Dynamic Capital Structure with Debt Renegotiation and Mean-Reverting Earnings

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Executive Summary

We develop a model of optimal capital structure with debt renegotiation and mean-reversion in earnings. Comparative statics are presented for optimal leverage, coupon choice and the renegotiation threshold, as well as related capital structure metrics. We also conduct a cross-model comparison with a framework featuring a conventional geometric Brownian motion-based state variable to examine implications of varying assumptions about the evolution of earnings.

We show that a manager who maximises firm value selects a higher initial leverage and that the leverage choice correlates negatively with earnings. This contrasts with predictions by earlier models of capital structure. In addition, we predict lower bond yields and higher recovery rates of debt compared to the benchmark case. The size of deviations from the absolute priority rule is equivalent across the models, given proportional bankruptcy costs and distribution of bargaining power between agents.

Our analysis brings capital structure research closer to the practical discussion by coupling two realistic elements. We let the diffusion of earnings align better with how they observably evolve and allow for restructuring of the firm at a sufficiently low earnings level.

Keywords: *Dynamic capital structure, contingent claims valuation, debt renegotiation, optimal restructuring, mean-reverting earnings.*

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Chapter 1

Introduction

"He who loves practice without theory is like the sailor who boards ship without a rudder and compass and never knows where he may cast."

- Leonardo da Vinci (1452-1519)

The famous publications of Modigliani and Miller (1958, 1963) (MM) are widely credited for marking the inception of the academic debate on the interplay between capital structure and fundamental firm value. An arguably more seminal contribution of the papers, however, was the operationalisation of general equilibrium analysis - once considered among the most abstract concepts in economic theory. The publications effectively explicated how capital structure is linked to the assumptions of efficient markets and rational agents. This illustration of the conceptual applicability of equilibrium theory in the capital structure discussion marked the backbone for a wave of subsequent advancements within this subfield. Debreu (1991) has later coined the period *'the mathematisation of economic theory'*.

The stream of literature that followed MM's rationalisation of the capital structure decision focused on the 1963 paper's heterodox conclusion about the optimality of full leverage. To explain the empirical and theoretical dissonance that arose, economists responded by introducing various frictions to the capital structure decision. One prominent expanse of the theory are the so-called asymmetric information models. Many of these succeed in demonstrating how frictions should reduce the theoretically predicted leverage ratios, but offer little practical guidance for business managers. On the other hand, later symmetric information models attempt to mend the problem by introducing other non-agency related model enhancements. In this way, a rational business manager without skewed incentives could be offered normative guidance in the capital structure decision. In practice, most managers recognise that earnings are not constant and that time allows for the possibility to revisit current decisions later. To incorporate these realities, more recent research, such as Leland (1994), has introduced stochastic firm value fundamentals. Furthermore, Fischer et al. (1989a) and Goldstein, Ju and Leland (2001) allow for the possibility to dynamically increase the debt level as a response to the firm's improving earnings capacity.

Following the introduction of dynamic models, the most recent contributions of Christensen, Flor, Lando and Miltersen (2002, 2014) have attempted to incorporate renegotiation and restructurings at a lower boundary threshold as well. Such analysis carries some similarities with the class of models collectively referred to as strategic debt service models (see e.g. Anderson and Sundaresan (1996)). Particularly, it is a joint conclusion that it will be optimal for equity and debt holders alike to avoid firm bankruptcy under certain assumptions even if it implies deviations from the absolute priority rule (APR).

Common for nearly all capital structure models introduced in a continuous-time setting is that they are inspired by the hallmark work of Black and Scholes (1973) and Merton (1974) in their application of contingent claims pricing. This has also caused a persistent convention of applying a geometric Brownian motion (GBM) as the diffusion process of the underlying state variable with the exception of a few studies such as Raymar (1991) and Sarkar and Zapatero (2003). However, both theoretical and empirical evidence suggest that considering alternative stochastic processes with mean-reverting properties will be a more realistic assumption for the evolution of a firm's value fundamental. For example, Bhattacharya (1978) argues that mean-reversion in cash flows is a more economically sound assumption than the alternative of a random walk process. The justification is the tendency of project cash flows to revert to levels where companies are indifferent about making further investments. The mean-reversion of earnings is further confirmed in various empirical tests (see e.g. Fama and French (2000)).

In this thesis we aspire to contribute to the latest research within capital structure theory by incorporating a mean-reverting earnings process in a framework that allows for debt renegotiation at a lower boundary of the underlying state variable. We develop the model in stages and proceed as follows: The rest of this chapter outlines our research objective and discusses related work in the field. Chapter 2 reviews essential theory of stochastic processes and contingent claims valuation underlying any satisfactory model of dynamic optimisation. In Chapter 3 we redevelop two previous models of optimal capital structure and assess their numerical performance. Chapter 4 analyses the debt renegotiation game by examining central aspects of game theory as well as two frameworks of strategic debt service. In Chapter 5 we develop a GBM-based benchmark model with debt renegotiation and consider an extension with callable debt. Chapter 6 augments the state variable in the model with debt renegotiation to include mean-reversion and conducts extensive comparative statics and cross-model comparison with the benchmark model. Chapter 7 concludes.

1.1 Research Objective

Our objective with the research undertaken in this paper is two-fold: Firstly, we set up a model to determine a firm's optimal capital structure when allowing for debt renegotiation under the assumption that the underlying state variable follows a geometric Ornstein-Uhlenbeck (GOU) process. Secondly, we conduct extensive comparative analysis between our model and a benchmark model with a state variable following a traditional GBM-based diffusion process.

In pursuing this research endeavour, we emphasise the findings of Sarkar and Zapatero (2003) on the empirical and theoretical attractiveness of considering a GOU process for the evolution of earnings. Accordingly, the model of capital structure is developed to examine the effect of imposing this assumption in the debt renegotiation setting considered by Christensen et al. (2014). A thorough comparative statics analysis is carried out in order to compare the implications for optimal capital structure in our GOU-based model with the ones obtained in a classical GBM-based framework. We restrict our model to consider possible restructuring at a lower boundary of the state variable.

As this research is largely motivated by the aim of adding another layer of realism to the ongoing academic discussion on capital structure, we analyse the findings with an emphasis on economic intuition and the ramifications for practical capital structure optimisation. Our model will however not be subjected to empirical tests, but instead be calibrated with empirically justified values for key parameters. Furthermore, the range of parameters considered will follow the historical discussion of dynamic capital structure literature, implicitly imposing a *ceteris paribus* assumption on other factors outside the model that might explain the optimal leverage choice.

1.1.1 Research Questions

In addressing the proposed research, we aim to provide answers to the following central questions:

RQ1 *How has the research on dynamic capital structure and debt renegotiation evolved, and what are the central requirements for the development of a satisfactory model of dynamic capital structure and debt renegotiation?*

RQ2 *How is the optimal capital structure decision affected by allowing for debt renegotiation vis-à-vis classical models of financing under uncertainty?*

RQ3 *How can the model of dynamic capital structure be modified to feature debt renegotiation and a GOU-based diffusion process of the driving state variable?*

RQ4 *What are the implications for optimal capital structure and related key metrics in the model with debt renegotiation of letting the state variable follow a GOU process?*

1.2 Literature Review

In their second publication on capital structure choice, Modigliani and Miller (1963) conjecture that firm value is a monotonically increasing function of the debt level due to the tax deductibility of interest payments. For many of the 50 years since, much of the academic effort devoted to capital structure choice has gone into challenging this theory. Researchers have approached the issue by attempting to model the costs that empirical observation suggests firms trade off against the corporate tax benefit. The most significant factors include agency costs, asymmetric information as well as direct and indirect costs of liquidation.

Jensen and Meckling (1976) pioneered the theory of agency by recognising that the determination of capital structure ought to be based on ameliorating conflicts of interests between the firm's stakeholders, particularly between management and equity owners. Subsequently, research moved towards modelling the effect of asymmetric information between the firm's insiders and outsiders. The aim was to examine the implications for capital structure choice of having to convey private information to capital markets. Most notably, Ross (1977) contends that - facing a bankruptcy penalty - management's choice of the level of debt will serve as a credible signal of the firm's future earnings potential. Conversely, Leland and Pyle (1977) argue that the retained equity by a firm indicates a future profit increase sufficient to offset the diversification that the owners forgo by not investing their funds elsewhere. Finally, Myers and Majluf (1984) link capital structure to investment choice by positing that firms favour internal funds over external financing, and debt over equity, due to the negative signalling effect of turning to capital markets for investment financing. This proposition has later been coined the *'pecking order theory'*.

However appealing the asymmetric information models seem, they possess a general problem rooted in their normative limitations and lack of appeal as a practical tool for capital structure determination. Another major school of capital structure theory presents the capital structure choice as a simple problem of balancing the benefits of tax savings with the direct and indirect costs of financial distress (see e.g. Kraus and Litzenberger (1973)). This *trade-off theory* has fared better in the research field and serves as the foundation upon which subsequent advancements of capital structure modelling have been based.

An initial extension of the trade-off theory was to account for uncertainty of future asset values. This lead to the development of stochastic models featuring static capital structure choice. In a representative contribution, Leland (1994) derives a closed-form solution for the optimal capital structure when the future firm value is uncertain. The applied valuation technique in stochastic models originates from the theories of option pricing and pricing of corporate liabilities as put forward by Black and Scholes (1973) and Merton (1974). Graham (2000) extends Leland's simple setting to consider a sophisticated taxation regime with tax benefits lost and carried forward. The paper provides evidence that firms could increase leverage substantially before the effective corporate tax rates start to decrease.

The static models see the establishment of an optimal capital structure as a single deterministic decision, which can never be re-considered in the face of changing states of nature. Consequently, a shortcoming of the static capital structure models is that they predict high leverage ratios. Based on empirical examination, it is evident that firms have a policy of leverage that the static models fail to account for. Thus; researchers recognised that a way to move closer to solving the *capital structure puzzle* (Myers, 1984) was to incorporate the role of time and dynamic choice.

To remedy the confounding effect of static optimisation, a strand of literature developing dynamic capital structure models emerged. The dynamic models allow the firm to re-optimise its capital structure in response to the diffusion of an underlying state variable, represented by either firm value, product price or earnings. The management maximises the firm value by repeatedly readjusting the firm's leverage. Enabling optimisation, the tax advantage is known and the risk of costly bankruptcy is estimated from the characteristics of the stochastic state variable. Constituting two of the first contributions to this academic field, Brennan and Schwartz (1978) as well as Kane, Marcus and McDonald (1984) develop continuous-time models that include randomness in firm value and dynamic refinancing. Brennan and Schwartz (1978) assume that the upper refinancing boundary is determined by an exogenous bond indenture, but are unable to obtain a closed-form solution and consequently use numerical techniques to analyse the firm's optimal capital structure choice. Kane et al. (1984) also consider an exogenous upper refinancing boundary but do in fact obtain a closed-form solution. However, the model is particular in the sense that the firm can not enter bankruptcy and has a firm value that follows a mixed jump-diffusion process. Interestingly, both papers allow their firms to re-optimise their leverage ratios continuously through time *without cost*. As a result, a suitable refinancing strategy can allow the firm to capture considerable tax benefits, while effectively retaining risk-less debt. The optimal leverage ratios in the two papers thus remain fairly high.

In fact, the presence or absence of frictions such as transaction costs should have a material impact on the optimal capital structure decision. With this motivation, Fischer et al. (1989a) further develop the model by Kane et al. (1984) to feature bankruptcy, callable debt, recapitalisation costs and a firm value process without jumps. Under this enhanced framework the authors are able to identify a region of the firm value where - for a given set of parameters - the benefit of readjustment does not justify the payment of the recapitalisation cost. The fundamental insight that follows is, accordingly, that transaction costs can warrant the choice of a seemingly sub-optimal capital structure. This could serve as a factor to explain why empirical cross-sectional tests for capital structure selection have failed to solidly corroborate the trade-off theory (see e.g. Shyam-Sunder and Myers (1999) or Graham (2000)).

Marking the next significant advancement within the field of dynamic capital structure theory, Goldstein et al. (2001) develop a similar model to Fischer et al. (1989a) and depict how the option value of levering up at a future juncture influences the capital structure decision. This is analysed by implementing callability of debt in the bond indenture and endogenising the bankruptcy threshold. The authors moreover criticise previous dynamic capital structure models, such as those of Fischer et al. (1989a) and Kane et al. (1984), for effectively reducing the capital structure problem to a single-period optimisation. It is argued that the boundary conditions applied in earlier literature are the same as those of a firm wishing to be optimally levered in solely one period. This might have led, among others, Fischer et al. (1989a) to conclude that the tax advantage is limited. In contrast to previous work, Goldstein et al. (2001) present a solution that allows for optimisation over an arbitrarily large number of upward restructurings. A key condition for attaining a dynamically consistent optimisation problem in the paper is the scaling feature inherent in the state variable evolution. The scaling feature is a result of the choice of modelling the state variable as log-normal. The paper is also the first to represent the state variable as earnings before interest and taxes (EBIT) rather than firm value. This choice of state variable offers several economically sensible advantages over the traditional form. Most notably, equity value will no longer be a monotonically increasing function of the corporate tax rate, as is the case in e.g. Leland (1994).

Following the work of Goldstein et al. (2001), multiple researchers have attempted to develop the basic model in various directions. Titman and Tsyplakov (2007) model a firm that can regulate both its capital structure and its investments when responding to stochastic changes in an underlying product price process that determines firm value and earnings. Importantly, however, in the context of this thesis, a few researchers have instead attempted to enhance the tractability of the model by altering the stochastic process. The vast majority of all classical papers assume that the state variable follows a geometric Brownian motion, which is in accordance with the original option pricing model of Black and Scholes (1973). Nevertheless, theorists such as Raymar (1991) and Sarkar and Zapatero (2003) as well as empirical researchers such as Lipe and Kormendi (1994) and Fama and French (2000) argue that earnings are more likely to exhibit some level of mean-reversion. Particularly, Sarkar and Zapatero (2003) contribute significantly to this notion by developing a static capital structure model that exhibits mean-reversion in earnings. A notable effect of this alternation is that the relationship between earnings and optimal leverage becomes negative. Hence, the authors remedy one of the major weaknesses of the standard trade-off theory, namely the prediction of a positive relationship between earnings and leverage. Additionally, the paper both predicts and verifies that the optimal leverage is an increasing function of the speed of mean-reversion. In a recent contribution to this subfield, Bjerrisgaard and Fedoryaev (2011) develop a dynamic model with callability and mean-reversion in earnings that maintains the scaling feature of Goldstein et al. (2001). They conduct a comparative statics analysis to verify the results of Sarkar and Zapatero (2003).

This expansion of the dynamic capital structure literature is one that is developed further in this thesis.

Goldstein et al. (2001) only consider re-optimisations of capital structure at the upper boundary with the motivation that adjustment at the lower boundary is unlikely to occur outside Chapter 11. It is argued that many new effects must be taken into account to appropriately model the incentives of equity holders in the default region. Posterior academic studies have, however, examined this scenario more in detail. Christensen et al. (2014) focus particularly on the renegotiation game that takes place in case of bankruptcy at the lower boundary. The authors manage to unite multiple strands of literature to form a dynamic capital structure model that incorporates refinancing frictions, callable debt, and a lower-boundary debt renegotiation game. By explicitly modelling the agents' behaviour at the lower boundary, the authors are able make conjectures about both the optimal leverage ratios and the size of *absolute priority rule* (APR) violations comparable to empirical observation.

Arguably, the implementation of an explicit renegotiation game in relation to capital structure optimisation is a relatively novel and underexplored field in academic research. In addition to the paper by Christensen et al. (2014) there are, however, a number of stochastic models dealing with debt renegotiation, relating to both corporate and sovereign default settings. The main distinction between these two settings is that there is no international bankruptcy law in the case of sovereign default, which alters the incentives of the negotiating agents. In a contribution to this field, Yue (2010) develops a strategic sovereign debt renegotiation model with stochastic endowments and employs Nash bargaining to settle terms in case of default. Despite the absence of a bankruptcy regulation, the country faces the threat of future exclusion from international capital markets if the parties cannot agree. By agreeing, creditors in turn obtain partial repayment of the face value, which would be withheld in case of disagreement. The model can be used to estimate credit spreads as well as the optimal debt-to-output ratio at the sovereign level.

In the corporate default setting Chapter 11 regulation determines the terms and bargaining power of the negotiation agents. A key feature of the regulation is the so-called absolute priority rules, specifying the order in which the various claimants rank in the case of bankruptcy. Senior creditors have first priority, while shareholders have lower priority. In this context, Anderson and Sundaresan (1996) construct a discrete-time binomial model where equity holders and creditors engage in a game about the size of the coupon payments. The equity holders have an option to refuse fully servicing their debt, while the creditors can choose to reject or accept the partial servicing. A rejection implies the start of a costly bankruptcy process, which can induce the creditors to accept less than full coupon payment. Under plausible parameter values, the model suggests realistic yield spreads and serves as a point of departure for more complex models. Mella-Barral and Perraudin (1997) set up a similar model to that of Anderson and Sundaresan (1996) but distinguish their work by instead considering a continuous-time setting. The output price follows a GBM and the agents bargain over the size of the future coupon payment in a way that allows for strategic debt servicing behaviour by the equity holders. While the problem considered in the strategic debt service (SDS) models is similar to that of Christensen et al. (2014), the latter stands out in the sense that equity holders are not assumed to be able to make take-it-or-leave-it offers to the creditors. The threat of bankruptcy must be a credible such in order for the creditors to allow concessions. Furthermore, the entire capital structure is renegotiable rather than merely the future coupon payments.

Chapter 2

Diffusion Processes and Contingent Claims Pricing

> *"When judged by its ability to explain the empirical data, option pricing theory is the most successful theory not only in finance, but in all of economics."*
>
> - Stephen A. Ross (1944-pres.)

The aim of this chapter is to provide a brief review of the theoretical underpinning for the models of dynamic capital structure that will be developed later in the thesis. To maintain tractability, the scope is restricted to elements of the applicable topics, which carry direct relevance for the derivation of the basic model. Particular emphasis is thus reserved for the central mathematical properties and techniques of stochastic calculus underlying earlier models of dynamic capital structure, which constitute the foundation upon which our model in this thesis is built.

Initially, we narrow in on the concept of diffusion processes with a focus on the fundamental differences between Brownian motions and Ornstein-Uhlenbeck processes. We also shed light on the attractiveness of using the latter to represent the dynamics of a firm's earnings. Subsequently, we focus on the tools for pricing of claims contingent on a value function of these stochastic processes. This is done to illustrate its use for dynamic optimisation and resultant applicability for the derivation of optimal capital structure. An understanding of the two boundary conditions imposed on the partial differential equation (PDE) will be integral in this regard. For this reason we carry out a particular exploration of the general properties of these[1].

[1] The chapter is based on Dixit and Pindyck (1994), Cochrane (2005) and Pennacchi (2008).

2.1 Diffusion Processes

A *diffusion process* can broadly be defined as a continuous-time stochastic process satisfying the Markov property. This requires the conditional probability distribution of future states to depend only on the present state for a given process (Cochrane, 2005).

A diffusion process can be developed by generalising a *Wiener process*, which is defined as the continuous-time limit of a normal discrete-time stochastic process. More formally, a variable z that follows a Wiener process satisfies two properties:

Property 2.1.1 *The change in $z(t)$ over the time interval Δt is given by*

$$z(t+\Delta t) - z(t) \equiv \Delta z = \sqrt{\Delta t}\epsilon \tag{2.1}$$

where $\epsilon \sim \mathcal{N}(0,1)$.

This property implies that Δz itself follows a Gaussian distribution where $\mathbb{E}[\Delta z] = 0$ and $Var[\Delta z] = \Delta t$.

Property 2.1.2 *The values of Δz for any two given intervals are independent, i.e.*

$$Cov[z(t+\Delta t) - z(t), z(s+\Delta t) - z(s)] = 0 \tag{2.2}$$

for the non-overlapping intervals $(t, t+\Delta t)$ and $(s, s+\Delta t)$.

It follows from this property that z satisfies the Markov property.

Next, consider a change in z from $t = 0$ over a longer time period to $t = T$ with $n = T/\Delta t$ time increments. This implies that

$$z(T) - z(0) = \sum_{i=1}^{n} \epsilon_i \sqrt{\Delta t}, \tag{2.3}$$

where ϵ_i is the value of ϵ over the i^{th} interval. The first two moments of (2.3) are $\mathbb{E}[z(T) - z(0)] = 0$ and $Var[z(T) - z(0)] = n\Delta t = T$. Holding the horizon T fixed, we see that the mean and variance of (2.3) are independent of the number of equidistant time steps n. Applying the central limit theorem (CLT) under the assumption that ϵ_i are independent and identically distributed, we can posit that

$$\operatorname*{plim}_{n\to\infty}[z(T) - z(0)] = \operatorname*{plim}_{\Delta t\to\ 0}[z(T) - z(0)] \sim \mathcal{N}(0, T). \tag{2.4}$$

The distribution of $z(t)$ over the interval $[0,T]$ can thus be thought of as the sum of $\Delta z_i = \sqrt{\Delta t}\epsilon_i$ when Δt becomes infinitely small. The Wiener process dz in continuous time can then be represented as

$$dz = \sqrt{dt}\epsilon_t \tag{2.5}$$

with $\mathbb{E}[dz] = 0$ and $Var[dz] = dt$. This notation implies that dz has the properties stated above for Δz in the limit as $\Delta t \to 0$, and we can thus write the change in $z(t)$ over $[0,T]$ as

$$z(T) - z(0) = \int_0^T dz(t) \sim \mathcal{N}(0,T), \tag{2.6}$$

where the integral sign on the right-hand side in (2.6) is a stochastic (Itô) integral[2].

Itô Process

The Wiener process that has been developed to this point has implicitly assumed a drift rate μ equal to zero and a variance rate σ equal to one. This is equivalent to stating that the expected value of z at any future state is equal to its current value, and that the variance of the change in z is equal to the length of the time interval.

We can generalise this Wiener process by considering a new process $x(t)$, which multiplies $dz(t)$ with the variance constant σ and adds a deterministic drift $\mu(t)$ per unit of time. Thus we obtain

$$dx = \mu(t)dt + \sigma dz(t), \tag{2.7}$$

for which the distribution over $[0,T]$ would then be

$$\int_0^T dx = \int_0^T \mu(t)dt + \int_0^T \sigma dz(t) \sim \mathcal{N}\left(\int_0^T \mu(t)dt, \sigma^2 T\right). \tag{2.8}$$

The expression in (2.8) is a generalised representation of the Wiener process where we can allow the variance and drift parameters to take on any desired value. Such a process is commonly referred to as an *Itô process*. We allow this process to be a function of t and the value of x at t, for which reason the stochastic differential equation (SDE) for $x(t)$ can be written on the form

[2] This defines the integral of the changes in continuous time as the equivalent of the sum of all changes in discrete time, cf. Equation (2.3).

$$dx(t) = \mu[x(t), t]dt + \sigma[x(t), t]dz. \tag{2.9}$$

The process in (2.9) satisfies the Markov property since the change in x only depends on its present value, and the process can thus be categorised as a diffusion process.

2.1.1 Geometric Brownian Motion

The expression in (2.9) with the adapted μ-term can also be classified as a *Brownian motion with drift.* A more common representation of (2.9) is however to allow the drift and variance rates to be proportional to the current level of x so that $\mu[x, t] \triangleq \mu x$ and $\sigma[x, t] \triangleq \sigma x$, i.e.

$$dx = \mu x dt + \sigma x dW \tag{2.10}$$

where dW denotes the increment of a Wiener process. This diffusion process, called a *Geometric* Brownian motion, is heavily applied to model an abundance of security prices, which is largely attributable to its elegant properties, including:

a. If the process x starts at a positive value $x > 0$, zero will be an absorbing barrier, i.e. the process will hit zero with measure zero probability.

b. The conditional distribution of $x(t)$ given $x(0)$ is log-normal, i.e. $log(x(t))$ will follow a Gaussian distribution.

c. Given $x(0)$, the expected value and variance of $log(x(t))$ are $\mathbb{E}[log(x(t))] = log(x(t) + (\mu - 1/2\sigma^2)(t)$ and $Var[log(x(t))] = \sigma^2(t)$.

d. Given $x(0)$, the expected value and variance of $x(t)$ are $\mathbb{E}[log(x(t))] = x(0)e^{\mu t}$ and $Var[log(x(t))] = x(0)^2 e^{2\mu t}(e^{\sigma^2 t} - 1)$.

e. Given $\mu > 0$, $\mathbb{E}[x(t)] \to \infty$ and $Var[x(t)] \to \infty$ as $T \to \infty$.

The predominant reason why the GBM is a popular tool to price e.g. stocks and interest rates is its mathematical tractability and the restriction of negative values imposed by a log-normal distribution. As indicated previously, the GBM is also the modelling convention for representing the state variable diffusion in the vast majority of dynamic capital structure models. One justification for this use, cited in much of the literature, is its homogeneity property (also referred to as scaling invariance). This property ensures that the values for debt and equity claims obtained in one solution will be repeated for other initial values of the state variable after the results have been re-scaled. The homogeneity property is examined and applied in more detail in the next chapters.

2.1.2 Geometric Ornstein-Uhlenbeck Process

An alternative diffusion process that admits a stationary probability distribution is the *Ornstein-Uhlenbeck process*, which is a *mean-reverting* process. The standard form can be represented as

$$dx = \kappa(\theta - x)dt + \sigma dW \tag{2.11}$$

where $\kappa \geq 0$ represents the speed of mean-reversion and $\theta > 0$ is the long-run equilibrium level to which x reverts. Again, this process can be modified to a *geometric* Ornstein-Uhlenbeck (GOU) process by allowing the volatility of the state variable to depend on its current value:

$$dx = \kappa(\theta - x)dt + \sigma x dW. \tag{2.12}$$

It is evident that the difference between (2.10) and (2.12) is in their drift term. Whereas the GBM has a constant drift term, the drift in the GOU is contingent on the current value of the process. The conditional mean is

$$\mathbb{E}[x(t+s)|x(t)] = \theta + e^{-\kappa s}(x(t) - \theta), \tag{2.13}$$

which implies that if $x(t) > \theta$, then $\mathbb{E}[x(t+1)|x(t)] < x(t)$; if the current value of the process is above θ, then next period is expected to be lower than the current price. In addition, the variability of the state variable under the GOU diffusion is a function of the mean-reversion parameter κ, as well as the variance parameter σ. The conditional variance of x after s periods is

$$Var[x(t+s)|x(t)] = \frac{\sigma^2 x(t)}{2\kappa}(1 - e^{-2\kappa s}). \tag{2.14}$$

Looking at the first-order condition with respect to the mean-reversion parameter

$$\frac{\partial Var[x(t+s)|x(t)]}{\partial \kappa} = \frac{-\sigma^2 x(t)}{2\kappa^2} + \frac{\sigma^2 x(t)}{2\kappa^2} e^{-2\kappa s} - \frac{2s\sigma^2 x(t)}{2\kappa} e^{-2\kappa s} < 0, \tag{2.15}$$

it is evident that the expected range of future values decreases with the level of mean-reversion. In addition, although increasing over time, the variance of the process reaches the long-run limiting level

$$\lim_{s\to\infty} Var[x(t+s)|x(t)] = \frac{\sigma^2 x(t)}{2\kappa}. \tag{2.16}$$

Thus it can be seen that, in contrast to the GBM, the GOU not only admits a stationary probability distribution but also has a bounded variance.

Due to these properties, many applications of stochastic models for e.g. commodity prices use a variation of the GOU process. The existence of a long-run mean level is an economically sound assumption for the forecasting of such processes. As discussed above - being the main motivation behind the development of our model below - this also holds true for any measure of corporate earnings. For this reason the application of the GOU process to dynamic capital structure modelling should add another layer of realism. This argument will be expanded in considerable detail in Chapter 6.

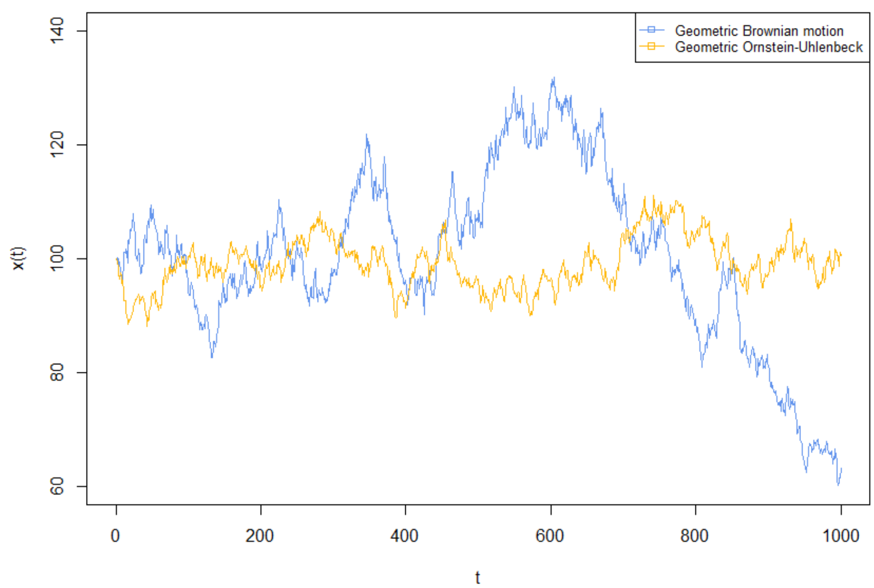


Figure 2.1: *Sample paths for $x(t)$ as a Geometric Brownian motion and a geometric Ornstein-Uhlenbeck process.* (Source: Own contribution)

2.2 Pricing of Contingent Claims

In order to achieve dynamic optimisation of capital structure it is first necessary to develop a methodology to model the total firm value. This value will in turn depend on the market values of debt and equity, which can be viewed as claims on the guiding state variable under the uncertainty of its future diffusion path. As highlighted by Dixit and Pindyck (1994), there are generally two techniques for dynamic optimisation of this kind: *dynamic programming* and *contingent claims pricing*. Although these two methodologies produce similar results, they differ in their assumptions about the completeness of financial markets and the discount rates that firms apply to value future streams of cash flows. While dynamic programming constitutes the general tool for optimisation under conventional assumptions of no arbitrage and absence of financial frictions, contingent claims pricing (or *options pricing*) uses more specific techniques of dynamic portfolio replication in a complete security price system.

This section focuses on developing the techniques of dynamic optimisation by means of contingent claims valuation. This is due to its direct applicability for issues in financial economics and the significance of its use in the existing capital structure modelling literature. This technique imposes a so-called *spanning* condition on financial markets, implying dynamic completeness of markets. Correspondingly, the risk for a given problem can be spanned by trading in existing securities. Given the spanning condition, we can construct a replicating portfolio, in order to derive the PDE for the valuation of a security that derives its value from the value of the firm (and time).

Itô's Lemma

In order to derive solutions for the values of claims on diffusion processes, it is necessary to work with functions of these. However, it is generally not possible to take differentials of continuous-time Itô processes. For this reason, we must make use of *Itô's Lemma*, which is an identity serving as the stochastic counterpart of the chain rule in ordinary calculus. Below, we state the result for the case of a single variable that follows a generic diffusion process; thus omitting the formal proof.

Let the variable $x(t)$ follow the SDE given in (2.9). Moreover, let $F(x(t), t)$ be an at least twice-differentiable function of the state variable and time. The differential of this function is then given by

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2 \tag{2.17}$$

where $(dx)^2 = \sigma(x,t)^2 dt$. Substituting for dx and $(dx)^2$ in (2.17), we obtain

$$dF = \left[\frac{\partial F}{\partial x}\mu(x,t) + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2(x,t)\right]dt + \frac{\partial F}{\partial x}\sigma(x,t)dW. \tag{2.18}$$

As we can see, a particular virtue of Itô's Lemma is that it dictates how to calculate the evolution path of a process that is defined as a function of another process. We will make extensive use of this property below and in the following chapters.

2.2.1 The Merton (1974) PDE Approach to Pricing Claims

The valuation methods based on option pricing methods generally employ one of two ways to price a contingent claim: Either by a PDE that stipulates the price of the claim as the solution, or by a probabilistic approach that calculates the price as an expectation under the risk-neutral measure $\mathbb{Q}$[3]. The former approach can be attributed to the seminal work of Merton (1974), in which he proposes a model for assessing the credit risk of a company by characterising the company's equity as a call option on its assets. He thus illustrates how corporate securities can be valued using the option pricing techniques developed by Black and Scholes (1973). The alternative probabilistic approach was first suggested by Black and Cox (1976).

In this section we focus on the former approach, as its general result will be central in the next chapters. Thus; following Merton (1974), assume that a firm can issue claims on the state variable x following a GBM process as in (2.10). Later the claims on the state variable will be classified as an equity claim E and a debt claim D on an EBIT process ξ, but for now we will follow the convention in this chapter and consider the claim on the process x and time t, i.e. $F(x(t), t)$. Given the assumed diffusion process for the state variable and assuming existence of an equivalent martingale measure $\mathbb{Q}$, we know that financial markets are complete and arbitrage-free by The First Fundamental Theorem of Asset Pricing. Thus F is a claim that can be replicated in the market and its value can be derived as a solution to a PDE[4]. To see this, we apply Itô's lemma:

[3]The Feynman-Kac result amounts to showing that these methods are in fact equivalent.

[4]Of course this solution only holds under the classic perfect market assumptions put forward in Merton (1974).

$$\begin{aligned} dF(x(t),t) &= \frac{\partial F(x(t),t)}{\partial x(t)}dx(t) + \frac{\partial F(x(t),t)}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F(x(t),t)}{\partial x(t)^2}dx(t) \\ &= \left[\frac{\partial F(x(t),t)}{\partial x(t)}\mu x(t) + \frac{\partial F(x(t),t)}{\partial t} + \frac{1}{2}\sigma^2 x(t)^2 \frac{\partial^2 F(x(t),t)}{\partial x(t)^2}\right] dt \\ &\quad + \frac{\partial F(x(t),t)}{\partial x(t)}\sigma x(t) dW(t) \\ &\triangleq \mu_F dt + \frac{\partial F(x(t),t)}{\partial x(t)}\sigma x(t) dW(t). \end{aligned} \tag{2.19}$$

Further, we assume the claim holder receives a continuous payout rate h with an associated cumulative dividend stream $H(t) = \int_0^t h(x(u),u)du$. We then have

$$dH(t) = h(x(t),t)dt. \tag{2.20}$$

Following portfolio replication theory, we know that the drift μ in (2.19) must equal the risk-free rate r corrected for the payout rate h in order to rule out arbitrage when F is a price process under the $\mathbb{Q}$-measure. Thus we can combine (2.19) and (2.20) to get

$$\mu_F = rF(x(t),t) - h(x(t),t). \tag{2.21}$$

Substituting this expression into (2.19) and rearranging terms, we get the final PDE:

$$\begin{aligned} &\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F(x(t),t)}{\partial x(t)^2} + \frac{\partial F(x(t),t)}{\partial x(t)}\mu x(t) + \frac{\partial F(x(t),t)}{\partial t} \\ &- rF(x(t),t) + h(x(t),t) = 0. \end{aligned} \tag{2.22}$$

The inclusion of an explicit dividend rate for the claim holder in the above derivation, as opposed to letting the rate exist affinely in the state variable, is a slight modification of the original Merton (1974) derivation. Following the traditional notation, $h_D(x(t),t) = C$ is the coupon that accrues to the debt holder and $h_E(x(t),t) = x(t) - C$ is the residual that accrues to the equity holder. As highlighted by Merton (1974), the PDE in (2.22) requires the addition of two boundary conditions to have a fully specified differential evolution. This will also become evident when the values of debt and equity claims are derived in the models of the next chapters.

2.2.2 Portfolio Replication and Optimal Boundary Conditions

In the class of models we will consider in the next chapters, equity holders - who hold the option on the firm's value - will face some choices at each period. These will generally be determined by some known boundary payoff as well as conditions that satisfy their incentive compatibility.

To examine this problem more generally, we can consider a firm whose profit flow is a function of the state variable x, which can be thought of as the firm's output price. For the sake of exposition, we will assume x follows

$$dx = \mu x dt + \sigma x dW, \tag{2.23}$$

i.e. a conventional GBM. Traditional finance theory suggests that x is only held if investors are provided with a sufficiently high return. This return is composed of the price appreciation μ (i.e. capital gain) and a dividend δ; thus $\gamma = \mu + \delta$ denotes the total expected return. Taking the risk-free rate r as exogenously given, we know from the Capital Asset Pricing Model (CAPM) that

$$\gamma = r + \phi\sigma\rho_{xm} \tag{2.24}$$

where ϕ denotes the market price of risk and ρ_{xm} is the return correlation between x and the market portfolio m.

Following portfolio replication theory, the value of the firm $F(x,t)$ with profit flow $\pi(x,t)$ is then found by replicating its mean-variance characteristics using assets of known value. Specifically, we can buy a portfolio consisting of 1 Danish krone (DKK) of the risk-less asset and n units of the firm's output for $(1+nx)$ DKK, which is held for an interval of time dt. In the same time, the risk-less asset pays a deterministic amount of rdt, while x pays a dividend $n\delta dt$ and a stochastic capital gain of $ndx = n\mu x dt + n\sigma x dW$. The total return per unit invested is thus

$$\frac{r + n(\mu+\delta)x}{1+nx}dt + \frac{\sigma n x}{1+nx}dW. \tag{2.25}$$

We need to compare (2.25) with the return to be earned by owning the firm in the same time interval. This involves a deterministic dividend of $\pi(x,t)dt$ and a stochastic capital gain, which can be calculated by applying Itô's lemma to F:

$$dF = \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\mu x + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2 x^2\right]dt + \sigma x \frac{\partial F}{\partial x}dW. \tag{2.26}$$

The comparable return per unit invested is thus

$$\frac{\pi(x,t)+\frac{\partial F}{\partial t}+\frac{\partial F}{\partial x}\mu x+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2x^2}{F(x,t)}dt+\frac{\sigma x\frac{\partial F}{\partial x}}{F(x,t)}dW, \tag{2.27}$$

as the $F(x,t)$ is the cost of the firm at t. For risk equivalence between owning the firm and the portfolio, we must choose such that

$$\frac{nx}{1+nx}=\frac{x\frac{\partial F}{\partial x}}{F(x,t)}. \tag{2.28}$$

In an arbitrage-free market, we must also ensure that two assets with identical risk yield the same return. Thus we further require

$$\frac{\pi(x,t)+\frac{\partial F}{\partial x}\mu x+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2x^2}{F(x,t)}=\frac{r+n(\mu+\delta)x}{1+nx} \tag{2.29}$$

Substituting (2.28) into (2.29) we get

$$\frac{\pi(x,t)+\frac{\partial F}{\partial x}\mu x+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2x^2}{F(x,t)}=r\left[1-\frac{x\frac{\partial F}{\partial x}}{F(x,t)}\right]+(\mu+\delta)\frac{x\frac{\partial F}{\partial x}}{F(x,t)}, \tag{2.30}$$

which upon simplification gives the PDE for the firm value:

$$rF(x,t)=\frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2x^2+\frac{\partial F}{\partial x}x(r-\delta)+\frac{\partial F}{\partial t}+\pi(x,t). \tag{2.31}$$

The virtue of deriving (2.31) with contingent claims pricing is that all coefficients are either specified by the model or observed directly in the market. However, many solutions to this PDE exist, and we thus need to consider two boundary conditions and the economics of the value function F to pick out a single, deterministic solution.

Boundary Conditions

In order to develop the boundary conditions needed to solve the PDE in (2.31), we must consider a time span beyond $t+dt$, which we have analysed above. If the firm's problem was subjected to a fixed time limit T at which point it would receive the terminal payout $\Omega(x,T)$, equation (2.31) would have the boundary condition

$$F(x,T)=\Omega(x,T)\quad\forall x. \tag{2.32}$$

Moreover, we can see that for each $t < T$, there will be a value of x, $x^*(t)$, where the firm will have to accept the termination payoff, i.e.

$$If \begin{cases} x \geq x^*(t), & \text{exercise} \\ x < x^*(t), & \text{continue.} \end{cases} \tag{2.33}$$

Thus; the boundary condition at each instant will be

$$F(x,t) = \Omega(x,t) \quad \forall(x,t) \quad \text{such that} \quad x(t) \geq x^*(t). \tag{2.34}$$

This condition is commonly referred to as the *value-matching condition*, as it matches the value of the unknown function F with the known termination payoff function Ω.

While the value-matching condition generally holds at any decision boundary, the boundary itself is however clearly unknown. Thus in order to determine the stopping time optimally, given the termination payoff $\Omega(x,t)$, we need to define a second condition. This will uncover the value of $x^*(t)$ that defines $F(x,t)$. The condition will require that at each instant t, the values of $F(x,t)$ and $\Omega(x,t)$ as functions of x must have matching slopes, i.e.

$$\frac{\partial F(x^*(t),t)}{\partial x} = \frac{\partial \Omega(x^*(t),t)}{\partial x} \quad \forall t. \tag{2.35}$$

The condition in (2.35) is often called the *smooth-pasting condition* due to its requirement of the slopes to match at the boundary $x^*(t)$. While the formal proof for this condition is fairly intricate, the intuition is straightforward. If the two functions do not smooth-paste at $x^*(t)$, then stopping at exactly $x^*(t)$ cannot be optimal - instead it would be optimal to stop an instant earlier or an instant later, depending on the curvature of the kink at the boundary. In Appendix A we have included an exposition of the value-matching and smooth-pasting conditions under the alternative use of dynamic programming for optimisation. Although these conditions will be similar, they are arguably more intuitively grasped under the so-called *optimal stopping problem* belonging to dynamic programming techniques. In any case, these conditions will be key in deriving values for the equity and debt claims in the models of dynamic capital structure presented in the next chapters.

Chapter 3

Previous Models of Optimal Capital Structure

> *"[T]he supposed trade-off between tax gains and bankruptcy costs looks suspiciously like the recipe for the fabled horse-and-rabbit stew - one horse and one rabbit."*
>
> - Merton H. Miller (1923-2000)

In this chapter we derive and present the results of two models of optimal capital structure. The first model by Leland (1994) represents an initial attempt to optimally balance the tax advantage to debt with the costs of financial distress in a simple static regime. This model is developed exactly along the lines of its original structure. The second model, a version of Goldstein et al. (2001), extends Leland (1994) to a dynamic setting with EBIT as the guiding state variable following a GBM. The company is allowed to take on more debt as the earnings capacity increases in order to better exploit the tax advantage to debt. This model is developed under slightly altered assumptions compared to those used by the original authors. By conducting these changes we show how two endogenous boundaries can be solved for and establish a solid foundation for our own model of debt renegotiation. In the static model, closed-form solutions for the optimal leverage are obtained, whereas the optimal firm value is found through numerical procedures over maximisation of the coupon in the dynamic setup. Both models originate from important contributions to practical optimisation policy. The publications represent two of the most significant advancements of the optimal capital structure discussion and serve as important building blocks for the models developed in Chapter 5 and 6.

3.1 A Static Model

In concert with the framework introduced by Merton (1974), Leland (1994) represents the value fundamental as unlevered firm value following a GBM, i.e.

$$dV = \mu V dt + \sigma V dW \tag{3.1}$$

where μ and σ are the constant drift and volatility parameters. Introducing debt as a perpetual bond which promises the holder a constant coupon $C > 0$, securities have no direct time dependence and only depend on x, i.e. $\frac{\partial F}{\partial t} = 0$. The general claim $H = F(V,t)$ receives the dividend stream υ_H when the firm is solvent. Based on the example introduced by Black and Cox (1976), Leland shows that the PDE in (2.22) becomes an ordinary differential equation (ODE) on the form

$$\frac{1}{2}\sigma^2 V^2 F_{VV}(V) + rVF_V(V) - rF(V) + \upsilon_H = 0 \tag{3.2}$$

where subscripts denote partial derivatives. The above expression assumes that the dividend rate from (2.22) is affine in earnings. The solution to the homogeneous part of (3.2) is given by

$$F(V) = k_1 e^{\beta_1 x} + k_2 e^{\beta_2 x} \tag{3.3}$$

with

$$\beta_1 = \frac{(\frac{1}{2}\sigma^2 - r) + \sqrt{(r + \frac{1}{2}\sigma^2)^2}}{\sigma^2} = 1$$
$$\beta_2 = \frac{(\frac{1}{2}\sigma^2 - r) - \sqrt{(r + \frac{1}{2}\sigma^2)^2}}{\sigma^2} = -\frac{2r}{\sigma^2}. \tag{3.4}$$

where the simple root expressions follow from Leland's simplifying assumption that $\mu = r$. As we will see, this assumption needs to be altered in later models in order to obtain finite expressions for the claim values. Since both roots are non-complex and $r > 0$, the complete solution to (3.2) is then given by

$$F(V) = k_0 + k_1 V + k_2 V^{-\frac{2r}{\sigma^2}} \tag{3.5}$$

where the constants k_0, k_1 and k_2 are to be determined by the particular boundary conditions.

3.1.1 Pricing of Debt and Equity Claims

We denote the debt and equity claims to be priced as $D(V)$ and $E(V)$, respectively. As noted previously, D promises a coupon $C > 0$ unless V reaches the bankruptcy level V_B. Representing the cost of distress, a fraction $0 \geq \alpha \geq 1$ is lost by the claimants in bankruptcy, leaving debt holders with $(1-\alpha)V_B$ and equity holders with nothing. The necessary value-matching conditions are then given by

$$\begin{aligned} D(V) &= (1-\alpha)V_B & & at \quad V = V_B \\ D(V) &\rightarrow \frac{C}{r} & & as \quad V \rightarrow \infty \\ E(V) &= 0 & & at \quad V = V_B \\ \frac{dE(V)}{dV} &\rightarrow (1-\tau)\frac{1}{r-\mu} & & as \quad V \rightarrow \infty. \end{aligned} \tag{3.6}$$

Note that if the debt was risk-free, its value would be $\int_0^\infty e^{-rt} C dt = \frac{C}{r}$. Debt must therefore approach the value of risk-free debt as earnings grow to infinity. Thus to satisfy the second and last expression in (3.6), we must let $k_1 = 0$ in (3.5). Furthermore, since $V^{-\frac{2r}{\sigma^2}} \rightarrow 0$ when $V \rightarrow \infty$, the second expression in (3.6) implies $k_0 = \frac{C}{r}$. Finally, it follows that $k_2 = [(1-\alpha)V_B - \frac{C}{r}]V_B^{\frac{2r}{\sigma^2}}$ from the first expression in (3.6). Substituting the conditions at the default boundary into (3.5), the ODEs for debt and equity are

$$\begin{aligned} D(V) &= \frac{C}{r} + \left[(1-\alpha)V_B - \frac{C}{r}\right]\left(\frac{V}{V_B}\right)^{-\frac{2r}{\sigma^2}} \\ E(V) &= V - (1-\tau)\frac{C}{r} + \left[(1-\tau)\frac{C}{r} - V_B\right]\left(\frac{V}{V_B}\right)^{-\frac{2r}{\sigma^2}}. \end{aligned} \tag{3.7}$$

Note that the second term in the expression for debt in (3.7) must be negative, as the first term C/r represents the maximum value of the bond in the absence of credit risk. Hence, since bond holders will not declare the firm bankrupt if they could receive more than the risk-free bond value, it must hold that $(1-\alpha)V_B < \frac{C}{r}$. Given the vales for debt and equity, the optimally levered firm value $A(V) = D(V) + E(V)$ is then given by

$$A(V) = V + \frac{\tau C}{r} - \left[\frac{\tau C}{r} + \alpha V_B\right]\left(\frac{V}{V_B}\right)^{-\frac{2r}{\sigma^2}}. \tag{3.8}$$

We see that the total value of the optimally levered firm is equal to the value of the unlevered firm plus tax benefits of debt and minus costs of financial distress.

The first two terms in (3.8) are the present value of earnings and tax shield in perpetuity and the last term is the loss at default, composed of the lost value of tax shield plus bankruptcy costs.

The derivation of the value of debt and equity claims directly from the value-matching conditions in (3.6) is a slight short-cut compared to the approach employed by Leland (1994). Instead, Leland evaluates the value of an artificial unlevered firm added to tax benefits TB minus bankruptcy costs BC at the boundary conditions. Total firm value is given by $\upsilon(V) = V + TB(V) - BC(V)$. The value of equity is then calculated as the residual $E(V) = \upsilon(V) - D(V)$ where $D(V)$ is derived from the conditions in (3.6) similar to the expression in (3.7). This however yields a result similar to (3.8).

3.1.2 Optimal Bankruptcy Level and Coupon

From the expression in (3.8) it is evident that we need to determine the bankruptcy level V_B as a function of the parameters, as this will allow for a complete expression of optimal firm value. To achieve this, we will argue that equity holders will maximise the value of their claim when selecting the bankruptcy threshold, imposing the smooth-pasting condition

$$\frac{dE(V)}{dV}\bigg|_{V=V_B} = 0 \ . \tag{3.9}$$

Solving for the optimal bankruptcy level, we obtain

$$V_B^* = \frac{(1-\tau)C}{r+\frac{1}{2}\sigma^2}. \tag{3.10}$$

Substituting (3.10) into (3.8), the expression for total firm value becomes

$$A(V) = V + \frac{\tau C}{r} - \left[\frac{\tau C}{r} + \alpha\frac{(1-\tau)C}{r+\frac{1}{2}\sigma^2}\right]\left(\frac{(r+\frac{1}{2}\sigma^2)V}{(1-\tau)C}\right)^{-\frac{2r}{\sigma^2}} . \tag{3.11}$$

As the expression in (3.11) is still defined over an arbitrary level of the optimal coupon, we then solve for C^* to get

$$C^*(V) = \frac{(r+\frac{1}{2}\sigma^2)Vs}{1-\tau} \tag{3.12}$$

where $s = \left(\frac{\tau\sigma^2}{d}\right)^{\frac{\sigma^2}{2r}}$ with $d = 2\tau r + \tau\sigma^2 + r\alpha - \tau 2r\alpha$. The optimal values for D and A are then given by

$$D^*(V) = \frac{(r+\frac{1}{2}\sigma^2)Vs}{(1-\tau)r} + \left[(1-\alpha)Vs - \frac{(r+\frac{1}{2}\sigma^2)Vs}{(1-\tau)r}\right] s^{\frac{2r}{\sigma^2}}$$

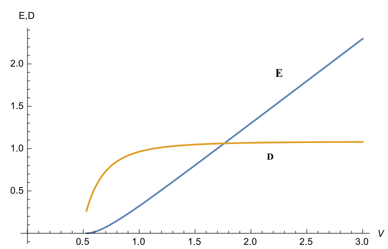
$$A^*(V) = V + \frac{\tau(r+\frac{1}{2}\sigma^2)Vs}{(1-\tau)r} - \left[\alpha Vs + \frac{\tau(r+\frac{1}{2}\sigma^2)Vs}{(1-\tau)r}\right] s^{\frac{2r}{\sigma^2}}, \qquad (3.13)$$

which completes our derivation of the Leland (1994) model. The reader should quickly observe that optimal coupon in (3.12) is a positive linear function of the level of the state variable. Thus; given a value for the state variable, we can immediately determine the level of optimal coupon. Moreover, we have obtained an optimal leverage ratio $D^*(V)/A^*(V)$, which can be tested against sensitivity to exogenous input variables, as illustrated in Figure 3.2 below.

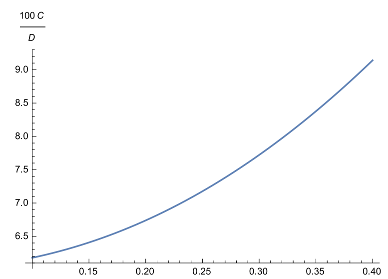
3.1.3 Model Performance

It can be calculated from (3.13) that the optimal coupon rate will increase with the effective tax rate on equity. This can be intuitively explained by the fact that the tax advantage of debt becomes larger as the tax rate increases. Under reasonable assumptions, an increase in the tax rate reduces the value of the equity claim proportionally less than it increases the value of debt. This in turn implies that the initial value of A could be increasing in the effective tax rate, which is clearly contradictory to economic intuition. This stems from the fact that Leland (1994) assumes a simplified tax structure, and is modified in Goldstein et al. (2001) and Christensen et al. (2014) by modelling the state variable as earnings and implementing a more sophisticated tax regime. Furthermore, these publications also implement a (partial) loss of the tax shelter when EBT is negative. Christensen et al. (2014) however recognise that the effect of a non-symmetric tax schedule for negative earnings has a limited effect on the total firm value since an increase in the value of the equity claim is almost fully offset by the decrease in the value of the debt claim. We have however included the equivalent optimisation problem for the Goldstein et al. (2001) model under partial loss of the tax shelter in Appendix B.

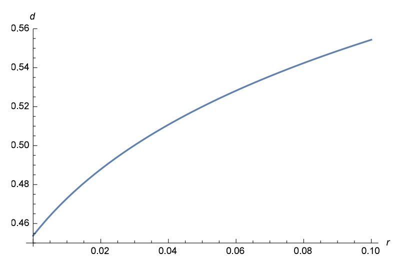
Furthermore, note in Chart 3.1b that the bond yield, defined as C/D, is increasing in the level of asset volatility. This is a consequence of the debt value decreasing in volatility. The convex relationship between C and σ implies that the coupon level does not necessarily correlate positively with the price of borrowing. This serves to illustrate that borrowing will always be more expensive for a firm whose future asset value becomes more uncertain, which is a realistic implication of the model. Chart 3.1d illustrates that the value of debt is an increasing function of the risk-free rate. As equity value will decrease due to the lower value of discounted earnings, the equity value at bankruptcy will be zero for higher levels of the value fundamental, increasing the value of debt.



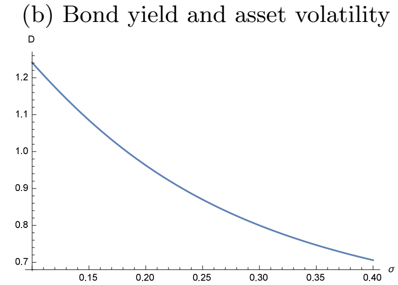
(a) Value functions and asset level



(b) Bond yield and asset volatility



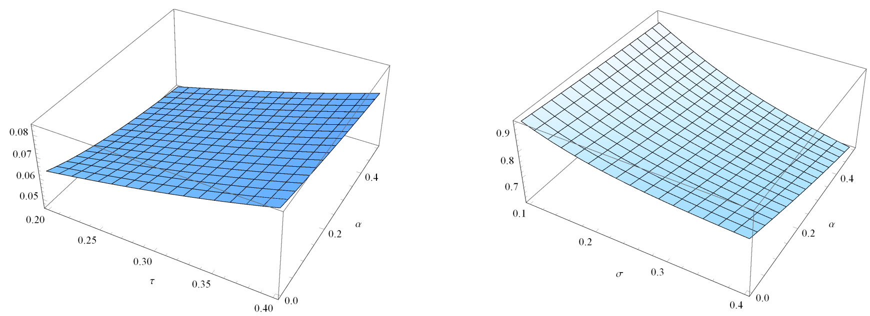
(c) Bankruptcy threshold and risk-free rate



(d) Debt value and asset volatility

Figure 3.1: *Sensitivity analysis of the Leland (1994) model.* $r = \mu = 6\%$, $\sigma = 20\%$, $\tau = 35\%$ *and* $\alpha = 50\%$. (Source: Own contribution)

We can observe a familiar relationship between the optimal coupon and the respective levels of bankruptcy costs and effective tax rate in Chart 3.2a. The display represents a graphic illustration of the key tenets belonging to the trade-off theory. Put differently, we see that the optimal coupon is an increasing function of the effective tax rate (or the (tax) advantage to debt), and a decreasing function of the bankruptcy cost (or the disadvantage to debt). Finally, observe that varying the bankruptcy cost and volatility of asset value will have a marked effect on the optimal leverage level. Faced by greater uncertainty about future asset values, management will choose a more conservative leverage when the choice of capital structure is irreversible once selected at $t = 0$. However, as we will see next, allowing for continuous increases in debt levels will lead to an even lower initial leverage choice.



(a) C as a function τ and α (b) $\frac{D}{D+E} = L^*$ as a function of σ and α

Figure 3.2: *Optimal coupon level and leverage ratio.* $r = \mu = 6\%$, $\sigma = 20\%$, $\tau = 35\%$ *and* $\alpha = 50\%$. (Source: Own contribution)

3.1.4 A Note on Continuous Restructuring

As noted above, the Leland model assumes a fixed capital structure after the initial bond issue and considers the firm liquidated at the default boundary. Thus; a straightforward critique of the model is its restriction of continuous readjustment of the capital structure that would allow for better exploitation of the tax advantage to debt. This is also the main motivation of the model in the section below. However, Leland (1994) does address this issue, arguing that either equity holders or debt holders will in fact block subsequent readjustments even in the absence of transaction costs. His contention is that existing debt holders will oppose issuance of new debt, as they will experience dilution of the value of their claim while having equal liquidation preference with new debt holders. Similarly, equity holders will refrain from buying debt with new equity issuance, as the remaining debt will consequently become safer - and thus more valuable - given a fixed initial coupon. This value is transferred from the existing equity holders, causing a negative net wealth effect of issuing more equity.

While the blocking of capital structure readjustments may hold true for small changes in the debt level, the argument fails to consider the case where all outstanding debt is retired before new debt is issued. As will become evident below, there can be ample scope for capital structure changes when agents take this possibility into account in their maximisation problem[5].

[5] Another way to introduce a scope for readjustments in this setup would be to impose maturity on the bonds, see e.g. Leland and Toft (1996)

3.2 A Dynamic Model with Callable Debt

From Chart 3.2b above it can be observed that the static model of optimal capital structure predicts a debt-to-value ratio of about 75% under the base case parameters assumed by Leland (1994). This is clearly in excess of empirically observed leverage ratios (see for example Bradley, Jarell and Kim (1984), Titman and Wessels (1988), Rajan and Zingales (1995) and Fama and French (2002)). In this section we consider a dynamic model introduced by Goldstein et al. (2001), which allows the firm to continuously increase debt levels. This enhances the tax advantage to debt significantly. It further improves the comparability with observable managerial conduct, which often supports frequent capital structure adjustments. Moreover, it will reduce the amount of debt issued upon the firm's inception, bringing the predicted leverage ratios closer to those observed in practice.

3.2.1 Operating Income as State Variable

Another major criticism that could be directed towards Leland (1994) is the underlying assumption of V being replicable with a traded asset after the initial bond issuance. As noted by e.g. Kane et al. (1984) and Fischer et al. (1989a), it must hold that the market value of real assets equal the market value of an optimally levered firm holding these assets in order to exclude arbitrage. However, if V would represent the value of a traded asset after the initial bond issue, an arbitrageur could buy the firm at V before the debt issue, and sell it for $A(V) > V$ afterwards, obtaining a risk-free profit of $A(V) - V$[6].

Goldstein et al. (2001) overcome this issue by introducing *earnings before interest and taxes* (EBIT), i.e. operating income, as the state variable ξ. Maintaining a GBM as the diffusion process, the EBIT process can be represented as

$$d\xi_t = \xi_t \mu dt + \xi_t \sigma dW_t \tag{3.14}$$

where the drift parameter μ and variance parameter σ are exogenously given. However, Goldstein et al. (2001) note that the drift rate is a function of the dividend payout ratio, which can be reasonably considered to depend on the level of the coupon rate. Nonetheless, the authors assume the payout ratio to be constant for simplicity purposes. The parameter r is the constant after-tax risk-free rate and the effective rate paid by the money market account. It assumed that $\mu < r$ such that the dividend payout ratio $\frac{\delta}{\xi} = r - \mu$ is positive.

[6]We ignore any effect of transaction costs to arrive at this conclusion.

Substituting unlevered firm value with EBIT, the state variable no longer represents a traded asset, but rather a claim to the entire payout from the output of the firm. As opposed to unlevered asset value, the value of EBIT does not depend on the capital structure. This also has the effect that the firm's investment policy is separated from its financing policy, in concurrence with the well-known Fisher separation theorem (Fisher, 1930). Later when we model the behaviour of equity holders and management acting in the interest of equity holders when the EBIT process approaches a bankruptcy threshold, this will turn out to be a strong assumption. That is, it could easily be argued that changing incentives of management renders it more likely to exhibit moral hazard in the realised investment policy when the firm approaches liquidation.

With the changed state variable, the servicing of debt is now financed by the EBIT process rather than with new equity issues. Finally, it becomes easier to study the effects of the tax shield when the value fundamental is represented as earnings before taxes.

Using EBIT as the value-generating mechanic of the underlying model, the distribution of earnings to claimants become directly comparable to that illustrated in a conventional corporate income statement:

Financial item	*Value*	*Payoff*
EBIT	ξ	
- Interest expenses	$-C$	$D(\xi) = (1 - \tau_i)C$
EBT	$\xi - C$	
- Income taxes	$-\tau_c(\xi - C)$	
Net income	$(1 - \tau_c)(\xi - C)$	$E(\xi) = (1 - \tau_e)(\xi - C)$

Table 3.1: *Earnings flow of the firm in Goldstein et al. (2001)*

The diffusion of the EBIT process ξ_t characterises earnings. The firm pays the coupon C on outstanding debt such that the after-tax payoff to debt holders is $(1 - \tau_i)C$. The firm deducts interest payments before paying corporate tax, for which reason the net income available for dividends is $(1 - \tau_c)(\xi - C)$. Assuming a dividend tax rate of τ_d, the after-tax payoff to equity holders will be $(1 - \tau_e)(\xi - C)$ where $\tau_e = \tau_c + (1 - \tau_c)\tau_d$. Note again that this assumes no loss of the tax shelter when $\xi < C$ and see Appendix B for maximisation under relaxation of this assumption.

3.2.2 Model Setup with Callable Debt

As in the Leland framework it is assumed that the firm can issue a single class of perpetual debt with a fixed instantaneous coupon C such that the debt and equity claims on the EBIT process become time-homogeneous. However, a call feature is imposed on the issued bonds. This will allow equity holders to increase leverage when earnings improve; thus better exploiting the tax advantage. Goldstein et al. (2001) restrict their attention to the case where the firm can only call the existing debt in its entirety and subsequently issue new debt. In addition to the setup in their model, we assume that the debt is callable at a premium λ to the par value P, and an issue cost q proportional to P is associated with raising debt financing. This assumption is in line with the assumed structure in later models developed in this thesis.

An important property from the Leland (1994) model, which is maintained in the current setup due to the characteristics of the GBM, is that the equity and debt price functions as well as the coupon rate are homogeneous of degree one in the EBIT process. From this follows a useful proposition, which we will make extensive use of in the solution to this model and the models in the following chapters. Suppose the initial coupon $C_0 = c_0\xi_0$. The positive homogeneity property then implies that if the firm has a policy of restructuring at $\xi = \bar{\xi} \triangleq u \cdot \xi_0$ for some fixed constant $u > 1$, then the new coupon will be given by $u \cdot C_0 = uc_0 \cdot \xi_0$ and the next restructuring according to the restructuring policy will occur at $\xi = u^2\xi_0$. Accordingly, with the initial optimal default level $\xi = \underline{\xi} \triangleq d \cdot \xi_0$ for a default policy $0 < d < 1$, the default level after the restructuring will be given by $d \cdot u \cdot \xi_0$. This feature of homogeneity will hold for debt and equity given that it is satisfied in the boundary conditions. In the static model setup this positive homogeneity can easily be seen from the expressions in (3.7) given proportionality between the default level and the coupon rate, as reflected in (3.10).

By extension, the log-normality of ξ allows the drift and variance rates of the diffusion to stay fixed across the upward restructurings; thus the EBIT process, and ultimately the capital structure policy, will remain unchanged.

A Note on the Restructuring Thresholds

In regards to the boundary conditions, the restructuring policy $(\underline{\xi}, \bar{\xi})$ of declaring bankruptcy at $d \cdot \xi_0$ and call existing debt at $u \cdot \xi_0$ could be determined either endogenously by the incentive compatibility constraints of the equity holders or exogenously by covenants given in the bond indenture. In Goldstein et al. (2001)

it is somewhat confusingly assumed that the bankruptcy threshold is determined by the smooth-pasting condition, whereas the upper restructuring boundary is committed to by management in the bond indenture. While the reason for this choice is unmotivated, we hypothesise that the authors resort to this option due to the fact that it is assumed that the firm's assets are sold off at their unlevered value at bankruptcy. Based on our trials, this assumption appears to prevent the authors from solving for two endogenous boundaries. This seeming methodological inconsistency is remedied in Christensen et al. (2014). In their context the firm is instead allowed to be taken over without loss of the tax advantage to debt in case of bankruptcy. This enables them to let both boundaries be derived endogenously from equity holders' incentives. Thus; the optimally levered firm value is included in both the upper and lower restructuring boundaries. We follow the methodology proposed by Christensen et al. (2014) in the derivation of the Goldstein et al. (2001) model below, arguing that retaining the assumption of the firm being taken over as a going concern at its optimally levered value constitutes a more economically sound setup. To the best of our knowledge, we are the first to consider the Goldstein et al. (2001) model under these refined conditions. However, a disadvantage of the resulting fixed-point problem is an increased computational complexity. For this reason, we will return to the Goldstein et al. (2001) definition of firm value at bankruptcy when considering the more complex stochastic process in Chapter 6. Also, the GBM-based model considered as a benchmark case in Chapter 5 will follow the same definition in order to have a clean laboratory for comparative analysis. For completeness of the setup for the model in this section, note that for the evolution of ξ in the range $\xi \in]d \cdot \xi_0, u \cdot \xi_0[$ the capital structure policy remains unchanged, implicitly assuming that adjustment costs will render changes unprofitable as suggested in Fischer et al. (1989a).

3.2.3 Solving the Model

The derivation of the value function on the EBIT process is very similar to the framework applied in the static framework. We must assume $\mu < r$ in order for the claims to have finite values. This is contrasted to Leland (1994) who is able to assume $\mu = r$, as he considers the unlevered firm value as the guiding state variable. He can thus neglect the requirement of finiteness in the value of the dividend stream, which becomes relevant for an earnings-based state variable (see

(3.17) below). The ODE from (3.2) under the EBIT process instead becomes

$$\frac{1}{2}\sigma^2\xi^2 F_{\xi\xi}(\xi) + \mu\xi F_{\xi}(\xi) - rF(\xi) + \upsilon_H = 0. \tag{3.15}$$

with the general solution to the homogeneous part given by

$$F(V) = k_1\xi^{\beta_1} + k_2\xi^{\beta_2} \tag{3.16}$$

and the roots β_1 and β_2 as the solutions to the fundamental quadratic, as given in (3.4). In contrast to the static setup, this dynamic version entails a set of equations with four value-matching conditions, two smooth-pasting conditions and two additional expressions for optimally levered firm value A and the value of the principal P. The value functions of debt and equity are given by

$$D(\xi) = b_1\xi^{\beta_1} + b_2\xi^{\beta_2} + (1-\tau_i)\frac{C}{r}$$

$$E(\xi) = e_1\xi^{\beta_1} + e_2\xi^{\beta_2} + (1-\tau_e)\frac{\xi}{r-\mu} - (1-\tau_e)\frac{C}{r} \tag{3.17}$$

where the τ_c term is included in the equity function in Goldstein et al. (2001) to correctly account for the relationship between equity value and effective tax rate. In the Leland (1994) model, the equity value is an increasing function of the effective tax rate, as the tax benefit increases monotonically with this rate. However, normally it is safe to assume that the present value of earnings is adversely affected by a higher tax rate; thus implying the opposite relationship. This is rectified with the inclusion of τ_e in (3.17).

Consider first the boundary conditions at the upper boundary. Assuming that the EBIT process starts at ξ_0, the existing debt is called at $u\xi_0$ at the premium λ, and new debt with higher coupon is issued to better match earnings with coupon payments for full tax shielding. The value of debt at the call boundary is thus

$$D(u\xi_0, \xi_0) = (1+\lambda)P \tag{3.18}$$

where $P = D(\xi_0; \xi_0)$ is the par value of existing debt. The corresponding value of the equity claim will equal the sum of the value of equity at the upper boundary and the newly issued debt less the cost of retiring the old debt:

$$
\begin{aligned}
E(u\xi_0, \xi_0) &= E(u\xi_0, u\xi_0) + (1-q)D(u\xi_0, u\xi_0) - (1-\lambda)P \\
&= u\xi_0 A - (1+\lambda)P
\end{aligned}
\tag{3.19}
$$

where A is a constant defined as $A = E(\xi_0; \xi_0) + (1-q)D(\xi_0; \xi_0)$. A then becomes an expression for the optimally levered firm per unit of EBIT, which at all times will equal the value of debt and equity of the newly optimally levered firm less the cost of retiring the debt belonging to the previous capital structure optimisation. An alternative interpretation of A is to consider it equal to the value of the total proceeds of issuing debt and equity that would accrue to an entrepreneur starting the firm with the sole claim on the EBIT process. This way of representing the firm's optimisation policy is in accordance with the research of e.g. Berens and Cuny (1995) and Graham (2000) on how firms should re-think the way of shielding themselves from corporate tax. Both studies support the notion that firms should aim at levelling coupon payments with earnings rather than focusing on the simple leverage ratio, defined as debt value to total firm value.

As previously stated, we here incorporate a smooth-pasting condition associated with the optimal call boundary in order to ensure consistency between equity value at the upper boundary and the equity holders incentive-compatibility level at the same boundary. We therefore require that equity holders will find it optimal to call the existing debt when

$$
\begin{aligned}
\frac{\partial E(u\xi_0, \xi_0)}{\partial \xi} &= E(\xi_0, \xi_0) + (1-q)D(\xi_0, \xi_0) \\
&= A.
\end{aligned}
\tag{3.20}
$$

That is; equity holders find it optimal to call the existing debt when firm value hits the value of an optimally levered firm for the given initial level of EBIT, ξ_0.

When the EBIT process hits $d\xi_0$ it as assumed that the equity holders withhold the coupon payment immediately and declare bankruptcy. The debt holders take over the company at a cost α of changing control, and equity holders get nothing. Note again the assumption that the tax shield is retained at bankruptcy, i.e. the firm is allowed to have debt in the event of default. This implies that the firm value will be equal to the value of its optimally levered assets, and the resultant value-matching conditions at bankruptcy are

$$
\begin{aligned}
D(d\xi_0, \xi_0) &= (1-\alpha)d\xi_0 A \\
E(d\xi_0, \xi_0) &= 0.
\end{aligned}
\tag{3.21}
$$

At the event of default, equity holders give up their claim. Thus; the accompanying smooth-pasting condition is

$$\frac{\partial E(d\xi_0, \xi_0)}{\partial \xi} = 0, \tag{3.22}$$

which closes the system of equations considered in the Goldstein et al. (2001) setup with upward restructurings. Including the expressions for A and P, the system of equations in (3.18)-(3.22) present a problem to solve eight equations with eight unknowns. The dimensionality of this problem can be reduced to a system of two equations and two unknowns by solving for the values of u and d given the exogenous parameters and the level of coupon. This system can however only be solved numerically as

$$C^* = \underset{C \in \mathbb{R}_+}{\operatorname{argmax}} A(\xi_0). \tag{3.23}$$

In other words, an entrepreneur who wants to maximise her proceeds from issuing debt and equity will determine the coupon rate to maximise A.

3.2.4 Model Performance

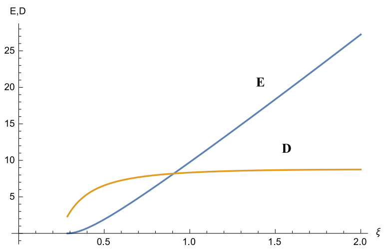
Arguably the most important result of allowing the firm to dynamically issue more debt is the lower initial leverage. This result is demonstrated below in Figure 3.4. Applying similar base case values for the input parameters, the optimal leverage level falls from 75% in the static setup to below 50% in the model with upward restructurings. The intuition behind this result is that the company will decide to issue debt with a lower coupon rate due to the fact that this can be adjusted if the EBIT process increases. And if the EBIT process decreases, a lower level of coupon could allow the company to avoid bankruptcy.

A few other results are worth noticing, however. Besides optimal leverage, many theorists within the field of dynamic capital structure are pre-occupied with predicting yield spreads. These will observably always increase with the coupon level as well as the magnitude of bankruptcy costs. Moreover, we saw in the static case that yield spreads were increasing in volatility. We see in Chart 3.3b that this is also the case in the dynamic setup. However, while always holding true for investment-grade firms, it is worth realising that an opposite relationship might hold true for high-yield - or *junk* - bonds. This dynamic is similar to the observation that interest rates on junk bonds may actually drop when the risk-free rate increases, as is also noted by Leland (1994). In any case, bonds issued with the possibility of upward debt restructurings are riskier for a given level of initial debt since the higher level of coupon will be associated with a higher default boundary. As a consequence, comparing Chart 3.3b with Chart 3.1b, we see that the bond yield for the given base case parameters is higher in the dynamic model. As highlighted by Goldstein et al. (2001), the predicted credit spreads in the dynamic setup accord better with empirically observed spreads. This is also consistent with practice where debt covenants rarely completely restrict new debt issues. In addition, new debt holders will typically rank *pari passu* with old debt holders in the priority of claims in liquidation, which renders current debt more risky.

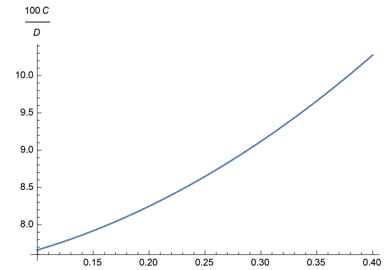
Moreover, as previously indicated, the dynamic model better exploits the tax benefit of debt financing. In accordance with the definition employed in Goldstein et al. (2001), we define the tax advantage to debt (TAD) as

$$TAD = \frac{E(\xi_0;\xi_0) + (1-q)D(\xi_0;\xi_0) - \frac{(1-\tau_e)\xi_0}{r-\mu}}{\frac{(1-\tau_e)\xi_0}{r-\mu}} = \frac{A}{\frac{1-\tau_e}{r-\mu}} - 1. \tag{3.24}$$

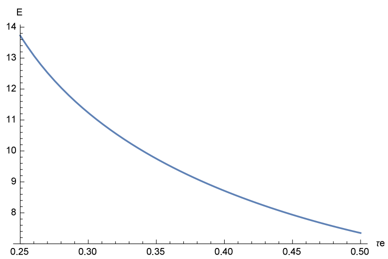
In words, the TAD is defined as the return to the initial firm owners from levering the firm relative to the value of the unlevered firm. Goldstein et al. (2001) find that, if the firm cannot increase its debt, the TAD is about 7%, which is increased slightly to about 9% in their own model. In the version of the model we have reviewed above, the TAD becomes even higher, as the adverse effect of personal taxes is excluded. It is however difficult to directly compare the tax benefit of debt across the two models due to the different assumptions about the effects of personal taxes and use of state variable. As previously stated, the static framework implies a positive relationship between the effective tax rate and the value of equity due to the exclusion of personal taxes, whereas this effect is reversed in the dynamic setup. This is also evidenced by the display in Chart 3.3c. These differing assumptions result in a measure of TAD that is not directly comparable. Obviously, though, the TAD diminishes as firm risk is increased since debt becomes more costly. This effect is illustrated in Chart 3.3d.



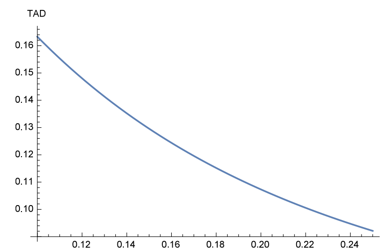
(a) Value functions and EBIT level



(b) Bond yield and EBIT volatility



(c) Equity and effective tax rate

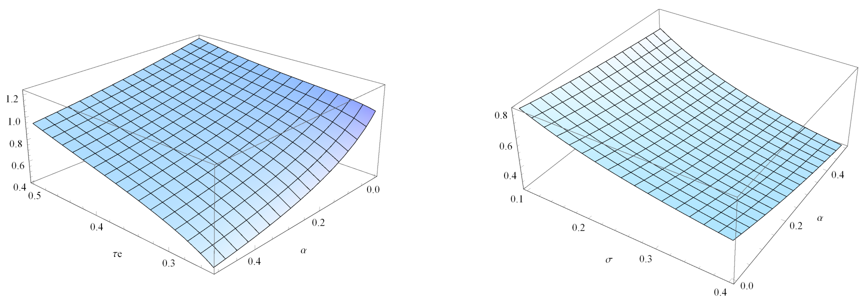


(d) TAD and EBIT volatility

Figure 3.3: *Sensitivity analysis of the Goldstein et al. (2001) model.* $r = 6\%$, $\mu = 2\%$, $\sigma = 20\%$, $\tau_e = 35\%$, $\tau_i = 20\%$, $\alpha = 50\%$, $q = 1\%$ *and* $\lambda = 5\%$. (Source: Own contribution)

As noted under the static setup, more firm risk also causes a substantial decrease in the optimal coupon for an otherwise identical firm. This effect is tractable to the decision of the initial firm owners who choose a lower coupon facing a higher risk of large decreases in earnings. On the other hand, the coupon level increases with the effective tax rate for equity holders, as this renders the attractiveness of debt financing higher. Volatility and tax benefit to debt thus have opposite effects on the optimal coupon as depicted in Figure 3.4.

In similar fashion, the optimal leverage level is negatively associated with an increase in the volatility of assets. Moreover, an increase in the price of bankruptcy as measured by the direct and indirect cost reflected by the level of α will also have an adverse impact on the optimal leverage level. As α increases, the coupon level will be lowered in response to the lower level of earnings for which bankruptcy will be triggered. As is illustrated in Chart 3.4b, fluctuations in these variables can have a significant impact on the optimal leverage level.



(a) C as a function τ_e and α

(b) $\frac{D}{D+E} = L^*$ as a function of σ and α

Figure 3.4: *Optimal coupon level and leverage ratio.* $r = 6\%$, $\mu = 2\%$, $\sigma = 25\%$, $\tau_e = 50\%$, $\tau_i = 35\%$, $\alpha = 25\%$, $q = 3\%$ *and* $\lambda = 5\%$. (Source: Own contribution)

3.2.5 Limitations

One issue in the Goldstein et al. (2001) framework, which merits a few comments, is the assumption about call of the entire debt in upward restructurings. A tendency fuelled by the proliferation of private equity as an asset class is to recapitalise the firm by releasing equity and replacing this with more debt on top of the existing bonds (Fraser-Sampson, 2010). This procedure has become a very significant contributor to leveraged buyout (LBO) returns. Thus; in models including callability of debt, one might instead consider a scenario where only a fraction of the debt is called or where the firm is re-levered without calling any existing debt.

Another evident limitation of the model in this section is its restriction of the possibility of restructuring the firm at a default threshold. While recognising that they disregard the downward restructuring option in their model, Goldstein et al. (2001) argue that a range of factors complicate the inclusion of this feature. The argument is that a number of effects, ignored in their model, will influence the modelling of equity holders' incentives in the lower restructuring region. One example of this is the issue of asset substitutability, which vanishes as the Fisher separation theorem is invoked. As rigorously expounded by Jensen and Meckling (1976), asset substitutability is likely to become a very significant risk for debt holders as the firm approaches the default threshold. A related issue considered by Myers (1977) is the tendency of shareholders to underinvest when the investment gains accrue to debt holders. Management acting in the interest of equity holders refrain from low-risk projects, as safe cash flows will not generate return for equity holders. Other theorists however claim that the potential moral hazard of management for low levels of earnings will be dampened by counter-effects such as the natural incentive to exert a good managerial effort in the presence of career concerns (see e.g. Gibbons and Murphy (1992)). In any case, the assumption of separation between investment and financing activities becomes less valid when the behaviour of the firm is modelled explicitly at the default threshold. Another example is the issue of asymmetric information. Myers and Majluf (1984) show that an incumbent management with private information about the quality of the firm might forgo positive-NPV projects if forced to turn to capital markets for financing due to the negative signalling effect of raising proceeds externally. This is again a violation of the Fisherian criterion for optimal investment. Finally, the recontracting framework of Chapter 11 that firms in restructuring encounter in practice could be considered too complex and lengthy for the development of any satisfactory model mimicking this procedure.

Notwithstanding these theoretical obstacles, solid empirical evidence exists to warrant the development of a model that includes the possibility of debt restructuring at the default threshold. In a study of 169 financially distressed U.S. firms, Gilson, John and Lang (1990) find that approximately half of the companies successfully restructure the debt before entering formal Chapter 11 proceedings. This is attributed to the fact that more of the firm's going-concern value is likely to be lost in Chapter 11, e.g. through asset sales, for which reason debt holders will accept restructurings sooner. Additional publications by Franks and Torous (1989), Eberhart, Moore and Roenfeldt (1990) and Weiss (1990), collectively called *the priority papers*, document absolute priority rule (APR) deviations in bankruptcy negotiations. These papers show that many restructurings will see equity holders participate in a reorganisation that does not provide for full payment of more senior (debt) claims. Again, senior claim holders accept this, as the absolute value of their share of the going concern value exceeds the one that can be expected in case of liquidation. In Chapter 4 we proceed to discuss some of the first serious attempts to include restructurings at the lower boundary.

Other issues in the Goldstein et al. (2001) paper relate to e.g. the state variable dynamics and the infinity of debt maturity. However, these subjects will be treated separately later in this thesis.

Chapter 4

Strategic Debt Renegotiation

> *"Unless the debtor pays the amount of the judgment or somebody guarantees his debt, the creditor shall take him home and fasten him in stocks or fetters. He shall fasten him with no less than fifteen pounds of weight."*
>
> - Laws of the Twelve Tables (451-450 B.C.)

In this chapter we aim to discuss the essential theory required for the development of a satisfactory model of debt renegotiation. A general conception of debt renegotiation modelling is a fundamental building block for understanding the dynamic capital structure models with a continuous restructuring feature. We review the results of three previous models of debt renegotiation and their methodologies, shedding light on fundamental contrasts and resemblances as well as the progression between them. The three selected models demonstrate distinctive perspectives and practices that produce different conclusions with regards to security pricing and optimal capital structure in the presence of debt renegotiation. The first model of Anderson and Sundaresan (1996) is a discrete-time binomial model where equity and debt holders bargain about the size of the future coupon payments. In the second model by Mella-Barral and Perraudin (1997), the agents still bargain about the size of the future coupon payments, however in a continuous-time setting. These two models arguably represent the most significant contributions to the category of models generally termed *strategic debt service* (SDS) models. Finally, we will discuss the approach of Christensen et al. (2014), which is fundamental for the method employed in later chapters of this thesis. In contrast to previous models, this last paper allows for re-optimisation of the entire capital structure rather than merely readjusting future coupon payments. Additionally, it is based upon more sound economic assumptions about the rational behaviour of the bargaining agents. We will focus exclusively on the game-theoretical aspects of the Christensen et al. (2014) model in this chapter. A more elaborate discussion of the framework specifics will be presented in Chapter 5 when we develop a version of the model to benchmark against our own model in Chapter 6.

4.1 Game-Theoretical Considerations

As a preface, we will consider a number of fundamental economic concepts that are essential when distinguishing between the models in this chapter. More specifically, we aim to emphasise some basic game-theoretical arguments that constitute sound economic behaviour. The models reviewed in this chapter will prove to exhibit varying degrees of compliance with these assumptions. This exposition will drive transparency and allow us to understand the catalysts of the various results.

4.1.1 Rationality and Utility

One of the most fundamental concepts in game theory is the notion of *rationality*. An abundance of literature has been devoted to the development of a generally acceptable definition of rationality. Still, no universally acknowledged characterisation of the concept can be said to have emerged. According to Blume and Easley (2008), the rationality principle most commonly assumed by working economists holds that *"individuals act in their best interest as they perceive it"*. In theory, an individual's best interest is in turn usually determined by the maximisation of his or her utility function. For the case of choice under uncertain circumstances, an agent would maximise their *expected* utility. Clearly, empirical research has indicated that individuals do not always act rationally according to this definition (Pennacchi, 2008). However, the expected utility paradigm constitutes the foundation upon which most of the classical work within economics is based. The first comprehensive formulation of the axioms defining the expected utility framework was developed by von Neumann and Morgenstern (1944). The theories developed in their publication can be argued to have marked the inception of the entire field of game theory. The foundation for any strategy in game theory is accordingly defined by agents maximising their utility functions.

As is well-known, the utility function can take on multiple forms depending on the agent's assumed relation to wealth. In practical terms, a risk-averse agent would have a concave utility function, whereas a risk-seeking agent would have a convex utility function. A risk-neutral agent possesses a special type of utility function that is linearly dependent on wealth. The assumption of risk-neutral agents is particularly prevalent in option pricing theory. It is commonly incorporated by computing the security prices under an equivalent martingale measure. Even though investors are known to be risk-averse in reality, the analysis can be successfully developed under a risk-neutrality assumption (Harrison and Kreps, 1979).

Rational and utility-maximising agents are essential for the creation of a satisfactory model of debt renegotiation. Later in this chapter, we will however demonstrate that certain previous models of debt renegotiation are, in fact, not fully consistent with these assumptions. We infer that the challenge for today's researchers is to structure a game-theoretically compliant, stochastic model of debt renegotiation and dynamic capital structure choice. This is indeed one of the main objectives of this thesis.

4.1.2 Strategic Bargaining

In the strategic situations considered in this paper, the agents are comprised by equity holders and debt holders. These two types of agents are traditionally assumed to act in rational self-interest without concern for their counterparty. In our circumstances, the agents will engage in bargaining over future coupon payments or over the firm's entire capital structure. They are thus faced with the challenge of agreeing upon a reasonable compromise, all while maximising their own payoff. Two fundamental bargaining models addressing this issue are Rubinstein's (1982) sequential bargaining game and Nash's (1950) axiomatic bargaining solution. Thus; gaining a full grasp of these will be necessary for establishing the infrastructure in later models.

Rubinstein's (1982) Bargaining Game

Rubinstein (1982) has developed a model of sequential bargaining where two agents negotiate the division of some surplus during a number of discrete time periods. Thanks to its intuitive format and result, the model has had great influence in economics since it was first proposed. In the generalised version of Rubinstein's (1982) model, the number of bargaining periods approaches infinity. For our purposes, it is more applicable to examine a case with a finite number of bargaining periods. The reason for this will become clear as we develop the model of debt renegotiation in Chapter 5.

Following Gibbons (1994), the setting of the Rubinstein game can be summarised as follows: Assume a three-period game played by two agents who bargain over the division of one Danish krone. Symmetric information is assumed, as the players are informed of their counterparty's payoff and incentives. Initially, player 1 will make a proposal that player 2 can either accept or reject. If player 2 agrees, the game ends. Otherwise, the game continues to the next round where player 2 makes a proposal. Time value of money is accounted for by introducing a per-period discount factor δ where $0 < \delta < 1$. This presents a critical friction in the model that encourages the agents to minimise the time spent haggling. In the absence of this feature, the agents could easily go on bargaining forever. The timing of the game unfolds in three stages:

1.1) Player 1 proposes to divide the krone so that player 1 and player 2 receive the shares $(s_1, 1 - s_1)$, respectively.

1.2) Player 2 chooses whether to accept or reject the offer. Acceptance implies immediate payout of the players' respective shares. Rejection results in the game proceeding to the next round.

2.1) Player 2 proposes an allocation of $(s_2, 1 - s_2)$ to player 1 and herself, respectively.

2.2) Player 1 decides whether to accept the offer or to proceed to the next round.

3.0) Player 1 and player 2 immediately receive the exogenously given allocations $(s, 1 - s)$, respectively. No further bargaining takes place.

Games of Rubinstein's form are conventionally solved by the use of backwards induction. Hence, we must start by computing the optimal offer of player 2 if the second period is reached. Player 1 can realise a payoff δs by rejecting player 2's offer. This can be seen as player 1's rejection payoff. It is thus known that player 1 will accept any offer where $s_2 \geq \delta s$, given that indifference implies acceptance. If player 2 awaits the exogenous payoff in the third period, she will realise a payoff $\delta(1 - s)$. Observably, the payoff from offering player 1 the minimum acceptable share in the current period is higher than the payoff from waiting. Formally, $\delta(1 - s) < (1 - \delta s)$. Player 2's optimal offer is therefore $s_2 = \delta s$. With symmetric information, player 1 can foresee player 2's second-period decision problem. Therefore, player 1 is aware that player 2 will accept any allocation $1 - s_1 \geq \delta(1 - s_2^*)$. The alternatives available to player 1 are to realise $1 - \delta(1 - s_2^*)$ in this period or $\delta s_2^* = \delta^2 s$ by waiting. Player 1's optimal response is thus to offer $s_1^* = 1 - \delta(1 - s_2^*) = 1 - \delta(1 - \delta s)$.

The backwards induction equilibrium in this game corresponds to player 1 choosing an optimal offer, $s_1^* = 1 - \delta(1 - \delta s)$, in the first period, which warrants player 2's immediate acceptance. The game illustrates how a situation of sequential bargaining can be resolved by the application of backwards induction. The setup is general in nature, but offers the opportunity for modification to suit more specific circumstances.

Nash's Bargaining Solution

Nash's (1950, 1953) model is attractive in applications because of its economically sound approach to bargaining. The *Nash bargaining solution* (NBS) ensures that the model satisfies certain desirable properties. These axioms apply to agents with *von Neumann-Morgenstern* utility functions and are summarised by Muthoo (1999) as follows:

Axiom 1 *Invariance to Equivalent Utility Representations*

Axiom 2 *Pareto Efficiency*

Axiom 3 *Symmetry*

Axiom 4 *Independence of Irrelevant Alternatives.*

Axiom 1 simply states that the result should be based on the individual's preferences, independent of equivalent specifications of the utility function. Axiom 2 refers to the classical concept of Pareto optimality. This implies that it should not be possible to make one agent better off without making the other worse off. By Axiom 3 it is suggested that the solution is independent of which player makes the initial proposal. Lastly, Axiom 4 states that the players do not allow remote alternatives to influence their decision in the game. A NBS would be one that satisfies all of the conditions above. Any model of debt renegotiation should arguably also satisfy these axioms.

The agents in the NBS situation are assumed to bargain over the partition of a certain surplus. To examine this more formally, consider the surplus Ψ which is divided such that player 1 receives a share x_1 and player 2 receives the corresponding share $x_2 = \Psi - x_1$. Both players have von Neumann-Morgenstern utility functions, $U_n(x_n) = U_n$. If the players are unable to agree upon a solution, they will realise the utilities F_1 and F_2, respectively. The solution is characterised by a pair of utilities $[U_1, U_2]$ that is obtained by maximising the function

$$\max_{(U_1,U_2)}(U_1 - F_1)(U_2 - F_2). \tag{4.1}$$

This expression is known as the *Nash product* and has a unique solution. Clearly, the players would not be interested in bargaining if the proceeds from doing so were negative. Hence, it is assumed that $U_1 \geq F1$ and $U_2 \geq F_2$ in order for a solution to exist.

To illustrate the implications of this model we can set up a simple example. Firstly, the players' utility functions can be defined as $U_1(x_1) = x_1$ for all $x_1 \in [0, \Psi]$ and $U_2(x_2) = x_2$ for all $x_2 \in [0, \Psi]$. This definition implies linearity between utility and wealth and thus risk-neutral agents. Assuming that the utility functions are differentiable in x_n, the obtained utilities can be characterised as $U_1^{NBS} = \frac{1}{2}(\Psi - F_2 + F_1)$ and $U_2^{NBS} = \frac{1}{2}(\Psi - F_2 + F_1)$. Furthermore, the respective shares of the surplus would equal

$$x_1^{NBS} = F_1 + \frac{1}{2}(\Psi - F_1 - F_2) \tag{4.2}$$

$$x_2^{NBS} = F_2 + \frac{1}{2}(\Psi - F_1 - F_2). \tag{4.3}$$

Here, it can be seen that player 1's share is strictly increasing in F_1 and strictly decreasing in F_2 . This is equivalent to saying that the bargaining power of a player increases with the utility value of his or her alternative payoff, F_n. In Rubinstein's (1982) bargaining game this alternative payoff was constituted by the next period payoff discounted back to present. In order to reach an agreement, the players must make sure that they both receive at least as much as they would have received if they failed to agree. Any remaining part of the surplus will be split equally if both agents are risk-neutral.

An alternative assumption could be that at least one of the agents is risk-averse. Let λ denote the risk-aversion parameter such that $U_1(x_1) = x_1^{\lambda}$ for all $x_1 \in [0, \Psi]$ and $U_2(x_2) = x_2$ for all $x_2 \in [0, \Psi]$. It can then be shown that the share of the remaining surplus is decreasing in risk-aversion. If, for example, player 1 were risk-averse, he would receive a share of the entire surplus equal to

$$x_1^{NBS} = F_1 + \frac{\lambda(\Psi - F_1 - F_2)}{1 + \lambda}. \tag{4.4}$$

Furthermore, the risk-neutral player 2 would claim the share

$$x_2^{NBS} = F_2 + \frac{(\Psi - F_1 - F_2)}{1 + \lambda}. \tag{4.5}$$

As λ approaches zero, the risk-neutral player assumes the entire share of the remaining surplus. Conversely, as λ turns to one, we obtain the risk-neutral result seen in (4.2) and (4.3).

Nash's framework lays the foundation for more complicated bargaining models and is very applicable in debt renegotiation settings, as will be explicated later. It is nonetheless worth noting that Nash does not explicitly model the bargaining process, as opposed to Rubinstein (1982).

4.2 Discrete-Time SDS Model

Due to its simplistic setup, the discrete-time model by Anderson and Sundaresan (1996) offers a great starting point for analysis of SDS. Hence, we commence with this model to demonstrate intuition for the process of renegotiating a firm's debt terms. The paper assumes a simple random-walk process for the firm value V_t. The process is modelled as a binomial process with up-ticks of size u and down-ticks of size $d = 1/u$. The cash flows f_t for the equity holders are proportional to the value of the firm such that $f_t = \beta V_t$, where β is the pay-out ratio. As in any binomial security pricing model, the expected value of cash flows is calculated under risk-neutral probabilities.

The game that unfolds under this setting is a symmetric information game that is ongoing at every time node. The equity holders will, based on the realised value of their cash flow, select some level of debt service S_t at each point in time. Conditional on full coupon payment $S_t = CS_t$, the creditors will accept the offer from the equity holders. Should the coupon payment fall short of the contracted amount, the creditors must choose whether to accept partial payment or to enforce a liquidation of the firm[7]. Liquidation is subject to a cost K that is deducted from the prevailing firm value such that the creditors' payoff equals $(V_t - K)$. The subgame perfect equilibrium can then be established by applying backwards induction from the terminal date T.

If the service payment is accepted at time T, the payoffs for the debt and equity holders, respectively, are given as $(S_T, V_T - S_T)$. If the service payment is not accepted, the payoffs are instead $max[(V_T - K, 0), 0]$. As long as $S_T < CS_T$, it is optimal for the debt holders to still accept the offer when $S_T \geq max(V_T - K, 0)$, and to decline otherwise. Given symmetric information, the equity holders foresee the best-response function of the creditors and choose to set $S_T = CS_T$

[7]Note that no bargaining process ensues if the offer is rejected. The alternative that equity holders submit a new offer, therefore, does not exist.

if $V_T - K > CS_T$, and otherwise to set $S_T = max(V_T - K, 0)$. In sum, the equilibrium terminal payoffs are given by $B(V_T) = min(CS_T, max(V_T - K, 0))$ for debt and $E(V_T) = V_T - B(V_T)$ for equity. Receding backwards in the binomial tree, the agents will have to account for the possible future values of debt and equity, weighted by their martingale probabilities and discounted back to the current time period. The realisation of future cash flows is of course contingent on continuing firm operations. Formalising this extensive-form game, the service level chosen by the owners is given by

$$S(V_t) = min\left(CS_t, max(0, max(V_t - K, 0) - \frac{pB(uV_t) + (1-p)B(dV_t)}{r}\right). \quad (4.6)$$

The corresponding instantaneous values of debt and equity are thus

$$B(V_t) = S(V_t) + \frac{pB(uV_t) + (1-p)B(dV_t)}{r} \quad and \quad (4.7)$$

$$E(V_t) = f_t - S(V_t) + \frac{pE(uV_t) + (1-p)E(dV_t)}{r}. \quad (4.8)$$

It should be pointed out that forced liquidation occurs in states where the cash flows are insufficient to cover the minimal acceptable level of debt service. In this scenario, the value of debt is given by

$$B(V_t) = max(0, min(V_t - K, CS_t + P_t)). \quad (4.9)$$

Hereby, the entire strategy space and the resultant security values are defined. It is thus possible to derive a sub-game perfect equilibrium and use the framework for debt contract valuation. Observably, the equity holders will service their debt strategically by continuously evaluating the possibility for underperformance of their debt contract without provoking a firm liquidation. This behaviour will be priced into the debt using a binomial grid. A thorough review of the numerical results of the model is considered beyond the scope of this thesis. One conclusion of the paper is however that SDS can explain much of the discrepancy observed between empirically observed - and theoretically predicted credit spreads. The research suggests that possibilities for renegotiation of debt terms have material implications for the pricing of corporate securities and, thus, likely also for the optimal capital structure of a firm.

4.3 Continuous-Time SDS Model

As a natural continuation of the previous model, Mella-Barral and Perraudin (1997) extend the analysis to evaluating the debt of a firm that operates in continuous time. In a familiar fashion, the claimants in this model bargain over the level of debt servicing at each instant, and not over the firm's entire principal. As a consequence of the continuous-time environment, security prices are derived using contingent claims analysis as expounded in Chapter 2. As will be demonstrated, it is possible to find closed-form solutions for the value of debt and equity as well as an optimal SDS function in this setting. To develop the model, we start out by assessing the pre- and post-bankruptcy values of an unlevered firm. These solutions can subsequently be used to determine the values of the rejection payoffs that the agents are faced with when selecting their best responses in the debt-servicing game.

The authors assume that the firm incurs an instantaneous production cost w and that the product price p follows a GBM. The net earnings flow consequently equates

$$p_t - w, \tag{4.10}$$

which implies that EBIT is only a GBM in the unlikely case that $w = 0$. The value of an all-equity financed firm with this earnings capacity is given by a function $W(p)$. Should the firm go bankrupt, the post-bankruptcy prospect will be impaired. In this case the firm can only generate net earnings of

$$\xi_1 p_t - \xi_0 w \quad \text{where} \quad \xi_1 \leq 1 \quad \text{and} \quad \xi_0 \leq 1. \tag{4.11}$$

The value of the firm, given that the earnings capacity has been impaired, is denoted by the function $X(p)$. By subjecting the functions $X(p)$ and $W(p)$ to the appropriate boundary conditions, it can be shown that their closed-form solutions are given by

$$W(p) = \frac{p}{r-\mu} - \frac{w}{r} + \left[\gamma - \frac{p_c}{r-\mu} + \frac{w}{r}\right]\left(\frac{p}{p_c}\right)^{\beta} \quad \forall p \geq p_c \tag{4.12}$$

$$X(p) = \frac{\xi_1 p}{r-\mu} - \frac{\xi_0 w}{r} + \left[\gamma - \frac{\xi_1 p_x}{r-\mu} + \frac{\xi_0 w}{r}\right]\left(\frac{p}{p_x}\right)^{\beta} \quad \forall p \geq p_x \tag{4.13}$$

where p_c and p_x are the liquidation thresholds for the respective firms and are denoted by

$$p_c = -\frac{\beta}{1-\beta}\frac{w+r\gamma}{r}(r-\mu) \tag{4.14}$$

$$p_x = -\frac{\beta}{1-\beta}\frac{\xi_0 w+r\gamma}{\xi_1 r}(r-\mu). \tag{4.15}$$

These solutions may later be used as elements in the derivation of the solution for a leveraged firm with strategically acting equity holders. Particularly, it should be noted that $X(p)$ is the value that the creditors of a leveraged firm can extract by declaring the firm bankrupt.

When equity holders can make take-it-or-leave-it offers, strategic debt service implies that the owners will choose some level of optimal debt servicing for each level of p_t. Hence, a state-dependent service flow function $s(p)$ can be defined. The form this function must take can be derived by straightforward economic reasoning. For high states of p_t, it is optimal to service the debt fully. This holds, as the liquidation value of the firm is sufficient to motivate creditors to enforce immediate liquidation in case of underperformance. For medium to low states of p_t, say when p_t is less than some threshold p_s, partial debt servicing is optimal, as creditors find liquidation suboptimal for certain levels of underperformance. For sufficiently low values of p_t, liquidation of the firm will be optimal for the equity holders such that no servicing will ensue. To formalise this intuition, we can state that there exists a p_s and a p_c^* such that

1. liquidation is triggered when p_t hits p_c^* the first time,
2. $s(p) < b \quad \forall p < p_s$ and $L(p) = X(p)$,
3. $s(p) = b \quad \forall p \geq p_s$,

where b denotes the contracted coupon payment. The value of the debt contract $L(p)$ must satisfy an ODE that can be derived by applying Itô's lemma. We can use the facts that the coupon payments are uncertain and given by the service flow function $s(p)$, as well as $L(p) = X(p)$ when $p \in [p_c^*, p_S]$. Thus; we have that $s(p)$ must satisfy the ODE:

$$rX(p) = s(p) + \mu p X'(p) + \frac{\sigma^2}{p} p^2 X''(p). \tag{4.16}$$

Note that $X(p)$ on the left-hand side is already known from previous computations. For this reason, we are able to derive the functional form for $s(p)$:

$$s(p) = \begin{cases} r\gamma, & \text{for} \quad p \in [p_c^*, p_x x) \\ \xi_1 p - \xi_0 w, & \text{for} \quad p \in [p_x, p_s) \\ b, & \text{for} \quad p \in [p_s, \infty)\,. \end{cases} \tag{4.17}$$

In order to complete the full solution, we must find the values of debt and equity as well as the trigger barriers, $p*_c$ and p_s. Once again applying Itô's lemma, we have that debt and equity must satisfy

$$rL(p) = s(p) + \mu p L'(p) + \frac{\sigma^2}{2} p^2 L''(p) \tag{4.18}$$

$$rV(p) = p - w - s(p) + \mu p V'(p) + \frac{\sigma^2}{2} p^2 V''(p). \tag{4.19}$$

In the absence of arbitrage, the value-matching and smooth-pasting conditions for the bankruptcy threshold $p*_c$ must be

$$V(p_c^*) = 0 \tag{4.20}$$

$$L(p_c^*) = X(p_c^*) \tag{4.21}$$

$$V'(p_c^*) = 0. \tag{4.22}$$

Furthermore, we can define the value of the debt contract for the interval between the bankruptcy threshold and the service boundary according to (4.23) below. As the price process approaches infinity, we can use limit arguments similar to those of the dynamic model presented in Section 3.2 to establish the values of debt and equity. In (4.23) we see that the value of debt approaches its risk-free value. Similarly, (4.25) establishes the value of equity in this case by Gordon's growth formula,

$$L(p) = X(p) \quad \forall p \in [p_c^*, p_s], \tag{4.23}$$

$$\lim_{p \to \infty} L(p) = b/r \quad \textit{and} \tag{4.24}$$

$$\lim_{p \to \infty} V(p) = \frac{p}{r - \mu} - \frac{w + b}{b}. \tag{4.25}$$

Given these conditions, we are able to solve for p_c^*, p_s, $V(p)$ and $L(P)$, and thereby complete the solution of the model. The solution is characterised by

$$p_s = -\frac{\beta}{1-\beta}\frac{b+\xi_0 w}{\xi_1 r}(r-\mu) \tag{4.26}$$

$$p_c^* = -\frac{\beta}{1-\beta}\frac{w+r\gamma}{r}(r-\mu) \tag{4.27}$$

$$V(p) = W(p) - L(p) \tag{4.28}$$

$$L(p) = \begin{cases} b/r + [X(p_s) - b/r]\,(p/p_s)^{\beta}, & \text{if} \quad p > p_s \\ X(p_s), & \text{if} \quad p \leq p_s. \end{cases} \tag{4.29}$$

This model by Mella-Barral and Perraudin (1997) demonstrates how to combine a debt renegotiation framework with a classical asset pricing model in continuous time. Their work has several interesting implications. For certain realisations of the state variable, the level of debt servicing will vary stochastically and the value of the debt contract will equate the bankruptcy value of the firm. High bankruptcy costs can cause strategic debt servicing behaviour even for rather high levels of earnings. The analysis in the paper is developed assuming the absence of taxes, which renders it less useful for calculation of an optimal capital structure. The authors pre-empt this issue by suggesting that a tax advantage would imply that it becomes optimal to maximise the firm's leverage. However, expected future debt renegotiation would then place a binding constraint on the amount of possible debt issuance. The model has similar implications to that of Anderson and Sundaresan (1996). It is still assumed that the creditors' only alternative in case of rejection is to see the firm go bankrupt. This way, any actual negotiation can hardly be said to take place. In sum, the model developed in this subsection offers a useful perspective on methods of analysing debt renegotiation in a continuous time setting. In order to develop a truly dynamic model that can offer normative capital structure guidance, there are a number of additional considerations to be made. Furthermore, the model has some game-theoretical drawbacks, which should be addressed further. Conveniently, many of these issues will be examined in Section 4.4 below.

4.4 Full Debt Renegotiation Model

We conclude this chapter with a game-theoretical assessment of the Christensen et al. (2014) model. We do not aspire to complete a full review of the model since a similar model will be developed in Chapter 5. Nevertheless, the game-theoretical arguments in the model are important and much connected to the discussion developed in this chapter. Therefore, we will focus exclusively on the strategic elements present in the debt renegotiation phase of the model and compare these to the models of Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997).

Credible Threats

In their discussion of debt renegotiation modelling, Christensen et al. (2014) present an important criticism of existing SDS models. As demonstrated in the two previous sections, the SDS frameworks assume that equity holders are able to make take-it-or-leave-it debt servicing offers to the creditors. However, it is never questioned whether the threat of withholding coupon payments is in fact credible. As observed in Nash's solution, rational agents should consider their own and their counterparty's alternative payoffs before responding. We demonstrated how an agent's share of the surplus is strictly increasing in the value of his or her rejection payoff. This is the case because it would be irrational for an agent to accept a value corresponding to less than the value of their rejection payoffs. Under symmetric information, this optimal response behaviour can also be foreseen by the other bargaining agent. We find a similar concept in Rubinstein's (1982) bargaining game where agents' alternative payoffs are comprised by the next-period surplus realisations discounted back to present value.

In the SDS models presented, the rejection payoff accruing to the creditors is simply assumed to be the bankruptcy value less costs of liquidation. However, the rejection payoff of the equity holders is never really assessed. In fact, the equity holders should be faced with a dilemma in case of rejection. One alternative is to avoid paying coupon and declaring the firm bankrupt, as they promised when submitting the debt service offer. Otherwise, they could continue paying coupon, which would grant them the option value of the firm as a going concern. If it proves more profitable to keep the firm alive, any rational von Neumann-Morgenstern utility-maximising agent would choose the latter. The alternative would violate the rationality principle. Moreover, the creditors should be able to conjecture whether withheld coupon payments and thus bankruptcy are an optimal response for equity holders, given rejection. If the equity holders' payoff from withholding coupons is less than they would receive by continuing payments, the threat is *non-credible*. In this case, creditors should be able to force further concessions from the owners. In the SDS models this becomes problematic since they simply *assume* that equity holders can make take-it-or-leave-it offers. Consequently, the equity holders are able to extract a disproportionate amount of surplus from the debt holders. As previously noted, Anderson and Sundaresan (1996) find that SDS can explain much of the difference between empirical and theoretically predicted credit spreads. Conceivably, that finding might be contingent on violation of the rationality principle. Indeed, predicted credit spreads are bound to rise significantly if the equity holders frequently can extract abnormal surpluses from the creditors. Christensen et al. (2014) mediate this issue by allowing the equity holders to appreciate that the debt holders' acceptance of the offer depends on their prediction of the equity holders' subsequent rational response. Conclusively, we therefore argue that the model rests on a sounder economic foundation than the SDS models.

The Renegotiation Game

Christensen et al. (2014) consider a continuous-time setting where the entire capital structure is adjustable. The paper assumes that debt and equity are derivatives of a stochastic EBIT process, ξ, where the perpetual debt is both callable at an upper boundary and renegotiable at a lower boundary. In order to avoid a continuum of debt renegotiation proposals, the paper assumes that the firm possesses a finite number of renegotiation options. This assumption resembles the one made in Rubinstein's bargaining game with finite periods in Section 4.1.2. The solution procedure will thus also be an iterative process of calculating the agreement and disagreement payoffs in every period. In Rubinstein's game, two exogenously given payoffs are assumed in the final time period. The equivalence to these final payoffs in Christensen et al. (2014) are then given by the value of the debt and equity claims in case the final bargaining attempt fails. In case of several remaining bargaining options, the cost of haggling materialises in the form of risk of further deterioration of EBIT. In accordance with Christensen et al. (2014), we will consider the case of one remaining option in our analysis.

Applying backwards induction, the initial step in obtaining a solution to the game is to establish the values of debt and equity in the case of no remaining renegotiation options. In this situation, the sole alternative at the lower boundary is to declare the firm bankrupt immediately. The debt and equity values in this case can easily be derived by contingent claims analysis. In fact, the solution will essentially be identical to that derived in Chapter 3 as our version of Goldstein et al. (2001). We will therefore take the solution as given in this analysis. The solution is characterised by the constants; u_0, d_0 and $C*_0$ [8] as well as the claim values $E_0(1;1)$ and $D_0(1;1)$. Note that EBIT is assumed to start at the point $\xi = 1$. For notational ease we can denote the firm value in this time period by $A_0 = E_0(1;1) + D_0(1;1)$. The subscript "0" signifies zero remaining renegotiation options. If the very last renegotiation proposal is rejected, the equity holders will have to choose between continuing to pay the suboptimal coupon and effectively liquidating the firm by ceasing to pay coupons.

By the homogeneity property, the value of equity under suboptimal coupon can be valued as a claim with zero remaining renegotiation options, and where EBIT started at point C_1/C_0^* rather than at 1. Notice that this starting point would implicate an optimal coupon choice of C_1. Under continuing suboptimal coupon payments, the equity claim is then denoted by

$$E_1^c = E_0(\widetilde{d_0}\xi_0; C_1/C_0^*). \tag{4.30}$$

The value of this claim can thus be derived from the case where no renegotiation options remained.

The value of equity in case of liquidation is given by equation (4.31). It states that equity holders become residual claimants in case of bankruptcy. In line with APR, they receive anything that is left after bankruptcy costs have been covered and the debt repaid:

$$E_1^b = max\left[(1-\alpha)A_0 d_1 - D_1, 0\right]\xi_0. \tag{4.31}$$

The equity holders' choice in case of rejection thus consists of choosing between E_1^b and E_1^c. Notice the difference from the SDS setting where equity holders were assumed to be pre-committed to declaring bankruptcy in case of rejection. The rejection value of the equity claim can thus be denoted by (4.32). This claim value corresponds to the final payoff for equity holders if the agents fail to agree in the last bargaining attempt in Rubinstein's bargaining game:

$$E_1^r = max\left[E_1^c, E_1^b\right]. \tag{4.32}$$

The value of debt is conditional on the choice of the equity holders. If the equity holders continue to pay coupon, the debt claim is worth

[8] u_0: call boundary scaling factor; d_0: liquidation boundary scaling factor; $c*_0$: optimal coupon

$$D_1^c = D_0(\widetilde{d_0}\xi_0; C_1/C_0^*). \tag{4.33}$$

In accordance with APR, withheld coupons - and thereby bankruptcy - leaves the debt holders with

$$D_1^b = min\left[(1-\alpha)A_0 d_1, D_1\right]\xi_0. \tag{4.34}$$

Thus; the debt value of the debt claim can be summarised as

$$D_1^r = \begin{cases} D_1^c, & \text{for} \quad E_1^c \geq E_1^b \\ D_1^b, & \text{for} \quad E_1^c < E_1^b. \end{cases} \tag{4.35}$$

Again we see that under symmetric information, the creditors recognise that bankruptcy is not the only possible alternative in case of rejection. They can also identify the level of EBIT value and corresponding equity value for which it will be optimal to proceed to liquidation following a rejection. Of course, knowledge of the creditors' foresight is in turn also possessed by the equity holders. Under these circumstances, an optimal renegotiation threshold of EBIT can be established. The threshold will be one that actually makes the equity holders' threat of bankruptcy credible. Any attempt to threaten the creditors with bankruptcy before the credibility level of EBIT will fail.

Now, consider the scenario where the proposal is accepted. The total firm value will be that of an optimally levered firm with no remaining renegotiation options:

$$E_1^a(d_0\xi_0, \xi_0) + D_1^a(d_0\xi_0; \xi_0) = A_0 d_0 \xi_0. \tag{4.36}$$

The combined gain from renegotiation can be computed as the difference between the firm values in the cases of acceptance and rejection. Formally,

$$R_1 = A_0 d_0 \xi_0 - (E_1^r + D_1^r). \tag{4.37}$$

To determine the split of the agreement surplus, assume that the equity holders possess an exogenously given bargaining power γ, which determines the fraction of R_1 that they receive. At this point, it is worth highlighting the strong applicability of the NBS example presented in Subsection 4.1. The terms E_1^r and D_1^r represent the disagreement payoffs F_1 and F_2 in a NBS. The condition for agreement in the NBS was that each bargaining agent received at least as much as their disagreement payoff. The remaining proceeds would be split equally if both players were risk-neutral. In this debt renegotiation setting, R_1 is evidently the value of the remaining agreement proceeds. An apparent difference is that an exogenous bargaining power determines the split of proceeds rather than two explicit utility functions. However, this can be thought of as a reduced-form NBS. Consider that the values of debt and equity with one remaining renegotiation option can be determined as

$$E_1(d_1\xi_0; \xi_0) = \gamma R_1 + E_1^r \tag{4.38}$$

$$D_1(d_1\xi_0; \xi_0) = (1 - \gamma)R_1 + D_1^r. \tag{4.39}$$

From the assumed distribution of bargaining power we can make immediate inference about the functional forms of the implied utility functions. For example, a bargaining power of 0.5 suggests that both agents are risk-neutral or equally risk-averse. A bargaining power of zero implies that the equity holders are very risk-averse relative to creditors. Finally, a bargaining power of one implies the opposite relationship. In this way, the model accommodates for different assumptions about the utility of creditors and equity holders.

The expressions (4.38) and (4.39) are the value-matching conditions at the lower boundary in the case of one remaining renegotiation option. By subsequently defining the remaining smooth-pasting and value-matching conditions at both the upper and the lower boundaries, an optimal capital structure can be determined. Upon solving for the state with one remaining renegotiation option, the procedure can be extended to solve for states with additional remaining options. The agents' alternative payoffs can always be derived from the round with $n-1$ remaining options.

In order to visualise the renegotiation game, it is convenient to create two payoff trees. These binomial trees show how the logic of backwards induction used in Rubinstein's (1982) sequential bargaining game can be applied to this problem. Figure 4.1 demonstrates the game from the equity holders' point of view when one renegotiation option remains. Obviously, the decision of accepting or rejecting the renegotiation proposal belongs to the creditors. In Figure 4.1 it should thus not be mistaken for a decision belonging to the equity holders. Figure 4.2 illustrates the game from the creditors' perspective when one renegotiation option remains. Similarly, it should be noted that the debt holders are not in control over the decision of whether to continue coupon payments or withhold them after a rejection. For notational briefness, the variable D_1^r has been denoted in a slightly different way in Figure 4.2. The 1s are simply dummy variables that indicate the expected choice of the equity holders. The interpretation however remains the same as in equation (4.35).

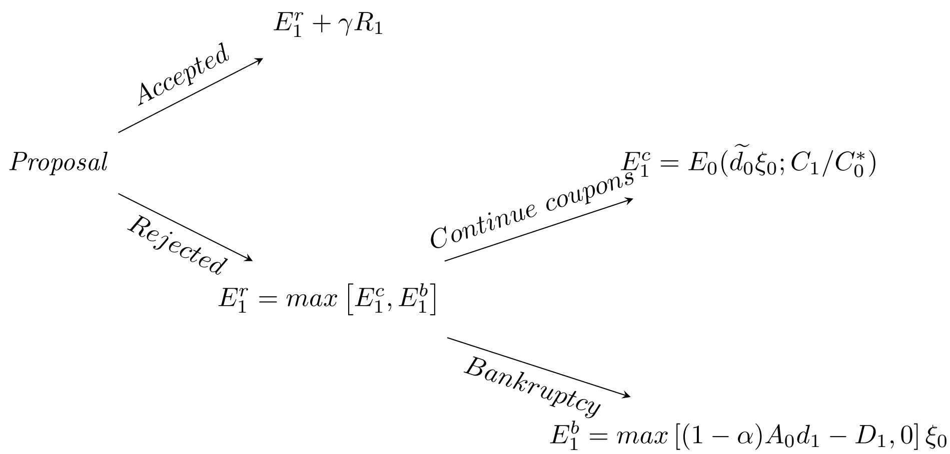


Figure 4.1: *Equity perspective.* (Source: Own contribution)

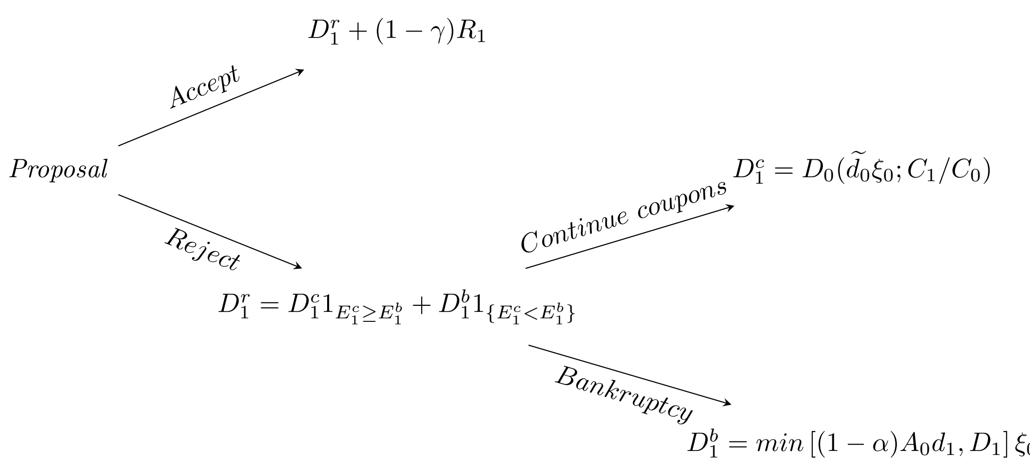


Figure 4.2: *Credit perspective.* (Source: Own contribution)

In sum, the model delivers a tractable way of modelling debt renegotiation. It further complies with the rationality assumption as well as the conventional game-theoretical axioms. The model applies backwards induction in a mode similar to that of Rubinstein (1982). The renegotiation gain is allocated according to an exogenous bargaining power, which can be seen as a reduced-form of Nash's bargaining solution. Furthermore, the model realistically allows for renegotiation of the entire capital structure rather than merely the future coupon payments. These constitute major differences compared to the SDS models, and can arguably be considered the main catalysts of the model's contribution to the capital structure discussion. The result is a model of dynamic capital structure and debt renegotiation resting on sound economic foundations. It is for this reason we find the model of Christensen et al. (2014) to be a useful platform upon which to build our proposed model in Chapter 6.

Chapter 5

A GBM-based Model with Debt Renegotiation

> *"[M]arriage involves an up-front cost of courtship, with uncertain future happiness or misery. It may be reversed by divorce, but only at a substantial cost."*
>
> - Avinash K. Dixit (1944-pres.) and Robert S. Pindyck (1945-pres.)

In this chapter we develop a GBM-based model of dynamic capital structure, allowing for debt renegotiation at a lower boundary of the EBIT process. The model is developed along the lines of Christensen et al. (2014) and will be the yardstick with which we will measure the impact of including mean-reversion in earnings in Chapter 6. As an intermediate discussion, we will expand the basic GBM-based model to feature callability of bonds, similar to the setup proposed by Goldstein et al. (2001). Since this feature should add realism to the model framework, it is interesting to assess its implications for optimal capital structure and related metrics. However, it also increases the computational complexity significantly. We will therefore consider non-callable debt in the model used for purposes of benchmarking against our own GOU-based model in the next chapter. As we will see, this enables us to obtain closed-form solutions for the optimal coupon and renegotiation threshold.

In the framework developed in this chapter we make a number of modifications to the basic assumptions compared to the Christensen et al. (2014) model. Firstly, we restrict ourselves to the case where only one renegotiation option exists. Secondly, we allow a symmetric tax schedule for negative earnings. Thirdly, we assume that the firm's assets are sold off at their unlevered value in case of bankruptcy.

The last alternation is in line with the assumption made in the original version of Goldstein et al. (2001). We discussed the implications of changing this assumption in Section 3.2.2.

5.1 The Benchmark Model

The model developed in this section will be the benchmark model when we later analyse the implications of having mean-reversion in EBIT. The model features a lower boundary where a debt renegotiation game takes place, and assumes non-callable debt. To explicitly model the game, it will be necessary to work with an iterative solution procedure. The optimal responses and available alternatives of the bargaining agents will depend upon the remaining number of bargaining options. If one option remains, the disagreement payoffs are derived from the case where no options remain. The section is therefore structured such that we first solve for the claim values in the case of no remaining options. Subsequently, we proceed to the case of one option. The game structure is similar to that of the model in Section 4.4. For an overview of the game it might therefore be convenient to refer to Figure 4.1 and 4.2.

The state variable considered in this setting is earnings before interest and taxes (EBIT). Under risk-neutral probabilities it is modelled as a GBM. That is,

$$d\xi_t = \xi_t \mu dt + \xi_t \sigma dW_t. \tag{5.1}$$

Naturally, ξ_t denotes EBIT at a time t. The drift under the risk-neutral measure is denoted by the constant μ, whereas σ is the constant volatility parameter. The firm can issue debt and equity, which are claims on the underlying earnings process ξ_t. Like in other GBM-based models, the homogeneity of degree one of the claims in the EBIT process is preserved. Furthermore, there exists a simliar tax regime including personal taxes τ_i, corporate taxes τ_c and dividend taxes τ_d. In this sense, the holders of a debt contract with a fixed instantaneous coupon C are entitled to a net cash flow of $(1 - \tau_i)C$. The flow to the equity holders is ξ_t after subtracting coupons, corporate taxes and taxes on dividends. Accordingly, the flow can be characterised as $(1 - \tau_e)(\xi - C)$, where $\tau_e = \tau_c + (1 - \tau_c)\tau_d$. The distribution of earnings is thus equivalent to the one illustrated in Table 3.1.

5.1.1 No Renegotiation Options

As is common practice in sequential bargaining settings, we apply backwards induction when solving the debt renegotiation game. Hence, we must first solve for the case with $(n-1)$ remaining renegotiation options. As we will see, this first step of excluding renegotiation of debt leads to convergence with the model developed in Chapter 3 along the lines of Leland (1994). Subsequently, we proceed to the case with n remaining options. In the setting without renegotiation, immediate liquidation will be the only possible option as ξ hits the lower boundary. In the notation that follows, the numerical subscript will denote the amount of remaining renegotiation options that a variable is calculated for.

In accordance with previous models, the general solutions to the values of the claims following theory of contingent claims pricing can be written on the form

$$D_0(\xi_t;\xi_s) = b_{01}\xi_t^{\beta_1} + b_{02}\xi_t^{\beta_2} + \frac{C_0(1-\tau_i)}{r} \tag{5.2}$$

$$E_0(\xi_t;\xi_s) = e_{01}\xi_t^{\beta_1} + e_{02}\xi_t^{\beta_1} + \frac{\xi(1-\tau_e)}{r-\mu} - \frac{C_0(1-\tau_e)}{r}. \tag{5.3}$$

These equations satisfy an ODE similar to (3.15). ξ_t is the current state of the EBIT process and ξ_s denotes its starting value. For simplicity the process will be started at $s = 0$. The parameters b_{01}, b_{02}, e_{01}, and e_{02} are constants to be determined. Furthermore, β_1 and β_2 are defined as the positive and negative roots of the fundamental quadratic equation, respectively. For more information on the specifics of this solution procedure, refer to Chapter 2 and 3.

In order to define the constants, we apply familiar economic intuition to derive the applicable boundary conditions. First of all, we can immediately see that the constants e_{01} and b_{01} belonging to the positive root β_1 must be zero. Otherwise the values of the claims would tend to infinity as EBIT increases. By eliminating e_{01} and b_{01} we ensure that the debt claim approaches $\frac{C(1-\tau_i)}{r}$ and that the equity claim approaches $\frac{\xi(1-\tau_e)}{r-\mu} - \frac{C(1-\tau_e)}{r}$ as EBIT grows without bound. Second, note that we are solving for the case where the firm will be liquidated immediately when EBIT hits a lower boundary. Following convention of previous capital structure models, bankruptcy is associated with a cost. Accordingly, a fraction α of the proceeds will be reserved for facilitating liquidation. Finally, the assumption that the assets are sold at their unlevered value in case of bankruptcy is carried over from previous models. The value of the debt contract when the firm defaults is thus given by

$$D_0(d_0\xi_0;\xi_0) = \frac{(1-\alpha)d_0\xi_0(1-\tau_e)}{r-\mu}. \tag{5.4}$$

Here we can see that the lower boundary value-matching condition differs from that of the model in Section 4.4. This stems from our differing assumptions about the going-concern value of the firm's assets in bankruptcy.

At the bankruptcy threshold the value of equity in the absence of renegotiation is zero;

$$E_0(d_0\xi_0;\xi_0) = 0. \tag{5.5}$$

With equations (5.4) and (5.5) it is possible to define algebraic solutions for the constants b_{02} and e_{02}. In this specific case, the expressions will be relatively neat. To stress the underlying calculations, we therefore include them in the notation here:

$$b_{02} = \frac{(d_0\xi_0)^{-\beta_2}((\alpha-1)d_0\xi_0 r(\tau_e-1) + C_0(\tau_i-1)(r-\mu))}{r(r-\mu)} \tag{5.6}$$

$$e_{02} = \frac{(\tau_e-1)(d_0\xi_0)^{-\beta_2}(d_0\xi_0 r + C_0(\mu-r))}{r(r-\mu)}. \tag{5.7}$$

In order to obtain a closed-form solution for the optimal bankruptcy threshold, we must proceed to identifying the applicable smooth-pasting condition. Since the equity holders choose when to declare bankruptcy, the condition will be related to the slope of the equity function. Differentiating the equity value-matching condition on both sides, we conclude bankruptcy is optimal when

$$\frac{\partial}{\partial\xi}E_0(d_0\xi_0;\xi_0) = 0. \tag{5.8}$$

We can then substitute (5.6) and (5.7) into the smooth-pasting condition and solve for the bankruptcy threshold to obtain

$$d_0\xi_0 = \frac{C_0\beta_2(r-\mu)}{r(\beta_2-1)}. \tag{5.9}$$

By calibrating the model parameters and taking the coupon rate as given, we can depict the firm value as a function of the selected coupon. Figure 5.1 illustrates the firm value as a function of the coupon rate for a given set of parameters. The function for the firm value[9] has a clearly indicated maximisation point, which highlights the possibility of solving for an optimal coupon rate.

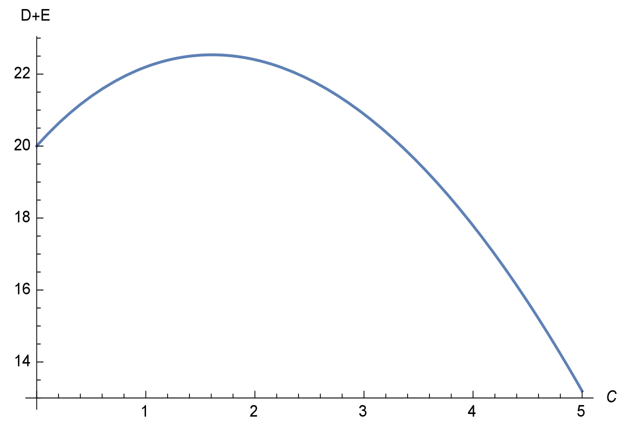


Figure 5.1: *Firm value as a function of coupon.* See Table 5.1 for base case parameter values. (Source: Own contribution)

Before deriving a solution, it is convenient to impose the simplifying assumption that the EBIT process starts at $\xi_s = 1$. Utilising the positive homogeneity property, we can define the initial debt and equity values as

$$D_0(\xi_s;\xi_s) = \xi_s D_0(1;1) \tag{5.10}$$

$$E_0(\xi_s;\xi_s) = \xi_s E_0(1;1). \tag{5.11}$$

Recognising that the firm incurs an issuance cost q when issuing new debt, the initial firm value can then be characterised as

$$A_0 = E_0(1;1) + (1-q)D_0(1;1). \tag{5.12}$$

We assume that the manager wishes to maximise the initial firm value as defined in equation (5.12). The manager's decisions thus amounts to choosing an optimal coupon:

$$C_0^* = \max_{C_0 \in \mathbb{R}_+} A_0. \tag{5.13}$$

With a closed-form system, the maximisation problem simply amounts to differentiating the firm value function with respect to C_0 and solving for the optimal coupon. This yields

[9]Note than firm value is calculated accounting for issuance costs: $FV = D(\cdot)(1-q) + E(\cdot)$. The notation is compressed in graphical output.

$$C_0^* = \frac{r(\beta_2 - 1)\left[1 - \frac{(q-1)\beta_2(\alpha(\tau_e - 1) - \tau_e + \tau_i}{q(\tau_i - 1) + \tau_e - \tau_i}\right]^{\frac{1}{\beta_2}}}{\beta_2(r - \mu)}. \tag{5.14}$$

Subsequently, the optimal coupon can be inserted into the expressions for the constants to define the values of debt and equity. This will determine the optimal capital structure under the optimised coupon with no possibility to renegotiate debt.

5.1.2 The Option to Renegotiate

In the previous subsection we obtained a solution for the case with no remaining renegotiation options. Hence, we can proceed to solve for the case with one remaining renegotiation option. We can use the results from Section 5.1.1 to determine the agents' disagreement payoffs when one renegotiation attempt remains. In theory, we could continue this procedure for an infinite amount of renegotiation options. However, most of the interesting model features will be captured by solving for the case with one remaining option. We therefore limit the analysis to this specific case[10].

As a first step, we must consider what the alternatives to a successful renegotiation attempt are. If the agents fail to renegotiate the debt, the equity holders will be faced with a dilemma. One alternative is to continue paying the suboptimal coupon even though EBIT is now at a lower point. To evaluate the equity claim under this scenario, we compute its value at the bankruptcy threshold under suboptimal coupon and zero renegotiation options. We perform a very similar derivation to that of the preceding section. Making use of the homogeneity property, we can value this claim as one that would have started at the point $\xi_0 = C_1/C_0^*$. The equity claim would have a general solution corresponding to

$$E_0(\widetilde{d}_0\xi_0; C_1/C_0^*) = e_{21}\xi_t^{\beta_1} + e_{22}\xi_t^{\beta_2} + \frac{\xi(1-\tau_e)}{r-\mu} - \frac{C_1(1-\tau_e)}{r}. \tag{5.15}$$

The debt claim in a situation of continued coupon payments would have a general solution given by

$$D_0(\widetilde{d}_0\xi_0; C_1/C_0^*) = d_{21}\xi_t^{\beta_1} + d_{22}\xi_t^{\beta_2} + \frac{C_1(1-\tau_i)}{r}. \tag{5.16}$$

[10]We refer to Christensen et al. (2002) for an analysis of the situation with $n > 1$ renegotiation options.

Solving for the values of the claims is again going to involve determination of the constants by using the same value-matching conditions as in the previous case with no possibility of restructuring. The sole difference is that EBIT has deteriorated and that the firm operates under a suboptimal coupon, C_1, that was determined when one renegotiation option remained. The scaling factor of the bankruptcy threshold under this coupon and EBIT level is denoted by $\widetilde{d}_0$. For conciseness, we can denote the equity value in case of continued coupon payments by

$$E_1^c = E_0(\widetilde{d}_0\xi_0; C_1/C_0^*). \tag{5.17}$$

The alternative to continue paying coupons is to cease the coupon payments. This will allow the creditors to enforce an immediate liquidation of the firm. The equity claim will then assume the value

$$E_1^b = 0. \tag{5.18}$$

The equity holders will attempt to maximise the value of their own claim. Their dilemma can thus be summarised accordingly:

$$E_1^r = max\left[E_1^c, E_1^b\right]. \tag{5.19}$$

The value derived by the creditors will be contingent on the choice of the equity holders. Under a scenario of continued debt servicing, the value of the debt contract will be given by

$$D_1^c = D_0(d_0\xi_0; C_1/C_0^*). \tag{5.20}$$

On the other hand, the value of the debt contract in case of liquidation will be the remaining value of the unlevered firm at bankruptcy less bankruptcy costs. This condition can be denoted

$$D_1^b = \frac{(1-\alpha)d_1\xi_0(1-\tau_e)}{r-\mu}. \tag{5.21}$$

The value of the debt contract can then be summarised as

$$D_1^r \begin{cases} D_1^c, & \text{for} \quad E_1^c \geq E_1^b \\ D_1^b, & \text{for} \quad E_1^c < E_1^b. \end{cases} \tag{5.22}$$

The attentive reader will notice that the notation is slightly altered from that of the Christensen et al. (2014) model in Section 4.4. More specifically, the bankruptcy conditions E_1^b and D_1^b do not contain any minimum or maximum statements. This stems from the fact that we simply assume that bankruptcy will always leave the equity holders with zero proceeds, whereas debt holders will take over the remnants of the unlevered firm. On the contrary, Christensen et al. (2014) account for the possibility that value will be left to equity claimants even after senior claim holders have received the full value of their claims in a small fraction of bankruptcy proceedings.

With the above notation established, we can identify an expression for the joint gain following a successful renegotiation attempt. Intuitively, the total gain will be the value of the optimally levered firm at the lower boundary with no remaining renegotiation options less the value of the firm if the renegotiation proposal had been rejected. The joint gain can then be formalised as [11]

$$R_1 = A_0 d_1 \xi_0 - (E_1^r + D_1^r). \tag{5.23}$$

The above expression implies that the explicit form for the value-matching condition will differ depending on the relative level of the renegotiation threshold. If $E_1^c > E_1^b$, it implies that $d_1 < \widetilde{d}_0$. This would suggest that the value function for equity under one remaining renegotiation option makes it optimal to wait *even* longer than when no renegotiation option remains before initiating renegotiation. Conversely, if $E_1^c < E_1^b$, the bankruptcy thresholds would be located such that $d_1 > \widetilde{d}_0$, and it would then be optimal to declare bankruptcy at a higher level of EBIT when one option remains. A last possibility would be that $E_1^c = E_1^b$ and the bankruptcy thresholds are located at the same level of EBIT, $d_1\xi_0 = \widetilde{d}_0\xi_0$. It would seem intuitive that $d_1 = \widetilde{d}_0$, but this can of course only be accepted by considering all scenarios in turn. Thus to avoid unsubstantiated exclusion, we investigate all three cases and verify the correct solution. First, we must define the share of the renegotiation gain attributable to the equity holders. The fraction is defined by an exogenous bargaining power, $\gamma \in [0, 1]$. This division is again reminiscent of a reduced-form NBS. The lower boundary value-matching condition for equity can be written on the following form:

$$E_1(d_1\xi_0; \xi_0) = \gamma R_1 + E_1^r. \tag{5.24}$$

Since the value of the debt claim depends on the choices of the equity holders, we can define it in a similar manner:

[11] Note that the value of A_0 was obtained from the setting with no renegotiation options.

$$D_1(d_1\xi_0;\xi_0) = (1-\gamma)R_1 + D_1^r. \tag{5.25}$$

In the base case it is assumed that $\gamma = (1-\gamma) = 0.5$. As discussed in Chapter 4, this can be attributed to risk-neutrality of the bargaining agents, but will hold in any case where the agents exhibit equal risk-aversion. Although this equality could be discussed from the perspective of typical investor risk profiles, we will simply assume risk-neutrality in our base case.

The applicable lower boundary smooth-pasting condition is found by differentiating the equity value-matching condition on both sides. Hence, it can be defined as a contingency on the relative levels of the lower boundaries as well. In general, it always holds that the smooth-pasting condition equals

$$\frac{\partial E_1(d_1\xi_0;\xi_0)}{\partial \xi} = \frac{\partial(\gamma R_1 + E_1^r)}{\partial \xi}. \tag{5.26}$$

To determine the relative level of d_1, we can plot the lower boundary smooth-pasting condition as a function of the bankruptcy threshold. There will be discontinuity in the function at the point where $d_1 = \widetilde{d}_0$. If the jump is located such that no solution for $\frac{\partial E_1(d_1\xi_0;\xi_0)}{\partial \xi} = \frac{\partial(\gamma R_1 + E_1^r)}{\partial \xi} = 0$ exists, we can conclude that $d_1 = \widetilde{d}_0$ is a viable solution. If there is a point where the smooth-pasting condition crosses the horizontal axis, we can see that there exists a solution where d_1 is either larger or smaller than $\widetilde{d}_0$. It will be necessary to calibrate the model numerically in order to plot the smooth-pasting condition. We refer to Table 5.1 for the selected numerical values of the parameters. Figure 5.2 illustrates the smooth-pasting condition for the calibrated parameter values.

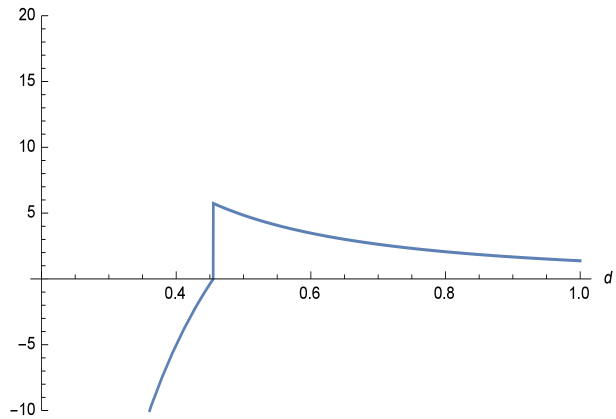


Figure 5.2: *Smooth-pasting condition.* Parameter values are defined in Table 5.1. (Source: Own contribution)

It becomes apparent from the figure that the discontinuity is in fact located such that $d_1 = \widetilde{d}_0$. Hence, the equity holders will initiate renegotiation at exactly the bankruptcy level of EBIT under coupon C_1 and zero renegotiation options. This follows intuitively from the need for credibility of their threat before starting negotiations. The applicable smooth-pasting condition will be $\frac{\partial E_1(d_1\xi_0;\xi_0)}{\partial \xi} = \frac{\partial(\gamma R_1 + E_1^C)}{\partial \xi}$.

The equity and debt claims in this period $E_1(d_1\xi_0;\xi_0)$ and $D_1(d_1\xi_0;\xi_0)$ are again defined by functions similar to (5.15) and (5.16). With all necessary conditions established, we can maximise the firm value by selecting an optimal coupon. This in turn determines the optimal capital structure when one renegotiation option remains, which completes the solution procedure.

5.1.3 Numerical Results for the Benchmark Model

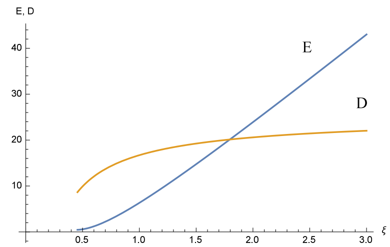
This section calibrates the GBM-based model that has been developed in Section 5.1. The numerical results will be analysed along two lines. Firstly, we assess intra-model performance by examining sensitives to exogenous parameters. Secondly, we compare the results to the case with no possibility of debt renegotiation. In this manner, we identify the general implications of the model as well as isolate the specific impact of debt renegotiation. The base case parameter values are defined in Table 5.1 below.

	Symbol	Value
Drift of EBIT	μ	2%
Risk-free rate	r	5%
Volatility of EBIT	σ	30%
Effective dividend tax rate	τ_e	40%
Tax rate on interest payments	τ_i	20%
Bankruptcy cost	α	10%
Issuance cost of debt	q	3%
Call premium	λ	5%
Equity holders' bargaining power	γ	50%

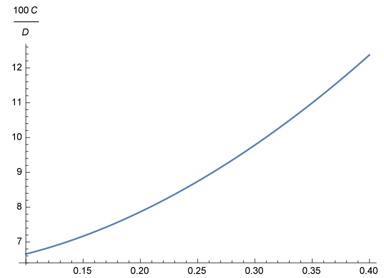
Table 5.1: *Base case parameter values applied to sensitivity testing.*

In Figure 5.3 we conduct sensitivity analysis of a number of metrics with respect to selected parameters. Chart 5.3a illustrates how the values of debt and equity change with EBIT. Debt converges to its risk-free value as EBIT grows, whereas equity becomes linearly increasing in EBIT. These findings are in line with the expected functional forms of debt and equity. Note also that the equity function equals its value-matching condition at the lower boundary and thus never takes on a value of zero. This is contrasted to the models with no renegotiation option.

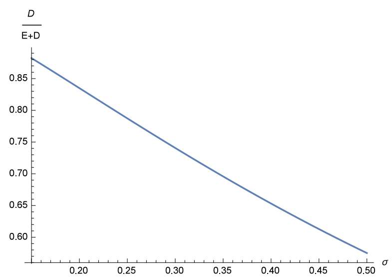
The bond yield as a function of the volatility of EBIT is depicted in Figure 5.3b. Comparing this figure with that of the Leland (1994) model (Chart 3.3b), we easily see that our model implies higher credit spreads. This prediction is satisfactory, as the prediction of too low credit spreads has long been a problem in classical static capital structure models. The result is also in concurrence with earlier models of dynamic re-optimisation such as Goldstein et al. (2001) and Christensen et al. (2014). The implications of earnings volatility is further examined in Figure 5.3c. We can see how the optimal debt-to-value ratio decreases linearly in volatility. The economic reason for this behaviour is that both gearing and earnings volatility increase the risk of bankruptcy for a given earnings level. Hence, increases in one must come at the expense of the other in order to maintain the firm's risk policy. Lastly, we find that the optimal leverage is strictly increasing in effective corporate tax rate. This relationship is depicted in Figure 5.3d and stems from the familiar effect of increasing tax advantage to debt.



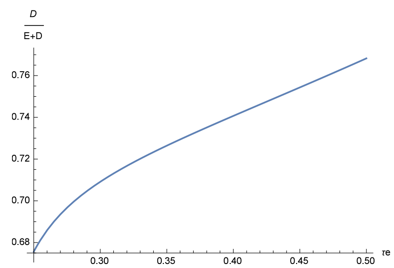
(a) Value functions and EBIT level



(b) Bond yield and EBIT volatility



(c) Leverage and EBIT volatility

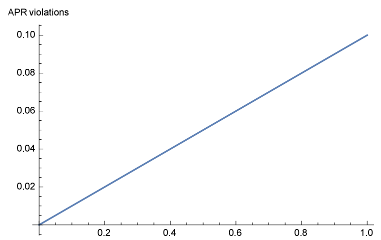


(d) Leverage and effective tax rate

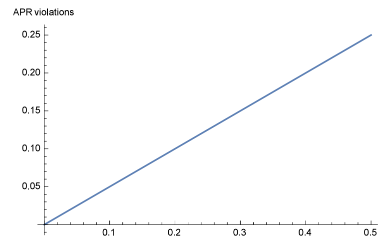
Figure 5.3: *Sensitivity analysis of the benchmark model.* See Table 5.1 for parameter values. (Source: Own contribution)

Another interesting aspect that arises in the presence of debt renegotiation is the existence of APR violations. We define APR violations as the share of the firm value that is attributable to the equity holders at the lower boundary. The level of APR violations is thus measured as $\frac{E_1(d_1\xi_0;\xi_0)}{E_1(d_1\xi_0;\xi_0)+D_1(d_1\xi_0;\xi_0)}$. The two variables that are most interesting to examine in the light of APR violation are bargaining power of equity holders and bankruptcy costs. In Figure 5.4d it is illustrated how APR violations are linearly increasing in equity bargaining power γ. This result is much expected due to the model specification. The exogenous bargaining power directly affects the share of the restructuring gain that is attributable to the equity holders. APR violations are non-existent when the equity holders have zero bargaining power. With maximum bargaining power, we can observe APR violations in the magnitude of 10% in this model. The effect of bankruptcy costs on APR violations is large and strictly increasing. As bankruptcy costs increase from 0% to 50%, we see in Figure 5.4b that the degree of APR deviations increases from 0% to 25%. We can thus infer that the existence of bankruptcy costs is critical in order for equity holders to be able to extract positive surplus from the creditors.

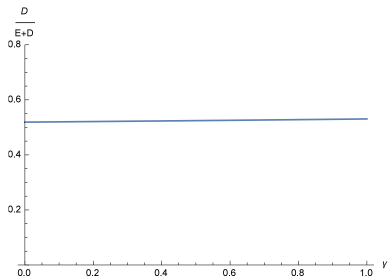
Figure 5.4 also shows how the recovery rate of debt holders varies with equity holder bargaining power and bankruptcy costs. We see that both variables have



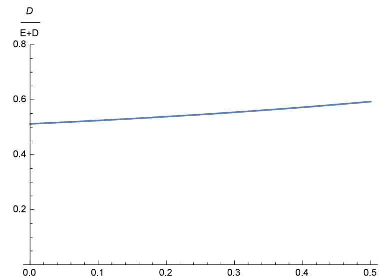
(a) APR violation and bargaining power



(b) APR violations and bankruptcy cost



(c) Recovery rate and bargaining power



(d) Recovery rate and bankruptcy costs

Figure 5.4: *Bargaining power and bankruptcy cost.* APR violation is defined as the share of the firm value that is attributable to equity holders at the bankruptcy threshold. Recovery rate is defined as the share of the principal that is reclaimed by debt holders in case of bankruptcy. See Table 5.1 for parameter values. (Source: Own contribution)

positive relationships to the recovery rate. Bargaining power however has a more limited effect than bankruptcy costs.

We present a more detailed comparative statics analysis in Table 5.2. Each exogenous variable is analysed for levels below and above the assumed base case specifications. All other variables are held fixed in order to isolate the analysed parameter's effect on key metrics. Observably, leverage and firm value are mostly sensitive to changes in the drift rate. Under a zero-drift assumption, the firm value is decreased sharply, whereas the leverage rate remains at a reasonable level. Debt and equity, thus, seem to deteriorate equally much in this case. However, when the drift rate is increased to four percent, the equity value increases dramatically. The total firm value jumps to a value of 66 and leverage decreases to 44%. This signifies that equity is strongly dependent on the long-term drift. A related observation is that a high drift rate decreases the TAD, which suggests an accelerating effect on the drop in optimal leverage. Under mean-reversion in EBIT, the positive drift is bound to disappear. Consequently, our observations in this section suggest strong implications for leverage and firm value when introducing mean-reversion in Chapter 6.

Regarding other relations, the bond yield mainly responds to volatility, the interest rate environment and changes in taxes, which appears reasonable. APR violations are affected by changes in the equity holder's bargaining power as well as the bankruptcy costs. The assumed level of risk-free rate appears to have rather large implications for the optimal coupon, the firm value and the bond yield. Lastly, issuance costs have only minor effects on the key metrics. Its chief influence is observed with respect to the TAD as well as the recovery rate.

	d	**C***	**FV**	**L***	**BY**	**RR**	**APRV**	**TAD**
$r = 4\%$	0.44	2.03	33.95	72.42%	7.43%	51.49%	5.00%	13.15%
$r = 6\%$	0.47	1.39	17.08	73.17%	10.33%	53.34%	5.00%	13.85%
$\sigma = 25\%$	0.49	1.47	22.71	72.82%	8.89%	52.50%	5.00%	13.53%
$\sigma = 35\%$	0.43	1.78	22.41	71.34%	11.11%	48.38%	5.00%	12.06%
$\mu = 0\%$	0.44	1.04	13.46	71.47%	10.80%	48.78%	5.00%	12.20%
$\mu = 4\%$	0.23	2.20	65.86	44.43%	7.53%	42.53%	5.00%	9.76%
$\alpha = 5\%$	0.46	1.65	22.60	73.62%	9.92%	50.18%	2.50%	13.02%
$\alpha = 15\%$	0.46	1.57	22.47	70.26%	9.92%	50.18%	7.50%	12.35%
$q = 0\%$	0.48	1.70	23.05	74.75%	9.86%	53.05%	5.00%	15.23%
$q = 6\%$	0.42	1.50	22.04	68.47%	9.94%	47.24%	5.00%	10.22%
$\gamma = 25\%$	0.46	1.65	22.60	73.62%	9.92%	50.18%	2.50%	13.02%
$\gamma = 75\%$	0.46	1.57	22.47	70.26%	9.92%	50.18%	7.50%	12.35%
$\tau_e = 30\%$	0.42	1.47	24.34	65.84%	9.18%	54.22%	5.00%	4.30%
$\tau_e = 50\%$	0.47	1.65	20.75	75.54%	10.52%	44.38%	5.00%	24.48%
$\tau_i = 10\%$	0.46	1.64	24.01	74.40%	9.18%	46.43%	5.00%	20.06%
$\tau_i = 30\%$	0.43	1.51	21.07	67.20%	10.65%	53.88%	5.00%	5.34%

Table 5.2: *Detailed sensitivity analysis of the benchmark model.* d = renegotiation threshold, C^* = optimal coupon, FV = firm value, L^* = optimal leverage, BY = bond yield, RR = recovery rate, APRV = APR violations, TAD = tax advantage to debt.

Comparison to a Static Setting

To better isolate the impact of introducing a scope for debt renegotiation, we compare the results to the ones obtained in a case with immediate bankruptcy. Thus; we can calibrate a model derived according to the procedure in Subsection 5.1.1. Essentially, we compare the case when no option to renegotiate exists to the case when one renegotiation option remains. The analysis is presented in Table 5.3 and assumes exogenous parameter values as in Table 5.1.

As a result of the renegotiation option, management selects a higher coupon and assumes more leverage. The riskier profile is motivated by the opportunity to avoid bankruptcy if earnings deteriorate. The tax advantage to debt is slightly boosted. The bond yield increases, as we would expect when comparing to a static model similar to Leland (1994). Management chooses a higher nominal value of coupon. This will demand a higher minimum EBIT level and raise the bankruptcy threshold. Consequently, the debt issued from the firm becomes riskier and rises in value proportionally less than the coupon, which increases the bond yield. Renegotiation implicates the appearance of APR violations, but at the same time the expected recovery rate of debt holders increases. This stems from the fact that both bargaining agents receive a positive renegotiation gain.

	No renegotiation	Renegotiation
Bankrupt./Reneg. threshold	0.39	0.46
Optimal Coupon	1.44	1.61
Firm value	22.27	22.53
Leverage	67.90%	71.94%
Bond yield	9.50%	9.91%
Recovery rate	46.02%	50.18%
APR Violations	0%	5.00%
Tax advantage to debt	11.34%	12.68%

Table 5.3: *The impact of debt renegotiation.* See Table 5.1 for base case parameter values.

5.2 Extension with Callable Debt

In this section we extend the model of the previous section to feature callability of debt at some upper boundary. This means that equity holders are offered the opportunity to restructure the firm's debt if EBIT reaches a sufficiently high level. As in Goldstein et al. (2001), restructuring is done by calling the firm's entire outstanding debt and then issuing new callable bonds. Admittedly, this is an expensive mode of levering up, but it offers the significant advantage of not dealing with different debt classes. A potential extension with several debt classes is instead examined in Section 6.6. In developing this model, we will take the specifications from the preceding section as given.

In theory, it is rather simple to add an upper boundary to the model developed in Section 5.1. However, the computational effort involved increases significantly. Consequently, this enhanced model is best presented under the less complex GBM-based paradigm.

5.2.1 Boundary Conditions

The values of the debt and equity claims are still given by the equations (5.2) and (5.3) when no renegotiation options remains. In the previous model, we set the constants e_{01} and b_{01} to zero in order for the expressions not to explode as EBIT grows. With both an upper and a lower boundary this restriction no longer applies. The value-matching conditions for equity and debt enable us to determine all the constants e_{01}, e_{02}, b_{01} and b_{02}. In order to be able to solve for two endogenous boundaries, we also find it necessary to change the assumption regarding the lower value-matching conditions from Section 5.1. We previously assumed that the firm's assets were sold off at their unlevered value in case of bankruptcy. However, in the computations we perform, we are unable to obtain satisfactory convergence when the upper boundary is solved for endogenously under this assumption. This finding is especially interesting considering that Goldstein et al. (2001) choose an exogenous upper boundary under the same assumption. As discussed in Section 3.2.2, this choice might stem from a similar issue in solving for two endogenous boundaries under their imposed assumptions. For our purposes, this means that the firm's assets are taken over as a going concern in case of bankruptcy. This new assumption is in line with that made in Christensen et al. (2014).

One drawback is that the alternation introduces some noise when assessing the impact of callability on capital structure. Additionally, the problem becomes computationally more heavy to solve. Notwithstanding these issues, the bankruptcy value conditions under the new assumption are

$$D_0(d_0\xi_0;\xi_0) = (1-\alpha)A_0 d_0 \xi_0 \tag{5.27}$$

$$E_0(d_0\xi_0;\xi_0) = 0 \tag{5.28}$$

$$D_1^b = (1-\alpha)A_0 d_1 \xi_0 \tag{5.29}$$

$$E_1^b = 0. \tag{5.30}$$

When debt is called, we assume that the firm will have to pay a proportional call premium λ. The call premium compensates the bond holders for an early termination of their contract. In this model the call premium, together with the issuance cost, deter management from continuous restructuring of the firm. This is an important friction - especially since we do not assume a limited amount of upper restructuring options. The problem of early recapitalisation and optimal call premium is discussed in more detail by Fischer et al. (1989b). Implementing the call premium, we have that the value of the original debt contract at the call boundary equals

$$D_0(u_0\xi_0;\xi_0) = (1+\lambda)D_0(\xi_0;\xi_0), \tag{5.31}$$

where $u_0\xi_0$ denotes the EBIT value at the call boundary. Furthermore, the value of the equity contract will equal the total firm value at the upper boundary less the price of calling existing debt. Formally, we have

$$E_0(u_0\xi_0;\xi_0) = A_0u_0 - (1+\lambda)D_0(\xi_0;\xi_0). \tag{5.32}$$

The nature of these value-matching conditions will essentially remain unchanged when we switch to the state where one renegotiation option exists. To be explicit, we can state the conditions in this situation as

$$D_1(u_1\xi_0;\xi_0) = (1+\lambda)D_1(\xi_0;\xi_0) \tag{5.33}$$

$$E_1(u_1\xi_0;\xi_0) = A_1u_1 - (1+\lambda)D_1(\xi_0;\xi_0). \tag{5.34}$$

As in previous cases, the smooth-pasting condition is found by differentiating the equity value-matching conditions on both sides with respect to $u_n\xi_0$. We have that the upper boundary smooth-pasting condition is

$$\frac{\partial}{\partial u_n\xi_0}E_n(u_n\xi_0;\xi_0) = A_n \tag{5.35}$$

where n denotes the number of remaining renegotiation options.

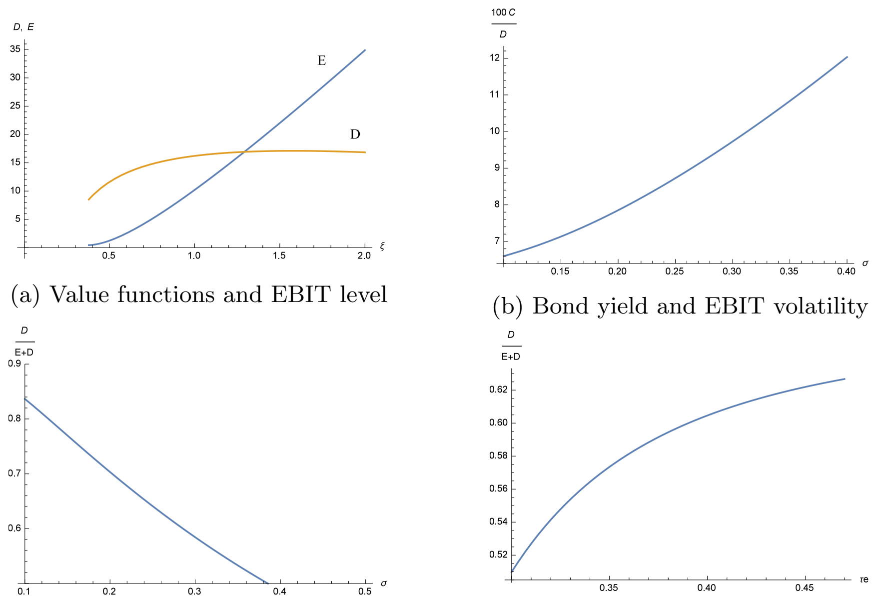
5.2.2 Solving the Model

The solution to this model is found in the same manner as in the previous model with non-calable debt. The first step is to solve for the case where no renegotiation options remains. Using the system of equations composed by (5.27), (5.28), (5.32), (5.31), (5.33), (5.34), (5.35) and (5.8) we can determine all constants and the optimal bankruptcy and call thresholds for the case with zero renegotiation options. We can also determine the value of the claims under suboptimal coupon, C_1, with zero remaining renegotiation options. This establishes the disagreement payoffs to the claimants in the bargaining game when one renegotiation options remains. Lastly, the optimal capital structure when one option remains can be completed in the same way as in Section 5.1.2, but adding the upper boundary conditions (5.33), (5.34), (5.35), and the new expressions for D_1^b and E_1^b. We of course still have to investigate the analytical form of the smooth-pasting condition due to the maximisation statement present in the equity value-matching condition. The optimal capital structure is found by maximising the firm value over an optimal coupon. This completes the solution procedure for the model featuring callable debt as well as debt renegotiation.

5.2.3 Model Performance

In this subsection we analyse the implications of the model with callability. As discussed in Section 5.2.2, the numerical comparison with the benchmark model will be slightly affected by the changed assumption about the firm's going concern value at bankruptcy. Despite this small violation of the *ceteris paribus* principle, comparison is meaningful - especially from a general model selection perspective.

In Figure 5.5 we conduct a sensitivity analysis for the same base parameters as in Figure 5.3. Evidently, the value functions of debt and equity still behave soundly. The equity function smooth-pastes at a slightly lower EBIT value than in Figure 5.3a. This indicates that the renegotiation threshold is moved further down in the extended model. The bond yield, as depicted in Figure 5.5b, has a similar functional relationship with volatility to that seen in Figure 5.3b with a marginally lower sensitivity. As before, leverage in 5.5a is negatively related to volatility. However, it decreases more in response to increasing volatility compared to the benchmark model. The leverage as a function of the effective tax rate still has a positive slope but diminishes at a higher marginal rate. Regarding bankruptcy costs and the equity holders' bargaining power, we find very similar relationships to those obtained in the benchmark model.



(a) Value functions and EBIT level

(b) Bond yield and EBIT volatility

(c) Leverage and EBIT volatility

(d) Leverage and effective tax rate

Figure 5.5: *Sensitivity analysis of the model with callable debt.* See Table 5.1 for parameter values. (Source: Own contribution)

To thoroughly examine the implications of the enhanced model, we also conduct an analysis of the key metrics of both models. Table 5.4 indicates the numerical results obtained in the models under the base case parameters in Table 5.1. As seen in the sensitivity analysis, the renegotiation threshold is lowered in the model with callable debt. In other words, it is optimal for equity holders in the model with callable debt to wait a bit longer than they would have done in the benchmark model before initiating renegotiation. This can be explained by the fact that equity holders now have the option to lever up at a later point in time. Thus; they choose a lower amount of debt initially, which implies reduced coupon payments to cover. The optimal renegotiation threshold will consequently be at a lower level of EBIT. This explanation is confirmed by the values in Table 5.4.

The firm is initially slightly more valuable when a call feature exists, which can be attributed to a higher equity value. The value of the debt contract only diminishes marginally. Moreover, we see a higher magnitude of the tax advantage to debt under the new regime. This result aligns with the results reported by Christensen et al. (2014) who obtain an increase in the TAD of 50% in comparison with the findings of Goldstein et al. (2001). Note however also that the assumption of a symmetric tax schedule in our setup provides for a larger TAD.

Lastly, the bond yield falls slightly. This effect results from the lower initial value of coupon. The lower coupon means that the bankruptcy threshold of EBIT falls such that the firm becomes less risky for a given level of initial debt.

	Benchmark Model	Callable Debt
Renegotiation threshold	0.46	0.38
Call threshold	-	1.81
Optimal Coupon	1.61	1.58
Firm value	22.53	25.93
Leverage	71.94%	60.66%
Bond yield	9.91%	9.44%
Recovery rate	50.18%	54.07%
APR Violations	5.00%	5.00%
Tax advantage to debt	12.68%	31.50%

Table 5.4: *Cross-model comparison of key metrics.* See Table 5.1 for base case parameter values.

5.3 Concluding Remarks

In this chapter, we have developed a GBM-based model of dynamic capital structure with debt renegotiation in two stages. In the first stage, the equity holders have the option to renegotiate their capital structure instead of going bankrupt. The renegotiation game is explicitly modelled in a way similar to the bargaining game introduced by Rubinstein (1982). In the second stage we allow the firm to issue callable debt. We obtain a solution to this extended model following a similar logic as in the basic one.

In the model with non-callable bonds and debt renegotiation we obtain a closed-form solution and identify a number of key relationships. The optimal leverage choice is mainly driven by the positive drift parameter, taxes, and volatility. The recovery rate and the size of APR violations are increasing in equity holder bargaining power as well as bankruptcy costs. The bond yield is mainly affected by volatility fluctuations in EBIT, changes in interest rates and taxes. The comparison with a static version of the model reveals that debt renegotiation increases leverage, enhances the tax-advantage and improves the recovery rate of debt holders. APR violations are a direct consequence of the possibility to renegotiate debt.

Once the analysis is extended to include callability of debt, we must use numerical methods to obtain a solution. Furthermore, the assumptions in the framework are slightly modified. The main quantitative implications of an added call feature are; decreased initial leverage and a higher tax advantage to debt. Additionally, we note that the optimal EBIT threshold for initiation of debt renegotiation is reduced in the model featuring callable debt.

The GBM-based model developed here is an excellent point of departure for further analysis. The assumption of earnings following a GBM is, however, mostly grounded on convenience, as detailed in Chapter 2. In reality, practitioners and academics alike probably do not expect a firm's earnings to drift constantly towards infinity or fluctuate in sync with a random-walk process. A more realistic stochastic earnings process would arguably be one reverting to some long-term mean level. With this motivation, we will carry the analysis of debt renegotiation and dynamic capital structure over to a framework with mean-reverting earnings in the next chapter. The game-theoretical infrastructure for debt renegotiation from Chapter 5 can be utilised directly under the new regime. Hence, Chapter 6 extends the analysis we have developed to a setting more aligned with the observed evolution of earnings.

Chapter 6

A GOU-based Model with Debt Renegotiation

> *"There is no more important proposition in economic theory than that, under competition, the rate of return on investment tends toward equality in all industries. Entrepreneurs will seek to leave relatively unprofitable industries and enter relatively profitable industries."*
>
> - George J. Stigler (1911 - 1991)

In this chapter we augment the model with debt renegotiation from Section 5.1 by letting EBIT follow a GOU process. We thus unify at least two strands of literature: We follow the recommendations of the research documenting the need for the application of mean-reverting processes in models with earnings as the guiding state variable. Moreover, we add to the current research of optimal capital structure optimisation by modelling a firm that can optimise under the possibility to restructure at a lower boundary of the state variable.

The chapter will proceed as follows. Firstly, we provide motivation, theoretical and empirical, for the imposition of mean-reversion on the EBIT process. Secondly, we illustrate how the ODE and general solutions for claims on the EBIT process change with the GOU modification. Thirdly, we solve the model under the alternative diffusion process and conduct extensive comparative statics analysis. We benchmark against the GBM version with downwards restructuring introduced in Chapter 5. Extensions and alternative modifications to the proposed model conclude the chapter.

6.1 Mean-Reversion in Earnings

As previously noted, the vast majority of dynamic capital structure models represent the state variable diffusion by a GBM. However, in terms of applying a GBM to the dynamics of earnings in a capital structure study, the mathematical properties of unbounded conditional mean and variance considered in Chapter 2 can be considered somewhat inconsistent with both economic intuition and empirical evidence. It is remarked by e.g. Bhattacharya (1978) and Sarkar and Zapatero (2003) that the imposition of a random walk property on corporate earnings is of limited economic relevance. They highlight that cash flows will intuitively revert around levels that make firms indifferent about making new investments in the particular opportunity that one given project represents. Put differently, a project can be expected to have a positive NPV when it earns an economic profit in product or factor markets. A natural corollary of this is that other firms will replicate the investment decision, eventually eroding the profit opportunity. This dynamic is well summarised by Nobel laureate George Stigler (1963) in the quotation introducing this chapter. Evidently the constant (positive) drift rate assumed in the GBM framework implies unconstrained upwards-drifting earnings. This is inconsistent with economic intuition of a certain long-term mean level, which Stigler alludes to. Furthermore, the GBM properties of unbounded variance and unstationary probability distribution seem unattractive once it is accepted that the conditional mean is bounded.

The empirical corroboration of introducing a GOU process as the evolution path of EBIT in dynamic capital structure frameworks can be found in an array of studies that find mean-reversion in corporate earnings. Here we will just include a few of the most significant contributions. Freeman, Ohlson and Penman (1982) find that book rates of return exhibit mean-reversion and function as a predictor variable for changes in earnings. Lipe and Kormendi (1987) and Easton and Zmijewski (1989) examine earnings in related contexts, finding that stock market reactions to profit warnings are greater when a firm's earnings show a higher degree of persistence. This is completed under the assumption that earnings are autoregressive and thus carries the implication that mean-reverting earnings is a more integrated notion in the general market sentiment *vis-à-vis* an extrapolative random walk process. Using a partial adjustment model, Fama and French (2000) estimate a rate of mean-reversion in earnings of about 38% for U.S. based firms with a sample time period from 1964-1996 and +2,000 firms sampled per year. The authors also find that the mean-reversion trend is somewhat non-linear with faster mean-reversion for below-mean and far-above-the-mean values. However, such findings still provide impetus for the use of GOU for dynamic earnings evolution under uncertainty, as they are facilitated by the conditional mean property in (2.13).

Following the logical arguments for mean-reversion and the empirical support for its presence, a number of theoretical models have been developed to capture the phenomenon. Pioneering the discussion, Bhattacharya (1978) develops a continuous-time valuation model for a firm with a GOU-based cash flow process. In a similar spirit, Raymar (1991) argues for a discrete-time model with mean-reversion in earnings. Both papers make simple predictions about how the speed of mean reversion is related to various valuation metrics. In a more recent contribution, Sarkar and Zapatero (2003) develop a model featuring mean-reversion in earnings and test their predictions on empirical samples. The applied stochastic process is, indeed, a GOU process and the empirical results generally support the predictions of the model. Finally, Titman and Tsyplakov (2002) consider a model where firm value is determined by earnings, which in turn is a function of exogenous price changes in the firm's product market. This price process is assumed to follow a mean-reverting process. As a consequence, the firm adjusts its leverage over time according to the firm value supported by the earnings capacity. While trivially finding evidence that a firm facing a below-mean price chooses a lower debt-to-value ratio, the authors also find that it chooses a higher debt coverage ratio. Intuitively this is due to the fact that such a firm should expect higher prices and - by extension - higher earnings in the future, giving it a higher debt capacity. Following this dynamic, both optimal leverage and earnings level will be expected to increase in expectation for a firm facing a below-mean price level.

In sum, the tractability of modelling earnings as mean-reverting has been established in both empirical and theoretical literature. The argument is constructed from fundamental economic principles of investments and competition. Approximating the theoretical and practical capital structure discussions, we couple mean-reverting earnings with a framework of dynamic capital structure that allows for debt renegotiation.

6.2 Derivation of the General Solution for a GOU Process

In this section we will examine the value of the firm as an investment opportunity under the modified stochastic process as well as the general solutions to the ODE satisfied by the firm value function. It is assumed that the EBIT stream follows a mean-reverting stochastic process with proportional volatility:

$$d\xi_t = \kappa(\theta - \xi_t)dt + \sigma \xi_t dW_t \tag{6.1}$$

where κ represents the speed of mean-reversion, θ is the long-term mean level and dW is the usual increment of a Wiener process. This is equivalent to the GOU process introduced in (2.12). As can be seen from the expression in (6.1), the process will converge to a GBM under two circumstances: *i)* if $\kappa = 0$ it becomes a GBM with zero drift, and *ii)* if $\theta = 0$ it becomes a GBM with drift equal to $-\kappa$. Thus; although the stochastic process has been altered, the log-normal earnings property of the GBM can still be re-established as a special case of the GOU process.

The assumption of the proportional variance is easily seen as a more realistic description of earnings volatility. Moreover, as we demonstrate below, an additional benefit is that the complexity of the obtained solutions will be significantly reduced. One economic issue, however, which persists under the defined diffusion process is that the earnings will be strictly positive at all times. This is a potentially important limitation, which is discussed further in Section 6.6.

6.2.1 Firm Value

Apart from the altered diffusion process, the general setup is carried over from the benchmark model developed in Chapter 5. Thus; we continue to consider a firm that issues perpetual debt with a continuous coupon C, such that claims on the earnings process are time-homogeneous.

Following Sarkar and Zapatero (2003), we can characterise the unlevered firm value as the equivalent of a firm that consists of one project and where earnings are not affected by leverage. In this simplistic setup we can define the firm value for all $s > t$ as the present value of a perpetual earnings stream:

$$V(\xi) = \mathbb{E}_t \left[\int_t^\infty \xi_s e^{-r(s-t)} ds \right] = \int_t^\infty \mathbb{E}_t[\xi_s] e^{-r(s-t)} ds. \tag{6.2}$$

The equality between the two expressions follows from Fubini's theorem. From (2.13) we know the conditional mean of ξ_s. Substituting for this we obtain

$$\begin{aligned} V(\xi) &= \int_t^{\infty} e^{-r(t-s)}[\theta + e^{-\kappa(s-t)}(\xi_t - \theta)]ds \\ &= \int_t^{\infty} \left[e^{r(t-s)}\theta + e^{(t-s)(r+\kappa)}(\xi_t - \theta)\right] ds \\ &= -\left[e^{r(t-s)}\frac{\theta}{r} + e^{(t-s)(r+\kappa)}\frac{\xi_t - \theta}{r+\kappa}\right]_t^{\infty} \\ &= \frac{\theta}{r} + \frac{\xi_t - \theta}{r+\kappa}. \end{aligned} \tag{6.3}$$

We immediately observe that the unlevered firm value has two components. The first term in (6.3) is a *permanent* component equal to the long-term mean parameter discounted in perpetuity. This term is unaffected by the fluctuations in the EBIT process. The second term is a *transitory* component whose value is a decreasing function of the speed of mean-reversion. This term catches the value effect of earnings deviations from the long-term mean. It can easily be seen that in the extreme case where $\kappa \to \infty$, this component disappears and θ becomes the deterministic EBIT level in every period.

6.2.2 Valuation of Claims

The general ODE satisfied by any claim on the process in (6.1) has a general form similar to the models developed in Chapter 3 and 5. Thus the value of a claim on the EBIT process following a GOU satisfies

$$\frac{1}{2}\sigma^2 F_{\xi\xi}(\xi) + \kappa(\theta - \xi)F_{\xi}(\xi) - rF(\xi) + \upsilon_H = 0. \tag{6.4}$$

The solution to (6.4) has a considerably more complex expression than the ones we have obtained previously with GBM-based state variables due to the implementation of the mean-reverting property. The general solution to the linear homogeneous part of the ODE is given by

$$k_1\xi^{\beta_1}M_1(\xi) + k_2\xi^{\beta_2}M_2(\xi), \tag{6.5}$$

where $M_1(\xi) = M(-\beta_1; 2-2\beta_1 + \frac{2\kappa}{\sigma^2}; \frac{2\kappa\theta}{\sigma^2\xi})$ and $M_2(\xi) = M(-\beta_2; 2-2\beta_2 + \frac{2\kappa}{\sigma^2}; \frac{2\kappa\theta}{\sigma^2\xi})$. The function $M = (\cdot;\cdot;\cdot)$ is the *confluent hypergeometric function* that has an infinite hypergeometric series given by

$$M(a;b;z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!} \tag{6.6}$$

where $(a)_n$ and $(b)_n$ are Pochhammer symbols used to denote rising factorials with $(a)_0 = 1$ and $(b)_0 = 1$[12]. Depending on the functional form, this series may have a known solution. β_1 and β_2 will be determined as the roots

$$\beta_{1,2} = \frac{2\kappa + \sigma^2 \pm \sqrt{8r\sigma^2 + (2\kappa + \sigma^2)^2}}{2\sigma^2} \tag{6.7}$$

to the quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + \kappa\beta - r = 0. \tag{6.8}$$

The polynomial in (6.8) is very similar to the fundamental quadratic used in the GBM-based models of previous chapters. The only difference is that the drift parameter, μ, has been exchanged for the parameter for the speed of mean-reversion, κ, in the multiplication with the root in the second term.

In our case, as previously, the upper and lower boundary conditions for the EBIT process will determine the constants k_1 and k_2. We omit the expression of the complete general solution to the inhomogeneous ODE (6.4). The inclusion of the dividend streams affine in earnings is relegated to the expressions for the debt and equity claims below.

6.2.3 Scaling Invariance

Before considering the claim values under the necessary boundary conditions, we include a comment on the property of homogeneity under the proposed GOU process. In Chapter 3 we discussed how homogeneity of degree one of the claims and the coupon in the EBIT process allowed the dynamics of the model to be unaltered in time. Put differently, the homogeneity implied a scaling invariance that allowed for transformation of the claim by some time-invariant factor. This meant, for example, that if the firm was restructured at the upper boundary and thus initiated with the EBIT value $\bar{\xi} = u\xi_0$, then the new boundary values would be $(u^2\xi_2, ud\xi_0)$. A related benefit of this scaling invariance, which is cited in most of the dynamic capital structure literature, is that the claim values are insensitive to changes in the underlying currency.

It turns out that the claims and coupon on the EBIT process under the GOU modification maintain this homogeneity property. The homogeneity is however extended under this setup such that the long-term mean level θ is scaled accordingly with earnings. This is easily illustrated by way of a simple example: Assume that

[12] We refer to Abramowitz and Stegun (1972) for extensive treatment of the confluent hypergeometric function and its related properties.

the earnings X of one firm exhibit mean-reversion to $\bar{X}$ and that the firm merges with an identical firm. We would naturally expect the joint earnings of these firms to revert to twice the earnings of the separate entities. By similar intuition, if the same firm has been recapitalised at some earnings level $\eta \cdot X$ where $\eta > 0$, its earnings will revert to $\eta \cdot \bar{X}$.

To formalise the intuition, suppose that X follows the generic mean-reverting stochastic process

$$dX_t = f(X_t, \bar{X})dt + g(X_t, \bar{X})dW_t. \tag{6.9}$$

Then for any $\eta > 0$, $Y_t \equiv \eta X_t$ should follow the process

$$dY_t = f(Y_t, \bar{Y})dt + g(Y_t, \bar{Y})dW_t \tag{6.10}$$

where $\bar{Y} = \eta\bar{X}$. We thus observe that the drift and diffusion of the process in (6.9) is homogeneous of degree one in the value of earnings and the long-term mean level. Equivalently, we can infer that the EBIT process in (6.1) is homogeneous of degree one in the pair (ξ, θ)[13].

6.3 The GOU-based Model

In this section, we set up the model with mean-reverting earnings and a lower restructuring boundary, equivalent to the setup assumed for the GBM-based model in Section 5.1. Thus; we restrict ourselves to the case with one remaining renegotiation option. As an intermediate step, we must again consider the scenario with zero remaining renegotiation options. In this case, the general setup will be comparable to the one assumed by Leland (1994) with two key differences: The state variable is represented by earnings rather than unlevered firm value, and is assumed to follow a GOU process rather than a GBM. Subsequently, we can proceed with the case where equity holders have a renegotiation option, similar to the framework introduced under a GBM in the previous chapter.

6.3.1 No Renegotiation

The setup for the solution to the step excluding debt renegotiation is effectively the same as in Section 5.1. However, the augmented stochastic process will alter the explicit forms of the value functions and the boundary conditions. Earnings

[13] See for example Bjerrisgaard and Fedoryaev (2011) for a formal proof of this lemma.

are taxed at τ_c, debt claims at τ_i and dividends at τ_d, such that the effective tax rate levied on firm owners is $\tau_e = \tau_c + (1-\tau_c)\tau_d$. Adding the particular cash flow solution to (6.5) we solve for the claims as

$$D_0(\xi_t;\xi_0) = b_{01}\xi_t^{\beta_1} M_1(\xi_t) + b_{02}\xi_t^{\beta_2} M_2(\xi_t) + (1-\tau_i)\frac{C_0}{r} \tag{6.11}$$
$$E_0(\xi_t;\xi_0) = e_{01}\xi_t^{\beta_1} M_1(\xi_t) + e_{02}\xi_t^{\beta_2} M_2(\xi_t) + (1-\tau_e)\left[\frac{\theta}{r} + \frac{\xi_t - \theta}{r+\kappa}\right] - (1-\tau_e)\frac{C_0}{r},$$

where $M_n(\cdot)$ denotes the confluent hypergeometric functions. Also, β_1 and β_2 denote the positive and negative roots of the fundamental quadratic equation, respectively.

We proceed by defining the boundary conditions to the setup with no possibility to restructure the firm. In the case of bankruptcy, the debt holders take over the unlevered firm after incurring the bankruptcy cost α. Equity holders are left with nothing. We define the value-matching conditions accordingly:

$$\begin{aligned} D_0(d_0\xi_0;\xi_0) &= (1-\alpha)(1-\tau_e)\left[\frac{\theta}{r} + \frac{d\xi_0 - \theta}{r+\kappa}\right] \\ E_0(d_0\xi;\xi_0) &= 0. \end{aligned} \tag{6.12}$$

With non-callable debt, the bond instrument will become risk-free as earnings grow to infinity, and the value of debt will equal the perpetuity coupon stream. By similar logic, the equity claim is defined by the perpetual dividend stream. The limit conditions are

$$\begin{aligned} E_0(\infty;\xi_0) &\to (1-\tau_e)\left[\frac{\theta}{r} + \frac{\xi - \theta}{r+\kappa}\right] - (1-\tau_e)\frac{C_0}{r} \\ D_0(\infty;\xi_0) &\to (1-\tau_i)\frac{C_0}{r}. \end{aligned} \tag{6.13}$$

To comply with the conditions in (6.13), we set the constants belonging to the negative root equal to zero. Moreover, we solve for the remaining constant by subjecting the value functions to the bankruptcy boundary conditions.

In order to identify the bankruptcy boundary, we subsequently define the smooth-pasting condition by differentiating the equity value-matching condition on both sides:

$$\frac{\partial E_0(d_0\xi_t;\xi_t)}{\partial \xi_t} = 0. \tag{6.14}$$

By inserting the constants into the smooth-pasting condition, we obtain an analytical expression. The solution procedure after this is exactly equivalent to that of Section 5.1. However, it is not possible to define closed-form solutions for the optimal coupon and bankruptcy threshold. A complete solution therefore requires the use of numerical techniques.

6.3.2 Including Renegotiation of Debt

The inclusion of a debt renegotiation option requires that we set up an explicit game. The disagreement payoffs from the game can be defined from the case of no renegotiation. The procedure is exactly equivalent to that of Section 5.1. However, the algebraic forms for the boundary conditions will of course differ. The applicable smooth-pasting condition must also be investigated due to the conditionality of the equity value-matching condition. Due to the similarity between the game settings, we omit the full notation of the bargaining structure and instead proceed to numerical analysis.

6.4 Comparative Statics and Cross-Model Comparison

In this section we conduct extensive comparative analysis in order to compare the performance of our GOU-based model *vis-à-vis* the GBM-based model of Section 5.1. We will aim at stressing both the analysis of the sensitivity to the various base case parameters of the model internally, as well as a cross-comparison with our benchmark setup. This structure allows us to illustrate the general model implications as well as the specific impact of mean-reversion in earnings. The analysis is followed by a section summarising key findings. Before conducting numerical analysis, we however consider some important preliminary remarks in relation to the interpretation and computation of our results.

6.4.1 A Comment on Comparative Statics

In the following part we examine empirical implications of our model results. When interpreting comparative statics such as the ones we produce, a few notes are worth considering. Strebulaev (2007) suggests some words of caution to keep in mind in empirical analysis of dynamic capital structure models, based upon observations of how these tests are typically carried out. He makes the noticeable comment that in any cross section firms are at various different stages of their refinancing cycle with very few being at or close to the theoretical refinancing point *date zero* that virtually all capital structure models initiate the firm at. This is because firms in a dynamic economy with frictions do not refinance as frequently as such an assumption would require in practice. This cyclicality arises because firms with different leverage and accumulation of financial decisions react differently to economic shocks, even if they are identical from a date zero perspective. As a consequence, even if firms do in fact follow a capital structure policy that aligns closely with the one proposed in any given dynamic model, a large discrepancy between optimal and actual leverage would still persist when conducting cross-sectional tests across a sample of multiple firms.

Evidently, the observations of Strebulaev do not just apply to the comparative statics considered in our model. If for example one were to consider comparative statics for any of the GBM-based models of the previous chapters, the caveat proposed would apply equally well. Thus; emphasising the recommendations proposed in Strebulaev (2007), empirical tests carried out on the results predicted by our model should take into account the firms' positioning in their refinancing cycle as well as the differing historical information for these firms arising from their different initiation points.

Procedure for Computations with Hypergeometric Functions

In addition to the general interpretation of the comparative statics, the computational procedure applied to problems involving hypergeometric functions further warrants a comment. The computation of such functions is a fairly intricate exercise. The non-trivial structure of the hypergeometric series that defines the function gives rise to an array of precision issues, which are more pronounced for certain ranges of the parameters. Thus; some numerical instability is an almost inevitable issue in most computations, except for the most elementary functions.

For this reason, all software programmes with multi-precision evaluator features to handle hypergeometric functions have some drawbacks. The main issue with *Mathematica*, which is the computation programme used for this dissertation, is related to convergence of the algorithm in root-finding and maximisation procedures for non-linear equation systems[14]. More specifically, the default method for root-finding and maximisation (FindRoot[] and FindMaximum[], respectively) in Mathematica is the so-called *Newton-Raphson Method*, which is based on a linear approximation. Omitting formal details related to this routine, the iteration procedure can be described briefly as follows: Let r be a root of a given function $f(x)$ such that $f(r) = 0$. Assume further that $f'(r) \neq 0$. Let then x_1 be a number in

[14] The issue extends to other computing environments, e.g. MATLAB (see Pearson (2009))

the vicinity of r. The tangent line to $f(x)$ at $(x_1, f(x_1))$ then has another point x_2 as its horizontal intercept, which is incrementally closer to the desired value r. We have that

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \tag{6.15}$$

We can continue by finding x_3 through the relation $x_3 = x_2 - f(x_2)/f'(x_2)$. The iterative procedure $x_{n+1} = x_n - f(x_n)/f'(x_n)$ then yields a sequence of x_n approximations to r.

The great advantage of the Newton-Raphson Method is its quadratic convergence near the root of an equation. For most well-behaved functions the results will stabilise after just a few iterations. However, the equations in our model contain hypergeometric functions, which render the system fairly unstable. The implications of this is that the *approximate zero*, or initial guess x_0, has to be very close to r in order to ensure the right convergence. In addition, the non-linearity implies that we do not have a procedure that finds all solutions. In fact, the default application of the aforementioned built-in functions returns just one solution and stops the iterative procedure when returning iterations outside the minimum and maximum values defined for the search interval, even if a correct solution might be inside the interval. Finally, in many of our computations we work with derivatives of functions, which may become zero over the iteration procedure. This renders the derivative at the root non-existent and the function will consequently not be defined for the given interval.

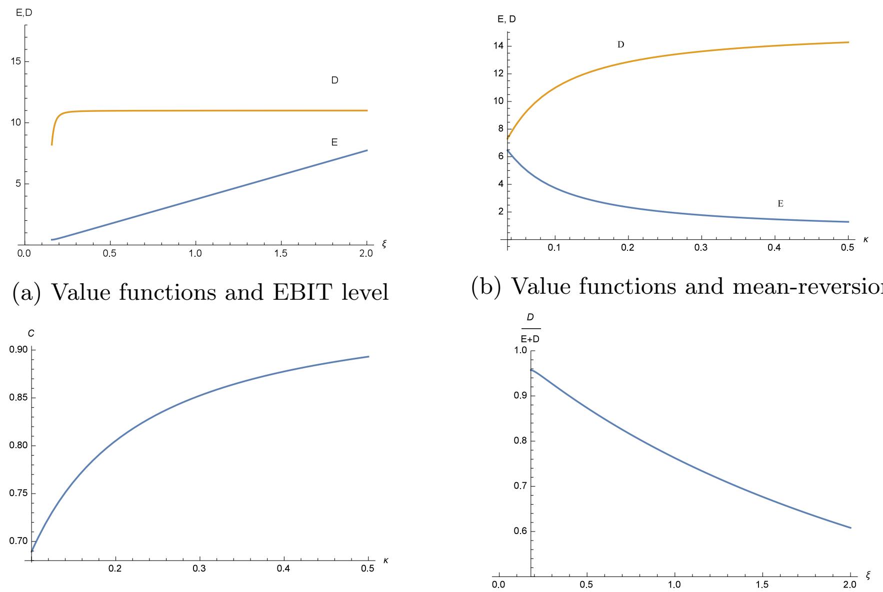
Notwithstanding these issues, we obtain satisfactory results for most of our simulations and can analyse comparative statics internally as well as conduct cross-model comparison with our benchmark model. The relatively high numerical stability we obtain is promoted by some simplifying assumptions detailed earlier. Specifically, both the exclusion of a call feature and the assumption that the tax advantage is lost at bankruptcy simplify the calculations. The computations will be performed under the base case values listed in Table 6.1. For tractability, the GOU model will be calibrated with the same parameter values as the benchmark case. However, for simulations of the GOU-based framework we must further assign values to κ and θ:

	Symbol	Value
Drift of EBIT	μ	2%
Risk-free rate	r	5%
Volatility of EBIT	σ	30%
Effective dividend tax rate	τ_e	40%
Tax rate on interest payments	τ_i	20%
Bankruptcy cost	α	10%
Issuance cost of debt	q	3%
Call premium	λ	5%
Equity holders' bargaining power	γ	50%
Speed of mean-reversion	κ	10%
Long-term mean of EBIT	θ	1

Table 6.1: *Base case parameter values under the GOU-based model.*

6.4.2 Model Performance with Mean-Reversion

We start our analysis by assessing the general model implications under inclusion of mean-reversion in earnings. The mosaic in Figure 6.1 summarises the findings graphically. In Figure 6.1a, we observe an altered curvature of the equity and debt claims as a function of a mean-reverting EBIT process. Most notably, we see that the debt function flattens out much quicker under the GOU modification. This is attributable to the fact that the volatility becomes bounded and thus that the value of the claim will tend quicker towards its risk-free value. Corroborating this finding, Figure 6.1b measures the impact of increases in the speed of mean-reversion on values of the debt and equity claims. Not surprisingly, the value of the debt claim is an increasing function of κ, while we observe a decay of the equity value. As more mean-reverting earnings ensure a higher certainty of coupon servicing given the lower probability of downward-drifting EBIT levels, creditors naturally enjoy a less risky claim. This increases its value for a given coupon. Conversely, equity holders, essentially owning a call option on the value of the firm, see their upside potential erode with earnings persistence. This economic intuition serves to explain the curvature of the claim functions.



(a) Value functions and EBIT level

(b) Value functions and mean-reversion

(c) Coupon and mean-reversion

(d) Leverage and EBIT level

Figure 6.1: *Sensitivity analysis of the GOU-based model.* See Table 6.1 for parameter values. (Source: Own contribution)

Moreover, in Figure 6.1c we can observe the familiar effect that the coupon level supported is strictly increasing in κ. Higher earnings stability reduces the risk of extended earnings shortfalls, which in turn allows the company to service a higher coupon for a given risk appetite. Equivalently, as we see in Figure 6.2 below, the property of earnings stability prompts the firm to select a higher leverage. A related result of this is an increase in the tax advantage to debt as mean-reversion increases.

Lastly, Figure 6.1d confirms an important result of Sarkar and Zapatero (2003). Indeed, their chief contribution is the finding that mean-reversion implies that leverage becomes a decreasing function of earnings. Our results support this proposal in the presence of GOU-based earnings and debt renegotiation possibilities. The result contrasts Leland (1994) where leverage is independent of earnings. Furthermore, studies by for example Titman and Wessels (1988), Rajan and Zingales (1995) and Fama and French (2002) have established empirically that a negative relationship between earnings and leverage exists. Consequently, a realistic model of dynamic capital structure ought to capture this effect.

Figure 6.2 gauges the sensitivity of optimal leverage to fluctuations in volatility and speed of mean-reversion. This illustrates a few insights complementary to the analysis of changes to the conventional costs and benefits of leverage. The optimal leverage peaks when mean-reversion is high and EBIT volatility is low. Moreover, we observe that the existence of mean-reversion has an offsetting effect on the impact of volatility on leverage. That is, for higher levels of mean-reversion the effect of higher uncertainty of EBIT has a lower influence on the management's initial choice of leverage.

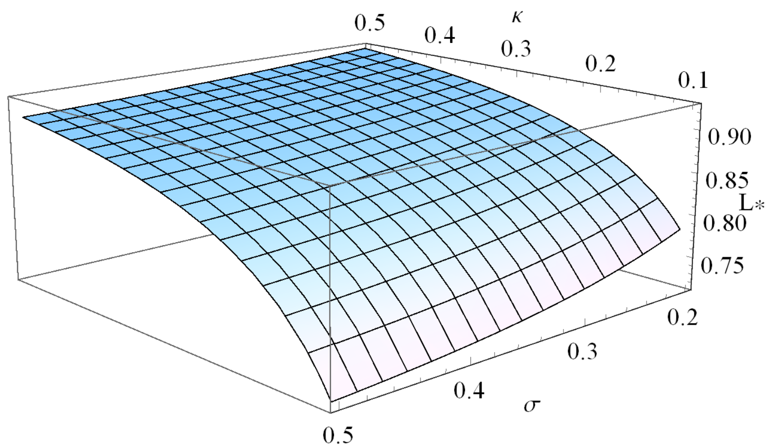
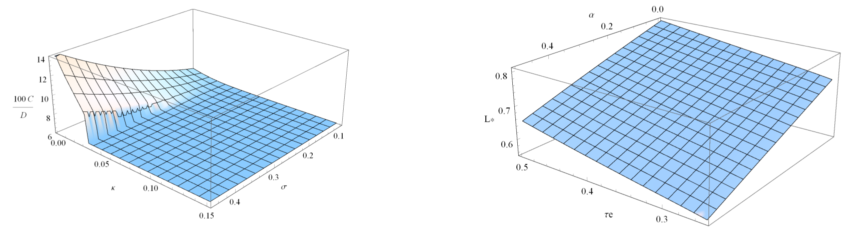


Figure 6.2: *Leverage as a function of volatility and mean-reversion.* See Table 6.1 for base case parameter values. (Source: Own contribution)

The analysis up until now suggests that the introduction of mean-reversion in EBIT could improve the firm's credit terms. In earlier models, we noticed how EBIT volatility had the opposite effect on the firm. Hence, it would be natural to investigate how the two parameters interact to establish the firm's overall credit profile. Figure 6.3a depicts how the bond yield is determined by the level of mean-reversion and the volatility. Interestingly, we can observe a highly non-linear relationship between the three parameters for certain realisations. For intermediate to large values of κ we see how the presence of mean-reversion essentially eradicates the effect of volatility. For example, for $\kappa = 10\%$ the bond yield is nonreactive to changes in volatility. However, as the speed of mean-reversion approaches zero, a familiar relationship between volatility and bond yield is re-established. This is a similar, but more pronounced, effect to that seen in the analysis of optimal leverage in Figure 6.2. This could be partially explained by the fact that volatility spikes pose a much smaller risk for creditors when coupled with the stabilising drift effect of $\kappa > 0$. From a practical perspective it is, nevertheless, questionable whether mean-reversion should be expected to have such major implications for the relationship between yield and volatility. For example, empirical tests (e.g Fama and French (2000)) indicate mean-reversion levels of about 38% in the US. Still, we do observe that yields are sensitive to volatility fluctuations.



(a) Bond yield as a function of σ and κ (b) Leverage as a function of τ_e and α

Figure 6.3: *Sensitivity analysis of the GOU-based model.* See Table 6.1 for parameter values. (Source: Own contribution)

Figure 6.3b presents the, nearly mandatory, analysis of the implications of bankruptcy costs and taxes on the choice of optimal capital structure. In harmony with the trade-off theory, our GOU-based model predicts that managers trade off the tax advantage to debt with the costs of bankruptcy. Compared to the GBM-based models presented in the chapters 3 and 5, we see that the relation between leverage and effective tax rate exhibits a higher degree of monotonicity.

In order to conduct a detailed analysis of the intercorrelations between our exogenous parameters and key metrics we have comprised a sensitivity table similar to the one produced in Section 5.1.3. Table 6.2 summarises the realisations of key metrics when one exogenous parameter is altered and all others equal those specified in Table 6.1. All exogenous parameters are varied to some levels below and above their base case specifications. Whereas most variables exhibit only moderate sensitivities to alternations in the base case parameters, the table enables us to observe a few interesting effects.

Firstly, optimal leverage appears be relatively insensitive to all parameters except for the speed of mean-reversion. A mere $5\%p$ change in κ implicates wide swings in the optimal leverage. We can thus claim that the degree of mean-reversion in earnings is of critical importance for any conclusion about optimal capital structure.

The recovery rate by debt holders exhibits high sensitivity to tax rates. The reasons for this might not be obvious initially, but is to be found in the specification of the debt claim. Equation (5.21) explains the strong dependence on the effective tax rate. The debt value at bankruptcy is defined by the unlevered value of the project, which is negatively related to τ_e. Similarly, the starting value of the debt claim is observed to be highly dependent on τ_i in (5.2).

The bond yield is in general only reactive to changes in the interest rate and the tax on interest income. This differs from the results of the benchmark model where the yield was more sensitive to movements in other variables.

Lastly, APR violations are insensitive to any moderate base case parameter fluctuations except for bankruptcy costs and equity holder bargaining power.

	d	**C***	**FV**	**L***	**BY**	**RR**	**APRV**	**TAD**
$r = 4\%$	0.15	0.72	18.16	77.06%	5.16%	77.03%	5.00%	21.05%
$r = 6\%$	0.17	0.66	11.91	71.54%	7.76%	76.73%	5.00%	19.12%
$\sigma = 25\%$	0.21	0.71	14.46	75.66%	6.46%	76.78%	5.00%	20.47%
$\sigma = 35\%$	0.12	0.85	14.36	72.88%	6.45%	76.99%	5.00%	19.67%
$\theta = 0.9$	0.14	0.62	13.36	71.88%	6.46%	76.89%	5.00%	19.30%
$\theta = 1.1$	0.17	0.76	15.44	75.92%	6.46%	76.90%	5.00%	20.62%
$\alpha = 5\%$	0.16	0.67	14.46	75.67%	6.46%	76.89%	2.50%	20.53%
$\alpha = 15\%$	0.16	0.69	14.34	72.42%	6.46%	76.89%	7.50%	19.47%
$q = 0\%$	0.16	0.69	14.73	74.71%	6.27%	74.59%	5.00%	22.75%
$q = 6\%$	0.16	0.85	14.07	73.35%	6.66%	79.35%	5.00%	17.25%
$\kappa = 5\%$	0.11	0.76	13.85	59.95%	6.48%	76.47%	5.00%	15.42%
$\kappa = 15\%$	0.19	0.71	14.67	80.54%	6.45%	77.03%	5.00%	22.21%
$\gamma = 25\%$	0.16	0.67	14.46	75.67%	6.46%	76.89%	2.50%	20.53%
$\gamma = 75\%$	0.16	0.69	14.34	72.42%	6.46%	76.89%	7.50%	19.48%
$\tau_e = 30\%$	0.15	0.69	15.04	70.87%	6.45%	89.63%	5.00%	7.39%
$\tau_e = 50\%$	0.16	0.69	13.77	77.41%	6.47%	64.16%	5.00%	37.66%
$\tau_i = 10\%$	0.16	0.69	15.72	76.26%	5.75%	68.41%	5.00%	31.03%
$\tau_i = 30\%$	0.16	0.69	13.08	71.33%	7.37%	87.81%	5.00%	8.97%

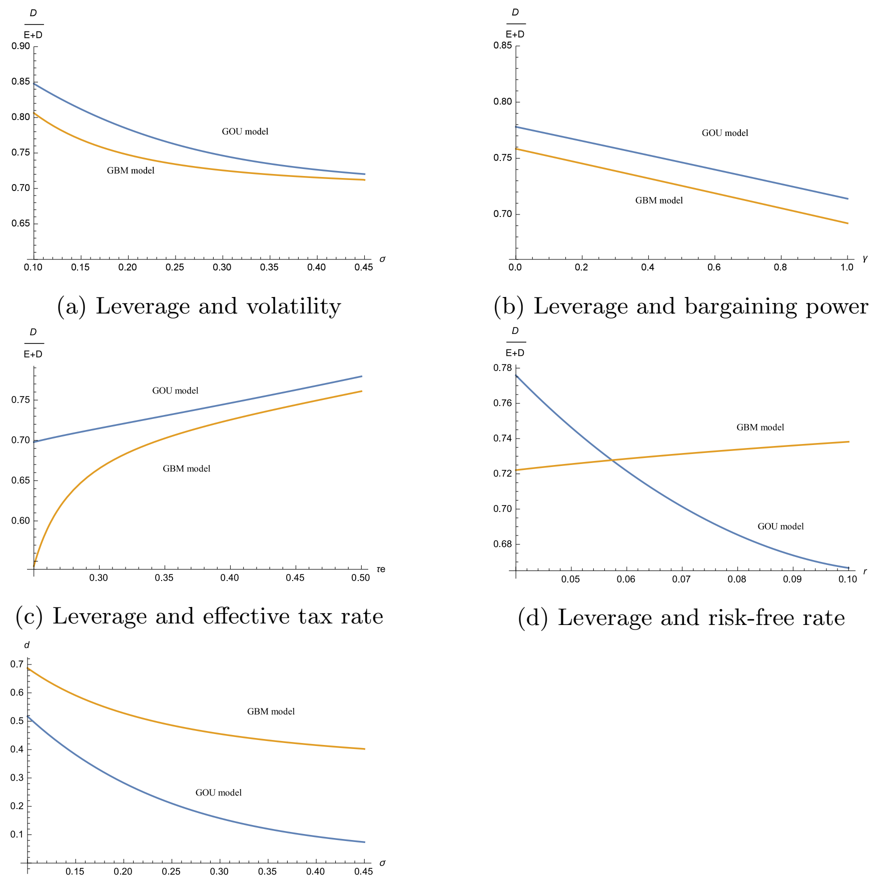
Table 6.2: *Detailed sensitivity analysis of the GOU-based model.* d = renegotiation threshold, C^* = optimal coupon, FV = firm value, L^* = optimal leverage, BY = bond yield, RR = recovery rate, APRV = APR violations, TAD = tax advantage to debt.

6.4.3 Cross-Model Comparison

Equipped with a suitable benchmark model from Chapter 5, we are now able to proceed to study how our GOU-based model compares with respect to general sensitivities and key metrics. As noted, the divergence between the two models exists in the specification of the stochastic earnings process. We are thus able to attribute any differences in sensitivities or metrics to the effect of mean-reversion. As a convenient point of departure, we graphically assess how the two models perform with respect to changes in a number of selected parameters.

The analysis is presented in Figure 6.4. In figures 6.4a and 6.4b we see how the optimal leverage per unit of volatility and bargaining power respectively is elevated in the GOU-based model. The higher level of leverage in the model with mean-reversion follows intuitively. As earnings become more persistent, the firm's managers should find that creditors offer better credit terms. This, should in turn encourage the use of debt financing.

The figure 6.4c examines how optimal leverage is affected by the effective corporate tax rate in the tax interval $25\% \leq \tau_e \leq 50\%$. As alluded to in the analysis of figure 6.3b, we can confirm that the relationship between taxes and leverage is, *ceteris paribus*, more linear under mean-reverting earnings than under a GBM-based regime. The policy implication of this follows logically; lowering the corporate tax rate below 35% would have marginally less negative effect on the use of debt financing in a GOU-based framework than in a GBM-based one. It appears the improvement in credit terms fully offsets the negative incentive effect of lower taxes. Mean-reversion in earnings thus proves of some significance for the conclusion about effects of changes in the tax regime.



(a) Leverage and volatility

(b) Leverage and bargaining power

(c) Leverage and effective tax rate

(d) Leverage and risk-free rate

(e) Renegotiation threshold and volatility

Figure 6.4: *Comparison of the GOU-based model and the GBM-based benchmark model.* See Table 6.1 for parameter values. (Source: Own contribution)

Another parameter for which we see a functionally altered relationship to leverage is the risk-free rate. The GBM-based benchmark model suggests that leverage is positively correlated to the risk-free rate. The economics behind this relationship appear somewhat unclear. In accordance with standard monetary policy theory, a higher risk-free rate should discourage borrowing. Figure 6.4d illustrates how the spurious relationship is mended by assuming that earnings are mean-reverting. The GOU-based model realistically predicts that the optimal leverage ratio is decreasing in r at a marginally diminishing rate.

Lastly, Figure 6.4e establishes that the renegotiation threshold is strictly lower in the mean-reversion model compared to the GBM benchmark. This originates in the fact that an EBIT level below the long-term mean in a GOU process provides for a stronger expectation of future earnings increases. On the contrary, a GBM process will not have any mean-reversion in its conditional expectation of the EBIT level, which introduces more uncertainty. Accordingly, the renegotiation threshold is higher for the GBM-based model. Moreover, we observe that increased volatility decreases the threshold, regardless of the diffusion process. This is intuitively explained by the lower resultant coupon and leverage.

	Benchmark Model	Callable Debt	Mean-reversion
Renegotiation threshold	0.46	0.38	0.16
Call threshold	-	1.81	-
Optimal Coupon	1.61	1.58	0.69
Firm value	22.53	25.93	14.40
Leverage	71.94%	60.66%	74.05%
Bond yield	9.91%	9.44%	6.45%
Recovery rate	50.18%	54.05%	76.89%
APR Violations	5.00%	5.00%	5.00%
Tax advantage to debt	12.68%	31.50%	20.00%

Table 6.3: *Cross-model comparison of key metrics.* See Table 6.1 for base case parameter values.

For a more extensive analysis of the implications of introducing mean-reversion in EBIT, we compare all key metrics of the three models that we have developed. Table 6.3, presents the values of the key metrics under the base case parameters specified in Table 6.1. The metrics of the benchmark model and the model featuring callable bonds correspond to those presented in Table 5.4.

We can re-confirm that we obtain a higher optimal leverage ratio under the base case parameters considered when comparing with the GBM-based benchmark model. As discussed, this is not surprising since the mean-reverting property of earnings reduces management's uncertainty about future income levels. In conjunction with better credit terms, this will provide impetus to select a higher initial leverage. As observed in Figure 6.2 and Table 6.2, the higher choice of leverage however decreases with the degree of mean-reversion and eventually converges with the result obtained for the GBM-based benchmark model. In line with these results, the tax advantage to debt is increased by about 5% compared to the benchmark model. Is is, however, still inferior to that observed when including callable debt in the extended GBM model.

The value of the firm decreases significantly compared to a GBM-based framework. This result follows from the lack of a positive upward drift in the earnings process when we apply a GOU. The strong effect of the drift can be seen in Table 5.2. As discussed with respect to Figure 6.1b, the value of the equity claim deteriorates with the introduction of mean-reversion and continues to fall as the speed of mean-reversion increases. The value of the debt claim also diminishes due to the lower value of the optimal coupon. The latter effect is however somewhat counterbalanced by safer earnings. Consequently, we also observe that bond yield falls compared to the benchmark.

A noteable difference is observed in the recovery rate of the debt holders. Under the new regime we see an increase of more than 20 percentage points in the recovery rate at bankruptcy. The initial value of debt will be less valuable under the lower coupon but will also lose relatively less value in case of bankruptcy. In essence, this stems from the fact that creditors overtake a less risky project than they do in the GBM case. APR violations, however, remain constant at around 5% as in previous models.

6.5 Summary of Key Results

In the preceding section we developed a model of dynamic capital structure with debt renegotiation and mean-reverting earnings. The model was developed similarly to the benchmark model of Chapter 5, but with a modified, GOU-based earnings process. Through numerical calibration we were able to identify a number of key findings resulting from the new model specification. Below we present the most important findings accompanied by brief elaboration.

Result I *Optimal leverage is increasing in mean-reversion and negatively correlated to earnings. Furthermore, the relation between leverage and effective tax rate is linearised compared to the benchmark model.*

The introduction of mean-reversion in EBIT increases the optimal leverage as cash flows are stabilised and credit terms improved. On a related note, the tax advantage to debt is increased by about 8%. The empirically justified finding that earnings and leverage correlate negatively is reconfirmed in our dynamic capital structure model. The result has previously been identified by Sarkar and Zapatero (2003) in a static framework. Lastly, the fundamental relationship between taxes and gearing is found positive but more linear than in the benchmark model. The finding implies that the existence of mean-reverting earnings can alter conclusions about a given tax policy.

Result II *The total firm value decreases under mean-reverting earnings. The value of debt is, ceteris paribus, increasing with the speed of mean-reversion, whereas the equity value decreases proportionately more.*

We observe a decrease in the initial firm value, which is driven by decreases in the values of both debt and equity. Even though debt increases with the persistence of earnings, the lower upside potential of the firm leads to the selection of a lower coupon. This prompts a slight decrease in the value of the debt claim compared to that of the benchmark model. The value of equity deteriorates due to the lack of a positive drift. Additionally, we note that debt converges to its risk-free value quicker than it does under the GBM-based regime. This is driven by the increased safety of cash flows that comes with mean-reversion.

Result III *The bond yield decreases under mean-reversion and becomes nonreactive to volatility fluctuations for medium-to-high values of the speed of mean-reversion.*

A significantly lower bond yield is observed under base case parameters in our GOU-based model compared to its benchmark. The lower borrowing cost is consistent with the notion of increased optimal leverage. Mean-reversion in EBIT appears to suppress the correlation between volatility and yield, which was seen in the GBM-based model. For moderate levels of mean-reversion, changes in volatility have no implications for the firm's borrowing costs.

Result IV *Mean-reversion in earnings establishes a negative relationship between leverage and the risk-free rate.*

In contrast to many previous GBM-based models, the GOU-driven model suggests a negative correlation between the risk-free rate and the optimal choice of leverage. The benchmark model indicated a slightly positive relation between the two variables. In accordance with standard theory of monetary policy, we would expect high risk-free rates to deter borrowing. The GOU-model thus creates a more satisfactory connection between interest rates and borrowing behaviour.

Result V *The size of APR violations is insensitive to our alternation of the stochastic process. Only changes in bankruptcy costs and equity holder bargaining power affect the magnitude of deviations from the APR.*

The size of APR violations proves to be contingent solely on equity holder bargaining power and bankruptcy costs. This originates from the model specification and was therefore expected. As discussed in Chapter 4, different levels of bargaining power implies different relative risk-aversion of creditors and equity holders. We can therefore establish a link between the size of APR deviations and the relative risk-aversions of the bargaining agents.

Result VI *The recovery rate of debt holders increases in our framework with mean-reverting earnings compared to the benchmark.*

Mean-reversion in earnings has a positive effect on the recovery rate of debt holders. The recovery rate is increased by more than 25 percentage points in the new framework. The relation stems from the fact that the initial debt value deteriorates, whereas the project value at bankruptcy decreases relatively less than in the benchmark case. Intuitively, the debt holders overtake a project carrying less risk at bankruptcy than they do when EBIT follows a GBM.

This summarises the main findings following the implementation of a model of dynamic capital structure and debt renegotiation with mean-reverting earnings.

6.6 Potential Modifications and Extensions to the Proposed Model

In order to conclude the treatment of our proposed model of dynamic capital structure, we will consider a number of alternative modifications to the model that would complement the proposed setup or extend it in certain directions. Note that we do not aspire to discuss the implications of these alternative features in their entirety, but rather discuss the incremental contribution of their inclusion under certain restrictions.

Evidently, as indicated in the development of the capital structure models of Chapter 3, 5 and 6, there is a wide array of lines along which a model of dynamic capital structure could be extended. Most pertinent is the addition of debt callability to the model setup proposed in this chapter. Moreover, we have noted how the log-normality property of both the GBM-based as well as the GOU-based model restrict the EBIT process from becoming negative. Therefore it could be considered a natural next step to consider other potential modifications of the earnings process that would relax this confounding assumption, and in fact this has already been investigated by several researchers. The model by Mella-Barral and Perraudin (1997), which was analysed in Chapter 4, models a GBM-based price process and considers a constant flow cost in production. In this way, EBIT can take on negative values. Similar results are obtained in related studies which consider the effect of *operating leverage*, i.e. having a higher degree of the cost base tied in fixed contracts. Berk, Stanton and Zechner (2010) for example consider the need to provide insurance by means of fixed wage contracts to employees in optimal employment contracting. This leads to a higher operational leverage and thus crowds out financial leverage, as the firm becomes more risky for a given level of earnings. Consequently, this is also a feature a dynamic capital structure modeller can consider for inclusion with the aim of obtaining lower optimal leverage ratios. Finally, one could also consider arithmetic versions of the Brownian Motions and Ornstein-Uhlenbeck processes discussed in Chapter 2, i.e. excluding the proportionality in the variance term. A model based on the former process is developed by Genser (2010). While this allows EBIT to attain values below zero, it also causes the homogeneity property to break down, significantly complicating the development of any satisfactory model with dynamic optimisation.

A range of other natural extensions could be included in this discussion. For example, a relevant but theoretically complex addition would be a framework accounting for private information on the part of equity holders[15]. In the following we will instead examine two potential modifications pertaining specifically to the renegotiation of debt, as this constitutes a main focus area of our model. We start out by considering the introduction of non-successful restructuring attempts between equity and debt holders. Subsequently, we analyse some implications of imposing finite maturity on the debt instrument.

6.6.1 Restructuring Failure

In the models developed in Chapter 5 and 6 it is always optimal for equity holders and debt holders alike to agree on renegotiation of the capital structure of the firm when the EBIT process hits a lower threshold. In other words, the going-concern value always exceeds the liquidation value. In practice, however, we clearly observe a large number of bankruptcies on a continuous basis. Thus; it seems more realistic to consider a framework with a positive probability of restructuring failure. In their suggestions for future research, Christensen et al. (2014) suggest intro-

[15]See Duffie and Lando (2001) for a first attempt in this direction.

ducing an exogenously given parameter denoting the probability that negotiations between debt holders and equity holders break down. While this would be a rather straightforward means of addressing the technical issue of no predicted bankruptcies, it only marginally adds more realism to the model setup. In fact, it would be hard to argue for the circumstances under which failed renegotiation would be plausible when the renegotiation gain is strictly positive.

Lehar (2015) instead suggests some requirements under which it would be possible to endogenise the positive probability of failed restructuring. He argues that the firm's chance of successful renegotiation is critically dependent on its debt structure. In our analysis we have refrained from considering multiple debt classes, implying that creditors are effectively united as one negotiating party. Lehar posits that under these circumstances restructuring will always be successful. The debt holders will collectively accept the restructuring proposal because they cannot increase their payoff by invoking bankruptcy. This corroborates our conclusion in this thesis. However, when introducing multiple debt classes there can be ample scope for failed restructuring. This is seen by assessing the situation where the most senior claimant (in Lehar (2015) the first claimant in the bankruptcy protocol) agrees to a debt reduction. Subsequent creditors will then realise that the value of their claim has appreciated, making them able to extract a higher premium for agreeing to restructure their claim. In other words, by agreeing to a debt reduction, the senior debt holder creates a positive externality for less senior debt holders. The more senior claim holder with information about the value of the firm's asset can rationally anticipate that the firm will not be able to make an acceptable offer to the less senior claim holders and therefore refuses the offer to restructure.

Lehar (2015) highlights two requirements that must hold in order to allow for restructuring failures. Firstly, the asset base must have a low value such that the minimum acceptable offers for all creditors cannot be met. Secondly, bankruptcy costs must be sufficiently low to render it cost-efficient for creditors to trigger default. It is easy to see how these two conditions have to be satisfied. In our model the debt holders will always be better off by accepting the restructuring proposal even in the extreme case that $\alpha = 0\%$. This is because a strictly positive restructuring gain is assumed. Thus; it must be coupled with the inclusion of several debt classes, which can cause a breakdown in negotiations as outlined above. Conversely, if liquidation costs are prohibitively costly, the chance of successful restructuring is proportionately higher. In the extreme case that $\alpha = 100\%$, the value of the creditors' rejection payoff will be zero and they should accept any proposal to restructure.

In order to implement restructuring failure in our model, we would need to consider the firm's asset value at the lower restructuring boundary. In concurrence with Lehar (2015), we can define it as

$$\underline{v} = (1-\phi)Ad\xi_0 \tag{6.16}$$

where ϕ is a fraction of the asset value assumed to be irreversibly lost to third parties upon entering renegotiation. The first requirement for restructuring failure thus restricts us to considering the case where $(1-\phi)Ad\xi_0 < P$. If we let p_i be the sequence of face values claimed by k creditors, then $P = \sum_{i=1}^{k} d_i$ denotes the total principal outstanding. In this case we can model the requirement for successful renegotiation as

$$\chi P \leq \underline{v}, \tag{6.17}$$

where $\chi \in [\underline{\chi}, \bar{\chi}]$ is the rate of exchange of old debt to new debt. This value is assumed to be given by the arrival order of creditors to the bankruptcy court in Lehar's (2015) setup. The restructuring gain from (5.23) accruing to equity holders can then be re-defined as

$$R_E^k = \gamma(\underline{v} - \chi P). \tag{6.18}$$

We can then proceed to define the conditions under which equity holders ex-post will find it optimal to enter rentegotiations. These will be the value-matching and smooth-pasting conditions from (5.25) and (5.26) substituting in the new restructuring gain. This will define a new lower restructuring threshold, which we will denote d_1^k. We can calculate that this new lower EBIT threshold will only be optimal ex-post when

$$E_1^k(d_1^k\xi_0;\xi_0) \geq E_1(d_1\xi_0;\xi_0). \tag{6.19}$$

This re-establishes the notion that equity holders will only renegotiate if their payoff keeps them at least as well off as the option to continue paying the suboptimal coupon.

Covenants as a Commitment Device

It is evident from the analysis of Lehar's setting that the equity holders choice of restructuring threshold can influence the outcome of renegotiations. Initiating restructuring earlier could ensure a sufficiently high asset value to cover the minimal acceptable offers for all creditors. Thus; as opposed to the model that we have proposed with no probability of bankruptcy, it can be optimal for equity holders to pre-commit to a specific threshold for restructuring. This would ensure a higher probability of a successful outcome of restructuring negotiations.

Lehar (2015) discusses the possibility to convey this credibly by including a covenant specifying an EBIT level for *technical* default in the bond indenture. Clearly, under the setup with no scope for failed restructurings, firm value would be strictly decreasing in any such covenant level. However, under this modified setup firm value can increase with the covenant level when negotiations are moved to a region where restructuring will be successful and 'dead-weight' bankruptcy costs are avoided. Importantly to note, however, the use of covenants as a commitment device should only be optimal for an intermediate level of α. If bankruptcy costs are sufficiently low, the benefit of saving liquidation costs does not outweigh the loss incurred for choosing a sub-optimal technical default boundary. Equivalently, if bankruptcy costs are sufficiently high, creditors will be willing to accept restructuring at any level of the restructuring threshold.

6.6.2 Finite Maturity of Debt

In all dynamic capital structure models developed throughout this thesis, a persistent assumption has been the issuance of perpetual bonds. The infinite maturity of debt is in fact an assumption that very few researchers have attempted to relax. The primary reason for this is the possibility to obtain neat and explicit solutions for the claims on a single state variable when these are time-independent. However, in practice perpetual bonds - or *consols* - are virtually non-existent, and it would thus be natural to investigate what the impact of the infinite-maturity assumption is on dynamic capital structure optimisation of the kind suggested in our model and the ones preceding it.

As stated in Footnote 5, Leland and Toft (1996) are among the few theorists who have introduced finite maturity. Their framework is developed along the lines of Leland (1994) with a simplistic modification to include expiration of bonds: The authors assume that new debt contracts are issued at every period in time with the same finite maturity T as well as a constant aggregate coupon C and principal P. In this manner, a new bond will have a fixed coupon rate $c = \frac{C}{T}$ and principal rate $p = \frac{P}{T}$. With constant debt service, the coupon payment will equal $C + \frac{P}{T}$ in every period. Aside from this alteration, the model is equivalent to

Leland (1994) and the authors in fact conclude that the optimal maturity is infinite. The paper however considers some important implications of finite-maturity debt for the issue of asset substitutability, which we discussed in Section 3.2.5. More precisely, one obtains reasonable incentive alignment between debt holders and equity holders with respect to risk-taking when debt is issued with short maturities. The paper thus challenges the sentiment of Jensen and Meckling (1976) that equity holders will find it optimal to increase risk. Only for longer maturities does the asset substitutability problem persist. Moreover, with this conclusion the paper establishes one potential agency-related explanation for why short-term maturity of debt is frequently observed in practice. It should be noted, however, that the assumption of a fixed coupon rate is an important driver of this apparent incentive alignment. In reality, creditors would most likely alter their return requirements for a given level of coupon in response to observed changes in the firm's riskiness. This is not taken into consideration by Leland and Toft (1996).

Flor and Lester (2002) develop an alternative model with endogenised determination of the debt maturity as a natural extension of the Leland and Toft (1996) framework. The article takes its point of departure in the model introduced by Goldstein et al. (2001), which was discussed in Chapter 3. The model results are analysed both with and without callability on debt, but an analysis of the possibility to renegotiate debt is excluded. For simplicity, we consider the version with non-callable debt here. This will imply that the capital structure can only be altered upon maturity of debt or in bankruptcy where the debt holders take over the firm and optimally re-lever the firm. The value-matching conditions at bankruptcy will thus be equivalent to (3.21).

Another set of value-matching conditions belonging to the claim values at debt maturity must be considered as well. At this point, debt holders' principal must be settled, after which new debt can be issued. Thus; at time T, given debt expiry without prior default, we have

$$\begin{aligned} E(\xi_T;\xi_0) &= \frac{\xi_T}{\xi_0}\xi_0 A - P \\ D(\xi_T;\xi_0) &= P. \end{aligned} \tag{6.20}$$

The conditions in (6.20) are reminiscent of the boundary conditions examined for the upper call boundary in (3.18) and (3.19). The smooth-pasting will also take on a similar form. However, with the finiteness of debt, we are no longer optimising time-homogeneous claims, for which reason the differential equation satisfied by claims on the firms remains a PDE. This way of solving PDEs for E and D numerically using the defined boundary conditions is a version of finite differencing also referred to as the *Crank-Nicolson Method*. The solution procedure is similar to the one of Section 3.2. In the absence of a call boundary, the maximisation in this problem is performed over the coupon C and time to maturity T. Defining $\varphi = (C, P, T)$ as the optimal dynamic debt policy, the maximisation problem is

$$\underset{\varphi=(C,P,T)}{\operatorname{argmax}} \; E(\xi_0, \varphi) + (1-q)D(\xi_0, \varphi) \tag{6.21}$$

where E and D solve the PDE in (2.22) subject to the given boundary conditions.

Analysing the implications of endogenising the time to maturity decision, Flor and Lester (2002) find that an interesting maturity trade-off. Particularly, the authors find that the firm will trade off lower aggregate restructuring costs for longer maturity of debt against the benefit of more frequent matching of the coupon with the current EBIT level in the case of shorter debt maturity. This trade-off would naturally be interesting to investigate in the context of our model with mean-reversion and a lower restructuring option. We will leave it to be established by future studies.

Chapter 7

Conclusion

> *"In the end, a theory is accepted not because it is confirmed by conventional empirical tests, but because researchers persuade one another that the theory is correct and relevant."*[16]
>
> - Fischer S. Black (1938-1995)

In this thesis we examine the optimal capital structure decision in the presence of debt renegotiation and mean-reverting earnings. Drawing on previous models of optimal capital structure and strategic debt service, we initially assemble a game-theoretically sound framework that features restructuring of debt and a GBM-based earnings process. The model serves as the yardstick with which we measure the impact of two specific augmentations. Firstly, we extend the model to include callable bonds. Subsequently, we enhance the stochastic earnings process to one characterised by reversion to a long-term mean. Focusing on the economic intuition and the ramifications for practical capital structure optimisation, we perform rigorous numerical analyses of the three models.

The previous models of optimal capital structure initially examined are the renowned works of Leland (1994) and Goldstein et al. (2001). To establish a solid foundation for our own capital structure model, we re-derive the results of the two publications. The Goldstein et al. model is adapted to our purposes by applying refined assumptions. Numerical simulations are performed to illuminate the implications and deficiencies of the previous models.

The analysis is extended by introducing two models of strategic debt service and assessing their compliance with game-theoretical principles. We observe how their results might be contingent on violation of the rationality assumption. Ultimately, a model by Christensen et al. (2014) is presented as the economically

[16] Fischer Black quoted by Derman (2004)

sound alternative. The model serves as the groundwork for our own models of debt renegotiation.

We proceed to develop a benchmark model that includes the option to renegotiate debt and assumes that EBIT follows a GBM. Extending the benchmark model, we analyse the possibility to include callability of debt. However, the computational complexity renders the extended model inelegant to work with when analysing the impact of mean-reverting earnings. The implications of the two models are examined through numerical tests, and conclusions regarding the effects of debt renegotiation and callability are made. The chief quantitative implications of debt renegotiation are; increased optimal leverage and a higher tax advantage to debt. Callability implies decreased initial leverage, a higher tax advantage to debt and a reduced optimal renegotiation threshold.

In a final advancement, we derive a model of dynamic capital structure and debt renegotiation with earnings following a GOU process. The model is calibrated and comparative statics are computed. A cross-model comparison is conducted to uncover further effects of mean-reverting earnings. We conclude that the optimal leverage increases and becomes negatively related to earnings as a consequence of mean-reversion. Correspondingly, the tax advantage to debt increases significantly. The debt contract benefits from increased earnings safety, whereas equity deteriorates by lost upside potential. Mean-reversion improves the firm's credit profile, which is reflected in a reduced bond yield. Lastly, the recovery rate is boosted, while the size of APR violations remains unchanged.

Our model of dynamic capital structure with debt renegotiation and mean-reverting earnings has several interesting implications for capital structure research. Considering the model's high leverage ratio in a wider model selection perspective, the modelling of dynamic capital structure choice with a GOU-based state variable diffusion poses a dilemma. While the imposition of mean-reversion brings the model closer to how earnings observably evolve, it does not constitute a means through which the prediction of optimal leverage can be better aligned with observed practice. We would however argue that any theoretical endeavour to develop a model that predicts lower leverage ratios should still support the implementation of mean-reversion in earnings. This assertion originates in the empirical corroboration of earnings persistence. A first step would arguably be to develop a GOU-based model that features callability of bonds, which was seen to decrease leverage in previous models.

References

Abramowitz, M. and Stegun, L. (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover Publications.

Anderson, R. and Sundaresan, S. (1996). *Design and valuation of debt contracts.* The Review of Financial Studies, 9 (1): 37–68.

Berens, J. and Cuny, C. (1995). *The capital structure puzzle revisited.* The Review of Financial Studies, 8 (4): 1185–1208.

Berk, J., Stanton, R. and Zechner, J. (2010). *Human capital, bankruptcy, and capital.* Journal of Finance, 65: 891–926.

Bhattacharya, S. (1978). *Project valuation with mean reverting cash flow streams.* Journal of Finance, 3: 1317–1331.

Bjerrisgaard, S. and Fedoryaev, D. (2011). *Dynamic capital structure modelling under alternative stochastic processes.* Master's thesis, Copenhagen Business School.

Black, F. and Cox, J. (1976). *Valuing corporate securities: Some effects of bond indenture provisions.* The Journal of Finance, 31 (2): 351–367.

Black, F. and Scholes, M. (1973). *The pricing of options and corporate liabilities.* The Journal of Political Economy, 81 (3): 637–654.

Blume, L. and Easley, D. (2008). *Rationality.* In *The New Palgrave Dictionary of Economics.* Palgrave Macmillian.

Bradley, M., Jarell, G. and Kim, E. (1984). *On the existence of an optimal capital structure: Theory and evidence.* The Journal of Finance, 39 (3): 857–878.

Brennan, M. and Schwartz, E. (1978). *Corporate income taxes, valuation, and the problem of optimal capital structure.* The Journal of Business, 51: 103–114.

Christensen, P., Flor, C., Lando, D. and Miltersen, K. (2002). *Dynamic Capital Structure with Callable Debt and Debt Renegotiations.* SSRN working paper series.

Christensen, P., Flor, C., Lando, D. and Miltersen, K. (2014). *Dynamic Capital Structure with Callable Debt and Debt Renegotiations.* Journal of Corporate Finance, 29 (1): 644–661.

Cochrane, J. (2005). *Asset Pricing.* Princeton University Press.

Debreu, G. (1991). *The mathematization of economic theory.* The American Economic Review, 81 (1): 1–7.

Derman, E. (2004). *My Life as a Quant: Reflections on Physics and Finance.* John Wiley and Sons, Inc.

Dixit, A. and Pindyck, R. (1994). *Investment under uncertainty.* Princeton University Press.

Duffie, J. and Lando, D. (2001). *Term structures of credit spreads with incomplete accounting information.* Econometrica, 69 (3): 633–664.

Easton, P. and Zmijewski, M. (1989). *Cross sectional variation in the stock market response to accounting earnings announcements.* Journal of Accounting and Economics, 11: 117–141.

Eberhart, A., Moore, W. and Roenfeldt, R. (1990). *Security pricing and deviations from the absolute priority rule in bankruptcy proceedings.* The Journal of Finance, 45 (5): 1457–1469.

Fama, E. and French, K. (2000). *Forecasting profitability and earnings.* Journal of Business, 73: 161–176.

Fama, E. and French, K. (2002). *Testing trade-off and pecking order predictions about dividends and debt.* The Review of Financial Studies, 15 (1): 1–33.

Fischer, E., Heinkel, R. and Zechner, J. (1989a). *Dynamic capital structure choice: Theory and tests.* The Journal of Finance, 44 (1): 19–40.

Fischer, E., Heinkel, R. and Zechner, J. (1989b). *Dynamic recapitalization policies and the role of call premia and issue discounts.* The Journal of Financial and Quantitative Analysis, 24 (4): 427–446.

Fisher, I. (1930). *The Theory of Interest: As determined by impatience to spend income and opportunity to invest it.* New York: Kelley and Millman (1954 reprint).

Flor, C. and Lester, J. (2002). *Debt maturity, callability, and dynamic capital structure.* University of Southern Denmark. Working paper.

Franks, J. and Torous, W. (1989). *An empirical investigation of U.S. firms in reorganization.* The Journal of Finance, 44 (3): 747–769.

Fraser-Sampson, G. (2010). *Private equity as an asset class.* Wiley Finance, 2nd edition.

Freeman, R., Ohlson, J. and Penman, S. (1982). *Book rate-of-return and prediction of earnings changes: An empirical investigation.* Journal of Accounting Research, 20 (2): 639–653.

Genser, M. (2010). *A Structural Framework for the Pricing of Corporate Securities. Lecture Notes in Economics and Mathematical Systems.* Springer-Verlag.

Gibbons, R. (1994). *A Primer in Game Theory.* Prentice Hall Books.

Gibbons, R. and Murphy, K. (1992). *Optimal incentive contracts in the presence of career concerns: Theory and evidence.* Journal of Political Economy, 100: 468–505.

Gilson, S., John, K. and Lang, H. (1990). *Troubled debt restructurings: An empirical study of private reorganization of firms in default.* Journal of Financial Economics, 27: 315–353.

Goldstein, R., Ju, N. and Leland, H. (2001). *An EBIT-based model of dynamic capital structure.* The Journal of Business, 74 (4): 483–512.

Graham, J. (2000). *How big are the tax benefits of debt?* The Journal of Finance, 56 (5): 1901–1942.

Harrison, J. and Kreps, D. (1979). *Martingales and arbitrage in multiperiod securities markets.* Journal of Economic Theory, 20: 381–480.

Jensen, M. and Meckling, W. (1976). *Theory of the firm: Managerial behavior, agency costs and ownership structure.* Journal of Financial Economics, 3: 305–360.

Kane, A., Marcus, A. and McDonald, R. (1984). *How big is the tax advantage to debt?* The Journal of Finance, 39 (3): 841–853.

Kraus, A. and Litzenberger, R. (1973). *A state-preference model of optimal financial leverage*. The Journal of Finance, 28: 911–922.

Lehar, A. (2015). *Restructuring failure and optimal capital structure*. University of Calgary. Working paper.

Leland, H. (1994). *Corporate debt value, bond covenants, and optimal capital structure*. The Journal of Finance, 49 (4): 1213–1252.

Leland, H. and Pyle, D. (1977). *Informational asymmetries, financial structure, and financial intermediation*. The Journal of Finance, 32: 371–387.

Leland, H. and Toft, K. (1996). *Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads*. Journal of Corporate Finance, 51 (3): 987–1019.

Lipe, R. and Kormendi, R. (1987). *Earnings innovations, earnings persistence, and stock returns*. Journal of Business, 60: 323–346.

Lipe, R. and Kormendi, R. (1994). *Mean reversion in annual earnings and its implications for security valuation*. Review of Quantitative Finance and Accounting, 4: 27–46.

Mella-Barral, P. and Perraudin, W. (1997). *Strategic debt service*. The Journal of Finance, 52 (2): 531–556.

Merton, R. (1974). *On the pricing of corporate debt: The risk structure of interest rates*. The Journal of Finance, 29 (2): 449–470.

Modigliani, F. and Miller, M. (1958). *The cost of capital, corporation finance and the theory of investment*. The American Economic Review, 48 (3): 261–297.

Modigliani, F. and Miller, M. (1963). *Corporate income taxes and the cost of capital: A correction*. The American Economic Review, 53 (3): 433–443.

Muthoo, A. (1999). *Bargaining Thoery with Applications*. Cambridge University Press, 1st edition.

Myers, S. (1977). *Determinants of corporate borrowing*. Journal of Financial Economics, 5: 147–175.

Myers, S. (1984). *The capital structure puzzle.* The Journal of Finance, 39 (3): 575–592.

Myers, S. and Majluf, N. (1984). *Corporate Financing and Investment Decisions When Firms Have Information That Investors Do Not Have.* Journal of Financial Economics, 13 (2): 187–221.

Nash, J. (1950). *The bargaining problem.* Econometrica, 18 (2): 155–162.

Nash, J. (1953). *Two-person cooperative games.* Econometrica, 21 (1): 128–140.

von Neumann, J. and Morgenstern, O. (1944). *Theory of Games and Economic Behaviour.* Princeton, N.J. ; Princeton University Press.

Pearson, J. (2009). *Computation of hypergeometric functions.* Master's thesis, University of Oxford.

Pennacchi, G. (2008). *Theory of Asset Pricing.* Pearson/Addison-Wesley.

Rajan, R. and Zingales, L. (1995). *What do we know about capital structure? some evidence from international data.* The Journal of Finance, 50 (5): 1421–1460.

Raymar, S. (1991). *A model of capital structure when earnings are mean-reverting.* Journal of Financial and Quantitative Analysis, 26 (3): 327–344.

Ross, S. (1977). *The determination of financial structure: The incentive-signalling approach.* The Bell Journal of Economics, 8: 23–40.

Rubinstein, A. (1982). *Perfect equilibirum in a bargaining model.* Econometrica, 50: 97–110.

Sarkar, S. and Zapatero, F. (2003). *The trade-off model with mean reverting earnings: Theory and empirical tests.* The Economic Journal, 113: 834–860.

Shyam-Sunder, L. and Myers, S. (1999). *Testing static tradeoff against pecking order models of capital structure.* Journal of Financial Economics, 51: 219–244.

Stigler, G. (1963). *Capital and rates of return in manufacturing industries.* Princeton, N.J: Princeton University Press.

Strebulaev, I. (2007). *Do tests of capital structure theory mean what they say?* The Journal of Finance, 62 (4): 1747–1787.

Titman, S. and Tsyplakov, S. (2002). *A dynamic model of optimal capital structure.* Working Paper, University of Texas at Austin.

Titman, S. and Tsyplakov, S. (2007). *A dynamic model of optimal capital structure.* McCombs research paper series, No. FIN-03-06.

Titman, S. and Wessels, R. (1988). *The determinants of capital structure choice.* The Journal of Finance, 43 (1): 1–19.

Weiss, L. (1990). *Bankruptcy resolution: direct costs and violations of priority of claims.* Journal of Financial Economics, 27: 285–314.

Yue, V. (2010). *Sovereign default and debt renegotiation.* Journal of International Economics, 80 (2): 176–187.

Appendix A

Boundary Conditions with Dynamic Programming

Consider a firm that faces some choices at each period t, which we can represent by the choice variable u. The state and control at time t affects the firm's profit flow, which accordingly can be denoted $\pi(x(t), u(t))$. Allowing the drift and variance parameters to depend on the choice variable u, the general diffusion of the state variable can then be represented as

$$dx = \mu(x, u, t)dt + \sigma(x, u, t)dW. \tag{A.1}$$

Note how dx depends on the choice u. We can apply Itô's lemma to $F(x, t)$ in order to derive the value function

$$\rho F(x, t) = \pi(x, u^*, t) + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\mu(x, u^*, t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma(x, u^*, t)^2 \tag{A.2}$$

where $u^* = u(x, t)$ is the optimal value of the control variable and ρ is the discount rate per unit of time. Equation (2.24) is a second-order PDE and a version of the *Hamilton-Jacobi-Bellman (HJB) equation.* Since many solutions exist to solve this PDE, we need to consider two boundary conditions and the economics of the value function F.

The Stationary ∞-Horizon Problem

If the firm's problem was subjected to a fixed time limit T at which point it would receive the terminal payout $\Omega(x,T)$, equation (2.24) would have the boundary condition

$$F(x,T) = \Omega(x,T) \quad \forall x. \tag{A.3}$$

This problem could be solved with techniques analogous to backward induction by finding $F(x,t)$ for all earlier instants. However, for an infinite-horrizon problem there is no fixed payoff point for the decision problem and thus no known value function from which we can apply backward induction. Instead the problem becomes recursive. Under the assumptions that the profit function π, the discount rate ρ and the probability distribution function are independent of the time increment, this causes the value function to be independent of time. Accordingly, we can express the modified HJB equation as

$$\rho F(x) = \pi(x,u^*) + \mu(x,u^*)F' + \frac{1}{2}\sigma(x,u^*)^2 F'', \tag{A.4}$$

which has now become an ODE with x as the only independent variable. In general, many functions are consistent with this ODE, for which reason we must consider some other constraints to pick out a single, deterministic solution.

This procedure is often referred to as *optimal stopping* of an Itô process. The easiest analogy to represent this problem is a firm, which faces the binary decision of continuing operations to retain a profit flow $\pi(x,t)dt$, or cease operations and earn a termination payoff $\Omega(x,t)$. Thus the HJB equation for the optimal stopping problem becomes

$$F(x,t) = max(\pi(x,t)dt + (1+\rho dt)^{-1}E[F(x+dx,t+dt)|x], \Omega(x,t)) \tag{A.5}$$

Intuitively we can see that for each t there will be a value for x, $x^*(t)$, where stopping will be optimal for lower values and continuation will optimal for higher values, i.e.

$$If \begin{cases} x > x^*(t), & \text{continue} \\ x \leq x^*(t), & \text{stop.} \end{cases} \tag{A.6}$$

It is thus evident that, in the continuation region, the first term of the maximisation problem in (2.27) will be higher. Expanding this term by Itô's lemma we obtain

$$\rho F(x,t)dt = \pi(x,t)dt + \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\mu(x,t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma(x,t)^2\right]dt, \tag{A.7}$$

which holds for $x > x^*(t)$. From (2.25) we know that $F(x,t) = \Omega(x,t)$ in the stopping region so that one set of boundary conditions is

$$F(x,t) = \Omega(x,t) \quad \forall(x,t) \quad \text{such that} \quad x(t) \leq x^*(t). \tag{A.8}$$

Equation (A.8) represents the value-matching condition. We need to introduce a smooth-pasting condition to uncover the value of the boundary in the region $x^*(t)$ in (x,t) space for which the PDE in (A.7) is valid. This condition will require the values $F(x,t)$ and $\Omega(x,t)$ as functions of x to meet tangentially, i.e.

$$\frac{\partial F(x^*(t),t)}{\partial x} = \frac{\partial \Omega_x(x^*(t),t)}{\partial x} \quad \forall t. \tag{A.9}$$

As it be seen by comparison with (2.34) and (2.35), the value-matching and smooth-pasting conditions will be similar, regardless of whether dynamic programming or contingent claims pricing is used for the dynamic optimisation problem.

Appendix B

Tax Shelter From Negative Earnings

In the version of the Goldstein et al. (2001) model that we consider in Chapter 3, the firm is able to deduct the entire coupon even if earnings are insufficient to cover the amount of the coupon payments. This is clearly inconsistent with how interest payments are deducted in practice. Normally the tax shelter is partially lost in case of negative earnings. This issue is expounded in depth by Graham (2000). Goldstein et al. (2001) extend this tax regime by allowing for a loss of the tax shelter when net income is negative. It assumed that the after-tax payout rate to equity holders $\delta\xi + b$ is given by

$$\delta\xi + b = \begin{cases} (1-\tau_e)(\xi - C), & for \quad \xi \geq C \\ (1-\tau_e)\xi - (1-\epsilon\tau_e)C, & for \quad \xi < C. \end{cases} \tag{B.1}$$

where $\epsilon \in [0,1]$ is an exogenously given parameter for the degree of tax shelter lost and carried forward. Implementing ϵ, the expressions for equity and debt given in (3.17) instead become

$$D(\xi) = d_1\xi^{\beta_1} + d_2\xi^{\beta_2} + (1-\tau_i)\frac{C}{r}$$

$$E(\xi) = \begin{cases} e_1\xi^{\beta_1} + e_2\xi^{\beta_2} + (1-\tau_e)\frac{\xi}{r-\mu} - (1-\tau_e)\frac{C}{r}, & for \quad \xi \geq C \\ e_1\xi^{\beta_1} + e_2\xi^{\beta_2} + (1-\tau_e)\frac{\xi}{r-\mu} - (1-\epsilon\tau_e)\frac{C}{r}, & for \quad \xi < C. \end{cases} \tag{B.2}$$

The boundary conditions for the maximisation will be the same as before with the additional requirement that equity value is continuous and differentiable at $\xi = C$, i.e. $E(C)_{\xi \geq C} = E(C)_{\xi < C}$ and $E'(C)_{\xi \geq C} = E'(C)_{\xi < C}$.