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The SABR-model

Pricing and Risk Management in Negative Interest Rate Environments

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Resumé

Denne afhandling undersøger udvidelser af SABR-modellen til at håndtere *implied* volatilitetssmil i et negativt rentemiljø. Der gennemgås først den underliggende teori for at kunne prise renteoptioner, hvorefter SABR-modellen, der kan modellere den ikke-konstante *implied* volatilitet, introduceres. Vi tester estimationsmetoder og undersøger egenskaber for SABR-modellen og dens parametre. Afhandlingen udvider den originale SABR-model med to videreudviklede modeller, der kan håndtere negative renter. De to nye modeller, Shifted SABR-model og Normal SABR-model, viser sig begge at kunne prise illikvide swaptioner med høj præcision. Normal SABR-modellen udviser bedre egenskaber til tolkning og kvotering af *implied* volatilitet i lavrentemiljøer og har desuden mere stabile parametre. Vi vælger derfor at bruge denne model til at vise, hvordan man udfører risikostyring af residuale risikokomponenter. Vi fokuserer især på delta som risikomål og bruger denne til at hedge og analysere en markedsposition. Vi konkluderer, at man kan opnå signifikante forbedringer ved at benytte Normal SABR-modellen til risikostyring, samt at denne model kan benyttes til at modellere volatilitetssmilet i negative rentemiljøer.

Contents

1 Introduction

The purpose of this thesis is to examine the SABR-model and analyse possible extensions to the model in order to price and manage risk precisely in a negative interest environment. In the original Black framework implied volatility was thought to be constant across different strikes (McDonald, 2013), this proved to be incorrect by empirical analysis of the implied volatility surface. Instead the volatility surface was curved and often resembled a smile, which is why we now refer to the volatility surface as the volatility smile. Several models were developed to incorporate the new reality and this is where the SABR-model was developed (Hagan et al., 2002). The SABR-model fits the implied volatility surface and produces an implied volatility quote ready to be used in the existing pricing formulas, which is why the SABR model soon became a market standard (Balland and Tran, 2013). The original work by Hagan was not designed to handle negative values. This was thought to be beneficial because neither stocks nor interest rates could become negative. This changed after the financial crisis in 2008, where central banks aimed to lower the interest rate to fuel the economy (Andrade, Phillippe, 2016). This finally resulted in negative interest rates in 2014, and the existing SABR model was no longer valid. This is still the issue we face today in Europe, which is why this thesis will examine the existing choices of SABR-models that can compute implied volatilities for negative strikes. Lastly, we will show why it is paramount to apply a correct model for the volatility smile when applying risk management to a long position in a swaption.

One of the challenges of this thesis has been to construct a computational module that was able to price fixed income products. This module works behind the curtains of the models and risk management analysis. The module handles everything from bootstrapping curves to rolling out payments schedules to fitting the volatility smiles for the time period of the analysis. The module utilises an Excel VBA library introduced in the M. Sc. course *Fixed Income Derivatives: Risk Management and Financial Institutions* at Copenhagen Business School. As an addition to this module, we have computed functions to fit SABR-models and compute residual risk measures in a negative interest rate environment.

We begin by setting up the fundamentals of interest rate derivatives in section 2 and laying down the groundwork for the products we will price later. This includes curve calibration, option pricing, mathematical framework and volatility. After having covered this theory, we move on to caps and floors which are the basis of swaptions – our main product to price. Once the groundwork is set up, we move to the volatility smile and the background for its existence in section 3. This leads to the main model of the thesis in section 4; the SABR-model. We define the SABR-model and explore the parameters in the model along with tests and a discussion on how to best estimate them. After we have chosen an estimation method, we introduce two new versions of the SABR-model; the Shifted SABR-model and the Normal SABR-model. Before we begin applying the models, we describe the data sets used in the thesis in section 5. This gives an idea of how the market has moved in the past years and why. The next step is to conduct an empirical analysis in section 6, where we compare the models' abilities to cope with negative interest rates. We test the models both on the robustness of their parameters and their pricing accuracy. We discuss the advantages and drawbacks of the models and choose one to use in section 7 on risk management. Here we see the effects of possible errors when estimating the parameters of the SABR-model. We explore the classic Greeks and apply our knowledge in a hedge, before we conclude on our results in section 8.

1.1 Methodology

The thesis processes information with methods commonly used within finance and economics (Ryan et al., 2002). We work with quantitative data on various financial contracts and interest rate derivatives, collected from the Bloomberg database. We see Bloomberg as the leader within financial data with its numerous data series which are updated frequently. In section 5 we will comment further on the data series used in the thesis.

We primarily use deduction in the thesis as we begin with a theory and seek to investigate it with empirical data. One work method of the financial field is to start with core models, test these on data and then move on to expand and adapt the models to better fit the current financial situation. In this thesis, we start by looking at the classic Black set-up with constant volatility, then see that constant volatility is not the market reality and expand to the SABR-model. The SABR-model is valid for a while, but when the negative interest rate environment sees the light of day, this model also needs renewal – introducing the Shifted SABR-model and the Normal SABR-model in section 4. We then use our empirical data to test these models and discuss their advantages and disadvantages in order to choose a model to continue the work with.

In general, we seek to lay down the basic theory for financial asset pricing and then build the thesis up by adding more expansions and models in the following section. We use the empirical data to validate the link between theory and practise and then use the positivistic and logical approach to argue our results and conclusions (Brier, 2005).

2 Setting up fundamentals

We begin the thesis by setting up the investment universe that we will be working in. In this section we will cover some of the fundamental theory that precedes the main theory on swaptions and the SABR-model. First we need to lay the groundwork for pricing by explaining interest rates, the products that use interest rates and option pricing. Once this is complete, we can combine our knowledge of swaps and options to price swaptions, which will be our main product to price. But first, lets set up our investment universe.

2.1 Interest rates

Most of the interest derivative products are contracts written on the xIBOR – either indirectly or directly. The xIBOR is a term for *InterBank Offered Rates* and is a set of rates in which major banks can borrow on an unsecured basis from each other. The rates are offered in different currencies and set by different organisations. The British Bankers Association sets for example the LIBOR every day at 11:00 GMT. These rates range from 1 business day to 12 months and are calculated as the truncated average of rates submitted by a group of panel banks. Note that the xIBOR rates are not actually a traded rate but rather a series of indicative rates. The following sections build on the lecture notes from the course *Fixed Income Derivatives: Risk Management and Financial Institutions* (Linderstøm, 2013).

When defining the xIBOR fixings, we first need to define some terms that are used; δ is the coverage, L is the xIBOR fixing for the product, N is the notional and $P(0,T)$ is the zero coupon bond (ZCB) running from time 0 to T. The ZCB is used to express the xIBOR rate, as we can get the same return from borrowing one unit at the xIBOR rate or sell $\frac{1}{P(0,T)}$ ZCB, assuming that we have no arbitrage. The xIBOR fixings follow the *Money Market Convention* as the interest paid a maturity on a notional N is calculated as $N \cdot \delta \cdot L$. The xIBOR is thus expressed as:

$$1 + \delta L(0,T) = \frac{1}{P(0,T)} \Leftrightarrow \tag{2.1}$$

$$L(0,T) = \frac{1}{\delta}\left(\frac{1}{P(0,T)} - 1\right) \tag{2.2}$$

This is also called the spot xIBOR, running from $t = 0$ to T. Using the above equation containing ZCBs, we can define the forward xIBOR rate, which describes the future interest rate contracted at time t and running from T to $T + \delta$. The forward rate $F(t,T,T+\delta)$

that can be contracted today at $t = 0$ is defined using ZCBs:

$$F(t, T, T + \delta) = \frac{1}{\delta}\left(\frac{P(t, T)}{P(t, T + \delta)} - 1\right) \tag{2.3}$$

The next step in pricing contracts written on xIBOR rates is to determine the present value of the future payments. The present value of future xIBOR payments is the discounted expected payoff under the forward Q-neutral probability measure:

$$\text{PV of xIBOR payments} = P(t, T + \delta) \cdot E_t^{Q_{T+\delta}}[(T, T + \delta)] \tag{2.4}$$

Using the Q-neutral probability measure means that the expected payoff is equal to the forward rate. Note that we will cover probability measures along with other mathematical framework in section 2.4.1.

$$E_t^{Q_{T+\delta}}[L(T, T + \delta)] = F(t, T, T + \delta) \tag{2.5}$$

This allows us to compute xIBOR payments using only ZCB prices. The xIBOR rates are essential in order to compute and use interest rate swaps and swaptions. This will be discussed in section 2.2.1.

2.2 Interest Rate Swaps

A plain vanilla interest rate swap (IRS) is the contract between two parties in which they agree to exchange a series of fixed interest rate payments for a series of floating interest rate payments over a pre-specified period of time. These two series are referred to as the legs of the contract, where the floating leg is linked to the appropriate xIBOR. When denoting an IRS, one refers to the fixed leg e.g. if one enters into a payer swap, one is paying the fixed leg and receiving the floating leg. The two legs do not necessarily have the same specifications, but the present value of all the payments must be the same for the contract to be fair.

Curr.	Index name	Spot start	Roll	Floating leg		Fixed leg	
				Freq.	Day count	Freq.	Day count
EUR	Euribor	2B	MF	S	30/360	A	30/360
USD	USD Libor	2B	MF	Q	30/360	S	30/360
GBP	GBP Libor	0B	MF	S	Act/360	S	Act/360
DKK	Cibor	2B	MF	S	30/360	A	30/360

Table 2.1: Standard market conventions of plain vanilla swaps, Linderstrøm (2015)

Table 2.1 shows the standard market conventions of plain vanilla swaps, which we will now explain further. These conventions are essential to agree upon when two counter parties are looking to agree on a price.

Spot start is present day in financial terms, meaning that if one enters into a spot starting EUR interest rate swap, then it will begin in two business days from today.

Roll is a term used to describe how non-business days are handled. If a payment is due on a non-business day then the payment day needs to be adjusted. The indices in table 2.1 are all Modified Following (MF), meaning that if an action is meant to occur on a non-business day this action should happen on the following business day, unless this day falls in the next month, then it should be the previous business day.

Frequency describes how often a payment is to be made on each leg. These are typically annual (A), semi-annual (S) or quarterly (Q) and do not necessarily have to be the same for the two legs.

Day count conventions determine how the coverage δ is calculated. Rates are usually quoted per year, so the coverage will be between 0 and 1. There are different conventions to use depending on the currency of the swap as shown in table 2.1, where the most common ones are:

- 30/360 assumes that a year has 360 days and each month is 30 days. The coverage is calculated as $\delta = \frac{1}{360}(\text{years} \cdot 360 + \text{months} \cdot 30 + \min[30, \text{days}])$
- Act/360 assumes that a year has 360 days and the coverage is calculated as $\delta = \frac{days}{360}$
- Act/365 is the same as Act/360 but assumes a year with 365 days, giving us $\delta = \frac{days}{365}$
- Act/365.25 accounts for leap years and the coverage is $\delta = \frac{days}{360.25}$

2.2.1 Pricing interest rate swaps

In the section we will cover how to price an interest rate swap. The swap starts at time T_S and matures at T_E. It is written on a notional N and discounted with ZCBs. The market standard is that swaps are traded with a spot start, meaning that $t \approx T_S$ (Linderstøm, 2013). This means that the contract will start after two business days. The coverage δ_i is the fraction of the year between the dates $T_S, ..., T_E$. Having defined the parameters of the IRS, we use them to define the present value of the swap as the sum of coverages times the forward rate times the notional discounted with the appropriate zero coupon price:

$$\mathrm{PV}_t^{\mathrm{Float}} = \sum_{i=S+1}^{E} \delta_i^{\mathrm{Float}} F(t, T_{i-1}, T_i) N_i P(t, T_i) \tag{2.6}$$

For the fixed leg, we have the same parameters except that here the rate to be paid is fixed as K instead of the forward rate as used in the floating leg. The coverage will of course be defined as $\delta_i^{\mathrm{Fixed}}$ as the payment dates do not need be equal. The fixed leg is discounted the same way as the floating leg – with the zero coupon price:

$$\mathrm{PV}_t^{\mathrm{Fixed}} = \sum_{i=S+1}^{E} \delta_i^{\mathrm{Fixed}} K N_i P(t, T_i) \tag{2.7}$$

Now that the two legs have been defined, we will look at the present value of the contract. For the payer, the fixed leg is a liability and the floating is an asset, since the owner will pay the fixed rate and receive the floating. Hence the value of the payer swap is:

$$\mathrm{PV}_t^{\mathrm{Payer}} = \sum_{i=S+1}^{E} \delta_i^{\mathrm{Float}} F(t, T_{i-1}, T_i) N_i P(t, T_i) - \sum_{i=S+1}^{E} \delta_i^{\mathrm{Fixed}} K N_i P(t, T_i) \tag{2.8}$$

The timeline for an interest rate swap is illustrated in figure 2.1 below. The contract is a payer swap, where the person who bought the contract is paying the fixed leg and receiving the floating leg.

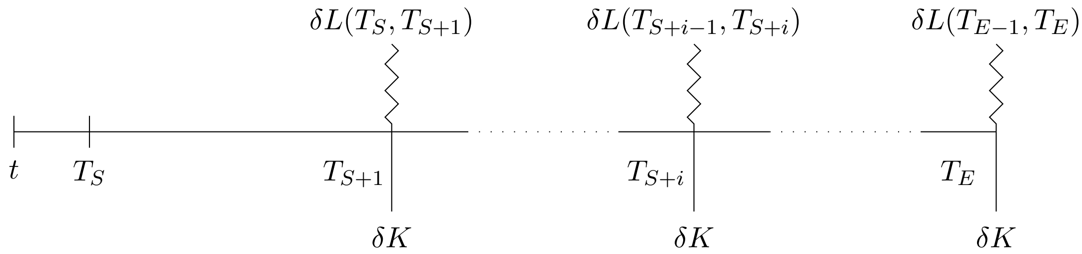


Figure 2.1: Timeline of a forward starting payer interest rate swap (own creation)

The counterparty, the receiver, has an opposite view of the contract, we get the following relationship between the two contracts: $\mathrm{PV}^{\mathrm{Payer}} = -\mathrm{PV}^{\mathrm{Receiver}}$.

In order for the present value of the two legs to be equal, we define the *par swap rate* $R(t, T_S, T_R)$ to be the fixed rate that ensures this condition:

$$R(t, T_S, T_E) = \frac{\sum_{i=S+1}^{E} \delta_i^{\mathrm{Float}} F(t, T_{i-1}, T_i) N_i P(t, T_i)}{\sum_{i=S+1}^{E} \delta_i^{\mathrm{Fixed}} N_i P(t, T_i)} \tag{2.9}$$

This leads to the party with the most favourable end of the contract to pay an upfront premium as compensation for the contract being unfair.

Using the par-swap rate in equation (2.9) and rearranging equation (2.8) we can write the price of a payer swap as

$$\mathrm{PV}_t^{\mathrm{Payer}} = A(t, T_S, T_E)(R(t, T_S, T_E) - K) \tag{2.10}$$

where $A(t, T_S, T_E) = \sum \delta_i^{\mathrm{Fixed}} P(t, T_i)$ constitutes the *annuity factor*[1]. To see why this way of stating the value of an IRS is smart, one can take the view of an fixed income trader. If a trader wishes to realise his position, he can either pay/receiver the value of the contract from the counterparty or he can enter into the corresponding receiver swap. When entering into the corresponding receiver IRS, the two floating rates will cancel out and the trader will be left with a series of fixed risk-free payments, also called an annuity. The value of the annuity will thus be the difference between fixed rate K of the original contract and the par-swap rate of the recently initiated IRS, discounted using the annuity factor which is exactly equation (2.10).

From equation 2.10 another interesting property can be seen. An IRS sensitivity towards changes in the par-swap rate is exactly the annuity factor, $\partial \mathrm{PV}_t^{\mathrm{Payer}} / \partial R = A$. This property thus ensures that entering into the opposite swap position is a risk-free way of closing a position (disregarding counterparty credit risk).

Interest rate swaps can be used to hedge loans with a floating rate. This is typically done by companies, where they have a loan at the xIBOR rate plus a credit spread. If a company wants to hedge the risk of an increase in the xIBOR rate, it can enter into a payer swap meaning that it will receive the xIBOR rate in exchange for a fixed rate. The company then uses the xIBOR payments for the interest on its loan and pays the fixed rate of the contract it just entered, resulting in a fixed loan.

2.3 Curve calibration

We have now seen how to price interest rate swaps using ZCBs prices observable in the market, but what about the cases when the market rates are not directly observable? This is where the zero coupon yield curve comes in handy. We can use observable par swap rates to calibrate a unique zero curve that can be used to compute forward rates for any

[1] Annuity factor can be viewed as a discounting function, that can be used to discount any form of cash flow.

given time. This is done with a set of knot points i.e. market rates for different maturities and a minimisation problem.

As an example to explain the calibration of a zero coupon yield curve (zero curve), we will use the case of pricing a 10Y EUR IRS. The payments for the fixed leg are annual and for the floating leg semi-annual, meaning that we have 10 fixed payments to discount and 20 floating payments which need a projection of the forward Euribor rates and discounting. As mentioned earlier, the problem is that we cannot observe all 20 rates that make up the interest rate swap and thus we need to calibrate the entire zero coupon yield curve. We use the knot points observed in the market to construct the zero curve and then the zero curve to interpolate between the knot points. It is also possible to extrapolate, meaning that we can use the zero curve to find rates beyond the last knot point. Using interpolation and extrapolation, it is possible to determine the entire zero coupon yield curve and thus compute any discount factor and forward rate we find necessary.

To construct the zero coupon yield curve, we need to solve a minimisation problem. The problem is formulated as a *least square minimisation problem* where we have observed the market quotes $\mathbf{A} = \{a_1, a_2, ..., a_N\}$ and a set of parameters $\mathbf{P} = \{p_1, p_2, ..., p_M\}$. The pricing method used in section 2.2.1 will be used here to compute a set of model quotes $\mathbf{B(P)} = \{b_1, b_2, ..., b_N\}$ depending on the parameters and the set of market quotes. The goal is now to minimise the squared difference between the market quotes and model quotes by altering the parameters, in this case the zero coupon bonds. The problem is formulated as:

$$\min_{\mathbf{P}} \sum_{i=1}^{N} (a_i - b_i)^2 \tag{2.11}$$

To ensure nice properties in the minimisation problem, we will choose $N = M$ i.e. the number of knot points is equal to the number of market quotes.

2.3.1 Dual curves

A natural extension to the single curve set-up is the dual curve set-up. Before the introduction of the dual curve set-up, the zero curve was used both to compute forward rates and discount factors. This implies that the underlying rate from which the zero-rate is derived of, in this case Euribor 6M, is risk-free. After the crash of Lehman Brothers in 2008, it was clear that the xIBOR rates were not risk-free but contained an element of credit risk. Therefore the market standard became to separate the zero curve and the discount curve.

The zero curve is constructed in the same way as in the single curve set-up from the par-swap rates that have the relevant xIBOR rate as the underlying asset. Remembering that the xIBOR rate is a non-traded rate at which a group of panel banks expect to be able to borrow money from one another. Therefore the zero curve contains an element of credit risk and will therefore only be used to compute forward rates and not discount rates.

The discount curve is calculated from *Overnight Index Swaps*[2] (OIS). The OIS rate is still tied up to xIBOR, but is settled each day. This minimises the credit risk as it is quite unlikely for bankruptcy to happen overnight. These rates are as close to a risk-free rate as we can hope to get and thus the curve constructed from them can be used to discount future cash flows. The methodology used to construct the discounting curve is identical to the zero curve.

2.3.2 An example of curve calibration

The curve calibration from section 2.3 has been implemented and will be described in this section. First the observable par swap rates $\mathbf{A} = \{a_1, a_2, ..., a_3\}$ have been collected from the Bloomberg database. We have collected two different series of par swap rates: par swap rates from swaps with 6M Euribor as the underlying rate and Overnight Index Swaps with 6M Euribor as the underlying rate. The data can be seen in table 2.2 below. The quotes cover a number of different maturities that are not perfectly overlapping.

Maturity	Par swap rates	Overnight par swap rate
1Y	-0.237%	-0.352%
2Y	-0.152%	-0.271%
3Y	-0.014%	-0.165%
4Y	0.129%	-0.011%
5Y	0.273%	0.094%
6Y	0.411%	0.252%
7Y	0.541%	0.377%
8Y	0.663%	0.497%
9Y	0.775%	0.606%
10Y	0.876%	0.707%
15Y	1.232%	1.069%
20Y	1.392%	1.237%
30Y	1.453%	1.317%

Table 2.2: Market quote inputs for calibration of the zero and discount curve, 03-12-2018 (Source: Bloomberg)

[2]The Overnight Index Swaps is a par swap rate settled every night, minimising the credit risk

Constructing the dual curve set-up is very similar to the single curve set-up. For the single curve set-up, the goal was to minimise the difference between model rates and market rates by changing the zero curve. In the dual curve set-up the goal is to minimise both the difference between the model rates based on 6M Euribor and market rates based on 6M Euribor while simultaneously minimising the difference between the model rates based on OIS and market rates based on OIS, by changing both the zero curve and discounting curve.

The model rates are based on equation (2.11) and the market rates can be seen in table 2.2. Note that the OIS rates are lower than the par swap rates, as they should be due to their minimised credit risk. The results are visualised in figure 2.2.

As we are not interested in rates beyond our last market rate (30 years), we will not be extrapolating. Interpolation is used to compute zero rates between the knot points and thus create a continuous zero curve and discount curve. The interpolation method in use is *hermite spline*. We use our set of knot points $\{t_1, t_2, ..., t_N\}$ to calculate the corresponding zero rates $\{r_1, r_2, ..., r_N\}$. The formula below lets us calculate any zero rate r_t between two knot points, where $t_i < t < t_{i+1}$

$$r(t) = a_i + b_i(t - t_i) + c_i(t - t_i)^2 + d_i(t - t_i)^3 \tag{2.12}$$

We use the Hermite Spline interpolation because it insures a smoother curve due to the polynomial functions compared to a standard linear interpolation.

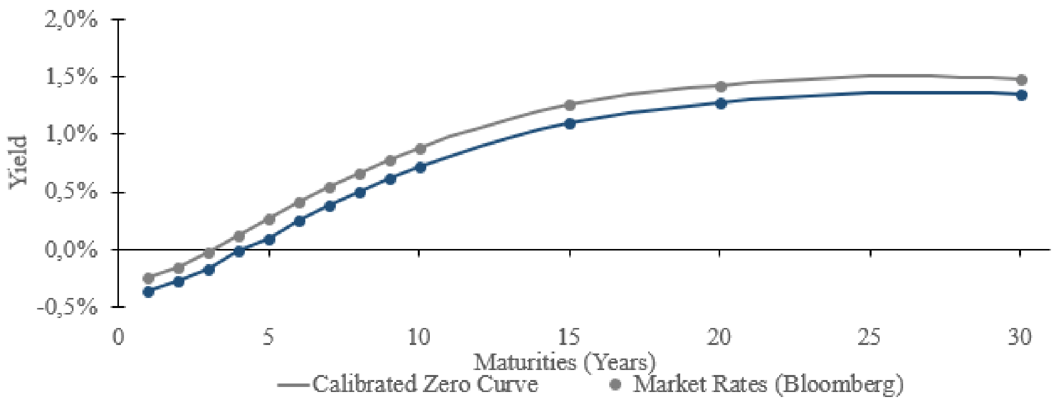


Figure 2.2: Calibrated curves

From figure 2.2, we can see that we have successfully calibrated a dual curve set-up. The curves are smooth which ensures arbitrage free pricing and there is a distinct difference between the zero curve and discounting curve indicating that we are now taking the credit spread into account when pricing interest rate products. The calibrated curves are essential to us when we move onto pricing swaptions, calibrating SABR-models and applying risk management in the empirical analysis of this thesis.

2.4 Option pricing

As we move towards using the SABR-model, we first need to know how to price options and then later swaptions as these are the financial contracts we will be pricing with SABR. This section covers option pricing theory, mathematical framework and two option pricing models; the Black model and the Normal model. This section is based on the option theory in (Linderstøm, 2013), (Hull, 2018) and (McDonald, 2013).

The two classic types of options are a call option and a put option. The call option gives the buyer the right, but not the obligation, to purchase the underlying asset at a prespecified price on a future date. A put option is the opposite, giving the buyer the right to sell the underlying asset at a prespecified price on a future date. The call option is purchased if an investor expects the price of the underlying asset to increase and vice versa for the put option. A call option is equivalent to holding a long position in the underlying asset, where a put option is holding a short position. The contract runs from t_0 to T, where the investor decides at time T whether he wants to exercise his right to purchase the underlying asset. If the investor decides to purchase the underlying asset, he will pay the strike price K for the underlying asset. The right to purchase the asset can only be exercised at T in European options, which is the type we will deal with. Another type of options are American, where the investor can exercise his option at any time from purchase of the option until it expires. The payoff for a call option is described in table 2.3 below.

	Call	Put
Payoff	$(V_T - K)^+$	$(K - V_T)^+$
Exercised when	$V_T > K$	$K > V_T$

Table 2.3: Payoff for put and call options

The price of the asset at time T is stochastic, which causes the payoff to be stochastic as well. While one could expect such a contract to be difficult to price, it is actually quite possible. The way to go is simply to compute the expected value of the option's payoff. Before we begin pricing options, we first need to elaborate on why this method is valid and introduce some probability measures and properties of stochastic processes.

2.4.1 Mathematical framework

Before we begin introducing some of the pricing models, we first need to cover some mathematical areas that help set up the theoretical framework.

Arbitrage

When using pricing models, it is important to secure that the model does not allow arbitrage. Arbitrage is when an investor can buy a contract at the price of zero and receive a non-negative payoff later for sure, and a positive payoff with a positive probability. The definition of arbitrage is:

An arbitrage is a value process $X(t)$ satisfying $X(0) = 0$ and for $t > 0$

$$P\{X(t) \leq 0\} = 1 \quad \text{and} \quad P\{X(t) > 0\} > 0 \tag{2.13}$$

For option pricing, arbitrage is described with a discount process $d(t)$. Arbitrage exists if an investor can start with $X(0)$ and later at time t, the investment will have the value of

$$P\left\{X(t) \geq \frac{X(0)}{d(t)}\right\} = 1 \quad \text{and} \quad P\left\{X(t) > \frac{X(0)}{d(t)}\right\} > 0 \tag{2.14}$$

When pricing financial products, the theoretical pricing models must be arbitrage-free. This means that every pricing process must satisfy the martingale measures.

Martingales

A stochastic process is a martingale if $\mathbb{E}_t^Q[f_T] = f_t$, meaning that the expected future value of a process is the value of today and that the process is drift-less. The process is under Q-probability, which means that we are in the risk-neutral world. In mathematical finance, we distinguish between the real world and the risk-neutral world and the probability that is valid in each world. In the real world we have P-probability and in the risk-free world Q-probability. These probability measures inform about the likeliness of different values in a random variable. If a process is a martingale, then it is easier for us to price because we know, what we expect of the future value.

Knowing our expectations of a martingale, then if f_T/g_T is a martingale, then it must hold that

$$\frac{f_t}{g_t} = E_t\left[\frac{f_T}{g_T}\right] \quad \Leftrightarrow \quad f_t = g_t E_t\left[\frac{f_T}{g_T}\right] \tag{2.15}$$

The last expression states that f_t is a martingale with g_t as numeraire. This property will be useful later when we cover caps and floors in section 2.5. The numeraire can be of our choice, which will help simplify pricing formulas later on.

With arbitrage and martingales in place, we have the foundation for the *First Fundamental Theorem of Asset Pricing:* "The model is free of arbitrage if and only if there exists an equivalent martingale measure". This is an important result because we can now price any financial derivative by calculating the expected value of the payoff.

Brownian Motion

A Brownian motion is one of the essential building blocks when it comes to derivatives pricing models. The process is also called a Wiener-process and is used to describe random movements in financial models e.g. the movements of a stock price. The process is used in the Bachelier model, which will be described later in section 2.4.2. The properties of a Brownian motion $X(t)$ are as follows:

- $X(0) = 0$
- For all $t_0 \leq t_1 \leq \cdots \leq t_n$, we have that $X(t_1) - X(t_0), X(t_2) - X(t_1), ..., X(t_n) - X(t_{n-1})$ are independent and random
- For any $t_{n-1} < t_n$, we have that $X_n - X_{n-1}$ is normally distributed with $N\ (0, (t_n - t_{n-1})\sigma^2)$
- $t \to X_t$ is a continuous function

Now with the properties in place, we can see that the process $X(t)$ is a martingale. This means that the expected value of the process tomorrow, given the value of today, is in fact the value of today – in mathematical terms $E[X(t_{n+1})|X(t_n) = X(t_n)$. The second property tells us that the Brownian motion follows the stochastic Markov process, meaning that the value tomorrow does not depend on history but simply what the value is today (Lawler, 2006).

2.4.2 Normal model

One of the simplest models for modelling negative interest rates is the Normal model, also called the Bachelier model, and was first introduced by Louis Bachelier in 1900. The model describes the movements of the forward rate f_t with a differential stochastic equation:

$$df_t = \sigma_N dW_t \tag{2.16}$$

where σ_N is the constant Normal volatility and W_t is a Wiener-process. Hence the movement of the underlying rate depends only on the volatility and the Wiener-process. The solution to the stochastic equation is given by:

$$f_t = f + \sigma_N W_t \quad \text{where } f = f_0 \tag{2.17}$$

We can price a call option with the Normal model with the following formula:

$$C = e^{-r(T-T)}\left[(F-K)\Phi(d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-d^2/2}\right] \tag{2.18}$$

where F is the underlying rate, K is the strike, $\Phi(\cdot)$ is the normal distribution function and

$$d = \frac{F-K}{\sigma\sqrt{T-t}}$$

The model gives us the possibility to work with negative interest rates because of the normal distribution, but this is also one of the disadvantages of the model. The forward rate may become very negative with a positive probability, which is in contrast to real-world expectations. Interest rates are unlikely to move far away from zero and towards high negative values.

Another drawback to the model is the assumption of constant volatility. This is a theme that will be addressed later in section 3.

2.4.3 Black Model

The Black model is a simpler version of the well-known Black-Scholes-Merton model (McDonald, 2013). The Black model is used to price European fixed income options and assumes that the option prices follow the lognormal process:

$$df_t = \sigma_{LN} f_t dW_t \tag{2.19}$$

where the movement of the underlying rate depends on a constant volatility σ_{LN}, the current forward par swap rate f_t and a Wiener-process W_t. We can use most of the BSM-equation, where the underlying asset is the futures prices and we discount both legs with the risk-free rate:

$$C = e^{-r(T-t)}[F\Phi(d_1) - K\Phi(d_2)] \tag{2.20}$$

where $\Phi(\cdot)$ is a cumulative probability distribution function for a standardised normal distribution and

$$d_1 = \frac{\ln(F/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

The lognormal distribution of prices makes extreme cases very unlikely together with the dependence of the current par swap rate, which was the problem with the Normal model. However, the Black model does not work for negative interest rates which have been occurring in current markets. The problem with the Black model is also that it assumes constant volatility, which is not the case in the real word – an issue we will address in section 3.

2.4.4 Volatility

This section will cover the two ways in which volatility can be quoted and an approximation to move between the two measures. We will introduce a second method to move between the two types of volatilities and conduct an example of the calculations needed to do so.

Black volatility
This volatility is used in the Black-model and is calculated relative to the forward rate. In practice this means that if a forward rate is e.g. 1% and changes to 1.1%, then the Black volatility is 10% as this is the relative change. As seen in section 2.4.3 the movements of the underlying forward par swap rate are assumed to have a lognormal distribution as seen in equation (2.19).

The problem with the Black volatility occurs when rates are close to zero. When a small change occurs in a forward par swap rate close to zero, then it will result in a dramatic Black volatility. We will later examine these large implied volatilities and their effect on the pricing models in section 6.

Normal volatility
The normalised volatility, or Normal volatility in daily speech, solves the problem of rates close to zero. Normal volatility is quoted in absolute terms and in basis points[3], meaning that if a forward rate is e.g. 1% and changes to 1.1%, then the Normal volatility is 0.1%

[3]One basis point (bps) is 1/10.000 or 1/100$^{\text{th}}$of 1% and commonly used as a unit of measure when it comes to changes in interest rates

or 100 bps. This type of volatility is used in the Normal model and assumes that the underlying par swap rate is normally distributed as seen in equation (2.16). The Normal volatility tends to be smaller than the Black volatility as it is calculated as the changes in actual basis points.

We will, however, have to use Black volatility when testing the original SABR and Shifted SABR model. We have collected both Normal volatility and Black volatility quotes from Bloomberg, but due to illiquidity, the implied Black volatility quotes lack data points for deep ITM and OTM strikes. We will use an iterative method to get the missing implied Black volatilities from the Normal volatilities.

Transforming Black volatility to Normal volatility

There exists an approximation for moving between Normal volatility and Black volatility for At The Money options (ATM). When examining equation 2.21 below, we can see the logic of this approximation. We transform the absolute Normal volatility to the relative volatility by dividing with the forward par swap rate (Linderstøm, 2013).

$$\sigma_{LN} \approx \frac{\sigma_N}{f_t} \tag{2.21}$$

where f_t is the forward par swap rate.

This approximation becomes increasingly imprecise when interest rates are around zero. This is obviously a problem for us, since we will be working in an interest rate environment with negative and close to zero rates. Additionally, the approximation only works for ATM volatilities, which is again a problem when computing a SABR model, which requires OTM and ITM volatility quotes. (Linderstøm, 2013).

As an alternative to the approximation, one can use the fact that the price of the option, when using the Normal model and the Black model, should be equal due to the no arbitrage argument.

$$C_N(F, K, t, T, \sigma_N) = C_B(F, K, t, T, \sigma_B^*) \tag{2.22}$$

From this relation, we can infer the Black volatility by solving the equation 2.22 in respect to σ_B^*. This has been done in practise by solving a least square minimisation problem through iterations.

In table 2.4 we computed the Black volatility using both the approximation and the iterative method. When comparing the computed implied Black volatilities with the market quotes, it is clear that the approximation becomes imprecise for away from ATM volatilities. When comparing the computed implied Black volatilities from the iterative method, we can conclude that this method is very precise. We will therefore use this method whenever we need to fill gaps in the implied Black volatility market data.

Bps away from ATM	Implied Nvol (Market quotes)	Implied Bvol (Market quotes)	Implied Bvol (Approximation)	Implied Bvol (Inferred from Prices)
-200	55.7	-	28.0%	-
-100	58.7	44.3%	29.5%	44.3%
-50	60.4	36.9%	30.3%	36.9%
-25	61.3	34.5%	30.8%	34.5%
0	62.2	32.6%	31.2%	32.6%
25	63.2	31.1%	31.7%	31.1%
50	64.2	29.8%	32.2%	29.8%
100	66.3	27.9%	33.3%	27.9%
200	71.1	25.4%	35.7%	25.4%
400	81.6	-	41.0%	23.0%

Table 2.4: Implied volatilities, (Source: Bloomberg and own calculations)

When examining the implied volatilities in table 2.4, we can see a very interesting relation between Black volatilities and Normal volatilities. While the Normal volatilities are greater for larger strikes, the Black volatilities are lower for larger strikes and vice versa for smaller strikes.

This difference arises because the implied Black volatilities are measured in relative terms and the implied Normal volatilities in absolute terms. This means that when strikes are low, the relative volatility increases. The volatility does not increase solely because low strikes exhibit higher volatility but also because of the fact that small changes in a low strike will have a much larger relative effect, than small changes in a higher strike. This is a major problem for using implied Black volatilities to interpret the volatility smile, since we do not know if the changes are due to the relative change or due to a "true" increase in volatility. We will therefore mainly use Normal volatilities when interpreting the shape of the volatility smile, but will have to use Black volatility for the models that require it.

2.5 Caps and Floors

Having covered option pricing theory, it is time to move on to interest rate option contracts. The theory in this section is based on (Linderstøm, 2013). A caplet is a call option with the forward xIBOR rate as the underlying asset. The caplet is written on $L(T, T+\delta)$

and can only be exercised at time T. The option payoff is fixed at the beginning of each accrual period, but not paid undtil the end of the period – i.e. fixed-in-advance and paid-in-arrears. We remember equation (2.5) from section 2.1 and can therefore value a caplet at time T:

$$\text{Caplet PV}_T = P(T, T+\delta)\delta(F(T, T, T+\delta) - K)^+ \tag{2.23}$$

To find the present value of the caplet we use the properties covered in section 2.4.1 on martingales. We use the ZCB as numeraire instead of trying to work out the expected value of two stochastic rates. We use a ZCB that matures at $T + \delta$ (T-forward measure) and thus obtain:

$$\begin{aligned} \text{Caplet PV}_t &= P(t, T+\delta)\delta E_t^{Q_{T+\delta}}\left[\frac{P(T, T+\delta)(F(T, T, T+\delta) - K)^+}{P(T, T+\delta)}\right] \\ &= P(t, T+\delta)\delta E_t^{Q_{T+\delta}}[F(T, T, T+\delta) - K)^+] \end{aligned} \tag{2.24}$$

We assume that $F(T, T, T + \delta)$ is normally distributed and we can therefore write the expected payoff using the probability properties of the Normal model to express the price as

$$\text{Caplet PV}_t = P(t, T+\delta)\delta\left[(F(t, T, T+\delta) - K)\Phi(d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-d^2/2}\right] \tag{2.25}$$

where

$$d = \frac{F(t, T, T+\delta) - K}{\sigma\sqrt{T-t}}$$

What we have in equation (2.25) above, is first discounting with the ZCB, the year fraction, in the brackets we have the normally distributed expected payoff and lastly the variance for the normal distribution. We can see from equation (2.25), that all we need to price the caplet is a ZCB, forward xIBOR payments, a strike, time to maturity and a volatility. The ZCB used for discounting and the forward xIBOR payment can be derived from the calibrated swap curves (covered in section 2.3). The strike and time to maturity are contract specific. The volatility is therefore the only unknown variable at this point. There exists a unique volatility that matches the market price of the caplet. We can therefore infer the volatility from the market price. When inferring the volatility from market prices, we refer to the volatility as the implied volatility. We stress that this is not the same as realised volatility.

Just as there is a put option for every call option, there is also a floorlet for every caplet. A floorlet is thus a put option with the forward xIBOR rate as the underlying asset. To price a floorlet, we can use the put-call-parity which ensures a fair price for a put and call option with the same strike K:

$$\text{Forward}(K) = \text{Call}(K) - \text{Put}(K) \tag{2.26}$$

Knowing the value of a forward contract is the present value of the discounted cash flow, $P(t, T+\delta)\delta(F(T,T,T+\delta) - K)$, we can insert this in the put-call-parity with the value of the caplet in equation (2.25) and solve for the floorlet (put), giving us the present value of a floorlet in the Normal model:

$$\text{Floorlet PV}_t = P(t, T+\delta)\delta\left[(K - F(t,T,T+\delta))\Phi(d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-d^2/2}\right] \tag{2.27}$$

where

$$d = \frac{F(t,T,T+\delta) - K}{\sigma\sqrt{T-t}}$$

A cap is a portfolio of caplets, and a floor is a portfolio of floorlets. To price these two portfolios we simply need to add all the caplets and floorlets from starting time T_S to the time they mature at T_E:

$$\text{Cap PV}_t = \sum_{i=S+1}^{E} P(t, T+\delta)\delta\left[(F(t,T,T+\delta) - K)\Phi(d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-d^2/2}\right] \tag{2.28}$$

$$\text{Floor PV}_t = \sum_{i=S+1}^{E} P(t, T+\delta)\delta\left[(K - F(t,T,T+\delta))\Phi(d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}}e^{-d^2/2}\right] \tag{2.29}$$

Caps and floors are Over The Counter (OTC) products. In theory caps and floors can be traded with any tenor xIBOR rate, the liquidity of the market is, however, concentrated around the main xIBOR rates from the interest swap market.
Caps and floors are very similar to swaptions, which will be the main products used for analysing the volatility smile in this thesis.

2.6 Swaptions

We have covered both swaps and options and now it is time to combine the two products to get a swaption. This section on swaptions is based on the swaption theory in (Linderstøm, 2013). A swaption is the right, but not the obligation, to enter into a spot starting interest rate swap with a specified fixed rate, called the strike, at a future point in time. An example of this contract could be that a company knows it is going to enter into a 5-year floating rate loan agreement, but prefers a fixed rate. Then it can buy a payer swaption starting in 6 months, where it pays a fixed rate, e.g. 3% per year and receives the xIBOR rate for 5 years. When the 6 months have passed the company can decide whether to exercise its right to buy the payer option. If the spot IRS is less than 3%, then the company will enter into a regular swap and not exercise its swaption. If the spot IRS is more than 3%, then the company will exercise its right to purchase the IRS in the swaption.

A payer swaption is having the option at time T_S to enter into an interest rate swap that matures at time T_E. If we assume that the swaption is entered at $t = 0$, then the standard notation of a swaption is that at time T_S the holder of the swaption decides whether he wants to execute his option to enter into the IRS. Note that he can only execute at the expiry and not before. If he executes, then he will pay the fixed rate K and receive the floating xIBOR rate for a period of $T_E - T_S$. An example is $T_S = 1Y$ and $T_E = 10Y$: the buyer will then have 1Y to determine if he wants to execute and if he does, he will enter into a swap for 9Y and thus has purchased a 1Y9Y swaption.

Swaptions can be settled in two ways: physical or cash settlement. The pricing is still the standard present value of the contract, but the calculation for the two types are different. If the holder of the swaption chooses the physical settlement, he will receive the underlying swap contract. The value of a payer swaption with physical settlement, using the pricing technique as with interest rate swap, can be expressed as:

$$\mathrm{PV}_{T_S}^{\mathrm{Physical}} = A(T_S, T_S, T_E)(R(T_S, T_S, T_E) - K)^+ \tag{2.30}$$

where

$$A(T_S, T_S, T_E) = \sum_{i=S+1}^{E} \delta_i P(T_S, T_i)$$

The owner of the payer swaption will choose to exercise the option if the present value is greater than zero. As an alternative to receiving the physical underlying interest rate swap, the owner of the option can choose a cash-settlement. When choosing the cash-settlement, the owner simply receives the present value of the underlying swap. It is beyond the scope of this thesis to analyse the difference in pricing between the two settlement types. We will, however, note that the main difference between physical and cash settlement is how future cash flows are discounted.

2.6.1 Swaption pricing

When pricing a swaption, we will use the same techniques as when pricing caplets and floorlets. When pricing caps and floors we used a zero-coupon bond as numeraire, but for swaptions we will use the swap annuity, $A(T_S, T_S, T_E)$, as numeraire.

$$\mathrm{PV}_t^{\text{Payer}} = A(t, T_S, T_E) E_t^A \left[\frac{A(T_S, T_S, T_E)(R(T_S, T_S, T_E) - K)^+}{A(T_S, T_S, T_E} \right] \tag{2.31}$$

Next we apply the probability distribution of the Normal model to quantify the expected present value for a payer swaption:

$$\mathrm{PV}_t^{\text{Payer}} = A(t, T_S, T_E) \left[(R(t, T_S, T_E) - K)\Phi(d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} e^{-d^2/2} \right] \tag{2.32}$$

where

$$d = \frac{R(t, T_S, T_E) - K}{\sigma\sqrt{T-t}}$$

As with caps and floors, we use the put-call parity to quantify the present value of a receiver swaption.

$$\text{Forward Starting Payer Swap}(K) = \text{Payer Swaption}(K) - \text{Receiver Swaption}(K) \tag{2.33}$$

And by isolation, we can calculate the present value of a receiver swaption.

$$\mathrm{PV}_t^{\text{Receiver}} = A(t, T_S, T_E) \left[(K - R(t, T_S, T_E))\Phi(d) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} e^{-d^2/2} \right] \tag{2.34}$$

2.6.2 An example of swaption pricing

Now we will price a swaption using the above formulas. We will be pricing in Excel and use the same data set and our calibrated curves from section 2.3.2. The swaption we will price is a spot starting interest rate swap written on Euribor 6M as the underlying asset.

The first step of pricing the swaption is to compute the fair value today of the underlying swap that we will later have the option to buy. To compute the fair value of the swap, we need to find the value of the floating and fixed leg. For the fixed leg we first need to calculate the par swap rate (rate for fixed leg) that guarantees the fairness of the contract and therefore a present value of zero for the swap. There are a few steps we need to go through in order to calculate the par swap rate. First we roll out a payment schedule and compute the coverages, then we construct a zero and discount curve to project future payments and then we can compute par-swap rate.

Coverage

The coverage is the time between payments for the two legs. In this example we have annual payments for the fixed leg and semi-annual payments for the floating leg. We construct a table that shows the payment dates and the coverage between the dates. If the payment date is not on a business day, we use our day count rule which is Modified Following.

Payment dates	Coverage	
	Fixed leg	Floating leg
03-06-20		0.5
03-12-20	1	0.5
03-06-21		0.5
03-12-21	1	0.5
03-06-22		0.5
05-12-22	1.00556	0.50556

Table 2.5: Date schedule and coverages for the two legs, start date 03-12-18

Curve calibration

We will use the zero and discount curves calibrated in section 2.3. From the zero curve we compute zero rates, and from the zero rates we compute for forward rates that constitute the forward xIBOR payments. From the discount curve we compute ZCB prices that we use to discount the future xIBOR payments.

Payment dates	Zero rates	Forward rate	ZCB price
03-06-20	-0.2020%	-0,1308%	1,0047
03-12-20	-0.1530%	-0.0064%	1.0055
03-06-21	-0.0885%	0.1701%	1.0056
03-12-21	-0.0163%	0.3456%	1.0050
03-06-22	0.0549%	0.4829%	1.0032
05-12-22	0.1283%	0.6375%	1.0005

Table 2.6: Zero rates, ZCB prices and forward rates for the floating leg of the IRS (Source: own calculations on data from Bloomberg)

We notice in table 2.6 that the ZCB prices are above zero. This is because the discounting curve is negative in the short term. This results in a positive discount effect. We have now calculated everything needed to compute the value of the floating leg. We input the information in equation (2.6) and compute the value as:

$$\mathrm{PV}_{\mathrm{Float}} = \sum_{i=S+1}^{E} \delta_i^{\mathrm{Float}} \cdot F(t, T_{i-1}, T) \cdot P(t, T_i) = 0.7549\% \tag{2.35}$$

The value of the fixed leg obviously depends on the strike. The goal here is to find a strike that ensures a swap with a value of zero. This strike is exactly the par swap rate. We use equation (2.9) to compute the par swap rate:

$$R(t, T_S, T_E) = \frac{\text{PV}^{\text{Float}}}{\sum_{i=S+1}^{E} \delta^{\text{Fixed}} \cdot P(t, T_i)} = 0.25026\% \tag{2.36}$$

We now have the par swap rate, which we will use as the strike for the fixed leg. The present value of the fixed leg is a calculation similar to the floating leg, except that we use the fixed rate instead of the forward rate. We set $K = R(t, T_S, T_E) = 0.25026\%$ and calculate the present value of the fixed leg:

$$\text{PV}_{\text{Fixed}} = \sum_{i=S+1}^{E} \delta_i^{\text{Fixed}} \cdot K \cdot P(t, T_i) = 0.7549\% \tag{2.37}$$

The two legs have the same present value, meaning that the contract has a present value of zero and is fair.

Pricing the swaption

Now that we have the fair price of the IRS, we move on to pricing the swaption i.e. the option to purchase the underlying IRS 1 year from the start date. We will use equation (2.32) to price our swaption. The equation uses a lot of the same inputs and also an implied volatility. The implied volatility for our swaption comes from the Bloomberg data base for a 1Y3Y EUR swaption on December 3$^{\text{rd}}$ 2018 and has a value of 0.3435%. The present value of the swaption ATM is

$$\text{PV}_t^{\text{Swaption}} = 0.4113\% \tag{2.38}$$

Note that even though we have priced a payer swaption, the price of a receiver swaption ATM is the same. The interpretation of the swaption price is that it costs us 0.41% of our notional to have the option to buy the underlying swap. As an alternative we could buy a forward starting swap, but then we would be forced to enter this swap one year from now.

We have now covered to fundamentals of pricing linear fixed income products and non-linear products e.g. options. We now move onto the volatility smile.

3 Volatility smile

The classic pricing models Black and the Normal model assume constant implied volatility, but empirical market data on implied volatility tells a different story – that implied volatility is not constant and instead resembles the shape of a smile as a function of moneyness e.g. in or out of the money (ITM/OTM). Implied volatility depends on different strikes and often has the shape seen in figure 3.1. In the following section we will cover the reasons for the existence of the smile and why the smile is important to include when pricing financial products.

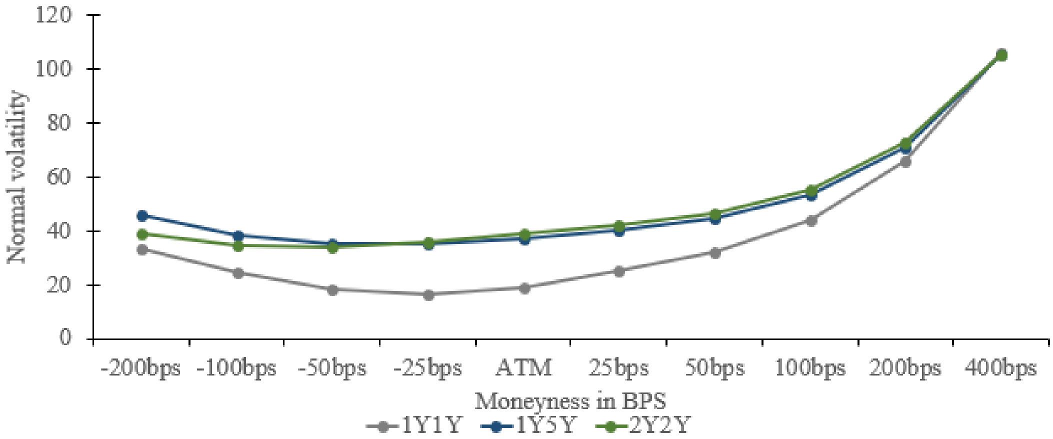


Figure 3.1: 6M Euribor swaption quoted in Normal volatility with different expiries and maturities, 28-01-2019, own creation (Data source: Bloomberg)

3.1 Reasons for the existence of the volatility smile

There are two main reasons for the existence of the volatility smile which we will cover in this section. It is important to understand the background for the smile before we move on to a pricing model, that takes the volatility smile into account. The two reasons are the probability distribution of the underlying asset and the existence of buying pressure for away from the money options (Hagan et al., 2002).

Probability distribution

When using a pricing model, whether it is the Normal model or the Black model, we assume that the underlying asset follows a specific probability distribution. This distribution is used to describe the probability of the underlying asset rising or falling. For the Normal model we use the standard normal distribution and for the Black model, we use the lognormal distribution. However, the empirical probability distribution does not necessarily look like the model standards. Figure 3.2 shows the standard normal distribution and an example of what an empirical distribution might look like. One of the problems lie within the tails of the distributions. If the underlying asset of an option is priced with the assumption that it follows the normal distribution but actually follows the empirical distribution of figure 3.2, then the probability of moving OTM is higher than anticipated and the pricing of the option will be incorrect.

Even though we suspect the theoretical probability distributions are not perfect representations of the reality, we still use them for two main reasons. Using these probability distributions results in nice closed-form solutions and are easy to work with, especially when computing risk measures like the Greeks and these models are already the market standard for pricing options. The solution is instead to model the volatility smile to correct the pricing of assets with a non-standard probability distribution.

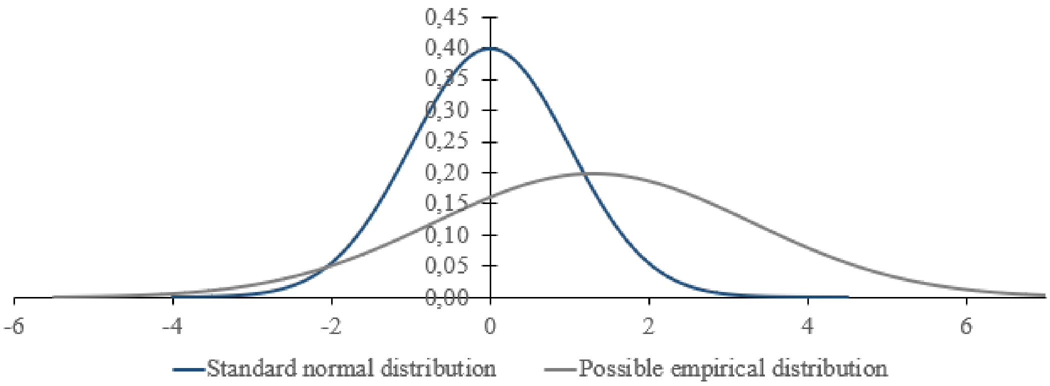


Figure 3.2: Example of probability distributions

Buying pressure

Options are often used as an insurance against "bad states" in the market. This creates a buying pressure on OTM options which causes the price to increase due to classic supply and demand (Bollen and Whaley, 2004). An example is pension funds that guarantee their clients a certain pay-off or rate on their pensions. The pension funds then buy OTM receiver swaptions to ensure that they are able to fulfil their contract with the clients even in case of large drops in the interest rate market. This demand for OTM swaptions drives the prices up and causes the volatility smile to often be higher for OTM swaptions compared to ATM options.

There is also a buying pressure on ITM options with rates being very low in the current market. Prices on options are a product of the expectations in the market and with rates being very low in the current market, rate hikes are expected. Companies with loans protect themselves against rate hikes by buying ITM payer swaptions, contributing to the right-hand-side of the smile in figure 3.1.

3.2 Importance of the volatility smile

We have now discussed the reasons for why the volatility smile exists and therefore it is time to cover the importance of the volatility smile and the difference it makes in risk management and pricing two types of illiquid options.

Risk Management
When working with options and investments, we often seek to hedge positions in order to manage risk. The process is basically changing a parameter in the pricing model and see how it affects the price of the product. This is a stress-testing of the parameters to see where most of the risk lies and what we need to do in order to minimise the risk of the investment. If an investor uses the classic Greeks to manage risk, he needs to use a model that can handle the volatility smile or he might risk hedging his position incorrectly. We will cover this topic thoroughly in section 7.

Pricing illiquid options
The importance of the volatility smile for illiquid options can be split into two groups: exotic options and illiquid OTM/ITM options.

Within exotic options, we could for example look at a barrier option. This type of contract has more than one strike and therefore raises the question of which implied volatility to use when pricing the option. It would be incorrect to use the ATM volatility after the barrier is hit, so we need an away from ATM volatility to calculate the correct price of the exotic option. Note that this type of option will not be covered in this thesis.

The other group is illiquid OTM/ITM options, where we cannot observe the market price and therefore we need an estimate of the implied volatility. Since we know that volatility is not constant and we need to estimate the price of the illiquid options, we look for the model that will give us the best possible estimate for the implied volatility.

We have discussed the underlying reasons for the existence of the volatility smile; buying pressure and non-standard probability distributions. We will now move on to our main model to model the implied volatility smile: the SABR-model.

4 The SABR-model

To manage the volatility smile, Hagan et. al. published a paper in 2002 introducing the *Stochastic Alpha Beta Rho Model*. The SABR-model provides a closed-form algebraic formula for the implied volatility depending on the strike K and the forward price f (Hagan et al., 2002). The SABR-model is the industry standard for modelling implied volatility for interest rate options and exhibits high flexibility due to its number of parameters that together shape the curvature and level of the smile (Balland and Tran, 2013). In this section we will cover the classic SABR-model, the parameters of the model and two expansions to deal with negative interest rates. We will analyse and discuss the different versions of the SABR-model in order to choose one to continue with when moving into further analysis and risk management.

4.1 The classic SABR-model

The SABR-model is a two factor model that describes the stochastic movements of both the forward rate f and the volatility $\hat{\alpha}$. The processes are described as

$$d\hat{F}_t = \hat{\alpha}_t \hat{F}_t^{\beta}\, dW_t^1, \qquad \hat{F}(0) = f \tag{4.1}$$

$$d\hat{\alpha}_t = v\hat{\alpha}_t\, dW_t^2, \qquad \hat{\alpha}(0) = \alpha \tag{4.2}$$

where W_t^1 and W_t^2 are Wiener-processes. The movement of the forward rate $\hat{F}_t$ depends on the volatility $\hat{\alpha}$ and the current forward rate with an exponent of β, which is between 0 and 1. The movement of the volatility $\hat{\alpha}$ depends on the current level of volatility and the parameter ν, which is the "volatility of the volatility". We will discuss these parameters in section 4.2 along with ρ which is the correlation between the two Wiener-processes:

$$dW_t^1 dW_t^2 = \rho dt \tag{4.3}$$

The SABR-model can accurately fit the implied volatility for any option that has a single exercise date t_{ex} meaning that it is very effective in pricing swaptions and caplets/floorlets. There is in extension of the SABR-model which can fit the volatility surface of European options with multiple exercise dates, called the dynamic SABR-model. Since we wish to price swaptions, we will stick to the classic SABR-model in this thesis. The SABR-model calculates an implied volatility for a given strike and forward rate. We can then use the implied volatility to correctly price options. The core SABR-model uses Black volatility, but it can actually also provide us with Normal volatility.

SABR-model with Black volatility

Equation 4.4 provides us with Black volatility, which we can use as input for implied volatility in the Black-formula to correctly price an option:

$$\sigma_B(K,f) = \frac{\alpha}{(fK)^{(1-\beta)/2}\left(1+\frac{(1-\beta)^2}{24}\log^2(f/K)+\frac{(1-\beta)^4}{1920}\log^4(f/K)+\cdots\right)}\cdot\left(\frac{z}{x(z)}\right)$$
$$\cdot\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta v\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}v^2\right]\cdot t_{ex}+\cdots\right\} \tag{4.4}$$

where

$$z = \frac{v}{\alpha}(fK)^{(1-\beta)/2}\log(f/K)$$
$$x(z) = \log\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}$$

When we have the case of ATM options, meaning that the option is struck at $K=f$, then the SABR-formula reduces to

$$\sigma_{ATM} = \sigma_B(f,f) = \frac{\alpha}{f^{1-\beta}}\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}}+\frac{1}{4}\frac{\rho\beta\alpha v}{f^{(1-\beta)}}+\frac{2-3\rho^2}{24}v^2\right]\cdot t_{ex}+\cdots\right\} \tag{4.5}$$

The omitted terms "$+\cdots$" are a part of the approximation that is the SABR-model, but are so small that they do not make a significant difference and thus we will not be implementing this expansion.

SABR-model with Normal volatility

The SABR-model providing us with Normal volatility is referred to as the stochastic β model in (Hagan et al., 2002). The processes are the same as for the SABR-model with Black volatility, but the formula providing the Normal volatility is different and expressed here:

$$\sigma_N = \frac{\alpha(1-\beta)(f-K)}{f^{1-\beta}-K^{1-\beta}}\cdot\left(\frac{z}{x(z)}\right)\cdot\left\{1+\left[\frac{-\beta(2-\beta)\alpha^2}{24\sqrt{fK}^{2-2\beta}}+\frac{\rho\alpha\nu\beta}{4\sqrt{fK}^{1-\beta}}+\frac{2-3\rho^2}{24}\nu^2\right]t_{ex}+\cdots\right\} \tag{4.6}$$

where

$$z = \frac{\nu}{\alpha} \frac{f - K}{\sqrt{fK}}$$

$$x(z) = \log\left(\frac{\sqrt{1 - 2\rho z + z^2} - \rho + z}{1 - \rho} \right)$$

The classic SABR formula using Normal volatility in equation (4.6) will constitute the starting point when we derive the Normal SABR-model in section 4.5.

4.2 Exploring the parameters

This section covers how the different parameters in the SABR-model affect the level, slope, and curvature of the volatility smile. The start value of each parameter is seen below and then changed both up and down in each testing of the parameter. The values are selected with inspiration from (Linderstøm, 2013).

$$f = 3\% \qquad \alpha = 0.03 \qquad \beta = 0.6 \qquad \rho = -0.3 \qquad v = 0.3$$

f In figure 4.1 three volatility smiles have been plotted with different levels of ATM. From the implied volatility of the ATM level, the backbone can be computed. The backbone is the movement of the ATM volatility when the underlying forward rates moves, where the skew and smile are the movements for a fixed strike K when the underlying forward rates moves. Therefore, the backbone describes the changes in volatility when forward rates are either high or low, where the skew and smile describe how much the volatility changes when a particular option moves in and out of the money.

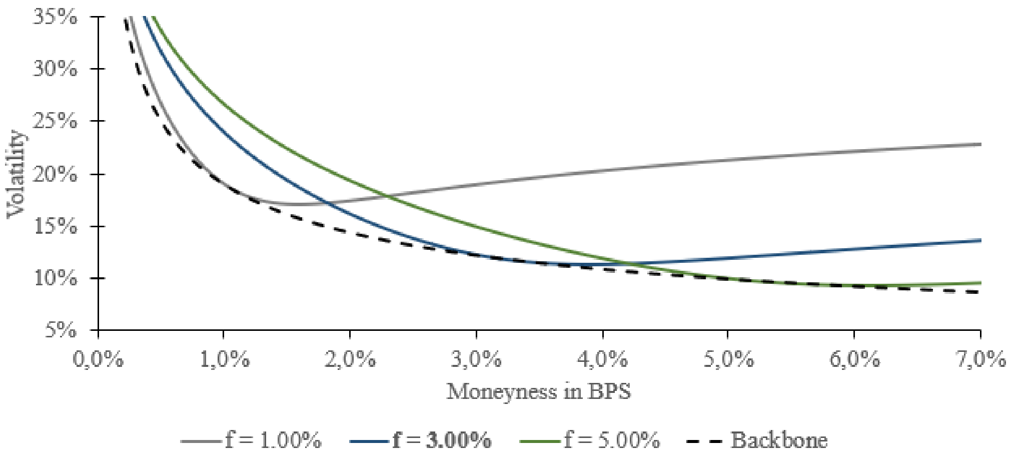


Figure 4.1: Shifted f and visualisation of the backbone effect with a $\beta = 0.6$ (own creation)

The backbone is almost entirely determined by the exponent β. As seen, when comparing figure 4.1 and figure 4.2, the greater the β, the flatter the backbone. We can therefore use the β exponent to analyse market properties. A low β would mean that the market exhibits high volatilities for low interest rates, while a high β indicates that the level of the underlying assets can be neglected.

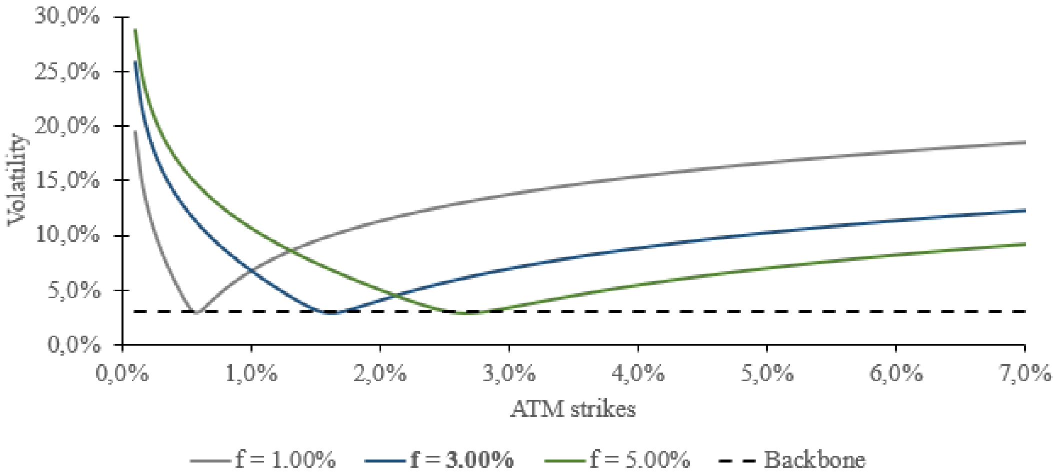


Figure 4.2: Shifted f and visualisation of the backbone effect with $\beta = 1$ (own creation)

α This parameter is the initial volatility in the model and is depicted in figure 4.3. The effect of this parameter is as can be expected – higher initial volatility results in higher implied volatility. The parameter primarily affects the level of the curve and secondarily the slope, meaning that the higher α, the greater the slope and the steeper the curvature of the graph. All three graphs have their minimum close to the ATM strike of 3%.

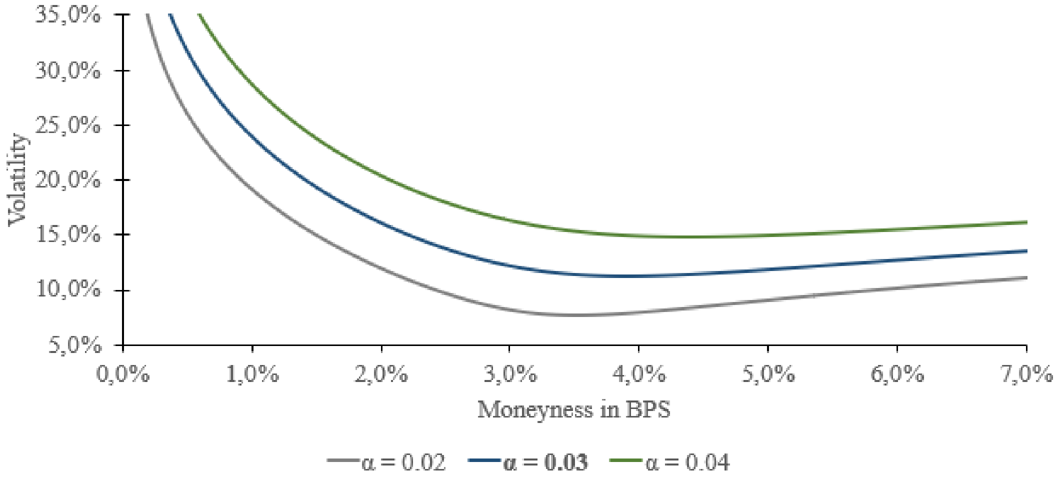


Figure 4.3: Effect on implied volatility for different values of α (own creation)

$\boldsymbol{\beta}$ From figure 4.4 we can see that β has the greatest impact on the level of the smile and a secondary effect on the curvature. From the figure we can see that the higher the β, the lower the level of the smile, and the higher the beta, the greater the curvature.

The effect of β is very similarity to the effect of ρ to the shape of the smile. This is why β is often fixed when estimating the parameters of the SABR model. This will be covered in depth in section 4.3.

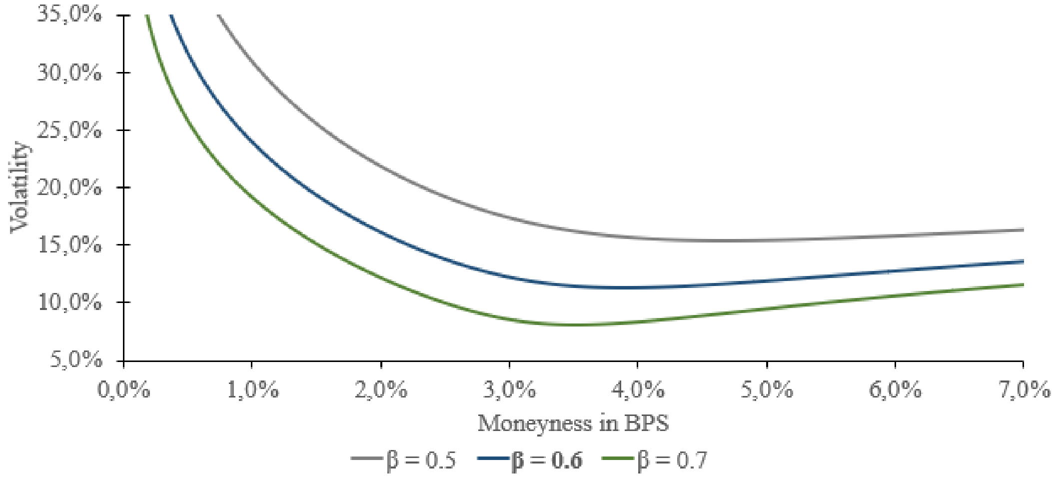


Figure 4.4: Effect on implied volatility for different values of β (own creation)

$\boldsymbol{\nu}$ ν is the "volatility of the volatility" and affects the curvature of the smile. For a higher value of ν, the smile becomes more pronounced and for a lower value it has a flatter structure. There is also an effect in the steepness of the curve, so a higher value for ν causes a steeper curve.

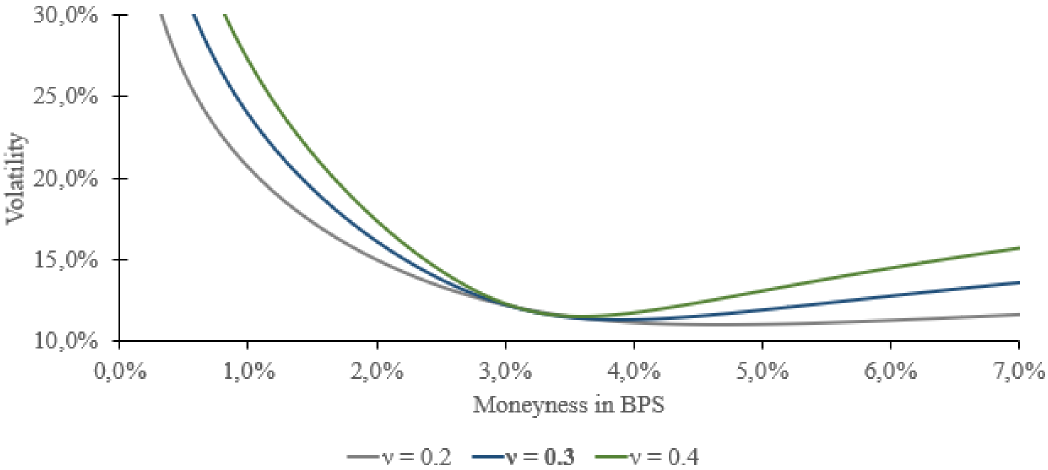


Figure 4.5: Effect on implied volatility for different values of ν (own creation)

$\boldsymbol{\rho}$ We know from equation (4.3) that ρ is the correlation between the movements of the forward rate and the volatility, meaning that we have $\rho \in [-1; 1]$. The starting point for ρ is a negative value of -0.3. We see the effect of a shifted ρ in figure 4.6 and notice that the smiles move around the ATM strike of 3%.

An interesting property of ρ, that is visible when shifting the parameter both in a positive and a negative direction, is that the operational sign dictates the movements of the volatility when bumping the underlying forwards rates. When the correlation is negative, the volatility decreases with an increase of the underlying forward rates, and vice versa. These movements relates to the intuition from equation (4.3), that describes the relationship between the forward rates and the volatility.

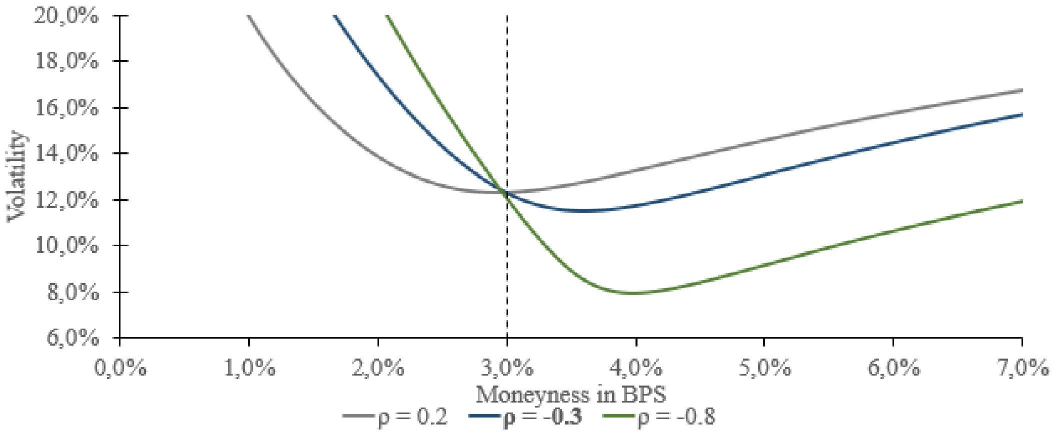


Figure 4.6: Effect on implied volatility for different values of ρ (own creation)

We have now seen how the different parameters in the SABR-model affect the shape and level of the smile. We will now move on to estimating the parameters.

4.3 Estimating SABR Parameters

There are several suggested methods of fitting the SABR-model. The different methods differentiate themselves on how to estimate or select β and α. β can either be fixed to a best guess or fitted along with the other parameters. α can either be estimated using equation (4.8) or by fitting it along with the other parameters.

All parameters α, β, ρ, ν can be estimated by solving a least square minimisation problem. To apply the minimisation problem we need a calibrated swap curve, implied volatility quotes $\hat{\sigma_i}$ and corresponding strikes K_i. Mathematically, we define the least square minimisation problem as follows (Linderstøm, 2013):

$$\begin{gathered}\min_{\Omega} = \sum_{i=1}^{n} (\hat{\sigma_i} - \sigma_N(K_i, f, \alpha, \beta, \rho, \nu))^2 \\ 0 \leq \beta \leq 1, \quad -1 \leq \rho \leq 1, \quad 0 \leq \nu, \quad 0 \leq \alpha\end{gathered} \tag{4.7}$$

As an example, we will test the different fitting methods on data from the 03-12-2018 on a 10Y10Y Euribor 6M swaption. We have calibrated a zero and discounting curve in the same fashion as in the example in section 2.3.2. For the standard SABR we need implied Black volatility quotes and their corresponding strike (See table 4.1 below).

Bps away from ATM	400	200	100	50	25	0	-25	-50	-100	-200
Strike in procent	7,5%	5,5%	4,5%	4,0%	3,7%	3,5%	3,2%	3,0%	2,5%	1,5%
Implied Black Volatility	15,6%	16,2%	16,8%	17,3%	17,6%	17,9%	18,4%	18,8%	20,0%	24,1%

Table 4.1: Shifted market quotes in implied Black volatility with different strike levels for a 10Y10Y Euribor 6M swaption on 03-12-2018 (Source: Bloomberg and own calculations)

We will now move on to testing the different methods of fitting the SABR parameters to match the implied volatility market quotes.

4.3.1 Influence of β

As stated when examining the properties of β, the effect of β to the volatility smile is very similar to the parameter ρ. It is therefore not crucial that the estimate of β is spot on when determining the shape of the smile. To demonstrate this, we have fitted the SABR parameters while fixing β to $\beta = \{0, 0.5, 1\}$ and then estimating the remaining three parameters by solving the least square minimisation problem. The results can be seen in the table 4.2 below.

β	0.50	0.00	1.00
α	0.03	0.01	0.88
ν	0.16	0.15	1.06
ρ	-0.11	0.57	-0.56

Table 4.2: Parameters for the SABR-model for $\beta = \{0, 0.5, 1\}$

In figure 4.7 we see the volatility smiles calculated from the parameters in table 4.2. Note that the smile stays almost the same, even though the parameters change. All three cases of β and the computed parameters fit the market data quite well. From the figure it appears that the choice of β does not have a big impact on the result close to ATM and deep OTM. If we look to deep ITM, we see that β has a slightly larger effect. However, pricing deep in or out of the money is not something we concern ourselves with in the short term, since the par swap rate then should decrease by 3% which is highly unlikely. If we were to look long term, we would recalibrate to get a more precise estimation. This all points to the possibility of fixing β in advance and not estimating it along with the other parameters, which we believe to be a smart move as an estimation of β with the other parameters could cause a risk of overfitting the smile. We will return to overfitting later in section 4.3.3.

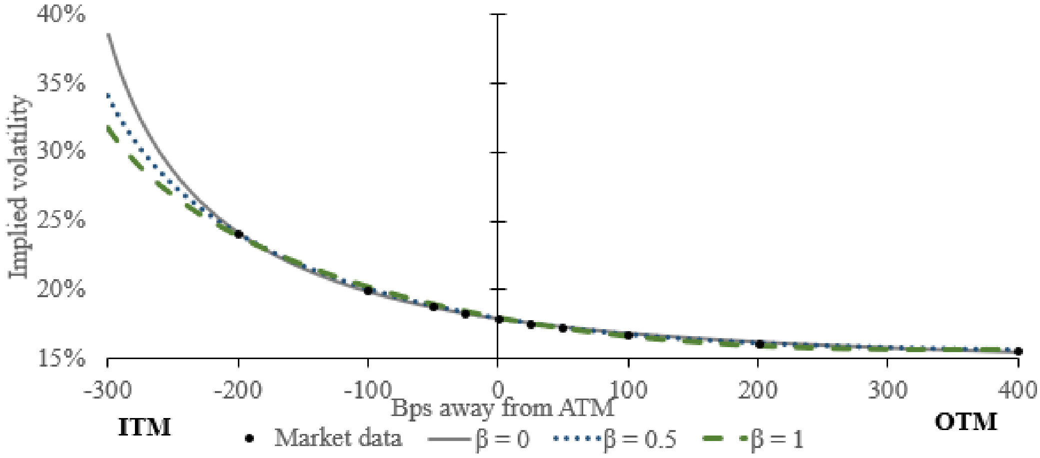


Figure 4.7: Smile fitting for different values of β. The other parameters can be seen in table 4.2, own creation (Data source: Bloomberg)

4.3.2 Two methods of estimating α

The volatility parameter α can be derived in two ways: by solving a minimisation problem and by inverting the ATM SABR-formula in equation (4.5). The first method is our classic minimisation problem, where we estimate α with equation (4.3). The second method is proposed in (Hagan et al., 2002), where the ATM SABR-formula is inverted numerically.

Computing α with minimisation

We use the minimisation problem from (4.3), where we set $\beta = 0.5$ and minimise the squared error between the SABR volatilities and the market volatilities with regards to α, ν and ρ. The estimation is performed for the base example, a 10Y10Y EUR swaption. The results and a comparison to the second method can be seen in table 4.3 and figure 4.8.

Solving for α

We use the ATM SABR-formula from equation (4.5) and use algebra to invert the equation and solve so that all variables on one side and are equal to zero.

$$\sigma_{\text{ATM}} = \frac{\alpha}{f^{1-\beta}}\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}} + \frac{\rho\beta\nu\alpha}{4f^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2\right]\cdot t_{ex}\right\} \tag{4.8}$$

$$\Leftrightarrow 0 = A\cdot\alpha^3 + B\cdot\alpha^2 + C\cdot\alpha - \sigma_{\text{ATM}}f^{1-\beta} \tag{4.9}$$

where

$$A = \left[\frac{(1-\beta)^2 T}{24 f^{2-2\beta}}\right], B = \left[\frac{\rho\beta\nu T}{4 f^{1-\beta}}\right] \text{ and } C = \left[1 + \frac{2-3\nu^2}{24}\nu^2 \cdot t_{ex}\right] \tag{4.10}$$

To solve this equation for α we need to find the root of the polynomial. There is a possibility of three roots, but for realistic parameter values we will typically only have one real root. If we get more than one, we will choose the smallest positive root (West, 2005). In practice the steps to take are as follows:

1. Choose initial values for β, ν and ρ
2. Solve equation (4.9) for α
3. Use α in the minimisation problem and minimise with regards to ν and ρ
4. Repeat step 2 and 3 until the sum of squared errors is at an acceptable level

The method is in steps, where we repeat some steps until the solution is acceptable. In table 4.3, we see some different values for the parameters and the squared error. We selected the start values to be the same as in the example from section 4.2 and $\beta = 0.5$. Then we solved equation (4.9) to get the α for the first estimation, which was used to estimate ρ and ν in the SABR set-up. The squared error is shown in the last column and improves every time the steps are repeated. We then used the new ρ and ν to estimate a new α and repeated these steps until the squared error was close to the minimisation solution. The results for the solving can be seen in the appendix in table 10.1.

	α	ρ	ν	Squared error
Start values	0.03	-0.3	0.3	
First estimation	0.06351	-0.6490	1.0965	0.0482
Final estimation	0.060123	-0.6376	1.0888	0.0469
Estimation by minimisation	0.0578	-0.6285	1.0836	0.0466

Table 4.3: Parameter values from estimating α for a 10Y10Y EUR swaption using minimisation and solving approach (own calculations)

The parameters in table 4.3 are depicted by the volatility smiles in figure 4.8. We see the improvement in the squared error deep ITM, but around ATM, the graphs are very similar. The estimation for α by minimisation gives the smallest squared error and is the simpler approach. Looking at the graph, the minimisation fit of α is also the one closest to the market data. Since there is not a big difference between the squared errors, we will use the simpler minimisation approach as estimation method for α.

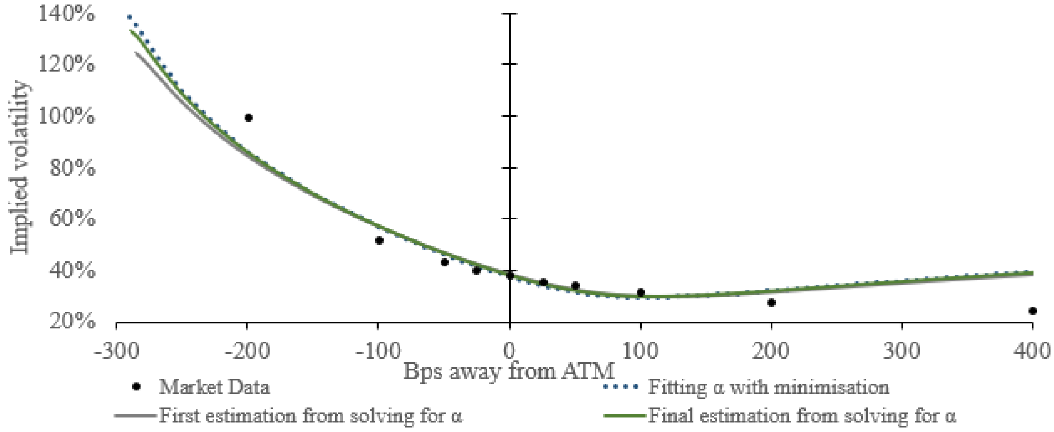


Figure 4.8: Volatility smiles from estimating α for a 10Y10Y EUR swaption using minimisation and solving approach (own calculations)

4.3.3 Estimation choice

We have gone through some choices to consider when estimating some of the parameters in the SABR-model. We will estimate α along with ν and ρ using the minimisation problem since it is a simpler approach and gives us a lower squared error than estimating α using the equation approach. We have looked at β and seen that it seems to be fine to select β in advance. However, we will try to fit a smile while estimating β alongside the other parameters.

This is done with the minimisation problem but now also with regards to β. The result of this fitting is visible in figure 4.9, where we see a graph that fits the market data very well. The squared error for estimation with free β is very low at 0.0081, where the squared error for the estimation with $\beta = 0.5$ is 0.0466. While it is tempting to let β be free and estimate it alongside the other parameters, it poses a risk of too much market noise in the estimation. The squared error indicates that the estimation fits this set of market data particularly well, meaning that it might be overfitting and thus have a problem being used on any other data set. If we use the overfitted model to price the same products a few days later, then we risk pricing them wrong because the market noise might be different. We need a more general estimation without modelling unnecessary market noise (Hagan et al., 2002), so we will proceed with setting $\beta = 0.5$ and estimate α, ν and ρ with the minimisation problem from equation (4.3).

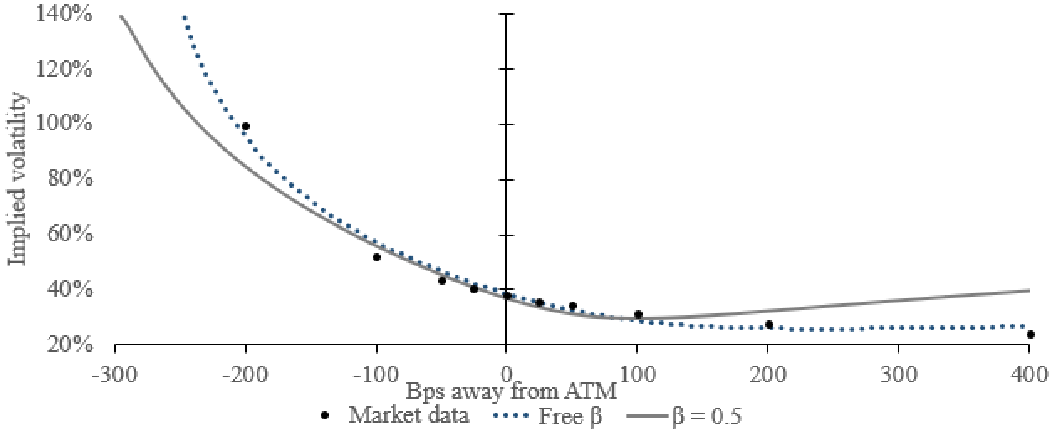


Figure 4.9: Estimation of the SABR-model, 10Y10Y EUR swaption

Showing the risk of overfitting β could be done with more frequently updated data, like daily or interdaily data, where we have weekly updates in our dataset. However, it is beyond the scope of this thesis to investigate this branch further, so we will put this subject to rest with the argument of fitting market noise made by (Hagan et al., 2002). Later in the thesis, we find that we will proceed to work with the Normal SABR-model, where $\beta = 0$, thus eliminating the subject of β estimation.

4.4 Shifted SABR-model

When SABR was originally developed, rates were very high and it was hard to imagine them ever becoming close to zero or even negative. Within the past years rates have dropped a great deal and even become negative, which causes problems for the classic SABR-model. We do not expect rates to drop forever, since very negative rates would cause money to be withdrawn from the banking system, creating a pressure on deposits. We expect there to be some lower barrier on negative rates, making the Shifted SABR-model one of the solutions to the problem of negative rates (Hagan et al., 2014).

With the shifted SABR-model, we add a positive constant s to the process describing the forward rate, so that the lower boundary for the forward rate F is $-s$. The processes in the shifted SABR-model are described as

$$d\hat{F} = \hat{\alpha}(\hat{F} + s)^{\beta}\, dW_1, \qquad \hat{F}(0) = f \tag{4.11}$$

$$d\hat{\alpha} = v\hat{\alpha}\, dW_2, \qquad \hat{\alpha}(0) = \alpha \tag{4.12}$$

where the two processes are correlated by

$$dW_1 dW_2 = \rho dt \tag{4.13}$$

We also add s to the strike K and the calibrated zero curve, but the rest of the SABR-model remains the same. The constant s needs to be selected in advance, which is one of the main drawbacks, because we do not know how negative the rates might be and this could result in a need for adjustment in s later on. This model will be tested later on alongside classic the SABR-model and the Normal SABR-model.

4.5 Normal SABR-model

Another extension of the SABR-model is the Normal SABR-model, which is in connection with the Normal model from section 2.4.2. This version can model negative rates and is derived by letting $\beta \to 0$ (Hagan et al., 2002). First we do this for the processes modelling the movements of the forward rate and the volatility, meaning that the process of the forward rate no longer depends on the current forward rate:

$$d\hat{F} = \hat{\alpha}\ dW_1, \qquad \hat{F}(0) = f \tag{4.14}$$

$$d\hat{\alpha} = v\hat{\alpha}\ dW_2, \qquad \hat{\alpha}(0) = \alpha \tag{4.15}$$

where the two processes are correlated by

$$dW_1 dW_2 = \rho dt \tag{4.16}$$

Now we recall equation (4.6) to find the formula for calculating Normal volatility in the SABR-model:

$$\sigma_N = \frac{\alpha(1-\beta)(f-K)}{f^{1-\beta} - K^{1-\beta}} \cdot \left(\frac{z}{x(z)}\right) \cdot \left\{1 + \left[\frac{-\beta(2-\beta)\alpha^2}{24\sqrt{fK}^{2-2\beta}} + \frac{\rho\alpha\nu\beta}{4\sqrt{fK}^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2\right] t_{ex} + \cdots\right\} \tag{4.17}$$

where

$$z = \frac{\nu}{\alpha}\frac{f-K}{\sqrt{fK}^{\beta}} \quad \text{and} \quad x(z) = \log\left(\frac{\sqrt{1-2\rho z + z^2} - \rho + z}{1-\rho}\right)$$

Now we let $\beta \to 0$ to obtain the Normal SABR-model. We start by looking at the first factor in equation (4.17) and see that for $\beta \to 0$, the numerator goes towards $\alpha(f-K)$ while the denominator goes towards $f - K$ and thus only α remains in the first factor as the rest is equal to 1. The second factor remains almost the same as β appears in the expression for z where the denominator goes towards 1 when $\beta \to 0$. Looking at the brackets, we see that the first and second term go towards zero, leaving the third term unchanged. This results in the Normal SABR-model, capable of modelling negative interest rates:

$$\sigma_N = \alpha \frac{z}{x(z)} \cdot \left\{ 1 + \frac{2-3\rho^2}{24} \cdot \nu^2 \cdot t_{ex} \right\} \tag{4.18}$$

where

$$z = \frac{\nu}{\alpha}(f - K) \quad \text{and} \quad x(z) = \log\left(\frac{\sqrt{1-2\rho z + z^2} - \rho + z}{1-\rho} \right)$$

We now have three versions of the SABR-model; the classic SABR-model, the Shifted SABR-model and the Normal SABR-model. These three models will be compared in section 6 to identify their strengths and weaknesses.

5 Data

The purpose of this section is to present and discuss the data sets used in this thesis. We have two types of data: swap data and swaption data. All data sets have been collected from the Bloomberg Terminal. As one of the most used data bases, we expect Bloomberg to give an accurate picture of the traded products in the market (Duyvesteyn and de Zwart, 2015).

5.1 Swap data

We have two sets of swap data. The first is a semi-annual interest rate swap written 6M Euribor data set and the second is an Overnight Index Swaps data set (OIS). Both swap rates are quoted in par-swap rates for a wide range of different maturities. The data sets will be referred to as EUR swap data and OIS data.

5.1.1 Euribor Swap

Our first swap data set consists of daily closing quotes for EUR swaps. The quotes are par-swap rates of a fixed-for-floating vanilla interest rate swap with the 6M Euribor rate as the underlying reference rate. We have collected swap rates from January 1st 2010 to January 1st 2019 with maturities of 1-10Y, 15Y, 20Y and 30Y. The EUR swaps are used to construct the zero curve and hence to construct expected future xIBOR payments, which is used to price swaptions.

	EUSA1	EUSA2	EUSA3	EUSA4	EUSA5
03-12-2018	-0.24%	-0.15%	-0.01%	0.13%	0.27%
04-12-2018	-0.24%	-0.16%	-0.03%	0.11%	0.25%
05-12-2018	-0.23%	-0.14%	-0.01%	0.13%	0.27%
06-12-2018	-0.23%	-0.14%	-0.02%	0.12%	0.26%
07-12-2018	-0.23%	-0.14%	-0.01%	0.12%	0.26%

Table 5.1: One week of 6M Euribor swaps quotes, including Bloomberg tickers (Source: Bloomberg)

When looking at table 5.1 we can see that the par swap rates increase with the maturity of the swap, which tells us that the zero curve should be upward sloping. This corresponds well with the constructed zero curve in section 2.3.

To show the development of the EUR swap, we have plotted the time series in figure 5.1 below.

Figure 5.1: Time series of a 2Y fixed-for-floating interest rate swap written on 6M Euribor (Data source: Bloomberg)

In figure 5.1 we see the par swap rates for a 2Y fixed-for-floating plain vanilla Euribor 6M swap. We see that rates started to move towards zero around 2014 and actually became negative in late 2014 and the beginning of 2015, which did not seem possible until that very time. Negative rates caused a lot of problems for the existing pricing models and especially the classic SABR-model, which was designed for an environment of 5% base rates and the lognormal distribution, meaning that negative rates were not even an option (Balland and Tran, 2013). This called for action to adapt the pricing models to this new low and negative interest rate environment, which is the main area this thesis seeks to discuss.

In addition, it is interesting for us to look at the volatility of the EUR swap. We also seek to accurately model realised volatilities in this thesis. Even though the realised volatility of the swap is not the same as the implied volatility of a swaption corresponding, there is still somewhat of a link, since the expectancy in implied volatility stems from the previously realised volatility and the future realised volatility has some of its base in implied volatility. It is important to note that the two are not the same, but are connected.

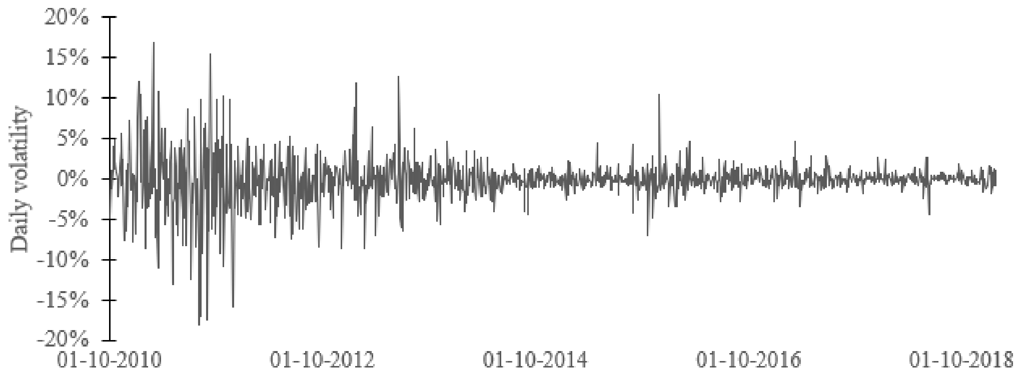


Figure 5.2: Daily realised volatility for a 2Y fixed-for-floating interest rate swap written on 6M Euribor (Data source: Bloomberg)

From figure 5.2 we can see that the realised volatility decreases in recent years. This should be visible when comparing realised volatilities of swaptions in 2010 and 2018. This is an effect of the actions the European Central Bank (ECB) has taken since the financial crisis in 2008 which will be described in section 5.3. The expectations to the volatility are both a product of previous realised volatility but also of the statements made by ECB. The target rate has both been decreased and kept steady by ECB, which results in the decreasing movements in figure 5.2.

5.1.2 Overnight Index Swap

The second swap data set consists of Overnight Index Swaps (OIS). OIS is, as the name dictates, a swap that settles over night. The fact that the swap settles over night is very important, since this property reduces the credit risk of the swap. This is why we use OIS when constructing the discounting curve.
The OIS from our dataset is written on 6M Euribor with the Bloomberg ticker *EUSWE*. We have collected EUR swaps from January 1st 2010 to January 1st 2019 for swaps with maturities of 1-10Y, 15Y, 20Y and 30Y.

	EUSWE1	EUSWE2	EUSWE3	EUSWE4	EUSWE5
03-12-2018	-0.352%	-0.271%	-0.165%	-0.011%	0.094%
04-12-2018	-0.348%	-0.278%	-0.161%	-0.032%	0.120%
05-12-2018	-0.348%	-0.265%	-0.144%	-0.011%	0.112%
06-12-2018	-0.348%	-0.269%	-0.149%	-0.018%	0.116%
07-12-2018	-0.340%	-0.264%	-0.143%	-0.014%	0.117%

Table 5.2: One week of 6M Euribor OIS quotes for 1Y-5Y, including Bloomberg tickers (Source: Bloomberg)

When comparing table 5.1 with table 5.2 above, one can indeed see that the OIS rates are lower than the EUR swap rates. This fits perfectly with the minimisation of credit risk in the OIS rates.

5.2 Swaption data

We have collected a series of daily mid-quote ATM swaptions quoted in both annual implied Normal and Black volatility. We have collected swaption quotes from January 1st 2010 to January 1st 2019 for swaptions with a wide range of combinations of expiries and maturities of 1-10Y, 15Y, 20Y and 30Y. This market data will be used when exploring the ATM volatility of swaptions.

In addition to ATM implied volatility quotes, we have also collected for different strikes both ITM and OTM. These quotes will be used to fit and analyse the volatility smile. The differences and levels of OTM/ITM implied Black and Normal volatilities have already been covered in section 2.4.4, we will therefore switch our focus to the ATM volatilities and what information these quotes can provide. Note that our data set containing OTM/ITM away from the money volatilities is weekly data.

	EUNE11	EUNE15	EUNE110	EUNE55	EUNE510
03-12-2018	21.81	41.07	43.115	61.89	59.99
04-12-2018	21.91	40.6	42.895	61.39	59.76
05-12-2018	21.36	40.88	42.9	61.39	59.48
06-12-2018	21.39	41.13	43.115	61.39	59.64
07-12-2018	21.28	40.97	43.03	61.39	59.44

Table 5.3: ATM implied Normal volatilities for 1 and 5 years expiries with 1, 5 and 10 year maturities. Source: Bloomberg

From table 5.2 we can see that volatility increases both with expiries and maturities. As all quotes are annualised, we can interpret the rising of volatilities to be because the market expects higher volatility in the coming years.

From figure 5.3 we can see that the implied Normal volatility decreases through the years. This corresponds very well with the realised volatility of the 2Y EUR swap in figure 5.2. We can therefore see that there is a link between implied and realised volatility.

Figure 5.3: Time series of 1Y2Y swaption quotes in implied Normal volatilities in bps (Data source: Bloomberg)

5.3 Macroeconomic perspective

We have now covered the data sets needed to do an empirical analysis of the SABR-model. The performance of a model is highly dependent on the data and we have therefore ensured that the data comes from a reliable source. In this section we will examine the underlying macroeconomic reasons for the historically low interest rate environment. A major driver for the European interest rate is the European Central Bank (ECB). We will therefore discuss the impact of the target rate of ECB and the *quantitative easing* program initiated by ECB in January 2015 (Andrade, Phillippe, 2016).

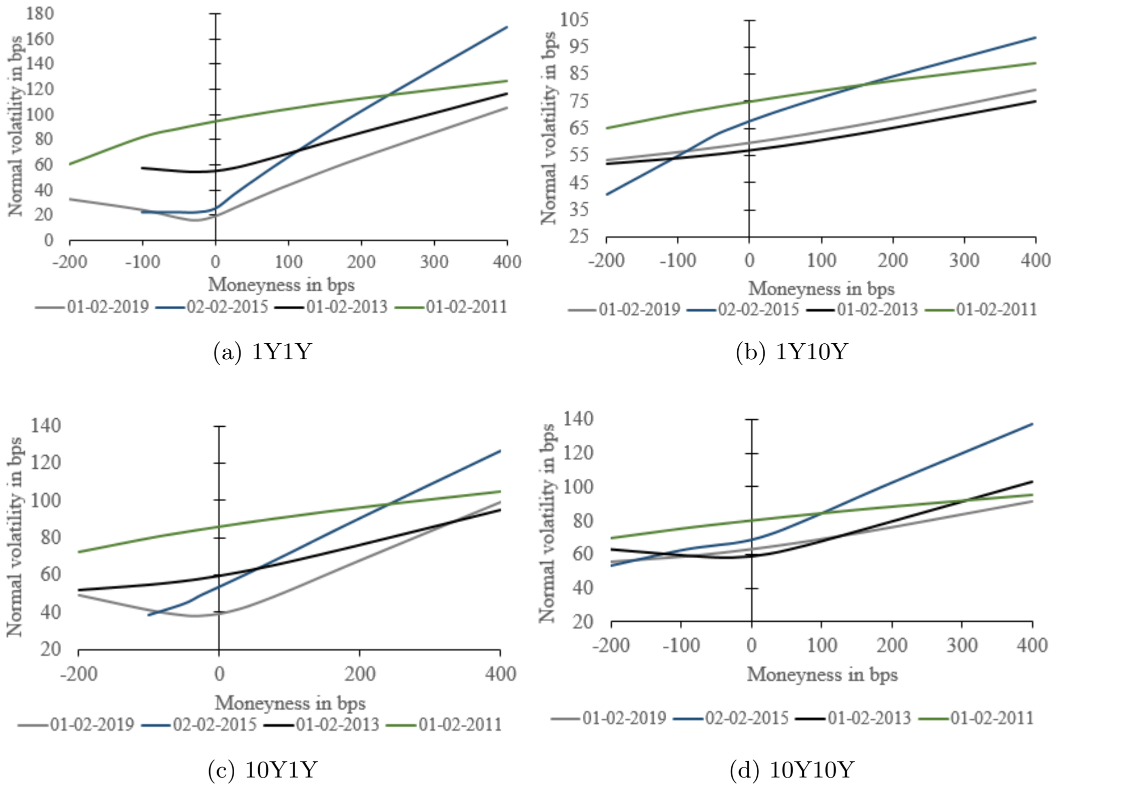
Firstly, ECB have continuously reduced their target rate since the financial crisis in 2008. This was done to accelerate the European economy. By reducing the interest rate, ECB made it cheaper for both nations and corporations to borrow money.
In addition to lowering the target rate, ECB implemented quantitative easing by initiating an *Asset Purchase Programme*. Through the program, ECB bought large amounts of treasury bills and corporate bonds. This was done to flush the market with cash, and as a result of this, the interest rate was lowered even further. Interest rates of bonds decline when there is a high demand, and the quantitative easing programme therefore reduced the cost of borrowing for both nations and corporations.

The effect of the target rate and the quantitative easing is clearly visible in our swap data. When looking at the 2Y Euribor 6M swap in figure 5.1, we can see that the par swap rate has been steadily declining throughout the entire period.
In addition, we can see the macroeconmic decisions when looking at implied volatilities

of our swaption data. We can see this effect in figure 5.3, that shows declining implied volatilities of a 1Y2Y swaption. There are two drivers behind the lowered implied volatility, one is obviously the low target interest rates, but also the signals and announcements from ECB play a key role for the implied volatility of swaptions. In September 2014 ECB cut the target rate by 20 basis points, but also stated that the target rate would remain steady for the foreseeable future. An announcement like this has a great impact on the volatility of swaptions, and when looking closely at figure 5.3, we can see that the implied volatility drops significantly after the announcement.

5.4 Historical volatility smiles

In section 3.1 we discussed the underlying reason for the existence of the volatility smile. In addition to the identified reasons i.e. underlying probability distribution of pricing models and buying pressure, we can also use the macroeconomic perspective and market expectations to explain and derive interesting properties of the market. This can be done by looking at the level and curvature of the volatility smile historically and across expiries and tenors.



(a) 1Y1Y

(b) 1Y10Y

(c) 10Y1Y

(d) 10Y10Y

Figure 5.4: Historical volatilities for different maturities and tenors (Data source: Bloomberg)

The graphs in figure 5.4 above show the volatility smiles from 2011 until today for a 1Y1Y, 1Y10Y, 10Y1Y and 10Y10Y Euribor 6M swaption. We have removed the smiles for the years, 2012, 2014, 2016, 2017 and 2018 because they were similar to some of the other smiles, and therefore a disturbing factor when examining the graphs.

The first thing we will look at is the level of the implied Normal volatilities. It is clear from the four graphs that the level of the implied volatility was higher in 2011 and then decreasing toward 2019. This development is clearly connected to the monetary policy of the European Central Bank (ECB) described in section 5.3. Back in 2011, ECB frequently lowered the interest rates to boost the economy and thus causing an increase in the realised volatility, while the current monetary policy is to keep the interest rates steady (European Central Bank, 2019). This statement is also supported by the realised volatility of a 2Y swap in figure 5.2.

The second interesting pattern we see is that the curvature of the smile seemed straighter and in an upwards sloping direction back in 2011 compared to 2019. This could be connected to market expectations, which could result in buying pressure. In 2011, most investors were confident that interest rates would be lowered, and market participants like pension funds could have applied buying pressure to swaptions with high strikes, and thus creating an upwards sloping volatility smile. In 2019, where the monetary policy in Europe is to keep the interest rates stable, the market participants like pension funds and large corporations might be applying buying pressure on both higher and lower strikes around ATM, thus creating, to some extent, a more classic volatility smile.

The third point we will make comes from comparing the smiles of swaptions with different expiries and tenors. In general, longer dated swaptions exhibit flatter smiles as can be seen in figures 5.4b and 5.4d. This is most likely a consequence of the market participants simply not knowing what will happen in 10 or 20 years. It is difficult for an investor to make a qualified guess on where the interest rates and their volatility will be so far in the future. The pricing of these options will therefore be flatter when the uncertainty of the market is so high.

Now that we know more about the data sets we will be using and the macroeconomic factors present, we will begin our empirical analysis of the three versions of the SABR-model.

6 Empirical analysis

In this section we will perform an empirical analysis of the three versions of the SABR-model. We will fit the models under negative interest rates, test parameter stability and do both an in-sample and out-of-sample test to compare the models. Based on the findings of the comparison, we will choose a model to use when applying risk management to swaptions in secition 7.

6.1 Comparing the models for negative interest rates

To test our three models in a negative interest rate environment, we will use the same calibrated curves as we found in section 2.3, where negative rates are present. Classic and shifted SABR will be computed in Black volatility and Normal SABR will be computed in Normal volatility. We will fit the three models using the selected estimation method from section 4.3.3 i.e. fixing $\beta = 0.5$ and computing α, ρ and ν using the minimisation approach.

6.1.1 Classic SABR-model

We first use our classic SABR-model to model the implied volatility. Even though we know that this version cannot proper model negative rates, it is important to do it and visualise the results to understand all aspects of the model. Figure 6.1.1 shows a 10Y10Y payer swaption modelled with the classic SABR-model and the problems caused by negative rates as we move deep ITM. As this is a payer swaption, we pay the fixed rate and receive the floating rate. We then change the strike by different amounts of basis points (from -300 to +400 basis points) to see how the volatility changes when we move in and out of the money. ATM is the par swap rate and when we move to higher strikes we are OTM, as we then would be paying a higher fixed rate than is fair and thus losing money. For the opposite situation, when the strike becomes lower we move ITM as we would be paying a lower fixed rate than the fair par swap rate and this would be in our favour, hence ITM.

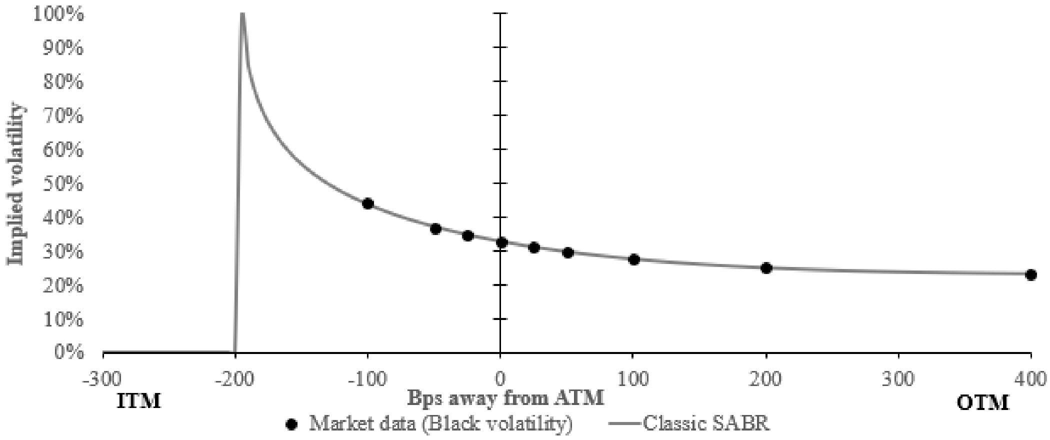


Figure 6.1: Classic SABR for 10Y10Y EUR swaption

The volatility smile is very high when we are deep ITM, because the volatility is quoted in Black volatility and is a relative size. As we move closer to zero, the movements have a larger impact on the implied volatility as they make up a bigger part of the very low strike. For higher strikes, the movements have a smaller impact on the implied volatility, causing the smile to be less expressed.

It is clear to see when the strike becomes negative, since the classic SABR-model cannot deal with negative rates. The graph for classic SABR dives around -200 bps, because the par swap rate is 1.99% and we hit negative strikes here.

6.1.2 Shifted SABR-model

The shifted SABR-model is shifted by $s = 1.5\%$ in both the strike and par swap rate as the model from equation (4.13) dictates. The method is identical to the classic SABR except for the shift in the model. This shift is also done for the calibrated zero curve that we use to price the fair par swap rate. Note that we do not shift the discount curve because this would cause a harder discounting and incorrect modelling of the implied volatility. The shift enables the model to deal with negative rates and it reacts differently than the original SABR-model, when we move close to -200 basis points. Shifted SABR-model provides an overall smooth volatility smile that fits the market volatilities nicely. It has the same curve upwards deep ITM as the original SABR-model, which is due to the same dynamics in Black volatility as mentioned in the previous section.

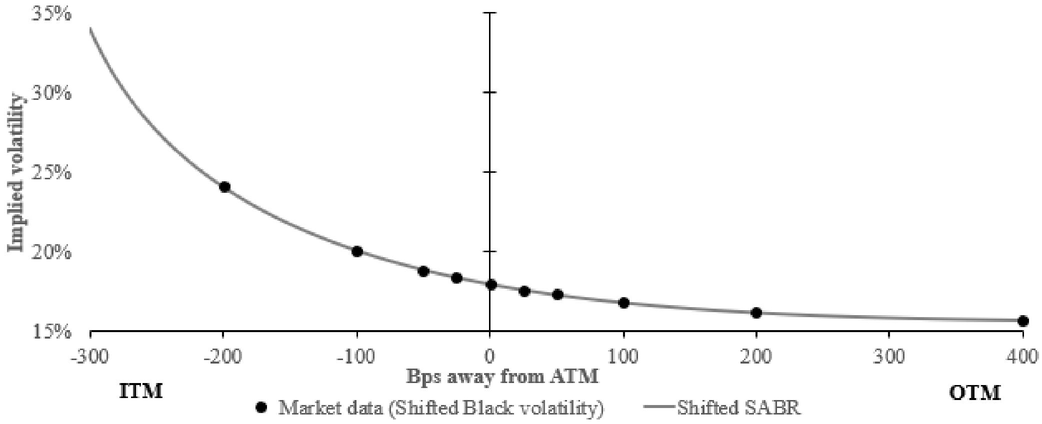


Figure 6.2: Shifted SABR for 10Y10Y EUR swaption

6.1.3 Normal SABR-model

The Normal SABR-model is estimated using the minimisation method for equation (4.18) and with $\beta = 0$. We use the data set containing away from the money Normal swaption volatilities and use our minimisation approach to compute the parameters for the Normal SABR-model. Then we compute all the implied volatilities for the different strikes and obtain graph depicted in figure 6.3.

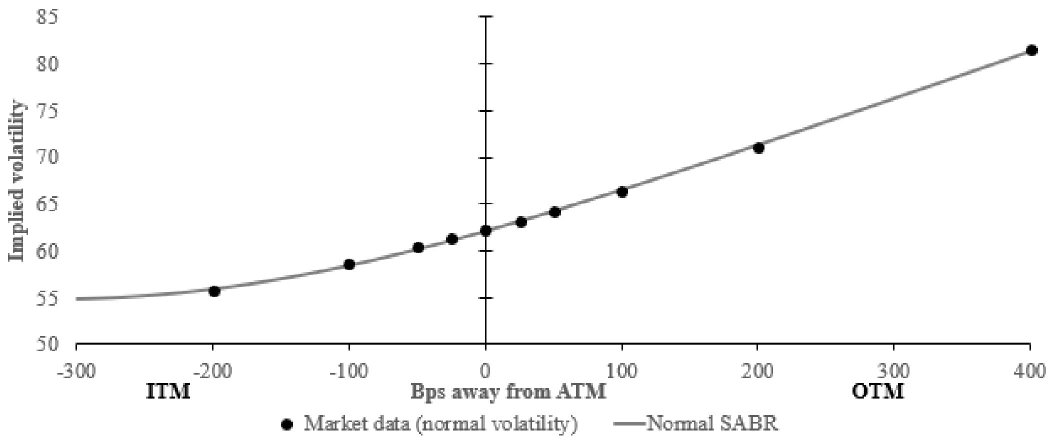


Figure 6.3: Normal SABR for 10Y10Y EUR swaption

We see in figure 6.3 that the calculated volatilities fit the market volatilities very well and express a smile curving upwards as we move out of the money. This dynamic fits well with the theory from section 3.1; a buying pressure on OTM options. With rates being very low in the current market, rate hikes are expected and thus create the buying pressure on OTM options. We notice, that it is more expensive to buy ATM options than ITM options. This is due to the low expectations for a decrease in rates since they are so low at the moment that expectations are only for them to stay the same or increase.

In this comparison for the three versions of the SABR-model, we have seen three very different smiles. The smiles for classic and Shifted SABR are sloping upwards when moving into the money, which contradicts our theory on OTM buying pressure. The shape of these smiles is due to the quotation of volatility. We have addressed this issue before, but there effect of the quotation is even more visible here. The volatilities for the classic SABR-model, as we move close to a par swap rate of zero, become extremely high; as the par swap rate becomes smaller, the relative change becomes bigger. The quotation of relative changes in volatility makes the smile difficult to interpret as one could mistakenly argue a buying pressure on ITM options, when in fact the buying pressure is on OTM options.

When we look at the Shifted SABR-model, the curve is still curving upwards in ITM, but not with volatilities as high as for classic SABR. This is due to the shift and the rates not being as close to zero as for the classic SABR. The Black volatility is very dependant on how big the shift is. If an investor chooses to use the Shifted SABR-model, he needs to agree on a shift with the counterparty of his financial contract and also make sure that the shift is big enough, so that he does not have to reshift later on.

The smile for the Normal SABR-model is curving upwards OTM as expected from the theory. We argue it easier to interpret and work with as there is no need to make any decisions prior to using the model.

For the next tests, we will exclude the Classic SABR-model as it cannot model negative interest rates. We will continue our work with both the Shifted SABR-model and Normal SABR-model and test their pricing accuracy in the next section.

6.2 Testing the models

For this section we will first test the stability of the parameters of the two remaining SABR-models. We do this to test how often one needs to calibrate the SABR-model in order to price accurately. Secondly, we will test the pricing accuracy of the models both in a *In-Sample Test* and in a *Out-Of-Sample Test*. Since the models are quoted in two different volatilities, we will test the models by comparing their pricing accuracy in Euro to see how well they perform.

6.2.1 Stability of parameters

We will test the stability of the parameters by fitting SABR models per week throughout 2018 and into the first month of 2019. We will continue to use our 10Y10Y EUR swaption for the tests.

Parameter stability in the Shifted SABR-model

In figure 6.4 below we can see the changes of the parameters in percentage from one week to the next. We see that the change to α is the lowest with a standard deviation of 0.6%, ν exhibits a standard deviation of 3.6% and ρ exhibits a standard deviation of 18.2%. The stability of the parameters in the Shifted SABR-model is thus quite different from the Normal SABR-model. What we see is that the parameters that influence the smile curvature, ν and ρ, exhibit larger standard deviations in the Shifted SABR-model compared to the Normal SABR-model. This is a natural consequence of the how Black volatility is measured in relative terms. When the strikes are close to zero, the relative volatility increases, and thus the parameters of the shifted SABR models will change to a greater extend to accommodate for these changes to the curvature of the smile.

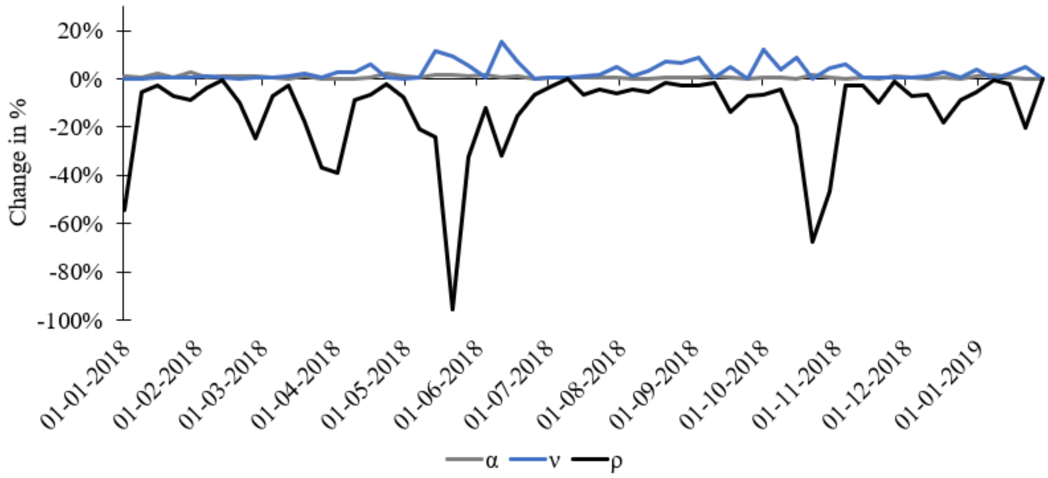


Figure 6.4: Change throughout the data period in Shifted SABR parameteres with a $\beta = 0.5$ for a 10Y10Y Euribor swaption (own calculation)

Parameter stability in the Normal SABR-model

In figure 6.5 below we can see the changes of the parameters in percentage from one week to the next. We can see that the biggest changes is to α and ρ, while ν is fairly stable. α exhibits a standard deviation of 6.2%, ρ exhibits a standard deviation of 5.9% while ν exhibits a standard deviation of 1.6% .

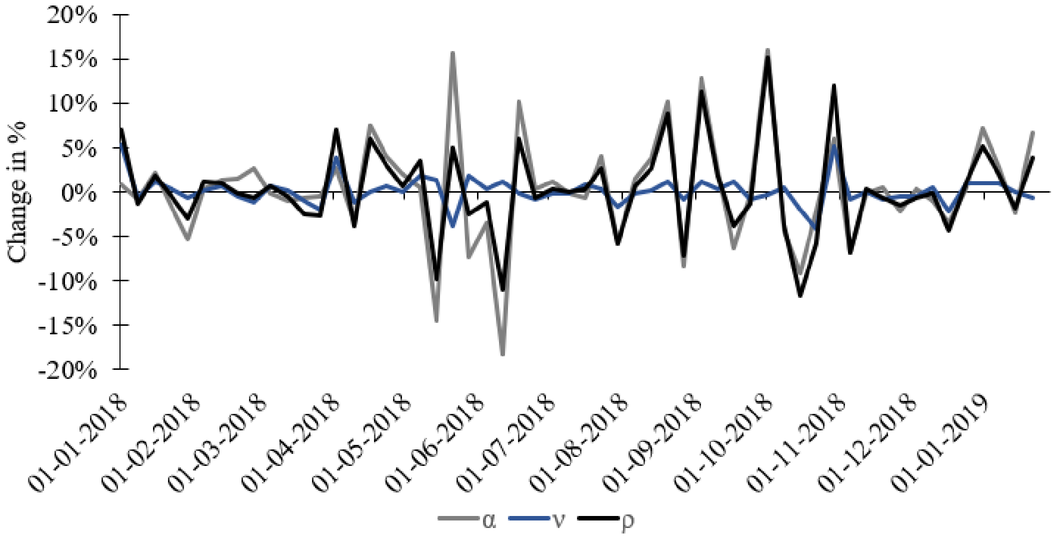


Figure 6.5: Change throughout the data period in Normal SABR parameters with a $\beta = 0.0$ for a 10Y10Y Euribor swaption (own calculations)

In (Hagan et al., 2002), Hagan argues that α, the ATM volatility, should be updated frequently in a fast paced market. How frequently depends on the market, but in some cases several times a day. He also argues that the remaining parameters should be updated at least once a week. Our analysis builds on weekly data, and from the magnitude of the changes, it seems reasonable to update every parameter at least every week.

From our analysis we see that we need to update the parameters of the SABR-model at least once a week. It would have been interesting to extend the analysis if we had daily or interdaily volatility quotes. We also see that the Shifted SABR-parameters exhibit a higher standard deviation for the parameters controlling the curvature of the smile, particularly ρ.

The parameter stability can however not stand alone. In section 7.1 we will examine how much the price changes when the parameters change.

6.2.2 In-Sample pricing accuracy

The second test we cover is an In-Sample test. To test the accuracy of the two models, we first fit a SABR-model to every Monday in 2018, then we calculate the price of a 10Y10Y swaption using the implied volatility from the SABR-model. Lastly we compare the computed price with the market price to see how accurately we can price using implied volatilities from the SABR-model.

In table 6.1 we see the prices for a 10Y10Y EUR payer swaption with a 100.000 Euro notional calculated with volatilities from the Shifted SABR-model. We compare the market prices with the model prices and see the difference in Euros below. The model prices and the market prices fit quite well, especially around ATM. When we move further away from ATM the difference becomes slightly larger, but overall we have been able to accurately price swaptions using the Shifted SABR-model.

Shifted SABR-model										
Basis points	400	200	100	50	25	ATM	-25	-50	-100	-200
Market prices	535.7	1,822.2	3,402.3	4,612.5	5,349.9	6,184.5	7,121.4	8,163.9	10,567.9	16,536.1
Model prices	540.7	1,814.4	3,391.7	4,605.6	5,346.2	6,184.7	7,125.6	8,172.0	10,580.2	16,532.3
Difference	5.0	-7.8	-10.5	-6.9	-3.7	0.1	4.2	8.1	12.3	-3.8

Table 6.1: Prices in Euro for Shifted SABR-model calculated with Black volatilities for a 10Y10Y EUR swaption with a 100.000 Euro notional on 03/12/2018 (Source: Bloomberg and own calculations)

In table 6.2 we see the same swaption as above but for the Normal SABR-model and calculated with Normal volatilities. The differences between the market prices and model prices are a bit bigger than for the Shifted SABR-model. We see again that the pattern of inaccuracies being larger as we move away from ATM, but in general the fit is also very good.

Normal SABR-model										
Basis points	400	200	100	50	25	ATM	-25	-50	-100	-200
Market prices	535.7	1,822.2	3,402.3	4,612.5	5,349.9	6,184.5	7,121.4	8,163.9	10,567.9	16,536.1
Model prices	529.1	1,841.4	3,420.7	4,621.3	5,351.9	6,179.2	7,109.2	8,146.2	10,547.1	16,548.0
Difference	-6.6	19.2	18.4	8.8	2.0	-5.3	-12.2	-17.7	-20.8	11.9

Table 6.2: Prices in Euro for Normal SABR-model calculated with Normal volatilities for a 10Y10Y EUR swaption with a 100.000 Euro notional on 03/12/2018 (Source: Bloomberg and own calculations)

We now compute the differences in pricing over a whole year to get a more thorough in-sample test. The differences between the market prices and model prices have been summed in absolute values, so that the negative and positive values do not cancel each other out. In table 6.3 we see the aggregated differences in the market and model prices for the two models. As earlier in the single-date example, we notice that the Shifted SABR-model has a lower error than the Normal SABR-model. However, we do not find this alarming since the difference is quite small for a sum over 53 data points for a 100.000 Euro notional.

Aggregated differences										
Basis points	400	200	100	50	25	ATM	-25	-50	-100	-200
Shifted SABR	284.2	449.4	607.2	397.0	211.1	14.9	248.4	465.8	701.1	217.8
Normal SABR	337.1	996.3	964.7	458.7	103.7	278.6	641.4	930.7	1,086.4	619.2
Difference	52.8	546.8	357.5	61.7	-107.5	263.7	393.0	464.8	385.3	401.4

Table 6.3: Aggregated absolute differences between market prices and model prices for a 10Y10Y EUR swaption with a 100.000 Euro notional for 2018 (Source: Own calculations)

While the In-Sample test provides some measure of the accuracy of the models, we will also conduct an Out-Of-Sample test to see how well the models perform when we remove one or more data points.

6.2.3 Out-Of-Sample pricing accuracy

For this test, we look at the accuracy of the pricing with the SABR-models, where we remove data points close to ATM and away from ATM. In practice, we conduct two versions of the test; one where we remove data points far away from ATM and another where we remove data points close to ATM. This will let us know more about how well the models do on pricing, when all data points are not available. This test is extremely relevant for the performance of the models, since the purpose of the models is to price products where we do not have market prices.

The approach is to remove the data point from the minimisation problem and solve the model as we have done before. Now we use the SABR-parameters to estimate the volatility for the data point we have removed. This will give us a new Out-Of-Sample SABR-volatility to use when pricing the swaption. The approach is the same for both examples below. We have selected an away from the money point at +400 basis points and a close to ATM point at -25 basis points. We will compute the differences in the prices, both for a single date and aggregated over a whole year.

Test for +400 basis points

The results for this example are visible in table 6.4. Here we see the results for both the Shifted SABR-model and the Normal SABR-model. If we compare the differences in prices for both Shifted and Normal from market to model, we see that the out-of-sample point has a higher difference than the in-sample case. The difference is not very big in the one day example, but aggregates to a more significant number over a whole year. We see that Shifted SABR fits better away from the money, but worse around ATM. For this test, we argue that both models work reasonably well.

Shifted SABR-model										
Basis points	**400**	200	100	50	25	ATM	-25	-50	-100	-200
Market prices	**535.7**	1,822.2	3,402.3	4,612.5	5,349.9	6,184.5	7,121.4	8,163.9	10,567.9	16,536.1
Model prices	**558.3**	1,828.8	3,397.3	4,606.2	5,344.8	6,181.6	7,121.6	8,167.6	10,577.0	16,534.2
Difference	**22.6**	6.5	-5.0	-6.3	-5.2	-2.9	0.2	3.7	9.1	-1.8
Aggregated diff.	**1,273.6**	371.4	288.7	362.0	295.4	163.5	16.2	215.1	518.1	105.7
Normal SABR-model										
Basis points	**400**	200	100	50	25	ATM	-25	-50	-100	-200
Market prices	**535.7**	1,822.2	3,402.3	4,612.5	5,349.9	6,184.5	7,121.4	8,163.9	10,567.9	16,536.1
Model prices	**499.2**	1,815.8	3,410.8	4,620.8	5,355.7	6,186.3	7,118.5	8,156.4	10,554.8	16,541.2
Difference	**-36.5**	-6.4	8.6	8.3	5.7	1.8	-2.8	-7.5	-13.1	5.1
Aggregated diff.	**1,871.2**	339.8	447.2	436.6	301.6	97.9	148.6	392.5	684.4	263.7

Table 6.4: Data for Out-Of-Sample test for 10Y10Y EUR swaption +400 bps on 03/12/2018 (Source: Bloomberg and own calculations)

Test for -25 basis points

Now we test closer to ATM to see what effect it has on the precision of the models. The first thing we notice from the results in table 6.5 is that the aggregated difference is much lower for this example close to ATM than it was for the previous example far from ATM. In fact, it would be difficult to spot which data points are the ones being tested as the results fit very nicely in with the remaining data points. It also seems that the models fit equally well in proportion to their own data sets.

Shifted SABR-model										
Basis points	400	200	100	50	25	ATM	**-25**	-50	-100	-200
Market prices	535.7	1,822.2	3,402.3	4,612.5	5,349.9	6,184.5	**7,121.4**	8,163.9	10,567.9	16,536.1
Model prices	540.5	1,814.7	3,392.4	4,606.4	5,347.0	6,185.5	**7,126.4**	8,172.7	10,580.7	16,532.3
Difference	4.8	-7.6	-9.9	-6.1	-2.9	1.0	**5.0**	8.8	12.8	-3.7
Aggregated diff.	271.9	436.1	570.5	351.8	163.9	59.6	**294.2**	507.9	731.2	214.5
Normal SABR-model										
Basis points	400	200	100	50	25	ATM	**-25**	-50	-100	-200
Market prices	535.7	1822.2	3402.3	4612.5	5349.9	6184.5	**7121.4**	8163.9	10567.9	16536.1
Model prices	529.4	1840.8	3419.2	4619.4	5349.9	6177.1	**7107.1**	8144.1	10545.3	16547.3
Difference	-6.3	18.6	17.0	6.9	0.0	-7.4	**-14.3**	-19.8	-22.7	11.2
Aggregated diff.	321.8	963.7	885.0	357.8	13.3	391.2	**754.4**	1039.7	1175.1	594.0

Table 6.5: Data for Out-Of-Sample test for 10Y10Y EUR swaption -25 bps on 03/12/2018 (Source: Bloomberg and own calculations)

Looking at the results from these tests, we can now conclude that both models manage to compute market data accurately. This is both for in-sample and out-of-sample test. The Shifted SABR-model has a slightly more accurate fit for the in-sample test with a lower difference to the market prices, both in the daily example and with aggregated data.

For out-of-sample, we also get good results especially close to ATM. Even though we get the highest differences in the example far away from the money, we still argue that both models can be used to price options where we do not have market data available. Before choosing a model to do further work with, we will do a final comparison.

6.3 Summing up the models

We have now completed a series of tests for the two versions of the SABR-model and now the time has come for us to choose which one we will continue to work with. None of the models are perfect and both show advantages and drawbacks. In this section we will make the final comparison and discuss which model shows the most promise and thus, which one we will use for section 7 on Risk Management.

6.3.1 Shifted SABR-model

This version of the SABR-model is simple in its adaptation and manages to model negative rates quite well. The lognormal probability distribution ensures realistic values because it removes the possibility of large negative values.

When testing the robustness of the parameters, we found that α and ν were stable, which is positive, but on the other hand ρ was very unstable and had a significantly higher standard deviation than any of the other parameters in both models. The Shifted SABR-model performed very well in the in-sample test as it was able to model prices with high accuracy. It also did well for the out-of-sample test close to ATM. The worst result was for the example far away from ATM, where it was easy to spot which data point was taken out of the sample. However, for a notional of that size, the result is still a success.

As mentioned in section 4.4, one of the drawbacks of the model is the need to select the shift s in advance since we need to select a shift big enough that the model does not need reshifting later on. We have tested the size of the shift and it does not make a big impact on the prices, so one can choose the shift quite freely. The results from a large shift are visible in the appendix in table 10.2. This is, however, something that needs to be addressed when entering into a contract using the Shifted SABR-model. The counterparties need to agree on the shift, as it has a big impact on the volatility when quoting in Black volatility as is custom in the Shifted SABR-model.

The relative quotation makes Black volatility difficult to interpret, because we need to read results with regards to the level of the volatility. When we work with interest rates close to zero, a small change in the rates has a much larger effect on volatility than if we were working with high levels of interest rates.

6.3.2 Normal SABR-model

The Normal SABR-model models both negative interest rates and rates close to zero without difficulty. It models after the normal distribution, which is drawback of the model because this makes it possible for rates to become infinitely negative.

The parameters of the Normal SABR-model were in general stable, where especially ν performed well. When we looked at the in-sample test, the model did well and managed to get close to the market data, which is very positive. It did not perform as well as the Shifted SABR-model for the aggregated differences, but considering the scale of the numbers, we would also argue that this test is a success. For the out-of-sample test the Normal SABR-model performed better for the example close to ATM than it did for the example far away from ATM – i.e. the same pattern as the Shifted SABR-model.

We see the quotation of Normal volatility being an advantage, since this type is easier to interpret as it does not change depending on the size of the volatility. We can thus interpret market expectations directly from the volatility smile quoted in Normal volatility. The curvature of the smile with Normal volatility also lines up nicely with the theory on the topic, being another advantage of using this type of volatility.

To sum up, both versions of the SABR-model manage to model negative interest rates with high precision but neither of them are perfect. Since we find it more convenient to not having to choose a shift and logical to interpret results quoted in Normal volatility, we will continue with the Normal SABR-model for section 7 on Risk Management.

7 Risk Management

This section will cover the risk management of swaptions. We will first show the sensitivity of the SABR parameters. We will then cover the classic Greek risk measures and show how to use the risk measures to reduce the risk of a long position in a swaption. We will do this by buying a swaption in the beginning of 2018, and then using a discrete delta hedge to show how to remove the delta risk of the swaption. We will see that using the SABR-model when computing risk measures is critical to identifying the proper risks of a position and in extension, the efficiency of a hedge.

7.1 Sensitivity of SABR parameters

This section is dedicated to examine the risk of an error when estimating the SABR parameters. We will test how much the price of a swaption changes when we change each of the SABR parameters individually. We have tested the sensitivity for a fixed expiry while varying the tenor and strike of the swaption.

To compute the sensitivity, we first calibrated the Normal SABR parameters, using the approach in section 4.3. Then we compute the swaption price using the SABR parameters to compute the implied Normal volatility. Next step is to increase the SABR parameter with 1% and recalculate the price. Mathematically we define the sensitivity, with respect to ν, by the equation below

$$\Delta^{\nu} = P_1^{Swaption}(F, K, t, T, \sigma(\alpha, \nu \cdot \delta, \rho)) - P_2^{Swaption}(F, K, t, T, \sigma(\alpha, \nu, \rho)) \tag{7.1}$$

Since α and the Greek vega describe the same movements, we will not explore α further in this section as its sensitivity will be covered in section 7.2. Instead we will analyse ν and ρ.

Sensitivity for ν

The first parameter we have tested the sensitivity for is ν, where we have increased the parameter in the Normal SABR-model, while maintaining α and ρ at their computed levels. We used the bumped ν to calculate new prices for the swaption and calculated the differences in the prices for the bumped and unbumped contract. The results are visible in figure 7.1.

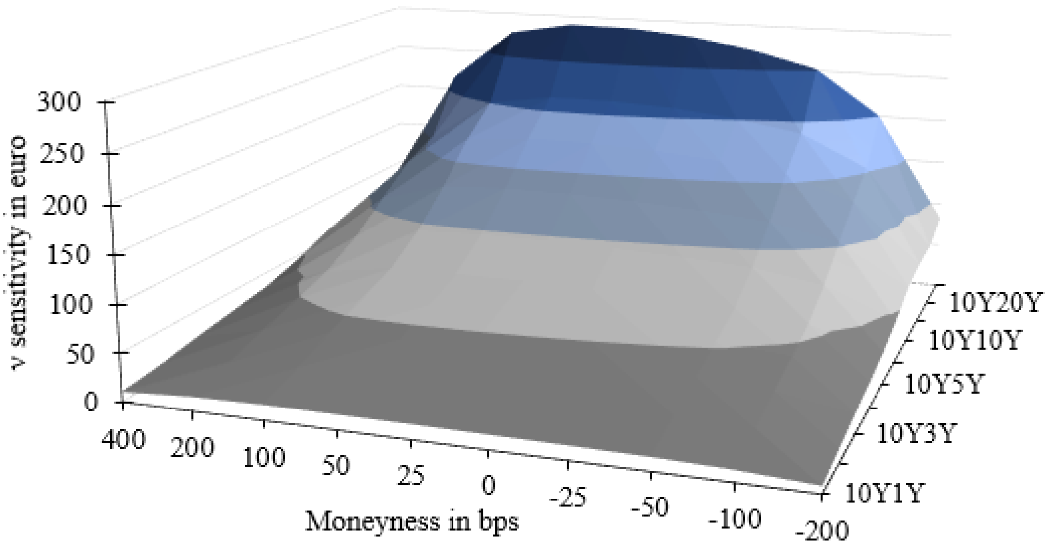


Figure 7.1: Change in the price of 10YXY swaption by bumping ν in the underlying Normal SABR-model by one percent

Here we see that ν is strictly positive, meaning that if we estimate ν higher than it should be, the price of the swaption will be higher. This fits well with our exploration of the parameters in section 4.2 where we saw that a higher ν caused an increase in the volatility.

Since α and ν have some of the same effects, we cannot discuss ν without comparing it to α. In the appendix, α is illustrated in figure 10.1, but the effect of α for a 10Y10Y swaption is also visible in figure 7.6 for Greek vega. Turning our attention back to ν, we see that ν has a bigger influence on strikes away from ATM and also a bigger influence than α does. This is because α causes a parallel change in the volatility smile, while ν has an effect on the curvature of the smile. Overall, ν has a bigger effect in nominal terms than α.

Sensitivity for ρ

Now we look at ρ and what happens to the price if it is estimated higher. To help us with this, we remember the formula for the Normal SABR-model in equation 4.18:

$$\sigma_N = \alpha \frac{z}{x(z)} \cdot \left\{ 1 + \frac{2 - 3\rho^2}{24} \cdot \nu^2 \cdot t_{ex} \right\} \tag{7.2}$$

where

$$z = \frac{\nu}{\alpha}(f - K) \text{ and } x(z) = \log\left(\frac{\sqrt{1 - 2\rho z + z^2} - \rho + z}{1 - \rho} \right)$$

We look at the effects of ρ on the volatility in equation (7.2) above. If we start by looking at the curly brackets, we have ρ in the numerator and if ρ increases, the numerator becomes smaller and the whole fraction becomes smaller as well, meaning that this will cause the volatility to decrease. Looking at the fraction in the second factor, we see that ρ is a part of $x(z)$ in the denominator and we turn to the expression for $x(z)$. The fraction in $x(z)$ is a bit tricky, as ρ is in both the numerator and the denominator, but we see that the effect of the numerator is larger, making the whole expression larger when ρ increases. Since $x(z)$ is in the denominator in our expression for the volatility, it will in the end have a decreasing effect on the volatility. Therefore we see that if ρ increases, the volatility will decrease causing the price to decrease. This effect is visible in figure 7.2 below.

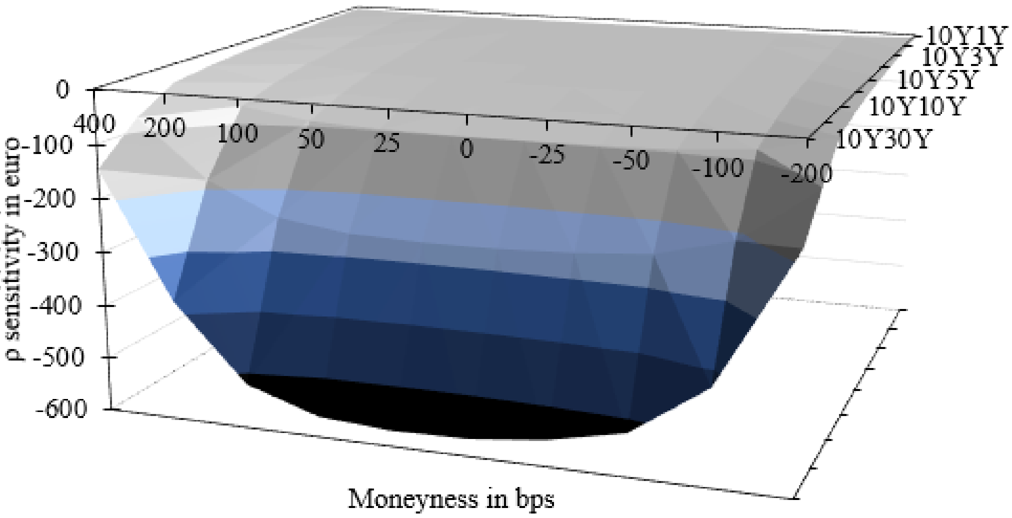


Figure 7.2: Change in the price of 10YXY swaption by bumping ρ in the underlying Normal SABR-model by one percent

The figure shows strictly negative values for ρ as expected from the analysis of the equation. The parameter has a larger effect on longer maturities as well as deep ITM and deep OTM options. So if we estimate ρ higher than it should be, the price of the option will be undervalued.

7.2 Greeks

In this section we will take a closer look at the risk measures in the SABR set-up for differering volatility, but also pay a brief visit to the classic set-up with constant volatility. The Greeks are the similar for both set-ups and we will briefly comment on the classic set-up before beginning our calculations in the SABR set-up, except for delta, where we will calculate both the classic delta and SABR delta. We will both cover formulas and compute examples of the relevant Greeks. For this section we will be using theory from (Linderstøm, 2013) and (Hagan et al., 2002).

Classic delta

The price sensitivity of an option for a movement in the underlying asset f is called delta (Δ). Delta is calculated by taking the first derivative of the option price with respect to the underlying asset. For swaption we differentiate equation (2.32) with respect to the par swap rate and obtain:

$$\frac{\partial PV_t(\cdot)}{\partial R(\cdot)} = A(t, T_S, T_E) \cdot \Phi(d) \tag{7.3}$$

We notice that the delta of our swaption is very similar to the delta of the underlying swap. The difference is the probability distribution chosen in the option pricing method. The delta risk measure can be used as information regarding the risk of a position, but can also be used to remove risk of changes in the underlying asset. This is called delta hedging, which is widely used in the industry. In section 7.3.3 we will apply delta hedging and show how it reduces risk.

When computing delta for a swaption, the common way to do it, is by computing the *Dollar Value of 1 Basis Points* (DV01). DV01 is an alternative to the differentiation in equation (7.3). The approach to this method is to *bump* each knot point in the zero and discount curve and compute the difference to the original price. Mathematically we can write the DV01 by

$$\text{DV01} = \frac{1}{10,000}\frac{\partial V(P)}{\partial P} \approx \frac{1}{10,000}\frac{V(P+\epsilon) - V(P)}{\epsilon} \tag{7.4}$$

where $V(P)$ is the value of the swaption depending on P which is the calibrated model quotes and ϵ is the value that we bump the curve with. Doing this calculation for all model quotes gives us a set of values for delta illustrated in figure 7.3.

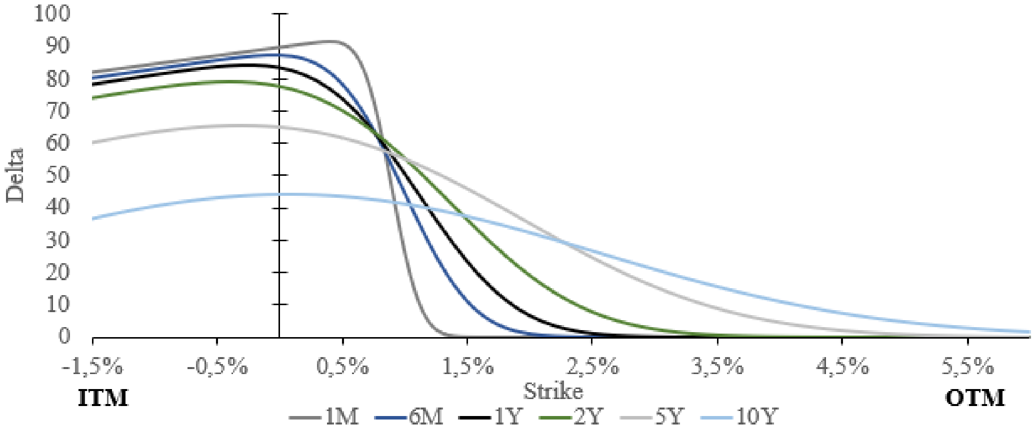


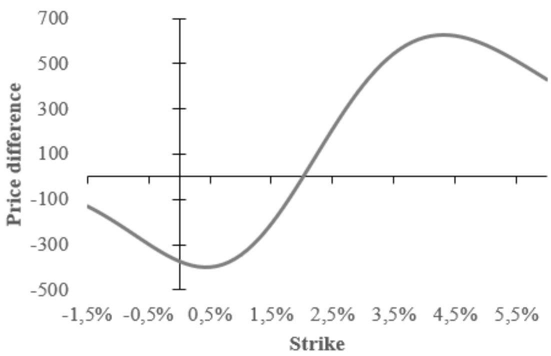
Figure 7.3: Delta for different expiries and a maturity of 10Y payer swaption with par swap rates depending on the expiry and a notional of 100,000 Euro

We see here that delta is positive for a payer swaption as an increase in the underlying rate will cause an increase in the expected payoff. Since we are bumping both the zero and discount curve, there are two effects accompanying an increase in the underlying asset. The first effect is the increase in the zero curve, which will give an increase in the forward rate and thus the expected payoff. The second effect is caused by the increase in the discount curve, meaning a harder discounting which will lower the expected payoff. The second effect is visible deep ITM as there is a decrease in delta, showing us that the discounting effect is dominant here. In general, when we are deep ITM, delta is close to the actual value of the increase in the underlying asset and will thus have a large effect on the swaption price since we are already sure we will be exercising the swaption. If we move towards ATM, we see that delta is the most sensitive here because an movement in the underlying asset could make the whole difference of whether the option is exercised or not. When we move to the right-hand side of ATM, we see that an increase in the underlying asset will have a smaller and smaller effect on the price and when we move deep OTM, delta is close to zero as we are so far away from ATM that we will not exercise the option.
Note that our par swap rates (ATM) here are different for each strike and will be lower for shorter expiries because our zero curve is upwards sloping.

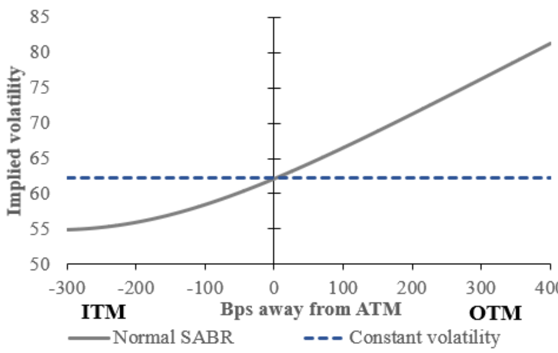
With longer time to expiry, the effect of delta flattens as there is more uncertainty in the future and there is still a lot of time for the underlying asset to move before it is time to exercise. A movement here is therefore not significant since the underlying asset could simply move back because of the long time to maturity.

SABR delta

Calculating the SABR delta is similar to calculating the Black delta except for one crucial step; changing volatility. In Black delta we used the ATM volatility for all strikes, but when calculating the SABR delta we need to have differing volatilities depending on the strikes. We use the SABR parameters computed in section 6.1.3 and the changing strikes to calculate the different Normal volatilities, then we calculate the original prices for the xY10Y swaptions. Next up, we need to bump the zero and discount curve by 1 basis point, calculate the bumped Normal volatilities and the new bumped prices as we did for the classic delta before. The difference between the original price and the bumped price is the SABR delta. The shape of the curves are very similar to the classic delta, therefore we will not spend time on a similar analysis. The figure for SABR delta can be seen in figure 10.2 in the appendix. Instead, we will look at the difference between the unbumped prices with constant volatility and the SABR delta, which can be seen in figure 7.4.



(a) Difference for bumped 10Y10Y EUR swaptions



(b) Normal volatility for SABR and constant

Figure 7.4: (Left) Difference in swaption price when using the SABR-model and constant volatility. (Right) Implied volatility curve

Figure 7.4 shows the effect of having non-constant volatility. We know figure 7.4b from section 6.1.3 and have added the level of the constant volatility to the graph. Figure 7.4a shows the difference in price between the original prices from the earlier section on classic delta and the price for the bumped 10Y10Y EUR swaption. Here we see the errors caused by constant volatility, where deep OTM options are priced too low and should be priced higher, while deep ITM options are priced too high. Since ATM are the same for the two pricing methods, the difference here is zero. We see an upward slope of the curve deep ITM in figure 7.4a because of the flattening of the curve in figure 7.4b.

Gamma

Gamma (Γ) is the second derivative of the swaption price with respect to the par swap rate and thus the sensitivity towards movements in delta. In practice, we get the following expression:

$$\text{Gamma DV01}^2 = \frac{1}{10,000^2}\frac{\partial^2 V(P)}{\partial P^2} \approx \frac{1}{10,000^2}\frac{V(P+\epsilon)+V(P-\epsilon)-2\cdot V(P)}{\epsilon^2} \tag{7.5}$$

In practice we bump the zero and discount curve both up and down with 1 basis point and use the new prices to calculate gamma for different strikes. The results are depicted in figure 7.5 below.

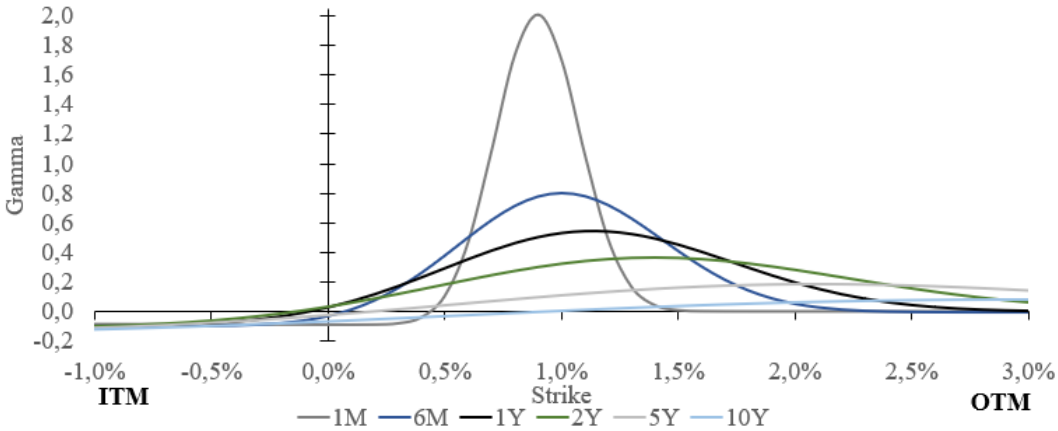


Figure 7.5: Gamma for xY10Y payer swaption with par swap rate depending on the expiry and a notional of 100,000

The graph for gamma is generally positive as seen in figure 7.5. The graph is interpreted while keeping the graph for delta in mind as gamma shows the movements of delta. If we look at close to ATM, we see that the shorter expiries have a large gamma here. This is due the large movement in delta as an increase in the underlying asset will make a big difference in whether we choose to exercise or not. We see that when we are deep ITM, we have negative values for gamma. As mentioned earlier, this is due to the dominating effect of the discount curve where the payoff will be discounted harder. The curves for gamma flatten as time to expiry increases as the uncertainty of gamma increases as well. Note that the classic gamma as described here and SABR gamma are very similar, which is why we have chosen not to do a section on both.

Vega

Vega is the sensitivity of a change in the implied volatility. In this section, we will not spend too much time on vega in the classic set-up, but briefly describe the method and then focus on vega in the SABR set-up. For the classic set-up with constant implied volatility, the way to calculate vega is by bumping the implied volatility by 1% for Black volatility and 1 basis point for Normal volatility. Then we calculate the bumped prices and compute vega by calculating the difference in the prices.

For the SABR set-up with non-constant implied volatility, we need to have a different approach due to the fact that the implied volatility is now a function of α, ρ and ν instead of a fixed implied volatility. Since the Black and Normal vega are computed by performing a parallel shift of the implied volatility by one unit (e.g. one percent or one basis point), we will mimic this approach by bumping our SABR parameters in a fashion that results in a parallel shift of the implied volatility. When recalling the effect of the different parameters to the volatility smile in section 4.3, it is apparent that we should bump the SABR parameter α to perform a parallel shift of the volatility curve. We bump α up by 1% for the expiries we have available (2-5Y, 7Y, 10Y and 15Y) and rerun the calculations to find the new bumped volatilities and prices (Linderstøm, 2013). The values for vega are shown in figure 7.6.

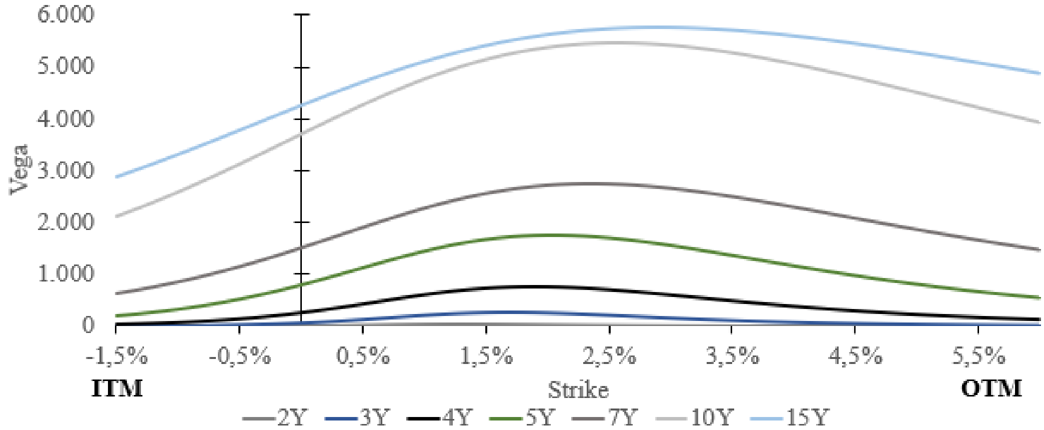


Figure 7.6: Vega for xY10Y payer swaption with par swap rate depending on expiry and a notional of 100,000

The first thing to notice is that vega is positive, which is due to the limited downside of a swaption. Vega is increasing for longer expiries due to the same effect, as we here have longer time to benefit from an increase in the volatility. If we look at around ATM, we see that vega is at its highest here for the individual curves. An increase in the volatility has the largest effect on the price here, because it could make the whole difference on whether we choose to exercise or not. The graphs all have a slight tilt, where deep ITM is lower than deep OTM. An increase in volatility is more needed deep OTM, as we need the volatility to move the underlying asset so much that it moves into the money. We also appreciate an increase in volatility for deep ITM as this could possibly move our contract even deeper into the money, but we still have something to loose when we are ITM. On the contrary, when we are deep OTM we have nothing to loose and everything to win, causing the higher value in vega in this area.

Theta

Theta (Θ) describes the sensitivity in the price depending on time to maturity. The calculation is based on what happens to the price when there is one business day less to expiry. In general, theta is negative due to the nature of the option; a limited downside and unlimited upside, so less time to use this dynamic is not preferable. For short expiries, theta can become positive due to an effect described earlier – the discounting effect. This effect is opposite for theta and short expiries as shorter time to expiry will cause lesser discounting of the expected payoff. We have calculated theta and the results are shown in figure 7.7

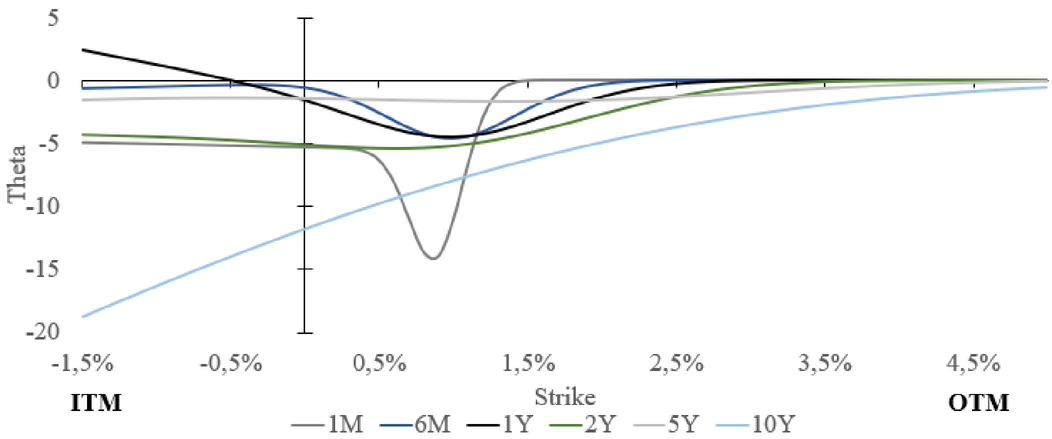


Figure 7.7: Theta for xY10Y payer swaption with par swap rate depending on expiry and a notional of 100,0000

If we start by looking at the right-most part of the graph in figure 7.7, where we are deep OTM, we see that theta is close to zero, especially for short expiries. A day less to expiry does not matter much here, because we know that will not be exercising anyway because we are so deep OTM. Moving towards ATM, theta becomes increasingly negative because with one day less, the underlying asset will have less time to move into the money and thus the value of the contract decreases. This effect is more visible for short expiries and flattens for longer expiries because of uncertainty in the long run. For the 1M expiry, theta is very negative for ATM because one day less can be determining whether we exercise or not. For long expiries deep ITM, theta is still negative because we wish for more time to benefit from the limited downside and thus have as much time as possible to move even deeper into the money. Looking at theta for short expiries deep ITM, we notice the positive value for 1Y which is due to the lessened discounting effect described earlier. This effect would normally also show for the shorter expiries of 1M and 6M, but due to negative interest rates this is not the case for this example.

We have now seen the effects of what happens if the model estimates the parameters incorrectly. The effects are bigger for long maturities and the further away from ATM we are. A further exploration of this will not be performed in this thesis, but we see the possibility for further analysis of this subject along with the parameter stability test conducted in section 6.2.1.

The Greeks gave us knowledge on the various risks in option pricing and lay ground for the next section on investment strategies and hedging. We especially found the SABR delta interesting and will continue the work with this greek in the next section.

7.3 Empirical example on risk management

Having covered different risk measures of the Normal SABR-model, we will now use this knowledge in practice with an empirical example. In the example, we will go through how to take a position with a financial contract and how to price the swaption and swap associated with it. To correctly price the financial products, we will use the Normal SABR-model to calculate the implied volatilities and finally use the Greeks to analyse the Profit & Loss (P&L) of the investment strategies along with a hedge to improve our investment.

7.3.1 Buy-and-hold strategy

In our example we will take a long position in a 1Y10Y payer swaption, since we would like to work with a product that expires within our data frame and is time relevant. The value of the swaption will depend on the forward rates F_t, the discount rates P_t, the expiry date T^E, the strike K and finally the implied Normal volatility σ_t^N. Note that the implied Normal volatilities will be computed using the Normal SABR-model and can change due to two reasons. One is that the general level of implied volatility of the 1Y10Y swaption can change. The other being that the volatility changes because the underlying swap moves in and out of the money and thus moves along the volatility smile. The rest of the variables will change with the market and thus be different for each passing day. The expiry date and the strike are constant, along with the maturity of the underlying asset due to the contract we have taken a long position in. Note that if it were a different financial contract, these variables would have different values. The price of the swaption, on the day it is purchased, is:

$$\mathrm{PS}_t^{\mathrm{1Y10Y}} = S_t^{\mathrm{1Y10Y}}(F_t, P_t, T^E, K, \sigma_t^N) \tag{7.6}$$

When one day has passed the price of the swaption will be:

$$\mathrm{PS}_{t+1}^{\mathrm{1Y10Y}} = S_{t+1}^{\mathrm{1Y10Y}}(F_{t+1}, P_{t+1}, T^E, K, \sigma_{t+1}^N) \tag{7.7}$$

The buy-and-hold strategy is simply to buy the 1Y10Y swaption and hold it until expiry. We compute the daily returns for each day in 2018 from 02/01/2018 where we take our long position:

$$\mathrm{P\&L}_t = \frac{\mathrm{PS}_t^{1Y10Y}}{\mathrm{PS}_{t-1}^{1Y10Y}} - 1 \tag{7.8}$$

The daily returns are accumulated to see what value our swaption has at expiry. The accumulated returns are visible in figure 7.8 below. In figure 7.8 below we have illustrated the accumulated return for the purchased swaption. Note that since we accumulate the returns, we assume that we are reinvesting potential earnings.

Figure 7.8: Accumulated return for a 1Y10Y EUR swaption starting on 02/01/2018 with a 100,000 euro notional, the last value is labelled

As the variables (forward rates, discount rates, time to expiry and implied volatility) move, so does the value of our swaption. Towards the end of the expiry, the swaption is OTM and has a value of zero, so we would not exercise our option to purchase the underlying swap. Before we look at P&L for the buy-and-hold position, we will investigate the classic risk factors. We have calculated the Greeks and will now analyse them using the graph in figure 7.9 below.

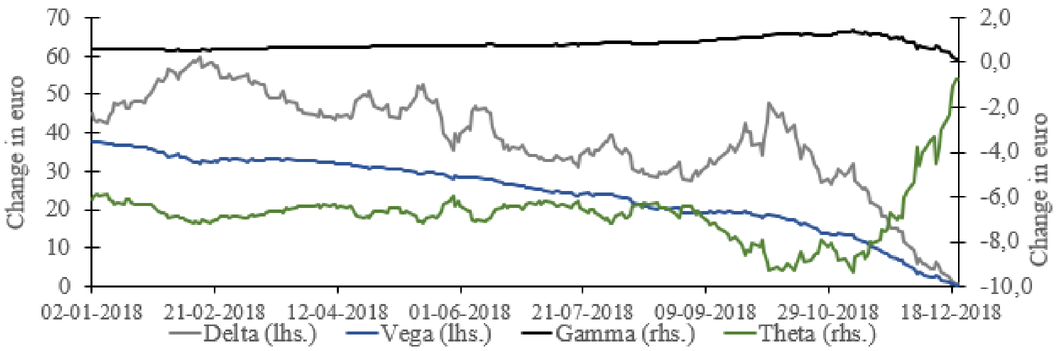


Figure 7.9: Greeks for a 1Y10Y EUR swaption starting on 02/01/2018 with a 100,000 euro notional. Delta and vega are on the left axis, gamma and theta on the right (own creation)

Starting with delta, we see that it is strictly positive and moving quite a lot. With a positive delta we benefit from an increase in the underlying swap and therefore we wish for delta to increase. Delta is mostly moving in values between 30 and 50, but then decreasing at the end. Towards expiry, the swaption is very deep OTM, meaning that delta is close to zero. Being close to zero when the swaption is deep OTM is expected delta behaviour and can also be seen in figure 7.3.

Moving on to vega, we also have a positive effect as we benefit from an increase in volatility. As mentioned earlier, this increase can come from two effects; changes in the market volatility or as the underlying swap moves in and out of the money along the volatility smile. In general, vega decreases when the underlying swap moves away from the ATM and as we are deep OTM close to expiry, vega is close to zero.

For the next Greek, we look at the secondary axis as these effects are smaller than the previous two Greeks. Gamma is the most steady of the Greeks, moving close to zero and thus suspected to be a low risk factor. We will therefore not spend much more time on gamma. One final comment on gamma is the slight increase as we move close to expiry which is common for gamma.

Last but not least, we have theta, which is our only negative greek. Theta has a negative carry[4] and consistently causes a decrease in the value of the swaption. Towards the expiry theta moves towards zero as the underlying swap is so deep OTM that theta has little to no effect on the value of the swaption. Looking back at figure 7.7, this is a general effect for theta for various expiries.

[4]Carry is the cost or benefit from holding an asset when nothing changes and a day goes by. Theta has a negative effect on the value of the swaption even though all other variables stay the same.

We suspect that delta is one of the biggest risk factors, but before we begin delta hedging we will do a decomposition of the P&L for the buy-and-hold strategy to see what the different risk factors are and how they impact the swaption.

7.3.2 Decomposing P&L for buy-and-hold

In this section we will take a closer look at why we experienced the loss on our buy-and-hold strategy. Note that it could just as well have been a gain, but for the scenario transpiring from the data for 2018, the swaption came out with a loss. We will look at the impacts caused by the different variables to explain why the graph in figure 7.8 looks the way it does and how we might pursue a different strategy to minimize the loss or even turn it into a gain. First we explain the different impacts of the variables and how to calculate dem.

To calculate the effect of the interest rates moving, we calculate the price of the swaption for time t and $t+1$, but a bit different than usual for $t+1$. We use all the same inputs as for t except for the zero curve and discount curve, where we use the ones for $t+1$. We then calculate the difference between the two prices for the swaption and are left with the impact of the interest rates. In practice the calculation is:

$$\text{Impact of interest rates} = \mathrm{P}_t^{\text{Payer}}(F_{t+1}, P_{t+1}) - \mathrm{P}_t^{\text{Payer}}(F_t, P_t) \tag{7.9}$$

For the impact of time, we simply use theta as we have calculated it in section 7.2. There is one small difference when it comes to calculating from Friday to Monday as we only use business days. Here we need to multiply theta by three to make sure we have accounted for the effect of all the days in a week:

$$\text{Impact of time} = \Theta_t \tag{7.10}$$

Lastly we have the impact of volatility which is calculated similar to the impact of the interest rates. We compute the price of the swaption for time t and $t+1$ using the inputs for t in both cases, but for the latter, we use the volatility for $t+1$ and calculate the difference. This gives us the impact of the volatility on the swaption:

$$\text{Impact of volatility} = \mathrm{P}_t^{\text{Payer}}(\sigma_{t+1}) - \mathrm{P}_t^{\text{Payer}}(\sigma_t) \tag{7.11}$$

The impacts of the variables have been summed monthly as a weekly summation made it difficult to see what was happening in the graph. The results can be seen in figure 7.10.

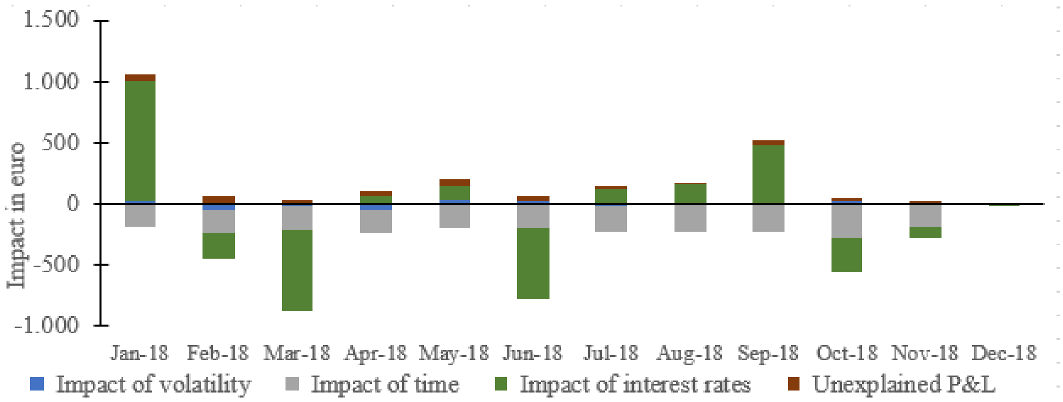


Figure 7.10: Impacts of variables for a 1Y10Y EUR swaption starting on 02/01/2018 with a 100,000 euro notional

We can see from figure 7.10 that our three different impacts explain the P&L well. The difference between the explained P&L and the realised P&L is called the unexplained P&L. The unexplained P&L is an effect of swaptions being non-linear products where second-order derivatives, like gamma, affect the price.

Starting with the impact of the volatility, we see that it is not very significant. In fact, the volatility is difficult to see in the graph, but nonetheless it is there, it just does not move very much. From the start of the swaption to expiry, the volatility moves from a value of 62.8 to 63.1 quoted in Normal volatility.

Next up, we have the impact of time which has a steady negative impact on the swaption value. This is a natural effect of having less time to utilise the optionality in the swaption and is known as theta decay[5].

As expected there is a big impact from the movements in interest rates. This impact is big on a month-to-month basis, but if we look at the total impact, the result is much smaller. If we take the absolute values of the impact of interest rates, we get a total movement of 16,581 euros, while the total impact of interest rates in fact is 75 euros. Even though the summed impact of interest rate is not great, due to the monthly impacts offsetting each other, we still recognise impact of interest rates as a major risk factor. In this case, the downwards movements of interest rates causes the swaption to end OTM, and is therefore worthless. There is a decrease of the interest rates when constructing the zero curves on the start date and expiry date of the swaption. This is visible in the appendix in figure 10.3.

[5]Theta decay or time decay is the rate of change in the option value as it comes closer to expiry

For this reason, we perform a delta hedge to reduce the exposure towards movements in the underlying interest rates.

7.3.3 Delta hedging

In this section we will take the view of a European investment bank. A client wishes to take a *short* position in a specific swaption, a 1Y10Y payer swaption, and we then take the opposite side of the deal; a *long* position in 1Y10Y the payer swaption. Now we are exposed to risk, having bought this swaption and seek to minimise that risk. The most straight-forward way to do this would be to take a short position the exact same swaption from another market participant, but this would not be the most profitable way to handle the risk. First of all, we would be paying extra due to the bid-ask spread in the swaption market and thus our potential profit would decrease – our profit currently being the bid-ask spread. Second, since swaptions are over-the-counter products (OTC), we might not be able to short the exact same swaption, let alone at a reasonable price. Instead we will use a combination of swaps and options to hedge our position. Swaps have a high delta and are very liquid, which means that the bid-ask spread will be smaller. Another reason to use swaps is that they are approximately a delta-one product[6]. Options can be used to hedge the non-linear risk factors such as gamma, vega and theta. One of the most ideal products to hedge these risk factors is an ATM straddle[7] since ATM straddles have high gamma, vega and theta and a low delta.

If we wish to hedge both linear and non-linear exposures, it is important that we hedge the non-linear exposures first. If we hedge delta-risk first and then our non-linear risks, then we will add additional delta risk and thus off setting our delta hedge. Instead, the proper way would be to first hedge the non-linear risk factors and then use a swap to hedge the delta risk. Using swaps to hedge the delta risk will not offset the non-linear risk factors because a swap has no gamma, vega or theta. In this thesis, we will limit ourselves to delta hedging and thus use swaps to do so.

As we seek to hedge a 1Y10Y payer swaption, we will hedge with a 1Y10Y forward starting payer swap. We could have used a different swap, but choosing one with the same expiry and maturity has the same delta ladder[8] as our swaption and this is in our favour as this

[6] A delta-one product is a product with delta equal to one. Swaps are only approximately equal to one due to discounting

[7] A straddle is an option on both a payer and receiver swaption with the same strike

[8] A delta ladder is the change in the option price when the underlying asset changes by 1 bps. The delta is calculated for each tenor of the option, which make up the delta ladder and summed is DV01 (Linderstøm, 2013)

puts less strain on our computations. Using the same forward starting swap also means that we are hedged against changes in the steepness of the zero curve. If we did not match the delta ladder, then we would only be hedged against parallel shifts in the zero curve (Linderstøm, 2013).

From calculations in section 7.3.1, we know that DV01 for the swaption on the first day is EUR 46.63, which means that we need to short the amount of underlying swap that increases with EUR 46.63 when the interest rate increases by 1 basis point. The DV01 of the swap on the first day is 97.92 and now need to calculate how much of the swap we need to short in order to have a value of 46.63. The amount needed is called w:

$$w = \frac{\text{DV01}^{\text{Swaption}}}{\text{DV01}^{\text{Swap}}} = \frac{46.63}{97.92} = 0.476256 \tag{7.12}$$

In order to hedge our swaption, we need to short $0.476256 \cdot N$. However, taking a short position in a payer swap gives the same hedge as taking a long position in a receiver swap and we prefer the latter due to common market practice (Linderstøm, 2013). Now we can calculate the P&L of the delta hedge:

$$\text{P\&L}^{\Delta \text{ hedge}} = (S_{t+1} - S_t) \cdot w = (257.7 - 0) \cdot 0.476256 = 122.716 \tag{7.13}$$

Taking a long position in a receiver swap on this amount will give our hedge a value of EUR 122.716 and the impact of the interest rate for the same day is -120.27, meaning that our whole portfolio has a value of EUR 2.44. We complete these calculations for the whole period and end up with the complete hedge displayed in figure 7.11.

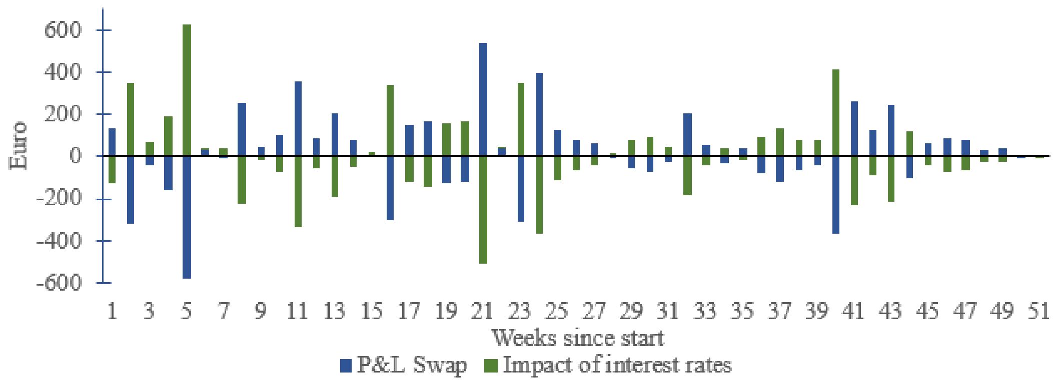


Figure 7.11: P&L for a delta hedged 1Y10Y EUR swaption starting on 02/01/2018 with a 100,000 euro notional

Figure 7.11 shows the quality of our delta hedge. The blue pins show the P&L of our delta hedge position i.e. our long position in a 1Y10Y receiver swap and the green pins show the impact of interest rates for our 1Y10Y payer swaption. Our hedge moves quite well with the impact of the interest rates, but is not perfect. This is due to swaptions being non-linear products. When the interest rates change, then delta will change and thus gamma as well, causing our delta hedge to not be perfect. This issue can be fixed with a delta-gamma hedge, but will not be covered in this thesis.

With our delta hedge we have achieved the effect we wished for; when the swaption looses money due to interest rate impacts, our hedge makes almost the same amount of money. The effect of the delta hedge can be seen in figure 7.12 where we have combined the P&L of the 1Y10Y payer swaption and 1Y10Y receiver swap.

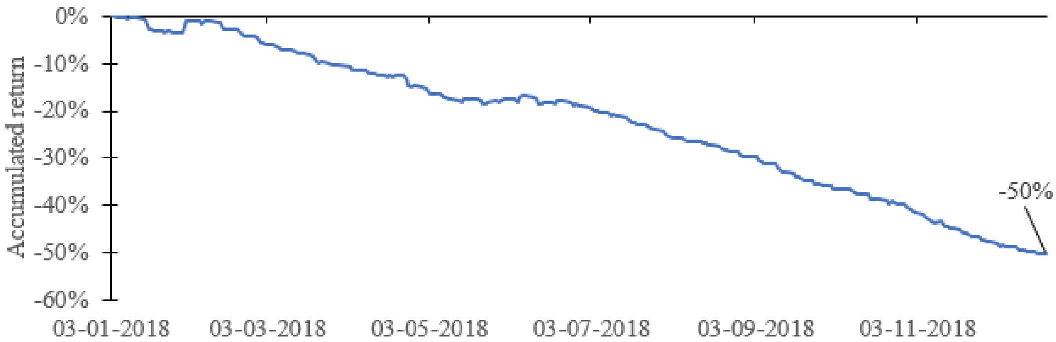


Figure 7.12: Accumulated return for a delta hedged 1Y10Y EUR swaption starting on 02/01/2018 with a 100,000 euro notional, the last value is labelled

We have not managed to turn the loss into profit for our long position in the 1Y10Y swaption, but we have cut our losses in half. Our delta hedge against the negative impact of interest rates has been a success, but we still loose money due to theta decay and the lack of volatility. We remember that buying an option is buying the opportunity to acquire the underlying asset, but if this opportunity does not turn out to be profitable, then we loose money.

Using all the theory from previous sections, we have successfully priced swaptions and swaps and shown how to delta hedge an investment in a realistic set-up with actual market data and strategies.

8 Conclusion

We have successfully adapted the classic SABR-model to price precisely and apply risk management in the current low and negative interest rate environment. The process started with the fact that implied volatility is not constant as first assumed in the original pricing models Black and Normal model. Implied volatility is shaped like a smile or a smirk depending on the market situation and for now it is a smile that is slightly more expressed for OTM options. The volatility smile is a product of the probability distribution of the underlying asset and a buying pressure on OTM options. In order to price options correctly, we need a model that gives us several implied volatilities depending on what the value of the strike is – enter the SABR-model. The classic SABR-model solves issue of constant volatility by providing a flexible and precise closed-form solution to calculate differing implied volatilities. The classic SABR-model worked for a period of time, but as the financial crisis came in 2008, the market changed and interest rates decreased ultimately hitting negative rates in 2014, even though this was unheard of until then. To solve the issue of negative rates, we chose to work with the Shifted SABR-model and the Normal SABR-model. An empirical comparison of the two models was performed. The result was that the models priced equally well both for an in-sample test and an out-of sample test. Even though the two models performed equally well, we chose to apply empirical risk management to the Normal SABR-model due to the advantages of working with Normal volatility. The empirical risk management analysis showed that the Normal SABR-model is subject to significant risk of errors when estimating the parameters. We therefore concluded that the parameters need to be updated at least weekly. When simulating a buy-and-hold position, we saw that a swaption position is heavily exposed toward movements in the underlying interest rates, that high volatility is significant for the success of a long position in a payer swaption and that theta decay reduces potential profits. Lastly we demonstrated how to reduce the delta risk of the 1Y10Y swaption position. This resulted in an almost delta neutral position that performed significantly better.

A major area in this thesis is volatility – both differing implied volatility leading to the volatility smile, but also the discussion of Black volatility versus Normal volatility. Through the comparison of Black and Normal volatility, we concluded that Black volatility results in quotation problems for low interest rates, which can result in very large implied volatilities. When looking at the historical volatility, we saw the decrease in volatility as an effect of the monetary policy of ECB and even though ECB announced the end of the Asset Purchasing Programme in December 2018, ECB still intends to keep interest rates steady and at a low level (European Central Bank, 2019).

We tested several different methods for estimating the SABR parameters, and found that a fixed beta and the minimisation approach for the remaining parameters, α, ρ and ν yielded the best results.

Both the Shifted SABR-model and Normal SABR-model performed well when testing in-sample and out-of-sample pricing, but the shape of their volatility smiles were different. While the Black volatility smile was curving upwards deep ITM, the Normal volatility smile was curving upwards deep OTM. This is due to the fact that Black volatility is quoted in relative volatility while Normal volatility is quoted in absolute volatility. Therefore, the implied Black volatility will increase for low strikes, even though the actual price decreases for low strikes. This constitutes a major problem for the Black volatility, because it makes it harder to interpret market conditions and prices.
Another issue arises when using Black volatility and the Shifted SABR-model. We argued that one must communicate the magnitude of the shift to align implied volatility quotes with the counterparty. Lastly, when using the Shifted SABR-model, one must make sure to shift far enough, so that an extra shift in the future will not be needed.

When applying risk management to swaptions, we concluded that it was paramount to factor in the volatility smile. This resulted in the difference between the classic delta and SABR delta. The SABR delta was used to perform the delta hedge to the buy-and-hold strategy, ultimately resulting in a successful hedge.

To sum up, the thesis demonstrated how to price and risk manage financial products in low and negative interest rate environments using extensions to the SABR-model and at the same time take non-constant volatility into account.

8.1 Further work

In this section we will briefly reflect on which topics and questions could be interesting to do further work with. While we limited ourselves to two extensions to the classic SABR-model, we could explore other extensions to find one that models implied volatilities even better than the ones in this thesis. This could be other extensions that can model negative interest rates e.g. the Free-Boundary SABR-model or the Mixed SABR-model.
Furthermore, additional interesting analyses could be performed if we acquired daily or interdaily swaption quotes and did an analysis of exactly how often the SABR-models need recalibrating in order to price accurately. Lastly, while we limited ourselves to delta hedging, another area to explore further could be delta-gamma hedging.

9 References

Andrade, Phillippe (2016). *The ECB's Assets Purchase Programme: An Early Assessement.* European Central Bank, 1956th edition.

Balland, P. and Tran, Q. (2013). SABR goes normal. *Risk*, 26(6):76–81.

Bartlett, B. (2006). Hedging under SABR model. *Wilmott magazine*, 4:2–4.

Bollen, N. P. and Whaley, R. E. (2004). Does Net Buying Pressure Affect the Shape of Implied Volatility Functions? *The Journal of Finance*, 59(2):711–753.

Brier, S. (2005). *Informationsvidenskabsteori.* Samfundslitteratur.

Duyvesteyn, J. and de Zwart, G. (2015). Riding the Swaption Curve. *Journal of Banking & Finance*, 59:57–75.

European Central Bank (2019). Monetary policy decisions. https://www.ecb.europa.eu/press/pr/date/2019/html/ecb.mp190410~3df2ed8a4c.en.html. Accessed: 10/05/2019.

Hagan, P. S., Kumar, D., Lesniewski, A. S., and Woodward, D. E. (2002). Managing Smile Risk. *Wilmott Magazine*, pages 84–108.

Hagan, P. S., Kumar, D., Lesniewski, A. S., and Woodward, D. E. (2014). Arbitrage-Free SABR. *Wilmott Magazine*, (69):60–75.

Hull, J. C. (2018). *Options, Futures, and Other Derivatives.* Pearson International Content, 9th edition.

Lawler, G. F. (2006). *Introduction to stochastic processes.* Chapman and Hall/CRC, 2nd edition.

Linderstøm, M. D. (2013). *Fixed Income Derivatives Lecture Notes.* Martin Dalskov Linderstøm.

McDonald, R. L. (2013). *Derivatives Markets.* Pearson, 3rd edition.

Ryan, B., Scapens, R. W., and Theobald, M. (2002). Research Method and Methodology in Finance and Accounting.

West, G. (2005). Calibration of the SABR model in illiquid markets. *Applied Mathematical Finance*, 12(4):371–385.