Factor-Based Mean-Variance Investing on the U.S. Stock Market

(Faktorbaseret Middelværdi-Varians Investering på det Amerikanske Aktiemarked)

Master Thesis

15th May, 2019

Cand.merc.FIN – Søren Gybel Frederiksen – 92816

&

Cand.merc.mat. – Julius Voigt Følsgaard – 91747

Characters Including Spaces: 182,554.
Number of Pages: 96

Supervisor: Claus Munk

Abstract

This thesis investigates how mean-variance asset allocation can benefit from factors in stock returns on the U.S. stock market. The performance of several portfolios formed upon a factor-based mean-variance analysis will thus be evaluated and compared to the performance of an equally weighted portfolio.

Based on this objective, the study first establishes a theoretical foundation for the identification of factors and the implementation of the empirical analysis. Having presented the theoretical foundation, a review of the literature within the field of factor models is conducted to investigate how factors in stock returns can be explained, and which factors have been found to explain variability in stock returns. Particular attention is paid to four well-established factor models within the literature. These include the factor models of Sharpe (1963), Fama & French (1993, 2015) and Carhart (1997).

Afterwards, the thesis presents a framework for the implementation of a factor-based mean-variance analysis. The framework is an attractive alternative to the traditional mean-variance analysis, as problems of singular variance-covariance matrices are not encountered in our implementation. The framework describes an unconstrained and a constrained solution to the mean-variance optimization procedure. Moreover, the framework is assessed in an econometric context.

The factor-based mean-variance analysis allows us to form portfolios on a dataset of stocks included in the S&P 500 Index. This provides the basis for the empirical analysis, which involves portfolio backtesting and performance evaluation from January 1979 to January 2019. The results are that the portfolios formed upon the factor-based mean-variance analysis do not outperform the equally weighted portfolio over the full evaluation period. However, separating the full evaluation period into several decades shows evidence for better performance of the factor-based portfolios relative to the equally weighted portfolio.

Several aspects of the thesis will form the basis of a discussion including causations for our findings, data biases and practical applicability. Along with the results of the empirical analysis, the discussion will lead to proposals of topics for further research.

Abstract *(Danish)*

Dette speciale undersøger hvordan middelværdi-varians aktiv allokering kan drage fordel af faktorer i aktieafkast på det amerikanske aktiemarked. Det vil således blive undersøgt hvordan porteføljer dannet på baggrund af en faktorbaseret middelværdi-varians analyse præsterer sammenlignet med en ligevægtet portefølje.

Ud fra denne målsætning, etableres først et teoretisk grundlag for identifikationen af faktorer og implementeringen af den empiriske analyse. Efter at have præsenteret det teoretiske grundlag, følger en gennemgang af litteraturen indenfor faktormodeller, for at undersøge hvordan faktorer i aktieafkast kan forklares, og hvilke faktorer, der har vist sig at forklare variabilitet i aktieafkast. Der lægges særlig vægt på fire veletablerede faktormodeller indenfor litteraturen. Disse omfatter faktormodellerne af Sharpe (1963), Fama & French (1993, 2015) og Carhart (1997).

Dernæst præsenteres et framework for implementeringen af en faktorbaseret middelværdi-varians analyse. Frameworket er et attraktivt alternativ til den traditionelle middelværdi-varians analyse, da problemer med singulære varians-kovariansmatricer ikke opstår i vores implementering. Specialet beskriver hvordan en ubegrænset og en begrænset løsning til middelværdi-varians optimeringen kan implementeres. Desuden vurderes frameworket i en økonometrisk kontekst.

Den faktorbaserede middelværdi-varians analyse anvendes til at danne porteføljer ud fra et datasæt af aktier, inkluderet i S&P 500 Indekset. Dette danner grundlaget for den empiriske analyse, der indebærer backtesting og præstationsevaluering af porteføljerne fra januar 1979 til januar 2019. Resultatet er, at porteføljerne dannet ud fra den faktorbaserede middelværdi-varians analyse ikke præsterer bedre end den ligevægtede portefølje i den fulde evalueringsperiode. Ved at opdele den fulde evalueringsperiode i årtier, viser det sig at de faktorbaserede porteføljer præsterer bedre end den ligevægtede portefølje.

Flere aspekter af specialet vil danne grundlag for en diskussion, herunder grundlaget for porteføljernes præstation, data bias, samt praktisk anvendelighed. Resultaterne af den empiriske analyse vil sammen med diskussionen føre til forslag af emner til videre forskning.

Table of Contents

1 – Introduction

The mean-variance framework proposed by Markowitz and the academic financial research on factor models are two important paradigms of modern finance.

The pioneering research of Markowitz was developed more than six decades ago and conceptualizes how an investor may allocate wealth across risky financial assets with different return and risk characteristics. The key contribution of Markowitz constitutes a strictly formulated analysis for portfolio choice. His theory paved the way for further research in financial economics, and his concepts are still frequently used in practical portfolio management today. For these reasons, he is deservedly regarded as *"the father of modern, scientific finance*".

On the other hand, factor models are merely econometric models relating the return on a financial asset to multiple explanatory factors. The publication of the three-factor model by Fama & French (1993) led off a huge literature on empirical factor modeling. Many factors have since been claimed to explain stocks returns significantly. Even though disagreement exists among theorists about the rationale behind these factors, factor models have undoubtedly had an enormous influence on the finance academia and the financial industry.

Therefore, we find it interesting to combine and apply these two concepts in portfolio construction. Hence, the research interest of this study shall be found where the academic and practical use of Markowitz' framework and factor models meet.

The main objective of this thesis is to implement a factor-based mean-variance analysis. We will do this in order to investigate the benefits of applying such an analysis compared to the traditional framework by Markowitz, where inputs often are based on sample estimates. We expect that adding more structure to the model, in the form of assuming that several factors explain all common variations in the returns of financial assets, will lead to fewer and more reliable inputs.

Initially, the purpose of this thesis included constructing a comparison portfolio based on the traditional Markowitz framework described above. Nevertheless, as we shall see moving forward, applying the traditional mean-variance analysis to a high number of assets will make such a comparison portfolio problematic to implement in practice.

Instead, we choose to construct portfolios based on several of the most well-established factor models of the finance literature. These include the factor models of Sharpe (1963), Fama & French

(1993, 2015) and Carhart (1997). In addition, we will construct a simple equally weighted portfolio as a comparison portfolio. With these portfolios at hand, we will compare the individual performance of each portfolio using several relevant performance measures. The market on which we implement our analysis is the U.S. stock market, where we consider securities included in the Standard & Poor's 500 Index (S&P 500 Index). This leads to the following problem statement.

1.1 - Problem statement

How can mean-variance asset allocation benefit from factors in stock returns on the U.S. stock market?

In order to answer the overall problem statement, some related issues have to be addressed, which will be done by answering the following supporting research questions:

- *How can factors in stock returns be explained, and which factors have been found to explain variability in stock returns?*
- *How can models based on these factors be implemented in the mean-variance analysis?*
- *How do several portfolios constructed upon a factor-based mean-variance analysis perform relative to an equally weighted portfolio, before transaction costs, in the period from January 1979 to January 2019?*

Moreover, the thesis will include a discussion of our findings, leading up to further research on the subject. Finally, a conclusion will summarize the results of this study in order to answer the overall problem statement.

1.2 – Delimitations

In spite of the narrow problem statement, several delimitations are necessary for the purpose of making a detailed analysis within the scope of this thesis.

Firstly, this study does not intend to conduct tests to assess the statistical validity of the factor models in the context of asset pricing. Instead, we will take the models as given since extensive literature has been devoted to this purpose. Hence, we assume that the factor models we consider are sufficiently powerful in explaining variability in stock returns. However, we will discuss various complications of the models highlighted by the literature and keep these in mind, when evaluating our results. Furthermore, we will consider the theoretical foundation for the factor models and assess the statistical implications with respect to implementing such models, thoroughly. The rationale for this relates to our desire of focusing more on portfolio construction and less on asset pricing in this thesis.

Secondly, since this study is intended as a financial study rather than a statistical study, the econometric implications of the applied statistical model (the OLS regression) will be discussed but not corrected for. Following this argument, the thesis will include a section where econometric issues that may arise in the application of the linear regression model on our data sample will be discussed. Moreover, a part of the discussion will address the problem of statistical significance. Throughout the thesis, a level of significance at 5% will be used.

We limit our investment universe of stocks to only encompass the Standard & Poor's 500 Index at a particular point in time. Lastly, for simplicity and as outlined in the problem statement, we will assume trading costs to be non-existent. Therefore, the results will not include any calculated costs incurred by transactions.

More exact delimitations regarding for example choice of factor models, length of estimation window and other appropriate restrictions will be presented through the relevant sections as we go along.

1.3 – Motivation

Through our academic time as business school students, we have repeatedly encountered the topic of Modern Portfolio Theory. Most students within finance programs are being taught Markowitz' mean-variance analysis both during their undergraduate studies, but also in their graduate studies.

Due to the popularity and applicability of the theoretical framework, we decided to make it one of the cornerstones of our thesis.

On the other hand, factor models have gained tremendous popularity and recognition in recent years. The seminal paper by Fama & French paved the way for research of new factors, and hundreds of potential factor candidates have since been proposed, giving rise to what Feng, Giglio, & Xiu (2019) deem a *"zoo"* of risk factors. We, therefore, find this area of finance interesting to incorporate as the second cornerstone of our thesis.

We recognize that these two major topics of finance are often introduced to students separately, but rarely used in conjunction. Hence, this is what our thesis intends to do.

The thesis is meant to contribute to the existing literature on portfolio theory by combining academic knowledge with a practical approach, leading to both alternations as well as supplements to the existing literature. The thesis sustains a focus on implications from already existing theories which are applied quantitatively.

The creation of this thesis is motivated by our academic background and aspirations for the future. Furthermore, our genuine interest in finance, portfolio analysis, and applied mathematics had led us to conduct this study and write this thesis.

1.4 – Methodology

To answer the problem statement, the thesis will follow a systematic structure and apply a quantitative methodology, where financial time series data will be used to construct portfolios and evaluate the performance of the portfolios. The results of the quantitative analysis will be used to examine and assess various approaches to portfolio construction.

An extensive selection of relevant academic literature and scientific research has been used as a basis for gaining insight into relevant financial topics and theory, as well as for inspiration to the research-design and methodology of this study. In particular, the analysis will be based on a theoretical framework consisting of established theories of portfolio optimization, asset pricing, factor modeling, and performance evaluation. Several literary sources will be used to build the theoretical foundation. These sources primarily include academic articles and textbooks. The articles have been found in well-known journals with publications of internationally acclaimed and frequently cited researchers. More practically, the articles have been retrieved from various databases such as Econ-

Papers, JSTOR, Social Science Research Network as well as through several author's webpages. The databases and sources we gather information from are all well-known and highly regarded sources of academic research and information. Therefore, we assume that they serve as high quality and reliable sources of information. Nonetheless, the references we make use of will be critically studied before being incorporated into the thesis.

A separate section has been devoted to describing the specific types of data employed in this thesis in detail. Consequently, this section will merely discuss the data from a general point of view. The financial time series data we use are retrieved from Thomson Reuters (2019), as well as from the data library of French (2019). These databases are internationally recognized and widely used in various scientific studies and tests on financial and economic issues and phenomena. Extensive screening and evaluation of our data have been conducted. Hence, we assess that the data in conjunction with the delimitations will form the foundation for highly reliable and valid analyses and conclusions. However, we recognize that the way we construct the data sample makes several data biases unavoidable. A discussion about such issues will thus follow towards the end of the thesis. Since the data is publicly available, in case someone wishes to replicate our results, this may be possible at any time, by retrieving the same financial time series data. Publicly available data sources also make it easier to perform this study using an extended or alternative data sample, opening up for further research. In this regard, a particular desire on our part has been to describe the implementation process of the analysis extensively. Accordingly, Section 4.2 will thoroughly describe how we form the analysis.

1.5 – Structure

While Section 1 introduced the thesis, Section 2 will provide the foundation for this study by describing the relevant theoretical frameworks we will employ. Furthermore, Section 3 will encompass various theory on factor models, including an identification of the most relevant factors for our study. Section 4 will outline the basis for the empirical study and describe in detail how we form the analysis to answer the research questions and the overall problem statement. This includes data sampling, portfolio construction as well as back-testing and performance evaluation. Moreover, Section 4 will consider possible econometric issues and make an initial assessment of our data. Section 5 will interpret the output of our analysis and conduct the portfolio backtesting, while Section 6 will include the performance evaluation of the constructed portfolios. Section 7 will discuss the findings of the two previous sections and lead to an overall conclusion to the problem statement in Section 8, where suggestions for further areas of research will be proposed.

2 – Theoretical Frameworks

2.1 – Mean-Variance Portfolio Theory

In this section, we will introduce the mean-variance analysis developed by Markowitz (1952). We will first consider the criterion for portfolio selection and then describe the idea of diversification. Moreover, we will elaborate on portfolio optimization with regards to mean-variance efficiency.

2.1.1 – Mean-Variance Rule

Markowitz (1952) presents his mean-variance framework for portfolio selection, where investors prefer higher expected return against lower expected return but favor lower risk against higher risk. Investors are thus return maximizing, and risk-averse and make decisions based on these preferences, known as the M*ean-Variance Rule*. Employing this criterion in the selection of portfolios, investors only evaluate the first two moments of the distribution, over a fixed, future period of time. These include the expectation of return on the portfolios, as the mean μ, and the risk of the portfolios as the variance σ^2.

We may illustrate investment decisions based on the Mean-Variance Rule by comparing two portfolios, Portfolio A and Portfolio B, with different subsets of securities. The idea of the Mean-Variance Rule is that Portfolio A is favored to Portfolio B if

$$\mu_A \geq \mu_B$$

and

$$\sigma_A^2 \leq \sigma_B^2$$

or equivalent

$$\sigma_A \leq \sigma_B$$

In the case that we have $\mu_A > \mu_B$, but also $\sigma_A^2 > \sigma_B^2$ we are not able to determine which portfolio investors favor based on the Mean-Variance Rule. More generally, a portfolio is deemed to be mean-variance efficient, in case it has the lowest variance among all the portfolios with the same expected return. In our example, Portfolio A is said to be mean-variance efficient and Portfolio B is said to be mean-variance inefficient, since Portfolio B both has lower expected return and higher risk than Portfolio A. In an investment universe of Portfolio A and Portfolio B, an investor would thus never be willing to hold Portfolio B.

2.1.2 – Diversification

The idea of diversification relates to the notion of "*not putting all of your eggs in one basket"*. Markowitz (1952) states that the Mean-Variance Rule implies that an investor should maximize expected return for a given variance and diversify the portfolio. We may illustrate the latter concept by considering Security 1 and Security 2 and forming a portfolio based on these securities. We have expectations of returns $\mu_1 = E[r_1]$, $\mu_2 = E[r_2]$ and variance of return $\sigma_i^2 = E[r_i - \mu_i]^2$. We have the expectation of return for the portfolio μ_p as

$$\mu_P = w_1\mu_1 + w_2\mu_2 \tag{2.1}$$

With w_i being the proportion (weight) of our wealth invested in security i. The sum of our weights must equal 1 and $w_i \geq 0 \; \forall \; i$.

We have the variance of return on the portfolio σ_p^2 as

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{1,2} \tag{2.2}$$

where $\rho_{1,2}$ refers to the correlation coefficient between the returns on Security 1 and Security 2, which can take a value in the interval $[-1; 1]$. The correlation coefficient can also be written as a function of the covariance between the two returns, $\sigma_{1,2}$ and the standard deviation of the returns on each security,

$$\rho_{1,2} = \frac{\sigma_{1,2}}{\sigma_1\sigma_2} \tag{2.3}$$

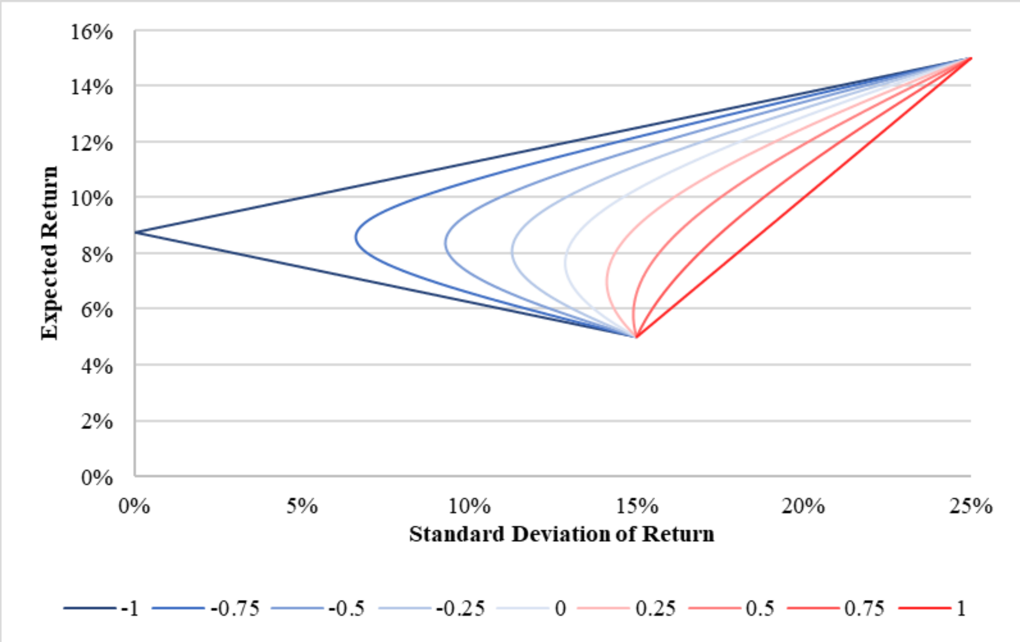
For $\rho = 1$ there exists a perfect positively linear relation between the returns on the two securities. The returns move in the same direction; thus, we are not able to benefit from diversification by combining the two securities.

If $\rho = -1$, there exists a perfectly negatively linear relation between the returns on the two securities. Since the returns move in opposite directions, we can eliminate portfolio risk.

For $\rho = 0$ no linear relation exists. With a non-perfect correlation, we are still able to reduce the overall portfolio risk with diversification, however we are not able to reduce it to zero. Hence, the riskiness of the portfolio critically depends on the sign of ρ.

In Figure (2.1) we simulate 2000 random portfolios consisting of Security 1 and Security 2 and evaluate Equation (2.2) in different instances of ρ. In the simulation, we choose the arbitrary values for means and variances of returns for the two securities, $\mu_1 = 5\%, \mu_2 = 15\%$ and $\sigma_1 = 15\%$, $\sigma_2 = 25\%$.

Figure 2.1 – Diversification Benefits for Different Correlation Coefficients:



Examining Figure (2.1), the advantages of diversification become clearer. As anticipated above, we see that in a decreasing correlation between returns, we are gaining the benefit of a reduced risk of the overall portfolio, for the same expected return. Thus, we are moving from the frontiers on the right to the frontiers on the left. We also see that by combining two securities with non-perfect correlation, we are able to construct a portfolio with a lower overall risk than that of each of the securities.

But what happens when we include more securities in the portfolio? Below we will consider this instance, through the findings of Munk (2017). In the case we extend our portfolio to include N number of securities, Equation (2.1) and Equation (2.2) can be generalized to

$$\mu_p = \sum_{i=1}^{N} w_i \mu_i = w^T \mu \tag{2.4}$$

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} w_i w_j \, \sigma_i \sigma_j \rho_{i,j} = w^T \mathbf{\Sigma} w \tag{2.5}$$

where $\sigma_{i,j} = \sigma_i \sigma_j \rho_{i,j}$. For a portfolio with securities held in equal proportions, $w_i = \frac{1}{N}$, we have

$$\sigma_p^2 = \frac{1}{N^2} \sum_{i=1}^{N} \sigma_i^2 + \frac{1}{N^2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \sigma_{i,j} \tag{2.6}$$

Across the N securities, we have the average variance as

$$\overline{\sigma_p^2} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 \tag{2.7}$$

and the average covariance as

$$\overline{\sigma_{i,j}} = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \sigma_{i,j} \tag{2.8}$$

Where $N(N-1)$ denotes the number of covariances.

By simply rewriting Equation (2.7) and Equation (2.8), and using them in Equation (2.6), we can state the variance of the portfolio as

$$\sigma_p^2 = \frac{1}{N^2} N * \overline{\sigma_p^2} + \frac{1}{N^2} * \big(N(N-1)\big) * \overline{\sigma_{i,j}} \Leftrightarrow$$

$$\sigma_p^2 = \frac{1}{N} * \overline{\sigma_p^2} + \frac{N^2 - N}{N^2} \overline{\sigma_{i,j}} \Leftrightarrow$$

$$\sigma_p^2 = \frac{1}{N} \overline{\sigma_i^2} + \left(1 - \frac{1}{N}\right) \overline{\sigma_{i,j}} \tag{2.9}$$

By including more securities in the portfolio, we see that the term $\frac{1}{N}\overline{\sigma_i^2}$ approaches zero. On the other hand, by increasing the number of securities the term $\left(1 - \frac{1}{N}\right)\overline{\sigma_{i,j}}$ approaches the average co-variance. Thus, for a portfolio with equal proportions invested in each security the total portfolio variance will approach the average covariance, when $N \rightarrow \infty$.

The total risk of a security can be decomposed to factors affecting all securities, as well as influences specific to the relevant firm. The former is known as market (or systematic) risk and may encompass economic impacts such as the growth of the economy, changes in inflation, etc. The latter is known as firm-specific (idiosyncratic) risk, where examples may include an unexpected strike amongst employees, unforeseen (major) damages to production plants, etc.

From Equation (2.9) we can conclude that by constructing portfolios with sufficiently low concentrations in each security, the asset specific risk can be diversified away. We denote such a portfolio as a *diversified portfolio*. Nevertheless, we are not able to eliminate any market risk through diversification, hence why the covariances across assets in Equation (2.9) remain when increasing the number of securities, N.

2.1.3 – Mean-Variance Efficient Portfolio Optimization

As concluded above, we say that a portfolio is mean-variance efficient if, for a given return, it has the minimum variance. Hence, we are interested in minimizing Equation (2.5) for an expected return μ in Equation (2.4). By matrix notation, we can formulate this as the following minimization problem

$$
\begin{gathered}
\min w^T \boldsymbol{\Sigma} w \\
\textit{subject to } w^T \mu = \mu_* \\
w^T \mathbf{1} = 1
\end{gathered}
\tag{2.10}
$$

where w refers to the $N \times 1$ vector of weights for each security, μ the $N \times 1$ vector of expected excess return on the securities, $\boldsymbol{\Sigma}$ the $N \times N$ variance-covariance matrix and $\mathbf{1}$ as a $N \times 1$ vector of ones, that act as a sum-operator, since we impose that the weights must sum to 1 as the only constraint. Solving this problem for a number of different target expected returns, μ_*, will thus produce an *efficient frontier* in a diagram like Figure (2.1) above.

The variance-covariance matrix in (2.10) must be non-singular. Non-singularity implies that the inverse of the variance-covariance matrix, denoted as $\boldsymbol{\Sigma}^{-1}$, exists. A more intuitive interpretation of the case with non-singularity is presented by Munk (2017). He states that in the presence of a non-singular variance-covariance matrix none of the risky assets are redundant, meaning that none of the individual risky assets can be replicated by a portfolio of the other risky assets. Therefore, these risky assets cannot be used to form a risk-free portfolio.

A closed-form solution exists to the optimization problem in (2.10). However, we will only consider the derivation for the tangency portfolio by Munk (2017). We can find this portfolio when the mean-variance analysis includes both risky assets and the risk-free asset. The tangency portfolio is located on the point, where a straight line drawn from $(0, r_f)$ in a μ, σ – diagram, is tangent to the efficient frontier. This straight line is also known as the Capital Allocation Line (CAL), and indicates combinations of the tangency portfolio and the risk-free asset where the highest Sharpe ratio is obtained (Bodie, Kane, & Marcus, 2009). The tangency portfolio is given by

$$w_{Tan} = \frac{1}{\mathbf{1}^T \mathbf{\Sigma}^{-1}(\mu - r_f \mathbf{1})} \mathbf{\Sigma}^{-1}(\mu - r_f \mathbf{1}) \tag{2.11}$$

In general, the inputs needed for mean-variance optimization comprise the expected excess returns and the variance-covariance matrix for all assets. The most common approach for obtaining these inputs is through estimation from historical sample data. The estimation of expected returns and the variance-covariance matrix is often based on the sample mean and sample variance-covariance matrix, respectively. This procedure can have estimation errors attached to it and the implementation may not be feasible in practice. These issues will be discussed in Section 4.2, including an alternative procedure for estimating the inputs by using factor models.

So far, no constraints have been imposed on the individual portfolio weights in (2.10). Nevertheless, as we shall also see in Section 4.2, this may be necessary due to extreme behavior of the portfolio weights and for practical applicability.

2.2 – Capital Asset Pricing Model (CAPM)

Sharpe (1964), Treynor (1961), Lintner (1965), and Mossin (1966) formalized the Capital Asset Pricing Model (CAPM), which builds upon the mean-variance framework of Markowitz (1952) described in Section 2.1. The central implication of the model is that the expected return of security i, $E[r_i]$, in excess of the risk free rate, r_f, can be written as a linear function of the expected return on the market, $E[r_m]$, in excess of the risk free rate. This is formalized as:

$$E[r_i] - r_f = \beta_i (E[r_m] - r_f) \tag{2.12}$$

where β_i (beta) is a risk measure capturing co-movement between the security and the market portfolio, defined as $\beta_i = \frac{Cov[r_i, r_m]}{Var[r_m]}$. That is, the relative risk of the particular security. The intuition behind Equation (2.12) is that each security is held as part of the market portfolio of risky securities, hence the expected return on the security should only depend on the risk it contributes to the market portfolio.

According to Bodie et al. (2009) the CAPM assumes the following conditions hold:

Assumption 1 – Complete Agreement

Complete agreement exists among investors; Thus, investors develop identical opinions about the parameter μ, r_f, σ, β and ρ. This leads to homogeneous expectations about the efficient frontier and the composition of the tangency portfolio.

Assumption 2 – Mean-Variance Preferences

Investors have mean-variance preferences and choose portfolio weights in order to maximize the Sharpe ratio, thereby arriving at the tangency portfolio.

Assumption 3 – Market Equilibrium

Investors are not themselves able to influence prices. Hence, they are price takers and the market is therefore always in equilibrium.

Assumption 4 – Frictionless Markets

Markets are frictionless, i.e., there are no taxes, no transaction costs and investors can access all assets in perfectly divisible portions.

Assumption 5 – Unlimited Borrowing and Lending

Investors can borrow and lend unlimitedly at the risk-free rate.

As evident, the CAPM adds two key assumptions to the framework provided by Markowitz (1952). These include *Complete Agreement* and *Unlimited Borrowing and Lending* and imply that all investors view the same opportunity set and invest in the same tangency portfolio. As a result of that all investors hold the same portfolio, this portfolio must equal the market portfolio of risky assets (Bodie et al., 2009).

More intuitively, the addition of risk-free borrowing and lending turns the efficient frontier into a straight line, since all investors will only hold a combination of the market portfolio and the risk-free asset. This straight line is also known as the Capital Market Line (CML). In the CAPM, it is assumed that investors may only differ with regards to their risk aversion. Hence, the overall optimal portfolio for an investor is determined by their degree of risk aversion (Munk, 2017).

Despite the theoretical appeal of the CAPM, Fama & French (2004) highlight the poor empirical record of the model. They attribute this to the strongly simplified assumptions brought forward by the model. In particular, they note that "*the failure of the CAPM in empirical tests implies that most applications of the model are invalid*". As we shall see in Section 3.2, the three-factor model of Fama & French (1993) was developed as a result of the market anomalies unexplained by the CAPM. In the late 1970s, studies found that market anomalies like size and certain price ratios add to the explanations of average returns (Fama & French, 2004). Size and value are thus added as additional factors in their three-factor model, besides the market factor.

However, other authors such as Roll (1977) suggest the practical impossibility of testing the CAPM, since the true market portfolio is unobservable (referred to as Roll's Critique). In theory, the market portfolio should include all invested assets. Empirically we are not able to observe such a portfolio and tests of the CAPM usually employ a stock index as a proxy for the market portfolio. Substituting the market portfolio with such a proxy may lead to false inferences about the CAPM's validity, which is what Roll (1977) emphasizes.

2.3 – Arbitrage Pricing Theory (APT)

The Arbitrage Pricing Theory (APT) was developed by Ross (1976) in response to the failure of the CAPM. The assumptions underlying the APT are very different from those of the CAPM. While the CAPM includes assumptions on complete agreement, mean-variance preferences and economy-wide equilibrium, the APT only assumes an absence of arbitrage and several other structural conditions. The following section will primarily make use of Cuthbertson & Nitzsche (2004), Skovmand (2015) and Ross (1976) as references.

The assumptions of the APT do more explicitly constitute:

Assumption 1 – Returns are generated by a factor model

In the APT, the return on security i, $r_{i,t}$, is priced as a linear product of K systematic risk factors, F, and an idiosyncratic risk factor, $e_{i,t}$.

$$r_{i,t} = a_i + \sum_{k=1}^{K} \beta_{i,k} F_{k,t} + e_{i,t} \tag{2.13}$$

where $E\left[e_{i,t}\right] = 0$, $cov\left[F_{k,t}, e_{i,t}\right] = 0$ and $cov\left[e_{i,t}, e_{j,t}\right] = 0$.

An equivalent way of writing Equation (2.13), that is also present within the literature on APT, is in terms of expectancy. We may take the expectation of Equation (2.13) and do some rewriting:

$$E[r_{i,t}] = E\left[a_i + \sum_{k=1}^{K} \beta_{i,k} F_{k,t} + e_{i,t}\right] \Leftrightarrow$$

$$E[r_{i,t}] = E[a_i] + E\left[\sum_{k=1}^{K} \beta_{i,k} F_{k,t}\right] + E[e_{i,t}] \Leftrightarrow$$

Because $E[a_i] = a_i$ and $E\left[e_{i,t}\right] = 0$ we get:

$$E[r_{i,t}] = a_i + E\left[\sum_{k=1}^{K} \beta_{i,k} F_{k,t}\right] \Leftrightarrow$$

$$E[r_{i,t}] = r_{i,t} - \left(\sum_{k=1}^{K} \beta_{i,k} F_{k,t} + e_{i,t}\right) + E\left[\sum_{k=1}^{K} \beta_{i,k} F_{k,t}\right] \Leftrightarrow$$

$$r_{i,t} = E\left[r_{i,t}\right] + \sum_{k=1}^{K} \beta_{i,k}\left(F_{k,t} - E\left[F_{k,t}\right]\right) + e_{i,t} \tag{2.14}$$

Where $\beta_{i,k}$ denotes the kth factor loading, $F_{k,t}$ is the systematic kth risk factor and $E\left[F_{k,t}\right]$ represents the expectation of the kth systematic risk factor. The expectation of the risk factor is with respect to information available at time $t-1$ or earlier. While the factors, $F_{k,t}$, are common impacts across the economy, and therefore are the same across securities, the sensitivity of each security towards the factors can differ. This sensitivity is given by the factor loading, $\beta_{i,k}$. One of the key assumptions of the factor structure underlying the APT is that the idiosyncratic risk components across securities and all time periods are uncorrelated, $cov[e_{i,t}, e_{j,t}] = 0$. Furthermore, the idiosyncratic risk component must also be independent of the risk factors, F, that is $cov\left[F_{k,t}, e_{i,t}\right] = 0$.

Nevertheless, one of the short comings of the APT as emphasized in the literature, is that the selection and number of risk factors in Equation (2.14) are ambiguous. As a consequence, Section 3.2 has been devoted to identifying the factors relevant for this study. Furthermore, Section 3.1 will discuss the statistical properties of models underlying a factor structure.

Assumption 2 – There exist enough stocks to eliminate idiosyncratic risk by diversification

The idea of diversification is one of the major premises of mean-variance portfolio theory as described in Section 2.1. In case security i is kept in a portfolio with securities that behave less alike (their price movements have low correlation with each other), we expect to see a reduction in total risk as a result of a decrease in the idiosyncratic risk component (Bodie et al., 2009). However, since we are not able to diversify the systematic risk component away, this is the only part we should be compensated for holding, i.e., the part that should be priced in an asset pricing model.

Consider the return on a portfolio, r_t^{Pf}, consisting of N securities, each with weight w_i, with K numbers of risk factors, $F_{k,t}$:

$$r_t^{Pf} = \sum_{i=1}^{N} w_i r_{i,t} = \sum_{i=1}^{N} w_i \left(E[r_{i,t}] + \sum_{k=1}^{K} \beta_{i,k}(F_{k,t} - E[F_{k,t}]) + e_{i,t} \right) \Leftrightarrow$$
$$r_t^{Pf} = \sum_{i=1}^{N} w_i E[r_{i,t}] + \left(\sum_{i=1}^{N} w_i \beta_{i,1}\right)(F_{1,t} - E[F_{1,t}]) +$$
$$\left(\sum_{i=1}^{N} w_i \beta_{i,2}\right)(F_{2,t} - E[F_{2,t}]) + \cdots + \sum_{i=1}^{N} w_i e_{i,t} \qquad (2.15)$$

We notice that the return on the portfolio, r_t^{Pf}, is a sum of the weighted average of expected returns, the weighted average of factor sensitivities times the innovation in the factors, and the weighted average of the idiosyncratic risk components.

Above, the residuals terms, e, were assumed to be uncorrelated across securities. Hence, for properly large values of N, $\sum_{i=1}^{N} w_i e_{i,t}$ will approach zero when $N \to \infty$ and thus be diversified away.

Therefore, we end up with an expression for the return on the portfolio:

$$r_t^{Pf} = \sum_{i=1}^{N} w_i r_{i,t} = \sum_{i=1}^{N} w_i E[r_{i,t}] + \left(\sum_{i=1}^{N} w_i \beta_{i,1}\right)(F_{1,t} - E[F_{1,t}]) +$$
$$\left(\sum_{i=1}^{N} w_i \beta_{i,2}\right)(F_{2,t} - E[F_{2,t}]) + \cdots + \left(\sum_{i=1}^{N} w_i \beta_{i,K}\right)(F_{K,t} - E[F_{K,t}]) \qquad (2.16)$$

Assumption 3 – Opportunities for arbitrage profits are traded away

The validity of the APT is proved by Ross (1976) through the absence of arbitrage, which implies that assets with similar cash flows in all states must have identical prices. The latter condition is also known as the Law of One Price, which is enforced by market participants engaging in arbitrage activity (arbitrageurs). More intuitively, in case arbitrageurs observe a violation of the Law of One Price they will simultaneously buy the asset where it is cheap and sell the asset where it is expensive. They will bid up the price where it is low and force it down where it is high, until the arbitrage opportunity is eliminated (Bodie et al., 2009).

Skovmand (2015) provides an intuitive derivation of the *No Arbitrage* condition. We apply his methodology below when proving the condition. Let us form a portfolio, Pf_0 with zero-investment, i.e., an arbitrage portfolio, meaning that

$$\sum_{i=1}^{N} w_i = 0 \tag{2.17}$$

because some stocks are held short, where the proceeds are invested in other securities. The return on the zero-investment portfolio must then be equivalent to Equation (2.16). Let us construct the portfolio to include no systematic risk:

$$\sum_{i} w_i \beta_{i,k} = 0 \quad \forall k \in [1, K] \tag{2.18}$$

This leaves us with the return on the portfolio as

$$r_t^{Pf_0} = \sum_{i=1}^{N} w_i r_{i,t} = \sum_{i=1}^{N} w_i E\left[r_{i,t}\right] \tag{2.19}$$

By construction the return $\sum_{i=1}^{N} w_i r_{i,t}$ is always equal to the expected return $\sum_{i=1}^{N} w_i E\left[r_{i,t}\right]$.

In the absence of arbitrage, a zero-investment portfolio with no systematic risk must have a zero return. Hence

$$\sum_{i=1}^{N} w_i E\left[r_{i,t}\right] = 0 \tag{2.20}$$

We are now able to define the exact condition that leads to the absence of arbitrage:

No Arbitrage Condition*: If we have a well-diversified portfolio with zero investment and zero systematic risk, then the return on the portfolio must be equal to zero.*

As a result of the no arbitrage condition, Campbell, Lo, & MacKinlay (1997) formulate the following model for the expected return underlying the APT:

$$E[r_{i,t}] = \lambda_0 + \sum_{k=1}^{K} \beta_{i,k}\,\lambda_k \tag{2.21}$$

Equation (2.21) can be derived based on linear algebra statements as shown by Danthine & Donaldson (2005). Equation (2.17), which concerns a zero-investment portfolio, implies that the vector of weights, w, is orthogonal to a vector of ones, **1**. Equation (2.18), which implies no systematic risk, states that the vector of weights, w, is orthogonal to the vectors of factor loadings, β. Therefore, it must true that the vector of ones, **1**, is orthogonal to the vectors of factor loadings, β. By assuming no arbitrage as in Equation (2.20), the vector of expected returns, $E[r]$, must be orthogonal to the vector of weights, w. A mathematical consequence is that the vector of expected returns, $E[r]$, can be expressed as a linear combination of the vector of ones, $\mathbf{1}$, and the vectors of factor loadings, β, as stated in Equation (2.21).

In Equation (2.21), λ_0 is the return on the risk-free asset, $\beta_{i,k}$, the sensitivity of asset i to the kth factor and λ_k, the risk premium associated with the kth factor. Ross (1976) states that in the case no risk-free asset exists, λ_0 is simply the return on a zero-beta portfolio, thus a portfolio where $\beta_{i,k} = 0$ for all i and k. One may notice that we did not include any i subscripts in λ_0 and λ_k, this is because these are constant across all securities and throughout time.

Campbell, Lo, & MacKinlay (1997) underline that for the relation in Equation (2.21) to strictly hold, the number of assets in the economy has to be approaching infinity. The model may therefore only be considered as approximate. They instead propose Equation (2.21) restriction as

$$E[r_{i,t}] \approx \lambda_0 + \sum_{k=1}^{K} \beta_{i,k}\,\lambda_k \tag{2.22}$$

Due to the fact that it is problematic to apply “approximate theories”, this thesis will assume that the strict, non-approximate version of Equation (2.22) holds, going forward.

3 – Factor Models

This section will elaborate on the factor structure underlying the APT as explained in Section 2.3. The section is structured first to describe the statistical implications of factor models and then identify a number of factor models, relevant for this study.

3.1 – Statistical Properties

First, it is essential to note that factor models have no theoretical foundation in the sense that they do not assume anything about a security's return in equilibrium. Bodie et al. (2009) suggest that factor models are merely a statistical assumption, hence why we need a theoretical framework to understand asset pricing in equilibrium. This is the reason why we choose first to explain APT and subsequently turn to the statistical properties of factor models.

With reference to Equation (2.13), a factor model explains the return on any security, $r_{i,t}$, as a linear function of K number of factors F, K number of factor loadings β, an intercept a_i and a residual $e_{i,t}$

$$r_{i,t} = a_i + \beta_{i,1}F_{1,t} + \cdots + \beta_{i,K}F_{K,t} + e_{i,t} \tag{3.1}$$

Equation (3.1) holds for all securities as well as for portfolios $i = 1,2, \dots, N$, and in all time periods $t = 1,2, \dots, T$. The factor, $F_{k,t}$, can relate to any economic variables found to explain $r_{i,t}$, where examples of variables may include growth in GDP, inflation and interest rates. Various ways exist for fitting factor models and Ruppert & Matteson (2010) note that conditional on the choice of the factor model, either the factor loadings, the factors, or both the factors and the loadings are unobservable parameters and must be estimated. Progression further, we have chosen to fit our factor models by time series regression due to its convenience in comparison to other approaches. In time series factor models, the factors are observable, while the factor loadings are the unobservable parameters to be estimated by regression.

As described in Section 2.3, the residuals of the factor model are mean zero random variables, which are assumed to be independent across securities and with the factors. The variance-covariance matrix of residuals, $\boldsymbol{\Sigma_e}$ is thus assumed to be diagonal. According to Ruppert & Matteson (2010), the variance-covariance matrix of returns in the factor model is derived as

$$\boldsymbol{\Sigma} = \boldsymbol{\beta}^{\boldsymbol{T}}\boldsymbol{\Sigma_F}\boldsymbol{\beta} + \boldsymbol{\Sigma_e} \tag{3.2}$$

With $\boldsymbol{\beta}$ being to the $K \times N$ matrix of the factor loadings, $\boldsymbol{\Sigma_F}$ the $K \times K$ variance-covariance matrix of the factors and $\boldsymbol{\Sigma_e}$ the $N \times N$ diagonal variance-covariance matrix of the residuals.

Throughout Section 2.1, we described Markowitz' mean-variance framework for determining optimal portfolios with respect to mean-variance efficiency. But what exactly are the benefits of using the factor structure for the estimation procedure of the inputs to the mean-variance Analysis?

First of all, Ruppert & Matteson (2010) suggest improved accuracy in the estimation of the variance-covariance matrix of returns. They imply that there exists a bias-variance tradeoff between estimating $\boldsymbol{\Sigma}$ using the factor structure, and not imposing a factor structure and merely using the sample variance-covariance matrix. While the sample variance-covariance matrix is unbiased, it consists of $N(N+1)/2$ estimates. Through each of these estimation procedures, we may incur errors. In the accumulation of these many errors, the result can be a considerable loss of precision.

In contrast, by estimating the variance-covariance matrix using the factor structure, we have to estimate $N * K$ factor loadings β, $K(K+1)/2$ parameters in the variance-covariance matrix of the factors $\boldsymbol{\Sigma_f}$ and N estimates in the diagonal variance-covariance matrix of residuals, $\boldsymbol{\Sigma_e}$. This sums to $N * K + N + K(K+1)/2$ parameter estimates. According to Ruppert & Matteson (2010), the number of stocks is usually large compared to the number of factors, $N > K$. The consequence is that $N * K + N + K(K+1)/2$ is much smaller than $N(N+1)/2$.

Based on our dataset, where $N = 188$, Table 3.1 shows the number of parameter estimates required for estimating the variance-covariance matrix using the factor structure (with $k = 1{,}3{,}4{,}5$) and the sample covariance matrix. We see that by imposing a factor structure on the returns, we get a significant reduction in the number of parameter estimates. Hence, without the application of a factor structure, we must estimate 17,766 parameters, while using the factor structure we only have to estimate 377, 758, 950 and 1,143 parameters for $k = 1, 3, 4$ and 5 factors respectively.

Table 3.1 - Number of Inputs to the Variance-Covariance Matrix:

Parameter Estimates:	**Total**
Sample Covariance	17,766
One Factor Structure (k = 1)	377
Three Factor Structure (k = 3)	758
Four Factor Structure (k = 4)	950
Five Factor Structure (k = 5)	1,143

Furthermore, Ruppert & Matteson (2010) advocate the advantage of expediency, understood as the convenience of having fewer parameters to estimate, and the ease with respect to updating these estimates. Suppose we have implemented a factor model including N securities and want to add another security to the model. In the case where no factor structure has been imposed on the variance covariance matrix, we must compute N sample covariances between the new security and the securities already included in the model. Thus, time series data for all of the N securities must be available. Evaluating Equation (3.2), we observe that in the case where a factor structure has been imposed on the variance-covariance matrix, we only require sufficient data to regress the new returns on the factors.

3.2 – Identification of Factors

Throughout the literature, various categories of factors in stock returns have been proposed. Nonetheless, academics do not agree on the theoretical justifications for the factors. According to Fama & French (2004), the explanations for the existence of factors commonly take two stances; the rational (risk-based) or the irrational (behavioral) explanation. In the first perspective, factors exist because asset pricing models are not capturing components of systematic risk. Abnormal returns thus exist owing to investors undertaking risks and capturing risk premia. In the second, factors exist because of investor overreaction. Thus, investors may act irrationally to information or possess certain psychological biases that affect how they process information (Fama & French, 2004). The theoretical framework that we will apply to form the analysis is the APT. Hence, we will focus more on the rational (risk-based) perspective going forward.

According to Munk (2017), two approaches commonly exist for identifying factors. The first includes pre-specifying theoretical factor models and testing whether they are empirically significant. The second constitutes identifying empirically significant factors and attempting to explain the rationale for why they should be priced factors.

Throughout Section 3.1, we specified that this thesis will only consider factor models fitted through time series regression. This has the consequence of limiting our universe of factors only to ones that are related to traded assets. Hence, in this section, we will only focus on factors that are tradable in financial markets. In this sense, the factor risk premiums are returns of *factor mimicking portfolios*, which are portfolios with returns that mimic the unexpected movements in the risk factors.

3.2.1 – Macroeconomic Factors

One may use economic theory to postulate a set of reasonable macro-economic variables that are good proxies for risks relevant to the investor. Chen, Roll, & Ross (1986) present such a factor model with macro-economic variables as common risk factors. They base their choice of factors on the formulation of the current price of a stock as the present value of the expected cash flows:

$$P_t = \sum_{\tau=1}^{\infty} \frac{E[CF_{t+\tau}]}{(1+r)^{\tau}} \tag{3.3}$$

Through this relation, Chen, Roll, & Ross (1986) suggest that common risk factors associated with returns should be variables which significantly impact expected cash flows, $E[CF_{t+\tau}]$, and/or discount rates, r. They consider inflation, interest rates, as well as indicators related to business-cycles as risk factors that may cause pervasive shocks to expected cash flows and/or discount rates.

Relevant common risk factors can also possibly be found in the literature for stock returns predictability. Rapach, Wohar, & Rangvid (2005) note that previously studied macro-economic variables among other things include the inflation rate, money stocks, aggregate output, the unemployment rate, interest rates, term spreads, and default spreads on bonds. Nevertheless, the macro-economic variables that are most commonly considered are the short-term interest rates (3-month T-bill and 1-month T-bill), the term spread and the default spread. Fama & French (1989) suggest that short-term interest rates can relate to current business conditions, while movements in the term spread and the default spread may relate to changes in the expectations about the short-term and long-term business conditions. Hence, these risk factors can be included in a factor model with great rationality.

Tradable representations of the macro-economic factors considered so far can, for example, be the return of inflation-linked bonds for a *real interest rate factor* or the return of a portfolio consisting of long nominal bonds and short inflation-linked bonds for an *inflation factor*. These representations can be found as Exchange Traded Funds (ETFs) of asset class indices. The benefits of specifying a factor model with such macroeconomic variables as factors include the theoretical attractiveness of explaining and interpreting the factors' influence on stock returns. Nevertheless, issues exist with regards to low explanatory power, which has been highlighted, as one of the major shortcomings of this class of factors (Ruppert & Matteson, 2015).

Single-Index Model

The Single-Index Model is a well-established macroeconomic factor model developed by Sharpe (1963).[1] The model is meant as an extension to the works of Markowitz (1952) to provide a simplified model of the relationship between stocks, and to provide evidence for the practical applications of the techniques of Markowitz (Sharpe, 1963). The model describes a linear relationship between the return of asset i and a single common underlying factor explaining the systematic risk affecting all asset returns. The single factor is assumed to equal the return on a market portfolio, or more specifically the return on a broad stock market index (as a proxy for the market). The formal definition of the Single-Index Model is

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_{i,M}\left(r_{M,t} - r_{f,t}\right) + e_{i,t} \tag{3.4}$$

Equation (3.4) states that the excess return on any stock, $r_{i,t} - r_{f,t}$, can be decomposed into an intercept, α_i, that is the excess return independent of the market's performance, the asset's sensitivity to market movements, $\beta_{i,M}$, the market risk premium, described by the excess return on the market, $r_{M,t} - r_{f,t}$, and the idiosyncratic risk due to firm-specific factors, $e_{i,t}$. The rationale for the Single-Index Model is the simple notion that stocks are driven by the same economic influences. Hence, Equation (3.4) provides a relatively simple tool to quantify the forces driving assets' returns. The specification in Equation (3.4) is closely related to that of the CAPM in Section 2.2, yet the Single-Index Model is merely a statistical technique, while the CAPM is an economic equilibrium theory as highlighted by Bodie et al. (2009).

Despite its popularity, Connor & Korajczyk (2010) suggest that in practice, the Single-Index Model does not describe all of the common variability across stocks. They point out that the simple separation of risk into two sources; systematic and idiosyncratic, may be an oversimplification. They suggest that there seem to be additional benefits from using a model with multiple factors. The following section will, therefore, describe several of the most well-known multi-factor models, belonging to the branch of fundamental factor models.

[1] While Sharpe (1963) denotes the Single-Index Model the *Diagonal Model*, the Single-Index Model is also often referred to as the *Market Model.*

3.2.2 – Fundamental Factors

Another way of modeling factors of security returns is by constructing a fundamental-based factor model. Fundamental-based factor models use observable asset characteristics (fundamentals) as factors. One of the first to suggest the use of standard accounting ratios (book-to-price ratios and market value of equity) in factor models were Rosenberg (1974).

Throughout the literature, other authors have also found that fundamental ratios may play a role as factors in explaining returns and that market beta is not the only risk factor. Banz (1981) notes that by sorting stocks on market capitalization, small stocks have higher average returns than what market betas suggest, calling for a size effect. Basu (1977) finds that when sorting stocks on earnings-to-price (E/P) ratios, future returns on stocks with high earnings-to-price ratios are higher than what market betas indicate. In addition, Stattman (1980) and Rosenberg, Reid, & Lanstein (1985) find that stocks with high book-to-market equity (BVE/MVE) ratios have higher average returns than what market betas suggest, calling for a value effect. The findings of Bhandari (1988) advocate that high debt-equity (BVD/MVE) ratios lead to returns that are too high relative to market betas.

The findings of the above-mentioned studies suggest that stock prices include information about expected returns left out by the market beta. This suggests that fundamental factors of E/P, BVE/MVE and BVD/MVE indeed play a role in explaining variability in stock returns and the possibility that multifactor models based on asset characteristics allow for a richer explanation of stock returns.

Fama-French Three-Factor Model

Fama & French (1993) find that stock returns of firms with a low market capitalization (small stocks) covary more with one another than stock returns of firms with a high market capitalization (big stocks). Moreover, they notice that returns on stocks with high book-to-market ratios (value stocks) covary more with one another than stocks with low book-to-market ratios (growth stocks). They argue that there are systematic risks involved with investing in the stocks of small firms and value stocks, not captured by the market factor, thus requiring additional risk factors.

In several influential papers, Fama & French (1992, 1993, 1996) propose an estimation procedure for fundamental factor models, where assets (or portfolios) are sorted into groups based on certain characteristics. More specifically they form six portfolios based on market capitalization (two groups) and book-to-market ratios (three groups). For the ease of understanding the sorting, the reader may see Figure 3.1 which illustrates the sorting methodology.

Figure 3.1 – Fama-French Three-Factor Model Portfolio Sort:

The SMB factor (Small minus Big) is constructed based on market capitalization as the difference between average returns of firms with a market capitalization below the median market capitalization (Small firms) and average returns of firms with a market capitalization above the median market capitalization (Big firms).

The HML factor (High minus Low) is constructed based on the book-to-market ratio as the difference between average returns of firms with a book-to-market ratio higher than the 70th percentile (Value), and average returns of firms with a book-to-market ratio lower than the 30th percentile (Growth).

Thus, Fama & French (1993) effectively construct mimicking portfolios for their factors since SMB and HML are the returns on portfolios that have long positions in one group of stocks and short positions in another group. As a result, they formulate the following three-factor model:

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_{i,M}\left(r_{M,t} - r_{f,t}\right) + \beta_{i,SMB}SMB_t + \beta_{i,HML}HML_t + e_{i,t} \tag{3.5}$$

where the factors constitute a market factor, a small-minus-big factor and a high-minus-low factor.

Fama & French (1996) find that their three-factor model fits the U.S. stock market well over the period 1963 – 1993, the same conclusion is found with regards to many other countries (Fama & French, 2012). Nonetheless, Fama & French (2004) acknowledge the shortcomings of the model from a theoretical perspective. These shortcomings include empirical motivation for the factors since their relation to risks relevant to the investor is unclear.

Fama & French (1996) suggest that the ability of the model to explain returns should be attributed to a premium on financial distress. There is a tendency that small value stocks are firms with poor past performance, thus more likely to experience future financial distress. More commonly, small firms may be more vulnerable to recessions. In the event of a recession, investors value high returns the most, hence investors would require a higher expected return on investments in stocks of small firms. In general, value firms will have a more significant part of tangible assets, meaning that in recessions they may experience excess capacity, which demands a return. In contrast, growth firms rely on growth prospects for the future, which can more easily be deferred. It is thus the pro-cyclicality of stocks on value firms that demands a higher expected return. Contrary, growth stocks and large stocks may have lower expected returns because they provide better protection against recessions (Munk, 2017).

Carhart Four-Factor Model

Jegadeesh & Titman (1993) discover that stocks which perform well relative to the market over the last 3 to 12 months tend to continue to perform well for the next few months, while the opposite applies for stocks that perform poorly. Hence, they conclude that stocks hold short-term momentum in returns. Acknowledging this, Carhart (1997) extends the Fama-French Three-Factor Model in Equation (3.5) to include a fourth momentum factor, UMD:

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_{i,M}\left(r_{M,t} - r_{f,t}\right) + \beta_{i,SMB}SMB_t + \beta_{i,HML}HML_t + \beta_{i,UMD}UMD_t + e_{i,t} \qquad (3.6)$$

The UMD factor (Up minus Down) is constructed based on stock returns over the last 2 to 12 months as the difference between average returns of the firms with stock returns higher than the 70th percentile (firms that have recently gone *up*) and average returns of the firms with stock returns lower than the 30th percentile (firms that have recently gone *down*). [2]

Hvidkjaer (2006) finds evidence for a relation between the momentum effect and liquidity problems. On the other hand, Avramov, Chordia, Jostova, & Philipov (2007) emphasize that momentum profitability is large and significant among firms of low credit quality. Hence, an economic rationale for the momentum factor may be that it captures liquidity risk or credit risk. Nevertheless, Daniel & Moskowitz (2016) underline the issues related to employing a momentum-based strategy. They find that such a strategy will eventually crash and give very negative returns following market downturns and through periods of heavy volatility.

[2] As we shall see in Section 4.1, we use the Momentum Factor (MOM) from the data library of French (2019), hence we describe his methodology for creating the factor.

Fama-French Five-Factor Model

Evidence provided by Novy-Marx (2013) and Titman, Wei, & Xie (2004) among others, highlight the inability of the three-factor model to explain average returns since the three factors do not account for variation in average returns related to profitability and investment. Novy-Marx (2013) finds that stocks of firms with high profitability have higher returns than stocks of firms with low profitability. Thus, he finds that gross profitability as measured by a firm's gross profits relative to its asset base is a powerful predictor of the cross-section of average returns and that it has as much explanatory power as the book-to-market ratio. Moreover, Titman, Wei, & Xie (2004) examine the relationship between increases in capital investments and succeeding stock returns. Their findings are that firms that increase their level of capital investment aggressively tend to have lower stock returns.

Motivated by this evidence Fama & French (2015) include two additional risk factors, *profitability*, and *investment* to their model extending the three-factor model in Equation (3.5) to a five-factor model:

$$\begin{aligned} r_{i,t} - r_{f,t} = \alpha_i + \beta_{i,M}\left(r_{M,t} - r_{f,t}\right) + \beta_{i,SMB}SMB_t + \\ \beta_{i,HML}HML_t + \beta_{i,RMW}RMW_t + \beta_{i,CMA}CMA_t + e_{i,t} \end{aligned} \tag{3.7}$$

The RMW factor (Robust minus Weak) is constructed based on profitability, as the difference between average returns of firms showing good profitability (robust firms) and average returns of firms showing bad profitability (weak firms). With reference to Fama & French (2015), the sorting ratio for profitability is calculated as

$$OP_{t-1} = \frac{(Revenue - COGS - SG\&A - Interest\ Expense)_{t-1}}{BVE_{t-1}} \tag{3.8}$$

and shows that profitability is measured as the ratio of revenue minus cost of goods sold (COGS), minus selling, general and administrative expenses, minus interest expenses relative to the book value of equity. The subscript $t - 1$ indicates that accounting data for the fiscal year ending in year $t - 1$ is used.

The CMA factor (Conservative minus Aggressive) is constructed based on firms' degree of investment, as the difference between average returns of firms with low investment (conservative firms) and average returns of firms with high investment (aggressive firms). With reference to Fama & French (2015), the sorting ratio for investment is calculated as

$$Inv_{t-1} = \frac{TA_{t-1} - TA_{t-2}}{TA_{t-2}} \tag{3.9}$$

and shows that investment is measured as the relative change in the firm's total assets from the fiscal year ending in year $t-2$ to the fiscal year ending in $t-1$.

When extending the three-factor model to include the RMW factor and the CMA factor, Fama & French (2015) note that the HML factor becomes redundant, in the sample they consider. This is because the average return of the HML factor is captured by its exposures to the other factors. In instances, where abnormal returns (as measured by the intercept, α) are the only concern, they suggest that a four-factor model excluding the HML factor may perform just about the same as the five-factor model.

3.3 – Sub-Conclusion

In the discussion of the prevailing explanation for the presence of factors in stock returns, two views are commonly taken within the finance literature. These include the rational (risk-based) explanation and the irrational (behavioral) explanation. The former encompasses that assets have different expected returns because they vary more or less with numerous priced factors, where the latter includes that factors are merely anomalies caused by investor overreaction.

Factors in stock returns can be identified by following various procedures. In this section, we have considered two approaches, which includes pre-specifying a theoretically founded factor model and testing it for empirically significance or discovering empirically significant factors and justifying them afterwards.

Macroeconomic factors are the first branch of factors that we have considered. While these factors seem like good candidates due to their strong intuitive appeal, they have generally been shown to explain stock returns poorly. Nevertheless, a macroeconomic factor model that we find relevant for this study is the Single-Index Model, which is an established factor model within the finance literature, that includes a broad market index as the single factor.

The second type of factors that we have reviewed is fundamental factors, which are factors based on simple accounting ratios and firm characteristics. A commonly applied approach for constructing such factors are the sorting methodology of Fama & French (1993). By sorting stocks on certain firm characteristic variables, factors are represented as returns of factor-mimicking portfolios. Fama & French (1993) specify their three-factor model to include a market, size and value factor. Carhart

(1997) extends this model to a four-factor model by including a momentum factor. More recently Fama & French (2015) have included a profitability factor and an investment factor in their original three-factor model. The models have been shown to explain average returns well and therefore possibly capture underlying risk premiums. However, these are more difficult to motivate theoretically.

In conclusion, we find that the Single-Index Model, the Fama-French Three-Factor Model, the Carhart Four-Factor Model, and the Fama-French Five-Factor Model pose relevant for the mean-variance factor framework that we intend to implement moving forward.

4 – Empirical Study

Following the theoretical frameworks and the review on factor models, this section will lay the foundation for answering the proposed research questions in order to answer the overall problem statement. We will do this by describing our data sample, how the data is gathered and how the data sample has been prepared for the analysis. Furthermore, this section will form the analysis by presenting the construction of our portfolios, and the basis for the portfolio backtesting and performance evaluation. Lastly, an initial assessment of the data sample will be made, where possible econometric issues will be considered.

4.1 – Data Sample

Since we have chosen to implement and test our framework on the U.S. stock market, we limit our stock-universe to only include components of the Standard & Poor's 500 Index (S&P 500 Index) at a particular point in time. The S&P 500 Index is an American stock market index that includes the 500 largest U.S. publicly traded companies. It is regarded as a good representation of large-cap U.S. equities and covers more than 80% of the U.S. equity market in terms of market capitalization (S&P Dow Jones, 2019a). The S&P 500 index is weighted based on market capitalization and the shares included in the index are "free-floating" shares, which means that they can be held by the general public.

According to S&P Dow Jones (2019b), the S&P 500 Index differs from other major indices by its diverse base of constituents and weighting practice. The constituents of the S&P 500 Index are chosen via a committee, and when considering a new candidate for the index, the committee employs eight primary criteria for inclusion. S&P Dow Jones (2019c) documents that these criteria comprise:

- Market Capitalization
- Liquidity
- Domicile
- Public Float
- Sector Classification
- Financial Viability
- Length of Time Publicly Traded
- Stock Exchange

S&P Dow Jones (2019c) reports that the companies in the S&P 500 Index are selected by the committee, in such a way that they are representative of the industries in the economy of the U.S. Furthermore, the S&P 500 Index is one of the most commonly followed equity indices and highly regarded as a proxy for the U.S. Equity Market (S&P Dow Jones, 2019b). For these reasons, we feel comfortable using the S&P 500 index as the investment universe of this study.

Similarly, we choose only to include factors related to the U.S. economy. These factors constitute the factors of the Fama-French Three-Factor Model, the Carhart Four-Factor Model, and the Fama-French Five-Factor Model. The methodology for the construction of these factors was described in Section 3.2 and will consequently not be presented here.

4.1.1 – Data Gathering

For the stock data, we gather a constituent list of the S&P 500 Index for the specific date of 01-01-2019. This is done using Thomson Reuters (2019) by applying the code *"S&PCOMP"*. Based on this list we extract a time series of daily closing prices and name for each respective constituent. We choose to use Thomson Reuters (2019) to extract the data, since this data-provider gives us the explicit option of adjusting prices for corporate actions (dividends, stock splits, share buyback, etc.), and thus obtain adjusted closing prices. The time interval for the time series has been chosen as the base date (earliest date available), 02-01-1973 to 01-01-2019. We take this approach to ensure that we have at least several periods of 60 months data when forming the analysis, to perform rolling-window regressions. The importance of using at least 60 months of data for each regression is suggested by Cuthbertson & Nitzsche (2004) in order to obtain reliable beta estimates.

Furthermore, we gather data on our factors from the data library of French (2019). From here, we obtain time series of returns of the *Fama/French 3 Factors*, *Momentum Factor (Mom) and Fama/French 5 Factors*. For the factors, we take a similar approach and extract the maximum data available. The earliest date available for the time series of returns is 07-1963, and the latest is 01-2019.

We also obtain time series on the risk-free rate from Kenneth French's data library. This risk-free rate is the one-month Treasury bill rate from Ibbotson Associates.

4.1.2 – Preparation of the Data Sample

The management of our data and preparation of the data sample have mainly been done in Excel.

The return r for stock i at time t is given by

$$r_{i,t} = \frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}} \tag{4.1}$$

Where $P_{i,t}$ and $P_{i,t-1}$refer to the closing prices of security i at time t and $t-1$, corresponding to the first day of month t and the first day of month $t-1$. Monthly returns are calculated throughout our data sample, from January 1974 to January 2019. Due to the fact that each parameter estimation requires 60 months of return observations, we will run the portfolio backtesting and performance evaluation from January 1979 to January 2019.

The constituents of the S&P 500 Index that we gather for the first day of January 2019 have a different number of past observations, due to differences in their time being included in the index. We have therefore chosen to reduce the data sample to only include stocks, with the same number of past returns. This has been done to make the rolling estimation procedure feasible and the construction of the portfolios within the scope of this thesis. As a consequence, the number of stocks in the data sample is reduced from 500 to 188. Nevertheless, we are aware of that this creates a potential survivorship bias in our data sample. This issue will be discussed in detail in Section 7.1. Likewise, we reduce the time series of the factors and the risk-free rate accordingly, so they have the same length as the time series of the stock returns.

As a result of the reduction, we find it relevant to gain an understanding of how the "prepared" data sample is structured and which stocks it includes. Firstly, the stocks do supposedly fulfill the selection criteria for the S&P 500 Index mentioned above. Hence, they must have a market capitalization above a certain level, be liquid stocks, be common stocks of U.S. Companies, be held in the hands of public investors, have a sector classification, be stocks of financially viable companies and have a particular history of being publicly traded. Secondly, Appendix A adds an industry classification to each stock in the final data sample and constructs an industry decomposition illustrated in a treemap. We observe that the final data sample is dominated by 36 companies in industrials (19%), 26 firms in financials (14%), 23 companies in consumer discretionary (12%), 22 firms in consumer staples (12%) and 22 companies in utilities (12%). Hence, the firms within these industries make up the majority of the final data sample.

Putting this into perspective, we find that the "unprepared" data sample does also include a high concentration of companies in financials. In contrast, the highest industry weights of the "unprepared" data sample are found within information technology and health care. The most significant changes in the industry composition, due to the way that we manage the data sample, are thus in the industries of information technology and health care. This may also seem obvious, since only including stocks with time series of returns dating back to the 1970s, will eliminate a large part of technology and healthcare firms. On the other hand, the "prepared" data sample includes a higher concentration of companies within industrials and utilities.

Nevertheless, we still deem the final data sample representative and appropriate for answering the problem statement of this study. These insights will pose valuable in the backtesting and performance evaluation when we will make inferences about the portfolios we intend to construct. We will move on to form the analysis.

4.2 – Forming the Analysis

4.2.1 – Portfolio Construction

The following section will present how we construct our portfolios, where the derivation of expected returns, the variance-covariance matrix, and portfolio weights are described in detail. Thus, for each portfolio, we will describe the procedure for estimating the parameters and afterwards portfolio choice. The portfolios we want to construct include a portfolio based on the traditional approach to mean-variance analysis, which we denote the *naïve Markowitz portfolio*, a number of portfolios based on a factor approach to mean-variance analysis, which we call *factor Markowitz portfolios* and a portfolio based on equal portfolio weights, which we denote the *equally weighted portfolio*.

For each of the portfolios, we intend to end up with a time series of returns in order to backtest and evaluate the performance of the portfolios. To arrive at this output, we apply a rolling input-estimation procedure on 60 months of past data, which is equivalent to five years. The argument for applying a window of this length was proposed in Section 4.1, as a way to get more reliable parameter estimates. Perhaps, another more economically intuitive rationale for taking such an approach is that companies may undergo restructuring, for instance through mergers and acquisitions, divestments, etc. Companies can change significantly over time, and we would like to have our estimates account for this fact.

The implementation of the portfolio construction is done in the programming language R for statistical computing from R Core Team (2019), and the programming code has been included in Appendix J. Nonetheless, the objective of this section is to describe the implementation in such a matter, that it could be implemented in the reader's software of choice.

For the construction of either of our portfolios, we define a matrix of returns for the 188 stocks in our dataset. The columns of the matrix correspond to a series of returns for each of the N stocks, while the rows represent time:

$$\boldsymbol{r} = \begin{bmatrix} r_{1,1} & \cdots & r_{1,N} \\ \vdots & \ddots & \vdots \\ r_{T,1} & \cdots & r_{T,N} \end{bmatrix} \tag{4.2}$$

The rationale for doing this is the possibility of conducting matrix operations, and because it makes the process of doing rolling operations through loops easier.

Following these definitions, we will elaborate on the procedure for each of the portfolios separately.

The Naïve Markowitz Portfolio

The construction of the naïve Markowitz portfolio is based on the optimization problem formulated in Section 2.1. Hence the inputs consist of the expected returns and the variance-covariance matrix for all assets. We will apply the most common approach for obtaining these inputs, which is through estimation from historical sample data. With reference to the portfolio being *naïve*, we will estimate the expected returns, and the variance-covariance matrix based on the sample mean and the sample variance-covariance matrix, respectively.

More specifically, the expected return of stock i is estimated as the simple arithmetic average of the past 60 monthly returns:

$$E[r_i] = \frac{1}{T}\sum_{t=1}^{T} r_{i,t} \tag{4.3}$$

which we bind into a vector of expected returns

$$\mu = \begin{bmatrix} E[r_1] \\ \vdots \\ E[r_N] \end{bmatrix} \tag{4.4}$$

Similarly, the variance-covariance matrix for all stocks is constructed based on the sample variance and sample covariance of the past 60 monthly returns. The sample variance of stock i is given by

$$\sigma_i^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(r_{i,t} - E[r_i]\right)^2 \tag{4.5}$$

while the sample covariance between stock i and stock j is computed by

$$\sigma_{i,j} = \frac{1}{T-1}\sum_{t=1}^{T}\left(r_{i,t} - E[r_i]\right)\left(r_{j,t} - E[r_j]\right) \tag{4.6}$$

The sample variance is located in the diagonal of the matrix, while the sample covariance is located in the off-diagonal, resulting in the following matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{1,1}^2 & \cdots & \sigma_{1,N} \\ \vdots & \ddots & \vdots \\ \sigma_{N,1} & \cdots & \sigma_{N,N}^2 \end{bmatrix} \tag{4.7}$$

With these inputs at hand, we utilize the closed-form solution for the tangency portfolio, formulated in Section 2.1, to compute the portfolio weights for the naïve Markowitz portfolio:

$$w_{NMP} = \frac{1}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\left(\mu - r_f\mathbf{1}\right)}\boldsymbol{\Sigma}^{-1}\left(\mu - r_f\mathbf{1}\right) \tag{4.8}$$

However, as we apply Equation (4.8), we encounter problems of singularity with respect to our variance-covariance matrix. As stated in Section 2.1, the derivation of the portfolio weights implies non-singularity of the variance-covariance matrix or said differently, the existence of an inverse of the matrix. To understand why singularity is a problem, we will define what a singular matrix is below.

Sydsæter, Strøm, & Berck (2005) state that a matrix $\boldsymbol{\Sigma}$ is said to be singular if its determinant is equal to zero, formulated as

$$|\boldsymbol{\Sigma}| = 0$$

The determinant of a matrix can be defined as:

$$|\boldsymbol{\Sigma}| = \phi_1 * \phi_2 \ldots \phi_{N-1} * \phi_N$$

Where ϕ is satisfying:

$$\boldsymbol{\Sigma}c = \phi c$$

The parameter ϕ is called the eigenvalue of matrix $\boldsymbol{\Sigma}$ and c is an eigenvector.

Matrix $\mathbf{\Sigma}$ is then said to be invertible ($\mathbf{\Sigma^{-1}}$) if the determinant of the matrix is different from zero, thus

$$|\mathbf{\Sigma}| \neq 0$$

Looping through our estimated variance-covariance matrices and applying Equation (4.8), we note that it is generally true that the inverse of the matrices does not exist due to singularity. In particular, we observe a limited number of positive eigenvalues. Hence, to make the optimization procedure feasible, we would have to reduce our data sample even further.

Ledoit & Wolf (2003) report similar problems with respect to the sample variance-covariance matrix. More specifically they underline that when the number of stocks N, is larger than the number of time series observations T (length of the estimation window), the sample variance-covariance matrix is always singular, although the true variance-covariance matrix is assumed to be non-singular. Likewise, Cornuejols & Tütüncü (2007) find that when solving large mean-variance optimization problems, the variance-covariance matrix is almost singular.

In our application, we have $N = 188$ and $T = 60$ for each estimation iteration, where N is particularly larger than T, which may be a reason for why we encounter problems. As a consequence, we will impose a factor structure on the variance-covariance matrix, a solution which is also suggested by Ledoit & Wolf (2003). In the following section, we will thus specify several factor models and describe the implementation of a factor-based mean-variance analysis.

The Factor Markowitz Portfolios

The construction of the factor Markowitz portfolios is based on the procedure of Section 3.1, where the statistical properties of factor models were reviewed. As noted previously, we intend to construct portfolios based on the Single-Index Model, the Fama-French Three-Factor Model, the Carhart Four-Factor Model, and the Fama-French Five-Factor Model. This will be done by imposing a factor structure on the expected returns and the variance-covariance matrix.

The first step in the estimation procedure is to run regressions of the excess returns of each stock on the relevant factors. The regressions will take the form of the definitions of the factor models presented in Section 3.2:

The Single-Index Model:

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_{i,M}(r_{M,t} - r_{f,t}) + e_{i,t} \tag{4.9}$$

The Fama-French Three-Factor Model:

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_{i,M}\left(r_{M,t} - r_{f,t}\right) + \beta_{i,SMB}SMB_t + \beta_{i,HML}HML_t + e_{i,t} \tag{4.10}$$

The Carhart Four-Factor Model:

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_{i,M}\left(r_{M,t} - r_{f,t}\right) + \beta_{i,SMB}SMB_t + \beta_{i,HML}HML_t + \beta_{i,UMD}UMD_t + e_{i,t} \tag{4.11}$$

The Fama-French Five-Factor Model:

$$\begin{gathered} r_{i,t} - r_{f,t} = \alpha_i + \beta_{i,M}\left(r_{M,t} - r_{f,t}\right) + \beta_{i,SMB}SMB_t + \\ \beta_{i,HML}HML_t + \beta_{i,RMW}RMW_t + \beta_{i,CMA}CMA_t + e_{i,t} \end{gathered} \tag{4.12}$$

Practically, we make use of (4.2) and construct a new matrix of excess returns by subtracting a vector of risk-free rates as

$$\boldsymbol{r^e} = \begin{bmatrix} r_{1,1} & \cdots & r_{1,N} \\ \vdots & \ddots & \vdots \\ r_{T,1} & \cdots & r_{T,N} \end{bmatrix} - \begin{bmatrix} r_{f,1} \\ \vdots \\ r_{f,T} \end{bmatrix} \tag{4.13}$$

Moreover, we define vectors of the relevant factors

$$\begin{bmatrix} r_{M,1} \\ \vdots \\ r_{M,T} \end{bmatrix} - \begin{bmatrix} r_{f,1} \\ \vdots \\ r_{f,T} \end{bmatrix} = M^e = \begin{bmatrix} r^e_{M,1} \\ \vdots \\ r^e_{M,T} \end{bmatrix}, SMB = \begin{bmatrix} SMB_1 \\ \vdots \\ SMB_T \end{bmatrix}, HML = \begin{bmatrix} HML_1 \\ \vdots \\ HML_T \end{bmatrix}$$

$$UMD = \begin{bmatrix} UMD_1 \\ \vdots \\ UMD_T \end{bmatrix}, RMW = \begin{bmatrix} RMW_1 \\ \vdots \\ RMW_T \end{bmatrix}, CMA = \begin{bmatrix} CMA_1 \\ \vdots \\ CMA_T \end{bmatrix}$$

We regress the past 60 months of excess returns for each stock onto the past 60 monthly observations of the factors. We follow this procedure until we have conducted 480 regressions, since we want to evaluate our portfolios from January 1979 to January 2019.

From each regression, we extract the factor loadings and bind these into the following matrix

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_{1,1} & \cdots & \beta_{1,N} \\ \vdots & \ddots & \vdots \\ \beta_{K,1} & \cdots & \beta_{K,N} \end{bmatrix} \tag{4.14}$$

In line with the theoretical premise of uncorrelated error terms, we also extract the residuals from each regression, calculate the sample variances of the residuals, and similarly bind these to produce the following diagonal matrix

$$\mathbf{\Sigma_e} = \begin{bmatrix} \sigma_{e_1}^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sigma_{e_N}^2 \end{bmatrix} \tag{4.15}$$

Where $\sigma_{e_i}^2$ denotes the variance of the residuals from regression i. Moreover, sample variances and sample covariances are computed for all relevant factors, and bound to construct the following matrix:

$$\mathbf{\Sigma_F} = \begin{bmatrix} \sigma_{F\,1,1}^{\;2} & \cdots & \sigma_{F\,1,K} \\ \vdots & \ddots & \vdots \\ \sigma_{F\,K,1} & \cdots & \sigma_{F\,K,K}^{\;2} \end{bmatrix} \tag{4.16}$$

With $\boldsymbol{\beta}$, $\mathbf{\Sigma_e}$and $\mathbf{\Sigma_F}$ at hand, we calculate the variance-covariance matrix of our stocks, by applying Equation (3.2):

$$\mathbf{\Sigma} = \boldsymbol{\beta}^T \mathbf{\Sigma_F} \boldsymbol{\beta} + \mathbf{\Sigma_e} = \begin{bmatrix} \sigma_{1,1}^2 & \cdots & \sigma_{1,N} \\ \vdots & \ddots & \vdots \\ \sigma_{N,1} & \cdots & \sigma_{N,N}^2 \end{bmatrix} \tag{4.17}$$

In order to compute the expected return for each stock, we must define APT equivalent formulations of the aforementioned factor models. A crucial assumption of the equilibrium versions of these models is that α_i will equal zero. In accordance with Equation (2.21), we therefore write

The Single-Index Model:

$$E[r_i] = r_f + \beta_{i,M}\big(E[r_M] - r_f\big) \tag{4.18}$$

The Fama-French Three-Factor Model:

$$E[r_i] = r_f + \beta_{i,M}\big(E[r_M] - r_f\big) + \beta_{i,SMB}E[SMB] + \beta_{i,HML}E[HML] \tag{4.19}$$

The Carhart Four-Factor Model:

$$E[r_i] = r_f + \beta_{i,M}\big(E[r_M] - r_f\big) + \beta_{i,SMB}E[SMB] + \beta_{i,HML}E[HML] + \beta_{i,UMD}E[UMD] \tag{4.20}$$

The Fama-French Five-Factor Model:

$$\begin{aligned} E[r_i] = r_f + \beta_{i,M}\big(E[r_M] - r_f\big) + \beta_{i,SMB}E[SMB] + \\ \beta_{i,HML}E[HML] + \ \beta_{i,RMW}E[RMW] + \beta_{i,CMA}E[CMA] \end{aligned} \tag{4.21}$$

To compute the expectation of each factor, we calculate the simple arithmetic average of the past 60 months of factor observations. With the factor loadings, the expectation of factors and the risk-free rate at hand, we calculate expected returns and bind these into a vector

$$\mu = \begin{bmatrix} E[r_1] \\ \vdots \\ E[r_N] \end{bmatrix} \tag{4.22}$$

Progressing further, we apply the closed-form solution for the tangency portfolio, to compute the portfolio weights for each factor Markowitz portfolio:

$$w_{FMP} = \frac{1}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1}\left(\mu - r_f \mathbf{1}\right)} \boldsymbol{\Sigma}^{-1}\left(\mu - r_f \mathbf{1}\right) \tag{4.23}$$

However, as we apply Equation (4.23) on the expected returns and the variance-covariance matrices through time, we observe questionable outputs. We see instances where the portfolio weights are very large with positive signs, and others are very large with negative signs.

Appendix B shows scatterplots of the weights for each factor Markowitz portfolio through time. While the portfolio weights of the Single-Index Markowitz portfolio do not show any extreme behavior, the weights across the multifactor Markowitz portfolios do indicate high positions in certain stocks, at specific points in time. Taking a practical perspective, such extreme weights would imply unrealistic use of leverage, despite that our weights are restricted to sum to one.

Feng & Palomar (2016) encounter similar problems and emphasize that in case short selling is allowed, one needs to limit the amount of leverage to avoid ridiculous weights. Therefore, we intend to implement an optimization procedure that will prohibit the extreme behavior of the portfolio weights by imposing certain constraints. Hence, in the next section, we will consider how this can be formulated and solved as a convex optimization problem.

The Constrained Factor Markowitz Portfolios

Henceforward, we will denote the former (unstable) solution, the *unconstrained* solution, and the implementation we consider below, the *constrained* solution. The constrained solution is carried out following Feng & Palomar (2016) and implemented using Fu, Narasimhan, & Boyd (2017).

We have decided to impose the constraint of not allowing short selling, in this case, the portfolio weights are restricted from being negative. We find this a not too unrealistic restriction since some investors face such a constraint in practice. We have also decided not to put further constraints on the portfolio weights, given that we want to end up with a solution as close to the solution given by Equation (4.23).

Continuing with the factor-based parameter estimates found above, we want to solve the following optimization problem:

$$\max_{w} \frac{w^T\mu - r_f}{\sqrt{w^T\mathbf{\Sigma}w}} \tag{4.24}$$

$$s.t.\ w^T\mathbf{1} = 1$$

Hence, the objective is to maximize the Sharp Ratio. Y. Feng & Palomar (2016) state that since the Sharpe ratio is non-concave, (4.24) is not a convex problem. Instead, they suggest rewriting the problem in convex form. Since we have $w^T\mathbf{1} = 1$, the problem can be expressed as

$$\max_{w} \frac{w^T(\mu - r_f\mathbf{1})}{\sqrt{w^T\mathbf{\Sigma}w}} \tag{4.25}$$

$$s.t.\ w^T\mathbf{1} = 1$$

Stated in this way, the objective function becomes scale invariant with respect to w. Hence, we can adjust the constraint $w^T\mathbf{1} = 1$ to $w^T\mathbf{1} > 0$ and set $w^T(\mu - r_f\mathbf{1})$ equal to an expected return target. We let the target be the mean of each period's expected returns. This is done to make the expected return target adjust in each period in order to prevent numerical errors and make the optimization feasible.

As a result of this, we can further rewrite the problem into a convex form:

$$\min_{w} w^T\mathbf{\Sigma}w \tag{4.26}$$

$$w^T(\mu - r_f\mathbf{1}) = \overline{(\mu - r_f\mathbf{1})}$$

$$w^T\mathbf{1} > 0$$

To impose the restriction of no short sales, we specify that $w \geq 0$ must apply to the solution. However, since we relaxed the constraint of $w^T\mathbf{1} = 1$, the weights we obtain may not necessarily sum to one. Recall, that we formulated the objective function to be scale invariant with respect to w. In order to force the weights to sum to one, the solution must be normalized through

$$\mathrm{w}_{CFMP} = \frac{w_i}{\sum_{i=1}^{N} w_i} \tag{4.27}$$

However, we must emphasize, that the solution is not the same tangency portfolio as found by Equation (4.23). The solution we obtain is a maximum Sharpe ratio portfolio that lies within the area of feasible portfolios. Naturally, in a $\mu, \sigma -$ diagram, the portfolio we find will be located to the right of the efficient frontier, which the tangency portfolio is tangent to. This is a consequence of the restriction we have imposed on the portfolio weights.

This procedure is applied on the expected returns and the variance-covariance matrices through time, for each factor portfolio respectively. With the portfolio weights at hand, we are finally able to construct a time series of realized portfolio returns, by multiplying each stock's weight in the factor Markowitz portfolio by its respective realized return.

As a final note, the constraints that one may impose on the convex optimization problem can easily be extended to restrictions of more practical concern. Y. Feng & Palomar (2016) note that beside capital budget ($w^T\mathbf{1} = 1$) and no-shorting ($w \geq 0$), the constraints can also include turnover (to control transaction costs) as well as upper and lower bounds on the portfolio weights. However, as noted above, our intention is to arrive at a solution as close to the unconstrained tangency portfolio as possible.

The Equally Weighted Portfolio

The construction of the equally weighted portfolio is based on (4.13) and a vector of portfolio weights, where the weights are simply calculated as $\frac{1}{N}$. Hence, we loop through the matrix of realized returns and apply the vector of portfolio weights through time to arrive at a timeseries of returns for the equally weighted portfolio.

4.2.2 – Portfolio Backtesting

Based on the historical returns of the proposed portfolios, we construct several measures as part of the portfolio backtesting. These include the cumulative return index, high-water mark, and drawdown as defined in Pedersen (2015). For each of these measures, we create time series and graph these to illustrate the variability and cyclicality of the portfolios. The cumulative return index of the portfolio is calculated as

$$RI_t = RI_{t-1} * R_{t-1,t} \tag{4.28}$$

where RI_t denotes the cumulative return index at time t and $R_{t-1,t}$ represents the gross-return from month $t - 1$ to t. We construct the index to have an initial value of 100 in January 1979.

The high-water mark is based on the changes in the cumulative return index and indicates the highest cumulative return index of the portfolio achieved in the past. It demonstrates whether the portfolio's ability to perform persists, and is calculated as

$$HWM_t = \max_{s \leq t} RI_S \tag{4.29}$$

where time s is a point in time earlier than time t. With the cumulative return index and high-water mark at hand, we can also calculate the drawdown of the portfolio. Drawdown is a central risk measure as it shows the cumulative loss since the losses initiated. Formulated differently, drawdown is the loss incurred since the high-water mark. The relative drawdown is given by

$$DD_t = \frac{HWM_t - RI_t}{HWM_t} \tag{4.30}$$

In case the portfolio is at its high-water mark, DD_t will have a value of zero, otherwise DD_t will be a positive number. Thus, the drawdown will persist as long as the cumulative return index remains below the historical peak. Equation (4.30) uses the most recent high-water mark as a metric for loss. Nonetheless, drawdown may also be calculated in relation to other points in time, for instance the cumulative return index at the start of the year. Applying the same logic of Equation (4.30), we may also consider the maximum drawdown as

$$MDD_T = \max_{t \leq T} DD_t \tag{4.31}$$

where time t is a point in time earlier than time T. The maximum drawdown is an indicator used to evaluate the downside risk of an investment.

4.2.3 – Performance Evaluation

In order to evaluate the performance of the portfolios, we will employ several relevant performance measures. Throughout this section, we will describe the theoretical rationale underlying these performance measures and how they may be implemented. Portfolio performance evaluation commonly refers to how a specific portfolio has performed in relation to a comparison portfolio. The evaluation may conclude that the portfolio has been overperforming, underperforming or has performed equally relative to its comparison portfolio.

Several reasons exist for why portfolio performance can be important to assess. An investor may want to evaluate the attractiveness of a specific portfolio composition. He/she may be holding a different portfolio and have an interest in the relative performance of his/her portfolio, to examine the need for rebalancing (Samarakoon & Hasan, 2006). Another, perhaps more practical reason can be that in case a portfolio is managed (actively) by a portfolio manager, reviewing performance can function to qualify how good a job the manager has done in managing the portfolio. For actively managed portfolios, it is not uncommon to link the manager's compensation to portfolio performance (Bodie et al., 2009).

According to Samarakoon & Hasan (2006) two standard methods for evaluating portfolio performance include *conventional methods* and *risk-adjusted methods*.

Conventional Methods

Conventional methods encompass comparing the periodic return on an investment portfolio with that of a particular benchmark portfolio, where the latter usually takes form of either a broad market index portfolio or a relevant style-benchmark portfolio. However, this method creates a problem with respect to differences in the level of risk. Better performance may be attributed to the investment portfolio carrying a higher level of risk than its benchmark portfolio. Hence, such an approach may produce invalid conclusions about the relative performance of an investment portfolio.

Risk-Adjusted Methods

Instead, returns have to be adjusted for the risk taken in order to make meaningful inferences about relative portfolio performance. But how can we adjust for the risk taken? We will turn to several risk-adjusted performance measures, which account for the differences in risk levels by adjusting returns for the risk taken. In their respective sections, we will describe the *Sharpe ratio*, the *Modigliani Risk-Adjusted Performance*, *Jensen's alpha*, and the *Information ratio*. We have chosen the Sharpe ratio as our first performance statistic since we assume full investment in each of our portfolios. Hence, we deem the total standard deviation of the portfolio the appropriate risk measure. Due to the limitations of the Sharpe ratio we include the Modigliani Risk-Adjusted Performance as well. To supplement these two statistics, we include Jensen's alpha to see how our portfolios perform relative to their market risk. Since our factor Markowitz portfolios may be more exposed to idiosyncratic risk, we also choose to include the Information ratio.

Sharpe Ratio

Through Section 2.1 on Mean-Variance Portfolio Theory, we documented the optimization procedure, where an investor chooses optimal weights to invest in risky assets, with regards to obtaining the maximum Sharpe ratio. This Sharpe ratio is the theoretical version of the performance measure, based on forecasts of expected returns and the portfolio variance, as described in Sharpe (1966). In this section, we will instead consider the empirical version based on excess returns. This performance metric is defined as

$$SR_p = \frac{\bar{r}_p - \bar{r}_f}{\sigma\left(r_p - r_f\right)} \tag{4.32}$$

where $\bar{r}_p$ denotes the average portfolio return, $\bar{r}_f$ the average risk-free rate, and $\sigma\left(r_p - r_f\right)$, the sample standard deviation of the excess returns of the portfolio. The average portfolio return is given by the arithmetic mean

$$\bar{r}_p = \frac{1}{T}\sum_{t=1}^{T} r_{p,t} \tag{4.33}$$

while the sample standard deviation of portfolio returns is defined as

$$\sigma_p = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(r_{p,t} - \bar{r}_p\right)^2} \tag{4.34}$$

The performance measure is a *reward-to-variability* ratio, evaluating historic average excess return per unit risk taken. One may compare the Sharpe ratio of two (or more) portfolios, where the highest ranked portfolio is assumed to be the one with the highest historical Sharpe ratio.

Modigliani Risk-Adjusted Performance

While the Sharpe ratio may be used for ranking the performance of several portfolios, Modigliani & Modigliani (1997) underline the difficulties of interpreting the numerical value of the Sharpe ratio. In response to this, they propose a related performance statistic that has a more economically intuitive interpretation. The measure is known as the Modigliani risk-adjusted performance (denoted by M^2 or M2), and defined as

$$M^2 = \bar{r}_{p^*} = \bar{r}_p \frac{\sigma_M}{\sigma_p} + \left(1 - \frac{\sigma_M}{\sigma_p}\right) \bar{r}_f \tag{4.35}$$

The intuition behind the performance statistic is to adjust the return of the portfolio for risk exposure. Equation (4.35) shows that the adjusted portfolio, p^*, is a combination of the unadjusted portfolio and the risk-free asset. The weights on the unadjusted portfolio and the risk-free asset are defined respectively as, $\frac{\sigma_M}{\sigma_p}$ and $\left(1 - \frac{\sigma_M}{\sigma_p}\right)$, in order to scale the portfolio to have the same standard deviation as the market portfolio, thus making these two comparable. We may also note that by construction

$$\sigma_{p^*} = w_p \sigma_p = \frac{\sigma_M}{\sigma_p} \sigma_p = \sigma_M \tag{4.36}$$

Going forward, we use the Modigliani risk-adjusted performance in excess of the average risk-free rate, $M^2 - \bar{r}_f$.

Jensen's Alpha

Jensen's alpha (Jensen, 1968) is one of the most frequently applied performance measures and measures the part of return that is left unexplained by the systematic risk of the portfolio. The empirical version of Jensen's alpha is commonly measured by the intercept, α_p, in the following regression

$$r_{p,t} - r_{f,t} = \alpha_p + \beta_p \left(r_{M,t} - r_{f,t}\right) + e_{p,t} \tag{4.37}$$

Jensen's alpha measures the average excess *abnormal* return, and in case we observe an $\alpha_p > 0$, the portfolio earns a return in excess of the systematic risk taken. Contrary, if $\alpha_p < 0$ the portfolio has underperformed in relation to the systematic risk of the portfolio. The regression in Equation (4.37) can be extended to either of the multifactor models explained in Section 3.2. However, the criteria for superior performance is still $\alpha_p > 0$.

The major short coming of Jensen's alpha is that it is an estimate, meaning that uncertainty is associated with the value of it. When we estimate α_p in Equation (4.37), we are interested in whether it is statistically different from zero, which can be determined by conducting a test of the following hypotheses

$$H_0\colon \alpha_p = 0$$

$$H_A: \alpha_p \neq 0$$

We compute a t-statistic for the estimate of alpha, $\hat{\alpha}_p$, as

$$t_{\hat{\alpha}_p} = \frac{\hat{\alpha}_p - \alpha_{H_0}}{\sigma(\hat{\alpha}_p)} \tag{4.38}$$

A t-statistic larger than 2 corresponds to an alpha that is statistically significant from zero. On the other hand, a t-statistic below 2 indicates an alpha that is not statistically significant from zero. In the latter case, a positive alpha could be achieved through luck (Pedersen, 2015).

Information Ratio

Portfolios that carry positive alpha may deviate significantly from the well-diversified market portfolio. This implies a reduction in diversification and more exposure to the idiosyncratic risk of certain assets. Hence, one must evaluate the tradeoff between abnormal excess return as measured by α_p and increased idiosyncratic risk exposure. According to Bodie et al. (2009) the information ratio quantifies this trade-off. The information ratio is also known as the *Risk-Adjusted Abnormal Return* or *Risk-Adjusted Alpha*, and calculated as

$$IR_p = \frac{\alpha_p}{\sigma(e_p)} \tag{4.39}$$

with α_p, being the intercept in Equation (4.37) and $\sigma(e_p)$, the sample standard deviation of the residuals in Equation (4.37).

4.3 – Econometric Theory and Initial Assessment

This section will evaluate the statistical techniques we employ to implement the portfolio construction of the factor models as outlined in Section 4.2. Therefore, the data sample will be assessed in the light of econometric theory, which includes an introduction to the Ordinary Least Square (OLS) regression, the assumptions underlying the OLS regression, as well as an evaluation of whether the assumptions are fulfilled. The assumptions are necessary to evaluate, because violations will lead to unreliable parameter estimates.

In Section 4.2 we explained how the portfolio construction concerned regressing the past 60 months of excess returns onto the past 60 monthly observations of the factors. This procedure will be done for each of the 188 stocks in our data sample, over each of the 480 rolling estimation windows. We will therefore end up running $480 * 188 = 90.240$ regressions in total. In theory, each of these regressions would require separate model checking. Obviously, this is not feasible or something that we intend to do. However, whenever it is possible, we will perform the model checking on the broadest level possible. As we shall see, this is possible when we assess the assumptions relating to the dependent variable and the explanatory variables. When we assess the assumptions relating to specific regressions, we will include three regressions for each factor model, made on different stocks, at different points in time. The rationale for doing this, relates to our ambition of assessing the econometric issues regarding the statistical tools we use, but not correcting for possible issues as stated in Section 1.2. Hence, we will assume that the conclusions of this section can be generalized to all the regressions we run as part of the portfolio construction.

The Assumptions Underlying the OLS Regression

A linear regression model relates the change in the dependent Y variable to one (or more) explanatory X variables. Formally, we may define the multivariate linear regression as follows:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_K X_K + e \tag{4.40}$$

With reference to Wooldridge (2016) and Fox (2016), the multivariate linear regression relies on the following assumptions:

Assumption 1 – Random Sampling

The multivariate linear regression in Equation (4.40) is based upon a random sample from the population of N observations.

Assumption 2 – Linearity in the Parameters

The model in the population is the linear model described by Equation (4.40), where the coefficients, β, are the unknown parameters. And therefore:

Assumption 3 – Zero Conditional Mean

$$E[e|X_k] = E[e] = 0, \qquad \forall k \in \ [0, K] \tag{4.41}$$

The explanatory X variables and the error term, e, are independent and the error term, e, has an expected value of zero.

Assumption 4 – Homoscedasticity

$$Var[e|X_k] = \sigma_e^2, \qquad \forall k \in [0, K] \tag{4.42}$$

The error term, e, has the same constant variance given all the explanatory variables.

In addition to Assumption 3 and Assumption 4, the error terms are assumed to be normally distributed with zero conditional mean and constant variance:

$$e \sim N(0, \sigma_e^2) \tag{4.43}$$

Assumption 5 – No Perfect Multicollinearity

No linear relationship exists between the explanatory X variables. If two or more explanatory variables have an exact linear relationship, then it is not possible to estimate the model using an OLS regression.

Wooldridge (2016) states that an estimated parameter fulfilling all five assumptions is said to be the "*best linear unbiased estimator*" (BLUE) of the theoretical parameter.

4.3.1 – OLS Regression

Before evaluating fulfillment of the assumptions outlined above, we will present a brief description of the OLS Regression and how the estimation technique works.

Taking the conditional expectation of Equation (4.40) leaves us with

$$E[Y|X_k] = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_K X_K, \qquad \forall k \in \ [0, K] \tag{4.44}$$

which is the formula used for deriving the fitted values of Y. According to Ruppert & Matteson (2015) the idea underlying OLS regression, is that $\hat{\beta}$ are the least square estimates that minimize the following relation:

$$\sum_{k}^{K} \left\{Y - \left(\hat{\beta}_0 + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 + \cdots + \hat{\beta}_K X_K\right)\right\}^2 \tag{4.45}$$

They state that the least square estimates, $\hat{\beta}$, are the ones that minimize the distance between the fitted value on the multivariate regression hyperplane and the actual value of Y.

4.3.2 – Econometric Issues

Following the description of the assumptions underlying the OLS regression and the concept behind the estimation procedure, we will now evaluate whether the conditions regarding the model are fulfilled. The evaluation will be structured into three sections. The first part will examine the data sample of excess returns in order to draw relevant conclusions. During this section, we will initially assess the statistical distribution of our data and the first assumption regarding *Random Sampling*. The second part will examine the regression residuals in order to evaluate the second, third and fourth assumptions regarding *Linearity in the Parameters, Zero Conditional Mean* and *Homoscedasticity*. The third part will evaluate the explanatory variables for each factor model in order to assess the fulfillment of the fifth assumption of *No Perfect Collinearity*.

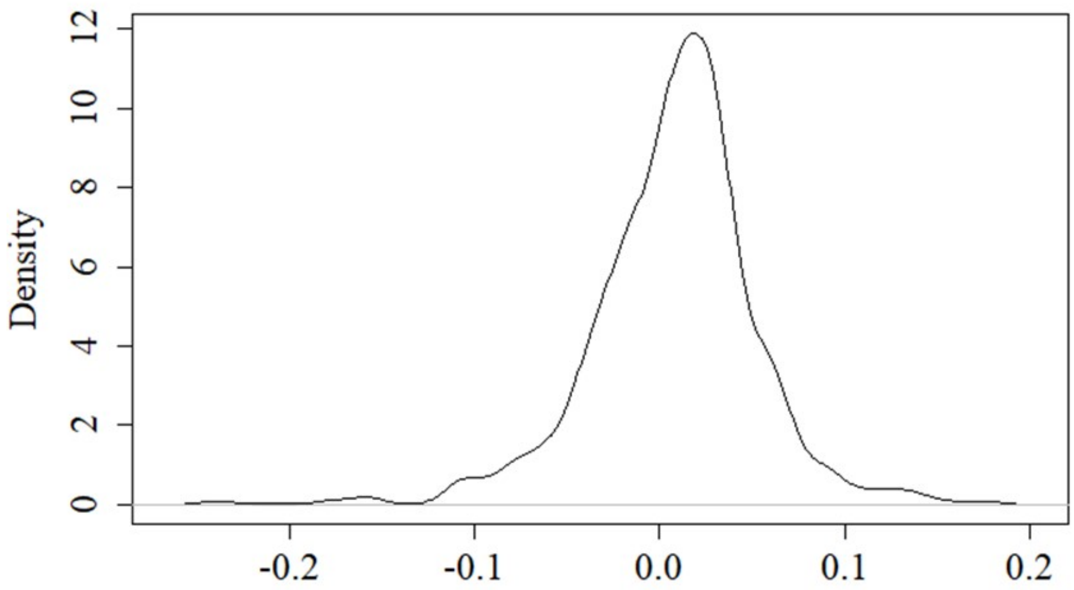
In statistics, one of the strongest techniques to analyze samples of data is merely to plot the data in various ways (Ruppert & Matteson, 2015). For simplicity and because this thesis is intended as a financial study rather than a statistical one, we will apply this technique to draw relevant conclusions.

The Excess Returns

Normality

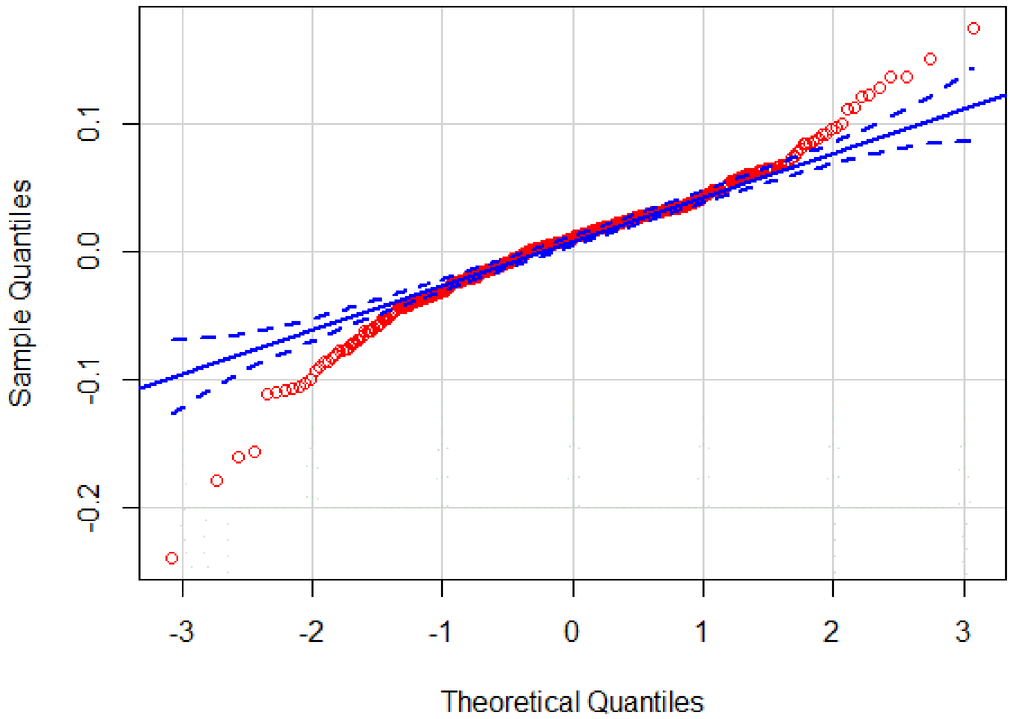
As a first impression of our data sample, we examine its statistical distribution. Thus, we create a time series by making a linear combination of all the excess returns on the stocks, at each point in time, throughout the full data sample. The rationale for doing this is that it gives us an efficient way of assessing the statistical distribution of the excess returns on each stock. In the case that the excess returns on each stock are normally distributed, then a linear combination of the excess returns will also be normally distributed. This is true since linear transformations preserve normality (Ruppert & Matteson, 2015).

Figure 4.1 – Density Plot of a Linear Combination of Excess Stock Returns



In Figure 4.1 a plot of the density of the linear combination of excess stock returns has been created using a kernel density estimator, due to the difficulties of interpreting a distribution through a histogram. Evaluating Figure 4.1, we see that the density is left skewed and appears to have fat tails. Hence, the distribution of the linear combination of excess stock returns does not seem to be normal.

Figure 4.2 – QQ-plot of a Linear Combination of Excess Stock Returns



To determine how the linear combination of the excess stock returns is distributed, we can also employ a so-called QQ-Plot, where "QQ" is an abbreviation for "Quantile-Quantile". As the name suggests, the plot compares quantiles of two different distributions against each other (Fox, 2016). In case the two distributions being compared are similar, the dots in the plot will lie approximately on a straight line. In Figure 4.2, a QQ-plot has been produced for the linear combination of excess stock returns, where the vertical axis displays the sample quantiles, and the horizontal axis shows the theoretical quantiles. Furthermore, confidence interval bands for the normal distribution have been added to the plot, as indicated by the dotted blue lines. The red dots in Figure 4.2 take a non-linear form, which suggest that the linear combination of excess stock returns is not normally distributed. Furthermore, it is apparent that the red dots are breaching the confidence interval bands in each tail, thus indicating a distribution with fatter tails than the normal distribution, as we similarly saw in Figure 4.1. The "convex-concave" curvature that we observe in Figure 4.2 is also noted by Ruppert & Matteson (2015) to be associated with heavier tails than the normal distribution.

We may also assess the properties of the distribution we observe graphically by considering the numerical measures of Skewness (Sk) and Kurtosis (Kur). With reference to Ruppert & Matteson (2015), these two statistics are defined as:

$$\widehat{Sk} = \frac{1}{T} * \sum_{t=1}^{T} \left(\frac{Y_t - \bar{Y}}{\sigma} \right)^3 \tag{4.46}$$

$$\widehat{Kur} = \frac{1}{T} * \sum_{t=1}^{T} \left(\frac{Y_t - \bar{Y}}{\sigma} \right)^4 \tag{4.47}$$

where Y_t denotes the observation at time t, $\bar{Y}$ the sample mean and σ the sample standard deviation. Skewness measures the symmetry in the distribution; hence perfect symmetry implies zero skewness. Kurtosis measures how the probability mass is concentrated in the center of the distribution, and how much of the probability mass that is located in the tails of the distribution. The normal distribution has a kurtosis of 3 and for this reason, most statistical software reports the *excess kurtosis*, which is the kurtosis in excess of 3. The linear combination of the excess stock returns has a skewness of -0.595 and an excess kurtosis of 3.377, thus indicating a distribution different from the normal distribution and adding to the conclusions suggested by Figure 4.1 and Figure 4.2.

Random Sampling

The first assumption underlying the OLS regression formally states that we should have a random data sample. When observing the linear combination of the excess stock returns, we examine whether independence exists between the current value, Y_t and the next value, Y_{t+1}. Such an independence is the idea of randomness. To assess whether this assumption is fulfilled, we will create a plot of autocorrelations between the data points in the time series. This is also referred to as an autocorrelation (ACF) plot.

However, before constructing the ACF plot, we acknowledge the fact that stock returns from time to time are dependent on each other, thus exhibiting autocorrelation. Campbell, Lo, & MacKinlay (1997) find that returns over different periods are not statistically independent. They emphasize that returns on stock portfolios show positive autocorrelation over short time periods, i.e., daily, weekly and monthly periods. They note that this is particularly true for returns over very short time periods.

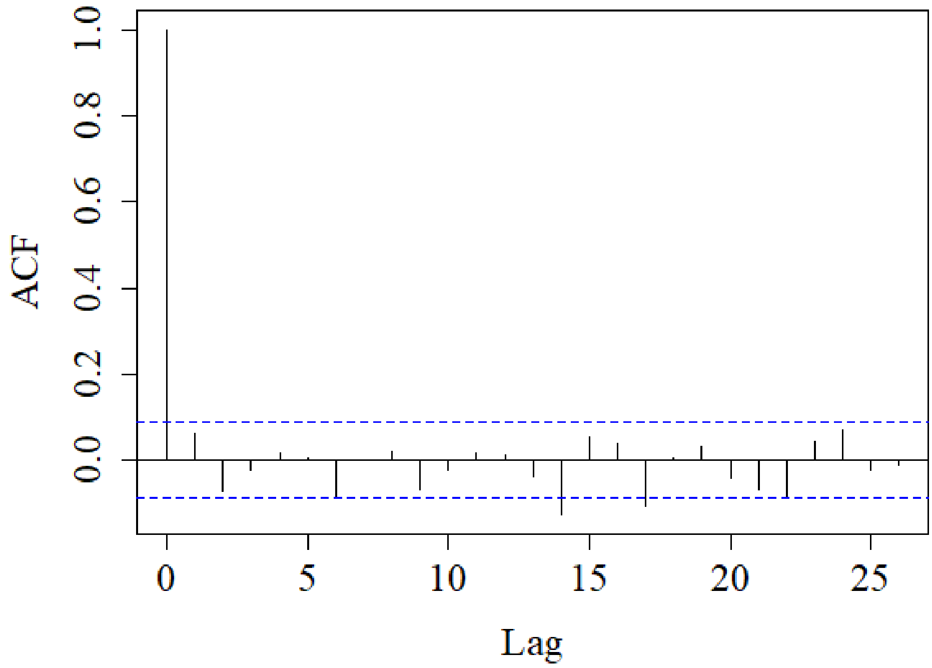
We construct the ACF plot following Ruppert & Matteson (2015). First, we define the sample autocovariance function, $\hat{\gamma}(h)$, as:

$$\hat{\gamma}(h) = T^{-1} \sum_{t=h+1}^{T} (Y_t - \bar{Y})(Y_{t-h} - \bar{Y}) \tag{4.48}$$

where T denotes the number of time periods and h is the time lag, Equation (4.48) gives us the autocovariance between Y_t and Y_{t-h}. With the autocovariance function at hand, we define the sample autocorrelation function, $\hat{\rho}(h)$, as:

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} \tag{4.49}$$

Figure 4.3 – ACF Plot of a Linear Combination of Excess Stock Returns



It becomes evident from the ACF plot in Figure 4.3 that the autocorrelation at time lag 0 is 1, which is always true, since an observation has a correlation of 1 with itself. This may also be inferred from the autocorrelation function in Equation (4.49), by setting h equal to zero. We can conclude that almost all of the autocorrelations are under the confidence limits (indicated by the dotted blue lines). Moreover, we do not observe any apparent patterns in the autocorrelations. If the data are in fact random, we expect to see no such patterns. The autocorrelation around time lag 14 and time lag 16 seem to be slightly outside the confidence limits. However, this is not something that should draw any concern with respect to whether we have a random data sample or not. We can thus conclude that there are no significant autocorrelations and that the linear combination of the excess stock returns is indeed independent. Hence, we have a random data sample.

The Residuals

Linearity in the Parameters, Zero Conditional Mean and Homoscedasticity

To assess whether the second assumption of linearity in the parameters, the third assumption of zero conditional mean and the fourth assumption of homoscedasticity are satisfied, we turn to examine the regression residuals. Fulfillment of the third and fourth assumptions will also indicate whether the second assumption is satisfied. Checking for zero conditional mean and homoscedasticity in the regression residuals can be done by plotting the fitted, $\hat{Y}$, values against the standardized regression residuals to see if any patterns emerge and detect possible anomalies. This procedure is well described in both Ruppert & Matteson (2015) and Fox (2016), which we will use as references throughout this section. They describe that the reason for standardizing the residuals is that the raw residuals are measured in the same units as the dependent variable and thus difficult to interpret across different regression models. A standardized residual is defined as

$$\frac{\widehat{e_t}}{\sigma_e \sqrt{1 - h_{s,t}}} \tag{4.50}$$

Here $h_{s,t}$ is called a "hat" value. The value can be interpreted from the equation below:

$$\widehat{Y_t} = \sum_{s=1}^{T} h_{s,t} Y_s \tag{4.51}$$

where $h_{s,t}$ is a measure of the contribution of the observed value, Y_s, on the fitted value, $\widehat{Y_t}$.

In Appendix C, plots of the fitted values, $\hat{Y}$, against the standardized residuals have been created. For each of the four factor models described in Section 4.2, three plots have been made, where each plot is based on a regression done on different stocks, at different points in time. Observing the individual data points in the plots, we can conclude that no odd trends or patterns exist that should raise our concern. By smoothing the data points (as indicated by the blue line in the plots), we see that they almost pass through zero, which indicates that the residuals have zero conditional mean and that the linear model in Equation (4.40) fits our data relatively well. Furthermore, we see that the dots form a horizontal band, which according to Johnson & Wichern (2007) suggest homoscedasticity. An explanation for why we do not see entirely straight smoothing lines is due to the influence of outliers. The outliers are pulling the smoothing line away from zero. As a consequence of that the OLS regression is being done on stock market data, we are not able to correct for the outliers by removing them. Moving forward, we therefore assume that the outliers do not cause any econometric issues for the OLS regression.

In conclusion, we assume that the residuals have zero conditional mean and constant variance (homoscedasticity), thus satisfying the third and fourth assumptions of the OLS regression. As a result, we can also infer that the assumption of linearity in the parameters is satisfied.

Normally Distributed Residuals

To evaluate the distribution of the residuals, we can apply the same method used to determine the distribution of the linear combination of excess stock returns. QQ-plots with confidence interval bands for the normal distribution are presented in Appendix D. In the same way as previously, three plots have been made for each of the four factor models, where each plot is based on a regression done on different stocks, at different points in time. Examining the plots in Appendix D, we observe that the data points (as indicated by the red dots) for the most part are not breaching the confidence interval bands. As emphasized in the previous section, several outliers do exist among the residuals, which we also observe in the QQ-plots. Similarly, we will assume that the outliers do not cause any issues for the OLS regression. We thus assume that the residuals are normally distributed.

No Perfect Multicollinearity

The notion of multicollinearity is that we cannot include two or more explanatory variables in the OLS regression that are highly correlated.[3] In the case where a high degree of multicollinearity is present, it becomes hard to estimate the separate effects of the explanatory variables (Ruppert & Matteson, 2015). To check for multicollinearity in the OLS regression, we apply the methodology by Ruppert & Matteson (2015). As a first step, we define the variance inflation factor (VIF) as:

$$VIF = \frac{1}{1 - R^2} \tag{4.52}$$

where R^2 is defined as:

$$R^2 = \frac{\sum_{t=1}^{T}\left(\hat{Y}_t - \bar{Y}\right)^2}{\sum_{t=1}^{T}(Y_t - \bar{Y})^2} \tag{4.53}$$

[3] We note that for regressions specified upon the Single-Index Model, the condition of multicollinearity is not an issue since the model does only include one explanatory variable, which is the excess return on the market.

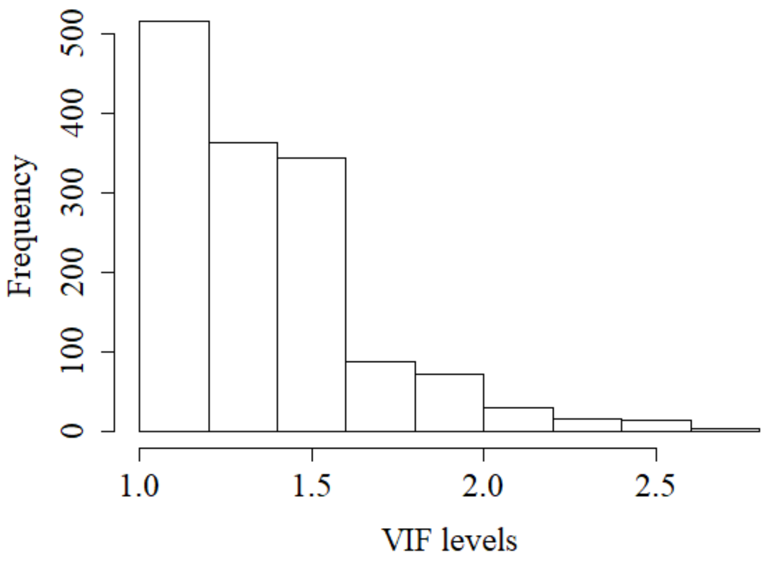
The idea of using the VIF is that if the OLS regression includes p explanatory variables, we will regress one of the explanatory variables onto the $p - 1$ other explanatory variables. We will report the R^2 of this regression and calculate the VIF using Equation (4.52).

Evaluating Equation (4.52), we can conclude that

$$\lim_{R^2 \to 1} VIF = \infty$$

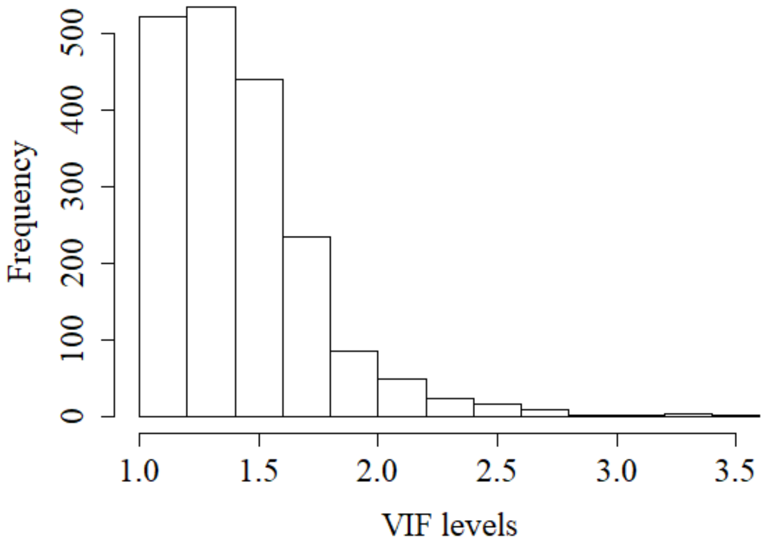
Hence, a VIF close to 1 is a good sign of no perfect multicollinearity.

Figure 4.4 – Variance Inflation Factors for the Explanatory Variables in the Fama-French Three-Factor Model



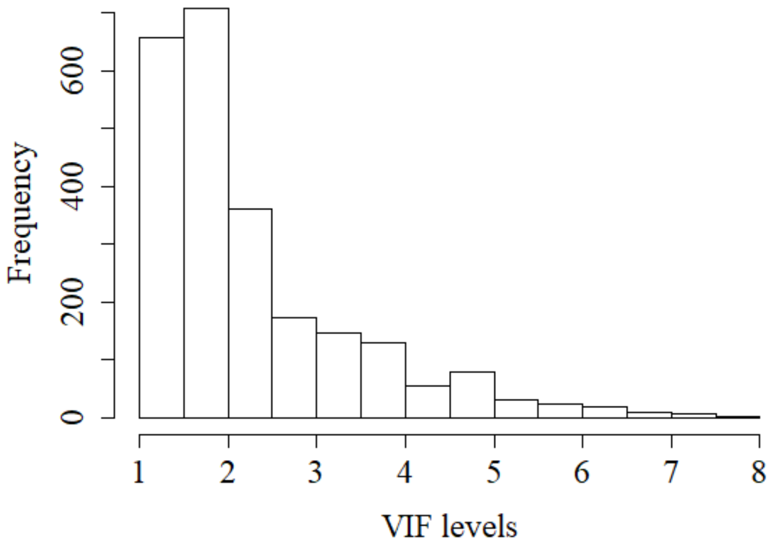
In Figure 4.4, the VIF levels have been calculated for the explanatory variables of the Fama-French Three-Factor Model and illustrated in a histogram. Examining the figure, we see that the explanatory variables of the Fama-French Three-Factor Model demonstrate relatively low VIF levels. More specifically, the majority of the VIF levels are below 2, indicating a R^2 below 0.5 according to Equation (4.52).

Figure 4.5 – Variance Inflation Factors for the Explanatory Variables in the Carhart Four-Factor Model



Similarly, Figure 4.5 illustrates the VIF levels for the explanatory variables of the Carhart Four-Factor Model. We observe an almost identical pattern to that of Figure 4.4, with similar VIF levels. However, we note the presence of slightly higher VIF levels compared to Figure 4.4.

Figure 4.6 – Variance Inflation Factors for the Explanatory Variables in the Fama-French Five-Factor Model



Lastly, the VIF levels for the explanatory variables of the Fama-French Five-Factor Model have been produced in Figure 4.6. Despite a moderate number of VIF levels below 2, we generally observe a higher level of VIFs with this model specification. According to Equation (4.52), a VIF of 8 equals a R^2 of 0.875. This suggests that several of the explanatory variables of regressions specified upon the Fama-French Five-Factor Model exhibit multicollinearity. From an econometric point of view, this model specification may thus be problematic. These findings can possibly relate to the issue of redundancy of the HML factor, when including the RMW and CMA factors in the Fama-French Three-Factor Model, as reported by Fama & French (2015).

Finally, the plot in Appendix E illustrates that the highest VIF levels occur for the explanatory variables of HML, RMW, CMA. By specifying regressions based upon the Fama-French Five-Factor Model, we can conclude that it is mainly for the HML, RMW and CMA factors that multicollinearity appear to be an issue.

A model with a high degree of multicollinearity may explain variability in the dependent variable more poorly, than a model where the explanatory variables with a high degree of multicollinearity are left out. Hence, Ruppert & Matteson (2015) suggest that the usual way of correcting for multicollinearity is to reduce the number of explanatory variables. However, because we take the models as given, we acknowledge the issues, but do not intend to correct for them. In conclusion, we assume that the condition of no perfect multicollinearity is generally satisfied across the OLS regressions we run.

4.4 – Sub-Conclusion

In this section, we have described how several factor models can be implemented in the mean-variance analysis. The expected returns and the variance-covariance matrix are calculated based on the factor loadings, the expected returns and the variance-covariance matrix of the factors, as well as the diagonal variance-covariance matrix of the residuals.

We have discovered that an unconstrained mean-variance optimization procedure produces extreme weights. Therefore, we also describe how a constrained procedure can be implemented. More specifically, the constrained procedure involves the realistic restriction of not allowing short sales. In addition, we have found the traditional mean-variance analysis unfeasible to implement, due to singular variance-covariance matrices, when applied on a high number of assets.

Furthermore, we have evaluated the assumptions underlying the OLS regression that we use to estimate the input parameters for the factor-based mean-variance analysis. We have shown that the dependent variable, which can be represented as a linear combination of the excess stock returns is not normally distributed. Additionally, we can conclude that the data are randomly sampled. By examining the regression residuals, we infer that the assumptions of zero conditional mean and constant variance (homoscedasticity) are fulfilled. Moreover, we find that the condition of normally distributed residuals is fulfilled. Nevertheless, multicollinearity in the regressions specified upon the Fama-French Five-Factor Model is recognized as a problem. The issue arises because of the relationship between the HML, RMW and CMA factors, which is also previously recognized in the literature. However, we assume that the assumption of no perfect multicollinearity is fulfilled across the regressions we run.

In conclusion, the econometric issues that we find are assumed to be of lower importance, and the OLS regressions and application of them are considered valid. We will apply the framework we have proposed and implement the analysis. Hence, the next section will describe the initial interpretation of the output and perform the portfolio backtesting.

5 – Analysis and Results

5.1 – Initial Interpretation of Output

This section will function as an initial interpretation of the output we obtain from following the implementation described in the Section 4.2. The output encompasses portfolio weights and portfolio excess returns for the constructed portfolios, for the time period from January 1979 to January 2019.

5.1.1 – Portfolios Weights

As a first step, we examine the constrained factor Markowitz portfolio weights, illustrated in the scatterplots of Appendix F. At first glance, we see differences in the dispersion of the weights for each portfolio, across the time period.

Table 5.1 – Summary Statistics for the Portfolio Weights

	S.I.	FF3	C4	FF5	E.W.
Average	0.53%	0.53%	0.53%	0.53%	0.53%
Standard Devation	0.31%	1.56%	1.84%	2.04%	0.00%
Minimum	0.00%	0.00%	0.00%	0.00%	0.53%
Maximum	2.38%	44.28%	56.69%	52.75%	0.53%

For the portfolio based on the Single-Index Model, relatively small concentrations in individual stocks are seen throughout the whole time period compared to the other portfolios. The average portfolio weight is 0.53%, while the standard deviation is 0.31%. The lowest portfolio weight is zero, due to the constraint of no short selling, while the highest is 2.38%. Thus, the weights of this portfolio are the ones that look most similar to how the weights of an equally weighted portfolio would look.

In contrast, the portfolio based on the Fama-French Three-Factor Model includes several points in time, where considerable positions are taken in individual stocks. This is particularly true for the years around 1982, 1991, 2004 and in the years from 2009 to 2013. The average portfolio weight is 0.53%, while the standard deviation is 1.56%. The lowest portfolio weight is zero, while the highest is 44.28%.

For the portfolio based on the Carhart Four-Factor Model, we similarly observe certain stock positions of significant size. This is particularly true during the beginning of the time period. However, the highest weights occur at similar points in time as for the portfolio's three-factor counterpart. The average portfolio weight is 0.53%, while the standard deviation is 1.84%. The lowest portfolio weight is zero, while the highest is 56.69%.

The portfolio based on the Fama-French Five-Factor model does also have substantial weights in the first part of the time period, where an even higher dispersion is seen. High weights are also observed in the years from 2009 to 2013. The average portfolio weight is 0.53%, while the standard deviation is 2.04%. The lowest weight is zero, while the highest is 52.75%.

With identical average weights, we are not able to conclude much about the overall level of weights across the portfolios. However, the portfolio based on the Fama-French Five-Factor Model shows the highest dispersion in its portfolio weights, while the portfolio based on the Single-Index Model shows the lowest dispersion in portfolio weights, as indicated by the standard deviation.

Nevertheless, the weights of the factor Markowitz portfolios should be seen in the light of the equally weighted portfolio. Over the entire time period, this portfolio is constructed to have positions of $1/N$ in all stocks, hence 0.53% with the 188 stocks included in our data sample. The major difference, relative to most of the factor Markowitz portfolios, is that the small weights throughout the time period make the portfolio extremely diversified. Therefore, it will not suffer from specific stocks performing bad, but on the other hand, it will not benefit as much from individual stocks performing good. The interesting question is then, whether the model we have implemented is able to estimate the expected returns and the variance-covariance matrix more precisely, and process signals in such a way that we benefit from holding specific stock positions. This is an important consideration for the factor Markowitz portfolios, because of the more concentrated positions in specific stocks, as observed in Table 5.1.

Furthermore, we may assess the monthly changes in the portfolio weights, as this can provide us with an indication about the possible turnover in the portfolios and need for rebalancing over time. The changes in the portfolio weights (denoted as $\Delta Weights$) are illustrated through scatterplots in Appendix G. As expected, the changes in the weights of the portfolio based on the Single-Index Model are minimal. Across the remaining factor Markowitz portfolios, we generally observe a similar level of changes in the portfolio weights, with absolute values under 10%. However, at certain points throughout the time period, substantial changes in the portfolio weights occur. Larger changes tend to cluster around specific moments in time, with values ranging from 10% to 40%, in absolute terms.

The scatterplots seem to suggest relatively high turnover for the portfolio based on the Carhart Four-Factor Model and relatively low turnover for the portfolio based on the Single-Index Model across time. In perspective, the equally weighted portfolio is constructed to have zero ΔWeights at all times.[4] In the case of high turnover and frequent need for rebalancing, the portfolios may incur higher (unaccounted-for) transaction costs. Hence, we desire a relatively stable portfolio composition over time. However, since only gross performance is considered in this thesis, the practical implications related to portfolio turnover and associated trading costs will be left for a discussion in Section 7.1

5.1.2 – Portfolios Excess Returns

For each of the constructed portfolios, returns in excess of the risk-free rate have been calculated in the tables of Appendix H, for the time period from January 1979 to January 2019. Recall, that the portfolios are constructed to have monthly holding periods. The tables, therefore, show monthly excess returns in percentages. The tables apply conditional formatting, by which excess returns found to be relatively high takes on a blue contrast, while returns found to be relatively low takes on a red contrast.

At first sight, we observe that the excess returns vary considerably from month to month, across the constructed portfolios. Through specific holding periods the portfolios deliver positive excess returns, wherein others they deliver negative excess returns. The tables include several periods with positive and negative continuity in excess returns. Thus, in the next section, we will consider how variability in excess returns coincide with shifting market conditions and turbulent periods of the stock market.

[4] This does not necessarily mean that the equally weighted portfolio does not require rebalancing. Consider the case where specific stocks have risen or fallen significantly, in such a case the investor must rebalance the portfolio accordingly, so the wealth invested in each stock still represents $1/N$ of the portfolio's total value. Therefore, ΔWeights may only provide us with an indication of the actual turnover in the portfolio.

Monthly statistics for each portfolio have been summarized in Table 5.2, shown below

Table 5.2 – Summary Statistics for the Portfolio Excess Returns

	S.I.	FF3	C4	FF5	E.W.
Average	0.73%	0.66%	0.77%	0.68%	0.79%
Standard Devation	4.62%	4.42%	4.55%	4.15%	4.49%
Minimum	-24.38%	-23.79%	-16.52%	-19.01%	-23.89%
Maximum	16.89%	19.75%	18.42%	16.40%	17.52%
Skewness	-0.50	-0.61	-0.47	-0.64	-0.59
Excess Kurtosis	3.19	3.74	1.74	2.84	3.45

Starting at the Single-Index Markowitz portfolio, the average monthly excess return is 0.73%, which corresponds to an annual excess return of $12 * 0.73\% \approx 8.78\%$. The monthly standard deviation of the excess returns is 4.62%, scaled to $\sqrt{12} * 4.62\% \approx 16.01\%$ annually.[5] During the time period, the highest observed monthly excess return is 16.89%, while the lowest is -24.38 %. The skewness and excess kurtosis of the monthly excess returns are -0.50 and 3.19 respectively.

For the Fama-French Three-Factor Markowitz portfolio, the average monthly excess return equals 0.66%, corresponding to 7.97% yearly. The portfolio's monthly standard deviation is 4.42%, scaled to 15.32% annually. During the time period, the highest observed monthly excess return is 19.75%, while the lowest is -23.79%. The skewness and excess kurtosis of the monthly excess returns are -0.61 and 3.74 respectively.

The Carhart Four-Factor Markowitz portfolio delivers an average monthly excess return of 0.77%, annualized to 9.26%. The portfolio's monthly standard deviation is 4.55%, corresponding to an annual standard deviation of 15.77%. The highest monthly excess return is 18.42%, with the lowest being -16.52%. Skewness and excess kurtosis of the monthly excess returns amount to -0.47 and 1.74 respectively.

The average monthly excess return of the Fama-French Five-Factor Markowitz portfolio is 0.68%, which corresponds to 8.15% on a yearly basis. The monthly standard deviation of the portfolio is 4.15%, equivalent to an annual standard deviation of 14.38%. The highest monthly excess return is

[5] More practically, we multiply the average monthly excess return by 12 to get the average annual excess return (thus, ignoring compounding). To get the annual volatility, we multiply the monthly volatility by $\sqrt{12}$, as a result of that volatility scales with $\sqrt{T}$.

16.40%, while the lowest is -19.01%. The skewness and excess kurtosis of the monthly excess returns are -0.64 and 2.48 respectively.

Lastly, the equally weighted portfolio provides an average monthly excess return of 0.79%, equivalent to a yearly excess return of 9.54%. The portfolio's monthly standard deviation is 4.49%, corresponding to a yearly standard deviation of 15.55%. The highest monthly excess return is 17.52%, with the lowest being -23.89%. Skewness and excess kurtosis of the monthly excess returns amount to -0.59 and 3.45 respectively.

It becomes apparent that during the time period, the equally weighted portfolio has the highest average excess return, while the lowest average excess return is observed for the Fama-French Three-Factor Markowitz portfolio. On the other hand, the highest standard deviation is seen for the Single-Index Markowitz portfolio, while the Fama-French Five-Factor Markowitz portfolio has the lowest standard deviation. In addition, we note that the excess returns across all the portfolios possess negative skewness. More specifically, if we were to observe the distribution of the excess returns for each portfolio, more distribution-mass would be in the left tail of the distribution, suggesting more extreme excess returns, especially on the downside. This is particularly true for the excess returns of the Fama-French Five-Factor Markowitz portfolio, which has the highest negative skewness. Furthermore, we observe positive excess kurtosis across all the portfolios. This suggests that the distributions of portfolio excess returns have fatter tails. We may therefore occasionally experience outliers (either positive or negative excess returns). On the other hand, the Fama-French Three-Factor Markowitz portfolio holds the highest positive excess kurtosis.

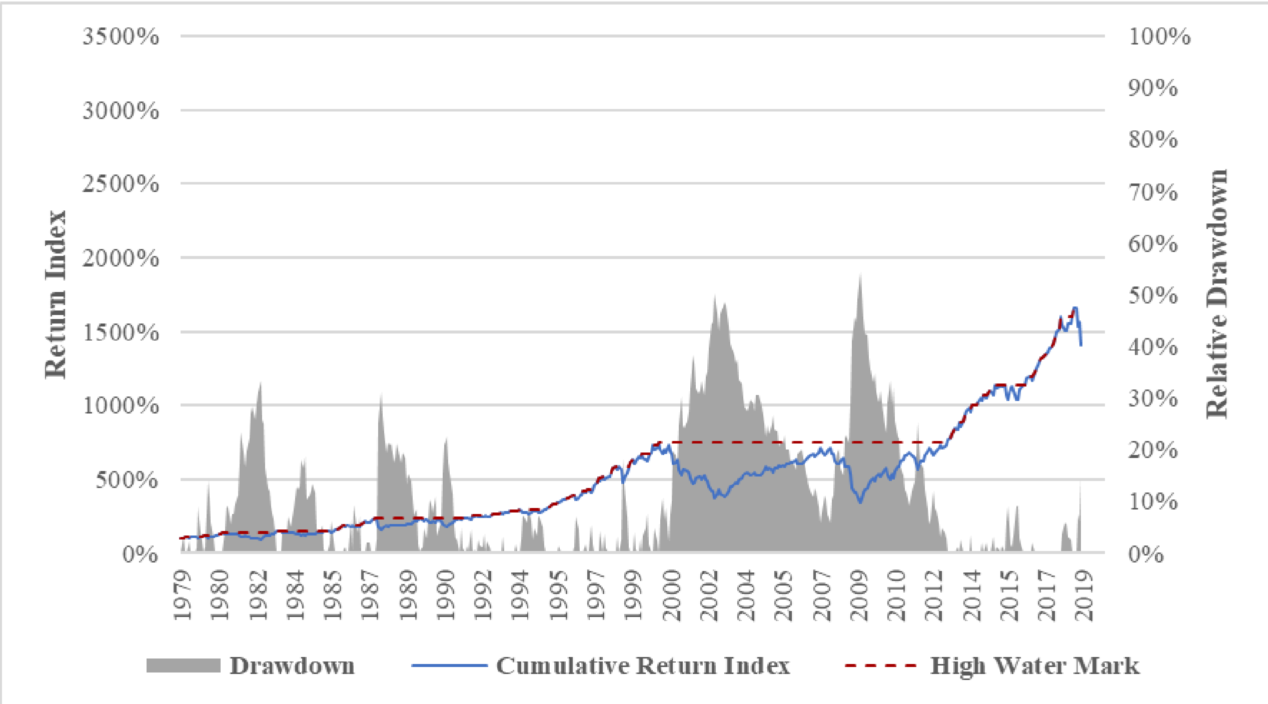
In contrast to merely interpreting the monthly excess returns of the portfolios independently over time and through summary statistics, the next section will put the returns into perspective by conducting a backtest for each of the constructed portfolios, in perspective to the overall stock market.

5.2 – Portfolio Backtesting

The following section will conduct a backtest for each of the constructed portfolios, from January 1979 to January 2019. As described in Section 4.2 the portfolio backtesting will consist of graphing the cumulative return index, high-water marks and drawdowns over time. This ought to give an impression of how the portfolios have done during periods of market turbulence, both alone but also in comparison to each other. Even though an assessment of the market portfolio will be done, emphasis will be put on the comparative performance between the factor Markowitz portfolios and the equally weighted portfolio, as a consequence of the thesis' overall problem statement. Accordingly, the first part of this section will observe how a portfolio formed on the market has fared throughout the evaluation period and elaborate on certain periods of market turmoil in order to understand the behavior and the market risk of the portfolios we have constructed.

The Market Portfolio

Figure 5.1 – Cumulative Return Index, High-Water Mark, and Drawdown for the Market Portfolio (Excess of the Risk-Free Rate)



Examining Figure 5.1, we see a gradually increasing cumulative return index from the beginning of the evaluation period, with specific periods of market disturbance during the first few decades. The first considerable drawdown occurs around the beginning of the year 1981, where the stock market suffers due to the early 1980s recessions in the US. For several years the US experience a severe aggravation of economic conditions, which becomes evident in the market portfolio through the decrease in cumulative return index and the drawdown of relatively high magnitude and duration.

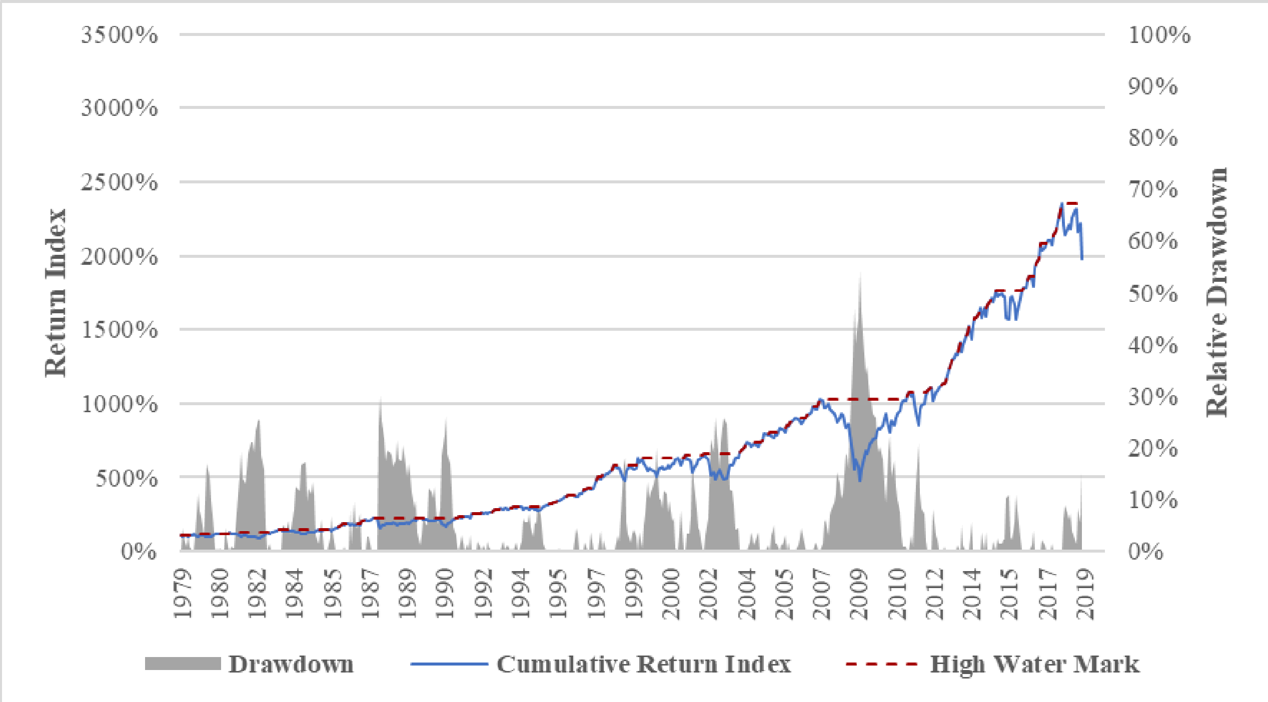
In the following years, the stock market experiences steady growth towards the global stock market crash of the year 1987, also referred to as *Black Monday*. In contrast, this downturn presents itself as a steep and rapid decline in the stock market, yet over a short time span of several weeks. Hence, the market portfolio's cumulative return index takes a sharp dip, as suggested by the large but short-lived drawdown, followed by a modest recovery.

Afterwards, the cumulative return index of the market portfolio progressively increases, with a lower and brief drawdown around the year 1990 supposedly due to the Persian Gulf War. For over a decade and towards the burst of the dot-com bubble the market portfolio dramatically appreciates. The significant rise in the stock market during the 1990s is fueled by favorable economic conditions as well as a boom in technology firms. At the peak, the market portfolio sets an all-time high-water mark of 751% at the beginning of the year 2000, which will not be surpassed until the year 2013. Following the burst of the dot-com bubble in the year 2000, the market portfolio experiences the most prolonged period of a falling cumulative return index. This also becomes evident in the drawdown reaching 50% around the year 2002 which, in terms of magnitude, is the second highest during the 40-year backtesting period.

From the bottom and up until the financial crisis, the market portfolio establishes a foothold and recovers. However, at the height of the year 2007, the market portfolio's cumulative return index declines sharply, while its drawdown increases towards its maximum. As apparent in Figure 5.1, the drawdown reaches above 50% in the trough.

Since then, the cumulative return index of the market portfolio has steeply increased, with several corrections underway. The portfolio experiences a moderate decline around the year 2011, which perhaps can be attributed uncertainty about the European Sovereign debt crisis' influence on the US economy, as well as the downgrading of the US Federal Government's Credit Rating. Nonetheless, the period moving forward is characterized by a positive uptrend in the stock market. Hence the market portfolio reaches a maximum high-water mark around 1600% in 2018. From here, stock market volatility starts to erupt, and fears of economic slowdown and rising interest rates begin to emerge explaining the variability in the cumulative return index seen towards the end of Figure 5.1. The market portfolio ends at a cumulative return index of 1412% in January 2019.

Figure 5.2 – Cumulative Return Index, High-Water Mark, and Drawdown for the Single-Index Markowitz Portfolio (Excess of the Risk-Free Rate)



From the onset of the evaluation period, the Single-Index Markowitz portfolio experiences several drawdowns during the first few decades, supposedly due to the same events affecting the market portfolio. The portfolio shows stability in the progression of the cumulative return index and has quite a similar performance to that of the general stock market until the end of the 1990s. The portfolio does not seem to catch the last part of the positive trend apparent in the stock market towards the end of the dot-com bubble. From the beginning of the 1990s and towards the burst of the dot-com bubble, the Single-Index Markowitz portfolio sees an increase in the cumulative return index from 216% to 564%.

This may perhaps be attributed to the composition of stocks in the data sample that the portfolios are constructed upon. Recall, the examination of the data sample in Section 4.1, where we found a lower concentration of stocks in the information technology industry. The boom of the 1990s was among other things fueled by the exponential growth in internet and technology stocks, which our data sample to some degree shows an absence of. In support of this proposition, the portfolio does not suffer as much by the burst of the bubble in the year 2000 as the stock market.

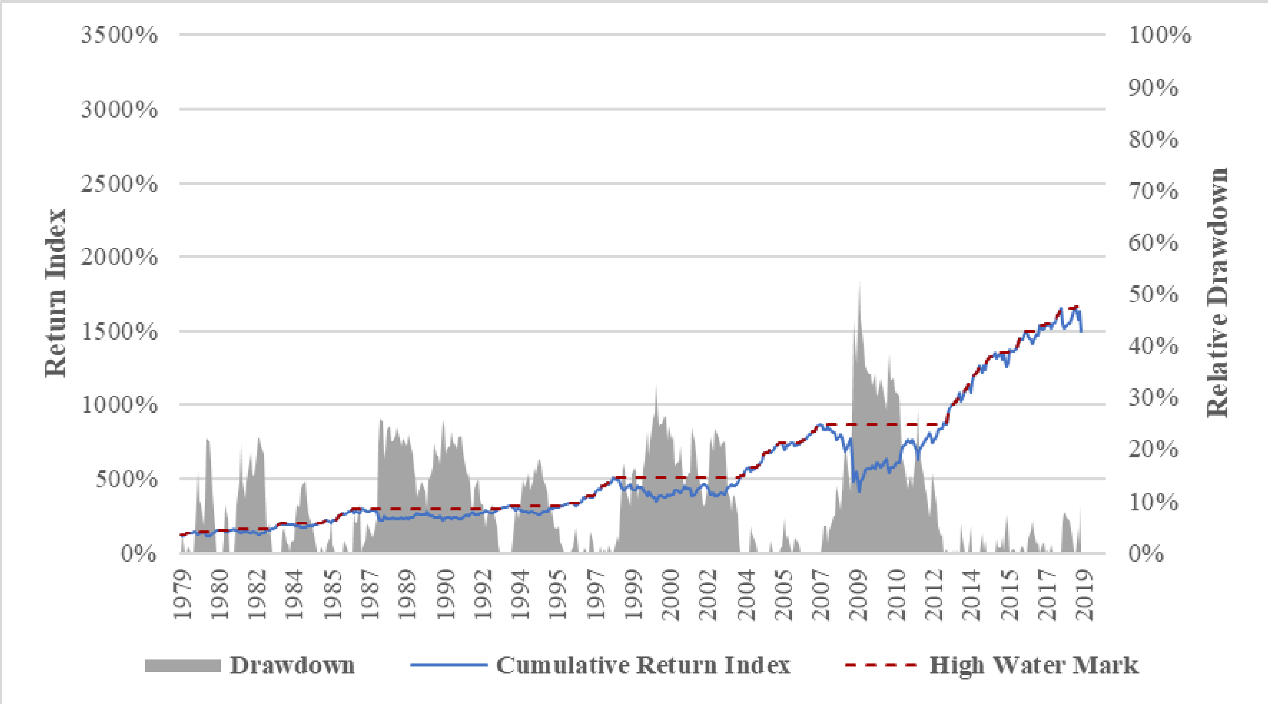
From the early years of the 2000s and up until the financial crisis, the cumulative return index of the portfolio steadily builds up. However, the portfolio encounters a moderate drawdown around the beginning of the year 2003. Besides attributing this to the aftermath of the dot-com crash, another possible cause could be the various accounting scandals hitting major U.S. companies, damaging investor confidence and leading to nervous stock markets.

At the peak of the U.S. credit bubble, the cumulative return index of the portfolio has climbed steadily to a new high-water mark of 1030%. Nonetheless, the market turbulence of 2007–2009 makes the drawdown of the Single-Index Markowitz portfolio surge and reach its maximum of 54% in 2009, which causes a decline in the cumulative return index to 473%.

In the period from the trough and towards the end of the time period, the cumulative return index of the portfolio sees a steep but volatile increase with several corrections along the way. Thus, higher variability in the cumulative return is observed towards the end. This may relate to the conclusion of Table 5.2, where the Single-Index Markowitz portfolio shows the highest standard deviation. Throughout the latter part of the backtesting period, the Single-Index Markowitz portfolio achieves its maximum high-water mark of 2353% in early 2018 and ends at a cumulative return index of 1975% in January 2019.

The Fama-French Three-Factor Markowitz Portfolio

Figure 5.3 – Cumulative Return Index, High-Water Mark, and Drawdown for the Fama-French Three-Factor Markowitz Portfolio (Excess of the Risk-Free Rate)



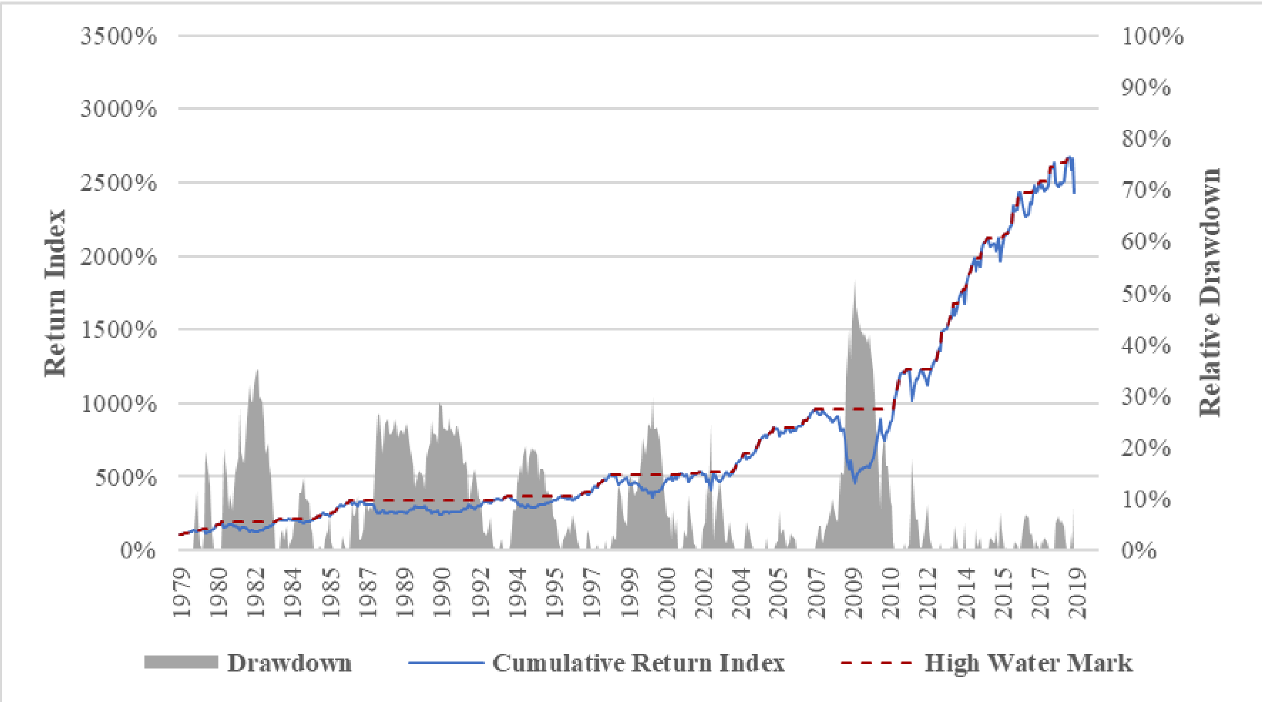
From the beginning of the backtest, the cumulative return index of the Fama-French Three-Factor portfolio fares almost equal to that of the Single-Index Markowitz portfolio and the general stock market. However, the cumulative return index generally sees a slightly higher level, while the drawdowns experienced in these years are of lower magnitude but longer duration.

From the 1990s and towards the turn of the millennium the cumulative return index sees a relatively low increase from 272% to 374% (perhaps due to the same reasons as explained for the Single-Index Markowitz portfolio). In Figure 5.3, we observe that when the stock market is peaking, as a result of the dot-com bubble, the Fama-French Three-Factor portfolio experiences an extended drawdown, which the portfolio will first recover from in the year 2004.

Until the burst of the U.S. credit bubble, the portfolio reaches a high-water mark of 871%. Nevertheless, from this height, the cumulative return index drops, and the drawdown rises towards its maximum of 52% in the year 2009. From the trough and towards the end of the time period, the Fama-French Three-Factor portfolio has a harder time recovering, which can be seen by the flatter growth in the cumulative return index. However, in comparison to the Single-Index Markowitz portfolio, the fare through the latter part of the backtesting period is characterized as less volatile, with fewer spikes in the cumulative return index. Thus, the Fama-French Three-Factor Markowitz portfolio achieves its maximum high-water mark of 1658% in late 2018 and ends at a cumulative return index of 1495% in January 2019.

The Carhart Four-Factor Markowitz Portfolio

Figure 5.4 – Cumulative Return Index, High-Water Mark, and Drawdown for the Carhart Four-Factor Markowitz Portfolio (Excess of the Risk-Free Rate)



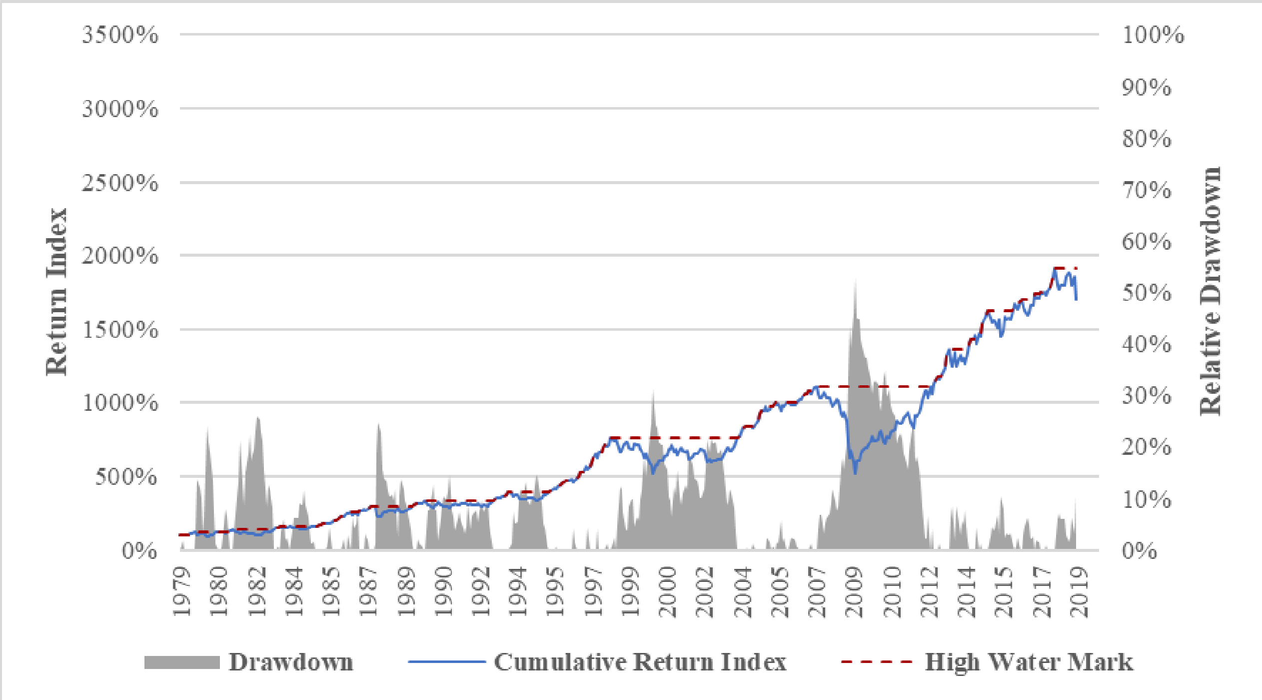
The Carhart Four-Factor Markowitz portfolio shows a consistently increasing cumulative return index with similar developments to that of the portfolio's three-factor counterpart, from the start of the backtesting period. From the end of the 1990s and through the dot-com bubble, the portfolio does also suffer from an extended drawdown. From the beginning of the year 2000 the cumulative return index increases from 389% to 963% before the onset of the financial crisis. As anticipated, the drawdown of the Carhart Four-Factor Markowitz portfolio rises, peaking at its maximum of 52% around the year 2009, due to the burst of the U.S. credit bubble.

An essential difference between the Carhart Four-Factor Markowitz portfolio and the other factor Markowitz portfolios becomes visible in the aftermath of the financial crisis. The portfolio yet re-gains ground, and towards the end of the backtesting period, the portfolio delivers a period of high growth in the cumulative return index, exceeding the other factor Markowitz portfolios by far. Moving forward, the cumulative return index surges, where earlier high-water marks continuously are being burst. The positive trend is initiated at 457% in the year 2009, from where the portfolio reaches its maximum high-water mark of 2674% in late 2018. However, as we observed for the Single-Index Markowitz portfolio, the cumulative return index shows higher fluctuations towards the end of the backtesting period. The portfolio thus finishes at a cumulative return index of 2432% in January 2019.

Putting Figure 5.4 into perspective, the backtest can perhaps be explained by Table 5.2. The Carhart Four-Factor Markowitz portfolio has the highest average monthly excess return across all factor Markowitz portfolios. Despite the relatively high standard deviation of the portfolio, the portfolio carries the lowest negative skewness. This may suggest a relatively lower frequency of extreme negative excess returns, which can explain the portfolio's sound performance. The performance of the Carhart Four-Factor Markowitz portfolio may also be attributed to the inclusion of the momentum (UMD) factor. Perhaps, the portfolio is able to process signals of short-term momentum and assigning higher weights to recent winners and lower weights to recent losers.

The Fama-French Five-Factor Markowitz Portfolio

Figure 5.5 – Cumulative Return Index, High-Water Mark, and Drawdown for the Fama-French Five-Factor Markowitz Portfolio (Excess of the Risk-Free Rate)



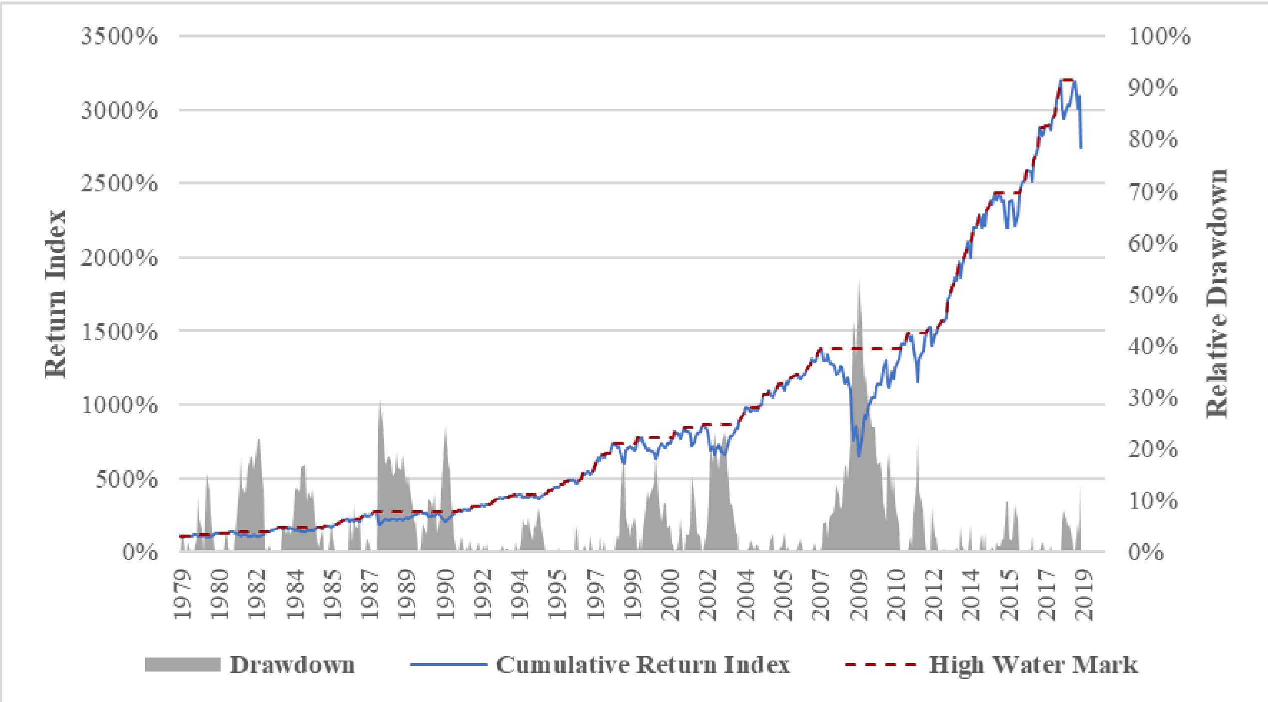
In Figure 5.5 we observe that the Fama-French Five-Factor portfolio shows similar performance to that of the previous portfolios, during the first decade. From the end of the 1980s and towards the end of the 1990s, the cumulative return index of the portfolio gradually increases slightly above that of the other factor Markowitz portfolios. The Fama-French Five-Factor Markowitz portfolio seems better at catching the general upturn in the stock market compared to the other factor Markowitz portfolios. Hence, the cumulative return index increases from 331% in the year 1990 to 716% just before the turn of the millennium. As a result, the portfolio is more resistant to the significant down-turns occurring around the year 2000 and year 2003 and retains a relatively high cumulative return index throughout the early 2000s.

In the following years and until the burst of the U.S. credit bubble, the Fama-French Five-Factor Markowitz portfolio experiences a steady and steep increase in the cumulative return index the to 1066% in the year 2007, surpassing the preceding portfolios in the same time frame. From this height, the drawdown of the portfolio surges towards its maximum of 53% in the year 2009, as a result of the market turbulence of 2007 - 2009.

Following the trough and onwards, the cumulative return index of the portfolio sees a flatter, but stable increase compared to Single-Index- and Carhart Four-Factor Markowitz portfolios. The fare of the portfolio towards the end of the time period is less volatile, where the cumulative return index goes from 522% in 2009 to 1911% in the start of the year 2018, which also marks the maximum high-water mark across the backtesting period. The portfolio ends at a cumulative return index of 1699% in January 2019.

The Equally Weighted Portfolio

Figure 5.6 – Cumulative Return Index, High-Water Mark, and Drawdown for the Equally Weighted Portfolio (Excess of the Risk-Free Rate)



Lastly, we consider the equally weighted portfolio in Figure 5.6. During the initial part of the backtesting period, much of the same variability encountered by the former portfolios also affects this portfolio. Nevertheless, the equally weighted portfolio outclasses the previous portfolios with respect to the cumulative return index, and until the end of the 1990s, the portfolio accumulates a return index of 781%.

Through the peak and burst of the dot-com bubble, the progression in the portfolio's cumulative return index begins to flatten, and increased volatility is witnessed. The equally weighted portfolio thus experiences a few years of turmoil during the start 2000s, yet it recovers and takes another steep leap forward towards the peak of the U.S. credit bubble. From the early 2000s, the cumulative return index of the portfolio rises to an impressive 1381% in 2007.

Throughout the financial crisis, the drawdown of the equally weighted portfolio reaches a maximum of 53%. Contrary to the portfolios previously considered, the duration of the drawdown is lower, suggesting a better ability to recover, which perhaps can be attributed to a diversification benefit resulting from the lower portfolio weights. This may thus explain the superior performance of the portfolio so far.

Following the bottom of the crisis, the portfolio experiences a decade of rapid growth in the cumulative return index. During the last decade, several corrections are observed along the way. Nonetheless, the development of the cumulative return index is steepening even further following every correction. In the latter part of the backtesting period, the cumulative return index of the equally weighted portfolio surges from 651% in the year 2009 to 3200% at the beginning of the year 2018, which is the highest high-water mark across all portfolios. The portfolio ends at a superior cumulative return index of 2748% in January 2019.

The findings of the backtest are consistent with the results of Table 5.2. The equally weighted portfolio has the highest average monthly excess return among the constructed portfolios, yet it also carries a relatively high standard deviation. In spite of the expected diversification benefits as a result of lower concentrations in each stock, the volatile development towards the end of the backtest may explain the higher standard deviation of the portfolio.

5.3 – Sub-Conclusion

In conclusion, the weights across the constructed factor Markowitz portfolios vary considerably throughout the time period. The Single-Index Markowitz portfolio generally shows the lowest portfolios weights, while the multifactor Markowitz portfolios take considerable positions in specific stocks, at certain points in time. The portfolio weights of the Carhart Four-Factor Markowitz portfolio see the highest fluctuations during the time period, suggesting a potentially high turnover. In contrast, the weights of the Single-Index Markowitz portfolio are more stable and less disperse.

Judging by the summary statistics for each portfolio, we may conclude that the equally weighted portfolio and the Carhart Four-Factor Markowitz portfolio have the highest average monthly excess returns across the entire sample. While the Single-Index Markowitz portfolio displays the highest standard deviation, the Fama-French Five-Factor Markowitz portfolio demonstrates the lowest standard deviation. In spite of this, it will be interesting to examine the portfolios' relative risk-adjusted performance in the next section, where we will adjust the average excess returns for the volatility of the portfolios. Furthermore, we can conclude that it is a general premise that all portfolios are left skewed and exhibit positive excess kurtosis.

Finally, based on portfolio backtesting from January 1979 to January 2019, we discover that the portfolios fare quite differently through shifting and adverse market conditions. In some periods the portfolios are more cyclical, whereas in others they are not. In terms of cumulative excess return, the equally weighted portfolio exhibits the best performance across all the constructed portfolios, while the Carhart Four-Factor Markowitz portfolio outperforms the other factor Markowitz portfolios.

However, as emphasized previously, returns in isolation do not tell the complete story about performance. Although the cumulative return index illustrates what a portfolio potentially could have earned and drawdown brings along viable insights about downside risk, returns must be adjusted for risk in order to make meaningful assessments of performance across different portfolios. Therefore, the next section will complete a performance evaluation based on several well-known performance metrics.

6 – Performance Evaluation

6.1 – Performance Evaluation

Following the initial interpretation of the output from the portfolio construction and the backtesting of the portfolios, this section will conduct a performance evaluation based on the performance metrics described in Section 4.2. The section is structured to first conduct the performance evaluation over the full time period, followed by evaluations over several sub-periods.

6.1.1 – Full Evaluation Period

In Table 6.1 below, the performance metrics for evaluating the portfolios' performance can be found. The metrics have been calculated for the time period from January 1979 to January 2019, which constitutes the full evaluation period. The average excess returns and volatilities have been annualized in order to arrive at yearly performance metrics for the portfolios.

Table 6.1 – Performance Evaluation, 1979 – 2019 (Full Sample)[6]

	S.I.	FF3	C4	FF5	E.W.	Market
Average exReturn	8.78%	7.97%	9.26%	8.15%	9.54%	**7.82%**
Volatility	16.01%	15.32%	15.77%	14.38%	15.55%	**15.23%**
Alpha	1.56%	2.46%	3.63%	2.94%	2.57%	**-**
(t-stat)	*1.27*	*1.41*	*2.00*	*1.80*	*2.12*	*-*
Portfolio Beta	0.99	0.70	0.65	0.75	0.91	**1**
Sharpe Ratio	0.55	0.52	0.59	0.57	0.61	**0.51**
$M^2 - r_f$	8.35%	7.91%	8.94%	8.62%	9.34%	**7.82%**
Information Ratio	0.20	0.22	0.32	0.29	0.34	**-**

In Section 5 it already became apparent that the equally weighted portfolio carried the highest average excess return over the full time period, and in terms of excess returns ranked the best performing portfolio. Adjusting the average excess return for the risk of the portfolio, we see that this is still

[6] The portfolio beta reported in this section is not the beta from Equation (4.37). Instead it is constructed as a periodic average of $\beta_{pf,M} = \sum_{i=1}^{N} w_i \beta_{i,M}$. For comparability, the portfolio beta of the equally weighted portfolio uses $\beta_{i,M}$ from the Single-Index Model and $w_i = 1/N$ for each stock. For future reference, the factor loadings for each portfolio are calculated in Appendix I.

the case since the portfolio has the highest Sharpe ratio of 0.61. In the light of the portfolio's level of average excess return, the volatility is thus relatively low. As suggested previously, this may be attributed to some form a diversification benefit stemming from the low and constant portfolio weights. The portfolio beta of the equally weighted portfolio at 0.91 is close to 1, suggesting co-movement with the market.

By comparing the Sharpe ratios across the factor Markowitz portfolios, the Carhart Four-Factor Markowitz portfolio outperforms with the highest Sharpe ratio of 0.59. Nonetheless, the portfolio provides the least market sensitive portfolio due to its low portfolio beta of 0.65. Scaling the two aforementioned portfolios to have the same risk as the market, the portfolios deliver excess returns of 8.94% and 9.34% respectively, which are approximately 1 percentage points above that of the market. Moreover, both portfolios achieve positive and statistically significant alphas in contrast to the other portfolios. The alphas constitute 3.63% and 2.57% respectively. Adjusting the alphas for the idiosyncratic risk of the portfolios, the conclusion remains the same. The information ratio of the equally weighted portfolio is higher than that of the Carhart Four-Factor Markowitz portfolio, amounting to 0.32 and 0.34 respectively. Hence, in terms of risk-adjusted alpha, the equally weighted portfolio also demonstrates superiority.

Table 6.1 reports an identical performance of the Fama-French Five-Factor and the Single-Index Markowitz portfolios over the evaluation period, where the Single-Index portfolio is the most cyclical portfolio as indicated by its portfolio beta close to 1. Furthermore, we note that it is a general premise that the multifactor Markowitz portfolios have a lower sensitivity towards the market.

The portfolio demonstrating the worst performance is the Fama-French Three-Factor Markowitz portfolio, which becomes evident through its low Sharpe ratio of 0.52. The Sharpe ratio translates into a $M^2 - r_f$ of 7.91%, which is only 0.1 percentage points above that of the market. Relative to the volatility of the other portfolios, we may conclude that the poor performance is due to the portfolio's low average excess return of 7.97%.

6.1.1 – Sub-Periods

The most common interpretation of performance measures and evaluation of their statistical significance implicitly assume that return observations are independently and identically distributed. More specifically this assumption implies that observations of returns are independently drawn from the same distribution with constant mean and variance (Bodie et al., 2009). For a passive or relatively stable investment strategy, this condition may not be too implausible. However, Munk (2017) suggests that in case an investment strategy varies considerably during the performance evaluation period, it makes little sense to calculate performance measures for the full period. Another, perhaps more intuitive rationale for separating the entire sample into smaller periods is that market conditions can change significantly over the 40 years that our data span. Evaluating portfolio performance during periods with different investment environments can thus add robustness to the conclusion. Accordingly, we have divided the full evaluation period from January 1979 to January 2019 into four sub-periods of 10 years in order to make meaningful inferences about the performance of the portfolios. Hence, the following section will evaluate the first and earliest period, which is from January 1979 to January 1989.

Table 6.2 – Performance Evaluation, 1979 – 1989

	S.I.	**FF3**	**C4**	**FF5**	**E.W.**	**Market**
Average exReturn	7.65%	9.97%	11.30%	11.27%	9.55%	**7.93%**
Volatility	17.06%	17.38%	18.37%	17.77%	16.97%	**17.11%**
Alpha	0.21%	3.81%	4.52%	4.79%	2.13%	**-**
(t-stat)	*0.11*	*1.06*	*1.27*	*1.36*	*1.18*	*-*
Portfolio Beta	0.98	0.65	0.64	0.71	0.94	**1**
Sharpe Ratio	0.45	0.57	0.61	0.63	0.56	**0.46**
M^2 - r_f	7.71%	9.84%	10.55%	10.91%	9.66%	**7.93%**
Information Ratio	0.04	0.34	0.41	0.44	0.38	**-**

According to Table 6.2, the Fama-French Five-Factor portfolio outcompetes the other portfolios over the evaluation period from January 1979 to January 1989 with respect to the Sharpe ratio. The portfolio has a Sharpe ratio of 0.63, and in terms of $M^2 - r_f$ earns 10.91%, which is about 3 percentage points higher than the excess return of the market. Based on the same metrics, the Carhart Four-Factor Markowitz portfolio demonstrates an almost identical performance. However, this portfolio is affected by its higher volatility of 18.37%, which in fact is the highest across all portfolios. Moreover, Table 6.2 shows a similar performance between the Fama-French Three-Factor portfolio and the equally weighted portfolio.

The Single-Index Markowitz portfolio has the least attractive performance during this evaluation period, as judged by its relatively low Sharpe ratio. While the other portfolios demonstrate a $M^2 -$

r_f above that of the market, this portfolio returns approximately 0.2 percentage points below that of the market. This is a consequence of the portfolio's relatively low average excess return of 7.65%.

Lastly, we note that alphas across all portfolios are positive, yet none of them are statistically significant. In spite of this, the alphas as well as the information ratios lead to similar rankings for the portfolios.

Table 6.3 – Performance Evaluation, 1989 – 1999

	S.I.	FF3	C4	FF5	E.W.	Market
Average exReturn	12.09%	7.43%	6.98%	10.87%	12.54%	**12.67%**
Volatility	12.85%	11.05%	11.05%	10.67%	12.60%	**13.57%**
Alpha	1.24%	0.55%	1.13%	3.73%	1.77%	**-**
(t-stat)	*0.69*	*0.20*	*0.38*	*1.53*	*1.06*	*-*
Portfolio Beta	0.98	0.75	0.58	0.85	0.92	**1**
Sharpe Ratio	0.94	0.67	0.63	1.02	0.99	**0.93**
M^2 - r_f	12.78%	9.11%	8.56%	13.80%	13.51%	**12.67%**
Information Ratio	0.23	0.07	0.12	0.50	0.35	**-**

In the portfolio backtesting we find that the period from January 1989 to January 1999 is characterized as an extended period of economic prosperity. This also becomes apparent in Table 6.3 through the generally higher performance metrics, compared to the ten-year evaluation period considered earlier. In general, we recognize that the best performing portfolios do also exhibit the highest exposure to the market.

Despite the high Sharpe ratio of the market, the Fama-French Five-Factor Markowitz portfolio demonstrates the best performance across all portfolios through its Sharpe ratio just above 1. This indicates that the portfolio more than compensates the investor for the total risk taken. Regarding the $M^2 - r_f$, the portfolio delivers 13.44%, which is about 1 percentage point higher than the excess return of the market. The equally weighted portfolio and the Single-Index Markowitz portfolio do also achieve high Sharpe ratios of 0.99 and 0.94 respectively. The portfolios have $M^2 - r_f$ above that of the market, however they carry relatively more systematic risk, as indicated by their higher portfolio betas.

Across all portfolios, positive but statistically insignificant alphas are observed. The information ratios lead to a similar ranking of the portfolios, and thus the same conclusion as the one we derive from the Sharpe ratios and the $M^2 - r_f$.

The Carhart Four-Factor Markowitz portfolio exhibits the lowest average excess return, resulting in a Sharpe ratio of 0.63 and a $M^2 - r_f$ about 4 percentage points below that of the market. Consequently, this portfolio demonstrates the worst performance across the portfolios in this evaluation period. In addition, the portfolio has the lowest co-movement with the market, as indicated by the low portfolio beta of 0.58.

Table 6.4 – Performance Evaluation, 1999 – 2009

	S.I.	**FF3**	**C4**	**FF5**	**E.W.**	**Market**
Average exReturn	2.26%	3.28%	3.48%	0.11%	2.91%	**-2.60%**
Volatility	16.77%	16.76%	16.54%	13.62%	15.55%	**15.73%**
Alpha	4.38%	4.79%	5.16%	1.37%	4.78%	**-**
(t-stat)	*1.27*	*1.07*	*1.24*	*0.38*	*1.41*	*-*
Portfolio Beta	0.87	0.59	0.56	0.72	0.68	**1**
Sharpe Ratio	0.13	0.20	0.21	0.01	0.19	**-0.17**
M^2 - r_f	2.12%	3.08%	3.31%	0.12%	2.94%	**-2.60%**
Information Ratio	0.67	0.58	0.49	0.14	0.74	**-**

In Table 6.4 we observe that all portfolios have very high risk and low average excess returns, which translate into very low risk-adjusted performance metrics, compared to the performance evaluations of the preceding sub-periods. During the time span from January 1999 to January 2009, two major events struck the U.S. stock market resulting in severe declines, including the dot-com crash as well as the financial crisis. The effects of these events are particularly apparent in the performance numbers of the market, where very high volatility and a negative average excess return are present, resulting in a Sharpe ratio of -0.17.

As we elaborated upon in the portfolio backtesting, the lack of upturn during the 1990s and absence of high drawdowns during the early 2000s make it likely that the way we prepare the data sample leads to a reduction in companies that pose to be particularly vulnerable during this time period. The existence of such a bias in the data sample and its implications will be discussed in Section 7.1. For this reason, the performance evaluation over this evaluation period will focus more on the relative performance between the factor Markowitz portfolios and the equally weighted portfolio, and less on the performance of the portfolios in relation to that of the overall stock market.

Given the highest average excess return of 3.48% and moderate volatility of 16.54%, the Carhart Four-Factor Markowitz portfolio delivers the highest Sharpe ratio of 0.21 and thus the best performance among the constructed portfolios. In addition, the Fama-French Three-Factor Markowitz portfolio and the equally weighted portfolio do also show relatively high Sharpe ratios of 0.20 and 0.19. On the other hand, the Fama-French Five-Factor Markowitz portfolio demonstrates the worst performance due to its minimal average excess return, despite it having the lowest volatility across the portfolios.

Finally, the alphas across all portfolios are above zero. However, since none of them are statistically significant, the alphas along with the information ratios may not lead to any reliable inferences about performance.

Table 6.5 – Performance Evaluation, 2009 – 2019

	S.I.	FF3	C4	FF5	E.W.	Market
Average exReturn	13.11%	11.22%	15.27%	10.34%	13.16%	**13.28%**
Volatility	16.98%	15.41%	16.17%	14.48%	16.71%	**14.00%**
Alpha	-1.74%	-1.12%	3.66%	0.18%	-1.45%	**-**
(t-stat)	*-0.80*	*-0.41*	*1.05*	*0.06*	*-0.68*	*-*
Portfolio Beta	1.14	0.83	0.82	0.69	1.08	**1**
Sharpe Ratio	0.77	0.73	0.94	0.71	0.79	**0.95**
M^2 - r_f	10.81%	10.19%	13.22%	10.00%	11.02%	**13.28%**
Information Ratio	-0.26	-0.14	0.35	0.02	-0.22	**-**

The time period from January 2009 towards the end of the sample marks a lengthy bull run, where the stock market experiences a sharp upswing. This is also evident in the performance of the market, where Table 6.5 shows a Sharpe ratio of 0.95.

Across the constructed portfolios, the Carhart Four-Factor Markowitz portfolio demonstrates the best performance numbers. The portfolio exhibits the highest Sharpe ratio of 0.94 primarily due to its very high average excess return of 15.27%. However, if the portfolio is scaled to have the same risk as the market, the excess return amounts to 13.22%, which is just below the excess return of the market.

Table 6.5 displays a quite similar performance between the Single-Index Markowitz portfolio and the equally weighted portfolio, with Sharpe ratios amounting to 0.77 and 0.79 respectively. The portfolio betas of these portfolios indicate a higher sensitivity towards the market, at 1.14 and 1.08 correspondingly. Since these portfolios deliver lower average excess returns than the market, this may be an explanation for the presence of negative alphas.

The Fama-French Five-Factor Markowitz portfolio displays the lowest average excess return of 10.34%. Despite its relatively low volatility, the portfolio ends at Sharpe ratio of 0.71 and has thus demonstrated the worst performance during this evaluation period.

6.2 – Sub-Conclusion

Based on the performance evaluation for the full evaluation period, which is over the time period from January 1979 to January 2019, we conclude that the equally weighted portfolio exhibits the best risk-adjusted performance. Across the factor Markowitz portfolios, the portfolio constructed upon the Carhart Four-Factor Model demonstrates excellent performance as well. It is generally true that all portfolios carry substantial volatility and that the best performing portfolios only compensate the investor for a little more than half of the risk taken. Nevertheless, the multifactor Markowitz portfolios generally display a lower level of systematic risk.

Separating the full sample into several sub-periods allow us to track the portfolios through different investment climates. Across all portfolios, we discover that the Fama-French Five-Factor Markowitz portfolio overperforms in the first two sub-periods. During the last two sub-periods, the Carhart Four-Factor Markowitz portfolio exhibits the greatest performance, while the Fama-French Five-Factor Markowitz portfolio demonstrates the worst. These findings are consistent with the conclusion of the portfolio backtesting in Section 5.2.

In general, we may conclude that the Single-Index Markowitz and the Fama-French Three-Factor portfolio demonstrate a modest performance. The full evaluation period, as well as the sub-periods provide evidence for this. Hence, portfolios constructed upon factor structures of fewer factors, may possibly forego important signals that some of the extended factor models are able to capture.

7 – Discussion

7.1 – Discussion

Following the analysis, this section will put the results into perspective by discussing several potential issues with regards to the implementation and data sampling. Furthermore, the practical relevance and implications in this context will also be discussed, for perspective.

7.1.1 – Statistical Significance

As underlined in Section 3.1, we choose to estimate factor models by time series regression. Therefore, an expected source of uncertainty relates to the accuracy of the estimates of the factor loadings. The analyses of Section 5 and Section 6 are based on the premise, that we can rely on the individual stock's sensitivity to the relevant factors, as indicated by their factor loadings. In case the estimates of the factor loadings are not statistically significant, then wide confidence intervals are associated with the estimates. This has consequences for the estimates of the expected returns and the variance-covariance matrix. In the worst case, this will influence how the mean-variance optimization procedure assigns weights to specific stocks and thus how the portfolios are composed.

In order to evaluate the accuracy of the factor loadings, we will observe their associated p-values and thus test whether they are statistically significantly different from zero. For a factor loading to be statistically significant, the associated p-value must be below a certain level of significance.

Table 7.1 – Number of Statistically Significant Factor Loadings for Each Factor Portfolio (in %)

	ExMkt	SMB	HML	UMD	RMW	CMA	Total
S.I.	85,22%	-	-	-	-	-	85,22%
FF3	86,17%	19,14%	27,83%	-	-	-	44,38%
C4	85,36%	18,61%	24,64%	19,54%	-	-	37,04%
FF5	83,92%	17,62%	19,18%	-	12,32%	13,64%	29,34%

Table 7.1 illustrates the relative number of statistically significant factor loadings across all regressions we run, for each of the four factor Markowitz portfolios. Each column indicates the percentage of statistically significant loadings, for a particular factor loading, while the *Total* column represents the total part of statistically significant factor loadings. In Table 7.1 we observe that across all factor Markowitz portfolios, the market factor loading carries the highest explanatory power since

around 85% of the loadings are found to be statistically significant. This suggests that the excess market return does seem to explain the variation of excess returns in our data sample quite well.

On the other hand, we notice that as the factor models are extended to include more factors, the total number of statistically significant factor loadings decline. Moving from the Single-Index- to the Fama-French Five-Factor Markowitz portfolio, the total level of statistically significant factor loadings declines from 85.22% to 29.34%. The immediate explanation for the limited explanatory power could be due to "overfitting". Hence, the inclusion of additional factors leads to the regression model fitting random noise in the data sample (Ruppert & Matteson, 2015). This may cause problems for the estimates of the expected returns and the variance-covariance matrix, since these are derived directly from the factor loadings as we saw in Section 4.2. Unreliable signals are thus conveyed to the optimization algorithm, which potentially could lead to misallocation of wealth between the stocks in the portfolio.

Conversely, the portfolios constructed upon a higher number of factors, did fare well through several periods of the portfolio backtesting and performance evaluation. From the previous section, we recall that the Fama-French Five-Factor Markowitz portfolio did demonstrate the best performance during the first two decades, while the Carhart Four-Factor Markowitz portfolio exhibited the best performance during the last two decades. This challenge the idea about the multifactor models not being parsimonious, since the additional factors could perhaps capture important signals and allow for better stock picking.

7.1.2 – Data Biases

A well-known disclaimer of many investment firms is that past performance is never an indicator of future performance. Because the analysis and results are based on historical data, it is hard to say how the portfolios will fare in the future. The intention of using 40 years of data was to get a general idea about how several portfolios constructed upon different factor-based mean-variance analyses performed through various market conditions.

When evaluating a trading strategy, it is of great importance that the data on which the strategy is tested on, possess as few biases as possible. Two biases do potentially exist within our data sample. These include survivorship bias and lookahead bias. Survivorship bias arises when performance evaluations only include companies, which have survived but do not account for firms that no longer exist. Lookahead bias stems from using information that is not available at the time of the trade.

On the other hand, the way we have structured the portfolio implementation works as out-of-sample testing in contrast to merely conducting in-sample testing, and therefore mitigates an optimization bias.

Survivorship Bias

As described in Section 4.1, we use the current index membership of the S&P 500 Index at a specific point in time for the implementation of the analysis. This has the consequence of imposing a survivorship bias in the data sample. Survivorship bias arises when only firms that have survived are included in a back-test or performance evaluation (Bodie et al., 2009). By only using the current members of the S&P 500 Index, the firms that have gone bankrupt are implicitly removed from the analysis. Furthermore, as we prepare the data sample to only include firms with equally long time series of returns, the bias is heightened.

Section 4.1 found that the committee behind the S&P 500 Index has an extensive screening process in place to maintain an index of healthy companies. Companies that no longer meet their selection process as a representative of the large-cap U.S. stock market are removed from the index. The data sample does therefore not only include firms which have survived during the 40 years our sample runs, but also robust and financially viable firms, that have shown excellent performance. This biases the portfolio backtesting and performance evaluation to look better.

A solution to mitigate the survivorship bias is to base the portfolio construction on constituents of the S&P 500 Index at every point in time. Hence, the portfolio from January 1979 to February 1979 should be based on the index constituents with 60 months of past return observations. Moving one month forward, the portfolio should be restructured with the new constituents, that have 60 months of past return observations and so forth. Given the limited scope of this thesis, we did not follow such an approach due to data availability and time resources.

In any case, we would not be able to eliminate the bias in our data sample completely. Recall, that part of the portfolio construction involves running regressions of the past 60 months of return data to get statistically reliable, but also economically meaningful factor loadings. Thus, firms must have survived for 5 years to be included in the analysis. It is important to underline that this may not be as big of a problem when forming portfolios based on stocks of the S&P 500 Index. However, applying the factor-based mean-variance analysis that we have proposed, on other investment universes with less rigorous screening mechanisms, this can be an issue. A potential trade-off therefore exists between minimizing the bias and getting more reliable parameter estimates. Reducing the length of the estimation window can mitigate the bias but produce less reliable factor loadings. In contrast, increasing the estimation window may lead to more reliable factor loadings, but heighten the bias. The conclusion must arguably be that a survivorship bias to some degree is inevitable when this approach is taken towards portfolio construction.

Look-ahead Bias

Another bias that may be present within our data sample is look-ahead bias, which occurs when information or data is being used that would not have been available during the period being analyzed (Ang, 2014). At the portfolio formation in 1979, we would have no chance of choosing the particular firms in our data sample. Furthermore, we would not have known that these firms would show particularly good performance going forward. As a consequence of this bias, the results of the portfolio backtesting and performance evaluation may look better than the trading strategy would do going forward. The solution we presented above to mitigate the survivorship bias can also function to reduce the look-ahead bias. Rather than using the current membership of the S&P 500 Index, the data sample should be based on the actual constituent changes over time.

Lastly, we may recognize that information issues do exist regarding the use of the factor data from French (2019) for the historical backtest and performance evaluations. While the idea of using a broad market index as the only factor was presented by Sharpe in 1963, the value and the size factor were first discovered by Fama & French in 1993. The momentum factor was identified by Carhart in 1997, while the profitability and investment factors were first discovered by Fama & French in 2015. In the time periods where we evaluate the portfolio’s performance, some of the factors can thus be argued not to have been discovered, and it may be unrealistic to assume that we can trade on them.

Optimization Bias

In the case where parameters have been optimized, there is a possibility that an optimization bias may arise. Pedersen (2015) states that when a trading strategy has been optimized and backtested within the same time period, the performance of the strategy is biased to look better since the parameters are optimal for that particular period. This is also known as in-sample testing in contrast to out-of-sample testing.

In out-of-sample testing, the parameters have been optimized on one data sample and tested on another. In Section 4.2 we described how we apply a rolling window estimation as part of the portfolio construction. Taking such an approach works like out-of-sample testing, which mitigates the optimization bias mentioned above. Each time period, we estimate parameters based on older data, simulate the portfolio's performance over the next month, estimate new parameters based on a shifted time window, simulate the portfolio's performance over the next month and so forth. In other words, we take a segment of our data to optimize and another segment of the data to validate the optimization. This leads to added robustness since the performance of the portfolio is validated across many sub-samples and thus through different market conditions and periods of time.

7.1.3 – Practical Issues

Transaction Costs

Recall, the analysis of the changes in the portfolio weights of Section 5.1. In the case of extensive rebalancing, the portfolios may suffer from higher trading costs. As outlined in the problem statement, this study intends to examine portfolio performance, without taking transaction costs into account. Nonetheless, transaction costs are inevitably an important aspect of portfolio evaluation in practice and can influence performance greatly.

According to Collins & Fabozzi (1991), transaction costs may include direct costs like broker commissions, for example, intermediaries charging fees for buying and selling securities, exchange fees, settlement fees, clearing fees, etc. On the other hand, they may also encompass more indirect costs like market impact, i.e., the deviation between the actual execution price and the price that would have existed in the absence of the trade. Collins & Fabozzi (1991) state that while commissions and fees are fixed and readily observable, market impact is neither fixed nor measurable. Nonetheless, they find that the former type of transaction costs is much smaller than the latter.

Perold & Robert S. Salomon (1991) report that costs related to market impact is proportional to the size of the trade being executed. When the size of the trade is substantial and represents a big part of the total trading volume in the stock, costs related to market impact can be significant. This is a consequence of that large trades may move the stock price substantially. Section 4.1 found that the committee behind the S&P 500 Index employs the criteria of *market capitalization* and *liquidity* when considering a new candidate for the index. Hence, costs related to market impact may not pose that big of an issue for the stocks included in our portfolios.

However, the naïve assumption of being able to transact at constant prices are somewhat unrealistic. We must therefore always anticipate that when we trade, the price will move against us. This is particularly relevant for the portfolios that see the highest changes in portfolio weights, as this may indicate trades of larger size.

In Section 5.1 we found that the multifactor Markowitz portfolios generally experience higher fluctuations in their portfolio weights, while the Single-Index Markowitz portfolio achieves more stable portfolio weights.[7] The Single-Index Markowitz portfolio would thus incur lower transaction costs, which could perhaps suggest better applicability of the portfolio in practice. Nevertheless, we must expect that the portfolios will encounter direct but also indirect transaction costs, which will affect their performance.

On a final note, we recognize that changes in the portfolio composition could in principle be explicitly modeled through the convex optimization technique described in Section 4.2. However, this must be a start for further research.

7.2 – Sub-Conclusion

Throughout the earlier parts of this study, it became clear that several of the multifactor Markowitz portfolios did relatively well both during the full evaluation period, but also through several subperiods. However, we may conclude that the high level of statistically insignificant factor loadings challenges the rationale for the excellent performance of these portfolios. The factor loadings form the foundation for the estimation of the expected returns and the variance-covariance matrix. Hence, they greatly influence how the mean-variance optimization procedure derives the portfolio weights.

Furthermore, we have discussed several relevant data biases, where survivorship bias and lookahead bias are present within our data sample. An immediate solution to mitigate these biases would be to consider historical constituents of the S&P 500 Index and include firms that have gone bankrupt. Additionally, the investment universe could be extended to include stocks with different characteristics than that of the S&P 500 members. Despite the existence of such biases, it must be emphasized that we investigate the relative performance between portfolios based on the same biased data sample. Therefore, we consider this issue less decisive for answering the problem statement.

[7] We also note that this is the case for the unconstrained solution to Single-Index Markowitz portfolio, where the portfolio weights are stable and exhibit non-extreme behavior.

Lastly, we have discussed the practical implications related to portfolio turnover and associated trading costs. Despite that transaction costs are unaccounted for in this study, they do inevitably pose an important consideration in practice. The portfolio compositions of the multifactor Markowitz portfolios are generally less stable, which presumably heightens transaction costs. Despite the modest performance of the Single-Index Markowitz portfolio, the fact that it requires less extensive rebalancing may imply lower transaction costs, and thus better practical applicability.

Nonetheless, the implications we have discussed so far may influence some of the results of the study. However, since it is hardly possible to consider all aspects when forming the analysis, these findings form an interesting foundation for further research.

8 – Conclusion

8.1 – Conclusion

The primary aim of this thesis was to investigate how mean-variance asset allocation can benefit from factors in stock returns on the U.S. stock market.

Based on a review of the academic literature within the field of factor models, we consider two classes of relevant factors, namely macroeconomic factors and fundamental factors. While macroeconomic factors have a strong theoretical appeal, they have generally been found to explain variability in stock returns insufficiently. On the other hand, the explanation for the ability of fundamental factors to explain returns are more unclear, yet they have been more successful in explaining variability in stock returns. Hence, we choose several well-known factor models for the analysis with regards to constructing portfolios. These factor models include Sharpe (1963), Fama & French (1993, 2015) and Carhart (1997).

We investigate and describe a framework for implementing factor models into the mean-variance analysis and evaluate the framework in an econometric context. We calculate the expected returns and the variance-covariance matrix based on factor loadings, the expected returns and the variance-covariance matrix of the factors, as well as the diagonal variance-covariance matrix of the residuals. Our findings constitute that an unconstrained solution to the factor-based mean-variance optimization procedure produces unrealistic results. Hence, we propose a constrained and more realistic solution. Moreover, we find no econometric issues regarding the implementation of the framework.

Applying the proposed factor-based mean-variance analysis, we construct portfolios that are rebalanced monthly and evaluate their performance from January 1979 to January 2019. Based on portfolio backtesting we discover that the portfolios fare differently through changing market conditions. In conclusion, the equally weighted portfolio yields the highest cumulative return index, while the Carhart Four-Factor Markowitz portfolio does also provide an attractive alternative. In terms of risk-to-reward, the conclusion remains. The highest risk-adjusted return is achieved by the equally weighted portfolio, followed by the Carhart Four-Factor Markowitz portfolio. However, separating the full evaluation period into several sub-periods results in additional findings. The Fama-French Five-Factor Markowitz portfolio demonstrates superior performance on a risk-adjusted basis during the first two decades. During the last two decades, the Carhart Four-Factor Markowitz portfolio achieves the highest risk-adjusted performance.

The evaluations over the sub-periods are indicators of that a factor-based mean-variance analysis can lead to better performance than an equally weighted portfolio, and thus benefit from relevant factors. However, the justification for the better performance has been a matter of discussion in this thesis.

We have discussed the statistically significance of the factor loadings and raised the question of whether the portfolios constructed upon a higher number of factors are able to capture important signals and thus pick stocks better. Moreover, several data biases have been discussed that might cause concern for our results. Hence, mitigating these biases completely would heighten the validity of our results. Furthermore, we have considered the aspect of transaction costs, which has led us to believe that the Single-Index Markowitz portfolio perhaps have good applicability in practice. This is a result of the portfolio's more stable composition reducing turnover and thus transaction costs.

In conclusion, we have proposed a more stable alternative to the traditional mean-variance analysis. As we have seen throughout this study, the traditional mean-variance analysis regularly encounters problems with respect to singular variance-covariance matrices, when applied on a high number of assets. Such problems did not occur in our implementation, as none of the variance-covariance matrices estimated across the factor portfolios turned out to be singular. For these reasons, the factor-based mean-variance analysis appears to be an attractive choice for practical applications rather than the traditional framework.

On a final note, we recommend using this study as a foundation for further research surrounding the mean-variance analysis and factor models through some suggested further areas of research that we believe would be interesting to expand upon. These are proposed in the following section.

8.2 – Further Research

Several parts of the thesis could be expanded through further research. These include the use of other factor models, forming portfolios upon different investment universes and imposing additional realistic constraints.

A reasonable start would be to apply the factor-based mean-variance analysis upon factor models, where several of the factors with less explanatory power have been removed. In particular, the value factor (HML) was found to be redundant when combined with the profitability (RMW) and investment (CMA) factors. In addition, Schwert (2003) reports that the size factor (SMB) has decreased considerably or even disappeared since it was first discovered. While this study has merely considered several of the most established factor models within the finance literature, evidence has been found for the existence of other factors in stock returns. Hence, forthcoming research could integrate other factors into the mean-variance analysis and investigate how portfolios formed upon these would perform. However, as emphasized by Feng, Giglio, & Xiu (2019) the vast number of potential risk factors proposed in the finance literature should be used with caution. Further research must therefore be aware of the explanatory power of the factors, and whether they truly capture new dimensions of risk.

Secondly, this study focusses solely on constructing portfolios from stocks of the S&P 500 Index. For future research, the investment universe could be extended to include stocks with different characteristics than that of the constituents of S&P 500 Index. Hence, it would be interesting to investigate the performance of portfolios formed upon less liquid, mid or small-cap stocks. Taking such an approach could potentially strengthen the conclusions we arrive at. Furthermore, Fama & French (2012) document their factors to be persistent in other countries, as well as globally. Therefore, another interesting way of expanding the analysis could be to apply the factor-based mean-variance analysis on stock markets in different countries. This could be done with ease since time series of developed market factors and returns are available from the data library of French (2019). Still, further research must be aware of differences in the explanatory power of the factors for different regions as underlined by Fama & French (2012).

Thirdly, in relation to practical applicability, further research could impose more realistic constraints on the convex optimization problem regarding regulations, capital budgets, investor preferences, etc. While the restriction of not allowing short sales may be relevant for many funds and institutional investors, turnover constraints can also be imposed to limit aforementioned transaction costs in the rebalancing. Lastly, upper and lower bounds on concentrations in specific stocks could be imposed to comply with fund policies and financial regulation.

Bibliography

Ang, A. (2014). *Asset Management: A Systematic Approach to Factor Investing*. Oxford University Press.

Avramov, D., Chordia, T., Jostova, G., & Philipov, A. (2007). Momentum and Credit Rating. *The Journal of Finance*, *62*(5), 2503–2520.

Banz, R. W. (1981). The relationship between return and market value of common stocks. *Journal of Financial Economics*, *9*(1), 3–18.

Basu, S. (1977). Investment Performance of Common Stocks in Relation to Their Price-Earnings Ratios: A Test of the Efficient Market Hypothesis. *Journal of Finance, 32*(3), 663–682.

Bhandari, L. C. (1988). Debt/Equity Ratio and Expected Common Stock Returns: Empirical Evidence. *The Journal of Finance*, *43*(2), 507–528.

Bodie, Z., Kane, A., & Marcus, A. J. (2009). *Investments* (8th ed.). McGraw-Hill/Irwin.

Campbell, J. Y., Lo, A. W., & MacKinlay, A. C. (1997). *The Econometrics of Financial Markets*. Princeton University Press.

Carhart, M. M. (1997). On Persistence in Mutual Fund Performance. *The Journal of Finance*, *52*(1), 57–82.

Chen, N.-F., Roll, R., & Ross, S. (1986). Economic Forces and the Stock Market. *The Journal of Business*, *59*(3), 383–403.

Collins, B. M., & Fabozzi, F. J. (1991). A Methodology for Measuring Transaction Costs. *Financial Analysts Journal, 47*(2), 27–44.

Connor, G., & Korajczyk, R. (2010). Factor Models in Portfolio and Asset Pricing Theory. In *Handbook of Portfolio Construction* (pp. 401–418). Springer.

Cornuejols, G., & Tütüncü, R. (2007). *Optimization Methods in Finance* (2nd ed.). Cambridge University Press.

Cuthbertson, K., & Nitzsche, D. (2004). *Quantitative Financial Economics: Stocks, Bonds and Foreign Exchange* (2nd ed.). Chichester, England: Wiley.

Daniel, K., & Moskowitz, T. J. (2016). Momentum crashes. *Journal of Financial Economics*, *122*(2), 221–247.

Danthine, J.-P., & Donaldson, J. B. (2005). *Intermediate financial theory* (2nd ed.). Elsevier.

Fama, E. F., & French, K. (1989). Business conditions and expected returns on stocks and bonds. *Journal of Financial Economics*, *25*(1), 23–49.

Fama, E. F., & French, K. R. (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics*, *33*(1), 3–56.

Fama, E. F., & French, K. R. (1996). Multifactor Explanations of Asset Pricing Anomalies. *The Journal of Finance, 51*(1), 55–84.

Fama, E. F., & French, K. R. (2004). The Capital Asset Pricing Model: Theory and Evidence. *Journal of Economic Perspectives*, *18*(3), 25–46.

Fama, E. F., & French, K. R. (2012). Size, value, and momentum in international stock returns. *Journal of Financial Economics*, *105*(3), 457–472.

Fama, E. F., & French, K. R. (2015). A five-factor asset pricing model. *Journal of Financial Economics*, *116*(1), 1–22.

Feng, G., Giglio, S., & Xiu, D. (2019). *Taming the Factor Zoo: A Test of New Factors*. National Bureau of Economic Research.

Feng, Y., & Palomar, D. P. (2016). *A Signal Processing Perspective on Financial Engineering*. *Foundations and Trends® in Signal Processing* (Vol. 9). Now Publishers, Inc.

Fox, J. (2016). *Applied Regression Analysis and Generalized Linear Models* (3rd ed.). London, England: SAGE.

French, K. (2019). U.S. Research Returns Data. Retrieved February 1, 2019, from http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

Fu, A., Narasimhan, B., & Boyd, S. (2017). CVXR: An R Package for Disciplined Convex Optimization.

Hvidkjaer, S. (2006). A Trade-Based Analysis of Momentum. *The Review of Financial Studies*, *19*(2), 457–491.

Jegadeesh, N., & Titman, S. (1993). Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency. *The Journal of Finance*, *48*(1), 65–91.

Jensen, M. C. (1968). The performance of mutual funds in the period 1945–1964. *The Journal of Finance*, *23*(2), 389–416.

Johnson, R. A., & Wichern, D. W. (2007). *Appled Multivariate Statistical Analysis* (6th ed.). Pearson.

Ledoit, O., & Wolf, M. (2003). Improved estimation of the covariance matrix of stock returns with an application to portfolio selection. *Journal of Empirical Finance*, *10*(5), 603–621.

Lintner, J. (1965). The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfoliosand Capital Budgets. *The Review of Economics and Statistics*, *47*(1), 13–37.

Markowitz, H. (1952). Portfolio selection. *Journal of Finance*, *7*(1), 77–91.

Modigliani, F., & Modigliani, L. (1997). Risk-Adjusted Performance. *The Journal of Portfolio Management*, *23*(2), 45–54.

Mossin, J. (1966). Equilibrium in a Capital Asset Market. *Econometrica*, *34*(4).

Munk, C. (2017). *Financial Markets and Investments* (Lecture Notes). Copenhagen, Denmark.

Novy-Marx, R. (2013). The other side of value: The gross profitability premium. *Journal of Financial Economics*, *108*(1), 1–28.

Pedersen, L. H. (2015). *Efficiently Inefficient: How Smart Money Invests and Market Prices Are Determined*. Princeton University Press.

Perold, A. F., & Robert S. Salomon, J. (1991). The Right Amount of Assets under Management. *Financial Analysts Journal*, *47*(3), 31–39.

R Core Team. (2019). R: A Language and Environment for Statistical Computing. Vienna, Austria: R Foundation for Statistical Computing. Retrieved from https://www.r-project.org

Rapach, D. E., Wohar, M. E., & Rangvid, J. (2005). Macro variables and international stock return predictability. *International Journal of Forecasting*, *21*(1), 137–166.

Roll, R. (1977). A Critique of the Asset Pricing Theory's Tests. *Journal of Financial Economics, 4*, 129–176.

Rosenberg, B. (1974). Extra-Market Components of Covariance in Security Returns. *The Journal of Financial and Quantitative Analysis*, *9*(2), 263–274.

Rosenberg, B., Reid, K., & Lanstein, R. (1985). Persuasive evidence of market inefficiency. *The Journal of Portfolio Management*, *11*(3), 9 LP-16.

Ross, S. (1976). The Arbitrage Theory of Capital Asset Pricing. *Journal of Economic Theory*, *13*(3), 341–360.

Ruppert, D., & Matteson, D. S. (2015). *Statistics and Data Analysis for Financial Engineering: With R Examples*. (Springer, Ed.) (2nd ed.).

S&P Dow Jones. (2019a). S&P 500 - Factsheet. Retrieved March 1, 2019, from https://us.spindices.com/indices/equity/sp-500

S&P Dow Jones. (2019b). S&P 500 - The Gauge of the Market Economy. Retrieved March 1, 2019, from https://us.spindices.com/indices/equity/sp-500

S&P Dow Jones. (2019c). S&P U.S. Indices - Methodology. Retrieved March 1, 2019, from https://us.spindices.com/documents/methodologies/methodology-sp-us-indices.pdf

Samarakoon, L. P., & Hasan, T. (2006). Portfolio performance evaluation. In C.-F. Lee & A. C. Lee (Eds.), *Encyclopedia of Finance* (pp. 617–622). Boston, MA: Springer US.

Schwert, G. W. (2003). Anomalies and market efficiency. *Handbook of the Economics of Finance*, *1*, 939–974.

Sharpe, W. F. (1963). A Simplified Model for Portfolio Analysis. *Management Science*, *9*(2), 277–293.

Sharpe, W. F. (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. *The Journal of Finance*, *19*(3).

Sharpe, W. F. (1966). Mutual Fund Performance. *The Journal of Business*, *39*(1), 119–138.

Skovmand, D. (2015). *Derivation of the APT restriction* (Lecture Notes). Copenhagen, Denmark.

Stattman, D. (1980). Book Values and Stock Returns. *A Journal of Selected Papers*, *4*, 22–45.

Sydsæter, K., Strøm, A., & Berck, P. (2005). *Economists' mathematical Manual* (4th ed.). Springer.

Thomson Reuters. (2019). Thomson Reuters Datastream Database. Accessed: February 1, 2019.

Titman, S., Wei, K. C. J., & Xie, F. (2004). Capital Investments and Stock Returns. *The Journal of Financial and Quantitative Analysis*, *39*(4), 677–700.

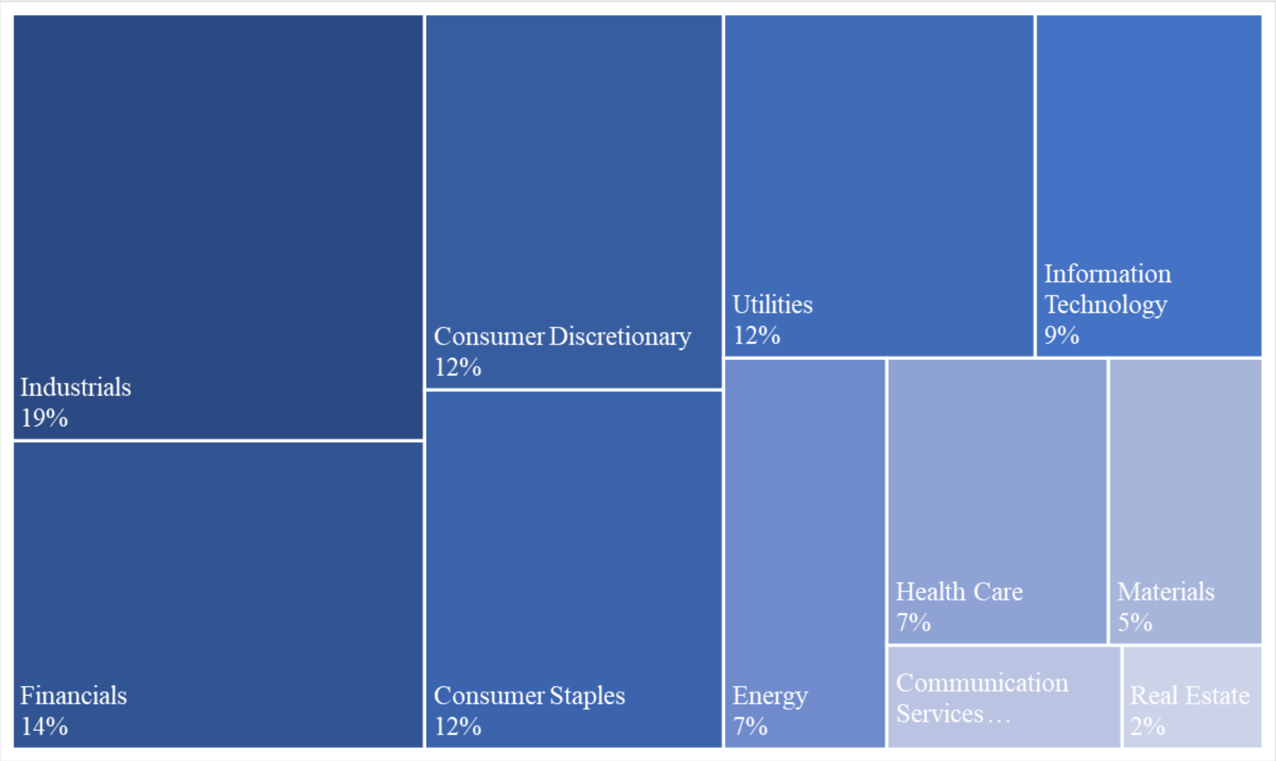
Treynor, J. (1961). Market Value, Time, and Risk.

Wooldridge, J. M. (2016). *Introductory Econometrics: A Modern Approach* (6th ed.). Boston, MA: Cengage Learning.

Appendices

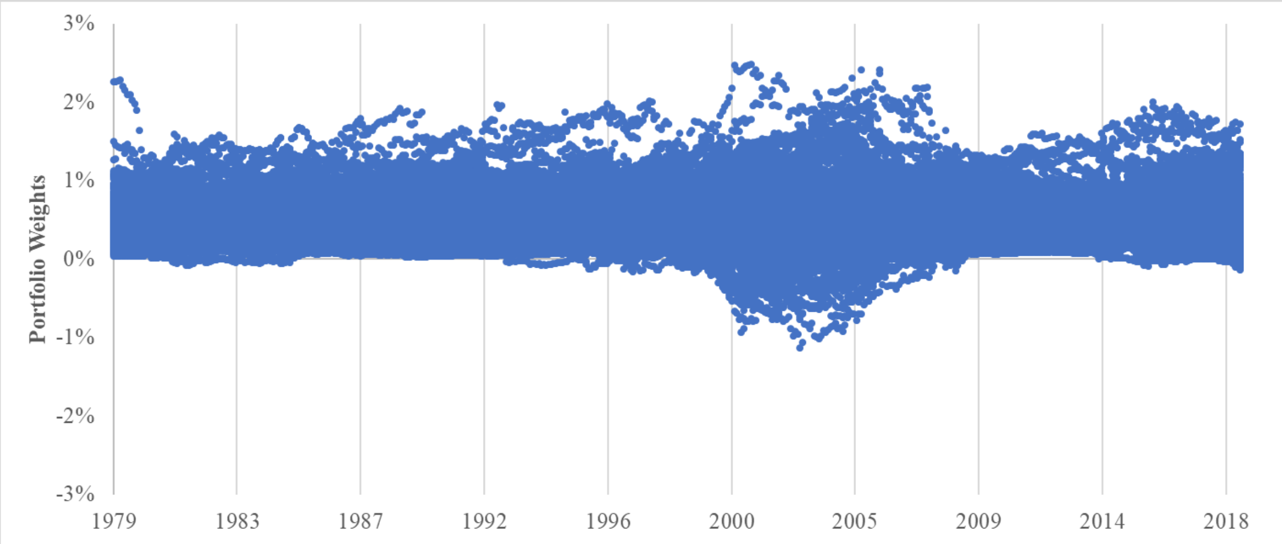
Appendix A – Data Sample Industry Decomposition

Industry Decomposition based on the Global Industry Classification Standard (GICS)

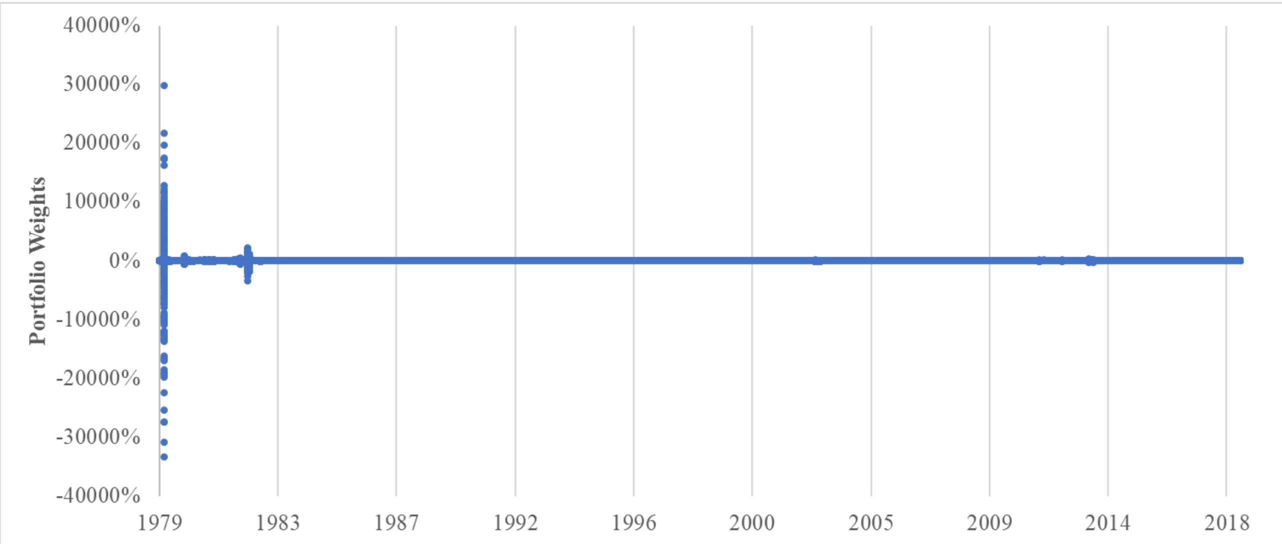


Appendix B – Unconstrained Factor Markowitz Portfolio Weights

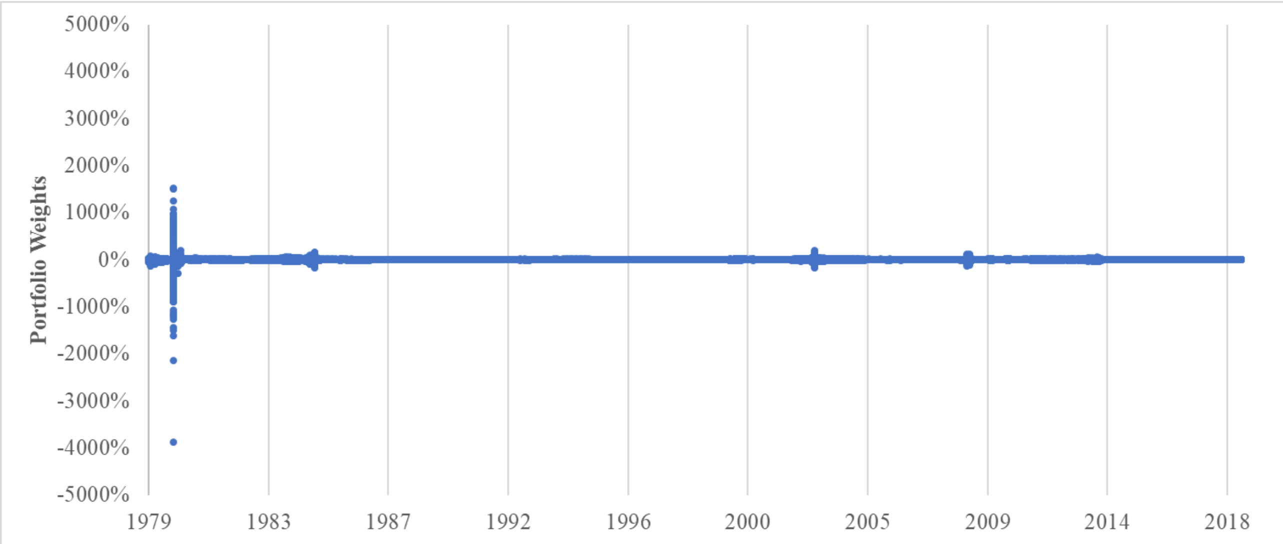
Unconstrained Portfolio Weights of the Single-Index Markowitz Portfolio



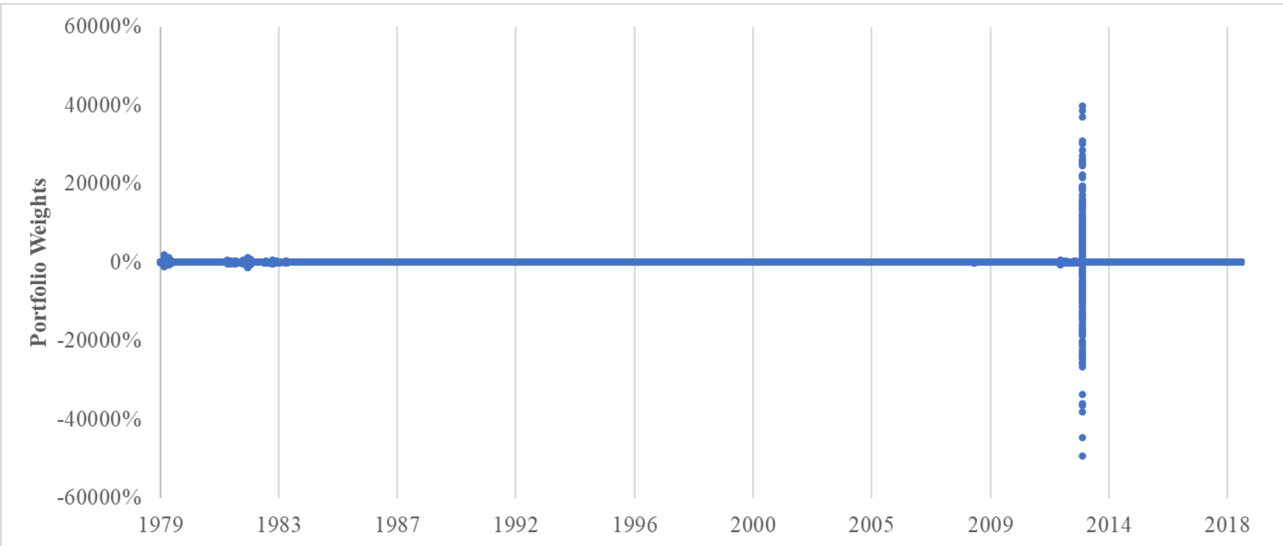
Unconstrained Portfolio Weights of the Fama-French Three-Factor Markowitz Portfolio



Unconstrained Portfolio Weights of the Carhart Four-Factor Markowitz Portfolio

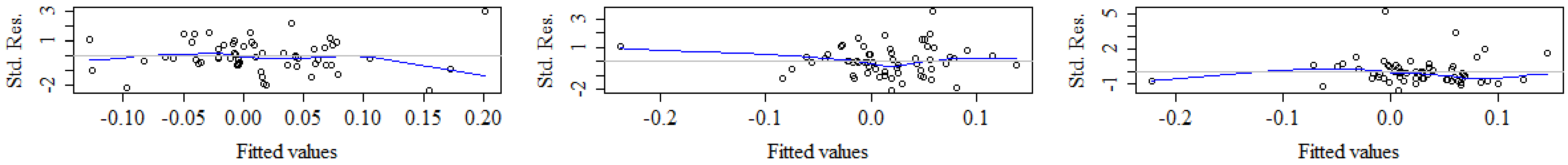


Unconstrained Portfolio Weights of the Fama-French Five-Factor Markowitz Portfolio

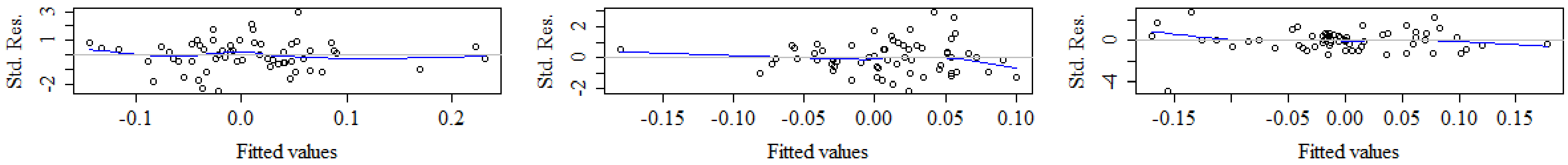


Appendix C – Fitted Values against Standardized Residuals for Each Factor Regression

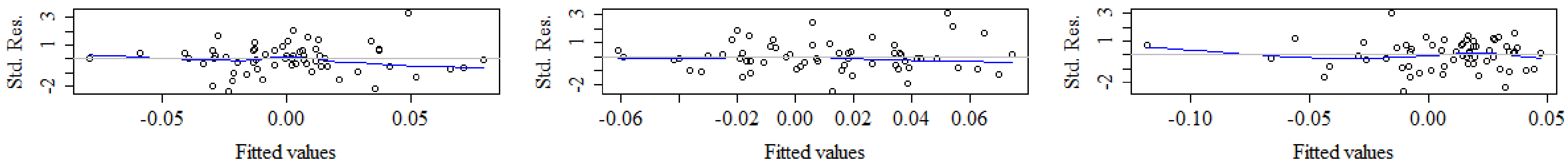
Single-Index Model



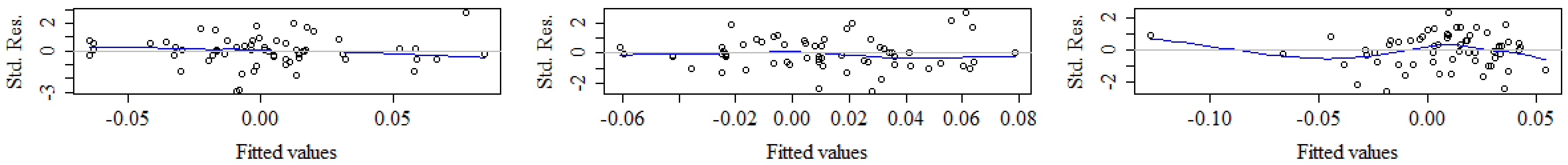
Fama-French Three-Factor Model



Carhart Four-Factor Model

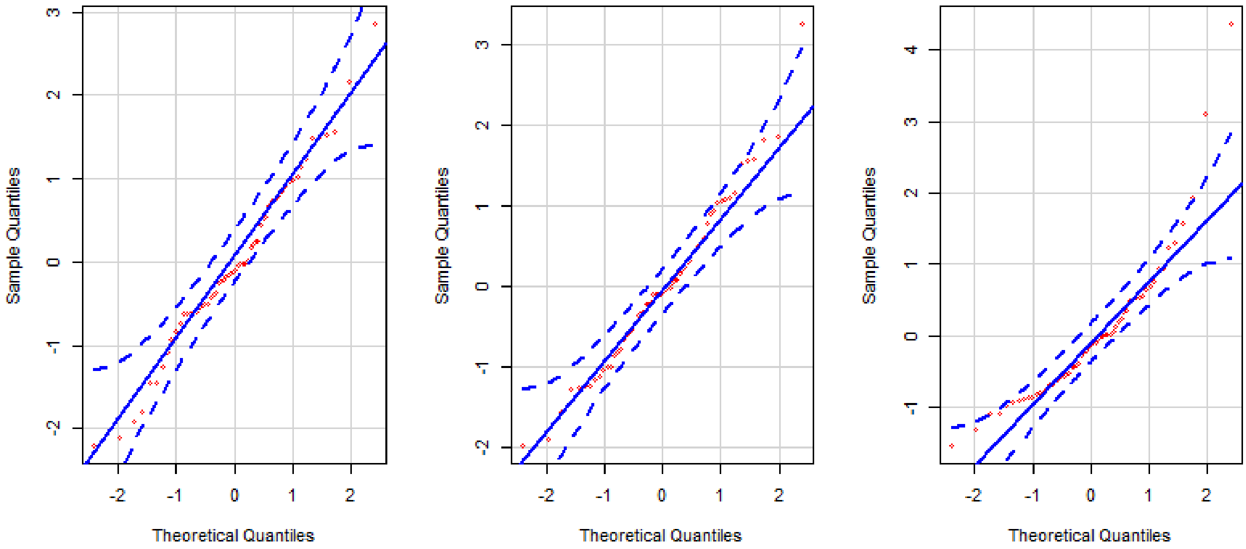


Fama-French Five-Factor Model

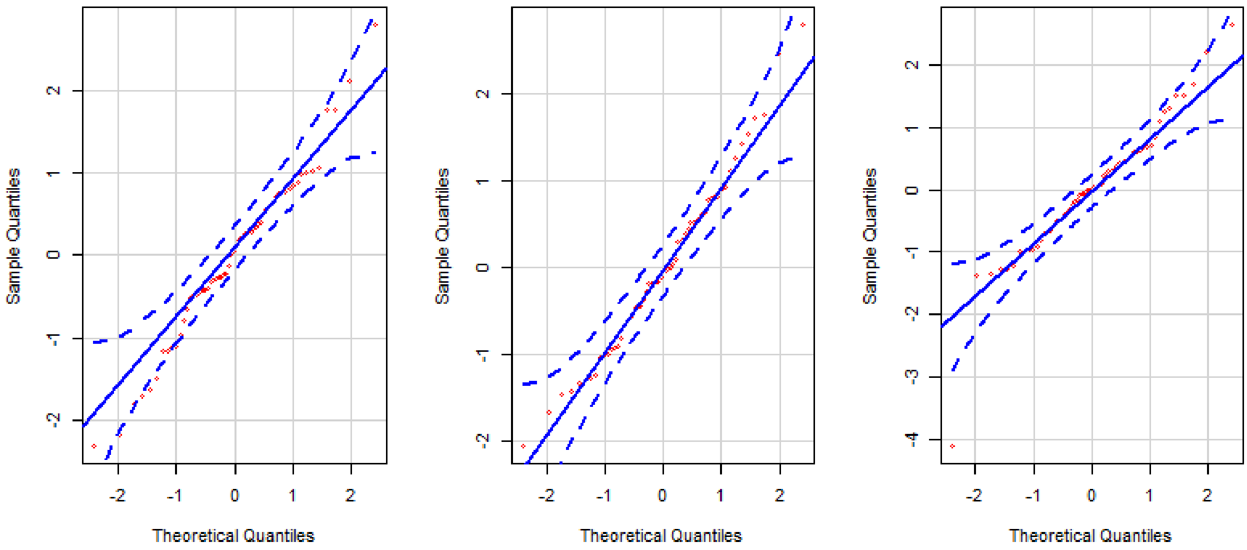


Appendix D – QQ-plots of Standardized Residuals for Each Factor Regression

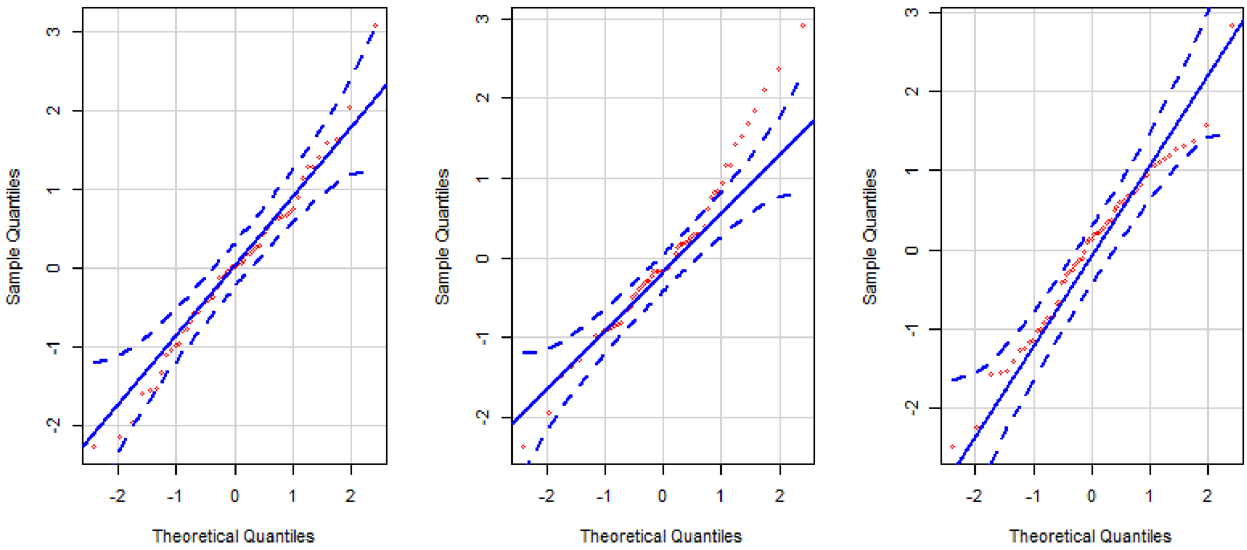
Single-Index Model



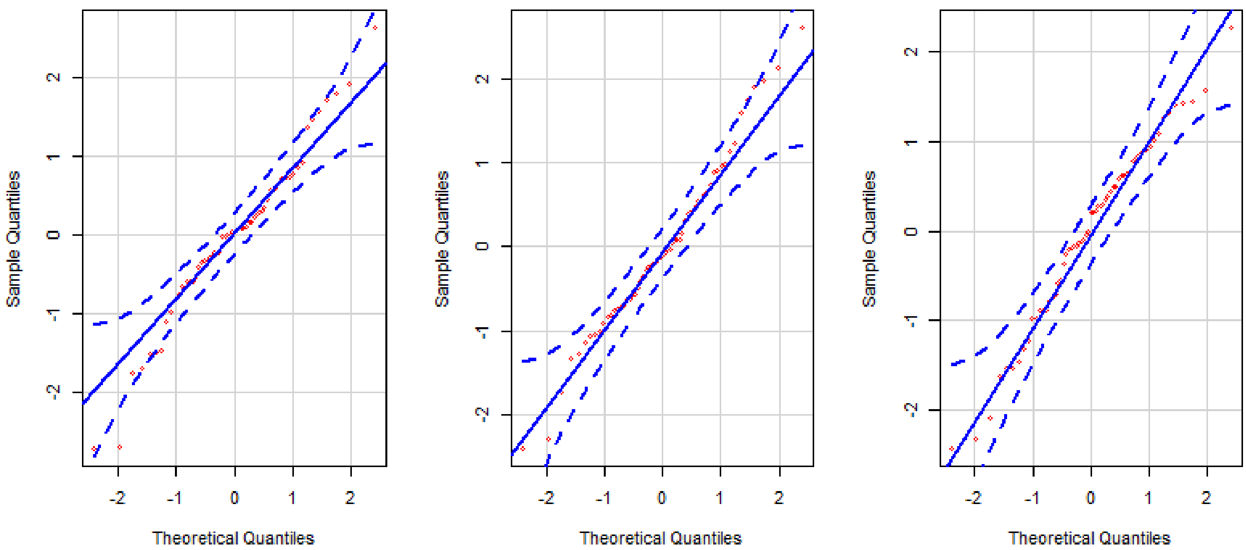
Fama-French Three-Factor Model



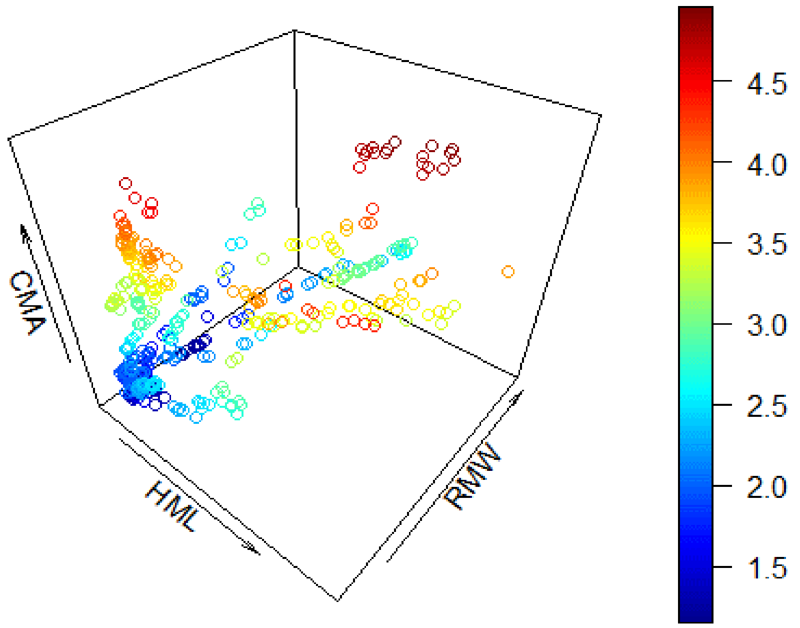
Carhart Four-Factor Model



Fama-French Five-Factor Model

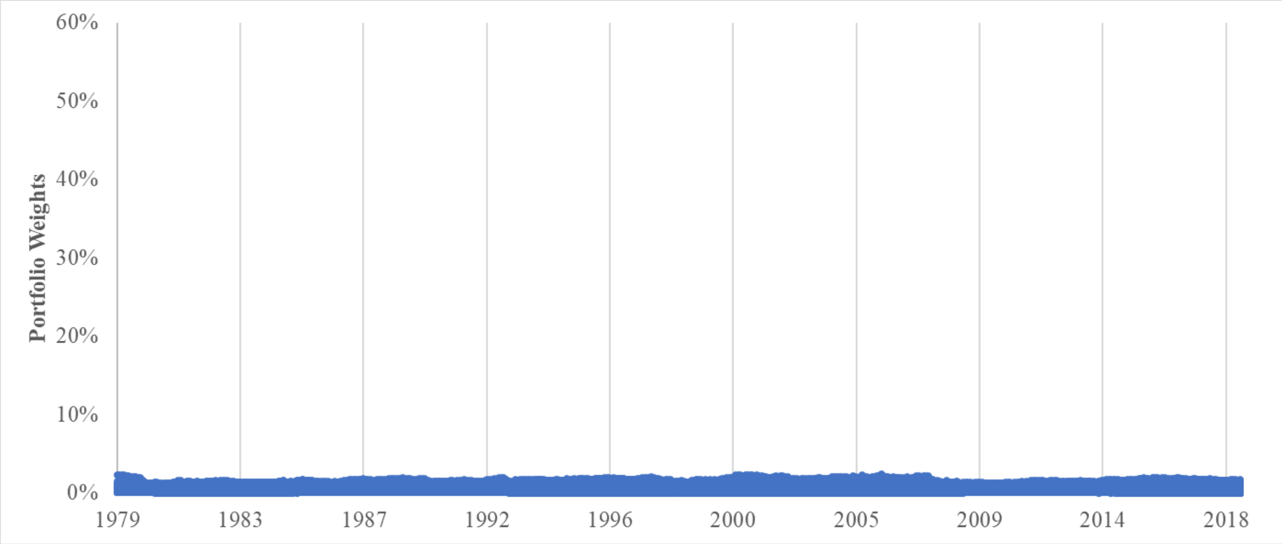


Appendix E – Variance Inflation Factors of the HML, CMA and RMW Factors

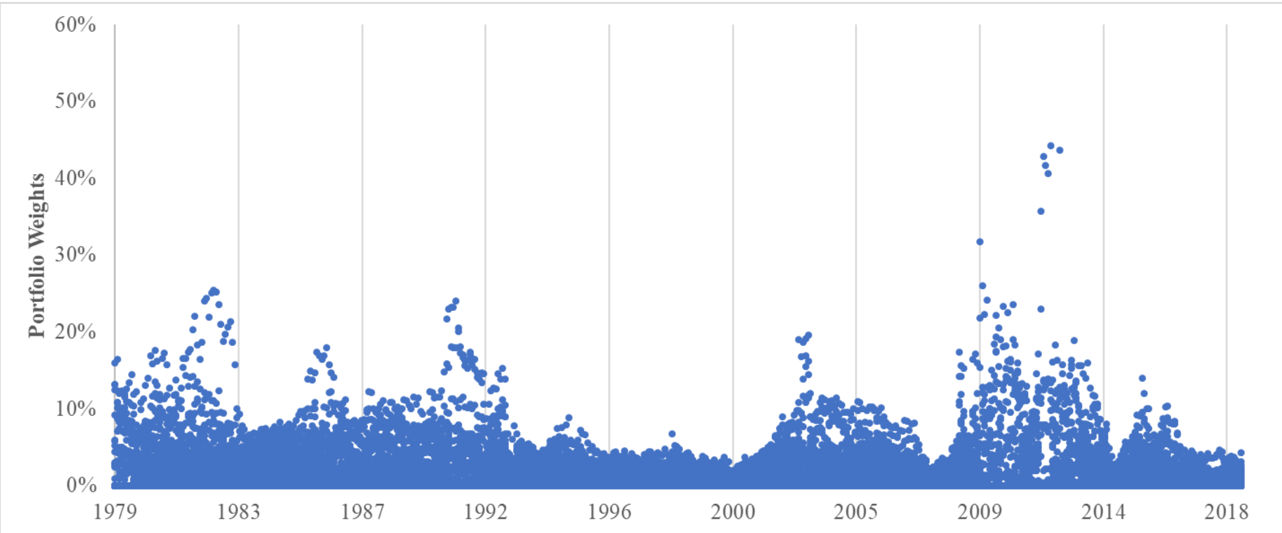


Appendix F – Constrained Factor Markowitz Portfolio Weights

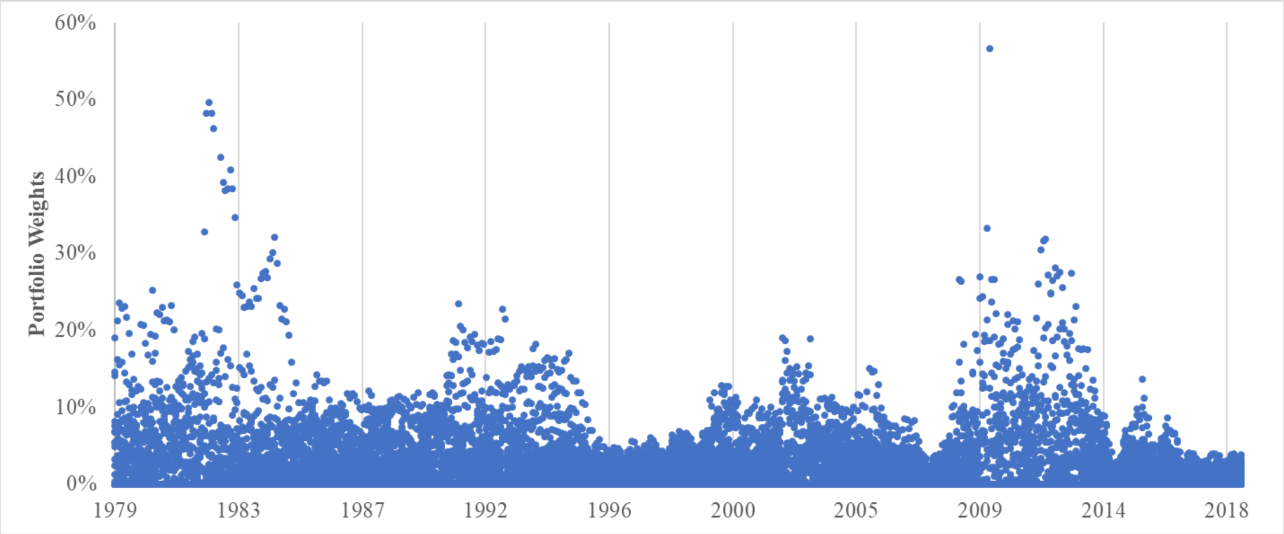
Constrained Portfolio Weights of the Single-Index Markowitz Portfolio



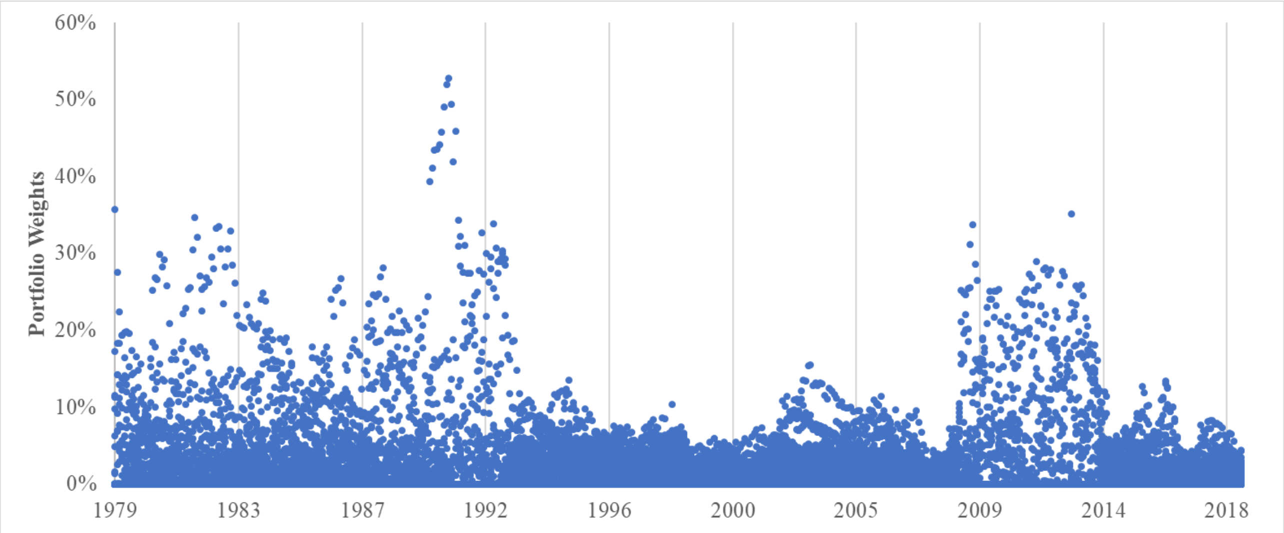
Constrained Portfolio Weights of the Fama-French Three-Factor Markowitz Portfolio



Constrained Portfolio Weights of the Carhart Four-Factor Markowitz Portfolio

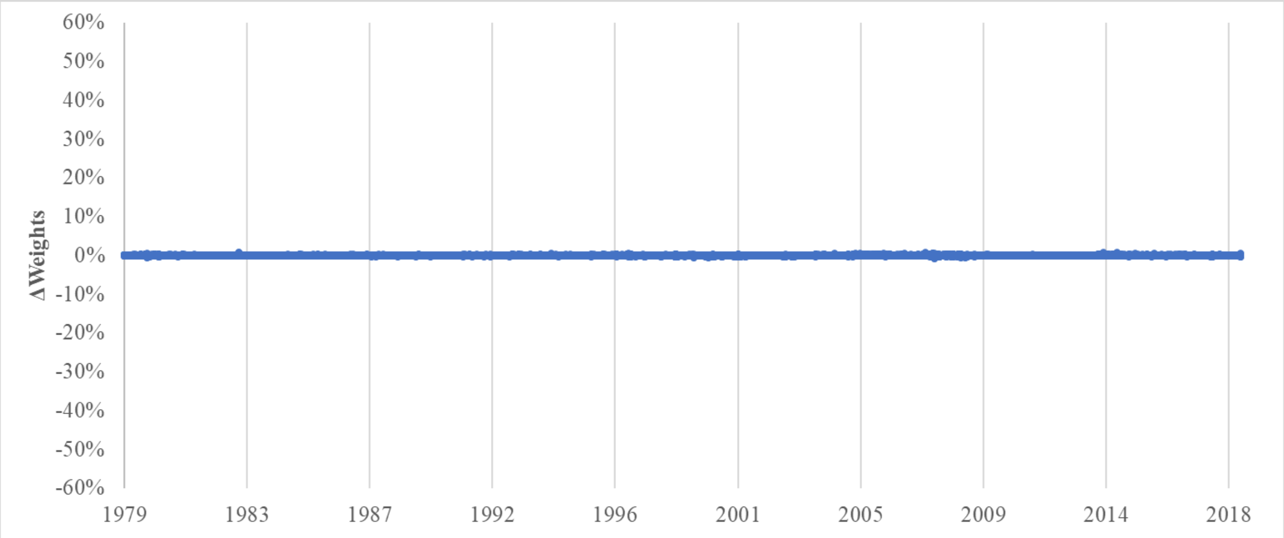


Constrained Portfolio Weights of the Fama-French Five-Factor Markowitz Portfolio

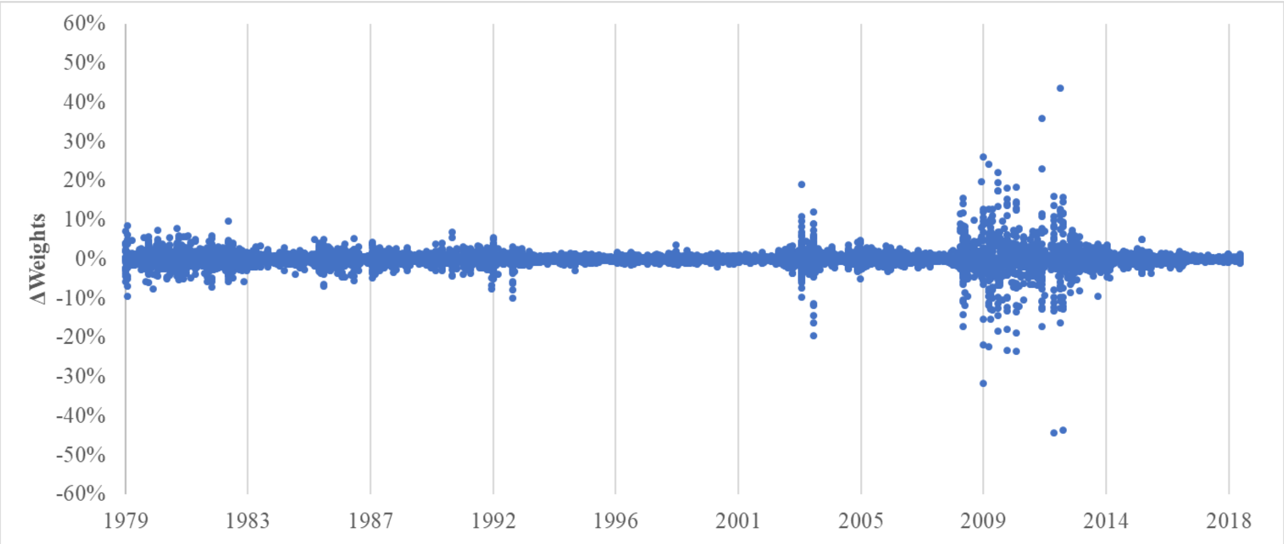


Appendix G – ΔWeights for the Factor Markowitz Portfolios

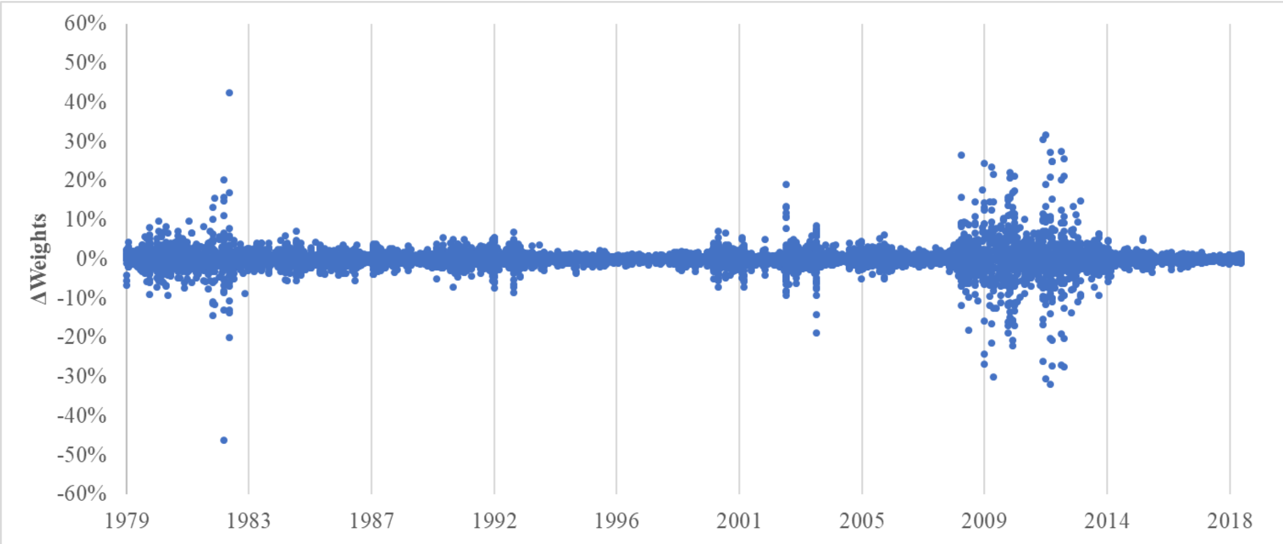
ΔWeights of the Single-Index Markowitz Portfolio



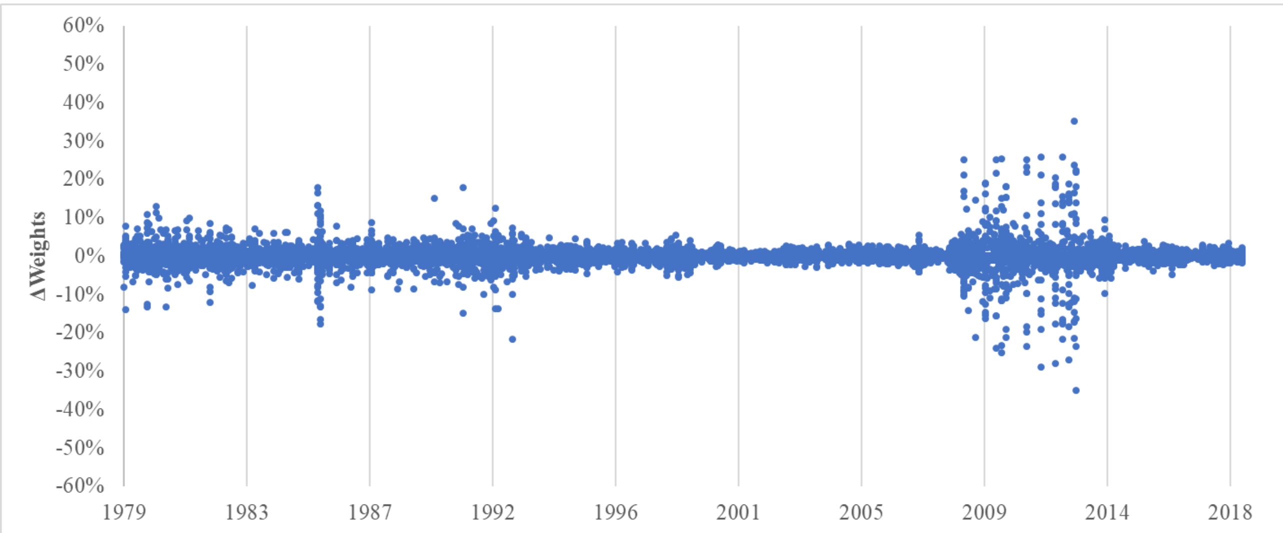
ΔWeights of the Fama-French Three-Factor Markowitz Portfolio



ΔWeights of the Carhart Four-Factor Markowitz Portfolio



ΔWeights of the Fama-French Five-Factor Markowitz Portfolio



Appendix H – Portfolio Excess Returns

Excess Returns of the Single-Index Markowitz Portfolio

	Monthly Returns (%)											
Year	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1979	-	4.2%	-4.5%	3.2%	0.7%	-2.2%	2.3%	2.7%	5.3%	-3.2%	-8.0%	3.5%
1980	1.3%	3.9%	-6.9%	-8.0%	3.1%	5.3%	3.3%	8.5%	-0.6%	2.1%	-0.7%	2.4%
1981	0.2%	-4.6%	3.9%	5.0%	-0.9%	0.2%	-3.7%	-3.3%	-7.3%	-5.7%	6.2%	1.2%
1982	-2.9%	-4.0%	-2.7%	0.1%	2.7%	-5.1%	-2.9%	0.1%	9.2%	3.1%	12.9%	5.3%
1983	-1.2%	2.9%	4.8%	1.9%	4.7%	2.0%	2.6%	-4.6%	-0.5%	1.7%	-2.5%	2.8%
1984	-1.1%	-3.8%	-4.9%	0.2%	0.7%	-5.7%	-0.2%	-0.3%	9.1%	-2.8%	1.4%	-2.6%
1985	2.8%	8.7%	1.8%	-2.2%	-2.7%	6.7%	1.6%	1.0%	-2.6%	-4.1%	3.2%	5.8%
1986	3.9%	3.2%	6.4%	4.3%	-0.7%	3.4%	1.5%	-7.7%	6.9%	-8.3%	5.0%	0.6%
1987	-2.9%	15.3%	3.2%	1.2%	-2.9%	0.1%	4.4%	3.6%	2.7%	0.5%	-24.4%	-7.3%
1988	6.8%	4.6%	5.6%	-2.6%	0.3%	0.7%	2.8%	-1.1%	-4.8%	4.5%	0.4%	-3.5%
1989	2.0%	5.4%	-3.1%	2.7%	4.0%	3.5%	-1.6%	6.7%	2.6%	-1.9%	-3.8%	1.4%
1990	0.3%	-7.5%	0.7%	1.3%	-2.4%	8.4%	-0.9%	-2.4%	-11.2%	-4.9%	-4.6%	9.8%
1991	2.5%	6.1%	8.0%	1.8%	2.1%	3.4%	-3.2%	1.7%	2.2%	-1.5%	1.1%	-2.6%
1992	9.7%	1.0%	1.8%	-1.8%	1.1%	0.9%	-1.2%	2.8%	-1.9%	0.9%	3.0%	2.8%
1993	1.5%	1.1%	0.0%	2.6%	-1.9%	1.7%	-0.7%	-0.4%	3.3%	-0.7%	1.3%	-1.8%
1994	1.6%	2.4%	-1.9%	-4.5%	1.0%	0.1%	-1.8%	2.5%	2.2%	-3.1%	0.6%	-4.1%
1995	2.3%	2.4%	3.4%	2.7%	0.8%	3.4%	1.3%	2.1%	0.7%	2.1%	-0.5%	4.3%
1996	1.0%	3.0%	0.8%	1.7%	-0.2%	0.8%	1.0%	-4.4%	1.1%	4.3%	1.6%	6.6%
1997	-1.8%	3.3%	1.3%	-3.6%	4.3%	5.2%	5.5%	5.3%	-3.9%	5.7%	-2.3%	4.1%
1998	0.4%	1.1%	4.6%	4.3%	-0.2%	-2.9%	1.3%	-5.1%	-10.5%	-1.5%	15.2%	2.3%
1999	1.8%	-1.9%	-0.8%	1.4%	11.7%	-3.6%	3.1%	-4.4%	-2.1%	-6.7%	3.5%	-1.1%
2000	-1.8%	-1.7%	-6.6%	12.3%	0.3%	1.8%	-3.6%	0.7%	4.1%	-2.4%	4.3%	-0.7%
2001	6.3%	1.7%	-3.3%	-4.7%	8.1%	2.4%	-3.4%	0.1%	-3.5%	-12.1%	5.8%	6.2%
2002	3.7%	0.4%	3.6%	2.5%	-3.0%	-3.4%	-7.2%	-9.9%	3.1%	-8.2%	4.7%	8.3%
2003	-6.2%	-3.5%	-2.2%	1.2%	8.1%	8.2%	0.4%	3.2%	4.9%	-0.3%	6.7%	1.4%
2004	3.4%	1.5%	3.5%	-1.6%	-2.1%	1.1%	1.7%	-1.7%	-1.0%	4.0%	0.3%	6.5%
2005	0.1%	-1.2%	2.4%	-3.0%	-2.0%	3.9%	-0.1%	5.5%	-1.4%	0.3%	-1.8%	5.7%
2006	-1.8%	3.5%	1.7%	0.9%	1.2%	-1.2%	-1.2%	-1.9%	2.8%	1.0%	2.4%	1.7%
2007	1.2%	3.1%	-1.7%	0.6%	3.8%	3.0%	-1.4%	-4.2%	-0.5%	3.4%	-4.1%	-1.2%
2008	-1.6%	-1.9%	-4.6%	2.8%	3.2%	-1.0%	-8.6%	0.3%	3.0%	-9.2%	-17.9%	-15.0%
2009	12.8%	-8.1%	-16.7%	16.9%	15.3%	6.0%	-2.3%	10.2%	0.8%	2.9%	1.1%	6.6%
2010	2.2%	-1.5%	3.4%	5.7%	4.1%	-11.0%	-3.6%	10.2%	-3.8%	6.2%	2.9%	2.7%
2011	4.9%	2.3%	-0.2%	3.5%	1.9%	-3.1%	2.3%	-6.5%	-5.9%	-9.2%	12.5%	2.9%
2012	1.4%	5.7%	2.5%	2.2%	-0.1%	-8.0%	5.5%	0.9%	1.9%	2.2%	1.0%	-0.7%
2013	2.0%	7.1%	1.2%	3.3%	1.2%	2.7%	-1.1%	6.6%	-5.0%	4.2%	4.1%	1.4%
2014	3.2%	-5.6%	6.5%	2.9%	-0.1%	2.0%	2.3%	-3.6%	3.9%	-3.7%	5.0%	1.9%
2015	1.3%	-1.9%	4.6%	-2.4%	1.0%	0.1%	-0.9%	-0.2%	-8.0%	-0.6%	8.9%	0.6%
2016	-3.5%	-5.4%	3.5%	5.6%	2.7%	1.1%	-0.3%	3.1%	1.3%	-1.1%	-2.7%	7.3%
2017	1.3%	1.8%	5.1%	-2.1%	0.9%	1.6%	0.6%	-0.1%	-1.4%	3.5%	1.1%	2.8%
2018	1.3%	4.3%	-6.3%	-2.9%	1.8%	1.2%	-1.2%	3.7%	1.8%	0.7%	-6.9%	2.9%
2019	-11.1%	-	-	-	-	-	-	-	-	-	-	-
	Month											

Excess Returns of the Fama-French-Three Factor Markowitz Portfolio

Year	Monthly Returns (%)											
	1	2	3	4	5	6	7	8	9	10	11	12
1979	-	19.8%	-3.9%	5.6%	6.6%	-1.3%	1.6%	2.6%	8.6%	-4.2%	-12.0%	6.6%
1980	0.9%	4.6%	-5.6%	-13.0%	0.9%	4.9%	7.2%	7.9%	7.0%	2.1%	-0.2%	0.8%
1981	-2.9%	-6.6%	3.0%	8.2%	1.4%	4.1%	-4.2%	-5.2%	-5.0%	-8.5%	9.3%	3.6%
1982	-3.7%	-2.3%	-4.0%	5.0%	0.2%	-5.1%	-3.9%	0.4%	1.7%	2.3%	10.2%	8.0%
1983	-2.1%	3.2%	2.9%	7.6%	5.5%	1.8%	3.6%	-3.9%	-1.0%	3.2%	0.3%	1.4%
1984	-1.8%	-0.3%	-4.7%	-2.4%	0.4%	-3.9%	-0.9%	-0.6%	4.3%	2.8%	1.6%	0.9%
1985	1.9%	3.0%	1.9%	2.5%	-1.2%	6.8%	2.4%	-1.7%	-1.3%	-4.0%	5.7%	3.4%
1986	4.0%	5.4%	5.7%	3.1%	-2.4%	1.5%	4.3%	0.0%	7.3%	-7.9%	2.3%	-0.1%
1987	-3.9%	11.5%	-0.9%	-1.5%	-3.5%	0.5%	2.0%	0.6%	-0.8%	-3.4%	-15.3%	-6.3%
1988	0.8%	10.0%	-2.1%	-4.9%	-1.2%	3.9%	0.1%	-1.3%	-2.2%	1.3%	3.5%	-1.7%
1989	0.3%	1.2%	-2.8%	2.0%	3.0%	3.8%	1.5%	4.8%	-2.4%	-1.0%	0.5%	1.6%
1990	3.8%	-6.1%	-1.6%	-1.8%	-5.1%	3.2%	-0.3%	2.2%	-10.2%	1.4%	7.3%	-1.3%
1991	-0.7%	-3.4%	2.5%	0.8%	0.9%	-2.8%	-0.3%	3.5%	2.3%	3.6%	-0.6%	2.4%
1992	7.9%	-6.1%	-1.1%	-1.3%	4.5%	1.3%	-0.2%	5.2%	-2.1%	-1.3%	-0.9%	-0.6%
1993	2.1%	2.8%	3.6%	2.3%	1.8%	0.6%	-0.3%	2.0%	2.0%	-0.2%	-2.4%	-6.5%
1994	1.3%	1.4%	-4.8%	-3.7%	1.9%	-1.8%	-1.4%	3.9%	-1.8%	-3.2%	1.6%	-3.7%
1995	0.5%	1.2%	3.2%	1.4%	1.4%	4.0%	-0.3%	1.7%	2.1%	0.4%	-0.6%	3.3%
1996	1.4%	2.2%	0.9%	2.4%	0.7%	-0.7%	-0.4%	-3.8%	2.3%	3.4%	2.1%	7.0%
1997	-1.0%	3.4%	1.4%	-3.9%	2.0%	4.4%	4.4%	4.5%	-1.3%	6.3%	-0.6%	2.4%
1998	2.4%	-1.6%	3.9%	5.5%	-0.6%	-2.6%	1.0%	-6.4%	-9.6%	1.4%	4.7%	2.1%
1999	1.2%	-6.5%	-1.6%	0.9%	6.1%	-2.8%	0.7%	-4.5%	-1.9%	-6.7%	6.0%	-3.8%
2000	-5.3%	-1.4%	-7.6%	12.1%	-0.4%	-0.1%	-2.3%	0.6%	5.7%	-3.5%	3.8%	-1.2%
2001	7.5%	-0.7%	-0.4%	-3.6%	8.4%	2.7%	-3.2%	-1.0%	-0.1%	-10.4%	3.7%	3.1%
2002	4.7%	0.2%	4.0%	2.6%	-0.1%	-3.6%	-3.9%	-7.5%	3.3%	-5.4%	0.8%	1.6%
2003	2.4%	-1.2%	-0.5%	5.5%	6.6%	1.5%	2.9%	-3.7%	0.6%	4.0%	5.1%	4.5%
2004	1.3%	4.9%	3.5%	2.6%	-5.2%	1.4%	1.5%	0.9%	2.0%	3.8%	2.0%	6.7%
2005	2.5%	0.1%	2.4%	-2.4%	2.6%	2.6%	1.1%	3.4%	-1.1%	0.2%	-5.9%	4.5%
2006	-0.9%	2.8%	0.9%	-0.8%	-2.0%	0.7%	0.7%	2.3%	0.8%	-0.2%	2.0%	3.5%
2007	0.1%	2.1%	-0.2%	3.6%	1.9%	1.3%	-1.9%	-3.2%	-0.1%	3.5%	-2.6%	-0.8%
2008	-1.9%	0.0%	-6.3%	3.1%	2.3%	-3.8%	-10.7%	2.1%	3.6%	5.7%	-17.6%	-23.8%
2009	14.2%	-10.0%	-16.2%	15.5%	9.2%	6.2%	1.4%	1.3%	-0.1%	3.3%	-3.2%	6.3%
2010	-0.2%	-4.6%	1.6%	2.7%	4.9%	-9.8%	-5.6%	7.8%	-0.2%	3.7%	0.4%	0.9%
2011	8.7%	7.5%	3.2%	2.5%	1.7%	-3.1%	2.9%	-2.6%	-6.4%	-9.3%	10.7%	2.7%
2012	3.8%	2.4%	1.6%	5.1%	-2.1%	-7.2%	4.7%	2.2%	4.7%	2.1%	-0.4%	3.9%
2013	-0.7%	9.2%	3.4%	3.0%	-0.5%	2.1%	-0.4%	6.0%	-5.7%	2.6%	4.7%	1.5%
2014	2.2%	-5.2%	7.1%	4.1%	1.0%	1.9%	2.5%	-3.8%	3.5%	-2.0%	4.4%	2.4%
2015	0.0%	0.6%	1.8%	-2.8%	1.8%	0.1%	-2.4%	3.1%	-7.1%	2.6%	7.1%	-0.6%
2016	-0.1%	0.3%	0.9%	5.1%	-1.1%	-0.1%	4.2%	0.4%	-2.8%	-1.7%	-1.9%	2.5%
2017	2.4%	-0.5%	4.8%	-1.9%	0.2%	2.0%	-0.4%	0.3%	-1.4%	1.4%	1.1%	2.8%
2018	0.4%	2.5%	-6.0%	-2.0%	1.2%	0.3%	0.3%	3.8%	2.3%	1.0%	-5.0%	3.6%
2019	-8.5%	-	-	-	-	-	-	-	-	-	-	-

Month

Excess Returns of the Carhart Four-Factor Markowitz Portfolio

Monthly Returns (%)

Year	1	2	3	4	5	6	7	8	9	10	11	12
1979	-	2.6%	2.2%	8.1%	4.5%	2.0%	5.2%	0.9%	9.1%	-3.9%	-7.8%	7.0%
1980	4.1%	5.8%	-3.0%	-16.5%	5.0%	3.0%	7.5%	8.3%	8.5%	6.2%	3.3%	7.9%
1981	-7.3%	-13.4%	6.9%	7.3%	-3.0%	3.2%	-4.8%	-3.5%	-3.2%	-11.2%	11.3%	2.8%
1982	-5.1%	-5.9%	-8.5%	4.3%	0.3%	-4.6%	-4.4%	0.5%	7.4%	1.7%	5.6%	7.9%
1983	-2.3%	5.7%	2.2%	10.9%	8.9%	2.9%	4.0%	-3.4%	-0.5%	2.1%	-2.8%	5.7%
1984	-1.0%	-1.5%	-2.6%	-1.1%	-0.3%	-5.0%	-0.2%	-3.4%	5.0%	0.1%	0.8%	5.2%
1985	1.1%	5.3%	2.8%	2.3%	-0.8%	9.5%	1.0%	-2.3%	-1.5%	-3.4%	5.7%	5.5%
1986	2.1%	4.7%	6.4%	4.7%	-2.7%	1.4%	6.4%	-0.5%	5.3%	-9.6%	3.2%	-0.1%
1987	-4.1%	9.6%	-0.3%	-1.5%	-4.5%	-1.9%	2.6%	-1.1%	0.9%	-2.2%	-13.8%	-5.5%
1988	0.5%	9.9%	-2.4%	-4.6%	-1.1%	3.8%	0.1%	-1.5%	-2.4%	1.6%	3.2%	-2.0%
1989	0.2%	1.1%	-2.8%	1.8%	2.8%	3.6%	2.2%	4.9%	-2.8%	-0.9%	0.4%	1.2%
1990	2.8%	-7.0%	-1.8%	-1.6%	-5.5%	3.1%	0.5%	1.9%	-10.2%	1.4%	5.7%	0.4%
1991	0.2%	-3.1%	2.6%	1.2%	0.9%	-2.5%	0.1%	3.4%	2.1%	3.8%	-0.8%	2.2%
1992	8.1%	-4.9%	-1.6%	-1.4%	4.4%	2.2%	0.0%	7.3%	0.1%	0.7%	-0.6%	-3.1%
1993	3.9%	1.9%	3.6%	0.8%	0.2%	-2.3%	2.1%	2.3%	1.4%	-0.1%	-1.0%	-7.1%
1994	0.3%	-0.1%	-5.5%	-4.1%	2.2%	-4.1%	-2.4%	6.9%	-3.8%	-2.2%	0.4%	-0.1%
1995	1.1%	6.2%	-2.4%	-0.1%	0.5%	4.9%	-0.4%	1.4%	-0.2%	4.1%	0.9%	1.7%
1996	3.1%	2.4%	-2.0%	-0.4%	-1.7%	-0.7%	1.8%	-4.1%	1.4%	2.2%	1.6%	5.3%
1997	0.7%	2.2%	0.8%	-3.8%	0.9%	3.9%	4.2%	4.6%	-1.2%	5.4%	0.0%	3.5%
1998	3.1%	-2.0%	3.2%	6.1%	-1.2%	-1.9%	1.2%	-6.5%	-5.0%	2.5%	3.7%	1.7%
1999	1.1%	-6.8%	-3.2%	0.1%	4.1%	-0.9%	-0.1%	-4.3%	-2.5%	-5.7%	4.5%	-2.9%
2000	-4.9%	1.5%	-7.9%	8.8%	-0.5%	1.6%	1.8%	3.3%	8.4%	2.5%	3.3%	0.0%
2001	4.5%	-5.5%	5.0%	-3.2%	6.9%	1.0%	-1.0%	-2.9%	1.4%	-8.1%	3.3%	1.9%
2002	4.8%	-0.2%	1.6%	1.7%	0.6%	-4.0%	-1.2%	-8.1%	3.1%	-16.0%	18.4%	9.7%
2003	-7.7%	-1.7%	-2.8%	2.1%	7.2%	3.6%	0.6%	-4.3%	2.1%	1.4%	7.5%	5.2%
2004	1.1%	3.5%	3.1%	2.1%	-5.6%	1.1%	1.3%	1.3%	2.3%	3.7%	2.3%	6.5%
2005	2.5%	0.2%	2.4%	-2.4%	2.6%	2.6%	0.6%	3.5%	-1.6%	0.0%	-5.8%	4.6%
2006	-1.1%	2.3%	1.5%	-0.8%	-2.0%	0.7%	0.6%	2.2%	0.9%	-0.2%	2.0%	3.4%
2007	0.1%	2.1%	-0.3%	3.5%	1.9%	1.3%	-1.8%	-3.1%	0.0%	3.6%	-2.7%	-0.8%
2008	-0.7%	-2.5%	-2.4%	1.7%	2.5%	0.5%	-5.4%	-5.0%	0.7%	-7.9%	-16.0%	-13.7%
2009	10.3%	-12.0%	-14.5%	11.1%	3.1%	5.6%	0.7%	1.4%	-0.9%	3.1%	-2.5%	5.2%
2010	6.5%	6.9%	8.8%	13.1%	7.3%	-10.9%	-5.7%	7.9%	0.1%	8.4%	1.0%	7.4%
2011	8.3%	9.1%	4.5%	2.8%	1.2%	-1.4%	2.9%	-1.2%	-6.6%	-10.9%	11.0%	3.2%
2012	-0.3%	5.6%	0.2%	-1.9%	-1.5%	-5.4%	5.2%	2.4%	5.4%	1.2%	-0.1%	7.0%
2013	-1.3%	9.3%	0.8%	0.2%	1.6%	4.1%	-0.7%	6.3%	-4.7%	4.0%	3.3%	1.0%
2014	2.3%	-5.3%	7.2%	4.3%	0.8%	2.1%	2.5%	-4.1%	3.5%	-1.9%	5.9%	1.8%
2015	0.6%	1.1%	-0.6%	-1.7%	0.7%	0.2%	-2.4%	4.2%	-7.3%	3.6%	5.6%	-0.7%
2016	0.5%	2.3%	0.3%	6.2%	-1.5%	0.3%	5.1%	-0.3%	-3.8%	-1.9%	-1.1%	0.6%
2017	3.6%	-0.4%	5.1%	-1.8%	0.6%	2.5%	-1.8%	0.7%	-1.3%	0.7%	1.2%	4.5%
2018	-0.2%	1.5%	-5.3%	-1.2%	1.5%	-0.7%	1.0%	3.2%	2.9%	0.2%	-3.3%	3.0%
2019	-8.7%	-	-	-	-	-	-	-	-	-	-	-

Month

Excess Returns of the Fama-French-Five Factor Markowitz Portfolio

Monthly Returns (%)

Year	1	2	3	4	5	6	7	8	9	10	11	12
1979	-	4.2%	-1.9%	4.8%	-0.2%	0.1%	2.7%	4.3%	6.3%	-3.7%	-10.3%	1.8%
1980	2.1%	7.9%	-3.1%	-19.0%	4.1%	2.5%	6.8%	7.3%	6.2%	3.1%	1.0%	-0.4%
1981	-4.4%	-3.3%	1.5%	11.2%	2.5%	3.8%	-3.8%	-6.5%	-3.6%	-9.2%	10.2%	5.1%
1982	-6.7%	-4.4%	-4.4%	6.5%	-2.8%	-4.2%	-3.6%	0.5%	3.6%	1.9%	7.6%	8.1%
1983	-4.6%	1.0%	4.1%	12.0%	5.5%	-0.6%	4.6%	-3.6%	-2.5%	4.6%	-0.8%	4.3%
1984	-1.9%	-1.7%	-2.8%	-0.1%	0.1%	-3.0%	0.3%	-3.0%	3.2%	2.6%	2.5%	3.1%
1985	-0.4%	2.9%	2.0%	2.0%	1.2%	7.5%	1.2%	-0.3%	-1.6%	-2.4%	5.2%	6.0%
1986	2.6%	1.8%	7.5%	5.1%	-2.0%	4.8%	5.1%	-2.9%	6.2%	-6.3%	2.2%	-1.0%
1987	-2.5%	12.5%	-0.1%	3.6%	-3.1%	1.2%	3.2%	3.5%	1.9%	-1.3%	-17.8%	-7.2%
1988	2.5%	9.2%	2.1%	0.6%	2.2%	1.4%	-0.5%	1.4%	-2.4%	10.3%	-8.3%	-3.4%
1989	2.3%	2.5%	1.0%	1.9%	3.3%	2.8%	3.9%	4.8%	-2.2%	1.1%	0.5%	2.7%
1990	1.4%	-7.5%	-0.2%	-2.1%	-3.4%	5.3%	0.4%	6.4%	-4.3%	-2.4%	-2.1%	0.6%
1991	-1.6%	-3.7%	8.2%	1.0%	3.1%	-2.5%	-1.4%	1.8%	1.6%	1.0%	-1.3%	-4.4%
1992	5.0%	-2.5%	-0.5%	-0.8%	2.4%	-2.0%	-2.5%	2.7%	-0.7%	-1.7%	4.7%	2.1%
1993	5.8%	3.1%	0.6%	2.1%	0.2%	0.9%	1.4%	0.9%	5.4%	-0.2%	-1.0%	-6.2%
1994	2.4%	-0.2%	-4.1%	-3.0%	1.5%	-1.6%	-1.1%	4.0%	0.8%	-1.8%	0.5%	-5.0%
1995	0.1%	2.3%	3.7%	4.5%	0.8%	3.7%	1.5%	2.7%	1.9%	1.7%	-0.6%	3.4%
1996	2.4%	1.6%	2.6%	2.2%	0.5%	0.3%	0.7%	-4.1%	2.6%	3.7%	2.9%	6.0%
1997	0.2%	3.7%	2.8%	-4.2%	4.8%	4.9%	4.3%	5.1%	-4.0%	5.3%	0.7%	3.8%
1998	2.6%	-1.2%	2.1%	6.1%	-0.6%	-1.7%	2.1%	-4.5%	-8.0%	0.6%	5.2%	3.5%
1999	0.3%	-4.9%	-1.6%	0.0%	5.8%	-2.0%	0.9%	-4.3%	-1.8%	-7.5%	4.7%	-3.6%
2000	-6.4%	-2.5%	-8.7%	10.1%	0.7%	3.6%	0.7%	-0.2%	4.6%	1.4%	6.2%	0.7%
2001	3.4%	-6.0%	1.7%	-5.2%	5.8%	2.1%	-2.5%	-1.7%	0.9%	-8.7%	2.5%	1.1%
2002	2.9%	0.6%	1.5%	2.2%	0.0%	-1.8%	-4.7%	-6.4%	3.1%	-3.5%	1.5%	-0.3%
2003	2.4%	0.5%	-1.2%	3.4%	2.9%	6.7%	-0.5%	-2.8%	1.0%	3.2%	5.0%	3.5%
2004	1.7%	3.4%	4.3%	0.9%	-0.4%	0.6%	0.1%	-1.2%	0.9%	3.0%	0.9%	6.5%
2005	1.8%	0.2%	2.3%	-2.3%	0.7%	2.3%	-0.7%	3.3%	-1.5%	0.3%	-4.5%	4.3%
2006	-0.6%	2.3%	0.4%	-1.2%	-1.1%	0.2%	0.3%	1.2%	2.4%	0.1%	1.4%	2.2%
2007	0.5%	1.3%	-1.4%	1.4%	2.2%	0.2%	-4.0%	-3.0%	-0.1%	3.9%	-1.8%	-1.2%
2008	-0.9%	-2.0%	-3.2%	2.7%	2.2%	-1.1%	-8.2%	-2.1%	2.5%	-5.4%	-16.9%	-14.4%
2009	7.2%	-11.0%	-12.8%	16.4%	0.6%	7.1%	2.3%	3.8%	-0.2%	3.3%	3.3%	4.0%
2010	-3.1%	-0.5%	0.9%	6.2%	1.5%	-8.9%	-2.0%	7.5%	-1.7%	4.3%	1.4%	1.7%
2011	7.0%	-1.5%	-1.0%	1.3%	3.7%	0.9%	2.7%	-4.1%	-2.9%	-5.0%	11.1%	-1.2%
2012	3.2%	3.2%	5.1%	6.9%	-0.3%	-4.1%	7.1%	-3.9%	5.8%	2.4%	2.0%	-1.4%
2013	1.2%	3.4%	1.9%	7.3%	2.9%	-6.2%	-2.5%	7.3%	-6.7%	1.8%	4.0%	-2.8%
2014	1.0%	-3.0%	6.2%	4.7%	2.0%	0.1%	2.2%	-4.3%	4.5%	-1.2%	6.2%	2.6%
2015	1.0%	1.9%	-2.5%	-2.2%	0.9%	-0.4%	-2.7%	3.6%	-7.2%	2.7%	5.9%	-0.7%
2016	0.2%	-0.2%	1.3%	4.9%	-0.8%	-1.5%	4.0%	0.5%	-2.9%	-1.6%	-1.7%	1.3%
2017	3.0%	-0.5%	4.9%	-1.8%	0.2%	2.1%	-0.1%	0.3%	-1.1%	1.5%	1.7%	2.5%
2018	1.4%	3.1%	-6.0%	-1.3%	1.4%	0.0%	-0.1%	3.4%	1.4%	-0.9%	-3.8%	3.2%
2019	-8.2%	-	-	-	-	-	-	-	-	-	-	-

Month

Excess Returns of the Equally Weighted Portfolio

Monthly Returns (%)

Year	1	2	3	4	5	6	7	8	9	10	11	12
1979	-	5.6%	-4.0%	4.5%	1.6%	-2.1%	3.1%	2.7%	6.1%	-2.6%	-8.5%	4.9%
1980	2.0%	4.8%	-6.8%	-9.2%	3.9%	5.6%	3.2%	9.6%	0.5%	1.9%	-0.6%	2.0%
1981	0.7%	-4.5%	3.3%	5.5%	-0.5%	0.8%	-3.6%	-3.2%	-6.9%	-5.8%	6.9%	1.4%
1982	-2.8%	-3.4%	-2.2%	0.3%	3.1%	-4.9%	-2.7%	0.2%	9.2%	3.1%	12.3%	6.1%
1983	-1.4%	2.8%	4.7%	1.7%	4.9%	2.2%	2.9%	-4.4%	-0.9%	1.8%	-2.2%	2.6%
1984	-1.4%	-3.3%	-4.6%	-0.4%	0.9%	-5.2%	-0.5%	-0.4%	8.5%	-2.2%	1.7%	-2.1%
1985	2.7%	7.8%	2.0%	-1.2%	-2.3%	6.8%	2.0%	0.5%	-2.3%	-3.9%	3.5%	5.6%
1986	3.6%	3.1%	6.3%	4.4%	-0.2%	3.3%	1.9%	-7.7%	6.6%	-7.9%	4.8%	0.3%
1987	-3.3%	15.1%	3.0%	1.2%	-2.6%	0.2%	3.8%	3.8%	2.5%	0.0%	-23.9%	-7.5%
1988	6.5%	4.8%	5.7%	-2.1%	0.5%	1.0%	3.2%	-1.0%	-4.7%	3.9%	0.6%	-3.4%
1989	2.2%	4.9%	-2.3%	3.2%	3.9%	3.7%	-1.0%	6.4%	2.6%	-1.5%	-4.1%	1.5%
1990	0.7%	-7.2%	0.9%	1.0%	-2.9%	8.7%	-1.1%	-2.1%	-10.9%	-5.2%	-4.3%	9.3%
1991	2.2%	6.5%	8.5%	1.5%	2.3%	3.4%	-3.3%	1.8%	2.1%	-1.1%	1.3%	-2.3%
1992	8.9%	1.4%	1.5%	-1.9%	1.4%	1.3%	-1.4%	3.2%	-1.9%	1.4%	3.4%	2.8%
1993	2.1%	1.8%	0.5%	2.8%	-1.3%	2.1%	-0.5%	-0.1%	4.1%	-0.4%	0.8%	-2.1%
1994	1.4%	3.0%	-2.1%	-4.3%	1.0%	0.1%	-1.5%	2.5%	2.1%	-3.0%	0.7%	-4.1%
1995	1.7%	2.6%	3.2%	2.9%	0.7%	3.5%	1.4%	2.4%	0.7%	1.7%	-0.8%	3.6%
1996	1.3%	2.8%	0.7%	1.5%	0.3%	0.3%	0.5%	-5.2%	1.3%	4.0%	1.8%	6.9%
1997	-1.4%	3.2%	0.9%	-3.5%	3.8%	5.6%	5.2%	6.1%	-3.1%	5.4%	-1.9%	3.4%
1998	0.3%	0.8%	4.8%	4.6%	-0.2%	-3.3%	1.3%	-5.8%	-10.6%	-0.5%	13.7%	2.4%
1999	1.9%	-2.0%	-2.0%	1.6%	11.2%	-2.8%	2.4%	-3.7%	-2.0%	-6.2%	2.2%	-1.2%
2000	-1.7%	-1.4%	-6.1%	11.3%	1.9%	2.5%	-4.0%	0.5%	4.8%	-0.5%	4.0%	0.2%
2001	6.1%	-0.9%	-1.8%	-3.8%	7.5%	2.3%	-3.4%	-0.1%	-1.9%	-10.2%	4.9%	4.5%
2002	3.3%	0.7%	3.5%	2.8%	-1.5%	-3.7%	-6.1%	-10.5%	3.7%	-7.1%	3.3%	6.3%
2003	-3.6%	-3.3%	-2.5%	2.0%	6.6%	8.5%	0.4%	1.3%	4.3%	0.7%	5.4%	1.9%
2004	3.5%	1.7%	3.4%	-0.9%	-1.5%	0.7%	1.3%	-1.1%	0.1%	3.1%	0.7%	6.3%
2005	0.6%	-0.4%	2.4%	-2.4%	-1.4%	3.5%	0.7%	4.3%	-1.2%	0.4%	-3.1%	4.9%
2006	-1.2%	3.7%	0.6%	0.8%	0.5%	-0.9%	-0.8%	-1.0%	2.7%	0.4%	2.5%	2.2%
2007	0.6%	3.0%	-1.7%	1.2%	3.3%	2.7%	-1.5%	-4.3%	-0.3%	3.4%	-3.1%	-1.2%
2008	-0.8%	-2.0%	-3.9%	2.2%	3.1%	-0.8%	-8.6%	0.7%	2.8%	-7.7%	-17.8%	-15.7%
2009	12.2%	-8.9%	-16.0%	17.5%	13.6%	6.7%	-2.2%	10.1%	1.6%	3.0%	0.3%	6.7%
2010	2.4%	-0.5%	3.1%	5.9%	4.3%	-11.0%	-3.5%	9.8%	-4.2%	6.2%	2.3%	2.7%
2011	5.8%	2.2%	-0.1%	3.3%	1.8%	-3.2%	2.0%	-6.1%	-6.6%	-9.9%	12.9%	2.9%
2012	1.6%	5.9%	2.8%	2.5%	-0.3%	-8.3%	6.0%	0.3%	2.4%	2.4%	1.1%	-0.7%
2013	2.0%	7.4%	1.1%	3.6%	1.0%	2.9%	-1.0%	6.6%	-5.0%	3.8%	4.0%	1.4%
2014	3.0%	-5.2%	6.8%	3.1%	0.1%	1.7%	2.3%	-3.8%	4.1%	-3.6%	4.7%	1.6%
2015	1.3%	-1.2%	3.5%	-2.0%	0.9%	0.0%	-1.4%	0.2%	-7.6%	-0.1%	8.0%	0.4%
2016	-3.2%	-4.1%	3.5%	6.1%	2.0%	0.8%	1.0%	2.7%	0.1%	-0.7%	-2.4%	5.7%
2017	1.9%	1.4%	5.0%	-2.0%	0.7%	1.5%	0.0%	0.4%	-1.4%	2.9%	0.8%	3.5%
2018	0.9%	3.5%	-5.9%	-2.3%	2.2%	1.0%	-0.3%	3.0%	2.0%	0.1%	-5.6%	2.6%
2019	-11.0%	-	-	-	-	-	-	-	-	-	-	-

Month

Appendix I – Portfolio Factor Loadings

Factor Loadings of the Equally Weighted Portfolio

Time Period	exMkt
1979 - 1989	0.94
1989 - 1999	0.92
1999 - 2009	0.68
2009 - 2019	1.08
Full Sample	0.91

Factor Loadings of the Single-Index Markowitz Portfolio

Time Period	exMkt
1979 - 1989	0.98
1989 - 1999	0.98
1999 - 2009	0.87
2009 - 2019	1.14
Full Sample	0.99

Factor Loadings of the Fama-French Three-Factor Markowitz Portfolio

Time Period	exMkt	SMB	HML
1979 - 1989	0.65	0.76	0.64
1989 - 1999	0.75	-0.27	0.67
1999 - 2009	0.59	0.14	0.75
2009 - 2019	0.83	-0.09	-0.11
Full Sample	0.70	0.13	0.49

Factor Loadings of the Carhart Four-Factor Markowitz Portfolio

Time Period	exMkt	SMB	HML	UMD
1979 - 1989	0.64	0.68	0.53	0.28
1989 - 1999	0.58	-0.25	0.54	0.36
1999 - 2009	0.56	0.12	0.61	0.05
2009 - 2019	0.82	0.07	-0.05	0.03
Full Sample	0.65	0.15	0.40	0.18

Factor Loadings of the Fama-French Five-Factor Markowitz Portfolio

Time Period	exMkt	SMB	HML	RMW	CMA
1979 - 1989	0.71	0.73	0.53	1.10	0.64
1989 - 1999	0.85	0.13	0.19	0.96	0.98
1999 - 2009	0.72	0.22	0.02	0.59	0.64
2009 - 2019	0.69	-0.18	0.05	0.07	0.08
Full Sample	0.75	0.23	0.20	0.68	0.59

Appendix J – R-Code

General Code

```
########################################################
## Data Analysis for Master Thesis By                 ##
## Julius Voigt Foelsgaard & Soeren Gybel Frederiksen ##
########################################################

## Clear Environment ##
rm(list=ls(all=TRUE))

## Install Packages for analysis ##
install.packages("readxl")
install.packages("MASS")
install.packages("CVXR")

## Attach packages ##
library(CVXR)
library(readxl)
library(MASS)

## Generate Empty Arrays ##
Roll_reg = NULL
BetaMatrix = NULL
Epsilonmatrix = NULL
Mean_factors = NULL
sigma_F = NULL
cov_equities = NULL
sigmamatrix = NULL
EXPECTED_RETURN = NULL
Markowitz_weights = NULL
Markowitz_weights_cons = NULL
Markowitz_exReturns_cons = NULL
Beta_portfolio=NULL
pvalues=NULL
portfolio_exReturns = NULL

################################
## Load data from excel file ##
################################
## Choose data file ##
exStock_returns<-read_excel(file.choose(),sheet = "SP500excessstockreturns")

## Generate excess return vectors for each stock ##
StockA<-exStock_returns$StockA

## Create excess return matrix for looping ##
exReturn_matrix<-cbind('vectors of excess stock returns')
```

```
## Create factor vectors ##
SI<-read_excel(file.choose(),sheet = "Single_Index");exMkt<-SI$exMkt

## Number of stocks & observations ##
nr_stocks<-ncol(exReturn_matrix)
nr.obs<-nrow(exReturn_matrix)
nr_loop_regression<-479

#####################################
## Rolling Regression Single-Index ##
#####################################

for (i in 0:nr_loop_regression) {
  for (j in 1:nr_stocks) {
    Roll_reg = lm(exReturn_matrix[(i+1):(i+60),j]~exMkt[(i+1):(i+60)])
    BetaMatrix = rbind(BetaMatrix,Roll_reg$coef)
    Epsilonmatrix = rbind(Epsilonmatrix,var(c(Roll_reg$resid)))
    pvalues = rbind(pvalues,summary(Roll_reg)$coef[,4])
  }
    Mean_factors = rbind(Mean_factors,mean(exMkt[(i+1):(i+60)]))
    sigma_F = rbind(sigma_F,var(exMkt[(i+1):(i+60)]))
}

##########################################################
## Create Expected Returns & Variance-Covariance Matrix ##
##########################################################

nr_stocksloop<-nrow(BetaMatrix)-nr_stocks

## Create diagonal variance-covariance matrix of residuals ##
for (i in seq(0,nr_stocksloop,nr_stocks)){
  sigmamatrix = rbind(sigmamatrix,diag(c(Epsilonmatrix[(i+1):(i+nr_stocks)])))
}

## Create variance-covariance matrix ##
## Please notice that in the beta-matrix we have the number of stocks (N) in the rows
and the number of
##factors (K) in the columns, therefore we transpose the left beta vector and not the
right as Equation (4.17) states.

for (k in seq(0,nr_stocksloop,nr_stocks)){
  cov_equities = rbind(cov_equities,(BetaMatrix[(k+1):(k+nr_stocks),2]
%*% t(BetaMatrix[(k+1):(k+nr_stocks),2]) * sigma_F[k/nr_stocks+1,1])
                          + sigmamatrix[(k+1):(k+nr_stocks),1:nr_stocks])
}
```

```
## Create expected returns ##
for (i in 0:nr_loop_regression) {
  for (j in 1:nr_stocks) {
    EXPECTED_RETURN = rbind(EXPECTED_RETURN,sum((BetaMatrix[j+(nr_stocks*i),2]
* Mean_factors[(i+1),1])))
  }
}

############################################
## Unconstrained Factor Markowitz Weights ##
############################################

ones<-rep(1,nr_stocks)

for (i in seq(0,nr_stocksloop,nr_stocks)){
    Markowitz_weights =
rbind(Markowitz_weights,(solve(cov_equities[(i+1):(i+nr_stocks),0:ncol(cov_equities)])
%*%                                                 EX-
PECTED_RETURN[(i+1):(i+nr_stocks)])/(as.numeric(t(EXPECTED_RETURN[(i+1):(i+nr_stocks)])
%*%
solve(cov_equities[(i+1):(i+nr_stocks),0:ncol(cov_equities)]) %*% ones)))
}

###########################################
##  Constrained Factor Markowitz Weights ##
###########################################

## Define optimization function & quadratic problem ##
portolioMaxSharpeRatio <- function(mu, Sigma) {
  w_ <- Variable(nrow(Sigma))
  prob <- Problem(Minimize(quad_form(w_, Sigma)),
                  constraints = list(w_ >= 0, t(mu) %*% w_ == mean(mu)))
  result <- solve(prob)
  return(as.vector(result$getValue(w_)/sum(result$getValue(w_))))
}

## Generate constrained Markowitz weights by solving quadratic problem ##

for (i in seq(0,nr_stocksloop,nr_stocks)) {
  Markowitz_weights_cons = cbind(Markowitz_weights_cons, portolioMaxSharpeRa-
tio(EXPECTED_RETURN[(i+1):(i+nr_stocks)],cov_equities[(i+1):(i+nr_stocks),1:ncol(cov_eq
uities)]))
}
```

```
############################
## Create Portfolio Betas ##
############################

for(i in seq(0,nr_stocksloop,nr_stocks)){
    Beta_portfolio = rbind(Beta_portfolio,t(BetaMatrix[(i+1):(i+nr_stocks),2]) %*% Mar-
kowitz_weights_cons[,(i+nr_stocks)/nr_stocks])
}

############################################
## Generate Time-series of Excess Returns ##
############################################

Markowitz_weights_cons_n = as.numeric(Markowitz_weights_cons)

## Generate portfolio excess returns ##
for (i in seq(0,nr_stocksloop,nr_stocks)){
  Markowitz_exReturns_cons = rbind(Markowitz_exReturns_cons,
exReturn_matrix[((i/nr_stocks)+62),0:ncol(exReturn_matrix)] %*% Marko-
witz_weights_cons_n[(i+1):(i+nr_stocks)])
}
```

```
## Create factor vectors ##
FF3<-read_excel(file.choose(),sheet = "FF3_m");SMB<-FF3$SMB;HML<-FF3$HML;exMkt<-
FF3$exMkt

## Number of stocks & observations ##
nr_stocks<-ncol(exReturn_matrix)
nr.obs<-nrow(exReturn_matrix)
nr_loop_regression<-479

############################
## Rolling Regression FF3 ##
############################

for (i in 0:nr_loop_regression) {
  for (j in 1:nr_stocks) {
    Roll_reg =
lm(exReturn_matrix[(i+1):(i+60),j]~exMkt[(i+1):(i+60)]+SMB[(i+1):(i+60)]+HML[(i+1):(i+6
0)])
    BetaMatrix = rbind(BetaMatrix,Roll_reg$coef)
    Epsilonmatrix = rbind(Epsilonmatrix,var(c(Roll_reg$resid)))
    pvalues = rbind(pvalues,summary(Roll_reg)$coef[,4])
  }
    Mean_factors =
rbind(Mean_factors,c(mean(exMkt[(i+1):(i+60)]),mean(SMB[(i+1):(i+60)]),mean(HML[(i+1):(
i+60)])))
    sigma_F =
rbind(sigma_F,var(cbind(exMkt[(i+1):(i+60)],SMB[(i+1):(i+60)],HML[(i+1):(i+60)])))
}

##########################################################
## Create Expected Returns & Variance-Covariance Matrix ##
##########################################################

nr_stocksloop<-nrow(BetaMatrix)-nr_stocks

## Create diagonal variance-covariance matrix of residuals ##
for (i in seq(0,nr_stocksloop,nr_stocks)){
  sigmamatrix =rbind(sigmamatrix,diag(c(Epsilonmatrix[(i+1):(i+nr_stocks)])))
}

## Create variance-covariance matrix ##
## Please notice that in the beta-matrix we have the number of stocks (N) in the rows
and the number of
##factors (K) in the columns, therefore we transpose the left beta vector and not the
right as Equation (4.17) states.
for (k in seq(0,nr_stocksloop,nr_stocks)){
  cov_equities = rbind(cov_equities,BetaMatrix[(k+1):(k+nr_stocks),2:ncol(BetaMatrix)]
%*%
```

```
sig-
ma_F[((k/(nr_stocks/ncol(sigma_F)))+1):((k+nr_stocks)/(nr_stocks/ncol(sigma_F))),1:3]
%*% t(BetaMatrix[(k+1):(k+nr_stocks),2:ncol(BetaMatrix)]) + sigmama-
trix[(k+1):(k+nr_stocks),1:nr_stocks])
}

## Create expected returns ##
for (i in 0:nr_loop_regression) {
  for (j in 1:nr_stocks) {
    EXPECTED_RETURN =
rbind(EXPECTED_RETURN,sum((BetaMatrix[j+(nr_stocks*i),2:ncol(BetaMatrix)] *
Mean_factors[(i+1),1:ncol(Mean_factors)])))
  }
}

############################################
## Unconstrained Factor Markowitz Weights ##
############################################

ones<-rep(1,nr_stocks)

for (i in seq(0,nr_stocksloop,nr_stocks)){
    Markowitz_weights =
rbind(Markowitz_weights,(solve(cov_equities[(i+1):(i+nr_stocks),0:ncol(cov_equities)])
%*%                                               EX-
PECTED_RETURN[(i+1):(i+nr_stocks)])/(as.numeric(t(EXPECTED_RETURN[(i+1):(i+nr_stocks)])
%*%
solve(cov_equities[(i+1):(i+nr_stocks),0:ncol(cov_equities)]) %*% ones)))
}

###########################################
##  Constrained Factor Markowitz Weights ##
###########################################

## Define optimization function & quadratic problem ##
portolioMaxSharpeRatio <- function(mu, Sigma) {
  w_ <- Variable(nrow(Sigma))
  prob <- Problem(Minimize(quad_form(w_, Sigma)),
                constraints = list(w_ >= 0, t(mu) %*% w_ == mean(mu)))
  result <- solve(prob)
  return(as.vector(result$getValue(w_)/sum(result$getValue(w_))))
}

## Generate constrained Markowitz weights by solving quadratic problem ##
for (i in seq(0,nr_stocksloop,nr_stocks)) {
  Markowitz_weights_cons = cbind(Markowitz_weights_cons, portolioMaxSharpeRa-
tio(EXPECTED_RETURN[(i+1):(i+nr_stocks)],cov_equities[(i+1):(i+nr_stocks),1:ncol(cov_eq
uities)]))
}
```

```
############################
## Create portfolio betas ##
############################

for(i in seq(0,nr_stocksloop,nr_stocks)){
    Beta_portfolio = rbind(Beta_portfolio,c(t(BetaMatrix[(i+1):(i+nr_stocks),2]) %*%
Markowitz_weights_cons[,(i+nr_stocks)/nr_stocks],
                                            t(BetaMatrix[(i+1):(i+nr_stocks),3]) %*%
Markowitz_weights_cons[,(i+nr_stocks)/nr_stocks],
                                            t(BetaMatrix[(i+1):(i+nr_stocks),4]) %*%
Markowitz_weights_cons[,(i+nr_stocks)/nr_stocks]))
}

############################################
## Generate Time-series of Excess Returns ##
############################################
Markowitz_weights_cons_n = as.numeric(Markowitz_weights_cons)

## Generate portfolio excess returns ##
for (i in seq(0,nr_stocksloop,nr_stocks)){
  Markowitz_exReturns_cons = rbind(Markowitz_exReturns_cons, exRe-
turn_matrix[((i/nr_stocks)+62),0:ncol(exReturn_matrix)] %*% Marko-
witz_weights_cons_n[(i+1):(i+nr_stocks)])
}
```

```
## Create factor vectors ##
FF3mom<-read_excel(file.choose(),sheet = "FF3mom_m");SMB<-FF3mom$SMB;HML<-
FF3mom$HML;exMkt<-FF3mom$exMkt;Mom<-FF3mom$Mom

## Number of stocks & observations ##
nr_stocks<-ncol(exReturn_matrix)
nr.obs<-nrow(exReturn_matrix)
nr_loop_regression<-479

##################################
## Rolling Regression FF3 + Mom ##
##################################

for (i in 0:nr_loop_regression) {
  for (j in 1:nr_stocks) {
    Roll_reg =
lm(exReturn_matrix[(i+1):(i+60),j]~exMkt[(i+1):(i+60)]+SMB[(i+1):(i+60)]+HML[(i+1):(i+6
0)]+Mom[(i+1):(i+60)])
    BetaMatrix = rbind(BetaMatrix,Roll_reg$coef)
    Epsilonmatrix = rbind(Epsilonmatrix,var(c(Roll_reg$resid)))
    pvalues = rbind(pvalues,summary(Roll_reg)$coef[,4])
  }
    Mean_factors =
rbind(Mean_factors,c(mean(exMkt[(i+1):(i+60)]),mean(SMB[(i+1):(i+60)]),mean(HML[(i+1):(
i+60)]),mean(Mom[(i+1):(i+60)])))
    sigma_F =
rbind(sigma_F,var(cbind(exMkt[(i+1):(i+60)],SMB[(i+1):(i+60)],HML[(i+1):(i+60)],Mom[(i+
1):(i+60)])))
}

##########################################################
## Create Expected Returns & Variance-Covariance Matrix ##
##########################################################

nr_stocksloop<-nrow(BetaMatrix)-nr_stocks

## Create diagonal variance-covariance matrix of residuals ##
for (i in seq(0,nr_stocksloop,nr_stocks)){
  sigmamatrix =rbind(sigmamatrix,diag(c(Epsilonmatrix[(i+1):(i+nr_stocks)])))
}

## Create variance-covariance matrix ##
## Please notice that in the beta-matrix we have the number of stocks (N) in the rows
and the number of
##factors (K) in the columns, therefore we transpose the left beta vector and not the
right as Equation (4.17) states.
for (k in seq(0,nr_stocksloop,nr_stocks)){
  cov_equities = rbind(cov_equities,BetaMatrix[(k+1):(k+nr_stocks),2:ncol(BetaMatrix)]
```

```
%*% sig-
ma_F[((k/(nr_stocks/ncol(sigma_F)))+1):((k+nr_stocks)/(nr_stocks/ncol(sigma_F))),1:4]
%*% t(BetaMatrix[(k+1):(k+nr_stocks),2:ncol(BetaMatrix)]) + sigmama-
trix[(k+1):(k+nr_stocks),1:nr_stocks])
}

## Create expected returns ##
for (i in 0:nr_loop_regression) {
  for (j in 1:nr_stocks) {
    EXPECTED_RETURN =
rbind(EXPECTED_RETURN,sum((BetaMatrix[j+(nr_stocks*i),2:ncol(BetaMatrix)] *
Mean_factors[(i+1),1:ncol(Mean_factors)])))
  }
}

############################################
## Unconstrained Factor Markowitz Weights ##
############################################

ones<-rep(1,nr_stocks)

for (i in seq(0,nr_stocksloop,nr_stocks)){
    Markowitz_weights =
rbind(Markowitz_weights,(solve(cov_equities[(i+1):(i+nr_stocks),0:ncol(cov_equities)])
%*%                                          EX-
PECTED_RETURN[(i+1):(i+nr_stocks)])/(as.numeric(t(EXPECTED_RETURN[(i+1):(i+nr_stocks)])
%*%
solve(cov_equities[(i+1):(i+nr_stocks),0:ncol(cov_equities)]) %*% ones)))
}

###########################################
##  Constrained Factor Markowitz Weights ##
###########################################

## Define optimization function & quadratic problem ##
portolioMaxSharpeRatio <- function(mu, Sigma) {
  w_ <- Variable(nrow(Sigma))
  prob <- Problem(Minimize(quad_form(w_, Sigma)),
                constraints = list(w_ >= 0, t(mu) %*% w_ == mean(mu)))
  result <- solve(prob)
  return(as.vector(result$getValue(w_)/sum(result$getValue(w_))))
}

## Generate constrained Markowitz weights by solving quadratic problem ##
for (i in seq(0,nr_stocksloop,nr_stocks)) {
  Markowitz_weights_cons = cbind(Markowitz_weights_cons, portolioMaxSharpeRa-
tio(EXPECTED_RETURN[(i+1):(i+nr_stocks)],cov_equities[(i+1):(i+nr_stocks),1:ncol(cov_eq
uities)]))
}
```

```
############################
## Create Portfolio Betas ##
############################

for(i in seq(0,nr_stocksloop,nr_stocks)){
  Beta_portfolio = rbind(Beta_portfolio,c(t(BetaMatrix[(i+1):(i+nr_stocks),2]) %*% Mar-
kowitz_weights_cons[,(i+nr_stocks)/nr_stocks],
                                          t(BetaMatrix[(i+1):(i+nr_stocks),3]) %*% Mar-
kowitz_weights_cons[,(i+nr_stocks)/nr_stocks],
                                          t(BetaMatrix[(i+1):(i+nr_stocks),4]) %*% Mar-
kowitz_weights_cons[,(i+nr_stocks)/nr_stocks],
                                          t(BetaMatrix[(i+1):(i+nr_stocks),5]) %*% Mar-
kowitz_weights_cons[,(i+nr_stocks)/nr_stocks]))
}

############################################
## Generate Time-series of Excess Returns ##
############################################

Markowitz_weights_cons_n = as.numeric(Markowitz_weights_cons)

## Generate portfolio excess returns ##
for (i in seq(0,nr_stocksloop,nr_stocks)){
  Markowitz_exReturns_cons = rbind(Markowitz_exReturns_cons, exRe-
turn_matrix[((i/nr_stocks)+62),0:ncol(exReturn_matrix)] %*% Marko-
witz_weights_cons_n[(i+1):(i+nr_stocks)])
}
```

```
## Create factor vectors ##
FF5<-read_excel(file.choose(),sheet = "FF5_m");exMkt<-FF5$exMkt;SMB<-FF5$SMB;HML<-
FF5$HML;RMW<-FF5$RMW;CMA<-FF5$CMA

## Number of stocks & observations ##
nr_stocks<-ncol(exReturn_matrix)
nr.obs<-nrow(exReturn_matrix)
nr_loop_regression<-479

############################
## Rolling Regression FF5 ##
############################

for (i in 0:nr_loop_regression) {
  for (j in 1:nr_stocks) {
    Roll_reg =
lm(exReturn_matrix[(i+1):(i+60),j]~exMkt[(i+1):(i+60)]+SMB[(i+1):(i+60)]+HML[(i+1):(i+6
0)]+RMW[(i+1):(i+60)]+CMA[(i+1):(i+60)])
    BetaMatrix = rbind(BetaMatrix,Roll_reg$coef)
    Epsilonmatrix = rbind(Epsilonmatrix,var(c(Roll_reg$resid)))
    pvalues = rbind(pvalues,summary(Roll_reg)$coef[,4])
  }
    Mean_factors =
rbind(Mean_factors,c(mean(exMkt[(i+1):(i+60)]),mean(SMB[(i+1):(i+60)]),mean(HML[(i+1):(
i+60)]),mean(RMW[(i+1):(i+60)]),mean(CMA[(i+1):(i+60)])))
    sigma_F =
rbind(sigma_F,var(cbind(exMkt[(i+1):(i+60)],SMB[(i+1):(i+60)],HML[(i+1):(i+60)],RMW[(i+
1):(i+60)],CMA[(i+1):(i+60)])))
}

##########################################################
## Create Expected Returns & Variance-Covariance Matrix ##
##########################################################

nr_stocksloop<-nrow(BetaMatrix)-nr_stocks

## Create diagonal variance-covariance matrix of residuals ##
for (i in seq(0,nr_stocksloop,nr_stocks)){
  sigmamatrix =rbind(sigmamatrix,diag(c(Epsilonmatrix[(i+1):(i+nr_stocks)])))
}

## Create variance-covariance matrix ##
## Please notice that in the beta-matrix we have the number of stocks (N) in the rows
and the number of
##factors (K) in the columns, therefore we transpose the left beta vector and not the
right as Equation (4.17) states.
for (k in seq(0,nr_stocksloop,nr_stocks)){
  cov_equities = rbind(cov_equities,BetaMatrix[(k+1):(k+nr_stocks),2:ncol(BetaMatrix)]
```

```
%*%         sig-
ma_F[((k/(nr_stocks/ncol(sigma_F)))+1):((k+nr_stocks)/(nr_stocks/ncol(sigma_F))),1:5]
%*% t(BetaMatrix[(k+1):(k+nr_stocks),2:ncol(BetaMatrix)]) + sigmama-
trix[(k+1):(k+nr_stocks),1:nr_stocks])
}

## Create expected returns ##
for (i in 0:nr_loop_regression) {
  for (j in 1:nr_stocks) {
    EXPECTED_RETURN =
rbind(EXPECTED_RETURN,sum((BetaMatrix[j+(nr_stocks*i),2:ncol(BetaMatrix)] *
Mean_factors[(i+1),1:ncol(Mean_factors)])))
  }
}

############################################
## Unconstrained Factor Markowitz Weights ##
############################################

ones<-rep(1,nr_stocks)

for (i in seq(0,nr_stocksloop,nr_stocks)){
    Markowitz_weights =
rbind(Markowitz_weights,(solve(cov_equities[(i+1):(i+nr_stocks),0:ncol(cov_equities)])
%*%                                         EX-
PECTED_RETURN[(i+1):(i+nr_stocks)])/(as.numeric(t(EXPECTED_RETURN[(i+1):(i+nr_stocks)])
%*%
solve(cov_equities[(i+1):(i+nr_stocks),0:ncol(cov_equities)]) %*% ones)))
}

###########################################
##  Constrained Factor Markowitz Weights ##
###########################################

## Define optimization function & quadratic problem ##
portolioMaxSharpeRatio <- function(mu, Sigma) {
  w_ <- Variable(nrow(Sigma))
  prob <- Problem(Minimize(quad_form(w_, Sigma)),
                constraints = list(w_ >= 0, t(mu) %*% w_ == mean(mu)))
  result <- solve(prob)
  return(as.vector(result$getValue(w_)/sum(result$getValue(w_))))
}

## Generate constrained Markowitz weights by solving quadratic problem ##
for (i in seq(0,nr_stocksloop,nr_stocks)) {
  Markowitz_weights_cons = cbind(Markowitz_weights_cons, portolioMaxSharpeRa-
tio(EXPECTED_RETURN[(i+1):(i+nr_stocks)],cov_equities[(i+1):(i+nr_stocks),1:ncol(cov_eq
uities)]))
}
```

```
############################
## Create Portfolio Betas ##
############################

for(i in seq(0,nr_stocksloop,nr_stocks)){
  Beta_portfolio = rbind(Beta_portfolio,c(t(BetaMatrix[(i+1):(i+nr_stocks),2]) %*% Mar-
kowitz_weights_cons[,(i+nr_stocks)/nr_stocks],
                                          t(BetaMatrix[(i+1):(i+nr_stocks),3]) %*% Mar-
kowitz_weights_cons[,(i+nr_stocks)/nr_stocks],
                                          t(BetaMatrix[(i+1):(i+nr_stocks),4]) %*% Mar-
kowitz_weights_cons[,(i+nr_stocks)/nr_stocks],
                                          t(BetaMatrix[(i+1):(i+nr_stocks),5]) %*% Mar-
kowitz_weights_cons[,(i+nr_stocks)/nr_stocks],
                                          t(BetaMatrix[(i+1):(i+nr_stocks),6]) %*% Mar-
kowitz_weights_cons[,(i+nr_stocks)/nr_stocks]))
}

############################################
## Generate Time-series of Excess Returns ##
############################################

Markowitz_weights_cons_n = as.numeric(Markowitz_weights_cons)

## Generate portfolio excess returns ##
for (i in seq(0,nr_stocksloop,nr_stocks)){
  Markowitz_exReturns_cons = rbind(Markowitz_exReturns_cons, exRe-
turn_matrix[((i/nr_stocks)+62),0:ncol(exReturn_matrix)] %*% Marko-
witz_weights_cons_n[(i+1):(i+nr_stocks)])
}
```

Code for the Equally Weighted Portfolio

```
##################################
## Equally Weighted Portfolio ##
##################################

nr_stocks<-ncol(exReturn_matrix)
nr.obs<-nrow(exReturn_matrix)
nr_loop<-479
portfolio_weights <- rep(1/nr_stocks,nr_stocks)

## Generate portfolio excess returns ##

for(i in (0:nr_loop)){
  portfolio_exReturns = rbind(portfolio_exReturns,
t(exReturn_matrix[(i+62),0:ncol(exReturn_matrix)]) %*% portfolio_weights)
}

##############################
## Create Portfolio Betas ##
##############################

SI<-read_excel(file.choose(),sheet = "Single_Index");exMkt<-SI$exMkt

for (i in 0:nr_loop) {
  for (j in 1:nr_stocks) {
    Roll_reg = lm(exReturn_matrix[(i+1):(i+60),j]~exMkt[(i+1):(i+60)])
    BetaMatrix = rbind(BetaMatrix,Roll_reg$coef)
  }
}

for(i in seq(0,(nrow(BetaMatrix)-nr_stocks),nr_stocks)){
  Beta_portfolio = rbind(Beta_portfolio,t(BetaMatrix[(i+1):(i+nr_stocks),2]) %*%
portfolio_weights)
}
```