and we fail to reject the hypothesis. The ALM model makes no restrictions on how the capital requirements should be invested. An increase in the SCR increases the total liabilities, thus the bonus potential is individualized less frequent, this provides a larger amount of assets to invest, providing a higher expected bonus potential in prospering markets. On average the SCR will account for 15-18 percent of the total value in the default ALM model, which might skew the results. In practice, the pension funds would still be able to make a small return on their capital requirements, but the effect would be lower as the assets would have to be invested in very low-risk assets.

The distributions, in figure 8.18 - 8.21, change significantly and for the -100% and +100% portfolios the increase in variance can be observed in the distribution. In the -100% scenario, an increase in variance is due to an increase in defaults, and in the +100% scenario the increase in variance is due to higher total assets. From the tables and figures, it is clear to see the effect of the SCR. The capital requirements prevent the defaults of the pension funds, thus the regulation has the desired effect in the ALM model. From the default rates, it does not seem like a higher capital requirement is needed, as the original ALM model already has zero defaults. The higher bonus potentials could instead be obtained by increasing the payout criteria or the payout amount for the bonus potential.

8.5 Payout Criteria

The pension fund should choose an optimal point of individualizing the collective bonus potential. In the ALM model, the individualization is considered a dividend, but in practice, when assets are individualized, the future benefits of the policyholders are increased and some of the assets remain in the pension fund to continue generating returns. The reserve, V(t), in equation (6.6) or (6.8), depending on whether the policyholder is in the retirement or saving phase, will increase in order to match the higher benefits the policyholder is expected to receive in the future.

The policyholders' desired payout criteria depend upon their age, which affects the future returns on their pension savings. Young policyholders have a long time to retirement and would prefer to receive individualization of bonus potential less frequent, as it provides the pension fund with more flexibility to achieve a higher expected return. In contrast, old policyholders would prefer frequent individualization of bonus potential to receive the increase in benefit immediate. By delaying the redistribution of bonus potential, the old policyholders receive an increased benefit for a shorter amount of time. The age of the policyholders is negatively correlated with the interest of receiving immediate increases in the benefit.

There exists no specific law in the contribution legislation *(kontributionsbekendtgørelse)* regarding the frequency or amount of the individualization of the bonus potential (Erhvervs- og Vækstministeriet, 2015). It is a decision made by the pension funds themselves. It can be risky to the pension fund if it individualizes bonus potential too often. Once individualized, the bonus potential no longer acts as a capital reserve, and negative results, for the with-profit policyholders, will have to be covered by the equity (Erhvervs- og Vækstministeriet, 2015). In the following section, different payout criteria will be tested to analyze their effect on the expected bonus potential. The original payout criteria of 140% is marked with an asterisk.

(1,000,000)	120%	140%*	160%	180%	200%
Mean	200,925	227,799	241,595	262,462	275,575
Std Dev	37,838	28,711	34,823	40,507	49,796
Default	2%	0%	0%	0%	0%
T-Test	6.9063	-	3.8336	8.3501	9.4387
P-Value	0.0000	-	0.0002	00000	00000

Table 8.17: Payout Criteria - Minimum Variance

(1,000,000)	120%	140%*	160%	180%	200%
Mean	249,657	284,627	310,068	334,887	358,283
Std Dev	57,203	33,111	39,312	45,046	53,985
Default	4%	0%	0%	0%	0%
T-Test	6.0134	-	6.2535	10.8678	13.3942
P-Value	0.0000	-	0.0000	00000	00000

Table 8.18: Payout Criteria - Tangency Portfolio

(1,000,000)	120%	140%*	160%	180%	200%
Mean	206,610	234,909	251,474	273,886	287,088
Std Dev	39,101	29,954	35,517	40,507	50,762
Default	2%	0%	0%	0%	0%
T-Test	7.034	-	4.5065	9.3695	10.1047
P-Value	0.0000	-	0.0000	00000	00000

Table 8.19: Payout Criteria - Risk Parity

(1,000,000)	120%	140%*	160%	180%	200%
Mean	257,412	290,288	312,954	343,214	363,720
Std Dev	53,229	35,171	42,938	49,994	52,932
Default	3%	0%	0%	0%	0%
T-Test	6.0458	-	5.1101	10.3338	13.5764
P-Value	0.0000	-	0.0000	00000	00000

Table 8.20: Payout Criteria - Sortino Ratio

The effect of individualizing the bonus potential less frequent is clearly seen in table 8.17 - 8.20, where the expected bonus potential for the higher criteria is considerably higher. The best performing portfolios are still the TP and ST by a large margin. All the null hypothesis are rejected at a 5 percent significance level. The effect of delaying the individualization is shown to increase the total expected bonus potential. By increasing the payout criteria, the pension fund is able to provide increased returns on the accumulated bonus potential. For the 120% payout criteria, the likelihood of default increases to 2-4%, as the expected bonus potential reduces the funding ratio to a drastically low level resulting in the pension fund being unable to withstand major shocks to the economy. The defaults in the test scenarios are primarily influenced by a simultaneous drop in both short rate and equity markets. The funding ratio is too low for the SCR to be sufficient to withstand difficult times in the economy. The shocks are caused by rare events, which could be caused by too few simulations or the incorrect individualization of bonus potential assumed in the model.

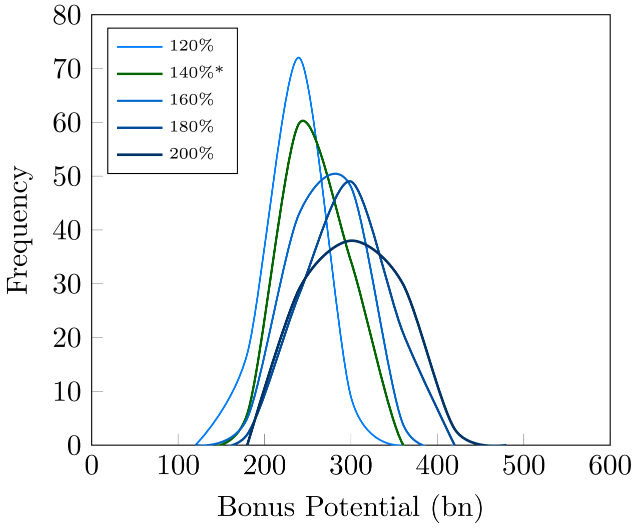

Figure 8.22: Payout Criteria - Minimum Variance

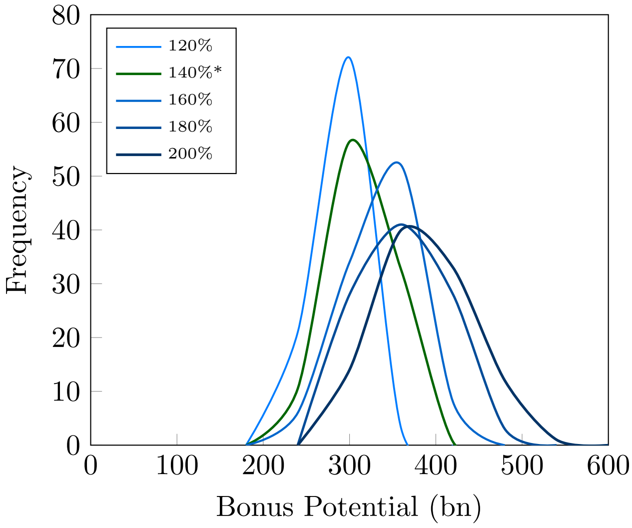

Figure 8.23: Payout Criteria - Tangency Portfolio

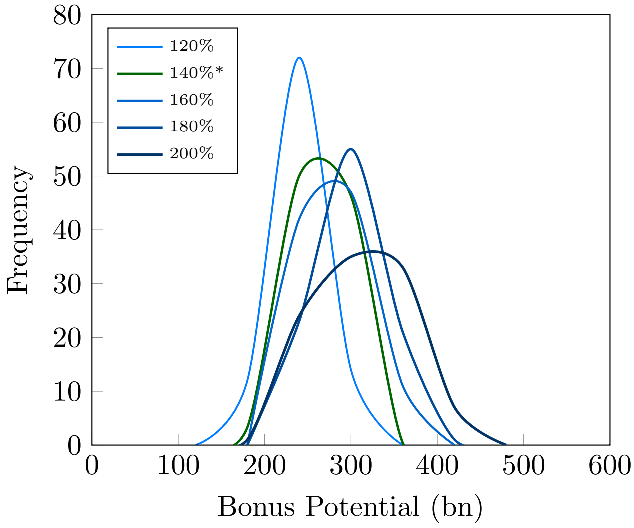

Figure 8.24: Payout Criteria - Risk Parity

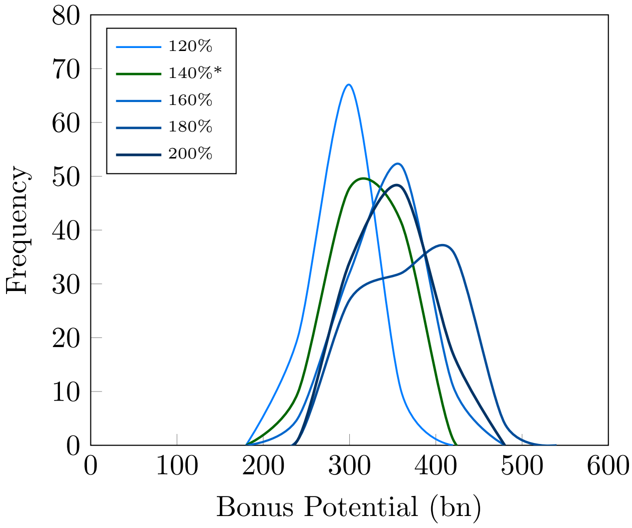

Figure 8.25: Payout Criteria - Sortino Ratio

For all test scenarios, there is a clear correlation between the payout criteria and the expected bonus potential, as the assets are able to accumulate a higher return over time. The same is evident for the variance, disregarding the cases with defaults. Specific to the 120% case, the variance increases drastically compared to the default case and is a result of the increase in defaults. The highest bonus potential is seen in the 200% test scenario for the ST portfolio, with the ST portfolio providing the highest bonus potential for all test scenarios.

All the distributions of the expected bonus potential changes significantly with the new payout criteria, which can be seen in figure 8.22 - 8.25. The higher variances widen the distributions. They are still normally distributed around the means but with larger tails. The effect of the payout criteria is clear, by delaying the redistribution of the bonus potential the pension fund is able to provide a higher expected return. Thus, the pension fund has to consider the payout criteria with respect to its policyholders.

8.6 Payout Amount

Identical to the payout criteria, no laws determine how much of the collective bonus potential should be individualized. Many of the same arguments apply to the payout amount; the young policyholders would prefer infrequent individualizations as the returns would accumulate over time and thus increase their benefits upon retirements, whereas the old policyholders would prefer a large proportion of the bonus to be redistributed immediately, as this would increase their current benefits.

(1,000,000)	25%	50%*	75%
Mean	220,966	227,799	216,788
Std Dev	30,183	28,711	26,611
Default	0%	0%	0%
T-Test	2.1679	0	3.9161
P-Value	0.0322	-	0.0001

Table 8.21: Payout Amount - Minimum Variance

(1,000,000)	25%	50%*	75%
Mean	277,179	284,627	264,339
Std Dev	36,948	33,111	57,103
Default	0%	0%	3%
T-Test	1.9394	0	3.4946
P-Value	0.0549	-	0.0007

Table 8.22: Payout Amount - Tangency Portfolio

(1,000,000)	25%	50%*	75%
Mean	227,220	234,909	221,095
Std Dev	31,386	29,954	42,217
Default	0%	0%	2%
T-Test	2.3453	0	3.1928
P-Value	0.0207	-	0.0018

Table 8.23: Payout Amount - Risk Parity

(1,000,000)	25%	50%*	75%
Mean	282,487	290,288	264,856
Std Dev	37,395	35,171	69,680
Default	0%	0%	5%
T-Test	1.9995	0	3.6042
P-Value	0.0479	-	0.0005

Table 8.24: Payout Amount - Sortino Ratio

In the following section, changes to the payout amount will be tested in order to determine the effect on the bonus potential and the default rate. The original payout amount was 50% which has been marked with an asterisk.

The results from the test scenarios can be seen in table 8.21 - 8.24. Compared to the default ALM model, the effect is a slight reduction in expected bonus potential for all payout amounts. All the expected bonus potentials are rejected at a 5 percent significance level, except for the TP 25% test scenario where we fail to reject the null hypothesis. The expected returns remain approximately the same, but the variances increase significantly.

By decreasing the payout amount to 25% of the funding ratio exceeding 1, the expected bonus potential will decrease. This is counter-intuitive, as the expected bonus potential should be negatively correlated with payout amount. A reduction in the individualization amount of bonus potential should provide the pension fund with an increase in the average funding ratio. Having more capital should provide a higher expected bonus potential if markets are rising and enable them to more frequently individualize the bonus potential. The findings could be limited by the sample size due to biased values of skewed results, or the reduction in payout amount more than but the variances increases significantly.

By increasing the payout amount, the default rate will increase for all portfolios, except for MV, which is unaffected in default scenarios by the change to 75%. This is driven by the MV minimizing the variance, which in this case is enough to withstand the higher risk. Intuitively, the expected bonus potential should become more risky with a higher payout amount, as the pension fund retain less capital. The average funding ratio is reduced as a high proportion of the pension fund's assets are redistributed to the policyholders. Identical to the 120% test scenario of the payout criteria, the pension funds end up distributing too much of their asset, and the SCR is not capable of withstanding a fall in both the equity market and interest rate, for multiple consecutive time periods.

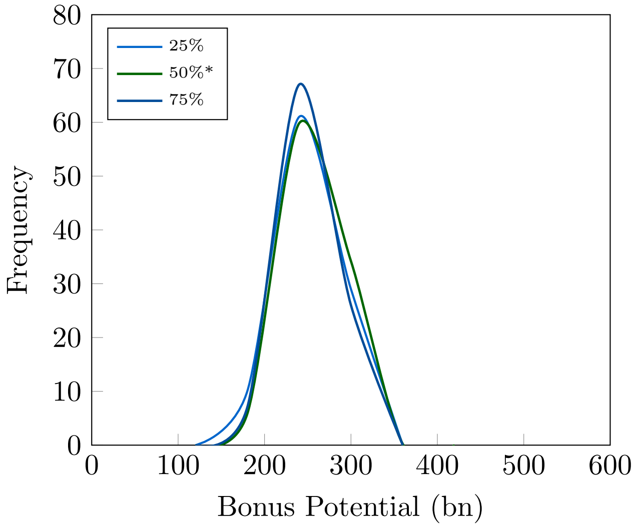

Figure 8.26: Payout Amount - Minimum Variance

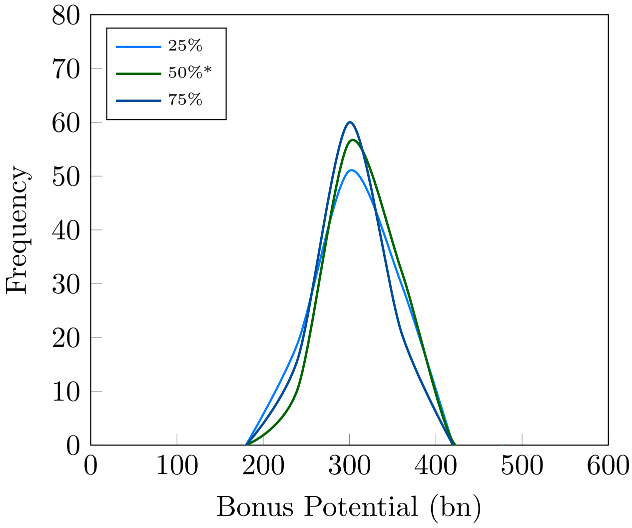

Figure 8.27: Payout Amount - Tangency Portfolio

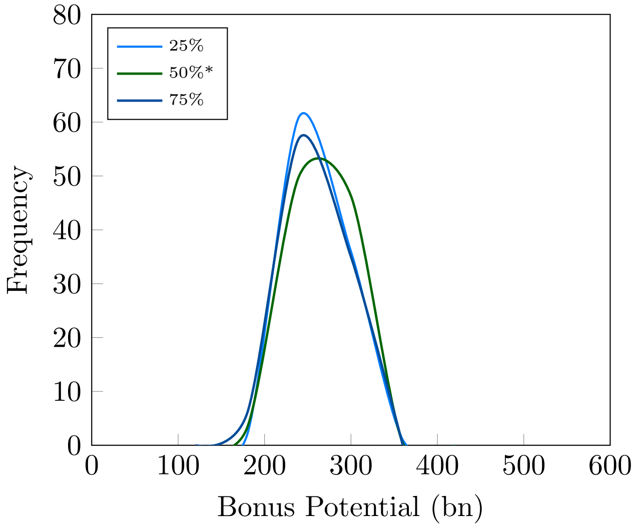

Figure 8.28: Payout Amount - Risk Parity

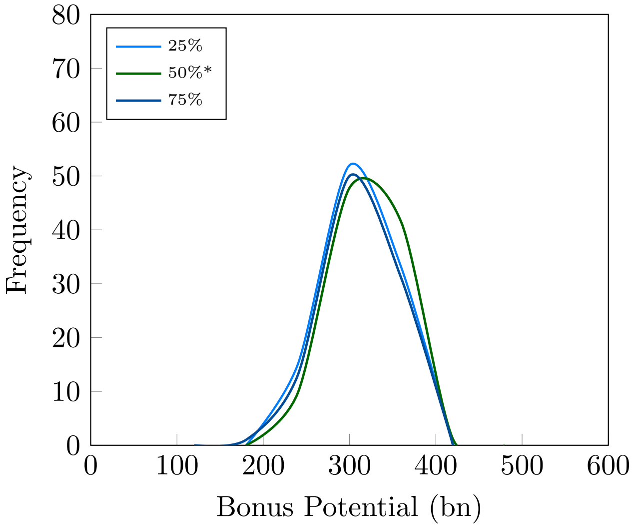

Figure 8.29: Payout Amount - Sortino Ratio

The mean and variance were slightly affected by the change in the payout amount, and the distributions in figure 8.26 - 8.29, are almost identical to the distribution from the original ALM model. It seems a change in payout amount provides1 little to no positive gain in the expected bonus potential.

8.7 Longevity

For pension funds, longevity is one of the most prominent risks and is the risk that the policyholders live longer than expected. The policyholder's whole life annuity, guarantee a series of benefits from retirement until death, an increase in the lifetime of these policyholders will obligate the pension funds to pay benefits for a longer time. When a contract is signed, the premiums are calculated using equation (6.5), with the expectation of longevity included in the calculations. The price of the insurance products and benefits will be set too low if the mortality is underestimated, and on average the pension fund would be subject to losses. The pension fund is supposed to use a conservative measurement of the mortality rate for calculation of the technical reserve, see section 6.2, but estimating the mortality rate for such a long time horizon is difficult, which pose a computational risk.

Every year a newborn person is expected to live 0.2-0.3 years longer than in the previous year (Steffensen & Ramlau-Hansen, 2017). Technological improvements and general health changes have a large effect on the improved lifetime. Within the last 20 years we have experienced considerable improvements in longevity, and whether these improvements can continue is uncertain. The mortality data from FSA is based on data from the last 30 years. Empirical data shows considerable improvements within the last 20 years and stationary improvements between 20 and 30 years go (Finanstilsynet, 2017a). The debate is whether the current estimations of improvements should be based on the last 20 or 30 years since this will have a major effect on the expected lifetime of individuals and future forecasts.

Regulators and pension funds are often wrong in their estimations of the expected lifetime of individuals and in the following section the affect of this will be tested. How will a longevity shock affect the expected lifetime improvements effect the expected bonus potential as an improved lifetime will have an effect on more than just the benefits?

(1,000,000)	-20%	-10%	0%*	+10%	+20%
Mean	253,675	260,401	227,799	253,911	257,300
Std Dev	29,966	36,083	28,711	29,181	35,721
Default	0%	0%	0%	0%	0%
T-Test	8.2641	8.7622	-	8.5443	8.0042
P-Value	0.0000	0.0000	-	0.0000	0.0000

Table 8.25: Longevity - Minimum Variance

(1,000,000)	-20%	-10%	0%*	+10%	+20%
Mean	312,555	325,582	284,627	313,108	318,722
Std Dev	37,441	41,301	33,111	39,982	41,042
Default	0%	0%	0%	0%	0%
T-Test	7.1836	9.6121	0	6.8911	8.0495
P-Value	0.0000	0.0000	-	0.0000	0.0000

Table 8.26: Longevity - Tangency Portfolio

(1,000,000)	-20%	-10%	0%*	+10%	+20%
Mean	261,849	266,889	234,909	260,565	262,408
Std Dev	32,803	37,428	29,954	35,611	32,617
Default	0%	0%	0%	0%	0%
T-Test	7.8903	8.2833	0	6.9624	8.0964
P-Value	0.0000	0.0000	-	0.0000	0.0000

Table 8.27: Longevity - Risk Parity

(1,000,000)	-20%	-10%	0%*	+10%	+20%
Mean	319,490	330,400	290,288	319,065	324,985
Std Dev	42,840	42,284	35,171	40,660	42,321
Default	0%	0%	0%	0%	0%
T-Test	6.5978	9.1743	0	6.8267	7.9293
P-Value	0.0000	0.0000	-	0.0000	0.0000

Table 8.28: Longevity - Sortino Ratio

By looking at table 8.25 - 8.28, the intuition behind the changes in the expected bonus potential is not clear. The effect of changes in longevity improvements leads to increases in expected bonus potential for both directions. The hypothesis is rejected for all bonus potentials at a 5 percent significance level. For all test simulations, the variance increases but the default rate remains at 0 percent.

The reasons the expected bonus potential increases is both due to the contracts and the benefits that the policyholders receive, and the construction of the ALM model and the individualization of the bonus potential. A reduction in expected lifetime affects both the term insurance and the benefits. If the mortality is expected to increase, the value of premiums decreases while the probability of the pension fund having to pay out terms insurances increases. At the same time, the value of the benefits fall. The effect is clearly seen for a drop of 10%, as a larger amount of policyholders will die before retirement and never receive any benefits. This results in the pension fund having less cash-outflow during retirement, and thus fewer costs, providing a higher expected bonus potential. When the expected lifetime improvements drop by 20 percent, a larger proportion is expected to die before retirement and a decrease in the expected bonus potential follow. This must be due to the increase in term insurances more than outweighing the reduction in benefits.

The exact opposite arguments are applied to increasing lifetime improvements. The policyholders are expected to live longer, thus receiving benefits for a longer time, while the probability of dying before retirement and receiving a term insurance, will decrease. The reduction in term insurance must more than outweigh the increase in benefits.

Another reason for the effects of the expected bonus potential given an increase in lifetime is due to the model's limitation of redistributing the bonus potential. As the bonus potential is individualized, the future benefits are supposed to increase, however the ALM model redistributes the bonus as a dividend. Had this not been the case, the effect should have been stronger and the output of the model would have been different.

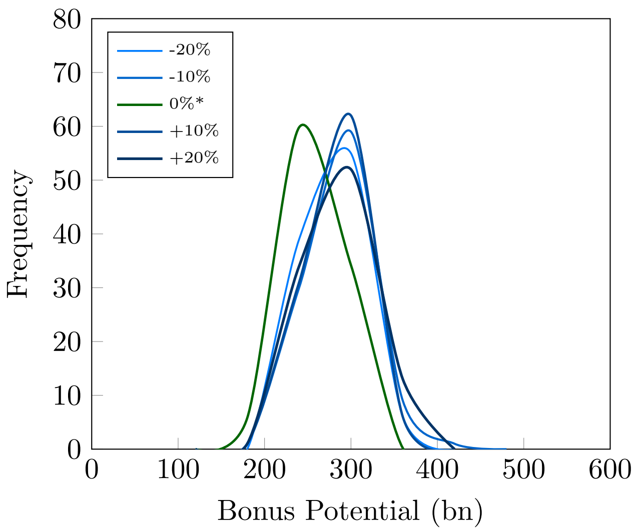

Figure 8.30: Longevity - Minimum Variance

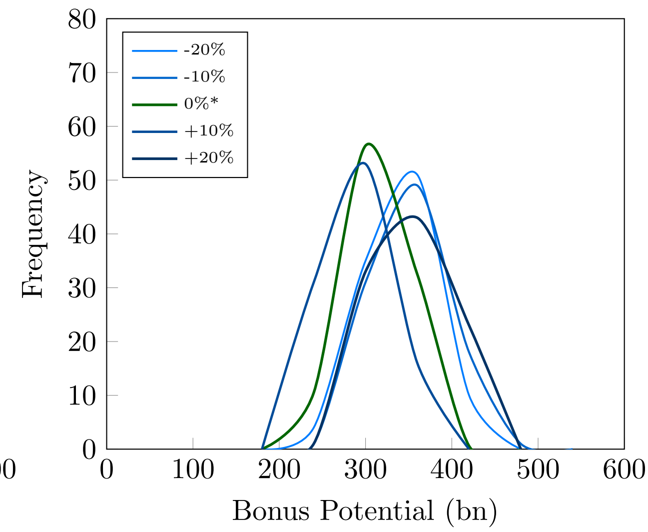

Figure 8.31: Longevity - Tangency Portfolio

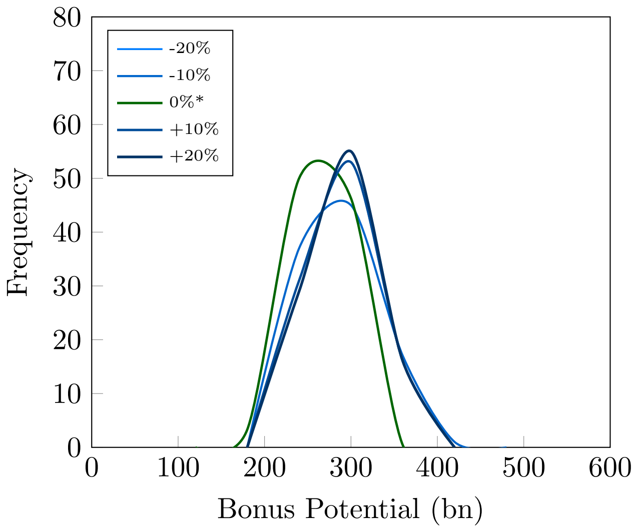

Figure 8.32: Longevity - Risk Parity

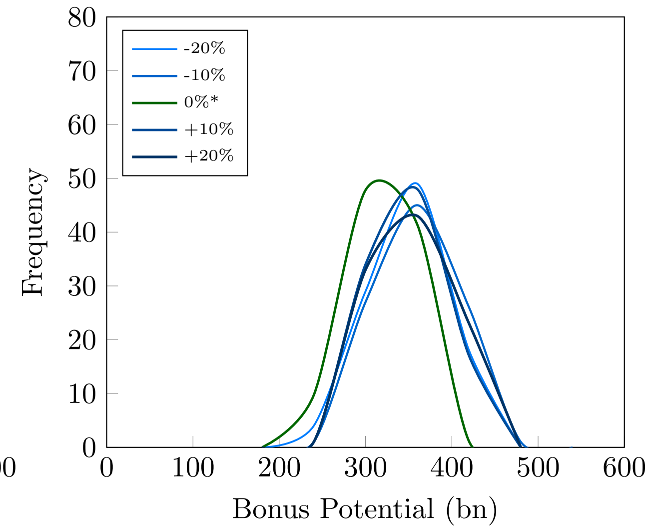

Figure 8.33: Longevity - Sortino Ratio

The distributions all move to the right with an increase in expected bonus potential. The variance increases for all cases yielding a slight widening of the distributions. The pension fund is unable to control changes in lifetime, but evidently, it is important to have a correct estimation of the mortality rate.

8.8 Time Horizon

From the previous sections and test scenarios, it is clear that even small changes in the variables and parameters in the ALM model can have a significant effect on the output and the bonus potential. The same is the case when parameters for the Geometric Brownian Motion model are estimated; the model can be largely affected by small changes in the parameters. When parameters for a model is estimated using historical data, a time period must be chosen that is thought to be a reflection of how the future prices will behave. If the time period is too short, the historical data might only capture an increasing stock market and not the economic downturns. The original ALM model used historical data from October 11, 1996, to December 29, 2017, as this is the longest time horizon with available data from our indcies. By using such a long time horizon, we ensure that both economic downturn and upturns are reflected in the model, gives a realistic view of how the future assets prices might behave.

As these indices are analyzed, different biases might affect the results of the model and skew the data. Survivorship bias is a major problem when indices are used to estimate historic returns since companies that perform poorly go bankrupt, are de-listed, or are removed from the indices. The companies remaining will be the best performing, and the estimated means will be higher than the true returns. The weights in most indices are determined by the market capitalization of the companies and will be biased towards the largest stocks, and these stocks tend to be overpriced (Fama & French, 1998).

In the following section, different time periods will be tested in order to see the effect of a different time horizon used to estimate the variables used in the GBM. New estimations for private equity and property are conducted as well, though these changes doos not affect the equity portfolio. The question is how this affects the bonus potential, the default rates and whether the performance of the portfolio methods had been different if other time horizons had been used in the original ALM model.

15 Years The first test scenario covers the 15 year period from January 6, 2003, to December 29, 2017. Compared to the original model, the dot-com bubble[24] is excluded from the historical data. The prices of the indices, at the beginning of the time horizon, were relatively low, followed by high returns in the following years. In table 8.29 the new estimates can be seen alongside the original parameters, marked with an asterisk.

	S&P 500	OMX C20	Nikkei 225	DAX	FTSE 100	Property	PE
$\mu*$	0.07054	0.10986	0.00433	0.08260	0.03405	0.13432	0.09786
$\sigma*$	0.20056	0.21256	0.25249	0.25452	0.19875	0.09629	0.10663
μ	0.0780	0.11751	0.07087	0.10397	0.04819	0.09422	0.14132
σ	0.19113	0.21184	0.24714	0.23057	0.18995	0.11133	0.08982
T-Test	0.3561	0.3433	2.5175	0.8072	0.6810	3.9113	3.9384
P-Value	0.7218	0.7314	0.0118	0.4196	0.4959	0.0002	0.0001

Table 8.29: Estimates of mean and standard deviation of all assets - 15 Years

	ZCB	S&P 500	OMX	Nikkei 225	DAX	FTSE 100	PO	PE
ZCB	1.0000	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500
S&P 500	0.2500	1.0000	0.4685	0.1942	0.6329	0.6047	0.7714	0.4935
C20	0.2500	0.4685	1.0000	0.4055	0.6807	0.7205	0.6434	0.4444
Nikkei 225	0.2500	0.1942	0.4055	1.0000	0.3629	0.3860	0.6103	0.3894
DAX	0.2500	0.6329	0.6807	0.3629	1.0000	0.8359	0.6992	0.4012
FTSE 100	0.2500	0.6047	0.7205	0.3860	0.8359	1.0000	0.7023	0.3900
Property	0.2500	0.7714	0.6434	0.6103	0.6992	0.7023	1.0000	0.8005
PE	0.2500	0.4935	0.4444	0.3894	0.4012	0.3900	0.8005	1.0000

Table 8.30: Correlations Matrix - 15 Years

The standard deviations have not been affected much, but the means increases for most of the indices. For the S&P 500, OMX C20, DAX and FTSE 100 indices we fail to reject the hypothesis at a 5 percent significance level and for the remaining indices, we reject the hypothesis and the means are significantly different. The most notable change

[24]The dot-com bubble occurred from 1997 to 2001 and was a period of excessive speculation due to the adaption of, and large growth in, the usage of the internet. Within a year the NASDAQ index doubled and reached the market peak on March 10, 2000, a few weeks later the market crashed and within a year trillions of dollars of investment capital had been lost (Ackert & Deaves, 2009).

is for Nikkei 225 with an increase of 6.6 percentage point, followed by private equity with an increase of 4.3 percentage point.

In table 8.30 the new correlations can be seen. Compared to the original correlations in table 7.6 the most notable change is between property or private equity and the equity indices, with an increase of up to 40 percentage points. The new mean, standard deviates and correlation will lead to a change in the weights for the different portfolio methods observed in table 8.4.

	S&P 500	OMX C20	Nikkei 225	DAX	FTSE 100
MV	43.14%	12.63%	23.70%	0.00%	20.53%
TP	23.29%	67.12%	5.05%	4.54%	0.00%
RP	23.68%	19.07%	21.02%	16.45%	19.77%
ST	3.64%	83.35%	0.00%	13.00%	0.00%

Table 8.31: Portfolio Weights - 15 Years

The portfolio weights of MV and RP have only changed slightly. The TP portfolio now includes both Nikkei 225 and DAX, while lowering the exposure to OMX C20. The highest Sharpe ratio is still obtained by OMX C20, but due to the correlation, the two indices are used to diversify the portfolio.

In the default ALM model, the TP and ST portfolio had almost the same weights, however, for the 15-year time horizon, the ST portfolio have significantly lower weights for the S&P 500 and DAX compared to the TP, which must be due to downside deviation in S&P 500 and DAX during this period.

(1,000,000)	MV	TP	RP	ST
Mean	244,095	285,932	257,144	299,013
Std Dev	27,774	34,459	29,564	34,431
Default	0%	0%	0%	0%
T-Test	5.4276	0.3744	7.0859	2.3698
P-Value	0.0000	0.7088	0.0000	0.0194

Table 8.32: ALM Simulations - 15 Years

In table 8.32, it is seen that for all portfolios, except for TP, the expected bonus potential is significantly different from the default ALM model in table 8.1, as we reject the hypothesis at a 5 percent significance level. For MV and RP, the change in the portfolio weights is relatively modest, and the increase in expected bonus potential must come from the higher estimated means. The ST portfolio still provides the highest expected bonus potential, though this time by a larger margin than in the default model.

10 Years The second test scenario covers the 9 year period from October 1, 2008, to December 29, 2017[25]. For this time period the majority of the financial crises is excluded and at the starting point of the historical data, the financial markets had almost reached the lowest point. The equity prices will be increasing for a large majority of the time period thus leading to higher estimated means in the ALM model.

	S&P 500	OMX C20	Nikkei 225	DAX	FTSE 100	Property	PE
$\mu*$	0.07054	0.10986	0.00433	0.08260	0.03405	0.13432	0.09786
$\sigma*$	0.20056	0.21256	0.25249	0.25452	0.19875	0.09629	0.10663
μ	0.10023	0.1285	0.08345	0.09609	0.05267	0.03968	0.11236
σ	0.20761	0.23004	0.26542	0.24364	0.20137	0.10817	0.08430
T-Test	1.4069	0.8297	2.9736	0.5073	0.8922	9.2616	1.3192
P-Value	0.1595	0.4068	0.0030	0.6120	0.3723	0.0000	0.1908

Table 8.33: Estimates of mean and standard deviation of all assets - 10 Years

As seen in table 8.33, all of the assets increase in the estimated mean compared to the default ALM model. If compared to the 15 years time horizon, only the DAX decreases in the estimated mean.

	ZCB	S&P 500	OMX	Nikkei 225	DAX	FTSE 100	PO	PE
	1.0000	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500
S&P 500	0.2500	1.0000	0.5023	0.2253	0.6709	0.6454	0.7617	0.1521
C20	0.2500	0.5023	1.0000	0.4093	0.7127	0.7405	0.5256	-0.034
Nikkei 225	0.2500	0.2253	0.4093	1.0000	0.3754	0.4074	0.5252	0.0777
DAX	0.2500	0.6709	0.7127	0.3754	1.0000	0.8573	0.7308	0.1491
FTSE 100	0.2500	0.6454	0.7405	0.4074	0.8573	1.0000	0.7361	0.116
Property	0.2500	0.7617	0.5256	0.5252	0.7308	0.7361	1.0000	0.3583
PE	0.2500	0.1521	-0.034	0.0777	0.1491	0.116	0.3583	1.0000

Table 8.34: Correlations Matrix - 10 Years

Most of the correlations remain almost identical to the default ALM model, as seen in table 8.34, where the only significant change is between property or private equity and the equity indices.

The hypothesis is tested and we fail to reject S&P 500, OMX C20, DAX, FTSE 100 and PE, almost identical to the 15 year time horizon. Nikkei 225 has a major increase in mean compared to the default ALM model with almost no change in the variance, while the property has a large reduction in mean with a minor increase in variance.

[25]Due to the properties in Cholesky's decomposition and Linear Algebra, this was the closest possible time horizon to 10 years. For certain matrices, the inverse of the matrix cannot be found and for a 10-year time horizon, the ALM model is not able to calculate the random numbers for the stochastic models.

	S&P 500	OMX C20	Nikkei 225	DAX	FTSE 100
MV	41.16%	10.69%	23.77%	0.00%	24.39%
TP	36.85%	54.68%	8.47%	0.00%	0.00%
RP	23.11%	18.95%	21.11%	16.76%	20.07%
ST	25.26%	73.33%	1.41%	0.00%	0.00%

Table 8.35: Portfolio Weight - 10 Years

The majority of the changes from the 15-year time horizon applies to the 10-year time horizon as well. The MV and RP portfolios have minor changes with increase weights in the FTSE 100. TP once again includes Nikkei 225 in the portfolio and have a higher weight in S&P 500. ST no longer includes DAX in the portfolio, which implies that the average downside deviation must have increased, likely due to the price increases prior to the financial crises no longer being included.

(1,000,000)	MV	TP	RP	ST
Mean	256,417	296,541	265,571	298,746
Std Dev	32,530	36,246	31,333	48,508
Default	0%	0%	0%	1%
T-Test	9.3835	3.4002	9.7185	2.2043
P-Value	0.0000	0.0009	0.0000	0.0296

Table 8.36: ALM Simulations - 10 Years

The same conclusions from the 15-year time horizon apply to the expected bonus potentials. MV and RP have had a slight increase in their expected bonus potential while TP and ST remain the same. ST now has a default rate of 1% and the portfolio is the only one to include such a large weight in the OMX C20 index. The ST portfolio has invested nearly 75 percent in the OMX C20 index, thus a shock in this index is likely the reason for the 1% default. This proves the portfolio is undiversified, and that the SCR is unable to protect against a shock in the index the portfolio is dependent on.

5 Years The third test scenario covers the 5 year period from January 4, 2013, to December 29, 2017. Compared to the 10-year time horizon the entire financial crises is now excluded and the period consists solely of price increases, which is clear from the estimates in table 8.37.

Of all the new test scenarios the 5-year time horizon provides by far the highest parameters for the stock indices and the standard deviation is lower than in the original ALM model and previous time horizons. Property has a small decrease in the mean but the variance has dropped substantially. The t-tests are lower and we only fail to reject DAX, FTSE 100 and PE. The correlations in table 8.38 have once again had very few changes, except for PE and property.

	S&P 500	OMX C20	Nikkei 225	DAX	FTSE 100	Property	PE
$\mu*$	0.07054	0.10986	0.00433	0.08260	0.03405	0.13432	0.09786
$\sigma*$	0.20056	0.21256	0.25249	0.25452	0.19875	0.09629	0.10663
μ	0.13358	0.15374	0.16817	0.11288	0.05183	0.11185	0.10611
σ	0.12444	0.17917	0.23239	0.18806	0.14489	0.02447	0.07201
T-Test	3.0844	1.9947	6.2304	1.1585	0.8717	2.3261	0.7566
P-Value	0.0021	0.0462	0.0000	0.2468	0.3835	0.0221	0.4540

Table 8.37: Estimates of mean and standard deviation of all assets - 5 Years

	ZCB	S&P 500	OMX	Nikkei 225	DAX	FTSE 100	PO	PE
ZCB	1.0000	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500
S&P 500	0.2500	1.0000	0.4232	0.1949	0.5674	0.5763	0.5728	0.5413
C20	0.2500	0.4232	1.0000	0.2840	0.6679	0.6212	0.2222	0.0338
Nikkei 225	0.2500	0.1949	0.2840	1.0000	0.2913	0.3242	0.4466	0.3353
DAX	0.2500	0.5674	0.6679	0.2913	1.0000	0.7929	0.4348	0.0531
FTSE 100	0.2500	0.5763	0.6212	0.3242	0.7929	1.0000	0.5929	-0.0318
Property	0.2500	0.5728	0.2222	0.4466	0.4348	0.5929	1.0000	0.3587
PE	0.2500	0.5413	0.0338	0.3353	0.0531	-0.0318	0.3587	1.0000

Table 8.38: Correlations Matrix - 5 Years

	S&P 500	OMX C20	Nikkei 225	DAX	FTSE 100
MV	61.38%	8.54%	11.88%	0.00%	18.19%
TP	60.86%	20.34%	18.8%	0.00%	0.00%
RP	27.96%	18.09%	17.87%	15.71%	20.36%
ST	56.80%	25.58%	17.62%	0.00%	0.00%

Table 8.39: Portfolio Weights - 5 Years

The portfolio weight have changed quite considerably for the last time horizon. THe MV portfolio now have a lower weight in OMX c20, while it increases the weight in both Nikkei 225 and FTSE 100. TP and ST have a higher weight in S& P 500 due its higher expected return relative to the variance. Both portfolio have a higher weight in Nikkei 225, due to the diversification effect, and have lowered the weight in OMX C20.

For the 5-year time horizon provides, by far, the largest bonus potentials. The expected bonus from the MV portfolio is higher than the bonus potential from the ST portfolio in the original ALM model. The low variance of the new estimates is reflected in the bonus potential, as this test scenario has the lowest variance in the expected bonus potential.

(1,000,000)	MV	TP	RP	ST
Mean	291,047	324,370	300,206	322,620
Std Dev	22,848	24,181	23,017	23,477
Default	0%	0%	0%	0%
T-Test	21.3631	11.6951	21.1827	8.9946
P-Value	0.0000	0.0000	0.0000	0.0000

Table 8.40: ALM Simulations - 5 Years

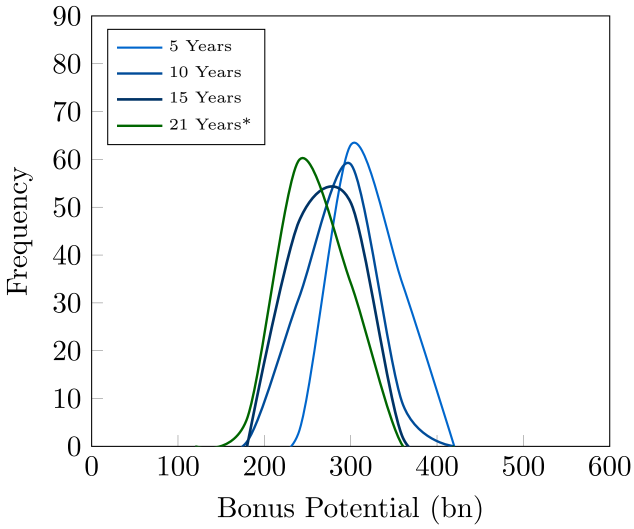

Figure 8.34: Time Horizon - Minimum-Variance

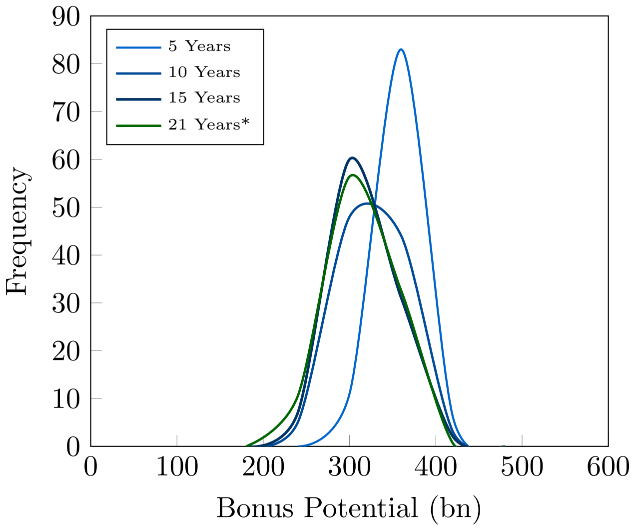

Figure 8.35: Time Horizon - Tangency Portfolio

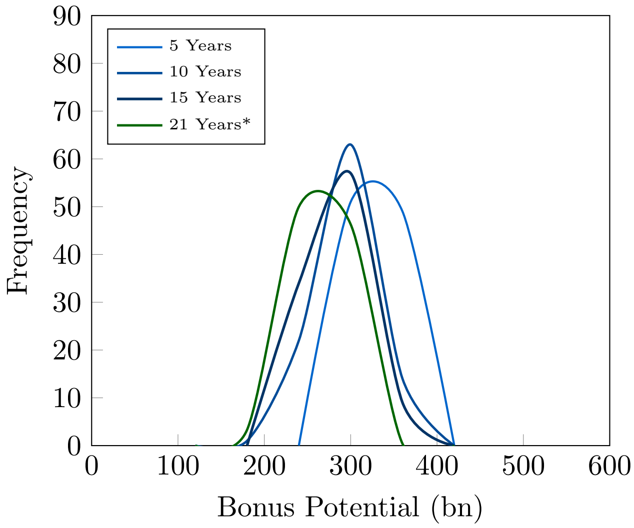

Figure 8.36: Time Horizon - Risk-Parity

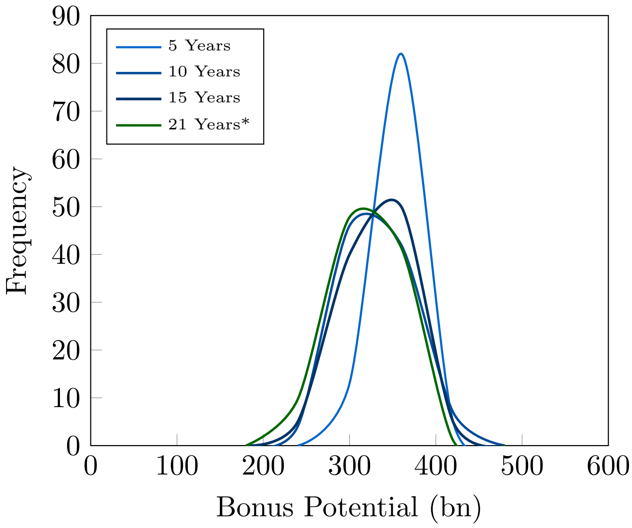

Figure 8.37: Time Horizon - Sortino Ratio

The distributions of the expected bonus potential changes significantly as the time horizon of the parameter estimation is changed, which can be seen in figure 8.34 - 8.37.

The model is very sensitive and even minor changes to the input variables affect the expected bonus potential. For the 5-year time horizon, the TP portfolio outperforms the ST portfolio by a small margin. This shows the importance of the time horizon used to estimate the parameters, as the results might change drastically. The goal of ST portfolio is to maximize the expected return, but it is done while lowering the risk to downside deviations of the assets while the goal of the TP portfolio is simply to maximize the expected excess return.

In the shorter time horizons, the rapid increases in asset prices have a significant influence on the parameters, and in such a market the TP portfolio will outperform the remaining portfolio methods. However, over longer time horizons the ST portfolio will dominate as it accounts for the downside deviation and will try to minimize the probability of negative returns below a target ratio. A significant problem with the test scenario and the time horizons is their possible lack of representativeness of how the future equity markets might evolve. Within the last 10-years, especially the last 5 years, we have seen considerable growth in the equity markets. By using this time horizon to estimate the parameters in the model, we implicitly assume that the future asset prices will behave in the same way, which is unlikely. Within the default ALM model, the time horizon used is 21 years, as this is the longest possible time horizon with historical data available for all indices. By expanding the time horizon we ensure that the estimate will capture both economic downturns and upturns, hence lowering the possible bias in the estimates.

8.9 Model Assessment

In the above analysis, the usefulness of the ALM model has been demonstrated; it is a very powerful tool that provides the pension fund with a deep understanding of their risks and how to manage it. The model demonstrates the pension funds' sensitivity to changes in both the market and variables, and how they are affected. This provides valuable insights, as a better understanding of the risks will, ceritus paribus, lead to better risk management. The future is unknown and as long as with-profit schemes make up the majority of accumulated pension savings, where the pension fund carries the risk, a need for this level of risk management is necessary. The goal of the ALM model is to maximize the expected bonus potential for the policyholders and the analysis have demonstrated how this is possible.

The different test scenarios have provided insight as to what variables significantly influence the pension fund; how an increase in risky assets, a change in longevity, tolerance band, etc. affect the pension fund, and which of these variables can be manipulated or adjusted, in order to increase the expected bonus potential, without risking the solvency of the pension fund.

Of the four portfolio theories the ST method proved to be the best, the portfolio provided the overall highest expected bonus potential without adding any unnecessary risk to the pension fund. The ST portfolio's assessment of the risk, in terms of the downside deviation, have proven very critical. The method uses a more intuitive approach when it comes to the understanding of risk, as it makes the distinction between downside and upside deviation. Over longer time horizons the portfolio method is able to construct a portfolio in such as way that the downside risk is lowered, while able to provide a high expected bonus potential for the policyholders.

9 Risk Management

The main focus of the ALM model is risk management; managing the unknown shocks of assets relative to liabilities, and ensuring the pension fund stay within the required solvency levels. In the pursuit of providing a maximum amount of collective bonus potential, the pension fund must consider risk factors not included in the model. They must understand each risk parameter efficiently and how they interact and correlate with the pension fund. Doing so will ensure an optimal risk management strategy, only accepting the desired risk the pension fund deems worth taking. Various risk parameters should be discussed to understand the interaction between the risk and the pension fund.

The ALM model constructed in section 7 was subject to various simplifications, given the restricted resources and scope of the paper. The purpose of the model was to get a better understanding of how pension funds uses a tool like this and how to maximize the expected bonus potential. The model tested four different portfolio theories and the performance model was determined. The effect of various changes in the model was tested and the effect on the expected bonus potential was analyzed in section 8. In risk management and through the use of an ALM model, many risk factors and considerations must be accounted. The models are not able to incorporate every risk, in the following section some of the most important risk parameters and the effect on Danish pension funds will be discussed.

9.1 Interest Rate Risk

The liabilities are heavily influenced by interest rates, and it is the most prominent risk factors that affect the pension fund, which was shown in section 6.1. The interest rate is an actuarial input used as the discount rate to calculate the present value of the liabilities. Estimating the discount rate too optimistically is costly, as too little benefit has been accrued over time and a realization of estimation errors is too late once the contract has been signed (Chen & Matkin, 2017). The consequences of an incorrect discount rate are significantly worse for a group of young policyholders, as the time of retirement is further away. The majority of new contracts are signed with unit-link contracts and risks, such as interest rate, are borne by the policyholders. The risk of estimation errors in the discount rate is decreasing as the number of policyholders in with-profit contracts is reduced over time. A study by Chen and Matkin (2017) found that "[...] if the median plan lowers its discount rate by one percent but investment returns and funding policies do not change, it may take about seven years before the funding ratio returns to its previous level [...]" (p. 11). The interest rate's influence on the funding ratio is greater for more well-funded pension funds. Pension funds with a high funding ratio are more sensitive to changes in the discount rate as discount rate changes the denominator of the ratio (Chen & Matkin, 2017), see table 9.1. Capital strong pension funds are more affected by reductions in discount rates, due to the high risk associated with changes in interest rate, it is typically hedged either through derivatives or investment policy. It is possible to avoid short-term interest rate risk by using financial instruments, which is done by Danica Pension who are hedging all of the interest rate risk relating to liabilities (Danske Bank Group, 2016). The same applies to PFA Pension who primarily uses swaps and swaptions to mitigate the exposure to the interest rate (PFA Pension, 2017a). while Nordea Liv & Pension protect the bonus potential by using a combination of financial instruments and bonds to continuously limit the exposure (Nordea Liv & Pension, 2017).

Interest rate risk also affects asset prices. Bonds are affected directly since the relation between bonds and interest rates are negatively correlated; as interest rates rise, the price of bonds fall, ceteris paribus, and vice versa. The correlation between liabilities and interest rates is also negatively correlated with falling interest rates increase the value of liabilities, and vice versa. This provides the strong relationship between bonds and liabilities, as changes in interest rates affect bonds and liabilities in the same direction.

Usually, the expected change in interest rate is priced in the value of interest related products through the forward rate. The relationship between bonds and the interest rates is more complicated than a simple negative correlation. The market's expectation to the change in interest rates is priced into the forward rate, thus a change in interest rate must be compared to the expected change via the forward rate (Collie, 2012). How the value of bonds correlates with changes in interest rates, depending on the realized change relative to the expected change. If the interest rate increases less than expected by the market, the correlation would be positive.

9.2 Funding Risk

The purpose of risk management is to consider potential challenges and manage future exposures. In pension funds, the liabilities are subject to changes in interest rates and mortality rates, thus the funding ratio is much more volatile than had the liabilities been constant. If the funding ratio approaches a critical level, funding risk brings various implications:

$$\lim_{A_t \to L_t} F_t = 1,$$

which initiates intervention from the FSA and impose heavy restrictions on the pension fund to restore capital reserves. The implications of poorly managed assets will increase

	High ratio	Low ratio	High ratio	Low ratio
Assets	2,000	2,000	2,000	2,000
Liabilities	1,000	1,500	1,100	1,650
Funding ratio	2.00	1.33	1.82	1.21
ΔL	*10%*	*10%*	-	-
Absolute ΔF	-	-	*-0.18*	*-0.12*

Table 9.1: Sensitivity in funding ratio to changes in liabilities.

	Falling interest rate		Rising interest rate	
	$\Delta IR > \Delta F$	$\Delta IR < \Delta F$	$\Delta IR > \Delta F$	$\Delta IR < \Delta F$
Correlation	Negative	Positive	Negative	Positive

Table 9.2: Correlation in interest rate related products dependent on the change relative to the expectations. IR = Interest rate, F = Forward rate.

the difficulties of improving the funding ratio as they are often affected by low liquidity, poor credit ratings, and financial distress.

Low liquidity brings the cost of missing out. When the pension funds lack sufficient capital, they have to forgo profitable investment opportunities. In severe cases the fund might be forced to sell their illiquid alternative investments at a discount, turning investments with a high expected return into potentially negative net present value projects.

Lack of liquidity is a major risk for the pension fund. If the liquidity reaches a critical level the pension fund must ensure the risk is managed appropriately with the purpose of restoring liquidity. Excessive risk-taking is heavily constrained in regulations and ensures that low liquidity does not make the pension fund shift their attitude towards excessive risk-taking in an attempt to restore liquidity (Rauh, 2008).

Liquidity is also affected by the choice of payout criteria and payout amount set by the pension fund. A low payout criteria produces frequent individualization of bonus potential, yielding lower liquidity, ceteris paribus. The inverse is the case for payout amount; intuitively, a low payout amount is better for liquidity. If the pension fund is faced with low liquidity, they could increase the payout criteria or lower the payout amount.

Poor credit rating is attributed to the default rating, as a high rating would limit the opportunities for short-term borrowings. Pension funds are not allowed to use long-term borrowing or leveraging but can utilize short-term borrowing. A reduction in credit rating would increase the cost of capital, ceteris paribus, hence while the likelihood of default rise as the funding ratio fall, it becomes more costly to use short-term borrowings. The pension fund must make cautious risk management decisions to avoid a decrease in credit ratings and ensure the option of short-term borrowing does not become more costly.

Financial distress follows a high likelihood of default, which is the result of a fall in the funding level of the pension fund. Finanical distress does not only trouble the pension fund financially but at various levels (Berk & DeMarzo, 2014), some of which are: loss of customers, loss of employees, and fire sales of assets.

If policyholders acquire the information that the pension fund is in financial troubles, they might choose to switch their savings to a different pension fund[26]. Financial distress

[26]This is only applicable to policyholders, whose pension savings are transferable.

might limit the potential of new policyholders, as investors would be reluctant to sign pension contracts with a pension fund in financial trouble.

Employees at the pension fund might fear for their employment status, and explore options at other pension funds and retaining key employees with know-how can be costly. At the same time, it can be difficult to hire new employees, as the potential outlook is unsafe, and a company in distress is unable to offer safe long-term employment contracts.

Pension funds are large investors of alternative investments with very illiquid nature. In an attempt to reduce the costs of financial distress or to avoid defaulting, the pension fund might have to sell the assets prematurely at a discount to raise capital. This could turn expected positive net present value projects into losses.

Financial distress could also turn healthy business relations fragile. Investors and business partners might turn to safer options, if they deem the distressed company too risky, giving rise to further difficulties for the pension fund. It could also limit their option to invest in highly sought after investment funds, as the investment fund would find it risky to accept a commitment from a company in distress. Financial distress could also prevent mergers or acquisitions, as a pension fund with large liabilities relative to assets would be a less attractive takeover target due to pension deficits being thought of as a form of debt (Mangiero, 2012).

The choice of payout criteria and payout amount applies to financial distress. Before the individualization of the collection bonus potential, the reserves are used to absorb negative returns. As the bonus potential is individualized the funding ratio decreases and puts the pension fund at risk. In section 8.5 and 8.6, it was shown that a reduction in payout criteria or an increase in payout amount would increase the default rate from 0 percent up to 5 percent. By delaying the individualization the pension could retain more funds and lower their overall risk. The same is true for the SCR and in section 8.4, it was shown that a complete elimination of the SCR would increase the default scenarios from 0 percent up to 10 percent. Even as the regulatory capital requirements assist the pension fund in limiting its financial distress costs and the capital requirements might prevent a default, though unknown changes to regulations still pose a risk.

9.3 Regulatory Risk

Regulations have changed the domain for pension funds over time. It is very difficult to predict regulatory changes, though is only relevant for drastic changes. Regulatory changes are usually corrective, to improve the security of the benefits, but changes can also be lenient to temporarily decrease the requirements of the pension funds (Blome et al., 2007). As regulatory constraints might significantly limit the flexibility of investment policies it is a risk that must be included in the risk management. Solvency II introduced complex risk-based constraints, making it more difficult to keep funding levels adequately ahead of liabilities (Schwaiger et al., 2010), and imposes an indirect risk aversion for the pension funds (Duarte et al., 2017), through the interconnection of risk and SCR.

Portfolio regulation is connected to investment policies to promote proper risk-taking and diversification. The two main categories of portfolio regulations are quantitative portfolio regulations and prudential regulations. Regulatory risk is applicable to both benefits accumulated by policyholders and the funding of these (Kemp & Patel, 2012). When assessing the risk of regulatory changes the pension fund must include the effect from regulation in funding level requirements, investment performance, through benefit levels, and the funding costs (Blome et al., 2007).

Quantitative portfolio regulations are measurable and impose limits on specific investment strategies. Solvency II heavily incentivize pension funds to use duration matching of assets and liabilities (European Commission, 2015), through a reduction in capital requirements, as it makes interest rates less risky. Alternatively, it penalizes risky investment strategies through an increase in capital requirements. The quantitative portfolio regulations prove to encourage safe investment decisions. The capital requirements serve to encourage safe investments through the positive correlation between risk and SCR.

Prudential regulations are non-measurable and a set of principles (Finanstilsynet, 2014). The pension fund must invest prudently through portfolio diversification and broad asset-liability matching (Davis, 2001). The idea behind prudential regulation is to allow pension funds to invest in any assets they desire, provided they understand and is able to manage the risk (European Commission, 2007). It provides the pension fund with more flexibility but ensures the nature of the investments follows the risk profile of the liabilities.

Funding level requirements are the easiest way to verify the solvency level of the pension fund. Regulations can require a strict approach to funding levels to ensure adequate assets relative to liabilities. It could also implement a flexible approach to funding levels, however, it would require the pension fund to insure itself against bankruptcy and loss of savings for the policyholders (Blome et al., 2007). The SCR and MCR are funding level requirements, as a breach of these activates a 'ladder of intervention' by the FSA. The capital requirements limit excessive risk-taking and provides a supervisory tool for the authorities and enables them to intervene as difficulties arise. The idea is to prevent defaults and protect the policyholders.

Investment performance is a measure of how well the pension fund provides collective bonus potential to its policyholders through an increase in benefit levels. In Denmark, current regulations state that the collective bonus potential must be individualized and redistributed to the policyholders and is added to the policyholder's guaranteed benefits. Prior to individualization, the collective bonus potential is used to cover losses incurred by the investments (Erhvervs- og Vækstministeriet, 2015). If regulatory changes were to limit the possibility of using the collective bonus potential to incur losses, it would drastically challenge the pension funds in times of negative returns. A stricter approach to dealing with negative returns would turn pension funds more risk-averse, as it would increase the funding costs of negative returns. Allowing the pension fund to cover their losses using the collective bonus potential allows the pension fund to aggregate more risk and provide the policyholders with a higher bonus potential. A regulatory system provides security at a cost; a higher level of protection of guarantees come at the cost of policyholders (European Commission, 2007).

Funding costs are the cost incurred by the owners of the pension fund through loss of equity[27]. Regulations might demand a specific level of equity relative to liabilities to reduce the risk of defaulting on the guarantees. An increase in such a criteria would be costly to the owners of the pension fund, as they would be required to inject additional equity to comply with the regulatory changes. In times of poor investments and decreasing asset values, the pension fund would have to inject new equity to comply with regulatory demands. Riskier investment strategies provide a higher expected return but result in a higher variance of the funding costs. The funding cost is another instrument in regulation to keep the risk-taking at appropriate levels, by aligning the interest of the owners with the policyholders.

The public interest theory states that the public needs to be protected by the government. Through regulation and the requirement of capital reserves, the pension fund will be able to lower the likelihood of default. This could also be done by increasing the equity, though it would imply an increase in funding cost to the owners.

9.4 Hedging Risk

Financial instruments are central to risk management, as they serve to transfer risk across multiple investors. The hedging risk is not influenced by external factors but borne my the pension fund itself. Good risk management will use hedging instruments to mitigate risk, but it should be done in accordance with the risk profile of the company. Using hedging instruments is not just about mitigating risk, it is about selectively keeping the desired risk, while reducing undesired exposure. Investors dislike uncertainty and by using derivatives they can reduce the volatility of assets and liabilities, limit the likelihood of unfavorable future events. Hedging is a trade-off of a reduction in downside potential at the cost of upside potential. Being too risk-averse could result in over-hedging and the pension fund must carefully consider which exposure is worth hedging and which is too expensive to mitigate.

Derivatives should be used as a value-enhancing strategy through a reduction in risk exposure of the company (Jorge & Augusto, 2016). Derivatives have the advantage of increasing and decreasing exposure, where using derivatives to profit off expected market imperfections would increase the risk and turn the strategy into speculation. This would have the opposite effect of hedging, as the investment instead of reducing the volatility of outcomes would further increase them.

[27] Funding costs are based on the book value of equity, excluding any potential changes in share price.

In pension funds, the use of derivatives for hedging purposes must be in alignment with its policyholders. The policyholder must benefit when using a hedging strategy to reduce exposure. If the strategy is used to solely reduce deficit volatility for the owners (Mangiero, 2012), as it might not follow the prudent-person principle. Derivatives have different characteristics and different uses depending on what type of risk is being hedged, the most well-known are futures, options, and swaps. Futures are the simplest derivative, agreeing to purchase/sell an underlying asset at a predetermined price and time, which is suitable to hedge against risks like future exchange-rates. Options provide the owner the possibility to purchase/sell an underlying asset at a predetermined price and time, which is useful to protect current holdings of assets through various strategies. Swaps is an agreement to exchange a payment of an underlying asset, which is suitable for hedging of interest rate risks. If the pension fund wishes to reduce exposure to fluctuations in interest rates, it can initiate a swap contract to exchange a floating rate for a fixed rate.

Over-hedging is a cause of excessive hedging. Most derivatives are acquired by paying a risk premium, and doing so excessively will inevitably reduce the expected upside potential. Some risks are more expensive to hedge, an example being longevity risk, which is traditionally hedged via reinsurance. Hedging longevity is expensive and a significant risk premium is paid to acquire such a hedge, but it might be a cheaper and more flexible option compared to reinsurance (Li, Waegenaere, & Melenberg, 2017). When hedging is used the pension fund must decide between static and dynamic hedging strategies. Dynamic hedging can follow the risk policy of the company, adjusting any changes that occur in the market, while static hedging is identical to a buy-and-hold strategy. Static hedging is cheaper but short-sighted and might be ineffective (Arbeleche, Dempster, Medova, Thompson, & Villaverde, 2003). Where dynamic hedging, especially for immature markets and contracts, can be very costly and might be unfeasible (Li et al., 2017).

The pension fund must consider every aspect of hedging with respect to the goals of the pension fund and be in alignment with the policyholders. The hedging strategies must match the investment policies with how often rebalance should occur, and what kind of risk is desired to keep or to mitigate.

9.5 Contract Risk

The length of the of pension contracts is a major risk for pension funds. Contracts range far into the future and upon initiation, all risks must be carefully considered. The future is uncertain and once a pension contract has been signed, the pension fund cannot alter it without the consent of the policyholder. This poses a major risk for the pension fund as the contracts can span far into the future. Within non-insurance, the contracts are renegotiated each year, which allows the insurance company to include unaccounted risks, that would otherwise result in a loss. When signing a contract between the pension fund and a policyholder, the type of products and insurances included in the contract will determine the risk to both parties. The pension fund can control the exposure to contract risk by limiting certain types of contracts it deems too risky. This is the case for with-profit contracts, which has exposed the pension funds to difficult times with low interest rates, as they have guaranteed a minimum return between one and five percent. The shift toward unit-link products has allowed the pension funds to reduce their contract risk, as the policyholders instead take on the risk of unfavorable future events.

The pension funds' current level of risk in with-profit contracts could be de-risked by allowing the policyholders to switch to unit-link. Danish pension funds already encourage the policyholders to switch from with-profit to a contract involving a greater degree of risk sharing. Unit-link drastically reduces the risk to the pension fund, but it does not eliminate it completely, as the policyholders still have insurances such as disability and term insurance.

Risks could also be attributed to contracts containing intervention options, through either a surrender option or the option to take a policy loan. The surrender option brings the risk of prematurely surrendering prior to the expenses having been recovered (Davis, 2001). With a surrender option, the policyholder is allowed to cancel the contract and receive an immediate surrender benefit sum[28] identical to the technical reserve, as mentioned in section 6.2. A policy loan is the option to borrow money from the pension fund with your own savings as collateral. Such an option in the contract could give rise to liquidity risk (Davis, 2001).

It is not possible to completely avoid contract risk by the design of their contracts. The pension fund must include insurances, thus it will always be faced with some kind of exposure. A term insurance in a unit-link contract would still be subject to interest rate and mortality rate changes, hence to limit the exposure, hedging instrument or investment policies should be utilized. The pension fund could also avoid options such as surrendering and loan policy to limit contract risk.

9.6 Market Risk

Pension funds are affected by market movements, where market risk is the risk of changes in value, which can skew the asset-liability matching issue, which it is unable to control or influence. Changes in market conditions affect both assets and liabilities and might also be referred to as asset-liability risk. It is not limited to factors resulting in a reduction of capital value, but the connective change between assets and liabilities (Kemp & Patel, 2012). The market risk factor provides a lot of uncertainty through unknown future movements. How the market will behave or change its beliefs towards the future is unpredictable. To capture such a risk in a model the pension fund must account for many possibilities of outcomes, possibly through investment strategies or even stochastic modeling. Some of the risks are shortfall risk or specific asset risks; bonds, equity, property and private equity.

[28]The surrender option is disincentivized by the government though a heavy taxation.

Shortfall risk is the risk of earning a return on investment lower than the risk-free interest rate, which is the result of poor investment decisions or unlucky market movements. Shortfall risk is a component of every risky investment and is what generate the expected excess return in accordance with the 'capital asset pricing model'. A higher standard deviation of an asset provides a higher expected return, ceteris paribus, and thus a higher shortfall risk. The shortfall risk is not identical to the standard deviation, as it only measures the downside risk, where the standard deviation does not differentiate between upside and downside potential, it just measures the possible deviation from the mean. Given a single asset the standard deviation and shortfall risk are positively correlated, as an increased shortfall risk follows an increased possible deviation from the mean.

Shortfall risk is undesirable but it is unavoidable when seeking an expected excess return. However, as diversification tend to limit the risk of a portfolio through a reduction in standard deviation, it limits the shortfall risk simultaneously. By diversifying to reduce the shortfall risk the pension fund would follow the prudent-person principle, as it decreases the exposure of the portfolio. One of the advantages of a pension fund is its ability to neglect short-term fluctuations caused by individual policyholders. The actuarial calculations of the liability side are based on the law of large numbers, meaning short-term fluctuations in the assets will not affect the overall fund (Ezra, 1980).

It is usually thought that a longer investment horizon provides safer investment return (Bodie, 1991), known as time diversification. This is due to negative outcomes potentially being balanced by future positive outcomes; a small sample is more likely to have a value below the mean due to negative shocks, where a larger sample will have a smoothing effect going towards the true distribution. This is equivalent to time horizons. By observing a small time horizon the investor might face a rare scenario, such as the financial crisis in 2008, resulting in a large loss in value. Thus, shortfall risk is negatively correlated with investment horizon, as investment horizon increases the shortfall risk declines (Bodie, 1991). The expected return of an asset, also known as the drift, is what contributes to the correlation between shortfall risk and investment horizon. The drift is the expected rise in the value of the asset over time. As the horizon increases the shortfall probability decreases, due to returns being normally, identically, and independently distributed (Cuthbertson & Nitzsche, 2005).

By following this notion, pension funds should invest a large amount in risky assets, particularly with respect to the young policyholders, and then increase the holdings of safe assets incrementally as time approaches maturity. Human capital is the ability to earn labor income (Munk, 2016), which further assist the idea of investing more aggressively as a young investor (The Vanguard Group, 2008), as one can easier recover from potential losses. In case of economic turmoil, with falling asset values, a young investor would be able to purchase the assets at a cheaper price, improving the likelihood of larger returns when the market recovers. This ability decreases as the investor approaches retirement, due to a reduction in human capital and a decrease in time horizon. In practice FSA regulates the individualization of bonus potential and the division of policyholders in groups, according to their risk (Erhvervs- og Vækstministeriet, 2015; Danica Pension, 2015).

To maximize the expected bonus potential, with respect to shortfall risk, the pension fund individualize their investment policies according to groups of policyholders dependent on age and risk preference. By dividing the investments into groups, the fund can tailor a strategy which would maximize the utility of the policyholders. It will then maximize the expected bonus potential by investing more aggressive for young groups while being very conservative towards the old or retired groups.

While the distribution of returns converges towards its expectations, with longer investment horizon, the worst-case scenario of terminal wealth rises, as does the best-case scenario (Kritzman, 2015). The increase in dispersion of the terminal wealth is an increase in variance and the link between expected mean, variance and standard deviation to the investment horizon is given by (Cuthbertson & Nitzsche, 2005)

$$\begin{aligned} E[\mu_k] &= k\mu, \\ E[\sigma_k^2] &= k\sigma^2, \\ E[\sigma_k] &= \sqrt{k}\sigma, \end{aligned} \tag{9.1}$$

with k being the time horizon in years and returns are assumed continuously compounded and identically, and independently distributed. The relationship clearly shows that the expected mean and variance increases linearly with time, while the standard deviation increases with the square root of time, and is evidently shown in table 9.3.

	$k=1$	$k=2$	$k=5$	$k=10$	$k=20$
μ_k	0.07	0.14	0.35	0.70	1.40
σ_k^2	0.20	0.40	1.00	2.00	4.00
σ_k	0.45	0.63	1.00	1.41	2.00
SR_k	0.16	0.22	0.35	0.49	0.70

Table 9.3: The effect of time on mean, variance, standard deviation and Sharpe ratio. The yearly estimates are assumed as $k = 1$.

Critics of time diversification argue, that investment horizon does not favor the expected return and the optimal investment strategy should be unaffected by the horizon. Investors are naturally risk-averse and, it is shown that, by assuming a typical logarithmic utility function, the investor would not be compensated by partaking in risky ventures (Kritzman, 2015).

Portfolio insurance relates to hedging strategies, as it attempts to avoid the downward scenarios of investments. Portfolio insurance differs from strategies like immunization through a focus on the left tail of the return distribution (Davis, 2001). The purpose is the same with options; to cap the potential loss while retaining the upside. Portfolio insurance should be used to ensure the pension fund stay within the minimum capital requirements set by regulations.

Due to the capital requirements set by regulations, it is assumed that Danish pension funds are always above a funding ratio of one otherwise the fund would have already defaulted. In a fully funded pension fund, the shortfall risk shifts the absorbing of loss towards the owners, while the potential gain is shared with the policyholders. If the collective bonus potential is assumed zero, equity will absorb all of the negative returns, while upside potential is instantly shared between equity and collective bonus potential (Bodie, 1991). This risk/return trade-off for the pension fund aligns its interest with the policyholders, to limit excessive risk-taking.

Asset risk can be caused by market imperfections resulting in the mispricing of assets, prediction errors towards the future yielding incorrect forward rates, irrational market movements, or stochastic change in prices. If the pension fund's sole goal was to maximize the value of the owners, it would be better off just securing the future benefits with no regards of providing collective bonus potential. The trade-off between risk and return of a capital sound fund is negative for the owners, hence investing in risky assets would reduce their expected value. A perfect immunization strategy could be a way to maximize owner value, ensuring any movement in liabilities is 100 percent correlated with asset movements, assuming a complete match is possible. This would disregard the policyholders, as such a strategy would provide only the conservatively guaranteed return since the prudent-person principle protects the policyholders against such behavior.

Passive asset management is usually thought to outperform active asset management. The pension fund might think it exhibits superior selection and timing skills, allowing it to beat the market (Bodie, 1991), but there is no evidence that suggests asset managers are able to outperform the market consistently, which is due to the markets generally being efficient in semi-strong form (Fama, 1998). Studies show that passive management generates a higher return on average (Malkiel, 2003), therefore the pension fund should not only trade actively selected assets, but should also follow the market as a whole through an index. Even if an investor trade an asset that is clearly mispriced, the market can stay irrational longer than an investor can stay solvent.

The pension fund can limit some of the asset risks through investment strategies. It can use hedging instruments to reduce the shortfall risk or limit the variance, and it can diversify across multiple assets converging towards a market portfolio. It could offset the stochastic change in value by investing in negatively correlated assets. In an attempt to reduce the capital requirements of Solvency II, pension funds could use a contingent immunization strategy (Bodie, 1991). It allows the pension fund to invest in risky assets as long as the value of the assets exceeds the liabilities. If the funding ratio approaches one it would switch to an immunized fixed income portfolio, allowing the fund to achieve a higher expected return using risky investments, but keeping a floor of safety in case asset returns are unfavorable.

9.7 Wealth vs. Income

In retirement saving two approaches can be utilized; income-driven strategy and wealth-maximization strategy. With-profit contracts are based on an income-driven strategy, where the policyholders are guaranteed a minimum level of retirement income with the potential for increases in benefit. It provides the fund with a magnitude of risk and to reduce it, pension funds turned to unit-link contracts. In unit-link the policyholders are not guaranteed a minimum level of retirement income, instead, the income is based the return of the investment decisions. The shift in contracts has turned the retirement saving strategy from an income-driven strategy to a wealth-maximization strategy. Unit-link contract focus on a return-driven investment policy; a higher expected return would provide the policyholder with a higher expected retirement benefit.

Wealth-maximization is not a direct indicator of income. To acquire an income stream the policyholder could use the terminal wealth to purchase an annuity paying a fixed coupon. Such a strategy is risky as the future income is dependent on the price at the time of purchase (Merton, 2014). The shift to unit-link turns the retirement account into an investment account, with the purpose of maximizing the terminal wealth. The interest of the policyholders should be a maximization of retirement income, which the pension fund can accomplish by changing the approach to risk. To provide the policyholder with an income-driven strategy, through a unit-link contract, the fund could invest the savings to ensure enough capital at retirement to purchase a replicating portfolio. A bond-like security that pays an annuity could be used as a replicating portfolio to ensure the income of the policyholder (Merton, 2014). The strategy would only take on risk until the point of the desired income has been achieved.

A major downside to this strategy is the increased risk perceived in price changes. As risk is managed to eliminate income volatility, the changes in asset prices are not reduced, which would make the return look much more volatile. This is due to the wealth of the account fluctuating with asset price changes. The interest of the owners, as well as the regulatory systems, do not allow such an income-driven strategy. The changes in asset value would become much more volatile and affect the capital requirements and funding ratio. Hence, it would become more costly to the pension fund to provide such income-driven strategy.

10 Further Studies

It is worth noting the full complexity of asset-liability management ranges further than what has been investigated through the analysis of this paper. Further developments of ALM models are important to keep learning and understanding the risk and factors influencing the pension funds. Intuitively, ALM models are more important when managing with-profit contracts, however, as has been shortly mentioned, the pension funds are still subject to risks in unit-link contracts. Hence, even as with-profit cease to be active, it is not an argument to reduce the importance of ALM models. Markets are ever evolving and such changes keep affecting the pension funds. The approach to risk management of pension fund should not be reflected by the de-risking of contract risk, but should also be ever evolving. The shift from with-profit to unit-link products could reverse, increasing the demand for with-profit again. Falling interest rates, product transparency, and competition in traditional life products are some of the reasons for turning to unit-link (Mueller, 2000). With the proper advancements in ALM models and/or potential rises in interest rates, combined with the attractiveness following traditional life products, a shift towards with-profit could be seen.

To encourage further studies of ALM models and to advocate a continuous importance of risk management in Danish pension funds, a series of suggested improvements based on the discoveries of the analysis, as well as the discussion of risk parameters relevant to Danish pension funds, will be presented.

10.1 Improving the ALM Model

The one-factor Vasicek model and GBM model were used to model the stochastic nature of interest rates and asset values, respectively. The ALM model could be improved by using models more in line with reality. Volatility is known to vary through time (J. C. Hull, 2012), thus the models should have incorporated a change in volatility to reflect realism. Both models assume the parameters constant over time. The GBM model also neglects mean-reversion resulting in an over-estimated growth in asset value and periodically simulates extreme cases. To improve the simulation precision, models providing more realistic correlation patterns should be used (Brigo & Mercurio, 2007). Multi-factor models capturing the stochastic changes in mean, variance and mean-reversion could be used for this. For completeness of an ALM model it should also include estimates of, but not limited to, sovereign interest rates and returns, corporate bond yields and returns, equity returns, foreign-exchange rates, inflation, GDP, unemployment, mortgage-backed securities, covered bonds, municipal bonds and derivatives (Society of Actuaries, 2016).

The asset allocation should incorporate the available assets to reflect the proper investment opportunities and should include the availability of investing in various equity markets and individual stocks. The model should include derivatives and a correct assessment of alternative investments. By adding financial instruments the shortfall risk could be reduced, either through limiting the standard deviation of the portfolio or just the downside risk. Using derivatives would reduce the capital requirements by Solvency II, ceteris paribus. It would also be possible to reduce the capital requirements through an optimization of diversification across the various risk classes in the basic SCR formula.

The objective function of the ALM model was to maximize the collective bonus potential. To use such a model in practice, the collective bonus potential and the individualizing of it must be calculated correctly according to actuary mathematics. The calculations should take into account the artificial parameter of the minimum investment return guaranteed to the policyholder, which is the most risky component in with-profit contracts. The same applies to the pension products, which should match the products provided to all of the policyholders. Our model assumed a very simplified version of product mix, which is much more extensive in practice. It should also be remembered, that incorporating the multiple risks associated with each pension product would further assist a reduction in the capital requirements through the SCR correlations.

Shorter time steps are paramount to achieve more realism. Being able to observe everything yearly is very inaccurate and provide very unreliable decision making. The models should reflect the possibility to make adjustments on a daily basis. Another important cost to include is transaction costs when trading assets. Being able to dynamically trade assets on a daily basis is limited by the potentially high costs following frequent trading. It was not included in the ALM model, but it is a crucial component of trading assets. Trading costs are a huge barrier to keeping a perfect hedge, assuming the necessary assets or instruments are available, as it requires repeated trading to keep the hedge.

The paper has focused the implementation of ALM models to with-profit schemes, but the risk of unit-link contract should also be included in the model. It is possible to split the contracts into two separate categories, as the characteristics and the risk factors associated are quite distinctive. The application of ALM toward unit-link contract would not be as extensive as with-profit, but as it does carry an element of risk, the ALM model should include it.

10.2 Back-testing

Back-test is an important tool to verify the use of the model. Some components of the ALM model is based on historical data and back-testing measures how well the model would have worked in the past (J. Hull, 2015). When applying back-testing to an ALM model, it could be done both at a micro level and a macro level. Micro back-testing would be applied to the inner stochastic models such as Vasicek and GBM, while macro back-testing would be applied to the complete ALM model.

The calibration of the stochastic models is dependent on the time horizon of the data and the results can vary tremendously. The ALM model is very sensitive to small changes in the parameters of the stochastic models. To understand the sensitivity, a stress-test should be conducted on the estimated parameters of the stochastic models. The stress-test of estimated parameters applies to any variable estimated based on historical data. The length of the time horizon drives the estimation, and it is no guarantee that history will repeat itself.

It is worth noting the back-test is not expected to match an exact replica of the historical data but should fall within an acceptable bound. The predefined acceptance bound is not limited to historical data, as it could also be internal projections or regulatory demands (Society of Actuaries, 2016). Back-test is used as a line of defense to ensure the validity of the model.

Back-testing can easily become subject to over-fitting, which is a result of over complicating the model to match the historical data (Bailey, Ger, de Prado, & Sim, 2016). The importance is not to match the historical data, but accurately simulating the future. A strategy to back-test the model is to use an out-of-sample test, once the parameters are estimated. A possible way of testing out-of-sample is by using less than all historical data, and then testing the model on the remainder of the data, to see if it would prove satisfactory results.

> In practice, the most effective validation is one that combines multiple modes of analysis. For real-world ESGs, point-in-time, in-sample and out-of-sample validation techniques, combined with well-defined acceptance criteria, should be used together to determine not just what is wrong with a model, but also what is right with the model. (Society of Actuaries, 2016, p.96)

11 Conclusion

Within the last decade there has been an increased focus on risk management due to implications in changing market conditions. The nature of the pension funds' liabilities makes them vulnerable to shocks in the interest rate and longevity, which are some of the most prominent risks faced by pension funds. With-profit contracts are characterized by a minimum guaranteed investment return and make up the majority of the accumulated reserves. During the 1980s and 1990s, the policyholders were guaranteed an annual return up to 4.5 percent on their pensions. These returns have become increasingly difficult to maintain as the interest rates started to decline. In the same period, the expected lifetime of the policyholders increased, further challenging the pension fund to maintain adequate funding ratios. Risk management has become increasingly important after the introduction of Solvency II, which imposed a series of risk-based constraints. The purpose of Solvency II is to establish a financially robust pension sector, and improve the pension fund's ability to withstand shocks to the assets and liabilities to ensure continued solvency. The asset-liability management model is an effective tool in risk management to deal with unknown future scenarios, by taking stochastic variables into account.

The asset-liability management model clearly benefit the pension fund in decision making under uncertainty. Four different portfolio theories were used to evaluate the effect of different approaches to investment on the expected bonus potential. The best performing method, in the default model, was the Sortino ratio, which is an extension of the Sharpe ratio but considers only the downside deviation. The tangency portfolio was the second best performing method, and the difference to the Sortino ratio was approximately two percent. Both methods yielded a default rate of 0 percent. A hypothesis test was conducted to investigate if the small difference was statistically significant. It provided a t-test of 3.7 validating the significance of the difference. The characteristics of Sortino ratio applies well to pension funds. It is a powerful strategy for asset allocation since it emphasizes the downside risk based on a target ratio. The collective bonus potentials are normally distributed with small tails, proving the exposure is well-managed by the model.

The value of the with-profit contracts were shown to fluctuate with changes in the short rate, proving the important link to proper risk management. The purpose of the paper was to examine how Danish pension funds utilize an asset-liability management model to maximize the collective bonus potential, and the sensitivity analysis showed how the pension fund is affected by external factors and how it can alter the expected bonus potential through decisions. An increase in the risk profile to 100 percent risky assets would boost the expected bonus potential of the Sortino ratio by approximately 62 percent, however, the variance increased by more than 3,900 percent yielding a default rate of 15 percent and a t-test of 8.0. The Solvency II framework states the yearly probability of default must not exceed 0.5 percent, equivalent to once in a 200-year period. A default of 15 percent in a 40-year period clearly violates Solvency II, which proves the policy of 100 percent risky assets is not viable. The solvency capital requirement is a risk-based constraint with the intentions of avoiding an increase in probability of default, with increases in exposure. Evidently, a complete elimination of the capital requirement increases the probability of default from 0 to 5-10 percent for all portfolio methods. It proves the use of the solvency capital requirement reduces default scenarios by increase the required capital reserve.

It was shown how the the liabilities in the synthetic fund fluctuates over time due to changes in the short rate. Given the paper excluded the use of derivatives, the full effect of the interest rates is captured. It confirms the inverse relationship connection between the liabilities and the interest rate. The stochastic estimation of the future price movements are very sensitive to the time horizon of the historical data. It was proved that a change in the time horizon of the data collection could have significant consequences to the expected bonus potential. A time horizon of 5, 10 and 15 years were tested, compared to 21 years in the default model, and they all produced an increase in collective bonus potential of approximately 11.1, 2.9 and 3.0 percent followed by a change in variance of -33.2, 37.9 and -2.1 percent, respectively. The findings are substantial and shed light on the importance of the time horizon in estimating the parameters of the stochastic models.

The implications of a series of risk factors were discussed. It was argued that hedging instruments are a crucial component of an asset-liability management model, as it can significantly reduce the capital requirements through a reduction in risk. It allows the pension fund to de-risk specific exposures that differ with its risk profile. The pension fund must carefully consider which exposure it wishes to mitigate, as excessive hedging brings significant costs. The shortfall risk is the risk of earning a return on investment lower than the interest rate. It was argued, that it is beneficial to focus on reducing the shortfall risk. It can be done through the use of financial instruments or by diversifying across assets or time horizon. Time diversification provides an increase in Sharpe ratio, and provide the pension fund with the opportunity to tailor the investments into groups of policyholders, thus maximizing the collective bonus potential.

An asset-liability management model provides Danish pension funds with an understanding of how different risk factors affect the expected bonus potential and their default risk. The model provides them with a foundation to make integrated financial decision under uncertainty. It allow the pension fund to stochastically model future market movements, and examine various investment strategies to discover the strategy yielding the maximum collective bonus potential. With the increased focus in risk management, challenging market conditions, and the implementation of Solvency II, the use of an asset-liability management model is a superior risk management tool in the current Danish market.

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Appendices

A ALM Model

The ALM model can be downloaded using the following link:
`https://tinyurl.com/ALMModel`

B Dynamics of the Reserve

Saving Phase

$$\begin{aligned} V(t) = b^a \int_m^T e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds + b^d \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds \\ -\pi \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds \end{aligned} \tag{6.5}$$

For simplicity when differentiating $V(t)$, every part of the equation is being split up and differentiated individually.

Deferred whole life annuity

$$b^a \int_m^T e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds = b^a e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \int_m^T e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds$$

$$\frac{d}{dt}\left(e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \right) = (r(t)+\mu(t))\, e^{\int_0^t (r(\tau)+\mu(\tau))d\tau}$$

$$\frac{d}{dt}\left(\int_m^T e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds \right) = 0$$

$$\frac{d}{dt}\left(b^a \int_m^T e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds \right) = b^a\,(r(t)+\mu(t))\, e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \int_m^T e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds$$

Term insurance

$$b^d \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds = b^d e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds$$

$$\frac{d}{dt}\left(e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \right) = (r(t)+\mu(t))\, e^{\int_0^t (r(\tau)+\mu(\tau))d\tau}$$

$$\frac{d}{dt}\left(\int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds \right) = -e^{-\int_0^t (r(\tau)+\mu(\tau))d\tau} \mu(t)$$

$$\begin{aligned} \frac{d}{dt}\left(b^d \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds \right) &= b^d\,(r(t)+\mu(t))\, e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \\ &\int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds - b^d e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} e^{-\int_0^t (r(\tau)+\mu(\tau))d\tau} \mu(t) \\ &= b^d\,(r(t)+\mu(t))\, e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \\ &\int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds - b^d \mu(t) \end{aligned}$$

Level premium

$$\pi \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds = \pi e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds$$

$$\frac{d}{dt}\left(e^{\int_0^t (r(\tau)+\mu(\tau))d\tau}\right) = (r(t)+\mu(t))\, e^{\int_0^t (r(\tau)+\mu(\tau))d\tau}$$

$$\frac{d}{dt}\left(\int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds\right) = -e^{-\int_0^t (r(\tau)+\mu(\tau))d\tau}$$

$$\begin{aligned}\frac{d}{dt}\left(\pi \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds\right) &= \pi\,(r(t)+\mu(t))\, e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \\ &\quad \int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds - \pi e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} e^{-\int_0^t (r(\tau)+\mu(\tau))d\tau} \\ &= \pi\,(r(t)+\mu(t))\, e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \\ &\quad \int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds - \pi\end{aligned}$$

Now all the individual parts are being put back together to form the dynamic of $V(t)$ in the saving phase.

Dynamics

$$\begin{aligned}\frac{d}{dt}V(t) &= \left(b^a\,(r(t)+\mu(t))\, e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \int_m^T e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds\right) \\ &\quad + \left(b^d\,(r(t)+\mu(t))\, e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds - b^d \mu(t)\right) \\ &\quad - \left(\pi\,(r(t)+\mu(t))\, e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds - \pi\right) \\ &= (r(t)+\mu(t))\, V(t) + \pi - b^d \mu(t)\end{aligned} \tag{6.6}$$

Retirement Phase

$$\begin{aligned}V(t) = b^a \int_m^T e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds + b^d \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds \\ -\pi \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds\end{aligned} \tag{6.5}$$

For simplicity when differentiating $V(t)$, every part of the equation is being split up and differentiated individually.

Benefit annuity

$$
\begin{aligned}
b^a \int_m^T e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds &= b^a e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \int_m^T e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds \\
\frac{d}{dt}\left(e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \right) &= \left(r(t)+\mu(t)\right) e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \\
\frac{d}{dt}\left(\int_m^T e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds \right) &= -e^{-\int_0^t (r(\tau)+\mu(\tau))d\tau} \\
\frac{d}{dt}\left(b^a \int_m^T e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds \right) &= b^a \left(r(t)+\mu(t)\right) e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \int_m^T e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds \\
&\quad - b^a e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} e^{-\int_0^t (r(\tau)+\mu(\tau))d\tau} \\
&= b^a \left(r(t)+\mu(t)\right) e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \\
&\quad \int_m^T e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds - b^a
\end{aligned}
$$

Term insurance

$$
\begin{aligned}
b^d \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds &= b^d e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds \\
\frac{d}{dt}\left(e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \right) &= \left(r(t)+\mu(t)\right) e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \\
\frac{d}{dt}\left(\int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds \right) &= 0 \\
\frac{d}{dt}\left(b^d \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds \right) &= b^d \left(r(t)+\mu(t)\right) e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \\
&\quad \int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} \mu(s) ds
\end{aligned}
$$

Level premium

$$
\begin{aligned}
\pi \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds &= \pi e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds \\
\frac{d}{dt}\left(e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \right) &= \left(r(t)+\mu(t)\right) e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \\
\frac{d}{dt}\left(\int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds \right) &= 0 \\
\frac{d}{dt}\left(\pi \int_t^m e^{-\int_t^s (r(\tau)+\mu(\tau))d\tau} ds \right) &= \pi \left(r(t)+\mu(t)\right) e^{\int_0^t (r(\tau)+\mu(\tau))d\tau} \int_t^m e^{-\int_0^s (r(\tau)+\mu(\tau))d\tau} ds
\end{aligned}
$$

Now all the individual parts are being put back together to form the dynamic of $V(t)$ in the retirement phase.

Dynamics

$$\begin{aligned}
\frac{d}{dt}V(t) &= \left(b^a\left(r(t)+\mu(t)\right)e^{\int_0^t(r(\tau)+\mu(\tau))d\tau}\int_m^T e^{-\int_0^s(r(\tau)+\mu(\tau))d\tau}ds - b^a\right)\\
&+\left(b^d\left(r(t)+\mu(t)\right)e^{\int_0^t(r(\tau)+\mu(\tau))d\tau}\int_t^m e^{-\int_0^s(r(\tau)+\mu(\tau))d\tau}\mu(s)ds\right)\\
&+\left(\pi\left(r(t)+\mu(t)\right)e^{\int_0^t(r(\tau)+\mu(\tau))d\tau}\int_t^m e^{-\int_0^s(r(\tau)+\mu(\tau))d\tau}ds\right)\\
&=\left(r(t)+\mu(t)\right)V(t)-b^a
\end{aligned} \tag{6.8}$$

C Mortality Table

	Observed mortality		Mortality improvements	
Age	Female	Male	Female	Male
0	0.003056	0.004002	0.027675	0.033344
1	0.000161	0.000230	0.050465	0.047407
2	0.000121	0.000143	0.046028	0.051605
3	0.000077	0.000117	0.053209	0.051799
4	0.000074	0.000116	0.055492	0.053372
5	0.000072	0.000107	0.050005	0.061894
6	0.000070	0.000096	0.046504	0.068393
7	0.000068	0.000089	0.044749	0.073212
8	0.000068	0.000081	0.043449	0.074923
9	0.000065	0.000076	0.044341	0.074622
10	0.000057	0.000073	0.047559	0.071303
11	0.000055	0.000076	0.047914	0.069817
12	0.000057	0.000077	0.044622	0.069408
13	0.000065	0.000083	0.041226	0.067451
14	0.000082	0.000091	0.034588	0.066323
15	0.000100	0.000102	0.029354	0.062091
16	0.000120	0.000123	0.027835	0.054805
17	0.000141	0.000155	0.028027	0.048375
18	0.000156	0.000198	0.028480	0.043746
19	0.000172	0.000244	0.030165	0.038487
20	0.000191	0.000294	0.030512	0.036390
21	0.000202	0.000343	0.028855	0.034770
22	0.000210	0.000383	0.027795	0.032629
23	0.000207	0.000404	0.026851	0.031183
24	0.000191	0.000409	0.027489	0.030325
25	0.000170	0.000394	0.028657	0.030272
26	0.000152	0.000370	0.030260	0.031067
27	0.000134	0.000356	0.031040	0.031993
28	0.000130	0.000350	0.030935	0.033055
29	0.000138	0.000358	0.030232	0.034345
30	0.000145	0.000368	0.030078	0.035328

	Observed mortality		Mortality improvements	
Age	Female	Male	Female	Male
31	0.000158	0.000375	0.030068	0.036378
32	0.000172	0.000378	0.030630	0.037258
33	0.000190	0.000380	0.032279	0.037153
34	0.000213	0.000389	0.033456	0.036287
35	0.000241	0.000406	0.034670	0.035541
36	0.000267	0.000440	0.035639	0.034244
37	0.000296	0.000482	0.035819	0.033418
38	0.000326	0.000525	0.035716	0.033377
39	0.000351	0.000567	0.035189	0.033116
40	0.000393	0.000615	0.034234	0.032816
41	0.000435	0.000664	0.033392	0.032333
42	0.000493	0.000728	0.032322	0.031120
43	0.000560	0.000833	0.031626	0.029292
44	0.000626	0.000981	0.031150	0.027457
45	0.000706	0.001148	0.030374	0.025643
46	0.000797	0.001321	0.029410	0.023910
47	0.000908	0.001501	0.028499	0.022801
48	0.001042	0.001676	0.027254	0.022020
49	0.001190	0.001833	0.026364	0.021398
50	0.001347	0.002006	0.025639	0.020695
51	0.001547	0.002194	0.024801	0.020176
52	0.001755	0.002370	0.024225	0.019735
53	0.001973	0.002597	0.023595	0.019458
54	0.002211	0.002886	0.023023	0.019631
55	0.002453	0.003221	0.022806	0.019912
56	0.002697	0.003613	0.022855	0.020306
57	0.002995	0.004111	0.023091	0.020841
58	0.003353	0.004634	0.023343	0.021551
59	0.003753	0.005199	0.023311	0.022176
60	0.004190	0.005837	0.023116	0.022806

	Observed mortality		Mortality improvements	
Age	Female	Male	Female	Male
61	0.004663	0.006546	0.022820	0.023403
62	0.005098	0.007279	0.022505	0.023862
63	0.005491	0.008061	0.022363	0.024294
64	0.005891	0.008866	0.022458	0.024804
65	0.006289	0.009654	0.022575	0.025282
66	0.006750	0.010411	0.022572	0.025818
67	0.007343	0.011186	0.022476	0.026238
68	0.008063	0.012136	0.022109	0.026506
69	0.008872	0.013202	0.021469	0.026635
70	0.009812	0.014492	0.020687	0.026498
71	0.010770	0.016117	0.019822	0.026204
72	0.011799	0.017896	0.018837	0.025779
73	0.013009	0.019738	0.017862	0.025351
74	0.014491	0.021809	0.016932	0.024794
75	0.016373	0.024093	0.015977	0.024271
76	0.018702	0.026684	0.015039	0.023582
77	0.021428	0.030139	0.014142	0.022751
78	0.024427	0.034460	0.013323	0.021769
79	0.027827	0.039420	0.012580	0.020707
80	0.031943	0.045360	0.011976	0.019546
81	0.037010	0.052199	0.011593	0.018307
82	0.042714	0.059632	0.011444	0.017052
83	0.049724	0.068376	0.011446	0.015692
84	0.057312	0.078289	0.011503	0.014305
85	0.065468	0.090198	0.011586	0.012962
86	0.074445	0.104116	0.011526	0.011756
87	0.084552	0.119732	0.011314	0.010720
88	0.095935	0.136974	0.010952	0.009749
89	0.109212	0.155493	0.010494	0.008808
90	0.124608	0.174893	0.009964	0.007891

	Observed mortality		Mortality improvements	
Age	Female	Male	Female	Male
91	0.141740	0.195759	0.009409	0.006847
92	0.160982	0.218697	0.008857	0.005877
93	0.181936	0.243366	0.008258	0.005129
94	0.204930	0.270744	0.007648	0.004543
95	0.229662	0.300300	0.007049	0.004218
96	0.256388	0.331591	0.006450	0.003760
97	0.285045	0.364421	0.005858	0.003203
98	0.315517	0.398544	0.005288	0.002558
99	0.347637	0.433669	0.004762	0.001782
100	0.381180	0.469464	0.004274	0.001192
101	0.415875	0.505575	0.003824	0.000943
102	0.451410	0.541633	0.003402	0.000759
103	0.487437	0.577271	0.002999	0.000680
104	0.523591	0.612140	0.002616	0.000703
105	0.559503	0.645919	0.002237	0.000464
106	0.594811	0.678327	0.001889	0.000257
107	0.629180	0.709133	0.001574	0.000081
108	0.663330	0.739454	0.001269	0.000000
109	0.695982	0.767635	0.001002	0.000000
110	0.726730	0.793481	0.000773	0.000000

D Data Input of Policyholders

Age	# People	Pension age	r^*
30	707	68	1%
31	696	68	1%
32	679	68	1%
33	654	68	1%
34	640	68	1%
35	653	68	1%
36	648	68	2%
37	683	68	2%
38	694	68	2%
39	712	68	2%
40	706	68	2%
41	733	68	2%
42	791	68	2%
43	778	68	2%
44	775	68	3%
45	809	68	3%
46	797	68	3%
47	763	68	3%
48	758	68	3%
49	784	68	3%
50	835	68	3%
51	893	68	3%
52	863	68	3%
53	839	68	3%
54	819	68	4%
55	771	68	4%
56	746	67	4%
57	743	67	4%
58	708	67	4%
59	708	67	4%
60	698	67	4%

Age	# People	Pension age	r^*
61	698	67	4%
62	684	67	4%
63	665	67	4%
64	671	66	4%
65	652	65	4%
66	636	65	4%
67	648	65	4%
68	637	65	4%
69	663	65	4%
70	695	65	4%
71	704	65	4%
72	666	65	4%
73	618	65	4%
74	558	65	4%
75	512	65	4%
76	442	65	4%
77	417	65	4%
78	383	65	4%
79	362	65	4%
80	334	65	4%
81	300	65	4%
82	268	65	4%
83	242	65	4%
84	214	65	4%
85	192	65	4%
86	170	65	4%
87	150	65	4%
88	132	65	4%
89	117	65	4%
90	98	65	4%

Age	# People	Pension age	r^*
91	84	65	4%
92	68	65	4%
93	55	65	4%
94	42	65	4%
95	29	65	4%
96	24	65	4%
97	18	65	4%
98	11	65	4%
99	7	65	4%
100	4	65	4%
101	3	65	4%
102	2	65	4%
103	1	65	4%
104	1	65	4%
105	0	65	4%
106	0	65	4%
107	0	65	4%
108	0	65	4%
109	0	65	4%
110	0	65	4%

E ALM Simulations

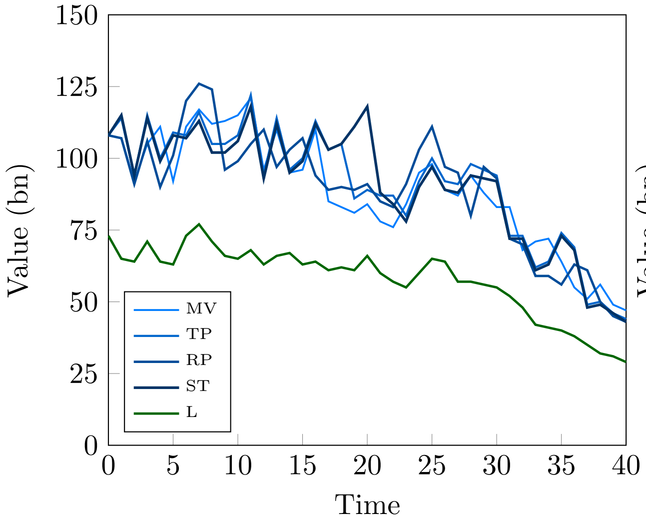

Figure E.1: Simulation 2 - Assets-Liabities

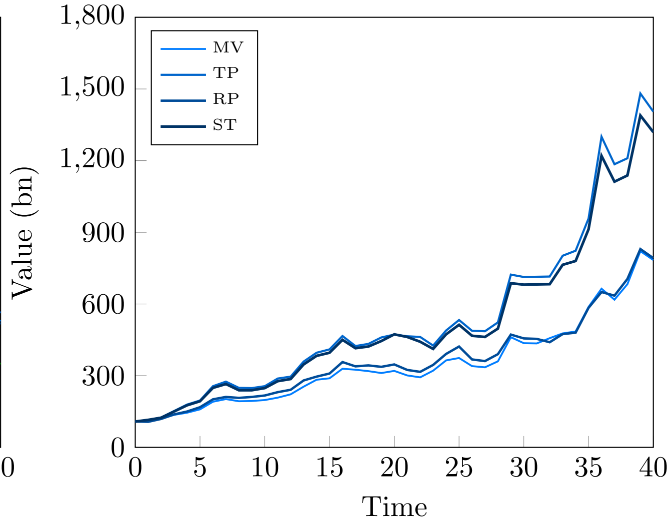

Figure E.2: Simulation 2 - Acc. Asset Values

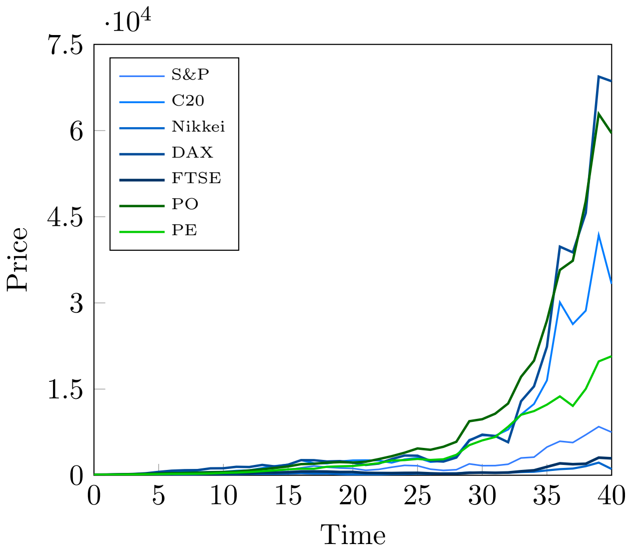

Figure E.3: Simulation 2 - Asset prices

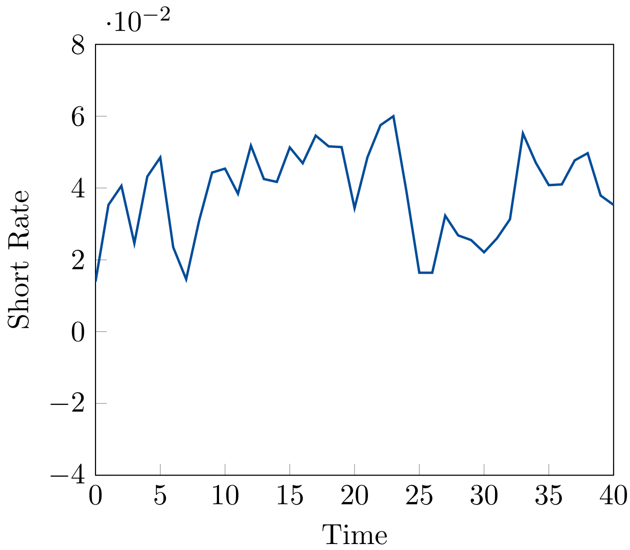

Figure E.4: Simulation 2 - Short Rate

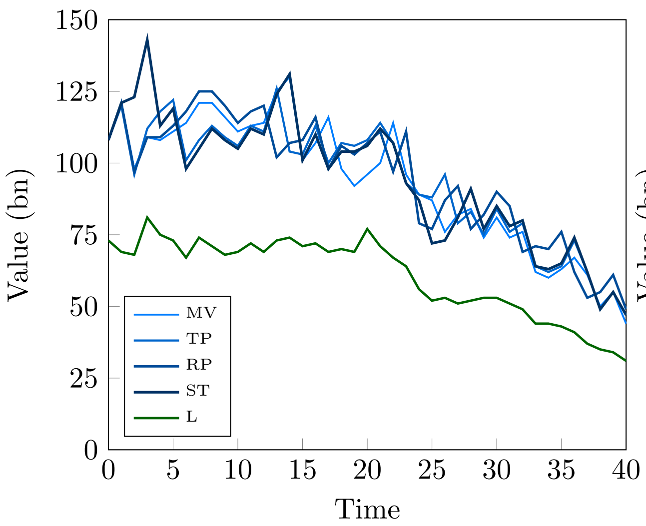

Figure E.5: Simulation 3 - Assets-Liabities

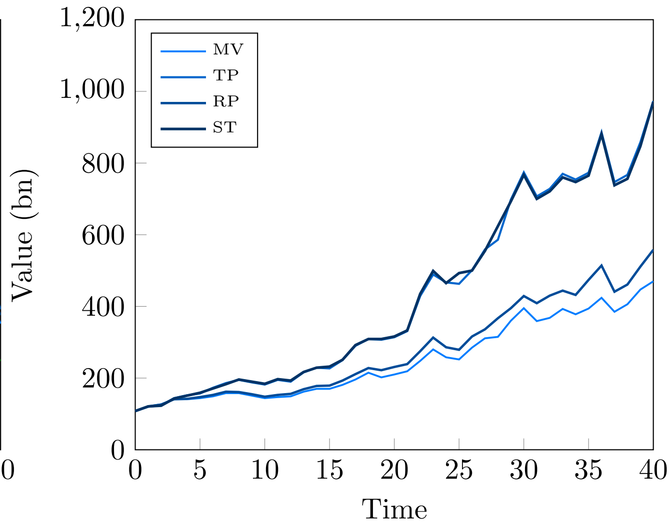

Figure E.6: Simulation 3 - Acc. Asset Values

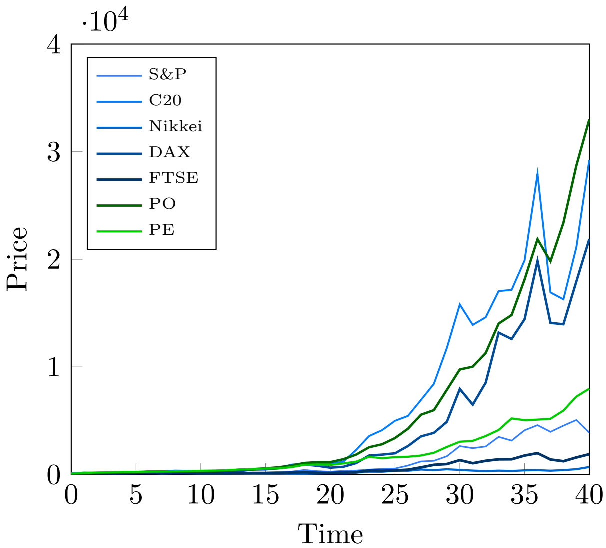

Figure E.7: Simulation 3 - Asset prices

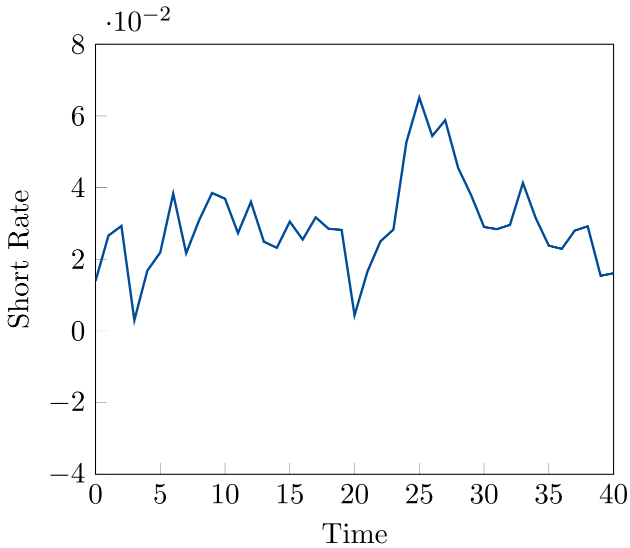

Figure E.8: Simulation 3 - Short Rate

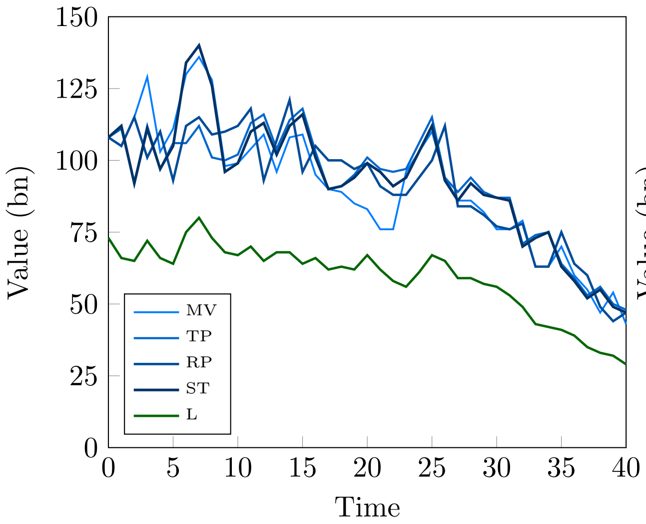

Figure E.9: Simulation 4 - Assets-Liabities

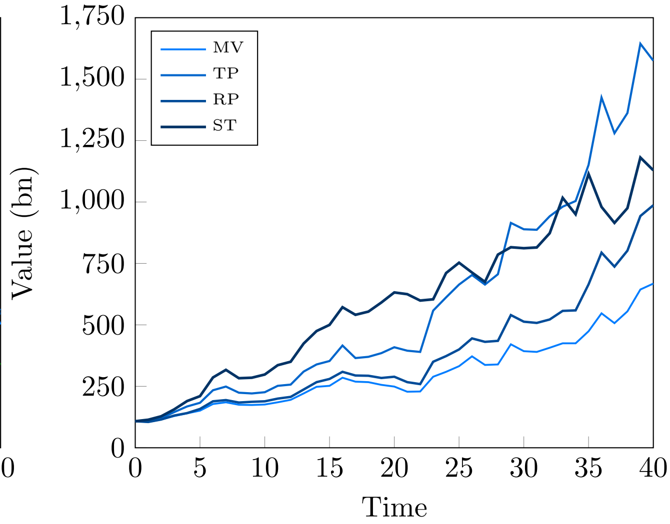

Figure E.10: Simulation 4 - Acc. Asset Values

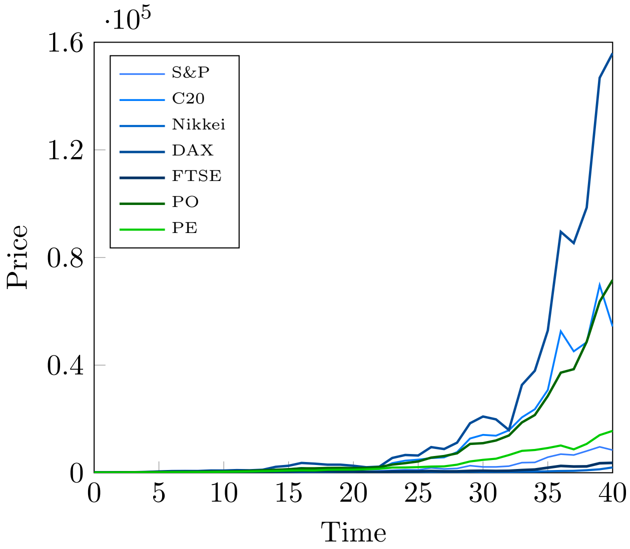

Figure E.11: Simulation 4 - Asset prices

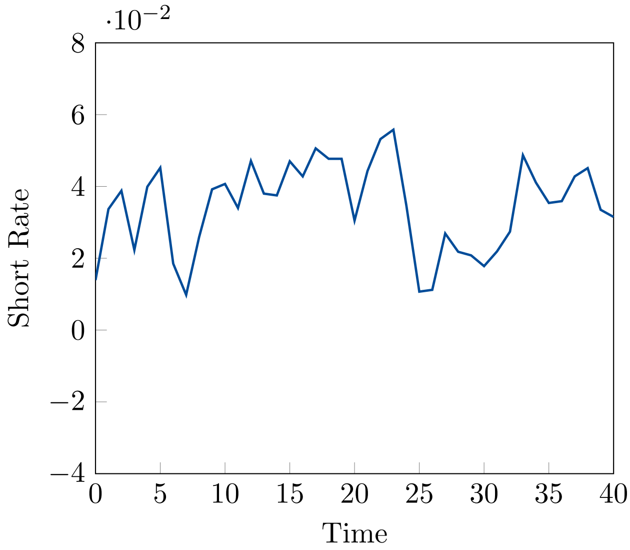

Figure E.12: Simulation 4 - Short Rate

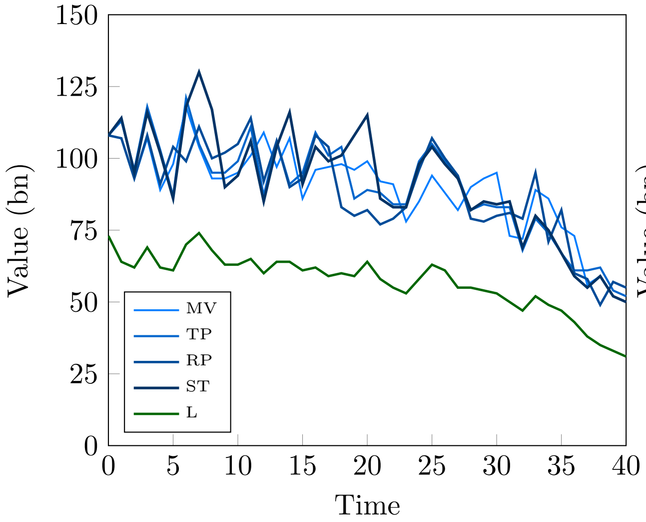

Figure E.13: Simulation 5 - Assets-Liabities

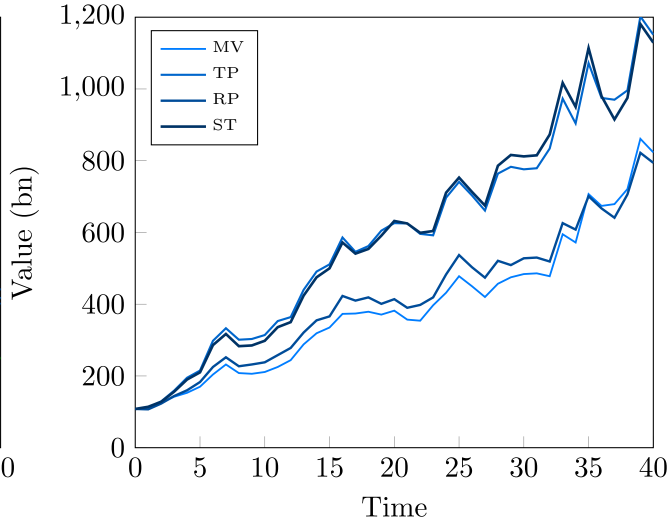

Figure E.14: Simulation 5 - Acc. Asset Values

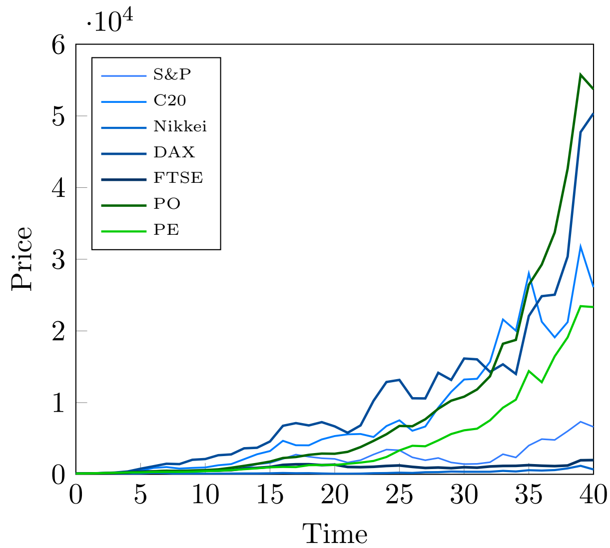

Figure E.15: Simulation 5 - Asset prices

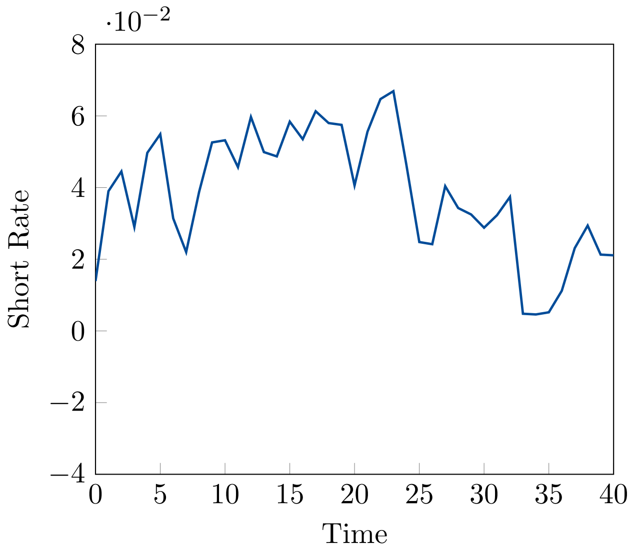

Figure E.16: Simulation 5 - Short Rate