Master Thesis

Optimal Allocation under a Stochastic Interest Rate and the Costs from Suboptimal Allocation

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17th May 2016

MSc Advanced Economics and Finance

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Number of pages	109
Number of characters	231,954

Abstract

The purpose of this thesis is to analyse investor's dynamic asset allocation strategies, when introducing a stochastic interest rate, and a non-constant market price of risk. This is followed by an evaluation of the costs of applying a suboptimal portfolio allocation strategy. The starting point is the classical static portfolio result of the mean-variance analysis developed by Markowitz (1952). This is followed by a demonstration of the intertemporal portfolio problem under constant investment opportunities from Merton (1969). The problem is solved by dynamic programming using the Hamiliton-Jacobian-Bellman equation to obtain portfolio result, the so-called myopic portfolio. The Merton's portfolio problem serves as foundation for the extensions in this thesis. In the first extension, a dynamic portfolio choice model includes the interest rate as state variable, where interest rate is modelled by a one-factor Vasicek model, which introduces the set of stochastic investment opportunities. The result of this model is a closed-form solution and is considered for a CRRA-investor, which shows that investors should hold the myopic portfolio and a hedging portfolio, which contains assets that are correlated with the state variable, the interest rate. From the analysis, the model results show that the stock allocation is time-invariant and decreasing in risk aversion. The bond allocation is increasing in investment horizon and risk aversion, since bonds through a hedging term is used to minimize exposure from the interest rate risk. In the second extension, a dynamic asset allocation model is developed again with a stochastic interest rate as a state variable, where the market price of risk is an affine function of the state variable. A closed-form solution is obtained for the optimal portfolio choice and applied for a CRRA-investor. The result shows that the stock allocation is still decreasing in risk aversion, but it also starts to vary over time. This is because the stochastic interest rate enters directly into the portfolio weight of stocks. The bond allocation under the second extension is different from the former. With the market price of risk as an affine function of the stochastic interest rate, the bond allocation fluctuates due to the changes in the interest rate. As the risk aversion increases, the fluctuations will be smaller, and the two models will converge to the same bond allocation, which is increasing in the investment horizon. To evaluate the portfolio choice models, two loss functions are considered. The loss function evaluates a suboptimal investment strategy under the optimal assumptions in terms of welfare losses for the investor. It is shown that applying the result from constant investment opportunities under the assumptions of the first extension leads to marginal welfare losses. However, using the investment strategy of the first extension under the assumptions of second extension shows a significant increase in welfare losses for the investor.

Acknowledgements

We wish to thank our supervisor Claus Munk for guiding us through the thesis process, for clarifying mathematical issues, and for the initial discussion of potential thesis topics within dynamic asset allocation. We also appreciate the inputs from Kasper Utoft Andersen and Steffen Dam Andersen. Further, we want to thank Joel Hartman for suggestions regarding R-coding.

Contents

Chapter 1

Introduction

Portfolio and consumption choice problems have been centric in the academic part of financial literature, since the important paper by Markowitz (1952), who introduced the concept of diversification in a tractable mathematical framework to understand the trade-offs between risk and return. However, the bridge between practitioners and academia were breached, when Merton (1969) analysed the portfolio-consumption choice problem in a continuous-time multi-period model. Merton derived a closed-form solution that confirmed Markowitz static portfolio result, that all investors should hold the same risky portfolio, also called the myopic portfolio. Along with this result, investors was ought to consume a fixed fraction of their wealth. As Merton later did, we also want to challenge precedent models under constant investment opportunities. In Section 1.1, we elaborate on the problem of interest, and explain our motivation for this interest. We formulate the thesis statement in detail and state a formal research question, we intend to answer with this thesis. Section 1.2 sets field of research with introducing the seminal and pivotal papers in dynamic asset allocation.

1.1 Problem of Interest and Research Question

We have restricted ourselves to only consider a portfolio choice problem instead of consumption-portfolio optimisation problem. This restriction will allow us to obtain tractable closed-form solutions in the most instances. We are applying the analogy of an institutional investor, since it confines us to only consider the utility, which the investor derives from terminal wealth. Even though this analogy is not completely characteristic for all investors who have similar investment behaviour, it is helpful to set the context of a discussion.

The myopic portfolio result was considered unrealistic by Merton (1973), and lead him to put forward the concept of stochastic investment opportunities. This concept implied that the opportunities were ought to be correlated with a set of state variables. His result under the stochastic investment opportunities showed that investors should invest in a hedge portfolio in addition to the myopic portfolio.

Our motivation is to apply the Merton's portfolio problem to a specific case, where we want to model the optimal portfolio choice under a stochastic interest rate as a state variable. We use a direct method in solving the portfolio choice problem, this imply that we do not derive a general solution for any stochastic state variable. The institutional investor analogy will help lead the discussion about how they are affected by the presence of a stochastic interest rate. In addition, we want to evaluate costs in terms of welfare losses when investors are employing suboptimal investment strategies under the given set of assumptions.

This thesis wants to investigate the effect of a stochastic interest rate on the investment opportunities and its influence on the optimal portfolio choice in a continuous-time setting. This is done by considering the model implications for institutional investors which is benchmarked against the myopic portfolio result under constant investment opportunities[1]. Moreover, this thesis also wants to evaluate the consequences of applying suboptimal investment strategies within this stochastic framework. From this, we can deduce a specific research question:

> *How does the introduction of a stochastic interest rate affect the optimal portfolio allocation, and what are the consequences of suboptimal portfolio allocation in terms of welfare loss for the institutional investors?*

To answer this question, we will consider the following five topics

- Main findings from the mean-variance analysis.
- Optimal allocation under constant investment opportunities.
- Extension of the allocation model with a stochastic interest rate.
- Allocation model with market price risk as an affine function of the stochastic interest rate.
- Welfare losses from suboptimal portfolio allocation.

[1]Parts of the initial Chapters 2, 3, and 4 are based on the bachelor thesis by Andersen and Nørgaard (2014)

To ensure clarity, the models presented in the following chapters are restricted to the following properties:

- The investor only gets utility from terminal wealth.
- The investor has Constant Relative Risk Aversion(CRRA)
- The investor receives no non-financial income.
- The investment horizon is known.
- The investor can allocate wealth in risky and risk-free assets.
- The time variation is driven by a single state variable.
- The risky assets are subject to time variation.
- There is no parameter uncertainty.
- There are no restrictions in trading assets.

These properties are common among various optimal portfolio problems, and have been studied extensively. The field of research is covered in the following section with a brief literature review.

1.2 Literature Review

A review of the literature is necessary in order to understand how the research field has been shaped through time and which important contributions to the literature we could build our analysis upon.

The foundation of optimal portfolio choice originates from the modern portfolio theory in the seminal paper by Markowitz (1952), he put forward a conceptual framework for portfolio management, namely the mean-variance analysis. The mean-variance analysis has the assumption that the investor's portfolio choice will only depend on the mean and the variance of their end-of-period wealth. The investor can with different combinations of these two moments form portfolios, where Markowitz defines an efficient portfolio to be either a portfolio with the lowest variance for a given expected return or the highest return for a given variance, depending on the investor's risk return preferences. The underlying idea of the theory was that assets should be chosen only based on inherent characteristics which were unique to the security. However, the investor should see how the single security co-moved with all other securities. If the investor accounted for these correlations among securities, then it would enable them to construct a portfolio that yielded the same expected return with less risk than a portfolio created without considering the correlations between each security, this is the benefits of diversification.

Tobin (1958) suggested an extension to Markowitz's mean-variance analysis, where an introduction of a risk-free asset into the investor's feasible set of allocation choices actually simplifies the investor's problem. Tobin showed that if a risk-free aaset existed and the investor had access to it, then the choice of the optimal portfolio of risky assets is clear and independent of the preferences of the investor for expected return and variance. This is Tobin's Separation Theorem, where the investor's problem is simplified into a choice between the portfolio, which maximises the ratio of expected return subtracted the return on the risk-free asset relative to the standard deviation, namely the tangency portfolio and the risk-free asset. Thereby will investors hold a given combination of the tangency portfolio and the risk-free asset matching their attitude toward risk, because it assumed that investors can lend and borrow at the risk-free rate. This separation was interpreted as investing in two funds, where investors could obtain the desired portfolio by holding a combination of the two, representing the tangency portfolio and the risk-free asset. This separation theorem is also known as the mutual fund theorem.

Sharpe (1964) and Lintner (1965) contributed to the work of Markowitz and Tobin. They added two assumptions in order to identify if a portfolio is mean-variance efficient, and turn the results from mean-variance model about each investor's investment behaviour into a testable hypothesis about the trade-off between expected return and risk in an equilibrium model, the Capital Asset Pricing Model (CAPM). The first assumption is complete agree, which imply that investors agree on the joint distribution of assets return from the current period to the next period. This is the true distribution, which imply that the distribution actually generates the asset returns. The second assumption is unlimited borrowing and lending at the risk-free rate, which does not depend on the amount. These assumptions insure that all investors will have same feasible set of efficient portfolios, and therefore will all use the tangency portfolio with the risk-free asset. Under these CAPM assumptions, the tangency portfolio is the market portfolio, and the risk-free rate is set to clear the market for borrowing and lending along with the prices of the risky assets. The market portfolio must be on the minimum variance frontier, if the markets are to clear.

Moving away from single-period models into a multi-period model of the portfolio problem, where the investment and consumption problem is solved simultaneously in a continuous-time formulation. This field was pioneered by the publications from Merton (1969, 1971, 1973) and Samuelson (1969). Merton's primary result of the portfolio problem in the continuous-time model is an optimal investment strategy, which is independent of the time horizon of the investor, and this aligns with the Markowitz-Tobin discrete mean-variance rules. This is done under the assumptions of log-normality of the distribution for assets prices instead of a normal distribution, and with a more general utility function than the quadratic utility function, which is used in mean-variance analysis. Similar to the myopic case of the mean-variance result, a higher risk aversion would lead to a lower fraction of wealth invested in tangency portfolio in the continuous-time model. The portfolio result in Merton (1973) separates itself from the single-period model, since it includes intertemporal hedging portfolios, which the investor uses to hedge her portfolio against shocks to the state variables. Samuelson (1969) considers a multi-period problem, which should correspond to a life-cycle problem with consumption and investment decisions both to be incorporated. He finds that the optimal fraction of wealth to invest in risky assets is constant under similar assumptions to Merton (1969). The life-cycle problem with consumption and investment produces the same result as Markowitz's static mean-variance result.

This thesis focuses on how a stochastic term-structure affects an investor's optimal portfolio choice in a continuous-time setting, where the investor only derives utility from terminal wealth. This area within dynamic asset allocation is particularly well-established, where we choose a few selected papers to lay the foundation. The predominately approach of solving these continuous-time consumption-investment choice problems is applying stochastic dynamic programming. This has been used in papers such as Sørensen (1999), Brennan and Xia (2000), and Munk et al. (2004), who also consider the problem under a stochastic interest rate. Sørensen (1999) uses a Vasicek (1977) model to describe the dynamics of interest rate, but Brennan and Xia (2000) consider the dynamics to be modelled by a two-factor Hull and White (1996) model. Both papers show that the wealth allocated in stocks and bonds is increasing in the expected returns, and the hedging portfolio is a zero-coupon bond with expiration at the investment horizon. The myopic portfolio is still prevalent under these models, since the stock allocation is constant over the investment horizon because equity is not a part of the hedging portfolio. If the bond's maturity is equal to the investment horizon, the optimal allocations are shown to be independent of investment horizon. The stock allocation is decreasing in risk aversion, while bond and cash allocations are increasing in it. In another paper, Munk et al. (2004), they look at the dynamic asset allocation problem under the assumptions of mean-reversion in returns, stochastic interest rates, and uncertainty about inflation. They consider an investor who wants to maximise her expected utility from real terminal wealth. In the solution of this they obtain three hedging terms, where an optimal hedge against changes in the interest is done by only investing in the bond, since it is perfectly negatively correlated with short interest rate. The optimal hedge against changes in the expected excess return of equity is obtained by only investing in stocks, because of the perfect negative correlation between processes of the stock and excess return. The optimal hedge for inflation is a combination of the stock and the bond.

Kim and Omberg (1996) model the portfolio choice problem differently, where they have the risk premium for a risky asset as the state variable. It is assumed to follow an Ornstein-Uhlenbeck process, and the risk premium is affine in the state variable. They find closed-form solutions to a portfolio choice problem with stochastic investment opportunities, where the investors have hyperbolic absolute risk aversion (HARA) and only derives utility from terminal wealth. Wachter (2002), on the other hand, considers a slightly different version of Kim and Omberg's portfolio choice problem. The investors are now modelled with constant relative risk aversion (CRRA) utility, but derive utility from both intermediate consumption and terminal wealth. Liu (2007) provides a general model that incorporates the aspects of Wachter, and Kim and Omberg while having special cases with time-varying volatility and inflation uncertainty as in Munk et al. (2004).

Evaluating the investor's investment strategies which the portfolio models have provided are important. Because it gives an understanding of how well or by how much an optimal investment strategy outperforms a suboptimal strategy. This issue is addressed in Larsen and Munk (2012), where they focus on the costs of suboptimal asset allocation. They provide a general theoretical framework to evaluate suboptimal investment strategies under the assumptions of the optimal investment strategy. Larsen and Munk do three applications of this general framework, which are focused on (i) interest rate risk, (ii) stochastic stock price volatility, and (iii) investing in value stocks and growth stocks. Larsen (2010) considers a portfolio choice problem with a dynamically complete international markets, and finds the wealth loss for an investor who do not include international allocation in the investment strategy. This brief review of literature entails an area which is vast in all its dimensions. In this context, we want to quote Wachter (2010):

> *"Ultimately, the goal of academic work on asset allocation is the conversion of the time series of observable returns and other variables of interest into a single number: Given the preferences and horizon of the investor, what fraction of her wealth should she put in stock? The aim is to answer this question in a "scientific" way, namely by clearly specifying the assumptions underlying the method and developing a consistent theory based on these assumptions. The very specificity of the assumptions and the resulting advice can seem dangerous, imputing more certainty to the models than the researcher can possibly possess. Yet, only by being so highly specific, does the theory turn into something that can be clearly debated and ultimately refuted in favor of an equally specific and hopefully better theory."*

From this quote, we continue by introducing institutional investors and their preferences towards risk, before we begin with allocation models, which are to answer our research question.

Chapter 2

Fundamental Theoretical Concepts

This chapter provides the fundamental theoretical building blocks underlying the area of dynamic asset allocation literature which is important to understand in order to do financial modelling. This is coupled with an formal introduction of the institutional investor types and their significant role in the financial sector. We have chosen to limited our problem of interest to only considering institutional investors rather than considering a private investor, which has modelling implications for the utility maximisation. The economic problem specifies itself to only consider utility maximisation over terminal wealth, instead of an economic problem which also includes consumption streams, and for example labour income over an investment horizon.

This chapter is organised as follows. Section 2.1 provides a formal description of the institutional investor and motivates why the institutional investor is of interest for financial modelling, while Section 2.2 introduces the concepts of risk aversion and utility functions.

2.1 Description of Institutional Investors

Institutional investors are organised as legal entities. According to Çelik and Isaksson (2014), the legal form varies across the institutional investors as well as their purposes, which could be from a profit maximising joint stock company to a limited liability partnerships (i.e. private equity funds), and in some cases sovereign wealth funds. Institutional investors may act independently or as part of a larger group of banks and insurance companies, which is the case of a mutual fund. Institutional investors and asset managers are often used synonymously, implying an institution that both manages and invests, but there are exceptions to this. U.S. Securities and Exchange Commission (2015) defines an institutional investor as an entity that exercises investment decisions over $ 100 million or more in securities.

The paper by Çelik and Isaksson (2014), provides evidence that pension funds, investment funds, and insurance companies in the OECD countries have increased their assets under management from $36 trillion in 2000 to $73.4 trillion in 2011. The largest increase of these three categories is seen in investment funds as they have increased by 121%. This made the relative share of total assets under management held by institutional investors increase from 37% in 2000 to 40% in 2011. However, both pension funds and insurance companies invest in mutual funds which are part of the investment fund category. In conclusion, the institutional investors role as financial intermediaries have a great influence on the investment strategies over recent years along with deregulation and globalisation of financial markets. Asset managers are also included under the general heading of institutional investors, where they have the day-to-day responsibility of managing investments. The capital under their management is provided by individuals and most types of institutional investors, implying asset managers invest on the behalf of pensions funds and mutual funds according to their investment policy. According to Investment and Pension Europe (2016), the top 400 asset managers have a total of €50.3 trillion worth of assets under management, where BlackRock is the company with the largest amount of assets under management, specifically €3,844,383 million.

Table 2.1 highlights the top asset managers in a global context as well as in a Danish context. This is aggregated numbers for assets under management, that illustrates the non-trivial sums of capital allocated by the institutional investors. However, doing a finer segmentation than overall assets under management can show the distribution of capital invested by the institutional investors. This will show that specific asset managers are favoured depending on the type of investment.

Institutional Investors and asset managers take part in the economic development and growth of the global economy. They are at the same time private companies which are interested in profit maximising and yielding high returns for the shareholders. However, these companies can of course also be focusing on other aspects in their investment strategies like environmental issues and corporate social responsibility, but their core business strategy still comes down to generate a high return to their shareholders.

Denmark	AUM €m	Global	AUM €m
Nordea IM	173,873	BlackRock	3,844,382.90
Danske Capital	107,413	Vanguard AM	2,577,380.10
PFA Pension	54,724	State Street Global Advisors	2,023,149
Nykredit AM	17,917	BNY Mellon IM EMEA	1,407,164
BankInvest	13,380	J.P. Morgan AM	1,266,805

Table 2.1: Asset under management (AUM) for the top 5 worldwide and Danish asset managers. **Source:** Investment and Pension Europe (2016) - Table: *Top Global AUM Table 2015 and Danish Asset Managers Table 2015*. Notes: IM:Investment Managment, AM: Asset Management.

To emphasise the large amounts of capital invested by the institutional investors, Çelik and Isaksson (2014) show that OECD countries like the Netherlands, Switzerland, Denmark, and the United Kingdoms have assets under management for more than twice their GDP. In countries such as Mexico and Czech Republic it however accounts for less than their GDP.

Due to their size and the goal of maximising return it is of the utmost importance how institutional investors conduct their investment strategies, and this is exactly what this thesis intends to investigate. What is the optimal investment strategy under a set of relaxed assumptions, and what are the costs to the investors if they choose an suboptimal investment strategy? The next sections will look into how the institutional investors derive utility and how their risk-taking takes form.

2.2 Risk Aversion and Utility

To model a dynamic asset allocation problem a measure of investor's attitude towards risk is required in order to rank portfolio choices. An investor's attitude towards risk can be represented by a utility function, $u(W)$, which contains all information about the investor's preferences and attitude towards risk. The utility function must incorporate that it is increasing in terminal wealth.

Furthermore, the different attitudes towards risk should be modelled into the utility function. This idea is normally presented by a fair game as in Bernoulli's Saint Petersburg Paradox described in Seidl (2013). The paradox describes how a risk-neutral investor will be indifferent between the expected value of the game and a certain payment of the same amount. On the other hand, a risk-averse investor will accept a certain payment, the certainty equivalent, which is smaller than the expected payment from the game, but certain. How large the certain payment must be to make the investor indifferent, depends on the level of risk aversion and the uncertainty in the game.

The difference between the expected value from the game and the certainty equivalent, when the investor is indifferent between those two, is the risk premium. It must compensate the risk-averse investor for the risk of the game. The risk premium will increase as the uncertainty in the game becomes larger, and the certainty equivalent thereby becomes smaller[1].

Traditionally in financial economics, investors are modelled with being risk averse, implying that investors should be compensated for their risk-taking. The utility function of wealth for a risk-averse investor must therefore be concave. Mathematically this implies that $u'(W) > 0$ and $u''(W) < 0$, where utility is strictly increasing in wealth, but increasing at a decreasing rate. The effect of the concave utility function shows that the risk averse investor will have a higher weight on losses than on winnings due to the positive but decreasing marginal effect.

In order to quantify the risk aversion, the Arrow-Pratt risk measures are used. Arrow (1970) defines two measures; the first measure is the absolute risk aversion (ARA), which quantifies the aversion towards risk to a monetary amount and is defined as follows

$$ARA(W) = -\frac{u''(W)}{u'(W)}. \tag{2.2.1}$$

The other measure is relative risk aversion (RRA), which is defined by taking wealth of the investor into consideration

$$RRA(W) = W \cdot ARA(W). \tag{2.2.2}$$

The relative risk aversion indicates the willingness of an investor to avoid a risky payment of a given size relative to the level of wealth. The established literature within this field tends to model investors with constant relative risk aversion (CRRA). To be more specific in the discussion of relative and absolute risk aversion, we will have to assume a form of the utility function. One could start a long and tedious discussion about the different kinds of utility functions and their advantages and dis-

[1] In some special cases such as insurance the certainty equivalent will be larger than the expected value Hiller et al. (2012).

advantages[2]. We simply move on by assuming the isoelastic power utility function that is defined as

$$u(W) = \frac{W^{1-\gamma}}{1-\gamma},$$

which is commonly used in the theory of asset pricing. It is a function of the investor's wealth, W, and the risk aversion parameter, γ. It exhibits the requirements for a risk averse investor with a concave utility function as $u'(W) = W^{-\gamma} > 0$ and $u''(W) = -\gamma W^{-\gamma-1} < 0$. With the definition of the utility function and the measures of risk aversion from Equation (2.2.1) and Equation (2.2.2). The absolute and relative risk aversion for this specific investor are

$$ARA(W) = \frac{\gamma}{W}$$
$$RRA(W) = \gamma.$$

From these risk measures of our specific investor assumptions, it is seen that relative risk aversion is constant when $\gamma > 0$. The absolute risk aversion is decreasing as the initial wealth is increasing. These measures of risk aversion makes an investor with an initial wealth of $1,000 less averse about betting $10 than an investor with initial wealth of $10. This seems as a fair assumption as the bet is a much larger fraction of the wealth for the second investor than for the first investor.

We will use this in modelling our problem in the following chapters. When considering the model with constant investment opportunities and the following model assuming a stochastic interest rate, the utility function will be used as a special case. From the risk aversion measures above, the special case will represent the allocation for an investor with decreasing absolute risk aversion, but a constant relative risk aversion. The utility function is an necessary part of the portfolio allocation problem as the investor is interested in maximisation of the utility.

[2]For a discussion of utility functions see for example Chapter 1 of Pennacchi (2008)

Chapter 3

Mean-Variance Analysis

In this chapter the one-period mean-variance analysis described by Markowitz (1952) is introduced. This chapter will mainly be based on Flor and Larsen (2011), but the focus here is on the portfolio allocation results and the intuition rather than the assumptions. This model serves as a good intuitive starting point for the effect of risk aversion and portfolio allocation. In Section 3.1 the model assumptions are discussed, while Section 3.2 continues with the general portfolio allocation result, and finally consider two specific cases of utility functions.

3.1 Introduction to the Model

Without specifying the utility function at this point, the investor is still assumed to be risk averse implying that the utility is increasing in wealth, but at a decreasing rate, $u'(W) > 0$ and $u''(W) < 0$. The investor can allocate wealth to either risky assets or a risk-free asset. The risk-free asset will have a certain return equal to the risk-free interest rate r. The risky assets will on the other hand not have a simple value for their return. The returns for the risky assets will be normally distributed, $\boldsymbol{R} \sim N(\boldsymbol{\mu}, \underline{\underline{\Sigma}})$, where the return vector $\boldsymbol{R}$ contains d risky assets. The expected returns are given by the vector $\boldsymbol{\mu}$ and the variance-covariance matrix $\underline{\underline{\Sigma}}$. The investor's terminal wealth, W_T, is the product of the initial wealth, W_0, and the returns from risk-free and risky assets

$$W_T = W_0[1 + r + \boldsymbol{\pi}^\top(\boldsymbol{\mu} - r\mathbf{1})]. \tag{3.1.1}$$

In Equation (3.1.1), $\boldsymbol{\pi}^\top$ is introduced, which is the vector with the weights in the d risky assets. The vector is defined as $\boldsymbol{\pi}^\top = (\pi_1, ..., \pi_d)$. These portfolio weights are important, as they are what the portfolios must be optimised with respect to.

3.2 The Optimisation Problem

In the mean-variance analysis the efficient portfolios are the ones that minimise the variance for a given return with the constraint that the portfolio weights must sum to one. Mathematically, the minimization problem is

$$\begin{aligned} \min \frac{1}{2}\boldsymbol{\pi}^\top \underline{\underline{\Sigma}}\boldsymbol{\pi} \\ s.t.\ \boldsymbol{\pi}^\top \boldsymbol{\mu} = \bar{\mu}, \\ \boldsymbol{\pi}^\top \mathbf{1} = 1. \end{aligned}$$

Given the two constraints, the minimisation can be solved by the use of Lagrange. The different portfolios for different given levels of return will produce an efficient frontier. The efficient frontier will be a hyperbola and consists of the portfolios that minimise the variance for a given return. The frontier consists of the most efficient combinations of assets.

A second frontier can be generated by introducing the risk-free asset, which the investor can combine in a portfolio with the risky assets. The efficient frontier from these portfolio will be a straight line. The straight line will at a certain point be tangent to the first efficient frontier from the case without a risk-free asset. The efficient frontiers; the case with and the case without a risk-free asset are illustrated in Figure 3.1. It also shows an example of the tangency portfolio and inefficient portfolios which are located below the efficient frontiers. At the tangency point, investors can by the use of only risky assets produce the same return and variance as in the case with a risk-free asset. The tangency portfolio can mathematically be defined by maximising the excess return relative to the variance and will be given as

$$\boldsymbol{\pi}_{tangency} = \frac{\underline{\underline{\Sigma}}^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{\mathbf{1}^\top \underline{\underline{\Sigma}}^{-1}(\boldsymbol{\mu} - r\mathbf{1})}. \tag{3.2.1}$$

The method to calculate the tangency portfolio leads to two important concepts about the tangency portfolio. First, it maximises the Sharpe-ratio which Sharpe (1966) introduced. Secondly, it is also linked to the famous Capital Asset Pricing Model by a direct link to the security market line where it has the value $\beta = 1$. The result is that the investor will use the tangency portfolio as part of a two-fund separation, where the investor combines an allocation in the risk-free with the rest of the wealth allocated in the tangent portfolio.

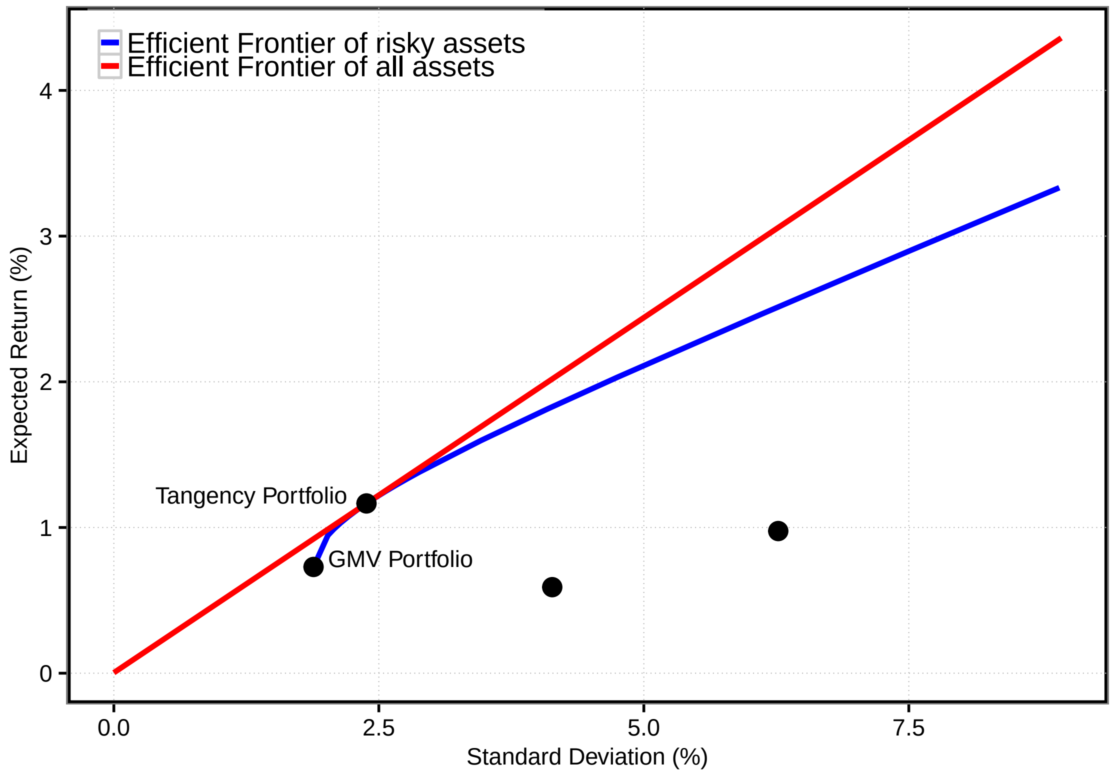

Figure 3.1: Efficient frontiers for the case with and without a risk-free asset. The blue line indicates the efficient portfolios, when there is no risk-free asset. The red line is the frontier, when there is a risk-free asset. The two dots without a label are indicating inefficient portfolios, whereas the two portfolios with labels indicate the tangency portfolio and the global minimum-variance portfolio.

The optimal combination between the two funds depend on the investor's utility function. In Section 2.2, we argued for using a utility function which would have CRRA. Unfortunately, this will not give a reasonable result in the mean-variance analysis. By the assumption of such a utility function, the investor will allocate all her wealth in the risk-free assets due to the probability of negative wealth, as a consequence of the normally distributed risky returns. An alternative is to assume a utility function, which will have constant absolute risk aversion (CARA) instead. This will, depending on the assumed parameters, make the investor allocate wealth in both of the two funds.

Besides the issues related to a realistic utility function with CRRA, the mean-variance analysis is a good starting point. It shows that the investor must combine the different assets to maximise the terminal wealth. Figure 3.1 shows how different levels of risk aversion, and thereby different locations on the straight efficient frontier, will yield different levels of return and variance.

Chapter 4

Dynamic Model with Constant Opportunities

To consider a more realistic model, we will go from the one-period mean-variance model into a dynamic model. This will make the wealth and its elements different. There will instead of a simple wealth formula be a description of wealth dynamics, which makes the calculation of the optimal allocation more complex. The chapter is based on Chapter 6 of Munk (2013).

As for the mean-variance analysis we will start by describing the different wealth elements before setting up the investors wealth. This is presented in Section 4.1. Afterwards we calculate the optimal allocation at a general level in Section 4.2, which we then use in Section 4.3 for the specific case of an investor with CRRA-utility.

4.1 Investor's Wealth

New assumptions are introduced for both the risk-free asset and the risky assets, respectively bonds and stocks. The asset types are presented individually and then combined in an expression for the investor's total wealth.

4.1.1 Bonds

For the risk-free, the return is still assumed to be the constant interest rate r. In this dynamic model the interest rate is assumed to be continuously compounded and the value of \$ 1 today will then be

$$e^{rT},$$

at time T, when the money has been invested in the risk-free asset for T years.

4.1.2 Stocks

Stocks serve in this model as the risky assets in which the investor can potentially invest. For this multi-period model a process to describe the development for the price of the stocks is needed. The stock returns are initially discretely defined to better understand the steps. First we define the stock price at time t as S_t. The return is given as

$$R_{t+\Delta t} = \frac{S_{t+\Delta t} - S_t}{S_t} = \mu\Delta t + \sigma\varepsilon_{t+\Delta t}\sqrt{\Delta t}, \tag{4.1.1}$$

from where we can see how the return will be the change in price of the stock over the time interval Δt relative to the price at time t. The return is dependent on the expected return μ, the independent shocks to the economy ε, and the stock volatility σ. However, moving from discrete time into a model in continuous time implies $\Delta t \to 0$. The returns in continuous time are thereby written as

$$\frac{dS_t}{S_t} = \mu dt + \sigma dz_t. \tag{4.1.2}$$

The expression is almost similar, but the assumption of a continuous change in prices will be more reasonable. According to Hull (2012) this is not only a common way to model the price; the modelling is also provides a good fit. For example are the continuous price changes used in the Black-Scholes-Merton model.

In Equation (4.1.2) we use dz_t as the expression for $z_{t+\Delta t} - z_t = \varepsilon\sqrt{\Delta t}$ as $\Delta t \to 0$. We assume dz_t to be a Brownian motion. For this to be true it must fulfil the following four assumptions, which are given in Øksendal (2003). For consistency we do however follow the notation as in Appendix B in Munk (2013). The assumptions are:

1. $z_0 = 0$,
2. $z_{t'} - z_t \sim N(0, t' - t)$, for all $t, t' \geq 0$, and $t' > t$,
3. $z_{t_1} - z_{t_0}, \ldots, z_{t_n} - z_{t_{n-1}}$ are mutually independent for all $0 \leq t_0 < t_1 < \ldots < t_n$,
4. z has continuous paths.

Then is dz_t normally distributed and the stock price will follow a Markov process and therefore be independent of the past as it is memoryless.

Itô's lemma

The stock prices could in the mean-variance analysis potentially take on negative values. That is unrealistic and changed in this model. We will by the use of Itô's lemma find the dynamics of the stock prices, from where we can define the mean and variance of the returns.

Following Øksendal (2003) for a process as $dx_t = \mu_t dt + \sigma_t dz_t$, the function $y_t = g(x_t, t)$ has the dynamics:

$$dy_t = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dx_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dx_t)^2.$$

Using this formula for the description of the stock returns in Equation (4.1.2), the stock prices can be shown to be log-normally distributed. The stock price will then be unable to take on negative values, which is a more realistic assumption than the alternative from the mean-variance analysis.

Assume $g(S,t) = \ln(S)$. This will give the partial derivatives

$$\frac{\partial g}{\partial t} = 0, \quad \frac{\partial g}{\partial S} = \frac{1}{S_t} \quad \text{and} \quad \frac{\partial^2 g}{\partial S^2} = -\frac{1}{S_t^2}.$$

We can then substitute the partial derivatives into the lemma from above. Afterwards it must be simplified. This is done by the use of a previous result, where dz is said to be normally distributed. Again with reference to Øksendal (2003) it can be more precisely defined as $dz_t \sim N(0, dt)$. From that we can use the rules $(dt)^2 = dt \cdot dz_t = 0$ and $(dz_t)^2 = dt$. The dynamics become

$$\begin{aligned} d\ln S_t &= \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial S^2}(dS_t)^2 \\ &= 0dt + \frac{1}{S_t}(\mu S_t dt + \sigma S_t dz) - \frac{1}{2}\frac{1}{S_t^2}(\mu S_t dt + \sigma S_t dz)^2 \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dz. \end{aligned}$$

This is the dynamics for the logarithm of the stock price at time t. To show the distribution we will use $d\ln S_t = \ln S_{t+\Delta t} - \ln S_t$, where the relative return in Equation (4.1.2) and Equation (4.1.1) is multiplied with the stock price S_t. When considering discrete time for the definition of the drift on the left-hand side, we must also change the definition on the right-hand side. The drift is then

$$\ln(S_{t+\Delta t}/S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma(z_{t+\Delta t} - z_t),$$
$$\ln S_{t+\Delta t} \sim N\left(\ln S_t + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t, \sigma^2 \Delta t\right)$$

where $\ln S_{t+\Delta t}$ afterwards is isolated and defined as a normal distribution. For the terminal stock price it is written as

$$\ln S_T \sim N\left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right),$$

which shows that the stock price is no longer normally distributed. The stock price is instead log-normally distributed and it is therefore impossible for it to become negative, which is the more realistic. From this we find the mean and variance for the returns, $\ln(S_T/S_0)$. The mean is given by

$$\mathrm{E}[\ln(S_T/S_0)] = (\mu - \frac{1}{2}\sigma^2)T$$

and the variance

$$\mathrm{Var}[\ln(S_T/S_0)] = \sigma^2 T.$$

For calculations of the stock price dynamics, or mean and variance of the returns, we refer to Appendix A.1.

4.1.3 Total Wealth

In Section 4.1.1 and 4.1.2 the two wealth elements have been described. These terms will be gathered in a term for the total wealth. The first step is to define the wealth as the sum of d assets multiplied with their individual price. $M^i_{t-\Delta t}$ is our number of asset i from time $t-\Delta t$ to t and P_t is the price of the asset at time t. By summation we then have the the wealth at time t:

$$W_t = \sum_{i=0}^{d} M^i_{t-\Delta t} P^i_t. \tag{4.1.3}$$

As we here work in a discrete set-up, we cannot have any change of assets between periods. There is no other income than initial wealth, but likewise there is no outflow of wealth before the terminal period. The only reason for changes in wealth between periods must therefore be changes in prices. The changes in wealth between periods can therefore be written as

$$W_{t+\Delta t} - W_t = \sum_{i=0}^{d} M_t^i (P_{t+\Delta t}^i - P_t^i).$$

Changes in prices must more specifically be a consequence of the returns. We want to rewrite the wealth changes by the use of returns instead. We therefore need some amounts which we can multiply with the risk-free and risky return assets. By the definitions from Equation (4.1.3) define element $\theta_t^i = M_t^i P_t^i$ as the value invested in asset i. The amount in a risk-free asset is defined as θ_t^0 and for the risky assets the vector is

$$\boldsymbol{\theta}_t = (\theta_t^1, \cdots, \theta_t^d)^\top.$$

To rewrite the changes in wealth use the previous definitions of returns for both bonds and stocks. The return on the risk-free asset is as before, r. The return for risky assets is now written in matrix-form

$$\boldsymbol{R}_{t+\Delta t} = \boldsymbol{\mu}\Delta t + \underline{\underline{\sigma}}\varepsilon_{t+\Delta t}\sqrt{\Delta t},$$

which is based on Equation (4.1.1). Bold notation, $\boldsymbol{x}$, indicates vectors, and underlining, $\underline{\underline{x}}$, indicates a matrix. The wealth changes are rewritten as

$$W_{t+\Delta t} - W_t = \theta_t^0 r\Delta t + \boldsymbol{\theta}_t^\top(\boldsymbol{\mu}\Delta t + \underline{\underline{\sigma}}\varepsilon_{t+\Delta t}\sqrt{\Delta t}).$$

The changes in wealth are due to returns on risk-free and risky assets. The terms in the discrete description can be rearranged into continuous time by $\Delta t \to 0$. We will thereby have the continuous description

$$dW_t = [\theta_t^0 r + \boldsymbol{\theta}_t^\top \boldsymbol{\mu}]dt + \boldsymbol{\theta}_t^\top \underline{\underline{\sigma}} d\boldsymbol{z}_t. \tag{4.1.4}$$

The wealth dynamics are to be changed again. Equation (4.1.4) is changed such that it depends on actual portfolio weights. It thereby presents the risk-free return and the market price of risk that shall compensate investors for taking risk. First, to define the market price of risk use the alternative description of price dynamics for the risky assets

$$d\boldsymbol{P}_t = diag(\boldsymbol{P}_t)[(r\mathbf{1} + \underline{\underline{\sigma}}\boldsymbol{\lambda})dt + \underline{\underline{\sigma}}d\boldsymbol{z}_t], \tag{4.1.5}$$

with $diag(\boldsymbol{P}_t)$ being a diagonal matrix of the prices. This introduces a new expression, $(r\mathbf{1} + \underline{\underline{\sigma}}\boldsymbol{\lambda})$, where we have the risk-free interest rate and $\underline{\underline{\sigma}}\boldsymbol{\lambda} = \boldsymbol{\mu} - r\mathbf{1}$. In the second term we isolate $\boldsymbol{\lambda}$ and then have the market price of risk given as $\boldsymbol{\lambda} = \underline{\underline{\sigma}}^{-1}\boldsymbol{\mu} - r\mathbf{1}$. Secondly, the portfolio weights are defined as the relative amount of the total wealth which is allocated in risky assets. In vector-form this is

$$\boldsymbol{\pi}_t = \frac{\boldsymbol{\theta}_t}{W_t}.$$

The final definition of the wealth dynamics under the assumption of constant investment opportunities is

$$dW_t = W_t[r + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}\boldsymbol{\lambda}]dt + W_t\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}d\boldsymbol{z}_t. \tag{4.1.6}$$

This definition is central for the rest of the chapter. The investor's utility is directly dependent on maximising the terminal wealth, and we want to do this maximisation with respect to the portfolio weights, $\boldsymbol{\pi}$, for risky assets. After including the portfolio weights in the definition of the wealth dynamics, we are therefore able to move on to the maximisation problem for the institutional investors.

4.2 Utility Maximization

Two different asset types have been gathered into a combined expression for the investor's wealth dynamics. The investor's maximization problem is then one step closer the solution for optimal allocation. For maximising the utility, there are still two steps left. First, the maximisation of the investors utility is done at a general level without specifying the utility function. It is however restricted to only focusing on terminal wealth as defined in Section 2.2. The final step is to define the utility function and use the general solution of the maximisation problem to define the optimal portfolio weights for the investor.

4.2.1 Utility Maximization

The investor wants to maximise the expected utility, where utility is only a function of the investor's terminal wealth as previously specified. We therefore have the maximization problem

$$J(W,t) = \sup_{\boldsymbol{\pi}} \quad E_{W,t}\left[u(W_T)\right].$$

As for the dynamics of the stock, Itô's lemma must also be applied to this maximization problem. These dynamics will be the Hamilton-Jacobi-Bellman (HJB) which must be solved for the optimal portfolio weights $\boldsymbol{\pi}$ in order to maximise the expected utility. By the use of Itô's lemma and substitution of the wealth dynamics, the HJB equation will be

$$0 = \sup_{\boldsymbol{\pi}} \left\{ \frac{\partial J}{\partial t}(W,t) + rWJ_W(W,t) + \underbrace{\frac{1}{2}J_{WW}(W,t)W^2\boldsymbol{\pi}^\top \underline{\underline{\sigma\sigma}}^\top \boldsymbol{\pi} + WJ_W(W,t)\boldsymbol{\pi}^\top \underline{\underline{\sigma}}\boldsymbol{\lambda}}_{\text{Portfolio weight dependent}} \right\}.$$

This HJB equation must be solved under the terminal condition $J(W,T) = \bar{u}(W)$. We maximise with respect to the portfolio weights, and the interest is therefore in the two terms on the right-hand side which are dependent on the portfolio weights. Following the literature these are defined as

$$\mathcal{L}^\pi J(W,t) = \sup_{\boldsymbol{\pi}} \left\{ WJ_W(W,t)\boldsymbol{\pi}^\top \underline{\underline{\sigma}}\boldsymbol{\lambda} + \frac{1}{2}J_{WW}(W,t)W^2\boldsymbol{\pi}^\top \underline{\underline{\sigma\sigma}}^\top \boldsymbol{\pi} \right\}. \tag{4.2.1}$$

Next step is to differentiate with respect to the portfolio weights and obtain

$$WJ_W(W,t)\underline{\underline{\sigma}}\boldsymbol{\lambda} + J_{WW}(W,t)W^2\underline{\underline{\sigma\sigma}}^\top \boldsymbol{\pi} = 0,$$

where we afterwards isolate $\boldsymbol{\pi}$

$$-\frac{J_W(W,t)}{J_{WW}(W,t)W}(\underline{\underline{\sigma\sigma}}^\top)^{-1}\underline{\underline{\sigma}}\boldsymbol{\lambda} = \boldsymbol{\pi}.$$

In this, the multiplication of the variance-covariance matrices is reduced such that the general result for optimal portfolio weights are given as

$$\boldsymbol{\pi} = -\frac{J_W(W,t)}{J_{WW}(W,t)W}(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\lambda}. \tag{4.2.2}$$

For a meaningful interpretation of the portfolio weights, a definition of the utility function is needed, which must be verified as a solution. Two comments can be made at this point; The fraction is the inverse of the relative risk aversion from Section 2.2, and the expression $(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\lambda}$ makes the investor follow the result from the mean-variance analysis in Chapter 3 with two-fund separation. The wealth in risky assets will be allocated among them according to the tangency portfolio

$$\boldsymbol{\pi}_{tangency} = \frac{(\underline{\underline{\sigma\sigma}}^\top)^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{\mathbf{1}^\top(\underline{\underline{\sigma\sigma}}^\top)^{-1}(\boldsymbol{\mu} - r\mathbf{1})}.$$

It is important to notice that even though the results are very similar, the assumptions are different.

The general descriptions of the optimal portfolio weights are then substituted back into Equation (4.2.1)

$$\mathcal{L}^{\pi} J(W,t) = W J_W(W,t) \left(-\frac{J_W(W,t)}{J_{WW}(W,t)W} (\underline{\underline{\sigma}}^{\top})^{-1} \boldsymbol{\lambda} \right) \underline{\underline{\sigma}} \boldsymbol{\lambda}$$
$$+ \frac{1}{2} J_{WW}(W,t) W^2 \left(-\frac{J_W(W,t)}{J_{WW}(W,t)W} (\underline{\underline{\sigma}}^{\top})^{-1} \boldsymbol{\lambda} \right) \underline{\underline{\sigma}\sigma}^{\top} \left(-\frac{J_W(W,t)}{J_{WW}(W,t)W} (\underline{\underline{\sigma}}^{\top})^{-1} \boldsymbol{\lambda} \right),$$

and it is then reduced to the following expression

$$\mathcal{L}^{\pi} J(W,t) = -\frac{1}{2} \left(\frac{J_W(W,t)^2}{J_{WW}(W,t)} \|\boldsymbol{\lambda}\|^2 \right).$$

The reduction of the expression is shown in more steps in Appendix A.2. The two terms on the right-hand side of the HJB equation, which are dependent on $\boldsymbol{\pi}$, are maximised and substituted back into the equation which is then

$$0 = -\frac{1}{2} \frac{J_W(W,t)^2}{J_{WW}(W,t)} \|\boldsymbol{\lambda}\|^2 + \frac{\partial J}{\partial t}(W,t) + rW J_W(W,t). \tag{4.2.3}$$

To maximise the investor's utility, we must as the next step define a utility function, which can solve the HJB equation. After verifying it to be a solution to the HJB equation it can be used to give a more specific definition of the portfolio weights, which were defined in Equation (4.2.2).

4.3 Findings for a CRRA-investor

As in Section 2.2, we follow the existing literature and suggest the isoelastic utility function

$$J(W,t) = \frac{g(t)^{\gamma} W^{1-\gamma}}{1-\gamma} \tag{4.3.1}$$

as a potential solution to the HJB equation in (4.2.3). It follows our previously defined criteria for a utility function; it only gives utility from terminal wealth, it is concave in wealth, and it can be shown to give the investor constant relative risk aversion. The partial derivatives are calculated in Appendix A.3, but in this case only the following three are necessary

$$\begin{aligned} J_W(W,t) &= g(t)^\gamma W^{-\gamma} \\ J_{WW}(W,t) &= -\gamma g(t)^\gamma W^{-\gamma-1} \\ \frac{\partial J}{\partial t}(W,t) &= \frac{\gamma}{1-\gamma} g(t)^{\gamma-1} g'(t) W^{1-\gamma}. \end{aligned} \tag{4.3.2}$$

By substitution of the three derivatives, the HJB equation will become

$$0 = -\frac{1}{2}\frac{(g(t)^\gamma W^{-\gamma})^2}{-\gamma g(t)^\gamma W^{-\gamma-1}}\|\boldsymbol{\lambda}\|^2 + \frac{\gamma}{1-\gamma} g(t)^{\gamma-1} g'(t) W^{1-\gamma} + rWg(t)^\gamma W^{-\gamma}, \tag{4.3.3}$$

which must be fulfilled with the terminal condition $g(T) = 1$. Equation (4.3.3) can be reduced. Create two terms within a parenthesis and take $W^{1-\gamma}g(t)^{\gamma-1}$ outside the parenthesis. It will not make sense for neither $W^{1-\gamma}$ nor $g(t)^{\gamma-1}$ be to zero. Therefore must the two terms

$$0 = \left(-r - \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2\right) g(t) - \frac{\gamma}{1-\gamma} g'(t),$$

which are left on the right-hand side of the HJB equation, have to be equal to zero. Define the constant $A = \frac{1-\gamma}{\gamma}\left(-r - \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2\right)$ such that the function $g(t)$ from the defined utility must solve the ordinary differential equation

$$g'(t) = A \cdot g(t), \tag{4.3.4}$$

with the terminal condition $g(T) = 1$. For this case with an investor, who only gets utility from terminal wealth, the ordinary differential equation is solved by

$$g(t) = \exp\{-A(T-t)\}.$$

This solution solves the ordinary differential equation, and the partial differential equation is thereby also solved. The suggested utility function from Equation (4.3.1) is therefore a solution to Equation (4.2.3) and we can continue by considering the optimal portfolio. Recall the general definition of the portfolio weights from Section 4.2.1:

$$\boldsymbol{\pi} = -\frac{J_W(W,t)}{J_{WW}(W,t)W}(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\lambda}.$$

Substituting the relevant partial derivatives of the utility function and rearranging will give

$$\mathbf{\Pi}(W,t) = \frac{1}{\gamma}(\underline{\underline{\sigma}}^{\top})^{-1}\boldsymbol{\lambda}. \tag{4.3.5}$$

This is the optimal portfolio weights of wealth to be allocated in risky assets for an investor, with utility from terminal wealth only, and who has constant relative risk aversion.

4.3.1 Implications for a CRRA-investor

This model will serve as our benchmark case in the world of dynamic portfolio models. It is the most simple case because of the assumption about constant investment opportunities.

For institutional investors it is clear how they should invest. As in the explanation of the intuition under the mean-variance analysis, the investor will have to allocate wealth between a risk-free asset and the tangency portfolio. Under both models the tangency portfolio is used, but the use of the dynamic model makes it possible to specify an allocation for the CRRA-investor between risk-free and risky assets, which was not possible in the previous model.

Realism of assumptions and results are the main reasons for going from a simple one-period model into a dynamic multi-period model. Assuming constant investment opportunities may not be very realistic. This is the topic for the next chapter, where more specifically the assumption of a constant interest rate is relaxed.

Chapter 5

Dynamic Model with a Stochastic Interest Rate

The concept of constant investment opportunities are illusory in the eyes of a financial professional. In this chapter, we therefore relax one of the underlying assumptions by making the interest rate stochastic.

In Section 5.1, we discuss why you should let the interest rate be stochastic and in Section 5.2, we introduce the Vasicek model that is used to determine the stochastic behaviour of interest rate. Section 5.3 deals with the new price dynamics under the stochastic interest rate. In Section 5.4 the results under the new model are calculated, but we do not introduce any specific utility function before going through the case of a CRRA-investor in Section 5.5. The allocation results are analysed in Section 5.6, and related to the real world in Section 5.7. Finally, Section 5.8 relates the findings to other interest rate models.

5.1 Motivation for a Stochastic Interest Rate

We choose to introduce a stochastic interest rate as both the nominal and the real interest rates are known to vary across time. In Figure 5.1, it is shown how the interest rate has behaved for Treasury Bonds with different maturities, respectively for 3 month, 10 years and 30 years. Incorporating the feature of a stochastic interest rate into the model for optimal portfolio choice imply relaxing the assumption regarding the constant interest rate, hence moving away from the constant investment opportunity set and toward the stochastic investment opportunity set.

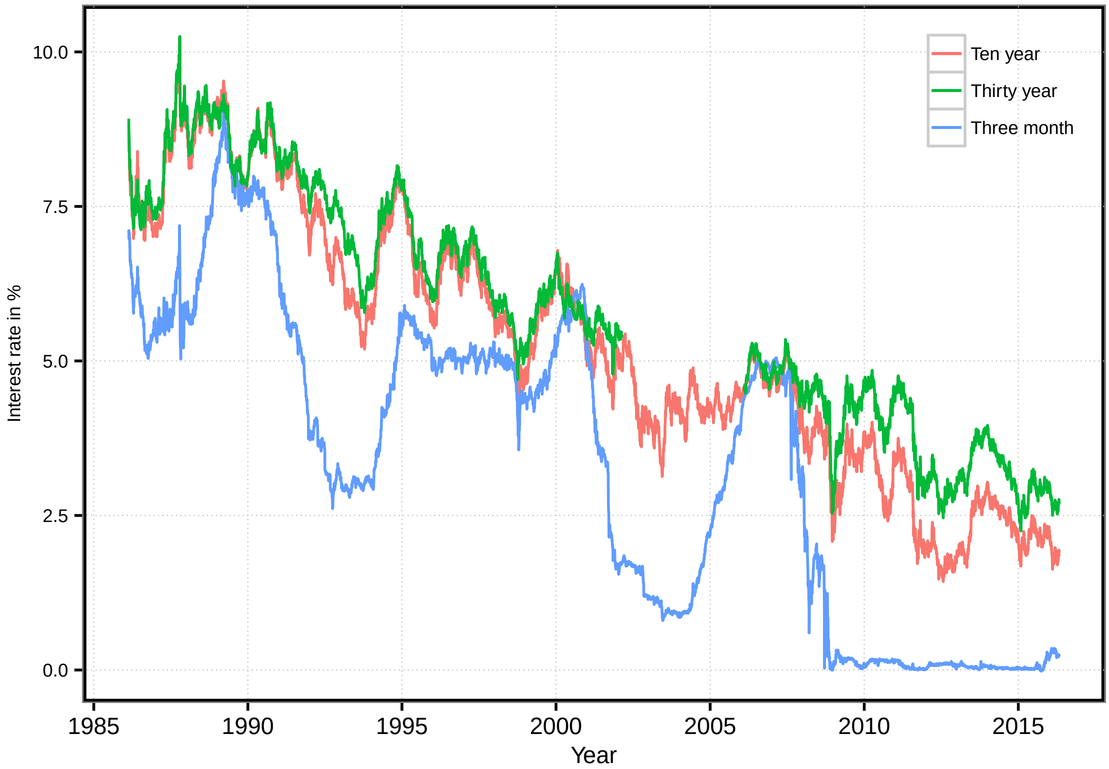

Figure 5.1: The figure presents the daily interest rates for three different U.S. Treasury Bonds from 1985 to 2015, which are not seasonally adjusted. **Data source:** FRED: DTB3, DGS10, and DGS30.

5.2 The Stochastic Interest Rate

In the seminal paper by Merton (1969), he looks into how an investor chooses the optimal investment strategy, specifically how many shares of which security the investor should over the given investment horizon to maximise expected utility from terminal wealth. In Merton's problem, essentially the investor can invest in a riskless money market account and n different stocks with different degrees of risk, however, a strong assumption in Merton's model is that the interest rates are deterministic. We want to introduce a stochastic process for the interest rate, and compare the analytical results to the case with constant investment opportunities.

We follow the existing literature such as Merton (1973) and Vasicek (1977) when modelling the stochastic interest rate, where the short rate is assumed to follow an Ornstein-Uhlenbeck process, which satisfies the stochastic differential equation:

$$dx_t = \theta[\mu - x_t]dt + \sigma dW_t$$

where $\theta > 0$, $\mu > 0$ and $\sigma > 0$ are parameters and W_t denotes a Wiener process. Moreover, the process has the characteristics of mean-reversion, implying that over time the process will drift towards the long-term mean.

We analyse portfolio problems in which the interest rate dynamics of the economy is described by the Vasicek model. In Vasicek (1977) the terminology is different, but it is here aligned with the notation used so far. The Vasicek model is a one-factor short rate model since all the behaviour of the interest rate is only determined by the market risk. The Vasicek model determines the instantaneous interest rate by the following stochastic differential equation:

$$dr_t = \kappa[\bar{r} - r_t]dt - \sigma_r dz_{1t}. \tag{5.2.1}$$

For the process it is here assumed that κ, $\bar{r}$ and σ_r are constant and positive. Further, mean-reversion is expected for r_t, which makes $r_t \neq \bar{r}$ lead to expected changes in the short rate. If the change will be upwards or downwards depends on the deviation from the mean. The mean-reversion in the process is due to the drift $\kappa[\bar{r} - r_t]$, which makes the process work towards the mean $\bar{r}$. The second term, σ_r, in the process is the stochastic element which leads the fluctuations around the long term mean $\bar{r}$.

In the literature the process is criticized for making the future short rate normally distributed, which makes it able to take on any negative value. The current monetary policies following the financial crisis in 2008 imply negative short-term interest rates by national banks in some countries and even the European Central Bank as presented in articles such as Randow and Kennedy (2015) and McAndrews (2015). This does however not fully justify the normal distribution as the distribution makes the short rate potentially take on any negative value. The previously mentioned properties are illustrated in the Figure 5.2 below, with different initial values and with parameter values of κ, $\bar{r}$ and σ_r.

Figure 5.2 shows a simulation of the short rate paths for the parameters of $r_0 = 0.03$, $\kappa = 0.03$, $\theta = 0.10$, and $\beta = 0.03$ for a Vasicek Model. The simulation is conducted with 10 trails with $T = 30$, where there is 200 sub-intervals. The expected value of r_t is included along two times the standard deviation to indicate how the short rate paths behave over time. It is seen there are paths that takes on negative values, which is not unnatural. Since the Vasicek model is based on a Ornstein-Uhlenbeck process, and it is likely for this process to generate negative values.

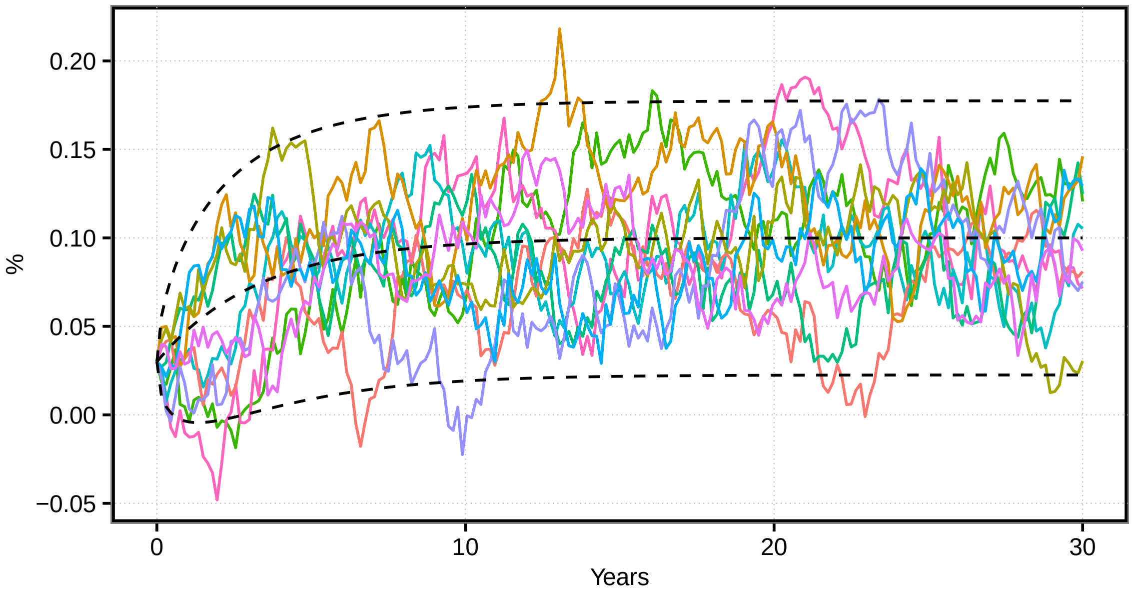

Figure 5.2: Illustrate simulated short rate paths of Vasicek One-Factor Model. The basic parameter values $r_0 = \kappa = 0.03$, $\theta = 0.10$, and $\beta = 0.03$

The interest rate process is used in the next section, where it is part of the wealth dynamics. The stochastic interest rate is included through the bond pricing and thereby also the bond dynamics.

5.3 New Description of Wealth Elements

A change of an assumption will lead to changes in the model. This section covers the descriptions of the wealth elements, because they have changed as a consequence of going from a constant to a stochastic interest rate. First we find the new dynamics for bonds and stocks, before combining them in the total dynamics for the investor's wealth.

5.3.1 Bond pricing and Dynamics

In Section 4.1.1, the terminal value of 1 monetary unit invested in a bond is given by e^{rT} when assuming a constant continuous interest rate. In this new set-up with a stochastic, but still continuous, interest rate, the price of a zero-coupon bond under a risk-neutral assumption is therefore written as

$$B(t,T) = E_t\left[e^{-\int_t^T r_\tau d\tau}\right]. \tag{5.3.1}$$

The price is the payment at maturity T discounted by the use of the definite integral of the interest rate from time t until maturity T. Subscript t indicates that this is the expected value at time t. Expectations are necessary as the interest rate is no longer constant nor certain.

This bond price is also the starting point for Vasicek's paper, where the finding for the specific case of an Ornstein-Uhlenbeck process is a zero-coupon bond price. We rewrite Vasicek's pricing equation to ensure consistency with our notation. For a step-by-step rewriting from one result to the other please refer to Appendix B.1. The calculations from the general price in Equation (5.3.1) to the zero coupon bond price, when assuming an Ornstein-Uhlenbeck process for the short-term interest rate, in Equation (5.3.2) are given in Appendix B.2. Following our notation the price is given as

$$\begin{aligned} B_t^{\bar{T}} &= e^{-y_\infty\left(\tau-\frac{1}{\kappa}(1-e^{-\kappa\tau})\right)-\frac{\sigma_r^2}{4\kappa^3}(1-e^{-\kappa\tau})^2-\left(\frac{1}{\kappa}(1-e^{-\kappa\tau})\right)r_t} \\ &= e^{-a(\bar{T}-t)-b(\bar{T}-t)r_t}, \end{aligned} \tag{5.3.2}$$

where

$$\begin{aligned} b(\tau) &= \frac{1}{\kappa}\left(1-e^{-\kappa\tau}\right) \\ a(\tau) &= y_\infty(\tau-b(\tau))+\frac{\sigma_r^2}{4\kappa}b(\tau)^2 \\ y_\infty &= \left(\bar{r}+\frac{\lambda_1\sigma_r}{\kappa}-\frac{\sigma_r^2}{2\kappa^2}\right). \end{aligned}$$

By the use of Itô's lemma the movements in the bond price can be found. As described in Section 4.1.2 it is known that when the interest rate follows a process $dx_t = \mu_t dt + \sigma_t dz_t$, then will the function $y_t = g(x_t, t)$ have the dynamics

$$dy_t = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dx_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dx_t)^2.$$

As seen from Equation (5.2.1), our interest rate follows a process similar to the one described just above. Our function $y_t = g(x_t, t)$ can then be the bond price. Using notation for the bond price, it will be

$$dB = \frac{\partial B}{\partial t}dt + \frac{\partial B}{\partial r_t}dr_t + \frac{1}{2}\frac{\partial^2 B}{\partial r_t^2}(dr_t)^2$$

Then substitute in the movement for the interest rate, which in a general form is given as $dr_t = \mu_t dt + \sigma_t dz_t$. This is of the same form as assumed in Equation (5.2.1). After the substitution and rearranging the process will be

$$dB = \left(\frac{\partial B}{\partial t} + \frac{\partial B}{\partial r_t}\mu\right) dt - \frac{\partial B}{\partial r_t}\sigma dz_t + \frac{1}{2}\frac{\partial^2 B}{\partial r_t^2}(\sigma^2 dz_t^2 + \mu^2 dt^2 - 2\sigma dz_t \mu dt)$$

Using the results $(dt)^2 = dt \cdot dz_t = 0$ and $(dz_t)^2 = dt$ the equation will be simplified and in a general form the bond price dynamics are given as

$$dB = \left(\frac{\partial B}{\partial t} + \mu\frac{\partial B}{\partial r} + \sigma^2 \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\right) dt + \sigma \frac{\partial B}{\partial r} dz_t.$$

In a model with Ornstein-Uhlenbeck as the interest rate process has the required form, the zero-coupon bond has the process

$$dB_t^{\bar{T}} = \left(\frac{\partial B_t^{\bar{T}}}{\partial t} + \kappa(\bar{r} - r_t)\frac{\partial B_t^{\bar{T}}}{\partial r} + \sigma_r^2 \frac{1}{2}\frac{\partial^2 B_t^{\bar{T}}}{\partial r^2}\right) dt + \sigma_r \frac{\partial B_t^{\bar{T}}}{\partial r} dz_{1t}.$$

The steps are to insert the first order partial derivative with respect to the interest rate, the second order derivative with respect to the interest rate, and the partial derivative with respect to time. This is followed by rearranging of the expression. Following the same steps as in Section 4.1.2, we can specify the function $g(x_t, t) = B_t^{\bar{T}}$. We therefore take the derivatives of Equation (5.3.2) with respect to time and the interest rate. They are

$$\begin{aligned}
\frac{\partial B_t^{\bar{T}}}{\partial t} &= \left(\frac{\sigma_r^2(1 - e^{-\kappa(T-t)})}{2\kappa^2 e^{\kappa(T-t)}} - \left(\frac{\lambda_1 \sigma_r}{\kappa} + \bar{r} - \frac{\sigma_r^2}{2\kappa^2}\right)\left(e^{-\kappa(T-t)} - 1\right) + r e^{-\kappa(T-t)}\right) B_t^{\bar{T}} \\
\frac{\partial B_t^{\bar{T}}}{\partial r_t} &= -\frac{1 - e^{-\kappa(T-t)}}{\kappa} B_t^{\bar{T}} \\
\frac{\partial^2 B_t^{\bar{T}}}{\partial r_t^2} &= \left(\frac{1 - e^{-\kappa(T-t)}}{\kappa}\right)^2 B_t^{\bar{T}}.
\end{aligned}$$

They can be substituted into the process for the bond price, which is shown above. With substitution we will have

$$\begin{aligned}
dB_t^{\bar{T}} = &\left(\frac{\sigma_r^2(1 - e^{-\kappa(T-t)})}{2\kappa^2 e^{\kappa(T-t)}} - \left(\frac{\lambda_1 \sigma_r}{\kappa} + \bar{r} - \frac{\sigma_r^2}{2\kappa^2}\right)\left(e^{-\kappa(T-t)} - 1\right) + r e^{-\kappa(T-t)}\right) B_t^{\bar{T}} dt \\
&- \left(\kappa(\bar{r} - r_t)\frac{1 - e^{-\kappa(T-t)}}{\kappa} - \sigma_r^2 \frac{1}{2}\left(\frac{1 - e^{-\kappa(T-t)}}{\kappa}\right)^2\right) B_t^{\bar{T}} dt \\
&+ \sigma_r \frac{1 - e^{-\kappa(T-t)}}{\kappa} B_t^{\bar{T}} dz_{1t}.
\end{aligned}$$

The bond price $B_t^{\bar{T}}$ is multiplied onto every term on the right-hand side of the equation, and we therefore divide by it. This reduces the right-hand side, which we are trying to simplify. It is then

$$\frac{dB_t^{\bar{T}}}{B_t^{\bar{T}}} = \left(\frac{\sigma_r^2(1-e^{-\kappa(T-t)})}{2\kappa^2 e^{\kappa(T-t)}} - \left(\frac{\lambda_1 \sigma_r}{\kappa} + \bar{r} - \frac{\sigma_r^2}{2\kappa^2} \right) \left(e^{-\kappa(T-t)} - 1 \right) + r e^{-\kappa(T-t)} \right) dt$$
$$- \left(\kappa(\bar{r} - r_t) \frac{1-e^{-\kappa(T-t)}}{\kappa} - \sigma_r^2 \frac{1}{2} \left(\frac{1-e^{-\kappa(T-t)}}{\kappa} \right)^2 \right) dt - \sigma_r \frac{1-e^{-\kappa(T-t)}}{\kappa} dz_{1t}.$$

We remove parentheses and terms which cancel out. Showing this many steps may seem tedious, but is done to ensure clarity.

$$= \left(\frac{\sigma_r^2(1-e^{-\kappa(T-t)})}{2\kappa^2 e^{\kappa(T-t)}} - \frac{\lambda_1 \sigma_r}{\kappa} \left(e^{-\kappa(T-t)} - 1 \right) - \bar{r} \left(e^{-\kappa(T-t)} - 1 \right) + \frac{\sigma_r^2 \left(e^{-\kappa(T-t)} - 1 \right)}{2\kappa^2} \right) dt$$
$$- \left(\bar{r} - r_t - \bar{r} e^{-\kappa(T-t)} + r_t e^{-\kappa(T-t)} - r e^{-\kappa(T-t)} - \frac{\sigma_r^2 \left(1 - e^{-\kappa(T-t)} \right)^2}{2\kappa^2} \right) dt$$
$$- \sigma_r \frac{1-e^{-\kappa(T-t)}}{\kappa} dz_{1t}$$

Several terms in the expression above cancel out and we can do some rearranging

$$\frac{dB_t^{\bar{T}}}{B_t^{\bar{T}}} = \left(\frac{\sigma_r^2(1-e^{-\kappa(T-t)})}{2\kappa^2 e^{\kappa(T-t)}} + \frac{\sigma_r^2}{2\kappa^2} \left(e^{-\kappa(T-t)} - 1 \right) + \frac{\sigma_r^2 \left(1 - e^{-\kappa(T-t)} \right)^2}{2\kappa^2} \right) dt$$
$$+ \left(r_t + \frac{\lambda_1 \sigma_r}{\kappa} \left(1 - e^{-\kappa(T-t)} \right) \right) dt + \sigma_r \frac{1-e^{-\kappa(T-t)}}{\kappa} dz_{1t}$$

The main focus is on the first parenthesis as it contains three fractions which will end up being equal to zero

$$\frac{dB_t^{\bar{T}}}{B_t^{\bar{T}}} = \underbrace{\left(\frac{-\sigma_r^2 - \sigma_r^2 e^{-2\kappa(T-t)} + 2\sigma_r^2 e^{-\kappa(T-t)}}{2\kappa^2} + \frac{\sigma_r^2 + \sigma_r^2 e^{-2\kappa(T-t)} - 2\sigma_r^2 e^{-\kappa(T-t)}}{2\kappa^2} \right)}_{=0} dt$$
$$+ \left(r_t + \frac{\lambda_1 \sigma_r}{\kappa} \left(1 - e^{-\kappa(T-t)} \right) \right) dt + \sigma_r \frac{1-e^{-\kappa(T-t)}}{\kappa} dz_{1t}.$$

Removing the first parenthesis simplifies the expression into

$$\frac{dB_t^{\bar{T}}}{B_t^{\bar{T}}} = \left(r_t + \frac{\lambda_1 \sigma_r}{\kappa} \left(1 - e^{-\kappa(T-t)} \right) \right) dt + \sigma_r \frac{1-e^{-\kappa(T-t)}}{\kappa} dz_{1t}.$$

Recall from the definition of the bond price above that $b(\bar{T}-t) = \dfrac{1-e^{-\kappa(\bar{T}-t)}}{\kappa}$, which makes it possible to simplify the movement of the zero-coupon bond even further into

$$\frac{dB_t^{\bar{T}}}{B_t^{\bar{T}}} = \big(r_t + \lambda_1 \sigma_r b(\bar{T}-t)\big)\, dt + \sigma_r b(\bar{T}-t) dz_{1t}$$

or equivalently

$$dB_t^{\bar{T}} = B_t^{\bar{T}} \left(\big(r_t + \lambda_1 \sigma_r b(\bar{T}-t)\big)\, dt + \sigma_r b(\bar{T}-t) dz_{1t}\right).$$

This is the dynamics of the price of a zero-coupon bond. Similarly, dynamics of any bond can be found. The more general version of the dynamics for a continuous coupon bond are found in Appendix B.3 and given as

$$dB_t = B_t \left((r_t + \lambda_1 \sigma_B(r_t, t))\, dt + \sigma_B(r_t, t) dz_{1t} \right). \tag{5.3.3}$$

5.3.2 Dynamics of the Stock Price

This section is inspired by the Appendix B in Munk (2013) and Pennacchi (2008). When determining the dynamics of the stock price, we often want to incorporate multiple price processes for different assets. We want to determine the covariances and correlations between the processes, since we are interested in knowing how these processes interact with each other. In our case, we are interested in seeing how the price processes of stocks and bonds respond to an exogenous shock, where the shock is defined as a one-dimensional process $z = (z_t)_{t\in[0,T]}$. But in this case the instantaneous increments of any two processes will be perfect correlated, which do not consider non-linear movements between the two processes. These processes are defined by two Itô processes as B and S

$$dB_t = \mu_{Bt} dt + \sigma_{Bt} dz_t, \qquad dS_t = \mu_{St} dt + \sigma_{St} dz_t$$

However, we are interested in having the characteristics of imperfect correlation in the changes over the shortest time period for the two processes. To obtain this, we must introduce an additional shock vector for the exogenous shock. By introducing a second exogenous shock $z_2 = (z_{2t})_{t\in[0,T]}$, we consider a two-dimensional set-up and we can thereby circumvent the issue of perfect correlation. The two processes for bonds and stocks can then be written as

$$\frac{dB_t}{B_t} = \mu_{Bt}dt + \sigma_{B1t}dz_{1t} + \sigma_{B2t}dz_{2t} \qquad \frac{dS_t}{S_t} = \mu_{St}dt + \sigma_{S1t}dz_{1t} + \sigma_{S2t}dz_{2t},$$

where $z_1 = (z_{1t})$ and $z_2 = (z_{2t})$ are independent Brownian motions. Having two-dimensional processes (dB_t, dS_t) we need to determine the first-order and second-order moments in order to find covariance and correlation for the two processes. The first-order moments are fully specified by μ_{Bt} and μ_{St}, where the four shock coefficients σ_{B1t}, σ_{B2t}, σ_{S1t}, and σ_{S2t} fully specify the second-order moments. The correlation function between the processes is

$$Corr_t[dB_t, dS_t] = \frac{\sigma_{B1t}\sigma_{S1t} + \sigma_{B2t}\sigma_{S2t}}{\sqrt{(\sigma_{B2t}^2 + \sigma_{B2t}^2) \cdot (\sigma_{S1t}^2 + \sigma_{S2t}^2)}}.$$

The two instantaneous variances and the instantaneous correlation are determined by the four shock coefficients, where different combinations of these coefficients give the same variances and correlation. This implies that we have an additional degree of freedom, and can therefore fix one of the four shock coefficients by setting it equal to zero. Since we are interested in how the stock price is affected by changes in the bond price, we choose to fix the shock coefficient, $\sigma_{B2t} = 0$. This will simplify the expressions for the two processes of B_t and S_t and yield the following dynamics

$$\frac{dB_t}{B_t} = \mu_{Bt}dt + \sigma_{Bt}dz_{1t} \qquad \frac{dS_t}{S_t} = \mu_{St}dt + \sigma_{St}(\rho_t dz_{1t} + \sqrt{1-\rho_t^2}dz_{2t}).$$

A different way to look at this is, the Weiner process dz_{2t} can be written as a linear combination of two other Weiner processes, one being dz_{1t}, and another process that is uncorrelated with dz_{1t}, such as dz_{2t}: $dz_{2t} = \rho dz_{1t} + \sqrt{1-\rho^2}dz_{2t}$.

Focusing on the process for the stock price, we can substitute in the definition of $\mu_{St} = r_t + \psi\sigma_{St}$, which gives

$$\frac{dS_t}{S_t} = (r_t + \psi_t\sigma_{St})dt + \sigma_{St}(\rho_t dz_{1t} + \sqrt{1-\rho_t^2}dz_{2t}). \tag{5.3.4}$$

The correlation parameter is ρ_t, that shows how the market returns of the stock and the bond is correlated over time, σ_S is the volatility of the stock, and the ψ is the Sharpe ratio of the stock which is assumed to be constant. The relationship between the two processes are now described as

$$\mathrm{Cov}_t[dB_t, dS_t] = \sigma_{Bt}\sigma_{Bt}\mathrm{Cov}_t\left(dz_{1t}, \rho_t dz_{1t} + \sqrt{1-\rho_t^2}dz_{2t}\right) = \sigma_{Bt}\sigma_{St}\rho_t dt.$$

If σ_{Bt} and σ_{St} are both positive, then the instantaneous correlation between the stock and bond processes is ρ_t. If σ_{Bt} and σ_{St} have opposite signs, then the instantaneous correlation is $-\rho_t$. The dynamics of the stock price now is dependent on how the bond dynamics evolve over time, and the correlation between these two price processes depends on the signs of the shock coefficients of each process. For detailed calculations, we refer to Appendix B.4.

5.3.3 Expression for Total Wealth Dynamics

A more in-depth description of the different steps in the process of combining the wealth dynamics is given in Section 4.1.3. We use the previous results and change them by describing how the different elements vary from before.

As defined in Section 4.1.3, we will use $\boldsymbol{\pi}$ as the vector containing the investments in risky assets and thereby being the fraction of the total wealth, which the investor has invested in risky assets. Previously it was said to represent the fraction of wealth in stocks, however this is misleading in this model with a stochastic interest rate, since the bonds are also uncertain. The price dynamics

$$d\boldsymbol{P}_t = \operatorname{diag}(\boldsymbol{P}_t)[r_t\mathbf{1} + \underline{\underline{\sigma}}(r_t,t)\boldsymbol{\lambda}_t)dt + \underline{\underline{\sigma}}(r_t,t)d\boldsymbol{z}_t]$$

will therefore also be related to bonds. The wealth dynamics does look similar to our previous finding in Equation (4.1.6) for constant investment opportunities. As mentioned, the portfolio weight vector $\boldsymbol{\pi}$ is changed, but the uncertain part represented by the matrix $\underline{\underline{\sigma}}_t(r,t)$ is also different. In this stochastic case we have shown that the uncertain part is not only related to the point in time, but also related to the interest rate r. We write the wealth dynamics as

$$dW_t = W_t[r_t + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)\boldsymbol{\lambda}_t)]dt + W_t\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)d\boldsymbol{z}_t, \tag{5.3.5}$$

based on a combination of the dynamics for a bond and for a stock, which are given in Equation (5.3.3) and Equation (5.3.4). We define the volatility matrix as

$$\underline{\underline{\sigma}}(r_t,t) = \begin{pmatrix} \sigma_B(r_t,t) & 0 \\ \rho\sigma_S & \sqrt{1-\rho^2}\sigma_S \end{pmatrix},$$

and the three vectors as

$$\boldsymbol{\pi} = \begin{pmatrix} \pi_B \\ \pi_S \end{pmatrix} \qquad \boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \qquad d\boldsymbol{z}_t \begin{pmatrix} dz_{1t} \\ dz_{2t} \end{pmatrix}.$$

5.4 Model under new Assumptions

This model is based on the same utility assumptions as we made in Section 4.2 for constant investment opportunities. Assume the utility function

$$J(W, r, t) = \sup_{\boldsymbol{\pi}} \quad E_{W,r,t}\left[u(W_T)\right].$$

As shown in Section 5.3.3, we are able to write the wealth dynamics as in Equation (5.3.5) and together with the stochastic interest rate in Equation (5.2.1), we will have to consider a case with two stochastic processes. Following Øksendal (2003) on Itô's lemma, we will in general terms for the function $Y_t = g(X_i, X_j, t)$, with two stochastic processes, have the dynamics

$$dY_t = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x_i}dX_i + \frac{1}{2}\frac{\partial^2 g}{\partial x_i^2}(dX_i)^2 + \frac{\partial g}{\partial x_j}dX_j + \frac{1}{2}\frac{\partial^2 g}{\partial x_j^2}(dX_j)^2 + \frac{\partial^2 g}{\partial x_i \partial x_j}(dX_i)(dX_j).$$

The dynamics are the same as used for the dynamics of wealth elements, but it includes a final term to consider the relation between the two stochastic processes. If this result is used for the indirect utility function $J(W, r, t)$ with wealth and interest rate as stochastic processes, the dynamics will be given as

$$\begin{aligned} dJ_t =& \frac{\partial J(W,r,t)}{\partial t}dt + \frac{\partial J(W,r,t)}{\partial r}dr_t + \frac{1}{2}\frac{\partial^2 J(W,r,t)}{\partial r^2}(dr_t)^2 \\ &+ \frac{\partial J(W,r,t)}{\partial W}dW_t + \frac{1}{2}\frac{\partial^2 J(W,r,t)}{\partial W^2}(dW_t)^2 + \frac{\partial^2 J(W,r,t)}{\partial r \partial W}(dr_t)(dW_t). \end{aligned}$$

Or written in another way to follow our notation:

$$\begin{aligned} dJ_t =& \frac{\partial J}{\partial t}(W,r,t)dt + J_r(W,r,t)dr_t + \frac{1}{2}J_{rr}(W,r,t)(dr_t)^2 + J_W(W,r,t)dW_t \\ &+ \frac{1}{2}J_{WW}(W,r,t)(dW_t)^2 + J_{rW}(W,r,t)(dr_t)(dW_t) \end{aligned}$$

Next is to substitute in the two stochastic processes. The wealth dynamics are

$$dW_t = W_t[r_t + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)\boldsymbol{\lambda}]dt + W_t\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)dz_t,$$

and the process for the interest rate is

$$dr_t = \kappa[\bar{r} - r_t]dt + \boldsymbol{\sigma}_r^\top d\boldsymbol{z}_t.$$

where the vector for interest rate volatility is defined as

$$\boldsymbol{\sigma}_r = \begin{pmatrix} -\sigma_r \\ 0 \end{pmatrix}.$$

After substitution, the dynamics will have the following expression

$$\begin{aligned} dJ_t =& \frac{\partial J}{\partial t}(W,r,t)dt + J_r(W,r,t)(\kappa[\bar{r} - r_t]dt + \boldsymbol{\sigma}_r^\top d\boldsymbol{z}_t) \\ &+ \frac{1}{2}J_{rr}(W,r,t)(\kappa[\bar{r} - r_t]dt + \boldsymbol{\sigma}_r^\top d\boldsymbol{z}_t)^2 \\ &+ J_W(W,r,t)(W_t[r_t + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)\boldsymbol{\lambda}]dt + W_t\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)d\boldsymbol{z}_t) \\ &+ \frac{1}{2}J_{WW}(W,r,t)(W_t[r_t + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)\boldsymbol{\lambda}]dt + W_t\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)d\boldsymbol{z}_t)^2 \\ &+ J_{rW}(W,r,t)(\kappa[\bar{r} - r_t]dt + \boldsymbol{\sigma}_r^\top d\boldsymbol{z}_t)(W_t[r_t + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)\boldsymbol{\lambda}]dt + W_t\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)d\boldsymbol{z}_t). \end{aligned}$$

Using the following results $(dt)^2 = 0$, $(dt) \cdot (dz) = 0$ and $(dz_t)^2 = dt$ in order to reduce the expression[1]

$$\begin{aligned} dJ_t =& \frac{\partial J}{\partial t}(W,r,t)dt + J_r(W,r,t)(\kappa[\bar{r} - r_t]dt + \boldsymbol{\sigma}_r^\top d\boldsymbol{z}_t) + \frac{1}{2}\|\boldsymbol{\sigma}_r\|^2 J_{rr}(W,r,t)dt \\ &+ J_W(W,r,t)(W_t[r_t + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)\boldsymbol{\lambda}]dt + W_t\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)d\boldsymbol{z}_t) \\ &+ \frac{1}{2}J_{WW}(W,r,t)W_t^2\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)\underline{\underline{\sigma}}(r_t,t)^\top \boldsymbol{\pi}_t dt \\ &+ J_{rW}(W,r,t)\boldsymbol{\sigma}_r W_t \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)dt. \end{aligned}$$

The focus is on the drift part of the stochastic process. From the dynamics, the drifts can be defined as

$$\begin{aligned} \text{Drift} =& \frac{\partial J}{\partial t}(W,r,t) + J_r(W,r,t)(\kappa[\bar{r} - r_t]) + J_W(W,r,t)W_t[r_t + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)\boldsymbol{\lambda}] \\ &+ \frac{1}{2}J_{WW}(W,r,t)W_t^2\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t)\underline{\underline{\sigma}}(r_t,t)^\top \boldsymbol{\pi}_t + J_{rW}(W,r,t)\boldsymbol{\sigma}_r W_t\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t,t) \\ &+ \frac{1}{2}\|\boldsymbol{\sigma}_r\|^2 J_{rr}(W,r,t). \end{aligned}$$

[1] In vector form: $(\underline{\underline{v}}^\top d\underline{\underline{z_t}})^2 = \underline{\underline{v}}^\top \underline{\underline{v}}dt$ and $(\underline{\underline{v_1}}^\top d\underline{\underline{z_t}})(\underline{\underline{v_2}}^\top d\underline{\underline{z_t}}) = \underline{\underline{v_1}}^\top \underline{\underline{v_2}}dt = \underline{\underline{v_2}}^\top \underline{\underline{v_1}}dt$

which leads to the HJB equation associated with our problem. The HJB equation that must be solved for this specific maximisation problem is

$$
\begin{aligned}
0 = \sup_{\boldsymbol{\pi}} \Bigg\{ & \frac{\partial J}{\partial t}(W,r,t) + J_r(W,r,t)(\kappa[\bar{r}-r]) + J_W(W,r,t)W[r + \boldsymbol{\pi}^\top \underline{\underline{\sigma}}(r,t)\boldsymbol{\lambda}] \\
& + \frac{1}{2} J_{WW}(W,r,t)W^2 \boldsymbol{\pi}^\top \underline{\underline{\sigma}}(r,t)\underline{\underline{\sigma}}(r,t)^\top \boldsymbol{\pi} + J_{rW}(W,r,t)\boldsymbol{\sigma}_r W \boldsymbol{\pi}^\top \underline{\underline{\sigma}}(r,t) \\
& + \frac{1}{2}\|\boldsymbol{\sigma}_r\|^2 J_{rr}(W,r,t) \Bigg\}.
\end{aligned}
$$

To find a potential candidate for the allocation, which will maximize the investor's utility, differentiate the right-hand side of the HJB-equation with respect to the portfolio weights

$$
\begin{aligned}
0 =& J_W(W,r,t)W\underline{\underline{\sigma}}(r_t,t)\boldsymbol{\lambda} + J_{WW}(W,r,t)W^2\underline{\underline{\sigma}}(r,t)\underline{\underline{\sigma}}(r,t)^\top \boldsymbol{\pi} \\
& + J_{rW}(W,r,t)\boldsymbol{\sigma}_r W \underline{\underline{\sigma}}(r,t).
\end{aligned}
$$

After isolating $\boldsymbol{\pi}$ the weights are

$$
\boldsymbol{\pi} = -\frac{J_W(W,r,t)}{J_{WW}(W,r,t)W}\left(\underline{\underline{\sigma}}(r,t)^\top\right)^{-1}\boldsymbol{\lambda} - \frac{J_{rW}(W,r,t)}{J_{WW}(W,r,t)W}\left(\underline{\underline{\sigma}}(r,t)^\top\right)^{-1}\boldsymbol{\sigma}_r. \quad (5.4.1)
$$

The expression is quite similar to portfolio weights under the assumption of constant investment opportunities in Equation (4.2.2), but an additional term appears in the portfolio weights. This second term is related to the stochastic interest rate. It considers the volatility and the second order partial derivative with respect to the short-term interest rate and the wealth. It is therefore a hedging term, which increases as the uncertainty regarding the interest rate increases.

To find a less general result, we continue by substituting the result from Equation (5.4.1) into the HJB equation. This is because the utility must be maximized with respect to the portfolio weights. After substitution of the portfolio weights it is seen how the volatility matrix is present in almost every term but ends up cancelling out, and it is therefore not part of the rewritten equation

$$
\begin{aligned}
0 =& \frac{\partial J}{\partial t}(W,r,t) + J_r(W,r,t)(\kappa[\bar{r}-r]) + \frac{1}{2}\|\boldsymbol{\sigma}_r\|^2 J_{rr}(W,r,t) + J_W(W,r,t)Wr \\
&+ J_W(W,r,t)W\left(-\frac{J_W(W,r,t)}{J_{WW}(W,r,t)W}\boldsymbol{\lambda}^\top - \frac{J_{rW}(W,r,t)}{J_{WW}(W,r,t)W}\boldsymbol{\sigma}_r^\top\right)\boldsymbol{\lambda} \\
&+ \frac{1}{2}J_{WW}(W,r,t)W^2\left(-\frac{J_W(W,r,t)}{J_{WW}(W,r,t)W}\boldsymbol{\lambda}^\top - \frac{J_{rW}(W,r,t)}{J_{WW}(W,r,t)W}\boldsymbol{\sigma}_r^\top\right) \\
&\cdot\left(-\frac{J_W(W,r,t)}{J_{WW}(W,r,t)W}\boldsymbol{\lambda} - \frac{J_{rW}(W,r,t)}{J_{WW}(W,r,t)W}\boldsymbol{\sigma}_r\right) \\
&+ J_{rW}(W,r,t)\boldsymbol{\sigma}_r W\left(-\frac{J_W(W,r,t)}{J_{WW}(W,r,t)W}\boldsymbol{\lambda}^\top - \frac{J_{rW}(W,r,t)}{J_{WW}(W,r,t)W}\boldsymbol{\sigma}_r^\top\right).
\end{aligned}
$$

Next step in the simplification of the equation is multiplication of the parentheses. This changes the equation and some terms are present both on the inside and the outside of the parentheses and thereby cancel out

$$
\begin{aligned}
0 =& \frac{\partial J}{\partial t}(W,r,t) + J_r(W,r,t)(\kappa[\bar{r}-r]) + \|\boldsymbol{\sigma}_r\|^2\frac{1}{2}J_{rr}(W,r,t) + J_W(W,r,t)Wr \\
&- \|\boldsymbol{\lambda}\|^2\frac{J_W(W,r,t)^2}{J_{WW}(W,r,t)} - \boldsymbol{\sigma}_r^\top\boldsymbol{\lambda}\frac{J_{rW}(W,r,t)J_W(W,r,t)}{J_{WW}(W,r,t)} + \|\boldsymbol{\lambda}\|^2\frac{J_W(W,r,t)^2}{2J_{WW}(W,r,t)} \\
&+ \boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\frac{J_{rW}(W,r,t)J_W(W,r,t)}{2J_{WW}(W,r,t)} + \boldsymbol{\sigma}_r^\top\boldsymbol{\lambda}\frac{J_{rW}(W,r,t)J_W(W,r,t)}{2J_{WW}(W,r,t)} \\
&+ \|\boldsymbol{\sigma}_r\|^2\frac{J_{rW}(W,r,t)^2}{2J_{WW}(W,r,t)} - \boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\frac{J_W(W,r,t)J_{rW}(W,r,t)}{J_{WW}(W,r,t)} - \|\boldsymbol{\sigma}_r\|^2\frac{J_{rW}(W,r,t)^2}{J_{WW}(W,r,t)}.
\end{aligned}
$$

After multiplying the parentheses it is seen from the expression above how some terms are identical except for their opposite signs, which means that the equation can be simplified even further into the partial differential equation

$$
\begin{aligned}
J(W,r,t) =& \frac{\partial J}{\partial t}(W,r,t) + J_r(W,r,t)\kappa[\bar{r}-r_t] + \frac{1}{2}J_{rr}(W,r,t)\|\boldsymbol{\sigma}_r\|^2 \\
&+ J_W(W,r,t)Wr - \frac{1}{2}\|\boldsymbol{\lambda}\|^2\frac{J_W(W,r,t)^2}{J_{WW}(W,r,t)} \\
&- \frac{1}{2}\|\boldsymbol{\sigma}_r\|^2\frac{J_{rW}(W,r,t)^2}{J_{WW}(W,r,t)} - \boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\frac{J_W(W,r,t)J_{rW}(W,r,t)}{J_{WW}(W,r,t)},
\end{aligned}
\tag{5.4.2}
$$

where the potential solution $J(W,r,t)$ as in Chapter 4 must satisfy the terminal condition $J(W,r,T) = \bar{u}(W)$. The definitions of the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}_r$ follows from the wealth dynamics of one bond and a stock index in Section 5.3.3. As it was also the case, when we considered constant investment opportunities it is not possible to come any further without assuming a utility function.

5.5 CRRA Utility and Stochastic Interest Rate

In this section, we will assume a specific utility function. The assumed utility function will, as in the case of constant investment opportunities, have to fulfil some mathematical criteria in order to be a potential solution. We therefore determine a partial differential equation based on our Hamilton-Jacobi-Bellman equation and find the ordinary differential equations to verify our suggested solution. After the verification of the suggested solution we determine the optimal portfolio weights in the case of a CRRA-investor.

5.5.1 Partial Differential Equation for CRRA-investor

In the process of finding a less general result, a certain utility function is assumed. We continue with the same utility function as in the case of constant investment opportunities. The utility function is given in equation (4.3.1) but for convenience restated here:

$$J(W,t) = \frac{g(r,t)^{\gamma} W^{1-\gamma}}{1-\gamma}. \tag{5.5.1}$$

Similar are some of the necessary partial derivatives previously found, but repeated here and also including the derivatives with respect to the interest rate. The calculations of the derivatives above are shown in Appendix A.3.

$$\begin{aligned}
J_W(W,t) &= g(r,t)^{\gamma} W^{-\gamma} \\
J_{WW}(W,t) &= -\gamma g(r,t)^{\gamma} W^{-\gamma-1} \\
\frac{\partial J}{\partial t}(W,r,t) &= \frac{\gamma}{1-\gamma} g(r,t)^{\gamma-1} g_t(r,t) W^{1-\gamma} \\
J_r(W,r,t) &= \frac{\gamma}{1-\gamma} g(r,t)^{\gamma-1} g_r(r,t) W^{1-\gamma} \\
J_{rr}(W,r,t) &= \frac{\gamma W^{1-\gamma}}{1-\gamma} \big((\gamma-1) g(r,t)^{\gamma-2} g_r^2(r,t) + g(r,t)^{\gamma-1} g_{rr}(r,t) \big) \\
J_{rW}(W,r,t) &= \gamma g(r,t)^{\gamma-1} g_r(r,t) W^{-\gamma}
\end{aligned} \tag{5.5.2}$$

These partial derivatives are substituted into the HJB equation. The substitution does not make the expression prettier, but it can be simplified. Initially it is:

$$
\begin{aligned}
0 =& \frac{\gamma}{1-\gamma} g(r,t)^{\gamma-1} g_t(t) W^{1-\gamma} + \frac{\gamma}{1-\gamma} g(r,t)^{\gamma-1} g_r(r,t) W^{1-\gamma} \kappa[\bar{r} - r] \\
&+ \frac{1}{2} \frac{\gamma W^{1-\gamma}}{1-\gamma} \big((\gamma-1) g(r,t)^{\gamma-2} g_r^2(r,t) + g(r,t)^{\gamma-1} g_{rr}(r,t) \big) \|\boldsymbol{\sigma}_r\|^2 + g(r,t)^{\gamma} W^{-\gamma} W r \\
&- \frac{1}{2} \|\boldsymbol{\lambda}\|^2 \frac{(g(r,t)^{\gamma} W^{-\gamma})^2}{-\gamma g(r,t)^{\gamma} W^{-\gamma-1}} - \frac{1}{2} \|\boldsymbol{\sigma}_r\|^2 \frac{(\gamma g(r,t)^{\gamma-1} g_r(r,t) W^{-\gamma})^2}{-\gamma g(r,t)^{\gamma} W^{-\gamma-1}} \\
&- \boldsymbol{\lambda}^\top \boldsymbol{\sigma}_r \frac{g(r,t)^{\gamma} W^{-\gamma} \gamma g(r,t)^{\gamma-1} g_r(r,t) W^{-\gamma}}{-\gamma g(r,t)^{\gamma} W^{-\gamma-1}}.
\end{aligned}
$$

The wealth $W^{1-\gamma}$ is part of all the terms in the equation and is therefore removed. The fractions are reduced by multiplication and it is then seen that two of the terms related to the volatility of the interest rate will cancel out. After these three steps, the partial differential equation is

$$
\begin{aligned}
0 =& \frac{\gamma}{1-\gamma} g(r,t)^{\gamma-1} g_t(t) + \frac{\gamma}{1-\gamma} g(r,t)^{\gamma-1} g_r(r,t) \kappa[\bar{r} - r_t] + \frac{1}{2} \|\boldsymbol{\lambda}\|^2 \frac{g(r,t)^{\gamma}}{\gamma} \\
&+ \frac{1}{2} \frac{\gamma}{1-\gamma} g(r,t)^{\gamma-1} g_{rr}(r,t) \|\boldsymbol{\sigma}_r\|^2 + g(r,t)^{\gamma} r + \boldsymbol{\lambda}^\top \boldsymbol{\sigma}_r \frac{\gamma g(r,t)^{\gamma-1} g_r(r,t)}{\gamma}.
\end{aligned}
$$

To reduce further we divide by $g(r,t)^{\gamma-1}$ and multiply with $\dfrac{1-\gamma}{\gamma}$. The equation is

$$
\begin{aligned}
0 =& g_t(t) + g_r(r,t) \kappa[\bar{r} - r_t] + \frac{1}{2} \|\boldsymbol{\lambda}\|^2 \frac{(1-\gamma) g(r,t)}{\gamma^2} \\
&+ \frac{1}{2} g_{rr}(r,t) \|\boldsymbol{\sigma}_r\|^2 + \frac{1-\gamma}{\gamma} g(r,t) r + \boldsymbol{\lambda}^\top \boldsymbol{\sigma}_r \frac{(1-\gamma) g_r(r,t)}{\gamma},
\end{aligned}
$$

which is simpler than the initial partial differential equation. To make the following steps faster, we can create parentheses for terms related to the same partial derivative. It then takes less substitution of the derivatives of $g(r,t)$, when we write the partial differential equation as

$$
\begin{aligned}
0 =& g_t(r,t) + \left(\kappa[\bar{r} - r_t] + \frac{1-\gamma}{\gamma} \boldsymbol{\lambda}^\top \boldsymbol{\sigma}_r \right) g_r(r,t) \\
&+ \left(\frac{1-\gamma}{\gamma} r + \frac{1-\gamma}{2\gamma^2} \|\boldsymbol{\lambda}\|^2 \right) g(r,t) + \frac{1}{2} g_{rr} \|\boldsymbol{\sigma}_r\|^2. \qquad (5.5.3)
\end{aligned}
$$

Compared to the case with constant investment opportunities, the partial differential equation that the function $g(r,t)$ has to solve now looks fairly more complicated. It is, however, still a necessity for $g(r,t)$ to solve the partial differential equation with the terminal condition $g(r,T) = 1$ for our guess from Equation (5.5.1) to be a solution to the HJB equation.

5.5.2 Ordinary Differential Equations

The partial differential equation to solve, the function $g(r,t)$, and the ordinary differential equations are more complex. Following literature such as Larsen (2010), Liu (2007), and Sørensen (1999) we will expect a solution to the partial differential equation with the terminal $g(r,T) = 1$ to be of the form

$$g(r,t) = \exp\left\{\frac{1-\gamma}{\gamma}A_0(T-t) + \frac{1-\gamma}{\gamma}A_1(T-t)r\right\},$$

where A_0 and A_1 are two ordinary differential equations. To move on in the process, the relevant derivatives of the function $g(r,t)$ are calculated and substituted into Equation (5.5.3). First, the relevant partial derivatives are

$$\begin{aligned}
\frac{\partial g(r,t)}{\partial t} &= \left(-\frac{1-\gamma}{\gamma}(rA_1'(\tau) + A_0'(\tau))\right) \cdot g(r,t) \\
\frac{\partial g(r,t)}{\partial r} &= \left(\frac{1-\gamma}{\gamma}A_1(\tau)\right) \cdot g(r,t) \\
\frac{\partial^2 g(r,t)}{\partial r^2} &= \left(\frac{(1-\gamma)^2}{\gamma^2}A_1^2(\tau)\right) \cdot g(r,t),
\end{aligned}$$

where the definition of $g(r,t)$ is used in the partial derivatives. After substitution, the partial differential equation will be

$$\begin{aligned}
0 = &\left(-\frac{1-\gamma}{\gamma}\left(rA_1'(\tau) + A_0'(\tau)\right)\right) \cdot g(r,t) \\
&+ \left(\kappa[\bar{r} - r_t] + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top \boldsymbol{\sigma}_r\right)\left(\frac{1-\gamma}{\gamma}A_1(\tau)\right) \cdot g(r,t) \\
&+ \left(\frac{1-\gamma}{\gamma}r + \frac{1-\gamma}{2\gamma^2}\|\boldsymbol{\lambda}\|^2\right) \cdot g(r,t) + \frac{1}{2}\left(\frac{(1-\gamma)^2}{\gamma^2}A_1^2(\tau)\right) \cdot g(r,t) \cdot \|\boldsymbol{\sigma}_r\|^2.
\end{aligned}$$

The function $g(r,t)$ is a part of all the terms. To simplify we divide all terms with $g(r,t)$, and thereby have the expression

$$\begin{aligned}
0 = &-\frac{1-\gamma}{\gamma}\left(rA_1'(\tau) + A_0'(\tau)\right) + \left(\kappa[\bar{r} - r_t] + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top \boldsymbol{\sigma}_r\right)\frac{1-\gamma}{\gamma}A_1(\tau) \\
&+ \frac{1-\gamma}{\gamma}r + \frac{1-\gamma}{2\gamma^2}\|\boldsymbol{\lambda}\|^2 + \frac{(1-\gamma)^2}{2\gamma^2}A_1^2(\tau) \cdot \|\boldsymbol{\sigma}_r\|^2.
\end{aligned}$$

We then multiply with the fraction $\frac{\gamma}{1-\gamma}$ and isolate $A_0'(\tau)$ as it is not multiplied with other terms. This is relevant as the function $A_0'(\tau)$ will be part of the system of ordinary differential equations

$$\begin{aligned}A_0'(\tau) =& - rA_1'(\tau) + \left(\kappa[\bar{r} - r_t] + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top \boldsymbol{\sigma}_r\right) A_1(\tau) \\ &+ r + \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2 + \frac{1-\gamma}{2\gamma} A_1^2(\tau) \cdot \|\boldsymbol{\sigma}_r\|^2.\end{aligned}$$

We gather terms which are related to the interest rate r by multiplication, such that we can put the interest rate outside a parentheses and thereby have the possibility to solve a second term for $A_1'(\tau)$.

$$\begin{aligned}A_0'(\tau) =& \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2 + \left(\kappa\bar{r} + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top \boldsymbol{\sigma}_r\right) A_1(\tau) + \frac{1-\gamma}{2\gamma} A_1^2(\tau) \cdot \|\boldsymbol{\sigma}_r\|^2 \\ &+ r(1 - A_1'(\tau) - \kappa A_1(\tau))\end{aligned}$$

We can then write the condition as two ordinary differential equations, which must be solved by function A_0 and A_1 in order to continue with the main partial differential equation problem. The two ordinary differential equations are

$$\begin{aligned}A_0'(\tau) =& \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2 + \left(\kappa\bar{r} + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top \boldsymbol{\sigma}_r\right) A_1(\tau) + \frac{1-\gamma}{2\gamma} A_1^2(\tau) \cdot \|\boldsymbol{\sigma}_r\|^2 \\ A_1'(\tau) =& 1 - \kappa A_1(\tau),\end{aligned} \tag{5.5.4}$$

with the initial conditions $A_0(0) = 0$ and $A_1(0) = 0$ because of the terminal condition $g(r, T) = 1$. When solving the ordinary differential equation $A_1'(\tau)$ with $A_1(0) = 0$, we can use the general solution given in theorem C.2 of Munk (2013) for a ordinary differential of the same shape as this problem. The solution is then given as

$$A_1(\tau) = \frac{1}{\kappa}(1 - e^{-\kappa\tau}). \tag{5.5.5}$$

For the other ordinary differential equation, A_0', we will use the initial condition $A_0(0) = 0$ such that it can be written as

$$\begin{aligned}A_0(0) &= A_0(\tau) - A_0(\tau) \\ A_0(\tau) &= A_0(\tau) - A_0(0) \\ A_0(\tau) &= \int_0^\tau A_0'(s)ds.\end{aligned}$$

From the function A_0' above, the function can be defined as

$$A_0(\tau) = \int_0^\tau \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2 + \int_0^\tau \left(\kappa\bar{r} + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right) A_1(\tau) + \int_0^\tau \frac{1-\gamma}{2\gamma}\cdot\|\boldsymbol{\sigma}_r\|^2 A_1^2(\tau),$$

where we have used the integral. As some values are constant and therefore does not depend on time, the expression can be rewritten as

$$A_0(\tau) = \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2\tau + \left(\kappa\bar{r} + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)\int_0^\tau A_1(s)ds + \frac{1-\gamma}{2\gamma}\cdot\|\boldsymbol{\sigma}_r\|^2\int_0^\tau A_1^2(s)ds.$$

We need expressions for the integrals of $A_1(\tau)$ and $A_1^2(\tau)$, and for this refer to Appendix C.2 in Munk (2013) and simply use the general result. We can then write the two integrals as

$$\int_0^\tau A_1(s)ds = \frac{\tau - A_1(\tau)}{\kappa} \quad \text{and} \quad \int_0^\tau A_1^2(s)ds = \frac{\tau - A_1(\tau)}{\kappa^2} - \frac{A_1^2(\tau)}{2\kappa}$$

and will then have the equation

$$\begin{aligned} A_0(\tau) =& \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2\tau + \left(\kappa\bar{r} + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)\frac{\tau - A_1(\tau)}{\kappa} \\ &+ \frac{1-\gamma}{2\gamma}\cdot\|\boldsymbol{\sigma}_r\|^2\left(\frac{\tau - A_1(\tau)}{\kappa^2} - \frac{A_1^2(\tau)}{2\kappa}\right). \end{aligned} \tag{5.5.6}$$

There are now solutions for the two ordinary differential equations presented above. These functions are used together with the function $g(r,t)$ to show that the suggestion is actually a solution to the Hamilton-Jacobi-Bellman equation.

5.5.3 Verifying Solution

In the following calculations the function $A_0(\tau)$ is not necessary, but the functions $A_1(\tau)$, $A_1'(\tau)$, and $A_0'(\tau)$ are needed. This is because we skip the step of finding the partial derivatives of $g(r,t)$ as this is already done at a general level above. After substitution of the general partial derivatives, we have the following partial differential equation to solve:

$$\begin{aligned} 0 =& -\frac{1-\gamma}{\gamma}\left(rA_1'(\tau) + A_0'(\tau)\right) + \left(\kappa[\bar{r} - r_t] + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)\left(\frac{1-\gamma}{\gamma}A_1(\tau)\right) \\ &+ \frac{1-\gamma}{\gamma}r + \frac{1-\gamma}{2\gamma^2}\|\boldsymbol{\lambda}\|^2 + \frac{1}{2}\left(\frac{(1-\gamma)^2}{\gamma^2}A_1^2(\tau)\right)\cdot\|\boldsymbol{\sigma}_r\|^2 \end{aligned}$$

Reduce by multiplying all terms with $\dfrac{\gamma}{1-\gamma}$ and substitute in the function $A_1(\tau)$

$$\begin{aligned}0 =& -rA_1'(\tau) - A_0'(\tau) + \left(\kappa[\bar{r} - r_t] + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)\frac{(1-e^{-\kappa\tau})}{\kappa} \\ &+ r + \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2 + \frac{1}{2}\|\boldsymbol{\sigma}_r\|^2\left(\frac{1-\gamma}{\gamma}\left(\frac{(1-e^{-\kappa\tau})}{\kappa}\right)^2\right).\end{aligned}$$

We then substitute in the two missing functions, $A_0'(\tau)$ and $A_1'(\tau)$, from Equation (5.5.4) and rearrange

$$\begin{aligned}0 =& -re^{-\kappa\tau} - \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2 + \left(r + \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2\right) + \left(\kappa[\bar{r} - r] + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)\frac{(1-e^{-\kappa\tau})}{\kappa} \\ &- \left(\kappa\bar{r} + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)\frac{1-e^{-\kappa\tau}}{\kappa} + \frac{1}{2}\left(\frac{(1-\gamma)}{\gamma}\left(\frac{(1-e^{-\kappa\tau})}{\kappa}\right)^2\right)\cdot\|\boldsymbol{\sigma}_r\|^2 \\ &- \frac{1-\gamma}{2\gamma}\cdot\|\boldsymbol{\sigma}_r\|^2\left(\frac{1-e^{-\kappa\tau}}{\kappa}\right)^2.\end{aligned}$$

Simplifying the equation and rearranging again gives the expression

$$\begin{aligned}0 =& r - re^{-\kappa\tau} - r(1-e^{-\kappa\tau}) + \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2 - \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2 + \left(\kappa\bar{r} + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)\frac{1-e^{-\kappa\tau}}{\kappa} \\ &- \left(\kappa\bar{r} + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)\frac{1-e^{-\kappa\tau}}{\kappa} + \frac{1-\gamma}{2\gamma}\cdot\|\boldsymbol{\sigma}_r\|^2\left(\left(\frac{1-e^{-\kappa\tau}}{\kappa}\right)^2 - \left(\frac{1-e^{-\kappa\tau}}{\kappa}\right)^2\right)\end{aligned}$$

From this expression it is seen that all terms cancel out and our suggested form of the indirect utility function is thereby verified, as it solves the partial differential equation based on the Hamilton-Jacobi-Bellman equation.

Another way to prove that the partial differential equation is solved is to take a step back and use Equation (5.5.3), which the partial derivatives have not yet been substituted into. From here it is possible to calculate the partial derivatives of the function $g(r,t)$, for which we have specified the functions $A_0(\tau)$ and $A_1(\tau)$, when the system of ordinary differential equations from Equation (5.5.4) was solved. The result will be the same; it will solve the partial differential equation.

5.5.4 Portfolio Weights for the CRRA-investor

As the suggested solution in Equation (5.5.1) is shown to actually be a solution to the HJB equation, we can use the general portfolio weights, and for shorthand notation use $\underline{\underline{\sigma}}(r_t,t) = \underline{\underline{\sigma}}$.

$$\boldsymbol{\pi} = -\frac{J_W(W,r,t)}{WJ_{WW}(W,r,t)}(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\lambda} - \frac{J_{rW}(W,r,t)}{WJ_{WW}(W,r,t)}(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\sigma}_r,$$

for which the partial derivatives of our suggested solution have to be used. The relevant partial derivatives are:

$$\begin{aligned} J_W(W,t) &= g(r,t)^\gamma W^{-\gamma} \\ J_{WW}(W,t) &= -\gamma g(r,t)^\gamma W^{-\gamma-1} \\ J_{rW}(W,r,t) &= \gamma g(r,t)^{\gamma-1} g_r(r,t) W^{-\gamma}. \end{aligned}$$

By substitution the expression for optimal portfolio weights will change into

$$\begin{aligned} \boldsymbol{\pi} =& \frac{g(r,t)^\gamma W^{-\gamma}}{W\gamma g(r,t)^\gamma W^{-\gamma-1}}(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\lambda} + \frac{\gamma g(r,t)^{\gamma-1} g_r(r,t) W^{-\gamma}}{W\gamma g(r,t)^\gamma W^{-\gamma-1}}(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\sigma}_r \\ =& \frac{1}{\gamma}(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\lambda} + \frac{\gamma g(r,t)^{\gamma-1} g_r(r,t) W^{-\gamma}}{\gamma g(r,t)^\gamma W^{-\gamma}}(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\sigma}_r \\ =& \frac{1}{\gamma}(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\lambda} + g(r,t)^{-1} g_r(r,t)(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\sigma}_r. \end{aligned} \tag{5.5.7}$$

To get further in the specification of the portfolio weights, we use some of the findings from the verification of the suggested solution to the partial differential equation. We will use

$$\begin{aligned} g(r,t) &= \exp\left\{\frac{1-\gamma}{\gamma}A_0(T-t) + \frac{1-\gamma}{\gamma}A_1(T-t)r\right\} \\ g_r(r,t) &= \frac{\partial g(r,t)}{\partial r} = \left(\frac{1-\gamma}{\gamma}A_1(\tau)\right)\cdot g(r,t) \\ A_1(\tau) &= \frac{(1-e^{-\kappa\tau})}{\kappa}. \end{aligned}$$

We substitute in the definitions and simplify the expression such that

$$\begin{aligned} \boldsymbol{\Pi}(W,r,t) &= \frac{1}{\gamma}(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\lambda} + \frac{1-\gamma}{\gamma}\frac{(1-e^{-\kappa\tau})}{\kappa}(\underline{\underline{\sigma}}^\top)^{-1}\begin{pmatrix}-\sigma_r \\ 0\end{pmatrix} \\ \boldsymbol{\Pi}(W,r,t) &= \frac{1}{\gamma}(\underline{\underline{\sigma}}^\top)^{-1}\boldsymbol{\lambda} + \frac{\gamma-1}{\gamma}(\underline{\underline{\sigma}}^\top)^{-1}\begin{pmatrix}\sigma_r \\ 0\end{pmatrix} b(T-t) \end{aligned}$$

Using definitions made in Section 5.3.3 the portfolio weights can be split up into stock and bond weights. The volatility matrix and the vector for the market price of risk were defined as

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_B(r_t,t) & 0 \\ \rho\sigma_S & \sqrt{1-\rho^2}\sigma_S \end{pmatrix} \quad \text{and} \quad \boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

In the expression for optimal portfolio the inverse of the transposed volatility matrix is used. After inverting and transposing the volatility it is together with the vector for market price of risk substituted into the portfolio weights. The portfolio weights are then

$$\begin{aligned} \begin{pmatrix} \Pi_B \\ \Pi_S \end{pmatrix} =& \frac{1}{\gamma} \begin{pmatrix} \frac{\sqrt{1-\rho^2}\sigma_S}{\sqrt{1-\rho^2}\sigma_S\sigma_B(r_t,t)}\lambda_1 + \frac{-\rho\sigma_s}{\sqrt{1-\rho^2}\sigma_S\sigma_B(r_t,t)}\lambda_2 \\ \frac{\sigma_B(r_t,t)}{\sqrt{1-\rho^2}\sigma_S\sigma_B(r_t,t)}\lambda_2 \end{pmatrix} \\ &+ \frac{\gamma-1}{\gamma} \begin{pmatrix} \frac{\sqrt{1-\rho^2}\sigma_S}{\sqrt{1-\rho^2}\sigma_S\sigma_B(r_t,t)}\sigma_r \\ 0 \end{pmatrix} b(T-t) \end{aligned}$$

where several terms cancel out and can be written as

$$\begin{pmatrix} \Pi_B(W,r,t) \\ \Pi_S(W,r,t) \end{pmatrix} = \frac{1}{\gamma} \begin{pmatrix} \frac{\lambda_1}{\sigma_B(r_t,t)} - \frac{\rho\lambda_2}{\sqrt{1-\rho^2}\sigma_B(r_t,t)} \\ \frac{\lambda_2}{\sqrt{1-\rho^2}\sigma_S} \end{pmatrix} + \frac{\gamma-1}{\gamma} \begin{pmatrix} \frac{\sigma_r}{\sigma_B(r_t,t)} \\ 0 \end{pmatrix} b(T-t).$$

Moving away from matrices, the fraction of the wealth which is invested in bonds is

$$\Pi_B(W,r,t) = \frac{1}{\gamma}\left(\frac{\lambda_1}{\sigma_B(r_t,t)} - \frac{\rho\lambda_2}{\sqrt{1-\rho^2}\sigma_B(r_t,t)}\right) + \frac{\gamma-1}{\gamma}\frac{\sigma_r b(T-t)}{\sigma_B(r_t,t)},$$

and the fraction of the wealth, which is allocated in stocks will be

$$\Pi_S(W,r,t) = \frac{1}{\gamma}\frac{\lambda_2}{\sqrt{1-\rho^2}\sigma_S}.$$

Having the portfolio weights, one can follow the structure in Munk (2013) to find the amount of wealth allocated into the locally risk-free asset and thereby the complete allocation strategy. By using the zero-coupon bond as the bond instrument, he shows how the investment in the locally risk-free will be

$$\begin{aligned}\Pi_0 &= 1 - \Pi_B - \Pi_S \\ &= \frac{1}{\gamma}\left(1 - \mathbf{1}^\top (\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\boldsymbol{\lambda}\right),\end{aligned}$$

which follows as we for the zero-coupon bond previously have defined $\sigma_B(r,t) = \sigma_r b(T-t)$. The bond allocation can be written as

$$\Pi_B(W,r,t) = \frac{1}{\gamma}\left(\frac{\lambda_1}{\sigma_B(r_t,t)} - \frac{\rho\lambda_2}{\sqrt{1-\rho^2}\sigma_B(r_t,t)}\right) + \frac{\gamma-1}{\gamma},$$

and the vector containing portfolio weights for both stocks and bonds will be

$$\begin{pmatrix}\Pi_B(W,r,t)\\ \Pi_S(W,r,t)\end{pmatrix} = \frac{1}{\gamma}\left(\underline{\underline{\sigma}}(r_t,t)^\top\right)^{-1}\boldsymbol{\lambda} + \frac{\gamma-1}{\gamma}\begin{pmatrix}1\\0\end{pmatrix}$$

Due to the rewritten bond weight, the total portfolio weights can therefore be written as a combination of the portfolio for an investor with a log-utility ($\gamma = 1$), who does not hedge, and the zero-coupon bond. The investment strategy is then defined as

$$\begin{pmatrix}\Pi_0\\ \Pi_B\\ \Pi_S\end{pmatrix} = \frac{1}{\gamma}\begin{pmatrix}\Pi_0^{log}\\ \Pi_B^{log}\\ \Pi_S^{log}\end{pmatrix}\frac{\gamma-1}{\gamma}\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

These results for a CRRA-investor are in the next section compared to the findings for the model in Chapter 4.

5.6 Results under new Assumptions

This section will mainly focus on the interpretation of the results. This is primarily done by comparing the recent findings with the results of the model with constant investment opportunities.

5.6.1 Assumptions

As mentioned at the beginning of this chapter, an extension of the model has been developed, since a stochastic interest rate seems more realistic than a constant interest rate.

When moving from the mean-variance model into the dynamic model with constant investment opportunities, the result was similar, but it was obtained under more realistic assumptions with stock prices being log-normally distributed so that they would not take on negative values. When extending with the stochastic interest rate, a model under a set of more realistic assumptions is developed, but the findings for this model are different than our previous results.

5.6.2 Mathematical Results

The difference in the results are best explained by considering the portfolio weights. In the model with constant investment opportunities the weights are

$$\mathbf{\Pi}(W,t) = \frac{1}{\gamma}(\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\boldsymbol{\lambda}$$

and in the model with a stochastic interest rate, the allocation was given as

$$\mathbf{\Pi}(W,r,t) = \frac{1}{\gamma}(\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\boldsymbol{\lambda} + \frac{\gamma-1}{\gamma}(\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\begin{pmatrix}\sigma_r\\0\end{pmatrix}b(T-t).$$

The model with stochastic interest rate has a new additional term compared to the model with constant investment opportunities, this is the hedging term. This changes the result from the previous two-fund separation into a result with three-fund separation. This new hedging term is interesting in several ways. The risk aversion parameter, γ, will have a different effect than previously. An increasing value of γ would previously have had a clear impact on the portfolio weights, where an increasing γ would reduce the allocations in risky assets. The second term in the new model makes the effect of γ ambiguous, as γ now has two opposite effects on the portfolio weights. An increasing γ will give the second term, which is the hedging part, a larger impact.

It works as a hedging term due to the effect of volatility. In comparison with the model with constant investment opportunities, we now consider the volatility of the interest rate, as the interest rate no longer is constant. Larger volatility on the interest rate will lead to larger uncertainty and because of the hedging term, the investor will allocate more wealth in bonds to hedge the interest rate risk. The specific expression for the hedging term is $\frac{\gamma-1}{\gamma}\frac{\sigma_r b(T-t)}{\sigma_B(r_t,t)}$. From this it is seen how it leads to a trade-off between hedging the interest rate risk or hedging the short-term risk. This is happening by moving wealth from cash towards bonds. We can also see the prioritization between the two sources of risk in the hedging term, where higher interest rate risk increases the hedge while higher volatility on the bonds will reduce the hedging.

Finally, $b(T-t)$, which was previously defined as $b(T-t) = \frac{1}{\kappa}(1 - e^{-\kappa(T-t)})$, has two interesting effects. First, an increase in the value of the constant κ would make the numerator smaller. Intuitively, this is because the short-term interest rate is faster at returning to its long-term mean and therefore there is less reason to hedge. Second, this introduces an element, which was not present in the case of constant investment opportunity set. Specifically, the horizon over which investors invest will now play a role when determining the optimal portfolio allocation. When investing over a longer time horizon the interest rate risk has an increasing importance relative to the bond volatility, and the investor will hedge more. A longer time horizon will increase the value of the function and thereby increase the numerator.

5.6.3 Difference in Implications

The mathematical descriptions of differences can also be shown by numerical examples. Inspired by other papers, we will for this numerical example and those in the following chapters use the estimates

$$\begin{aligned} &\mu_B = 2.1\% \qquad \sigma_B = 10\% \qquad \bar{r} = 1\% \qquad \sigma_r = 5\% \\ &\mu_S = 8.7\% \qquad \sigma_S = 20.2\% \qquad \rho = 0.2 \end{aligned}$$

where subscript r indicates interest rate, B indicates bond, and S indicates stock. The estimates are from the book Dimson et al. (2002). By the use of these estimates in the model, we can define the values

$$\psi = 0.3812 \qquad \lambda_1 = 0.11 \qquad \lambda_2 = 0.3666 \qquad \kappa = 0.4965$$

We present Table 5.1 and Table 5.2; one table for portfolio values when using the model for constant investment opportunities and a table for portfolio values, from the model with the assumption of a stochastic interest rate. We start with Table 5.1, which is presenting a numerical example for the CRRA-investor under the assumption of constant investment opportunities.

In Table 5.1 we present less figures than in the case with stochastic interest rate, since the results are independent of time as described earlier. The risk aversion will

γ	Stock	Bond	Cash	Exp. return	Std. dev.
0.5	3.7049	0.7032	-3.4081	0.3030	0.7656
1	1.8525	0.3516	-1.2041	0.1565	0.3828
2	0.9262	0.1758	-0.1020	0.0833	0.1914
5	0.3705	0.0703	0.5592	0.0393	0.0766
10	0.1852	0.0352	0.7796	0.0247	0.0383
20	0.0926	0.0176	0.8898	0.0173	0.0191
100	0.0185	0.0035	0.9780	0.0115	0.0038
150	0.0123	0.0023	0.9853	0.0110	0.0026

Table 5.1: Portfolio allocation for a CRRA-investor for different levels of risk aversion under constant investment opportunities.

however make the results change. Naturally, we see lower weights for stocks and bonds as the risk aversion is increasing. This reduction in the wealth allocated in the tangency portfolio will then lead to a higher allocation in our second fund. As one would expect, this will reduce the expected return, but most importantly for the risk averse investor, there will also be a reduction in the volatility of the investment.

T	γ	Stock	Bond	Cash	Hedge	Exp. return	Std. dev.
$T = 2.5$	0.5	3.7046	-0.0126	-2.6919	-0.7160	0.2951	0.7481
	1	1.8523	0.3517	-1.2040	0.0000	0.1565	0.3827
	2	0.9261	0.5338	-0.4600	0.3580	0.0872	0.2046
	5	0.3705	0.6431	-0.0136	0.5728	0.0456	0.1080
	10	0.1852	0.6796	0.1352	0.6444	0.0317	0.0839
	20	0.0926	0.6978	0.2096	0.6802	0.0248	0.0758
$T = 5$	0.5	3.7046	-0.2196	-2.4850	-0.9229	0.2928	0.7442
	1	1.8523	0.3517	-1.2040	0.0000	0.1565	0.3827
	2	0.9261	0.6373	-0.5634	0.4615	0.0883	0.2094
	5	0.3705	0.8087	-0.1791	0.7383	0.0474	0.1207
	10	0.1852	0.8658	-0.0510	0.8306	0.0338	0.1010
	20	0.0926	0.8944	0.0130	0.8768	0.0270	0.0950

Table 5.2: Portfolio allocation for a CRRA-investor for different investment horizons and risk aversion levels under stochastic investment opportunities. The portfolio now contains a hedging term, which is increasing in risk aversion and investment horizon.

Next, Table 5.2 with values from the model with stochastic short-term interest rate. We do again use the result for a CRRA-investor and the assumed figures described above. As mentioned, there is three-fund separation and the tangency portfolio will still be bonds and stocks, but it will have the hedging subtracted. The allocation in the tangency portfolio will be the same as in the case with constant investment opportunities, but the bond allocation will be different. The stock allocation will be identical to the previous one because the mathematical expressions are identical. The total allocation in bonds will be different, since it is implied that there will be a different fraction of the wealth remaining for the locally risk-free asset, than in the case of the constant investment opportunities. When working with the stochastic model, considering multiple time horizons is important because the hedging term makes the allocation result dependent on time.

The hedging term makes the long-term investors willing to take on more short-term risk, as the time horizon increases. This is because they become more focused on hedging the future interest rate risk. This is seen in Figure 5.3 where the efficient frontier moves towards the southeast as the time horizon increases.

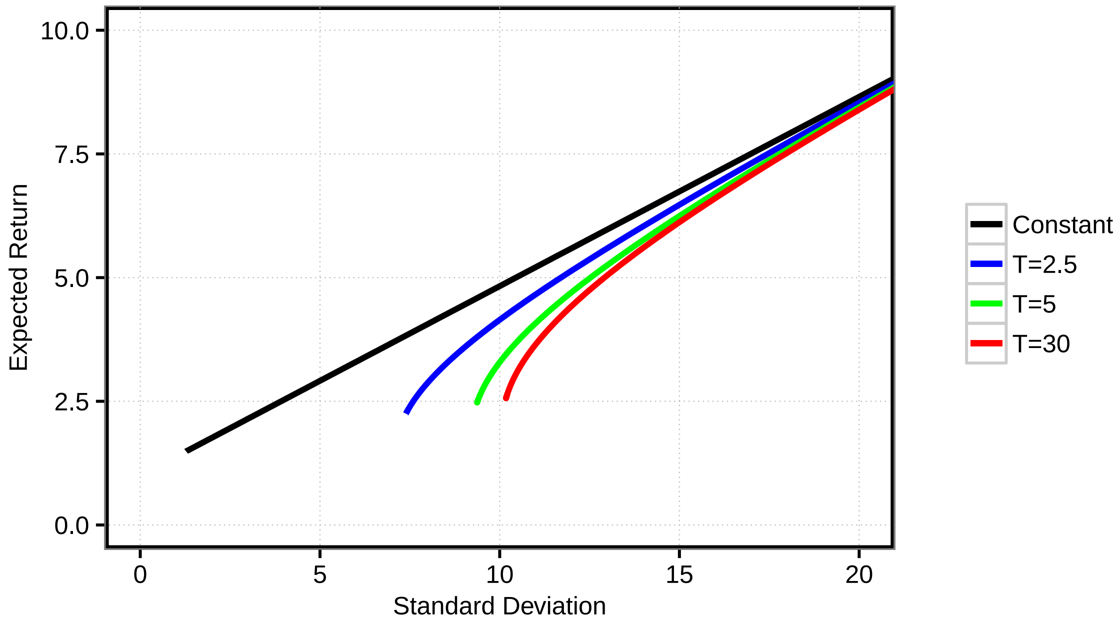

Figure 5.3: Efficient frontiers for different investment horizons. They represent return and variance for different portfolios chosen by an investor. The frontiers are created, as the investor chooses different allocations depending on the level of risk aversion.

5.6.4 The Effect of Risk Aversion

The tendency to focus more on hedging the future interest rate risk can both be seen from the figures in Table 5.2, but the tendencies are seen more clearly in Figure 5.4 for the allocation in bonds across different levels of risk aversion, γ. Each line represent either the allocation at different time horizons or the constant investment case. From this it is seen how the fraction of wealth allocated in bonds is increasing as the time horizon is increasing. The increasing allocation in bonds, must lead to a reduced allocation somewhere else. The allocation in stocks does not depend on the time horizon which is the case both when working with a constant or a stochastic interest rate. The increase in bond allocation must therefore lead to a reduction in the fraction on wealth which is held in the locally risk-free asset. Refer to Figure 5.5, where the allocation in cash across different levels of risk aversion is decreasing as the time horizon is increasing.

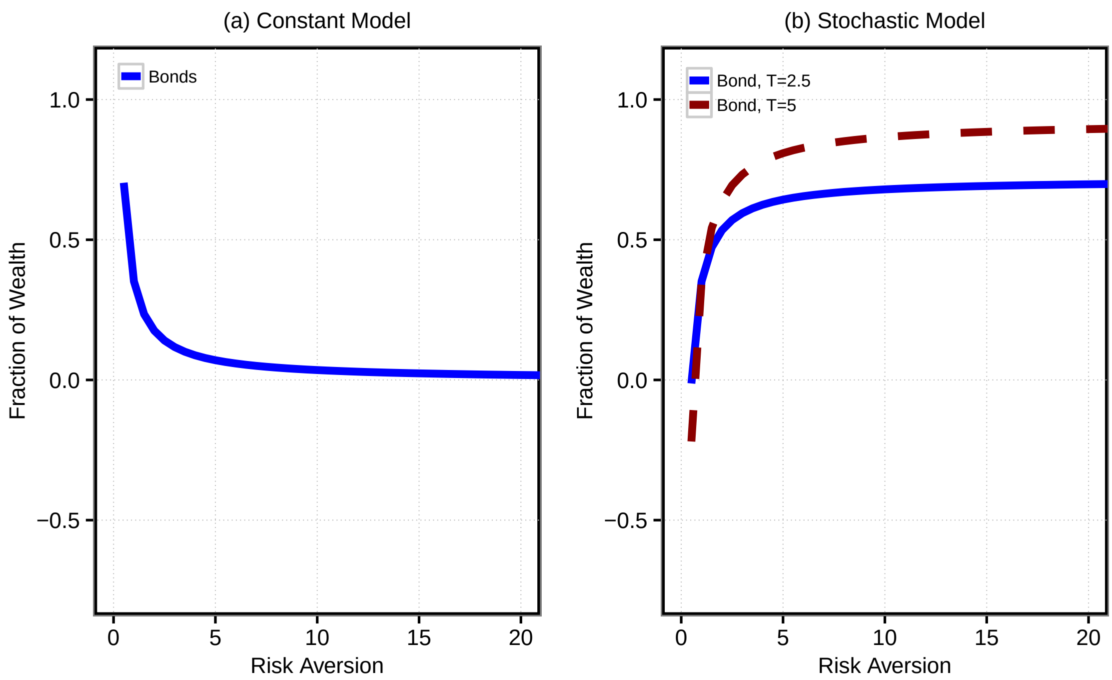

Figure 5.4: Illustrates the bond allocation for a CRRA-investor under the model with constant opportunities and the model with a stochastic interest rate for an investment horizon of $T = 2.5$ and $T = 5$.

This result is different from the case with constant investment opportunities. As mentioned, the result for allocation in stocks will be the same. However, in the case with constant investment opportunities, there is no variation in the allocation of wealth in bonds and cash across time. As previously shown in Section 4.3 the allocations, when considering constant investment opportunities, are constant. This is also presented in Table 5.1.

The two models differ, when looking at the allocations in bonds and locally risk-free asset across different levels of risk aversion. From Figure 5.4 and Figure 5.5, it is for the model with the constant investment opportunities seen that the allocation in the locally risk-free asset will increase, as the risk aversion is becoming higher. In the stochastic model an increase in risk aversion will also increase the locally risk-free asset. The main difference between the two models is the bond allocation.

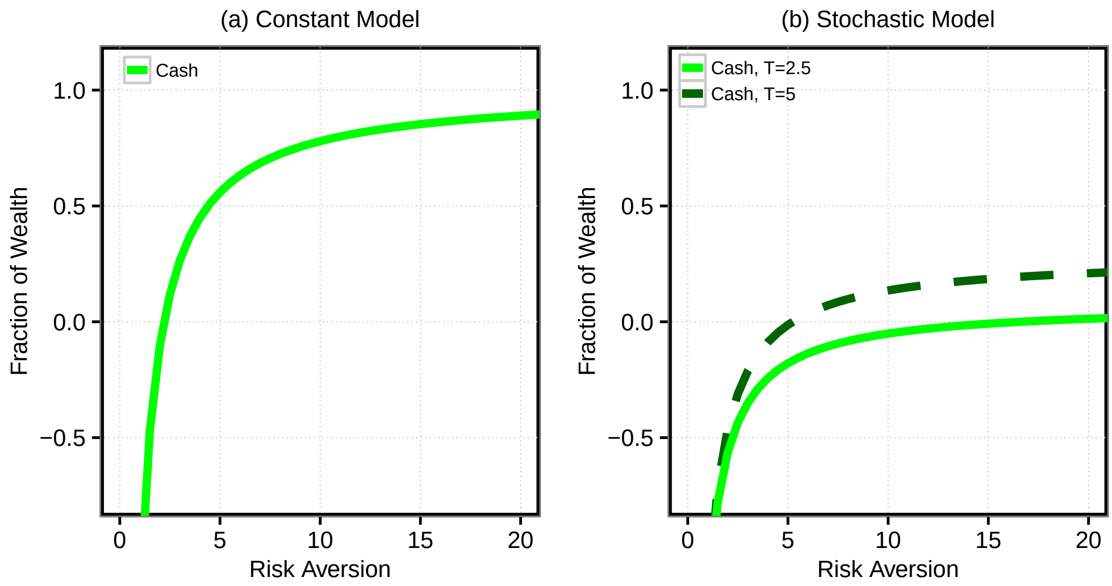

Figure 5.5: Illustrates the allocation in cash as the locally risk-free asset for a CRRA-investor. Panel (a) presents the model with constant opportunities. and Panel (b) is the model with a stochastic interest rate.

While the constant model will move towards an allocation of zero in bonds as the risk aversion increases. The stochastic model, on other hand, will move in the opposite direction with an increasing bond allocation as risk aversion is increasing. The difference is due to the hedging term, which is also pointed out in Section 5.5 and Section 5.6.2. There is, however, an upper limit to the bond allocation, but this limit is time dependent and will increase, as the time horizon gets expanded. This effect is shown in Figure 5.4.

5.6.5 The Effect of Time

Different from the model with constant investment opportunities it is now interesting to consider how the time horizon will affect the investment strategy. When the efficient frontiers for a couple of fixed time horizons were presented in Figure 5.3, the allocation in bonds and the locally risk-free asset would change if the time horizon was changed. Increase in time horizon will lead to a lower allocation in the locally risk-free asset. This is also seen in Figure 5.6, where the allocation in cash, bonds, and the specific hedging term is presented. We see how a longer time horizon will move the investor's focus from a local risk to hedging the potential future interest rate fluctuations. Allocation in hedging is the only factor that moves the bond allocation, which is why the two curves will increase in a parallel manner, when the investor has a risk aversion $\gamma > 1$. In the case with $\gamma < 1$ the investor would for an increasing time horizon allocate in the opposite direction, with a decrease in hedging

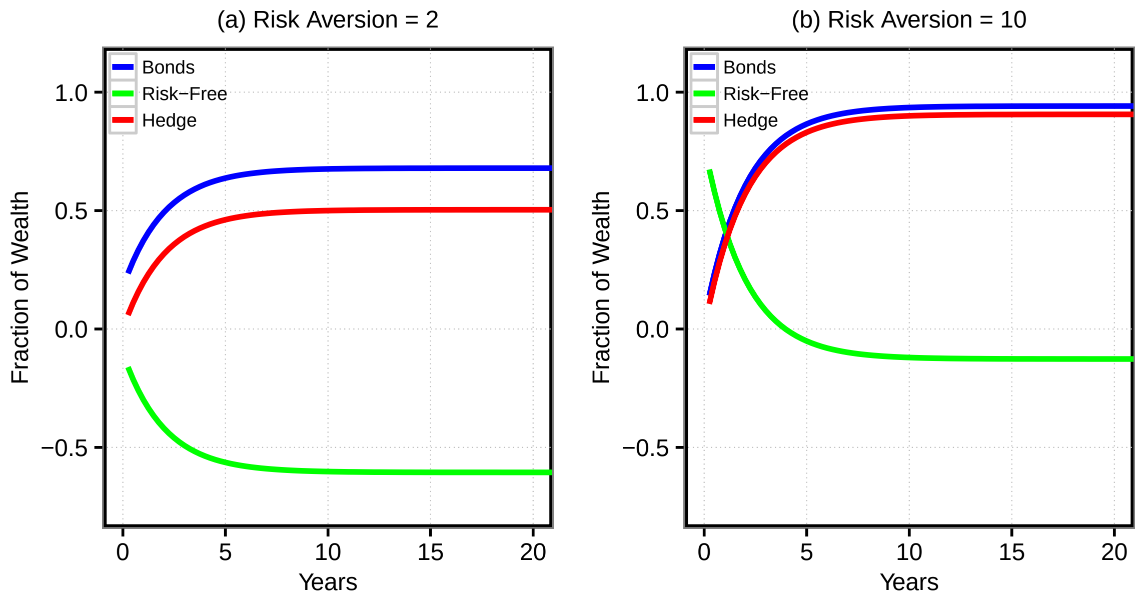

Figure 5.6: Illustrates the allocation for a CRRA-investor under the dynamic model with a stochastic interest rate over investment horizon, and for different risk aversion levels, $\gamma = 2$ and $\gamma = 10$.

but an increase the fraction of wealth allocated in cash.

An increasing time horizon will lead to more hedging, but the effect of time is not constant. As seen from the expression for bond allocation, the level of risk aversion will also play a role. As the risk aversion increases, the effect of time will also increase. The mathematical term is already presented, but Figure 5.6 shows that the allocation effect from a change in time will in absolute values have an increasing slope parameter as γ is increasing. For illustration we have plotted the functions for an investor with a risk aversion of $\gamma = 2$ and another investor with a risk aversion of $\gamma = 10$.

5.7 Real World Advice

There is not always congruence between suggestions from financial advisers and financial theory. Advisors sometimes use arguments for which there are no theoretical foundation and sometimes the theoretical models are to simple. In papers, such as Munk and Sørensen (2001), it is shown how a model with constant investment opportunities can be extended to also consider the individual's labour income. This makes the results more realistic, and approaches the advice which is given by practitioners.

When only considering institutional investors, it would not make sense to consider the labour income extension. Our extension with the assumption of a stochastic interest rate is, however, very relevant for institutional investors. It is also relevant for households as it changes an assumption about the investment opportunities which both groups of investors consider.

Assuming that institutional investors will have to follow the same advice as household investors, such as an increasing allocation in stocks when the investment horizon is increasing, we are able to relate our findings for the stochastic model to the real world advices.

Our comparison between the model with a constant interest rate and the model with a stochastic short-term interest rate has shown that the result does not change much. We still see exactly the same allocation in stocks. The fraction in stocks reacts in a natural way to the value of the risk aversion, but the fraction is still independent of time which is inconsistent with the standard advice for investors where an increasing time horizon should lead to a larger fraction in risky assets.

We do, however, see a more realistic fraction invested in bonds, as we now have a hedging term. This makes the fraction invested in bonds dependent on the time horizon. The effect of the hedging term combined with its effect of time makes the stochastic model more realistic. Risk aversion leads to a hedging against the future risk instead of just buying the locally risk-free asset, cash. There is still an increase in cash when risk aversion is increasing, but the model now includes hedging against future risk, which makes it closer to real-world advices.

The conclusion is that the extension changes the way in which institutional investors shall invest. It does, however, not affect the stock allocation.

5.8 Comparison to Other Interest Rate Models

This section intends to do an outward look to compare our findings to results under different interest rate models in a qualitative approach, namely (i) Two-Factor Vasicek Model, (ii) Cox-Ingorsoll-Ross (CIR), and (iii) Other interest rate models.

(i) Two-Factor Vasicek: In Brennan and Xia (2000), they consider the portfolio implication from a two-factor Vasicek interest rate model, where an investor only yield utility from terminal wealth. They assume the equity risk premium, in excess of the instantaneously risk-less interest rate, to be a constant. It is assumed to be a bivariate Markov process:

$$dr = [\theta(t) + \lambda_r + u - ar]dt + \sigma_r dz_r$$
$$du = [\lambda_u - bu]dt + \sigma_u dz_u$$

The new element, u, introduces variability in the long-run target for the short-term interest rate for which the short rate is adjusting. Specifically, this new process allows for independent variation in the short and long term of the yield curve. Future values of r and u are normally distributed.

In the paper, the optimal portfolio is found to be a weighted sum of three portfolios. The first portfolio is the mean-variance portfolio, which is optimal for investors characterized by log utility or a short investment horizon. The two other portfolios are constructed in order to hedge against the movements in the two state variables r and u.

Investors create a perfect hedge against the two variables in a stochastic investment opportunity set using two bonds and cash, which implies that the proportion of stock in the portfolio is solely determined by the investor's myopic demand as under constant investment opportunities.

The portfolio result implies that the bond-stock ratio increases with the risk aversion parameter, when one of the bonds that is available need to have a maturity equal to the investment horizon, and if the investor has a positive holding of stock. This is inherently similar to our result with the one-factor Vasicek model for the interest rate. However, if assuming the two-factor Vasicek and there is not a bond with maturity equal to the investment horizon, then the bond-stock ratio is not necessarily increasing in the risk aversion parameter. Therefore, the a two-factor interest model produces a qualitatively differently result than a one-factor model in its predictions regarding the relationship between risk aversion and the bond-stock ratio.

(ii) CIR: Some suggest the Cox-Ingersoll-Ross model as an alternative to the Vasicek model. It does for example deviate from the one-factor Vasicek model by having a volatility which is dependent on the interest rate as seen from the equation

$$dr_t = \kappa[\bar{r} - r_t]dt - \sigma_r \sqrt{r_t} dz_{1t}.$$

However, the results from the CIR model are not much different from the Vasicek model. In Deelstra et al. (2000), there is a comparison of the investment strategies under the one-factor Vasicek and the Cox-Ingersoll-Ross model. They have a set-up, where the interest rate models and investment strategies are comparable. Considering the stock weight first, we see that both interest rate models yield the same weight in stock. However, the hedging term and hence the total bond demand depends on the interest rate model. The hedging and thereby also the bond allocation is for $\gamma > 1$ increasing in the time horizon.

(iii) Other: Chan et al. (1992) did an empirical comparison of eight different models for modelling the short interest rate dynamics to see which model performs best or has the best fit regarding the 1 Month Treasury bond. Their results tell the story about the importance of correctly modelled volatility. The models performing best in describing the dynamics of interest rate over time allow the conditional volatility of interest rate changes to be highly correlated with the level of the interest rate. Despite expectations, they find that the widely used models Vasicek (1977) and Cox et al. (1985) perform relative poorly compared the other models such as Dothan (1978) and Black and Karasinski (1991). Since interest rate volatility is of crucial importance in hedging interest risk, and since the most commonly used models do not capture this dependence, this will have implication regarding the optimal portfolio choice.

In Kraft (2004), he considers the Black and Karasinski (1991) model for the short rate. The optimal portfolio yields the investment strategy where the investor should put all her wealth into the money market account. This is a so-called 'Passive' investment strategy and it is optimal both with and without investors having the ability to trade in other assets like stocks, bonds or any other asset. Remember our findings of the portfolio with the Vasicek model, then this result stands out. In an economy where investors can invest in assets such stocks and bond, but choose not to, implies that such a solution cannot be in a state of equilibrium.

Chapter 6

Dynamic Model with a Non-Constant Market Price of Risk

To extend the allocation model by introducing a stochastic interest rate, as in the previous chapter, is not the only solution. We have discussed that the interest rate can modelled in several ways. One can also change the modelling of other parameters. In this chapter, we will show how we can extend the model with a non-constant market price of risk in combination with a stochastic interest rate. Section 6.1 describes different ways to model a non-constant market price of risk. In Section 6.2 we motivate why the market price risk could be linked to the interest rate. We therefore suggest a new description of the market price of risk in Section 6.3. The optimal portfolio for the new dynamics are presented in Section 6.4, and the chapter ends with an analysis in Section 6.5.

6.1 Market Price of Risk

Extending the allocation problem with a non-constant market price of risk can be done in different ways, see for example Duarte (2004), Dai and Singleton (2000), Kim and Omberg (1996), and Duffee (2000) for different methods and tests. One of the approaches is to make the market price of risk, λ, stochastic. In papers such as Tanaka (2009) the model is extended by assuming a stochastic interest rate and a stochastic market price of risk. Here we choose another method to make the market price of risk non-constant. We find it realistic for it to be related to the interest rate, and we therefore choose to make it a function of the stochastic interest rate. We intend to keep the short-term interest rate following the same process as in the previous chapter, namely the Vasicek model, but now it will additionally also affect the market price of risk.

6.2 Interest Rate as a Market Predictor

According to Damodaran (2012), market timing is a practice of relying on a signal of when to enter or exit the market, and it stems from investor believe that markets do not account for all information available from the market fundamentals. In this section, we attempt to explain why the interest rate can potentially serve as a predictor for market movements. We will elaborate on why the interest rate is one of the strongest indicators for the market. As conventional wisdom dictates, an investor should sell stocks when the interest rates are low and buy when they are high. There is strong empirical evidence supporting this conventional advice that a decrease in the short interest rate seems to predict a high stock market return.

In Ang and Bekaert (2007), who looked at multiple predictors for the excess returns, they find that the interest rate has a predictive ability on the short-term horizon, since the short interest rate is strongly negative correlated with excess returns. A study by Breen et al. (1989) evaluated an investment strategy of alternating between stock and cash depending on the level of treasury bond rate and found that investors applying this strategy would see an additional 2% in excess return if the portfolio was actively managed.

However, there is some restrictions to this strategy, which a paper by Abhyankar and Davies (2002) investigated. For this, they used the correlation structure between the short interest rate and stock market returns from 1929 to 2000. They find limitations to the predictability of stock market return and it is only exhibited in 1950 to 1975. Afterwards the short interest rate has had a low predictive power on the general market. However, the predictability has some industry presence.

A recent study by Rapach et al. (2016), focuses on the short interest rate ability to predict the aggregate stock returns. They find evidence which shows that the interest rate does contain information about future market returns when it is an aggregate measure. They use their measure, the short interest index (SII). It predicts lower future returns if SII obtains higher values. They illustrate the power of their index by comparing it to the range of different predictors used in Welch and Goyal (2008), where it significantly outperforms all of them at different horizons. The SII performs well both in in-sample and out-of-sample tests and statistics. At the same time, the SII provide economic significance of its predictive ability, since a mean-variance investor is seen to generate large utility gains from allocating between stocks and bonds.

6.3 Changes in Dynamics

In this chapter, we will make the market price of risk dependent on the interest rate. For the interest rate dynamics, we will have the price of risk associated with z_1 given as

$$\lambda_{1t} = \bar{\lambda}_1 + \tilde{\lambda}_1 r_t.$$

As argued in Section 6.2, the expected return on the stock market is related to the interest rate. To capture the market price of risk associated with z_2 we therefore write

$$\lambda_{2t} = \bar{\lambda}_2 + \tilde{\lambda}_2 r_t.$$

6.3.1 Bond Price and Dynamics

For the new situation with $\lambda_{1t} = \bar{\lambda}_1 + \tilde{\lambda}_1 r_t$, we use the new price of the bond

$$B_t^{\bar{T}} = e^{-a(\bar{T}-t)-b(\bar{T}-t)r_t},$$

which is given in Munk (2013). It is almost the same as before but with $b(T-t) = \dfrac{1-e^{\tilde{\kappa}(T-t)}}{\tilde{\kappa}}$, where $\tilde{\kappa} = \kappa - \sigma_r \tilde{\lambda}_1$. In the previous model with constant market prices of risk, the dynamics of bond were described in the following way

$$\frac{dB_t^{\bar{T}}}{B_t^{\bar{T}}} = (r_t + \lambda_{1t}\sigma_r b(\bar{T}-t))dt + \sigma_r b(\bar{T}-t)dz_{1t}.$$

However, this model is differently defined, since $b(\bar{T}-t) = \dfrac{1-e^{-\tilde{\kappa}(\bar{T}-t)}}{\tilde{\kappa}}$ where $\tilde{\kappa} = \kappa - \sigma_r \tilde{\lambda}_1$ and $\lambda_{1t} = \bar{\lambda}_1 + \tilde{\lambda}_1 r_t$. $\bar{\lambda}_1$ and $\tilde{\lambda}_1$ are constants. The expression of the expected rate of return changes to account for the relationship between the interest rate and the market prices of risk

$$r_t + \sigma_r b(\bar{T}-t)\left(\bar{\lambda}_1 + \tilde{\lambda}_1 r_t\right) = \sigma_r b(\bar{T}-t)\bar{\lambda}_1 + \left(1 + \sigma_r b(\bar{T}-t)\tilde{\lambda}_1\right) r_t$$

Substituting this into expression of the zero-coupon bond dynamics

$$\frac{dB_t^{\bar{T}}}{B_t^{\bar{T}}} = \left[\sigma_r b(\bar{T}-t)\bar{\lambda}_1 + \left(1 + \sigma_r b(\bar{T}-t)\tilde{\lambda}_1\right) r_t\right] dt + \sigma_B dz_{1t}.$$

This is the new bond dynamic under the Vasicek model, where the market price of risk is an affine function of the interest rate. A high interest rate will in the Vasicek model with constant market prices of risk lead to a higher return. In this model, it still has the positive effect from the interest rate itself, but it does also have an effect via the relationship with the market price of risk, λ_{1t}. Depending on the sign and the size on the coefficient $\tilde{\lambda}_1$ and $b(\bar{T}-t)$, there will be either an increased or a decreased effect of the interest rate.

Assuming $\bar{\lambda}_1 + \tilde{\lambda}_1 r_t$ on average will be equal to the previously assumed constant value of λ_1 and $\tilde{\lambda}_1 > 0$, there will be a lower constant compensation for the risk. The investor will therefore have a part of the compensation, as dependent on the level of the interest rate.

6.3.2 Stock Price and Dynamics

In the Chapter 5, the stock dynamics were defined as

$$dS_t = S_t \left[(r_t + \sigma_S \psi_t)dt + \rho\sigma_S dz_{1t} + \sqrt{1-\rho^2}\sigma_S dz_{1t} \right].$$

Papers such as Fama and Schwert (1977) finds evidence for the excess return on the stock markets to vary negatively with the level of the interest rates. We therefore include the new definition of the market price in the bond price dynamics. The expected excess rate of return on an asset equals the product of its vector of sensitivities to the shocks and the vector of market prices of risk associated with the shocks. Applying this to the stock in our case gives

$$\sigma_S \psi_t = \rho\sigma_S \lambda_{1t} + \sqrt{1-\rho^2}\sigma_S \lambda_{2t}. \tag{6.3.1}$$

The benefit of having the market price of risk, as an affine function of interest rate is that the function enables the model to potentially capture the predictability in stocks. This predictability can be shown from Equation (6.3.1) in the following way

$$\begin{aligned} \psi_t =& \frac{1}{\sigma_S}\left(\rho\sigma_S\lambda_{1t} + \sqrt{1-\rho^2}\sigma_S\lambda_{2t}\right) \\ =& \left(\rho\bar{\lambda}_1 + \sqrt{1-\rho^2}\bar{\lambda}_2\right) + \left(\rho\tilde{\lambda}_1 + \sqrt{1-\rho^2}\tilde{\lambda}_2\right) r_t \end{aligned} \tag{6.3.2}$$

This can be substituted in and gives the dynamics of the stock process under this model, where market price of risk is an affine function of the interest rate

$$\frac{dS_t}{S_t} = \left(r_t + \sigma_S\left(\left(\rho\bar{\lambda}_1 + \sqrt{1-\rho^2}\bar{\lambda}_2\right) + \left(\rho\tilde{\lambda}_1 + \sqrt{1-\rho^2}\tilde{\lambda}_2\right) r_t\right)\right) dt \\ + \left(\rho + \sqrt{1-\rho^2}\right)\sigma_S dz_{1t}$$

From Equation (6.3.2), if $\rho\tilde{\lambda}_1 + \sqrt{1-\rho^2}\tilde{\lambda}_2 < 0$ the model would fit the observations that average excess stock return tend to be low when interest rates are high and the opposite case is also true.

6.3.3 New Wealth Dynamics

We have new dynamics of the wealth elements and as a consequence, we will have a new expression for the dynamics of the wealth. The expression is similar to Equation (5.3.5), where the dynamics were defined as

$$dW_t = W_t[r_t + \boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t, t)\boldsymbol{\lambda}_t)]dt + W_t\boldsymbol{\pi}_t^\top \underline{\underline{\sigma}}(r_t, t)dz_t. \tag{6.3.3}$$

When considering the different elements, most of the vectors and the matrix for volatility are unchanged. The difference is the new definition

$$\boldsymbol{\lambda}_t = \begin{pmatrix} \bar{\lambda}_1 + \tilde{\lambda}_1 r_t \\ \bar{\lambda}_2 + \tilde{\lambda}_2 r_t \end{pmatrix}.$$

6.4 Optimal Portfolio

The wealth is of the same form as before, when we write it in matrix form. The vector containing the market prices of risks is different, but besides that we will have the same HJB equation as presented in Equation (5.4.2). Due to the same indirect utility function, we will also have the same partial differential equation for the function $g(r,t)$ to solve. From Equation (5.5.3), we have the partial differential equation given as

$$0 = g_t(r,t) + \left(\kappa[\bar{r} - r] + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right) g_r(r,t) \\ + \left(\frac{1-\gamma}{\gamma} r + \frac{1-\gamma}{2\gamma^2}\|\boldsymbol{\lambda}\|^2\right) g(r,t) + \frac{1}{2} g_{rr}\|\boldsymbol{\sigma}_r\|^2,$$

with the terminal condition $g(r,T) = 1$. The form of the function $g(r,t)$ is this time different, as we are in a quadratic framework. We extend the previous form with a quadratic term such that we have a qualified guess of the form

$$g(r,t) = \exp\left\{\frac{1-\gamma}{\gamma}A_0(T-t) + \frac{1-\gamma}{\gamma}A_1(T-t)r + \frac{1}{2}\frac{1-\gamma}{\gamma}A_2(\tau)r^2\right\}.$$

Following the same procedure as in Chapter 5, we find the partial derivatives of the function, and substitute the results into the partial differential equation, which we will have to solve. The partial derivatives are

$$\begin{aligned}
\frac{\partial g(r,t)}{\partial t} &= \left(-\frac{1-\gamma}{\gamma}\left(\frac{1}{2}r^2A_2'(\tau) + rA_1'(\tau) + A_0'(\tau)\right)\right)\cdot g(r,t)\\
\frac{\partial g(r,t)}{\partial r} &= \left(\frac{1-\gamma}{\gamma}(A_1(\tau)+A_2(\tau)r)\right)\cdot g(r,t)\\
\frac{\partial^2 g(r,t)}{\partial r^2} &= \left(\left(\frac{1-\gamma}{\gamma}(A_1(\tau)+A_2(\tau)r)\right)^2 + \frac{1-\gamma}{\gamma}A_2(\tau)\right)\cdot g(r,t),
\end{aligned}$$

and after substituting the partial derivatives into the partial differential equation, we will have it given as

$$\begin{aligned}
0 = &-\frac{1-\gamma}{\gamma}\left(\frac{1}{2}r^2A_2'(\tau) + rA_1'(\tau) + A_0'(\tau)\right)\cdot g(r,t)\\
&+\left(\kappa[\bar{r}-r_t] + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)\cdot\left(\frac{1-\gamma}{\gamma}(A_1(\tau)+A_2(\tau)r)\right)\cdot g(r,t)\\
&+\frac{1}{2}\left(\left(\frac{1-\gamma}{\gamma}(A_1(\tau)+A_2(\tau)r)\right)^2 + \frac{1-\gamma}{\gamma}A_2(\tau)\right)\cdot g(r,t)\|\boldsymbol{\sigma}_r\|^2\\
&+\left(\frac{1-\gamma}{\gamma}r + \frac{1-\gamma}{2\gamma^2}\|\boldsymbol{\lambda}\|^2\right)\cdot g(r,t).
\end{aligned}$$

To simplify the equation, we remove the expression for $g(r,t)$ as done with the model in Chapter 5, and then divide both sides with $\frac{1-\gamma}{\gamma}$

$$\begin{aligned}
0 = &-\frac{1}{2}r^2A_2'(\tau) - rA_1'(\tau) - A_0'(\tau) + \left(\kappa[\bar{r}-r_t] + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)\cdot(A_1(\tau)+A_2(\tau)r)\\
&+r + \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2 + \frac{1}{2}\left(\frac{1-\gamma}{\gamma}(A_1(\tau)+A_2(\tau)r)^2 + A_2(\tau)\right)\|\boldsymbol{\sigma}_r\|^2. \qquad (6.4.1)
\end{aligned}$$

The parentheses are removed by rearranging and multiplying

$$\begin{aligned}A_0'(\tau) = &-\frac{1}{2}r^2A_2'(\tau) - rA_1'(\tau) + \kappa\bar{r}A_2(\tau)r - \kappa rA_1(\tau) - \kappa rA_2(\tau)r + \kappa\bar{r}A_1(\tau)\\ &+ \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_rA_2(\tau)r + \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_rA_1(\tau) + r + \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2 + \frac{1}{2}A_2(\tau)\|\boldsymbol{\sigma}_r\|^2\\ &+ \frac{1-\gamma}{2\gamma}A_1^2(\tau)\|\boldsymbol{\sigma}_r\|^2 + \frac{1-\gamma}{2\gamma}A_2^2(\tau)r^2\|\boldsymbol{\sigma}_r\|^2 + \frac{1-\gamma}{2\gamma}2A_1(\tau)A_2(\tau)r\|\boldsymbol{\sigma}_r\|^2,\end{aligned}$$

We will as the next step substitute in some of the vectors. This is a necessity in this model as the vector containing the market prices of risk is related to the interest rate. Following the previous procedure, we will afterwards isolate terms related to the short-term interest rate. We therefore have to know how the market price of risk will change this partial differential equation. It is relevant for the expression $\|\boldsymbol{\lambda}\|^2$ and the multiplication $\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r$. Remembering the definitions of the vectors

$$\boldsymbol{\lambda}_t = \begin{pmatrix}\bar{\lambda}_1 + \tilde{\lambda}_1 r_t\\ \bar{\lambda}_2 + \tilde{\lambda}_2 r_t\end{pmatrix} \quad \text{and} \quad \boldsymbol{\sigma}_r = \begin{pmatrix}-\sigma_r\\ 0\end{pmatrix}.$$

From these vectors we can find the following two expressions

$$\begin{aligned}\|\boldsymbol{\lambda}\|^2 &= \left(\bar{\lambda}_1 + \tilde{\lambda}_1 r\right)^2 + \left(\bar{\lambda}_2 + \tilde{\lambda}_2 r\right)^2 = \bar{\lambda}_1^2 + \tilde{\lambda}_1^2r^2 + 2\bar{\lambda}_1\tilde{\lambda}_1 r + \bar{\lambda}_2^2 + \tilde{\lambda}_2^2r^2 + 2\bar{\lambda}_2\tilde{\lambda}_2 r\\ \boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r &= \begin{pmatrix}\bar{\lambda}_1 + \tilde{\lambda}_1 r & \bar{\lambda}_2 + \tilde{\lambda}_2 r\end{pmatrix}\begin{pmatrix}-\sigma_r\\ 0\end{pmatrix} = -\sigma_r\bar{\lambda}_1 - \sigma_r\tilde{\lambda}_1 r,\end{aligned}$$

which are substituted in and we will have the following equation

$$\begin{aligned}A_0'(\tau) = &-\frac{1}{2}r^2A_2'(\tau) - rA_1'(\tau) + \kappa\bar{r}A_2(\tau)r - \kappa rA_1(\tau) - \kappa rA_2(\tau)r\\ &+ \frac{1-\gamma}{\gamma}\left(-\sigma_r\bar{\lambda}_1 - \sigma_r\tilde{\lambda}_1 r\right)A_2(\tau)r + \kappa\bar{r}A_1(\tau) + \frac{1-\gamma}{\gamma}\left(-\sigma_r\bar{\lambda}_1 - \sigma_r\tilde{\lambda}_1 r\right)A_1(\tau)\\ &+ r + \frac{1}{2\gamma}\left(\bar{\lambda}_1^2 + \tilde{\lambda}_1^2r^2 + 2\bar{\lambda}_1\tilde{\lambda}_1 r + \bar{\lambda}_2^2 + \tilde{\lambda}_2^2r^2 + 2\bar{\lambda}_2\tilde{\lambda}_2 r\right) + \frac{1}{2}A_2(\tau)\|\boldsymbol{\sigma}_r\|^2\\ &+ \frac{1-\gamma}{2\gamma}A_1^2(\tau)\|\boldsymbol{\sigma}_r\|^2 + \frac{1-\gamma}{2\gamma}A_2^2(\tau)r^2\|\boldsymbol{\sigma}_r\|^2 + \frac{1-\gamma}{2\gamma}2A_1(\tau)A_2(\tau)r\|\boldsymbol{\sigma}_r\|^2.\end{aligned}$$

The equation is rewritten, and terms related to the interest rate, r, are isolated in an order such that they fit three ordinary differential equations. This is done by creating parentheses for each of the ordinary differential equations in the same way, as we did in the previous chapter. We divide it into the following three functions

$$
\begin{aligned}
A_0'(\tau) =& \kappa\bar{r}A_1(\tau) - \frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1 A_1(\tau) + \frac{1}{2\gamma}\left(\bar{\lambda}_1^2 + \bar{\lambda}_2^2\right) + \frac{1}{2}A_2(\tau)\|\boldsymbol{\sigma}_r\|^2 + \frac{1-\gamma}{2\gamma}A_1^2(\tau)\|\boldsymbol{\sigma}_r\|^2 \\
0 =& r\left(1 - A_1'(\tau) + \kappa\bar{r}A_2(\tau) - \kappa A_1(\tau) - \frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1 A_1(\tau) - \frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1 A_2(\tau)\right) \\
&+ r\left(\frac{1}{2\gamma}\left(2\bar{\lambda}_1\tilde{\lambda}_1 + 2\bar{\lambda}_2\tilde{\lambda}_2\right) + \frac{1-\gamma}{2\gamma}2A_1(\tau)A_2(\tau)\|\boldsymbol{\sigma}_r\|^2\right) \\
0 =& r\left(-\frac{1}{2}rA_2'(\tau) + \frac{1}{2\gamma}\left(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2\right)r + \left(-\kappa - \frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)rA_2(\tau)\right) \\
&+ r\left(\frac{1-\gamma}{2\gamma}A_2^2(\tau)r\|\boldsymbol{\sigma}_r\|^2\right).
\end{aligned}
$$

For the second equation, we can remove r on the outside the parentheses, isolate $A_1'(\tau)$ and then reduce the parentheses until we have the ordinary differential equation given as

$$
\begin{aligned}
A_1'(\tau) =& 1 + \frac{1}{\gamma}\left(\bar{\lambda}_1\tilde{\lambda}_1 + \bar{\lambda}_2\tilde{\lambda}_2\right) + \left(\kappa\bar{r} - \frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1\right)A_2(\tau) \\
&+ \left(-\kappa - \frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1 + \frac{1-\gamma}{\gamma}A_2(\tau)\|\boldsymbol{\sigma}_r\|^2\right)A_1(\tau)
\end{aligned}
$$

If we now focus on the last equation, we can isolate $A_2'(\tau)$ by changing some of the parentheses and afterwards moving $A_2'(\tau)$ to the other side of the equality sign

$$
\frac{1}{2}rA_2'(\tau) = \frac{1}{2\gamma}\left(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2\right)r + \left(-\kappa r - \frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1 r\right)A_2(\tau) + \frac{1-\gamma}{2\gamma}A_2^2(\tau)r\|\boldsymbol{\sigma}_r\|^2
$$

We have $A_2'(\tau)$ isolated after multiplication such that the ordinary differential equation is given as

$$
A_2'(\tau) = \frac{1}{\gamma}\left(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2\right) + 2\left(-\kappa - \frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)A_2(\tau) + \frac{1-\gamma}{\gamma}A_2^2(\tau)\|\boldsymbol{\sigma}_r\|^2.
$$

The three ordinary differential equations are changed such that the terms $A_0'(\tau)$, $A_1'(\tau)$, and $A_1'(\tau)$ are isolated. We still have to solve this system of ordinary differential equations in our process of solving the partial differential equation. The three equations are

$$
\begin{aligned}
A_0'(\tau) =& \frac{1}{2\gamma}\left(\bar{\lambda}_1^2+\bar{\lambda}_2^2\right)+\left(\kappa\bar{r}-\frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1\right)A_1(\tau)+\frac{1}{2}\left(A_2(\tau)+\frac{1-\gamma}{\gamma}A_1^2(\tau)\right)\|\boldsymbol{\sigma}_r\|^2 \\
A_1'(\tau) =& 1+\frac{1}{\gamma}\left(\bar{\lambda}_1\tilde{\lambda}_1+\bar{\lambda}_2\tilde{\lambda}_2\right)+\left(\kappa\bar{r}-\frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1\right)A_2(\tau) \\
&+\left(-\kappa-\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1+\frac{1-\gamma}{\gamma}A_2(\tau)\|\boldsymbol{\sigma}_r\|^2\right)A_1(\tau) \qquad (6.4.2)\\
A_2'(\tau) =& \frac{1}{\gamma}\left(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2\right)+2\left(-\kappa-\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)A_2(\tau)+\frac{1-\gamma}{\gamma}A_2^2(\tau)\|\boldsymbol{\sigma}_r\|^2.
\end{aligned}
$$

Two of the ordinary differential equations which we have found are in the form similar to the ones in Appendix C.3 of Munk (2013). We therefore already have the solution to the system of ordinary differential equations, which consists of $A_1'(\tau)$ and $A_2'(\tau)$ from above and their initial conditions; $A_1(0)=0$ and $A_2(0)=0$. For a system of equations of the form

$$
\begin{aligned}
A_2'(\tau) &= a - bA_2(\tau) + cA_2(\tau)^2 \\
A_1'(\tau) &= d + fA_2(\tau) - \left(\frac{1}{2}b - cA_2(\tau)\right)A_1(\tau),
\end{aligned}
$$

we will have the following solutions

$$
\begin{aligned}
A_2(\tau) &= \frac{2a\left(e^{v\tau}-1\right)}{(v+b)\left(e^{v\tau}-1\right)+2v} \\
A_1(\tau) &= \frac{d}{a}A_2(\tau)+\frac{2}{v}(db+2fa)\frac{\left(e^{v\tau/2}-1\right)^2}{(v+b)\left(e^{v\tau}-1\right)+2v}
\end{aligned}
$$

with $v=\sqrt{b^2-4ac}$. The ordinary differential equations above fit into the form, which is necessary to use the solutions to the system. As the ordinary differential equations are written in the same order as the definition of the forms above, we can directly define the expressions

$$
a=\frac{1}{\gamma}\left(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2\right) \qquad b=2\left(\kappa+\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right) \qquad c=\frac{1-\gamma}{\gamma}\|\boldsymbol{\sigma}_r\|^2
$$

$$
d=1+\frac{1}{\gamma}\left(\bar{\lambda}_1\tilde{\lambda}_1+\bar{\lambda}_2\tilde{\lambda}_2\right) \qquad f=\left(\kappa\bar{r}-\frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1\right).
$$

The expressions are substituted into the general solution for this type of system. We have the following solutions to the system of ordinary differential equations

$$A_2(\tau) = \frac{2\left(\frac{1}{\gamma}\left(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2\right)\right)\left(e^{v\tau}-1\right)}{\left(v+2\left(\kappa+\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)\right)\left(e^{v\tau}-1\right)+2v} \tag{6.4.3}$$

$$\begin{aligned} A_1(\tau) =& \frac{1+\frac{1}{\gamma}\left(\bar{\lambda}_1\tilde{\lambda}_1+\bar{\lambda}_2\tilde{\lambda}_2\right)}{\left(\frac{1}{\gamma}\left(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2\right)\right)}A_2(\tau)+\frac{2}{v}\Bigg(\left(1+\frac{1}{\gamma}\left(\bar{\lambda}_1\tilde{\lambda}_1+\bar{\lambda}_2\tilde{\lambda}_2\right)\right) \\ &\cdot 2\left(\kappa+\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)+2\left(\kappa\bar{r}-\frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1\right)\left(\frac{1}{\gamma}\left(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2\right)\right)\Bigg) \\ &\cdot\frac{\left(e^{v\tau/2}-1\right)^2}{\left(v+2\left(\kappa+\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)\right)\left(e^{v\tau}-1\right)+2v}. \end{aligned} \tag{6.4.4}$$

$A_0(\tau)$ is not considered as it is unnecessary when solving the partial differential equation.

6.4.1 Verification of Solution

The next step is to verify our suggested form of the utility function by solving the partial differential equation from Equation (6.4.1). We will use the definitions of the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}_r$, as defined earlier, and the functions $A_0'(\tau)$, $A_1'(\tau)$, and $A_2'(\tau)$. First, substitute in the ordinary differential equations $A_0'(\tau)$, $A_1'(\tau)$, and $A_2'(\tau)$ which are given in Equation (6.4.2). The partial differential equation becomes

$$\begin{aligned} 0 =& -\frac{1}{2}r^2\left(\frac{1}{\gamma}\left(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2\right)+2\left(-\kappa-\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)A_2(\tau)+\frac{1-\gamma}{\gamma}A_2^2(\tau)\|\boldsymbol{\sigma}_r\|^2\right) \\ &-r\left(1+\frac{1}{\gamma}\left(\bar{\lambda}_1\tilde{\lambda}_1+\bar{\lambda}_2\tilde{\lambda}_2\right)+\left(\kappa\bar{r}-\frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1\right)A_2(\tau)\right)+\left(r+\frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2\right) \\ &-r\left(-\kappa-\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1+\frac{1-\gamma}{\gamma}A_2(\tau)\|\boldsymbol{\sigma}_r\|^2\right)A_1(\tau)-\frac{1}{2\gamma}\left(\bar{\lambda}_1^2+\bar{\lambda}_2^2\right) \\ &-\left(\kappa\bar{r}-\frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1\right)A_1(\tau)-\frac{1}{2}\left(A_2(\tau)+\frac{1-\gamma}{\gamma}A_1^2(\tau)\right)\|\boldsymbol{\sigma}_r\|^2 \\ &+\left(\kappa[\bar{r}-r_t]+\frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)A_1(\tau)+\left(\kappa[\bar{r}-r_t]+\frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)A_2(\tau)r \\ &+\frac{1}{2}\left(\frac{1-\gamma}{\gamma}(A_1(\tau)+A_2(\tau)r)^2\right)\|\boldsymbol{\sigma}_r\|^2+\frac{1}{2}A_2(\tau)\|\boldsymbol{\sigma}_r\|^2. \end{aligned}$$

Several terms cancel out, such that the equation is

$$\begin{aligned}
0 =& -\frac{1}{2}r^2\left(\frac{1}{\gamma}\left(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2\right)+2\left(-\kappa-\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)A_2(\tau)\right)\\
&-r\left(1+\frac{1}{\gamma}\left(\bar{\lambda}_1\tilde{\lambda}_1+\bar{\lambda}_2\tilde{\lambda}_2\right)+\left(\kappa\bar{r}-\frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1\right)A_2(\tau)\right)+\left(r+\frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2\right)\\
&-r\left(-\kappa-\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)A_1(\tau)-\frac{1}{2\gamma}\left(\bar{\lambda}_1^2+\bar{\lambda}_2^2\right)-\left(\kappa\bar{r}-\frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1\right)A_1(\tau)\\
&+\left(\kappa[\bar{r}-r_t]+\frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)A_1(\tau)+\left(\kappa[\bar{r}-r_t]+\frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)A_2(\tau)r.
\end{aligned}$$

Then substitute in the vector $\|\boldsymbol{\lambda}\|^2$, which we defined earlier when we found the ordinary differential equations. In the same step, we also remove r as it is present with both a positive and a negative sign, such that the equation becomes

$$\begin{aligned}
0 =& -r^2\left(-\kappa-\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)A_2(\tau)-r\left(\kappa\bar{r}-\frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1\right)A_2(\tau)\\
&-r\left(-\kappa-\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)A_1(\tau)-\left(\kappa\bar{r}-\frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1\right)A_1(\tau)\\
&+\left(\kappa[\bar{r}-r_t]+\frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)A_1(\tau)+\left(\kappa[\bar{r}-r_t]+\frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)A_2(\tau)r
\end{aligned}$$

The final step is to multiply the two vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\sigma}_r$ as we also did when finding the ordinary differential equations. It is thereby seen how all the terms cancel out, and we have proven that our suggestion is a possible solution to the partial differential equation.

6.4.2 Portfolio Weights

As mentioned earlier, we are using the same utility function and HJB equation, as in the previous model. When considering the portfolio weights, we will therefore only have a difference in the form of the function $g(r,t)$, which we defined in a new way for the case with non-constant market price of risk. We therefore use Equation (5.5.7) where we have the portfolio weights in the case of a CRRA investor. The portfolio weights are under this model given as

$$\boldsymbol{\pi} = \frac{1}{\gamma}(\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\boldsymbol{\lambda}+g(r,t)^{-1}g_r(r,t)(\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\boldsymbol{\sigma}_r.$$

for which we will use the function $g(r,t)$ and the partial derivative $g_r(r,t)$. Both are defined previously in this chapter. After substituting in these two functions we will have the portfolio weights given as

$$\boldsymbol{\pi} = \frac{1}{\gamma}(\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\boldsymbol{\lambda} + (\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\boldsymbol{\sigma}_r\left(\frac{1-\gamma}{\gamma}(A_1(\tau)+A_2(\tau)r)\right).$$

It does not seem very different from our previous result, when we write it in matrix form. When considering the elements of the equation it is however different from our previous findings. We still have the portfolio weights, the volatility matrix and vector for interest rate risk

$$\boldsymbol{\pi}_t = \begin{pmatrix}\pi_B\\ \pi_S\end{pmatrix} \qquad \underline{\underline{\sigma}}(r_t,t) = \begin{pmatrix}\sigma_B(r_t,t) & 0\\ \rho\sigma_S & \sqrt{1-\rho^2}\sigma_S\end{pmatrix} \qquad \boldsymbol{\sigma} = \begin{pmatrix}-\sigma_r\\ 0\end{pmatrix}.$$

The vector for the market price of risk is defined differently, which is the idea with this new model, and it is given as

$$\boldsymbol{\lambda}_t = \begin{pmatrix}\bar{\lambda}_1 + \tilde{\lambda}_1 r_t\\ \bar{\lambda}_2 + \tilde{\lambda}_2 r_t\end{pmatrix}.$$

Substituting in the definitions of the matrix and the vectors makes expression ugly. We therefore directly specify the allocation in the two types of assets, where we have the allocation in bonds given as

$$\pi_B = \frac{1}{\gamma}\left(\frac{\bar{\lambda}_1 + \tilde{\lambda}_1 r}{\sigma_B(r_t,t)} - \frac{\rho\left(\bar{\lambda}_2 + \tilde{\lambda}_2 r\right)}{\sqrt{1-\rho^2}\sigma_B(r_t,t)}\right) + \frac{\gamma-1}{\gamma}\frac{\sigma_r}{\sigma_B(r_t,t)}\left(A_1(\tau)+A_2(\tau)r\right),$$

and the weight for allocation in stocks will be

$$\pi_S = \frac{1}{\gamma}\frac{\bar{\lambda}_2 + \tilde{\lambda}_2 r}{\sqrt{1-\rho^2}\sigma_S}.$$

6.5 Analysis of Allocation Results

For the rest of this chapter, we will analyse the new portfolio results. As in Chapter 5, we will start with interpretation of the mathematical result. This is followed by a numerical analyses. Finally, we conduct an alternative calibration of the market price of risk.

6.5.1 Mathematical Implications

In this section, we will examine the mathematical implication of the newly developed model for optimal portfolio choice and how it differs from the former model. Both are repeated here for convenience. First, the model only extended with a stochastic interest rate is given as

$$\boldsymbol{\pi} = \frac{1}{\gamma}(\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\boldsymbol{\lambda} + (\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\boldsymbol{\sigma}_r\left(\frac{1-\gamma}{\gamma}A_1(\tau)\right), \text{ with } A_1(\tau) = \frac{1}{\kappa}(1-e^{-\kappa\tau}).$$

Secondly, the model with a stochastic interest rate and the market price of risk as an affine function of the interest rate given as

$$\boldsymbol{\pi} = \frac{1}{\gamma}(\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\boldsymbol{\lambda} + (\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\boldsymbol{\sigma}_r\left(\frac{1-\gamma}{\gamma}(A_1(\tau)+A_2(\tau)r)\right)$$

where $A_1(\tau)$ and $A_2(\tau)$ for the latter model are

$$\begin{aligned}
A_2(\tau) =& \frac{2\left(\frac{1}{\gamma}\left(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2\right)\right)(e^{v\tau}-1)}{\left(v+2\left(\kappa+\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)\right)(e^{v\tau}-1)+2v} \\
A_1(\tau) =& \frac{1+\frac{1}{\gamma}\left(\bar{\lambda}_1\tilde{\lambda}_1+\bar{\lambda}_2\tilde{\lambda}_2\right)}{\left(\frac{1}{\gamma}\left(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2\right)\right)}A_2(\tau) + \frac{2}{v}\left(1+\frac{1}{\gamma}\left(\bar{\lambda}_1\tilde{\lambda}_1+\bar{\lambda}_2\tilde{\lambda}_2\right)\right) \\
&\cdot\left(2\left(\kappa+\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)+2\left(\kappa\bar{r}-\frac{1-\gamma}{\gamma}\sigma_r\bar{\lambda}_1\right)\frac{1}{\gamma}\left(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2\right)\right) \\
&\cdot\frac{\left(e^{v\tau/2}-1\right)^2}{\left(v+2\left(\kappa+\frac{1-\gamma}{\gamma}\sigma_r\tilde{\lambda}_1\right)\right)(e^{v\tau}-1)+2v}.
\end{aligned}$$

The difference between the two models lies in the hedging term, and the direct effect of the interest rate in the fraction of wealth allocated to stocks and bonds. The hedging term is more complex in its interpretation, because of the functional expressions of the ordinary differential equations. The $A_1(\tau)$ is differently formulated in the models, while $A_2(\tau)$ is new in the model with affine market price of risk. If we consider the scenario, where investors can invest in stocks and bonds, we will have the following expressions

$$\pi_B = \frac{1}{\gamma\sigma_B}\left(\bar{\lambda}_1 + \tilde{\lambda}_1 r_t - \frac{\rho}{\sqrt{1-\rho^2}}(\bar{\lambda}_2 + \tilde{\lambda}_2 r_t)\right) + \frac{\gamma-1}{\gamma}\frac{\sigma_r}{\sigma_B}(A_1(\tau) + A_2(\tau)r)$$
$$\pi_S = \frac{1}{\gamma}\frac{\bar{\lambda}_2 + \tilde{\lambda}_2 r}{\sqrt{1-\rho^2}\sigma_S}$$

If we look at the fraction of wealth invested in bonds, the decomposed expression sheds light on how the hedging term can have a positive effect on the amount of wealth allocated to bonds under the assumption of $\gamma > 1$. However, it is still difficult to state any analytical results from this portfolio weight, because of the complexity of the ordinary differential equations. To disentangle this effect we will resort to a numerical analysis in Section 6.5.2. This is to analyse the effects of different parameters and how the model behave under the set of historical estimates, which we introduced in Section 5.6.3.

Considering the stock weight in the portfolio, the fraction of wealth invested in stock will be directly affected by the interest rate. This is due to the affine relationship between the market price of risk and the interest rate. This is a new effect, which differs from the model with a constant market price of risk. From this weight, the amount of wealth allocated to stock is increasing in the interest rate.

6.5.2 Model Implications and Comparison of Models

To generate the numerical results in a consistent fashion with our analysis of the model with only a stochastic interest, the same historical estimates are used to model the portfolio results. The parameters for the simulation of the interest rate are restated together with the estimates used for the portfolio weights

$$\begin{array}{lllll} r_t = 1\% & \bar{r} = 1\% & \kappa = 0.4965 & \rho = 0.2 & \lambda_1 = 0.11 \\ \bar{\lambda}_1 = 0.109 & \tilde{\lambda}_1 = 0.067 & \lambda_2 = 0.3666 & \bar{\lambda}_2 = 0.3650 & \tilde{\lambda}_2 = 0.06 \\ \sigma_s = 0.202 & \sigma_r = 5\% & \sigma_B = 0.1 & & \end{array}$$

The values of $\bar{\lambda}_i$ and $\tilde{\lambda}_i$ are calibrated to make the average value of the new description $\bar{\lambda}_i + \tilde{\lambda}_i r$ equal to the value of the constant market prices of risk, λ_1 and λ_2.

6.5.3 Wealth Invested in Bonds

Considering the allocation of wealth in bonds, in order to understand the implications of the model with affine market price of risk compared to the model with constant market price of risk. Figure 6.1 illustrates the two model's relationship between the amount of wealth invested in bonds and investment horizon for four different levels of risk aversion, and it shows that the models differ in terms of their bond allocation. The thick green function in all four panels indicates the allocation for the model with non-constant market price of risk. The thin blue function is the allocation in bonds from the model with constant market price of risk. The pink shaded area between the functions indicates the difference in the amount of wealth allocated to bonds between the new model and the previous model from Chapter 5.

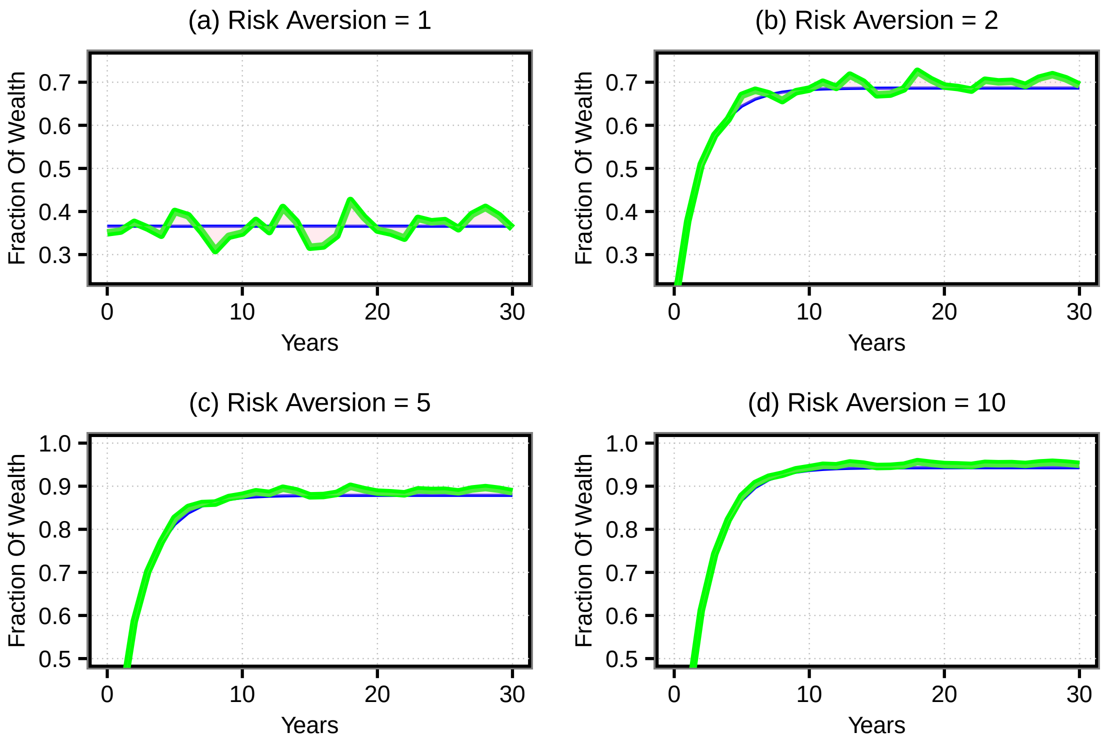

Figure 6.1: The four panels show the bond allocation for the two models with stochastic interest rate at four different levels of risk aversion. The thick blue functions is the model with non-constant market price of risk, whereas the thin blue function is assuming constant market price of risk. The pink shaded are indicates the difference in bond allocation from assuming a non-constant market price of risk.

The model with only a stochastic interest rate has a permanent increase in the amount of wealth invested in bonds over the entire investment horizon. Extending the model with an affine market price of risk makes the change in the bond allocation vary with the interest rate. There is still an increasing trend in the allocation of wealth towards more bonds, but the increase does not follow the same well-behaved increase as the model from Chapter 5.

The fluctuations in the bond allocation for the new model are decreasing in risk aversion. The two allocation models also converge towards the same bond allocation as the risk aversion is increasing. In order to shed more light on why the fluctuations decrease and why the two models converge, we again consider the weight of wealth allocated to the bonds

$$\pi_B = \underbrace{\frac{\bar{\lambda}_1 + \tilde{\lambda}_1 r_t}{\gamma \sigma_B}}_{\text{First term}} - \underbrace{\frac{1}{\gamma \sigma_B} \frac{\rho(\bar{\lambda}_2 + \tilde{\lambda}_2 r_t)}{\sqrt{1-\rho^2}}}_{\text{Second term}} + \underbrace{\frac{\gamma - 1}{\gamma} \frac{\sigma_r}{\sigma_B} (A_1(\tau) + A_2(\tau) r)}_{\text{Hedging term}} .$$

From this expression, the increase in bond allocation can only originate from two terms of the three terms, as the second term will have a negative impact. This is because the correlation between stocks and bonds is positive, $\rho = 0.2$. In Table 6.1, the expression for the bond has been decomposed into the different terms, namely the first term, the second term and hedging term. $A_1(\tau)$ and $A_2(\tau)$ are included as well. The first term will be defined as the investment incentive, which is the incentive to allocate wealth only due to investment purposes, whereas the following two terms are either related to the stock allocation or the hedging purpose.

From the terms in Table 6.1, it is seen that the investment incentive decrease with an increase in the risk aversion. The second term is decreasing if the risk aversion or the volatility of bonds are increasing. The decrease makes the correlation with stocks less influential. The hedging term is, on the other hand, increasing in risk aversion. This can imply that the investor employs bonds as a hedging instrument rather than a investment vehicle.

The overall increase in bonds in the model with an affine market price of risk can only arise from the investment incentive or the hedging incentive of the investor given that $\gamma > 1$. We already addressed that the investment incentive of bonds is decreasing in risk aversion, and that the hedging incentive is increasing in risk aversion. When the investor's risk aversion is increasing, we therefore see a shift from investment incentive to hedging incentive, which reduces the fluctuations in the bond allocation. This is because the interest rate has less impact in the hedging term than in the investment term.

This change is also the reason why the two models will converge in their bond

γ	T	1st	2nd	Hedge	$A_1(\tau)$	$A_2(\tau)$
0.5	1	2.193	-1.493	-1.058	2.117	0
	30	2.176	-1.489	-1.588	3.176	0.016
1	1	1.097	-0.746	0	2.058	0
	30	1.088	-0.745	0	3.095	0.008
2	1	0.548	-0.373	0.507	2.029	0
	30	0.544	-0.372	0.764	3.054	0.004
5	1	0.219	-0.149	0.805	2.012	0
	30	0.218	-0.149	1.212	3.030	0.002
10	1	0.110	-0.075	0.903	2.006	0
	30	0.109	-0.074	1.360	3.022	0.001
20	1	0.055	-0.037	0.951	2.003	0
	30	0.054	-0.037	1.434	3.018	0.0004

Table 6.1: A decomposed view of how the different terms affect the bond allocation. 1st and 2nd term should be constant over time but varies because of the interest rate simulation.

allocation, as the risk aversion is increasing. For an increasing risk aversion, the hedging term will in both models have an increasing impact. The two models are therefore converging in bond allocation, because their respective values of $A_1(\tau)$ and $A_2(\tau)$, which are used in the hedging terms, are converging.

If $\gamma = 1$, the hedging incentive is not present and if $\gamma < 1$ the hedging incentive is negative. For both cases, the investment incentive will be the only positive effect in bond allocation and investment incentive is now relevant. The overall effect of a lower risk aversion is a lower bond allocation. A larger fraction of the investor's wealth must therefore be allocated in stocks or the locally risk-free asset.

Under the model with affine market price of risk, the interest rate takes on a dual role in its analytical implication for the investment strategy. If we again consider the expression of wealth allocation in bonds, the interest rate enters in all terms. The first term, the investment incentive in bonds is increasing in the interest rate. The second term, that accounts for the correlation between stocks and bonds, is increasing in the interest rate. The hedging term is also increasing in the interest rate, but the effect of the interest rate is small, since it is multiplied with $A_2(\tau)$. The dual role comes into effect in the weight of bonds, where the interest rate is pulling the amount of wealth allocated in two opposite directions. The investment incentive and hedging incentive increase the amount of wealth allocated in bonds, while the term correlated with the stocks is reducing the amount of wealth in bonds. The effect of the interest rate's dual role is dependent on the parameter in the market price of risk, which is multiplied with the interest rate. If $\tilde{\lambda}_i > 1$, then the interest rate's effect becomes more pronounced, and if $\tilde{\lambda}_i < 1$ the effect becomes less pronounced.

6.5.4 Wealth Invested in Stocks

The weight of wealth invested in stock is given as

$$\pi_S = \frac{1}{\gamma} \frac{\bar{\lambda}_2 + \tilde{\lambda}_2 r}{\sqrt{1-\rho^2}\sigma_S}.$$

The weight shows that stocks are directly independent of time, but it will vary over time as the interest rate is stochastic. An example is shown in Figure 6.2, where the stock allocation is changing over time since the interest rate is. As mentioned earlier and as seen from the equation just above, the interest rate will in the model with an affine market price of risk have a direct positive effect on the amount of wealth allocated to stock. The weight is as in the previous model decreasing in risk aversion.

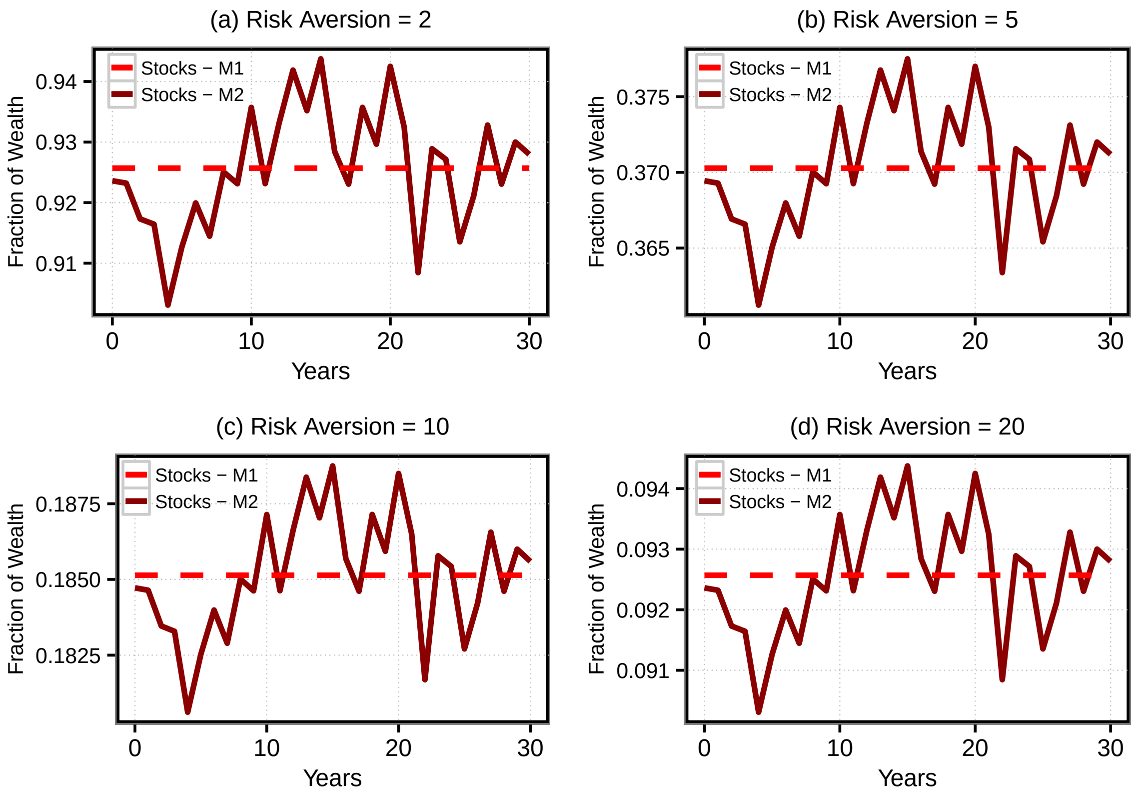

Figure 6.2: Allocation in stocks across a 30 year investment horizon. M2 indicates the extended model with an affine market price of risk. M1 indicates the model from Chapter 5, which is only extended with a stochastic interest rate.

In Section 6.2, we explained how investors use the interest rate as a predictor of where the market is heading in order to make their timing of the investment decision. Under this model with an affine market price of risk, Figure 6.2 shows the information content in the interest rate is rich in terms of market timing. We use the same interest rate simulation for the four panels in the figure. This makes the evolutions in the optimal stock allocations identical, but at different levels. At $T = 4$, the optimal portfolio weights reaches its minimum, whereas it at $T = 14$ reaches its maximum. This is because of the relation with the interest rate, and the fluctuations would therefore be even larger if $\tilde{\lambda}_2$ was increased.

An interesting feature of the model with affine market price of risk is that the mean value of the varying investment in stocks is equal to the weight of the former model, where the stock weight is constant over the investment horizon. This is because the portfolio weights for stocks in the two models become identical, due to the way we calibrate the market price of risk in the model, where it is non-constant.

6.5.5 Comparison of Models

In Table 6.2, we are considering the wealth allocation at two terminal periods between a stock index, a bond, and the locally risk-free asset, across different levels of risk aversion to compare the investment strategies between the models.

		Affine Market Price of Risk				**Stochastic Interest Rate**			
T	γ	Stock	Bond	Hedge	Cash	Stock	Bond	Hedge	Cash
T=1	0.5	3.692	0.280	-0.416	-2.972	3.705	0.309	-0.394	-3.014
	1	1.846	0.348	0.000	-1.194	1.852	0.352	0.000	-1.204
	2	0.923	0.374	0.200	-0.297	0.926	0.373	0.197	-0.299
	5	0.369	0.387	0.318	0.244	0.370	0.386	0.315	0.244
	10	0.185	0.391	0.356	0.424	0.185	0.390	0.355	0.425
	20	0.092	0.393	0.375	0.515	0.093	0.392	0.374	0.515
T=30	0.5	3.696	-0.355	-1.059	-2.341	3.705	-0.304	-1.007	-2.401
	1	1.848	0.352	0.000	-1.200	1.852	0.352	0.000	-1.204
	2	0.924	0.689	0.513	-0.613	0.926	0.679	0.504	-0.606
	5	0.370	0.885	0.815	-0.255	0.370	0.876	0.806	-0.246
	10	0.185	0.950	0.915	-0.135	0.185	0.942	0.906	-0.127
	20	0.092	0.982	0.964	-0.074	0.093	0.974	0.957	0.067

Table 6.2: A comparison between the two portfolio models, where the optimal portfolio choices are displayed across different risk aversion levels for two investment horizon, $T = 1$ and $T = 30$.

The allocation of wealth in stocks between the two models show to marginally differ. For a perfectly calibrated model the results should be the same because the affine market price of risk is calibrated such that it on average will equal the constant market price of risk. The additional assumption about affine market price of risk can therefore make a difference in the stock allocation, if calibrated differently. In Table 6.2, the stock allocation is decreasing in risk aversion. The effect of the investment horizon is a consequence of the simulation of the stochastic interest rate. This analytical result is expected, and states that an investor, who is becoming more averse to risk will reduce his allocation in stocks.

Table 6.2 should show how the bond allocations in the two models converge towards each other, as we also presented in Figure 6.1. The deviations from this are because of the imperfect calibration of the market price of risk. From the table it is however also seen that our argument of converging hedging terms is true. At both time time horizons, we see that the hedging terms converge towards the same values as the risk aversion is increasing.

The result from the comparison of the two extended models is that they are almost identical, where our calibration makes the hedging term the only difference between the two models. To show the effect of the calibration, the next section will show the allocation result under an alternative calibration.

6.5.6 Alternative Calibration of the Market Price of Risk

In Section 6.5.5, the market price of risk was calibrated such the fixed part of the market price of risk was having the largest weight, while the part which is multiplied with the interest rate was fairly small in magnitude. This will be referred as the base case specification.

In this section, an alternative calibration of market price of risk parameters is performed. In order to determine the pure effect of changing to an alternative specification, the interest rate is kept constant. This makes us able to calibrate the market price of risk perfectly. We also consider the alternative calibration in the situation with a stochastic interest rate, and its effect on the bond allocation.

The alternative specification is done by changing both λ_1 and λ_2. The estimates under this case are the following

$$\bar{\lambda}_1 = 0.06 \qquad \tilde{\lambda}_1 = 5$$
$$\bar{\lambda}_2 = 0.301 \qquad \tilde{\lambda}_2 = 6.5$$

Attaching a larger part of the market price of risk for both λ_1 and λ_2, which is multiplied with interest rate, implies that shocks to the interest rate have a greater impact on investors portfolio choice. In order to counter this effect, investors increase their long-term hedging and thereby increase the overall allocation to bonds.

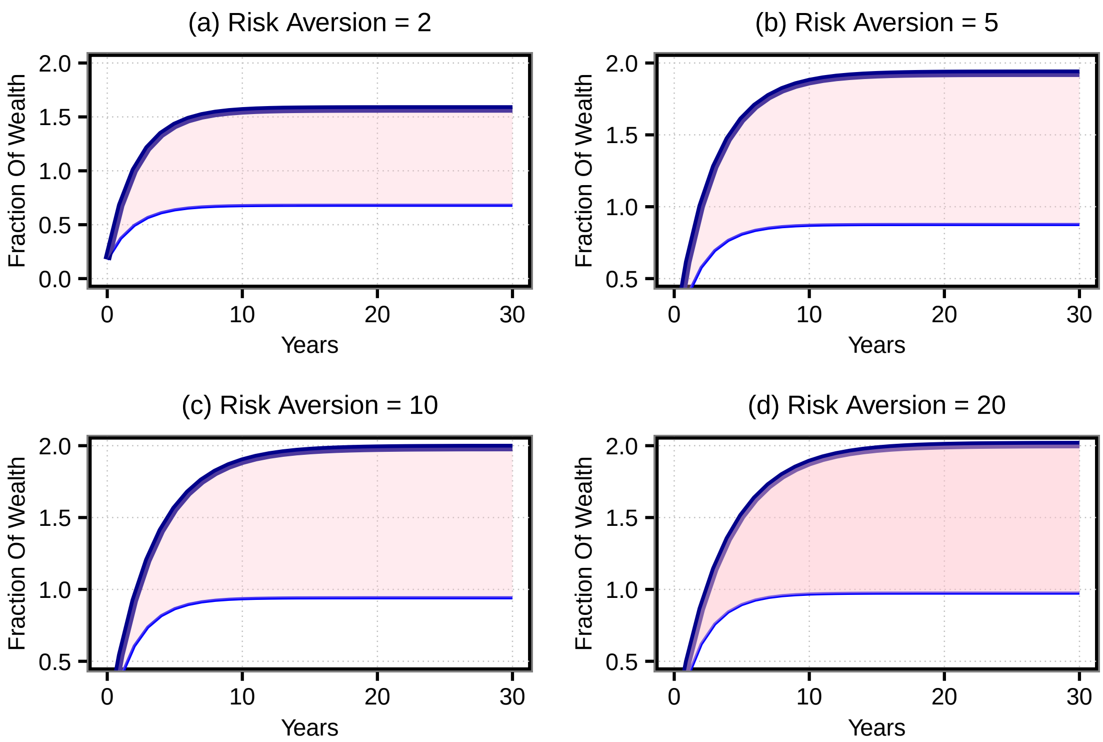

Figure 6.3: Illustrates the bond allocations of the two extended models, where the market price of risk is perfectly calibrated under a constant interest rate, $r = 0.01$. The upper thick blue function is the model with non-constant market price of risk. The lower thin blue function is the model extended with a stochastic interest rate

In Figure 6.3, there is a large difference between the bond allocations of the two models. In contrast to our previous findings with the base specification, we here see an increasing difference between the models' bond allocation, when the risk aversion is increasing. This is because the hedging terms no longer converge towards each other, since the differential equations, $A_1(\tau)$ and $A_2(\tau)$, in the model with an affine market price of risk are more sensitive to changes in the calibration, than $A_1(\tau)$ in the model which is only extended with a stochastic interest rate. As the risk aversion increases, the hedging term differences become more important, as the hedging term will account for a larger part of the bond allocation formula.

To compare this case with the case from Figure 6.1, we will have to present Figure 6.4, where the bond allocations for the alternative calibration is based on a stochastic interest rate. Due to the differences in the simulated interest rate, we cannot perfectly calibrate the market price of risk.

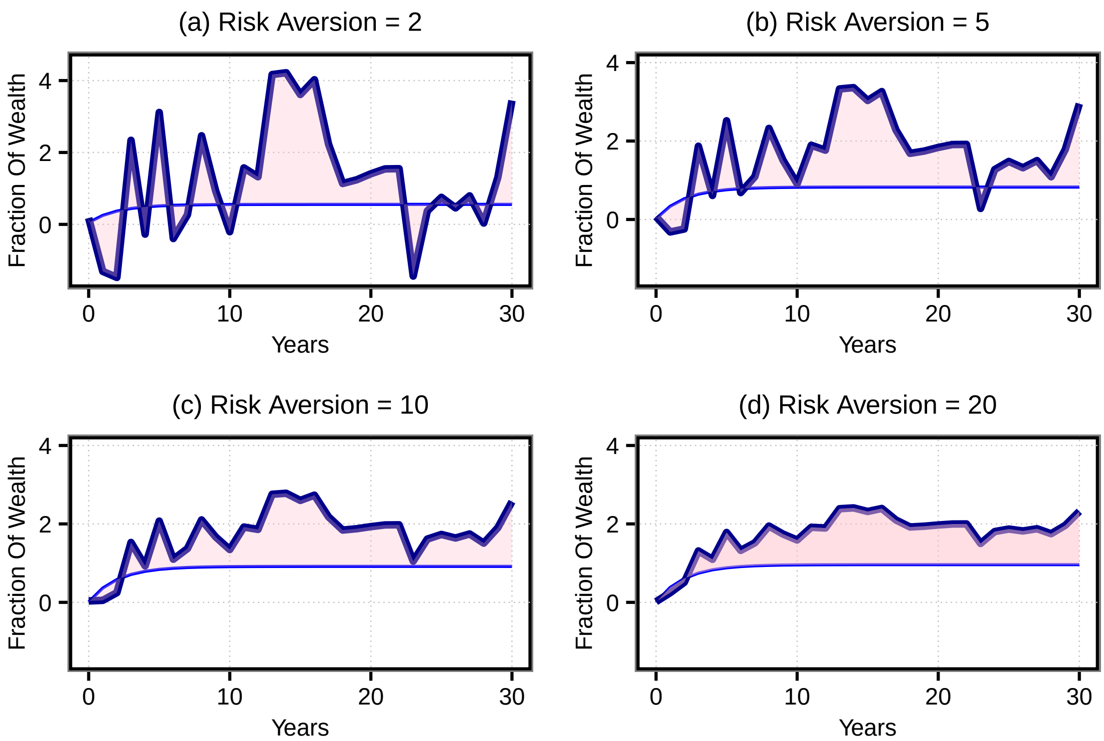

Figure 6.4: Illustrates the bond allocations of the two extended models, where the market price of risk is not perfectly calibrated, where the interest the interest rate follows a one-factor Vasicek model.

Comparison between the two figures shows how the fluctuations increase significantly, when the market price is highly dependent on the interest rate. The fluctuations are especially severe in panel (a) of Figure 6.4, in which we can see points through time where an investors will actually short bonds. As the risk aversion increases, the fluctuations become smaller, which was also the case under the base specification in Figure 6.1. However, the bond allocation converges towards the same level as in Figure 6.3, and thereby moves away from the model only extended with a stochastic interest rate. This is different from Figure 6.1, where the two allocation results converged.

Chapter 7

Suboptimal Allocation and its Costs

This chapter is focused on finding a function for the loss as a consequence of suboptimal portfolio allocation. The idea is to show how important it is to allocate optimally, and thereby show that the extensions in the previous chapters are relevant.

Suboptimal allocation can actually have greater importance than just represented by the utility loss. When considering institutional investors one could also consider the potential loss from providing a suboptimal return, which is lower than the returns from direct competitors. Annual comparisons between pension fund's returns and industry rankings could potentially increase the effect of suboptimal returns. We do, however, focus on the case where changes in the wealth from today and until its terminal period only happens as a result of changes in the different assets and prices.

We will start by defining the loss function at a general level in Section 7.1, and in Section 7.2 show the effect of deviating from the optimal portfolio under constant investment opportunities, where we follow Munk (2013). We continue with two specific cases. First, in Section 7.3 the loss from assuming constant investment opportunities, when the true model in fact has a stochastic interest rate. Secondly, we consider the loss from only extending the model with a stochastic interest rate, when the investor should follow a model with both a stochastic interest rate and a non-constant market price. This loss function and its results are presented in Section 7.4.

7.1 Loss Function at a General Level

We remember the utility function from the previous chapters

$$J(W, r, t) = \frac{g(r,t)^{\gamma} W^{1-\gamma}}{1-\gamma},$$

which we will use, when we define the loss function. We want to obtain the same utility level in both the optimal and suboptimal case and will therefore set the two utility functions equal. Due to the lower utility in the suboptimal case, the investor will have to put up additional wealth equal to $W \cdot \ell$ to make the utility levels equal. To find the percentage of extra wealth, ℓ, we set the two utility functions equal

$$\begin{aligned} \frac{\hat{g}^{\gamma}(W(1+\ell))^{1-\gamma}}{1-\gamma} &= \frac{g^{\gamma} W^{1-\gamma}}{1-\gamma} \\ \hat{g}^{\gamma}(W(1+\ell))^{1-\gamma} &= g^{\gamma} W^{1-\gamma} \\ (1+\ell)^{1-\gamma} &= \frac{g^{\gamma}}{\hat{g}^{\gamma}} \\ \ell &= \left(\frac{g^{\gamma}}{\hat{g}^{\gamma}}\right)^{\frac{1}{1-\gamma}} - 1, \end{aligned} \tag{7.1.1}$$

where we have used $g = g(r, t)$ for shorter notation, and the hat above g is used to indicate the suboptimal case.. We can see it as the percentage of extra wealth, which the investor will need to have when utility should be the same, as under the optimal strategy. We can switch the sign and we will have the loss function. Then will ℓ instead be the percentage of wealth, which the investor will pay for having the optimal investment strategy instead. The idea is similar to the certainty equivalent and risk premium, which we presented in Section 2.2 about utility and risk aversion.

7.2 Suboptimal Allocation for Constant Investment Opportunities

The model under constant investment opportunities has been solved in Chapter 4. From this model, we have the function

$$g(r,t) = e^{-\frac{\gamma-1}{\gamma}\left(r+\frac{1}{2\gamma}||\boldsymbol{\lambda}||^2\right)(T-t)},$$

which we will use as our optimal case. For some suboptimal case we have

$$\hat{g}(r,t) = e^{-\frac{\gamma-1}{\gamma}\left(r+\pi\sigma\lambda-\frac{\gamma}{2}\pi^2\sigma^2\right)(T-t)},$$

where the portfolio weights, π, can be set to different values to illustrate the consequences of suboptimality for extra wealth needed. When having both of these functions we can substitute them into the general description of the loss function. We can thereby calculate the function which will represent the loss from not following the optimal allocation result. By substitution, the loss function will be

$$\begin{aligned}
\ell &= \left(\left(e^{-\frac{\gamma-1}{\gamma}\left(r+\frac{1}{2\gamma}||\boldsymbol{\lambda}||^2\right)(T-t)}\right)^{\gamma}\cdot\left(e^{-\frac{\gamma-1}{\gamma}\left(r+\pi\sigma\lambda-\frac{\gamma}{2}\pi^2\sigma^2\right)(T-t)}\right)^{-\gamma}\right)^{\frac{1}{1-\gamma}} - 1\\
&= e^{\left(r+\frac{1}{2\gamma}||\boldsymbol{\lambda}||^2\right)(T-t)-\left(r+\pi\sigma\lambda-\frac{\gamma}{2}\pi^2\sigma^2\right)(T-t)} - 1\\
&= e^{\left(\frac{1}{2\gamma}||\boldsymbol{\lambda}||^2-\pi\sigma\lambda+\frac{\gamma}{2}\pi^2\sigma^2\right)(T-t)} - 1\\
\ell &= e^{\frac{1}{2\gamma}(\lambda-\gamma\pi\sigma)^2(T-t)} - 1. \qquad (7.2.1)
\end{aligned}$$

This loss function shows the loss from choosing an allocation which is different from the optimal allocation in Equation (4.3.5). The effect of suboptimal allocation does of course vary with the values of the other parameters in the loss function. We focus on the risk aversion and the time horizon. The loss for different levels of risk aversion is illustrated across deviating portfolio weights in panel (a) of Figure 7.1. Similarly, panel (b) of Figure 7.1 shows how the loss varies across portfolio weights for different lengths of the time horizon.

From the panel (a), it is clearly seen that under the presented levels of risk aversion, an increasing deviation will lead to an increasing loss. This increasing effect is getting even stronger, as the risk aversion increases. The same effect is seen for an increasing time horizon. The optimal portfolio does not vary, as time is irrelevant in the model with constant investment opportunities, but time definitely has an effect on the loss from deviating. Intuitively, it makes sense that an investor who invests suboptimally for several years will have a greater loss than an investor, who only invest for one year. We therefore have that, for the same deviation, an investor with a high risk aversion, and a long time horizon will have a greater loss than an investor with a lower risk aversion and a shorter time horizon.

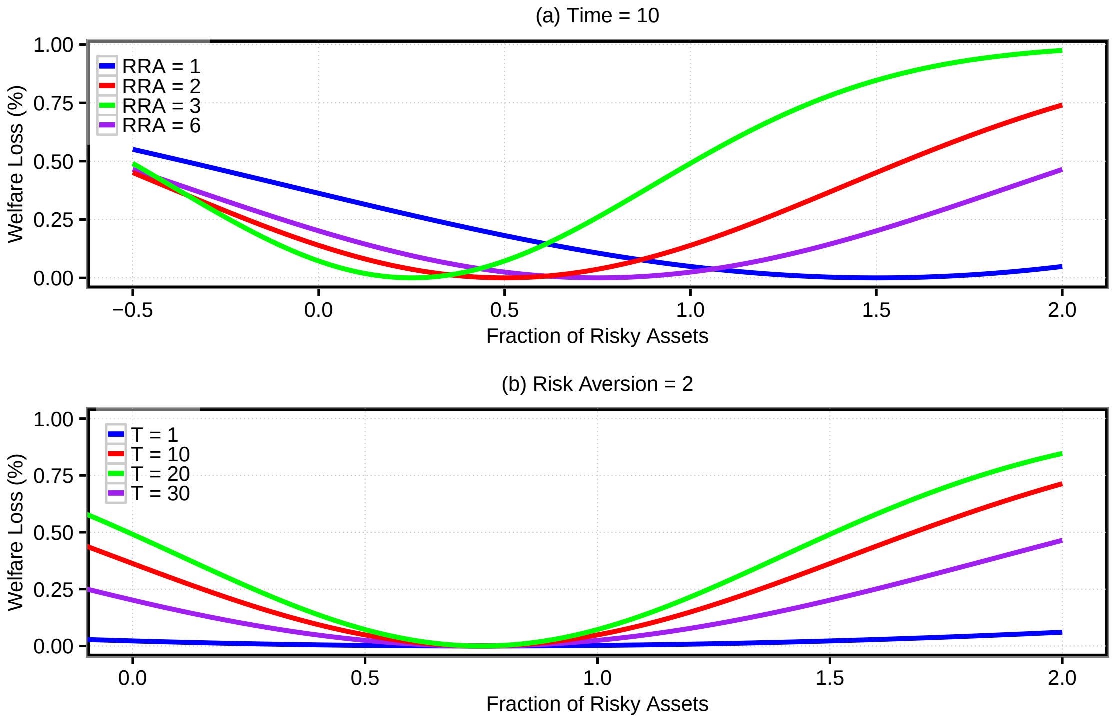

Figure 7.1: Loss at different levels of risk aversion and different time horizons. Panel (a) illustrates the additional wealth for different risk aversion levels across different portfolio weights. Panel (b) presents the additional wealth for different time horizons across different portfolio weights. Both are based on Equation (7.2.1) with $\sigma = 0.2$ and $\lambda = 0.3$.

7.3 Suboptimal Allocation in a Stochastic setting

We have extended the model with constant investment opportunities by introducing a stochastic interest rate. This new model is now assumed to be our optimal case. It is therefore interesting to see how the investor's wealth will behave if she does not recognize that the stochastic model is in fact the correct model. From Section 7.2 we expect the loss to be dependent on risk aversion and investment horizon. The loss function will also show if the extension of the model actually has a notable effect. The loss under a stochastic interest rate has also been considered by others. See for example Larsen (2010) who calculates how a local bias under a stochastic interest rate will lead to a wealth loss.

We know from the general finding of the loss function that we will need the two different expressions of $g(r,t)$. They are both of the form

$$g(r,t) = \exp\left\{\frac{1-\gamma}{\gamma}A_0(T-t) + \frac{1-\gamma}{\gamma}A_1(T-t)r\right\},$$

and it is then the ordinary differential equations which makes the difference. From Chapter 5 we have the optimal case, and the ordinary differential equations are therefore already given as

$$\begin{aligned} A_1(\tau) =& \frac{1}{\kappa}(1-e^{-\kappa\tau}) \\ A_0(\tau) =& \int_0^\tau \frac{1}{2\gamma}\|\boldsymbol{\lambda}\|^2 + \left(\kappa\bar{r} + \int_0^\tau \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}^\top\boldsymbol{\sigma}_r\right)A_1(\tau) + \int_0^\tau \frac{1-\gamma}{2\gamma}\cdot\|\boldsymbol{\sigma}_r\|^2 A_1^2(\tau). \end{aligned}$$

The next step is to find the ordinary differential equations in the suboptimal case.

7.3.1 Ordinary Differential Equations for the Suboptimal Case

We use the model with constant investment opportunities as our suboptimal situation. For the suboptimal case, we will have to evaluate the suboptimal portfolio weights under the optimal assumptions. We will therefore use the portfolio weights from the model with constant investment opportunities, which previously for a CRRA-investor were defined as

$$\boldsymbol{\pi}(W,t) = \frac{1}{\gamma}(\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\boldsymbol{\lambda},$$

but they will be substituted into the HJB equation from the model with a stochastic interest rate. For convenience, it is here restated:

$$\begin{aligned} 0 = \sup_{\boldsymbol{\pi}}\Bigg\{ & \frac{\partial J}{\partial t}(W,r,t) + J_r(W,r,t)\kappa[\bar{r}-r] + \frac{1}{2}J_{rr}(W,r,t)\|\boldsymbol{\sigma}_r\|^2 \\ & + J_W(W,r,t)W(r+\boldsymbol{\pi}^\top\underline{\underline{\sigma}}(r,t)\boldsymbol{\lambda}) + \frac{1}{2}J_{WW}(W,r,t)(W^2\boldsymbol{\pi}^\top\underline{\underline{\sigma}}(r,t)\underline{\underline{\sigma}}(r_t,t)^\top\boldsymbol{\pi}) \\ & + J_{rW}(W,r,t)\boldsymbol{\sigma}_r W\boldsymbol{\pi}^\top\underline{\underline{\sigma}}(r,t)\Bigg\}. \end{aligned}$$

Instead of maximizing by differentiating with respect to the portfolio weights we will simply substitute in the suboptimal weights from the model with constant investment opportunities. After substitution, the volatility matrix in every term will cancel out. We thereby have the new partial differential equation

$$\begin{aligned}0 =&\frac{\partial J}{\partial t}(W,r,t) + J_r(W,r,t)\kappa[\bar{r}-r] + \frac{1}{2}J_{rr}(W,r,t)\|\boldsymbol{\sigma}_r\|^2 + J_W(W,r,t)W\left(r+\frac{1}{\gamma}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\lambda}\right)\\&+\frac{1}{2}J_{WW}(W,r,t)\left(W^2\frac{1}{\gamma^2}\|\hat{\boldsymbol{\lambda}}\|^2\right) + J_{rW}(W,r,t)\left(W\frac{1}{\gamma}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\sigma}_r\right).\end{aligned}$$

The same utility function is assumed and we therefore have the same partial derivatives as when we considered the stochastic interest rate model. They are given in Equation (5.5.2) and written here again:

$$\begin{aligned}J_W(W,t) &= \hat{g}^\gamma W^{-\gamma}\\ J_{WW}(W,t) &= -\gamma\hat{g}^\gamma W^{-\gamma-1}\\ \frac{\partial J}{\partial t}(W,r,t) &= \frac{\gamma}{1-\gamma}\hat{g}^{\gamma-1}\hat{g}_t W^{1-\gamma}\\ J_r(W,r,t) &= \frac{\gamma}{1-\gamma}\hat{g}^{\gamma-1}\hat{g}_r W^{1-\gamma}\\ J_{rr}(W,r,t) &= \frac{\gamma W^{1-\gamma}}{1-\gamma}\left((\gamma-1)\hat{g}^{\gamma-2}\hat{g}_r^2 + \hat{g}^{\gamma-1}\hat{g}_{rr}\right)\\ J_{rW}(W,r,t) &= \gamma\hat{g}^{\gamma-1}\hat{g}_r W^{-\gamma}.\end{aligned}$$

With these partial derivatives substituted into the equation, the following expression is obtained

$$\begin{aligned}0 =&\frac{\gamma}{1-\gamma}\hat{g}^{\gamma-1}\hat{g}_t W^{1-\gamma} + \frac{\gamma}{1-\gamma}\hat{g}^{\gamma-1}\hat{g}_r W^{1-\gamma}\kappa[\bar{r}-r]\\&+\frac{1}{2}\frac{\gamma W^{1-\gamma}}{1-\gamma}\left((\gamma-1)\hat{g}^{\gamma-2}\hat{g}_r^2 + \hat{g}^{\gamma-1}\hat{g}_{rr}\right)\|\boldsymbol{\sigma}_r\|^2 + \hat{g}^\gamma W^{-\gamma}W\left(r+\frac{1}{\gamma}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\lambda}\right)\\&-\frac{1}{2}\gamma\hat{g}^\gamma W^{-\gamma-1}\left(W^2\frac{1}{\gamma^2}\|\hat{\boldsymbol{\lambda}}\|^2\right) + \gamma\hat{g}^{\gamma-1}\hat{g}_r W^{-\gamma}W\frac{1}{\gamma}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\sigma}_r,\end{aligned}$$

where the wealth, $W^{1-\gamma}$ is removed, as it is part of all terms. The equation is reduced to

$$\begin{aligned}0 =&\frac{\gamma}{1-\gamma}\hat{g}^{\gamma-1}\hat{g}_t + \frac{\gamma}{1-\gamma}\hat{g}^{\gamma-1}\hat{g}_r\kappa[\bar{r}-r] + \frac{1}{2}\frac{\gamma}{1-\gamma}\left((\gamma-1)\hat{g}^{\gamma-2}\hat{g}_r^2 + \hat{g}^{\gamma-1}\hat{g}_{rr}\right)\|\boldsymbol{\sigma}_r\|^2\\&+\hat{g}^\gamma\left(r+\frac{1}{\gamma}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\lambda}\right) - \hat{g}^\gamma\frac{1}{2\gamma}\|\hat{\boldsymbol{\lambda}}\|^2 + \hat{g}^{\gamma-1}\hat{g}_r\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\sigma}_r.\end{aligned}$$

We use the same partial derivatives of $\hat{g}(r,t) = \hat{g}$. We can reduce the derivatives by defining $\hat{g} = \exp\left\{\frac{1-\gamma}{\gamma}\hat{A}_0(T-t) + \frac{1-\gamma}{\gamma}\hat{A}_1(T-t)r\right\}$ and present them as

$$\begin{aligned}
\frac{\partial \hat{g}}{\partial t} &= \left(-\frac{1-\gamma}{\gamma}(r\hat{A}_1'(\tau) + \hat{A}_0'(\tau))\right) \cdot \hat{g} \\
\frac{\partial \hat{g}}{\partial r} &= \left(\frac{1-\gamma}{\gamma}\hat{A}_1(\tau)\right) \cdot \hat{g} \\
\frac{\partial^2 \hat{g}}{\partial r^2} &= \left(\frac{(1-\gamma)^2}{\gamma^2}\hat{A}_1^2(\tau)\right) \cdot \hat{g}.
\end{aligned}$$

This changes the partial differential equation, as it now contains both $\hat{A}_0'(\tau)$ and $\hat{A}_1'(\tau)$

$$\begin{aligned}
0 =& \frac{\gamma}{1-\gamma}\hat{g}^{\gamma-1}\left(-\frac{1-\gamma}{\gamma}(r\hat{A}_1'(\tau) + \hat{A}_0'(\tau))\right) \cdot \hat{g} + \frac{\gamma}{1-\gamma}\hat{g}^{\gamma-1}\left(\frac{1-\gamma}{\gamma}\hat{A}_1(\tau)\right) \cdot \hat{g}\kappa[\bar{r} - r] \\
&+ \frac{1}{2}\frac{\gamma}{1-\gamma}\left((\gamma-1)\hat{g}^{\gamma-2}\left(\left(\frac{1-\gamma}{\gamma}\hat{A}_1(\tau)\right) \cdot \hat{g}\right)^2\right)\|\boldsymbol{\sigma}_r\|^2 \\
&+ \frac{1}{2}\frac{\gamma}{1-\gamma}\left(\hat{g}^{\gamma-1}\left(\frac{(1-\gamma)^2}{\gamma^2}\hat{A}_1^2(\tau)\right) \cdot \hat{g}\right)\|\boldsymbol{\sigma}_r\|^2 + \hat{g}^{\gamma}\left(r + \frac{1}{\gamma}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\lambda}\right) - \hat{g}^{\gamma}\frac{1}{2\gamma}\|\hat{\boldsymbol{\lambda}}\|^2 \\
&+ \hat{g}^{\gamma-1}\left(\frac{1-\gamma}{\gamma}\hat{A}_1(\tau)\right)\hat{g}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\sigma}_r.
\end{aligned}$$

We divide the equation by $\hat{g}$. Where it is not present we will raise $\hat{g}$ to the power of -1. As a result there will be $\hat{g}^{\gamma-1}$ as a part of every term. The equation is therefore divided by $\hat{g}^{\gamma-1}$, as a second part of this step

$$\begin{aligned}
0 =& \frac{\gamma}{1-\gamma}\left(-\frac{1-\gamma}{\gamma}(r\hat{A}_1'(\tau) + \hat{A}_0'(\tau))\right) + \frac{\gamma}{1-\gamma}\left(\frac{1-\gamma}{\gamma}\hat{A}_1(\tau)\right)\kappa[\bar{r} - r] \\
&+ \frac{1}{2}\frac{\gamma}{1-\gamma}\left((\gamma-1)\left(\frac{1-\gamma}{\gamma}\hat{A}_1(\tau)\right)^2\right)\sigma_r^2 + \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(1-\gamma)^2}{\gamma^2}\hat{A}_1^2(\tau)\|\boldsymbol{\sigma}_r\|^2 \\
&+ \left(r + \frac{1}{\gamma}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\lambda}\right) - \frac{1}{2\gamma}\|\hat{\boldsymbol{\lambda}}\|^2 + \left(\frac{1-\gamma}{\gamma}\hat{A}_1(\tau)\right)\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\sigma}_r.
\end{aligned}$$

The next step is to reduce the equation by multiplying the fractions and parenthesis containing the risk aversion parameter, γ. Following this, the reduced equation is rearranged such that terms related to the interest rate are gathered

$$\begin{aligned}
0 =& -\hat{A}_0'(\tau) + \left(\kappa\bar{r} + \frac{1-\gamma}{\gamma}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\sigma}_r\right)\hat{A}_1(\tau) + \frac{1-\gamma}{2}\hat{A}_1^2(\tau)\|\boldsymbol{\sigma}_r\|^2 + \frac{1}{\gamma}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\lambda} - \frac{1}{2\gamma}\|\hat{\boldsymbol{\lambda}}\|^2 \\
&- r\hat{A}_1'(\tau) - \hat{A}_1(\tau)\kappa r + r.
\end{aligned}$$

The equation is then divided into two differential equations. They are

$$\begin{aligned}
\hat{A}_0'(\tau) =& \frac{1}{2\gamma}\|\hat{\boldsymbol{\lambda}}\|^2 + \left(\kappa\bar{r} + \frac{1-\gamma}{\gamma}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\sigma}_r\right)\hat{A}_1(\tau) + \frac{1-\gamma}{2}\hat{A}_1^2(\tau)\|\boldsymbol{\sigma}_r\|^2 \\
\hat{A}_1'(\tau) =& 1 - \hat{A}_1(\tau)\kappa,
\end{aligned}$$

which are almost similar to our previous ordinary differential equations from the optimal case. The only difference is the last term in the equation for $A_0'(\tau)$, which no longer is divided by γ. Due to the high degree of similarity to our previous equations, we simply state the solutions to the above system of ordinary differential equations:

$$\begin{aligned}
A_1(\tau) =& \frac{1}{\kappa}\left(1 - e^{-\kappa\tau}\right) \\
A_0(\tau) =& \frac{1}{2\gamma}\|\hat{\boldsymbol{\lambda}}\|^2\tau + \left(\kappa\bar{r} + \frac{1-\gamma}{\gamma}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\sigma}_r\right)\frac{\tau - \hat{A}_1(\tau)}{\kappa} \\
&+ \frac{1-\gamma}{2}\|\boldsymbol{\sigma}_r\|^2\left(\frac{\tau - \hat{A}_1(\tau)}{\kappa^2} - \frac{\hat{A}_1^2(\tau)}{2\kappa}\right)
\end{aligned} \tag{7.3.1}$$

7.3.2 The Loss Function

We have all the necessary information to determine the additional wealth needed to obtain the same utility level, when using the model with constant investment opportunities, even though the interest rate is stochastic. The general loss function is given in Equation (7.1.1), and we have the form

$$g(r,t) = \exp\left\{\frac{1-\gamma}{\gamma}A_0(T-t) + \frac{1-\gamma}{\gamma}A_1(T-t)r\right\}.$$

By substitution of $g(r,t)$, we will have the loss function

$$\begin{aligned}
\ell =& \left(\frac{\left(\exp\left\{\frac{1-\gamma}{\gamma}A_0(T-t) + \frac{1-\gamma}{\gamma}A_1(T-t)r\right\}\right)^\gamma}{\left(\exp\left\{\frac{1-\gamma}{\gamma}\hat{A}_0(T-t) + \frac{1-\gamma}{\gamma}\hat{A}_1(T-t)r\right\}\right)^\gamma}\right)^{\dfrac{1}{1-\gamma}} - 1 \\
=& \frac{\exp\left\{A_0(T-t) + A_1(T-t)r\right\}}{\exp\left\{\hat{A}_0(T-t) + \hat{A}_1(T-t)r\right\}} - 1 \\
=& \exp\left\{A_0(T-t) - \hat{A}_0(T-t) + A_1(T-t)r - \hat{A}_1(T-t)r\right\} - 1
\end{aligned}$$

By comparing definitions of $A_1(\tau)$ from the optimal case in Equation (5.5.5), and $\hat{A}_1(\tau)$ from the suboptimal case in Equation (7.3.1). The similarity is clear, and they therefore cancel out. The loss function is reduced to

$$\ell = \exp\left\{A_0(T-t) - \hat{A}_0(T-t)\right\} - 1,$$

and it is then the definition of $A_0(\tau)$ and $\hat{A}_0(\tau)$ that makes the difference. Again refer to the optimal case in Equation (5.5.6) and the suboptimal case in Equation (7.3.1) for definitions. Most terms cancel out, and the loss function is reduced even further

$$\begin{aligned}\ell = \exp\Bigg\{&\frac{1-\gamma}{2\gamma} \cdot \|\boldsymbol{\sigma}_r\|^2 \left(\frac{\tau - A_1(\tau)}{\kappa^2} - \frac{A_1^2(\tau)}{2\kappa}\right) \\ &- \frac{1-\gamma}{2} \cdot \|\boldsymbol{\sigma}_r\|^2 \left(\frac{\tau - \hat{A}_1(\tau)}{\kappa^2} - \frac{\hat{A}_1^2(\tau)}{2\kappa}\right)\Bigg\} - 1.\end{aligned}$$

Since $A_1(\tau)$ and $\hat{A}_1(\tau)$ are equal. We choose to write them both as $A_1(\tau)$ and can simplify the function

$$\ell = \exp\left\{\left(\frac{1}{\gamma} - 1\right)\frac{1-\gamma}{2} \cdot \|\boldsymbol{\sigma}_r\|^2 \left(\frac{\tau - A_1(\tau)}{\kappa^2} - \frac{A_1^2(\tau)}{2\kappa}\right) r\right\} - 1.$$

We can use the definition $A_1(\tau) = \frac{1}{\kappa}(1 - e^{-\kappa\tau})$ and substitute it into the loss function

$$\ell = \exp\left\{\left(\frac{1}{\gamma} - 1\right)\frac{1-\gamma}{2} \cdot \|\boldsymbol{\sigma}_r\|^2 \left(\frac{\kappa\tau - (1-e^{-\kappa\tau})}{\kappa^3} - \frac{(1-e^{-\kappa\tau})^2}{2\kappa^3}\right) r\right\} - 1. \quad (7.3.2)$$

We have hereby defined the loss function from following the suboptimal allocation result from the model with constant investment, when the interest rate is in fact stochastic.

7.3.3 Interpretation and Implications of the Loss Function

In this section, we have found an expression for the loss, which an investor incurs when she is applying the incorrect portfolio model given the assumptions of the market variables. Specifically, we have considered the scenario where the investor uses the portfolio model strategy with constant investment opportunities. This choice is evaluated under the assumptions of the optimal investment strategy with a stochastic interest rate.

In this analysis, we would expect that an investor committing the same mistake over a longer investment horizon would incur a larger welfare loss, implying that the welfare loss must be increasing in the investment horizon. However, the impact of risk aversion is not clear cut from the loss function mathematically defined above. Therefore, we use the historical estimates from the numerical analysis in Section 5.6.3 to illustrate the welfare losses in Figure 7.2.

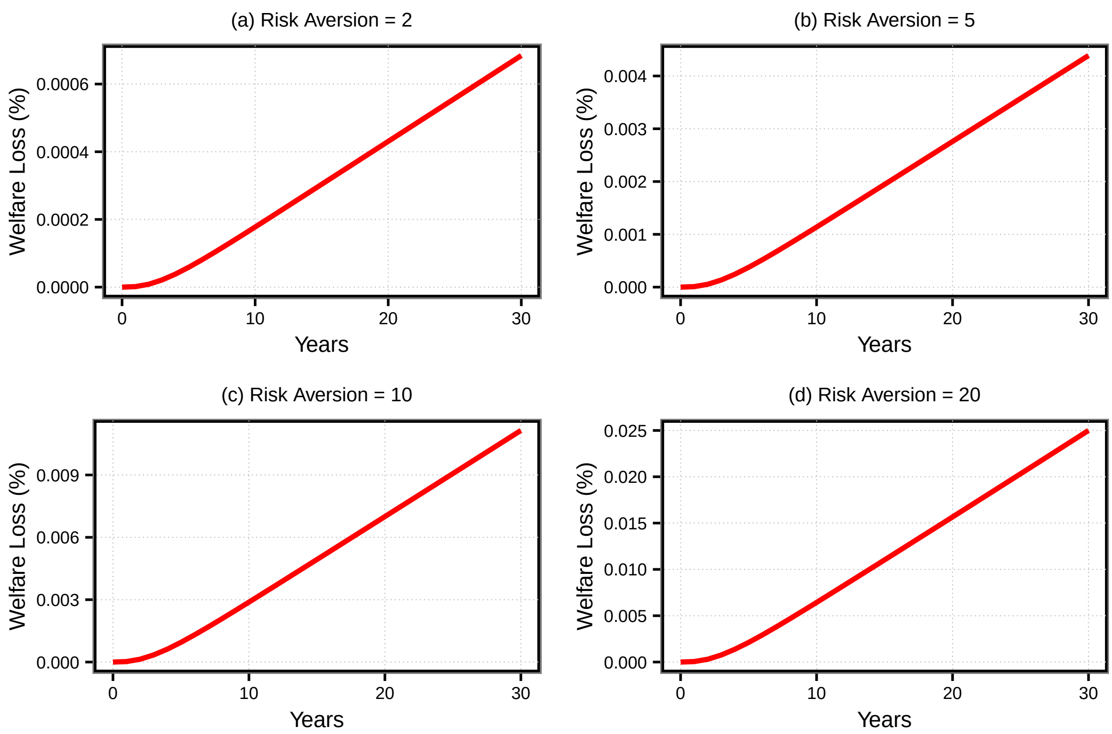

Figure 7.2: Illustrates the percentage of additional wealth needed for different values of risk aversion parameter, γ, for a CRRA-investor over an investment horizon of 30 years. The numerical results are based on the historical estimates previously used.

First, we consider the effect from the investment horizon on the additional wealth needed when using a suboptimal portfolio choice under this scenario. Figure 7.2 displays the percentage of additional wealth needed for different levels of the risk aversion, γ, over a investment horizon of 30 years. It is increasing in the investment horizon, which aids to the analogy of an investor, who has suboptimal investment strategies over a longer period of time will be punished more by committing to the same mistake for several periods.

The effect of risk aversion was unclear from Equation (7.3.2), but Figure 7.2 shows that the welfare loss is increasing in risk aversion. The increase in the additional wealth needed, as the risk aversion increases, is because the investor uses the suboptimal model with constant investment opportunities. In the optimal model with a stochastic interest rate, the investors will hedge more as the risk aversion increases, but the investor does not hedge in the suboptimal scenario and is punished for this lack of hedging. The needed wealth is increasing for higher levels of γ, since the hedging significantly increases, but is absent under the model with constant investment opportunities.

7.4 Loss when Assuming Constant Market Price of Risk

This section calculates the loss from only following the first extension with a stochastic interest rate against, when the true models is the second extension, where the interest rate is stochastic and the market price of risk is non-constant. The first model is assumed to yield a suboptimal allocation result, whereas the second model is assumed to be the true model and thereby give the optimal allocation result.

7.4.1 The New Set-up and Ordinary Differential Equations

The utility function and the wealth dynamics are the same as in Chapter 6. The definition is different for some of the wealth matrices. However, in matrix form, we will have the same HJB equation to solve

$$\begin{aligned} 0 = \sup_{\boldsymbol{\pi}} \Bigg\{ & \frac{\partial J}{\partial t}(W,r,t) + J_r(W,r,t)(\kappa[\bar{r} - r]) + J_W(W,r,t)W[r + \hat{\boldsymbol{\pi}}^\top \underline{\underline{\sigma}}(r,t)\boldsymbol{\lambda}] \\ & + \frac{1}{2} J_{WW}(W,r,t)W^2 \hat{\boldsymbol{\pi}}^\top \underline{\underline{\sigma}}(r,t)\underline{\underline{\sigma}}(r,t)^\top \hat{\boldsymbol{\pi}} + J_{rW}(W,r,t)\boldsymbol{\sigma}_r W \hat{\boldsymbol{\pi}}^\top \underline{\underline{\sigma}}(r,t) \\ & + \frac{1}{2}\|\boldsymbol{\sigma}_r\|^2 J_{rr}(W,r,t) \Bigg\}, \end{aligned}$$

with the terminal condition $\hat{g}(r,T) = 1$. The suboptimal portfolio weights from the case with constant market price of risk are

$$\hat{\boldsymbol{\pi}} = \frac{1}{\gamma}(\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\hat{\boldsymbol{\lambda}} + \frac{\gamma - 1}{\gamma}(\underline{\underline{\sigma}}(r_t,t)^\top)^{-1}\begin{pmatrix}\sigma_r \\ 0\end{pmatrix} b(T-t).$$

We substitute them into the HJB equation and reduce the expression by multiplying $\underline{\underline{\sigma}}(r_t, t)$ into the parentheses such that

$$\begin{aligned}
0 =& \frac{\partial \hat{J}}{\partial t}(W,r,t) + \hat{J}_r(W,r,t)\kappa[\bar{r} - r] + \frac{1}{2}\hat{J}_{rr}(W,r,t)\sigma_r^2 \\
&+ \hat{J}_W(W,r,t)W_t r + \hat{J}_W(W,r,t)W_t \left(\frac{1}{\gamma}\hat{\boldsymbol{\lambda}}^\top \boldsymbol{\lambda} - \frac{\gamma-1}{\gamma}\sigma_r(\bar{\lambda}_1 + \tilde{\lambda}_1 r)b(T-t)\right) \\
&+ \frac{1}{2}\hat{J}_{WW}(W,r,t)W_t^2 \left(\left(\frac{1}{\gamma^2}\hat{\boldsymbol{\lambda}}^\top \hat{\boldsymbol{\lambda}} + \frac{(\gamma-1)^2}{\gamma^2}\sigma_r^2 b(T-t)^2 + 2\frac{1}{\gamma}\frac{\gamma-1}{\gamma}\sigma_r\hat{\lambda}_1 b(T-t)\right)\right) \\
&- \hat{J}_{rW}(W,r,t)W_t\sigma_r\left(\frac{1}{\gamma}\hat{\lambda}_1 + \frac{\gamma-1}{\gamma}\sigma_r b(T-t)\right).
\end{aligned}$$

We use the partial derivatives which are similar to the ones before in the case with non-constant market price of risk. They are:

$$\begin{aligned}
\hat{J}_W(W,r,t) &= \hat{g}^\gamma W^{-\gamma} \\
\hat{J}_{WW}(W,r,t) &= -\gamma \hat{g}^\gamma W^{-\gamma-1} \\
\frac{\partial \hat{J}}{\partial t}(W,r,t) &= \frac{\gamma}{1-\gamma}\hat{g}^{\gamma-1}\hat{g}_t W^{1-\gamma} \\
\hat{J}_r(W,r,t) &= \frac{\gamma}{1-\gamma}\hat{g}^{\gamma-1}\hat{g}_r W^{1-\gamma} \\
\hat{J}_{rr}(W,r,t) &= \frac{\gamma W^{1-\gamma}}{1-\gamma}\left((\gamma-1)\hat{g}^{\gamma-2}\hat{g}_r^2 + \hat{g}^{\gamma-1}\hat{g}_{rr}\right) \\
\hat{J}_{rW}(W,r,t) &= \gamma \hat{g}^{\gamma-1}\hat{g}_r W^{-\gamma}.
\end{aligned}$$

The derivatives are substituted into the partial differential equation and it is seen how $W^{1-\gamma}$ appears in every term. We therefore divide the equation with $W^{1-\gamma}$ to remove it. After substitution, the partial differential equation is

$$\begin{aligned}
0 =& \frac{\gamma}{1-\gamma}\hat{g}^{\gamma-1}\hat{g}_t + \frac{\gamma}{1-\gamma}\hat{g}^{\gamma-1}\hat{g}_r\kappa[\bar{r}-r] + \frac{1}{2}\frac{\gamma}{1-\gamma}\left((\gamma-1)\hat{g}^{\gamma-2}\hat{g}_r^2 + \hat{g}^{\gamma-1}\hat{g}_{rr}\right)\sigma_r^2 \\
&+ \hat{g}^\gamma r + \hat{g}^\gamma\left(\frac{1}{\gamma}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\lambda} + \frac{\gamma-1}{\gamma}\sigma_r(\bar{\lambda}_1 + \tilde{\lambda}_1 r)b(T-t)\right) \\
&- \frac{1}{2}\gamma\hat{g}^\gamma\left(\frac{1}{\gamma^2}\hat{\boldsymbol{\lambda}}^\top\hat{\boldsymbol{\lambda}} + \frac{(\gamma-1)^2}{\gamma^2}\sigma_r^2 b(T-t)^2 + 2\frac{\gamma-1}{\gamma^2}\sigma_r\hat{\lambda}_1 b(T-t)\right) \\
&- \gamma\hat{g}^{\gamma-1}\hat{g}_r\sigma_r\left(\frac{1}{\gamma}\hat{\lambda}_1 + \frac{\gamma-1}{\gamma}\sigma_r b(T-t)\right),
\end{aligned}$$

Again, we use the definition

$$\hat{g} = \exp\left\{\frac{1-\gamma}{\gamma}\hat{A}_0(T-t) + \frac{1-\gamma}{\gamma}\hat{A}_1(T-t)r + \frac{1}{2}\frac{1-\gamma}{\gamma}\hat{A}_2(\tau)r^2\right\}$$

to find the partial derivatives and to shorten the expression for the derivatives

$$\begin{aligned}
\frac{\partial \hat{g}}{\partial t} &= \left(-\frac{1-\gamma}{\gamma}\left(\frac{1}{2}r^2\hat{A}_2'(\tau) + r\hat{A}_1'(\tau) + \hat{A}_0'(\tau)\right)\right)\cdot \hat{g} \\
\frac{\partial \hat{g}}{\partial r} &= \left(\frac{1-\gamma}{\gamma}(\hat{A}_1(\tau) + \hat{A}_2(\tau)r)\right)\cdot \hat{g} \\
\frac{\partial^2 \hat{g}}{\partial r^2} &= \left(\left(\frac{1-\gamma}{\gamma}(\hat{A}_1(\tau) + \hat{A}_2(\tau)r)\right)^2 + \frac{1-\gamma}{\gamma}\hat{A}_2(\tau)\right)\cdot \hat{g}.
\end{aligned}$$

After substitution, it is seen how we can remove $\hat{g}$ and afterwards remove $\hat{g}^{\gamma-1}$ from the partial differential equation. Following the same procedure, as in the previous chapters, we can continue by multiplying parentheses. This simplifies the expression by removing several of the risk aversion terms. After these four steps, the partial differential equation is given as

$$\begin{aligned}
0 = &-\frac{1}{2}r^2\hat{A}_2'(\tau) - r\hat{A}_1'(\tau) - \hat{A}_0'(\tau) + (\hat{A}_1(\tau) + \hat{A}_2(\tau)r)\kappa[\bar{r} - r] \\
&- \frac{1}{2}\frac{(\gamma-1)^2}{\gamma}\left(\hat{A}_1(\tau) + \hat{A}_2(\tau)r\right)^2\sigma_r^2 + \frac{1}{2}\left(\frac{1-\gamma}{\gamma}\left(\hat{A}_1(\tau) + \hat{A}_2(\tau)r\right)^2 + \hat{A}_2(\tau)\right)\sigma_r^2 \\
&+ r + \frac{1}{\gamma}\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\lambda} + \frac{\gamma-1}{\gamma}\sigma_r(\bar{\lambda}_1 + \tilde{\lambda}_1 r)b(T-t) - \frac{1}{2\gamma}\hat{\boldsymbol{\lambda}}^\top\hat{\boldsymbol{\lambda}} - \frac{(\gamma-1)^2}{2\gamma}\sigma_r^2 b(T-t)^2 \\
&- 2\frac{\gamma-1}{2\gamma}\sigma_r\hat{\lambda}_1 b(T-t) + (\gamma-1)(\hat{A}_1(\tau) + \hat{A}_2(\tau)r)\sigma_r\left(\frac{1}{\gamma}\hat{\lambda}_1 + \frac{\gamma-1}{\gamma}\sigma_r b(T-t)\right).
\end{aligned}$$

The vectors for the market price of risk were in the suboptimal and optimal case defined as

$$\hat{\boldsymbol{\lambda}} = \begin{pmatrix}\hat{\lambda}_1\\ \hat{\lambda}_2\end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\lambda} = \begin{pmatrix}\bar{\lambda}_1 + \tilde{\lambda}_1 r\\ \bar{\lambda}_2 + \tilde{\lambda}_2 r\end{pmatrix},$$

which makes it possible to write out some of the terms in the expression above. We will use the result

$$\hat{\boldsymbol{\lambda}}^\top\boldsymbol{\lambda} = \hat{\lambda}_1\bar{\lambda}_1 + \hat{\lambda}_2\bar{\lambda}_2 + \hat{\lambda}_1\tilde{\lambda}_1 r + \hat{\lambda}_2\tilde{\lambda}_2 r,$$

which is substituted into the expression. The expression is now

$$\begin{aligned}
0 =& -\left(\frac{1}{2}r^2\hat{A}_2'(\tau) + r\hat{A}_1'(\tau) + \hat{A}_0'(\tau)\right) + (\hat{A}_1(\tau) + \hat{A}_2(\tau)r)\kappa[\bar{r} - r] \\
& - \frac{1}{2}\frac{(\gamma-1)^2}{\gamma}\left(\hat{A}_1(\tau) + \hat{A}_2(\tau)r\right)^2\sigma_r^2 + \frac{1}{2}\left(\frac{1-\gamma}{\gamma}\left(\hat{A}_1(\tau) + \hat{A}_2(\tau)r\right)^2 + \hat{A}_2(\tau)\right)\sigma_r^2 \\
& + r + \left(\frac{1}{\gamma}\left(\hat{\lambda}_1\bar{\lambda}_1 + \hat{\lambda}_2\bar{\lambda}_2 + \hat{\lambda}_1\tilde{\lambda}_1 r + \hat{\lambda}_2\tilde{\lambda}_2 r\right) + \frac{\gamma-1}{\gamma}\sigma_r\left(\bar{\lambda}_1 + \tilde{\lambda}_1 r\right)b(T-t)\right) \\
& - \frac{1}{2\gamma}\hat{\boldsymbol{\lambda}}^\top\hat{\boldsymbol{\lambda}} - \frac{(\gamma-1)^2}{2\gamma}\sigma_r^2 b(T-t)^2 - 2\frac{\gamma-1}{2\gamma}\sigma_r\hat{\lambda}_1 b(T-t) \\
& + (\gamma-1)(\hat{A}_1(\tau) + \hat{A}_2(\tau)r)\sigma_r\left(\frac{1}{\gamma}\hat{\lambda}_1 + \frac{\gamma-1}{\gamma}\sigma_r b(T-t)\right).
\end{aligned}$$

The equation can be rearranged by isolating the terms depending on, if they are related to r^2, r, or not related to the interest rate

$$\begin{aligned}
0 =& -\frac{1}{2}r^2\hat{A}_2'(\tau) - A_2(\tau)r^2\kappa - \frac{(\gamma-1)^2}{2\gamma}\hat{A}_2^2(\tau)r^2\sigma_r^2 + \frac{1-\gamma}{2\gamma}A_2^2(\tau)r^2\sigma_r^2 \\
& + r - r\hat{A}_1'(\tau) - \hat{A}_1(\tau)\kappa r + \hat{A}_2(\tau)r\kappa\bar{r} - \frac{(\gamma-1)^2}{\gamma}\hat{A}_1(\tau)\hat{A}_2(\tau)r\sigma_r^2 \\
& + \frac{1-\gamma}{\gamma}\hat{A}_1(\tau)\hat{A}_2(\tau)r\sigma_r^2 + \frac{\gamma-1}{\gamma}\tilde{\lambda}_1 r\sigma_r b(T-t) + \frac{1}{\gamma}\hat{\lambda}_1\tilde{\lambda}_1 r + \frac{1}{\gamma}\hat{\lambda}_2\tilde{\lambda}_2 r \\
& + (\gamma-1)\hat{A}_2(\tau)r\sigma_r\left(\frac{1}{\gamma}\hat{\lambda}_1 + \frac{\gamma-1}{\gamma}\sigma_r b(T-t)\right) \\
& - \hat{A}_0'(\tau) + \hat{A}_1(\tau)\kappa\bar{r} - \frac{(\gamma-1)^2}{2\gamma}\hat{A}_1^2(\tau)\sigma_r^2 + \frac{1}{2}\hat{A}_2(\tau)\sigma_r^2 + \frac{1-\gamma}{2\gamma}\hat{A}_1^2(\tau)\sigma_r^2 \\
& + \frac{\gamma-1}{\gamma}\bar{\lambda}_1\sigma_r b(T-t) + \frac{1}{\gamma}\hat{\lambda}_1\bar{\lambda}_1 + \frac{1}{\gamma}\hat{\lambda}_2\bar{\lambda}_2 - \frac{1}{2\gamma}\hat{\boldsymbol{\lambda}}^\top\hat{\boldsymbol{\lambda}} - \frac{(\gamma-1)^2}{2\gamma}\sigma_r^2 b(T-t)^2 \\
& - 2\frac{\gamma-1}{2\gamma}\sigma_r\hat{\lambda}_1 b(T-t) + (\gamma-1)\hat{A}_1(\tau)\sigma_r\left(\frac{1}{\gamma}\hat{\lambda}_1 + \frac{\gamma-1}{\gamma}\sigma_r b(T-t)\right)
\end{aligned}$$

From the rearranging above, the partial differential equation can be divided into three ordinary differential equations. As before, we want an expression for $\hat{A}_0'(\tau)$, $\hat{A}_1'(\tau)$, and $\hat{A}_2'(\tau)$. We find these from the equation above, and simplify the expressions for ordinary differential equations such that we can write them as

$$\hat{A}_2'(\tau) = -2A_2(\tau)\kappa + (1-\gamma)\,A_2^2(\tau)\sigma_r^2$$

$$\begin{aligned}\hat{A}_1'(\tau) =& 1 + \frac{1}{\gamma}\left(\hat{\lambda}_1\tilde{\lambda}_1 + \hat{\lambda}_2\tilde{\lambda}_2\right) + \frac{\gamma-1}{\gamma}\tilde{\lambda}_1\sigma_r b(T-t) - \left(\kappa + (1-\gamma)\,\hat{A}_2(\tau)\sigma_r^2\right)\hat{A}_1(\tau)\\ &+ \left(\kappa\bar{r} + \sigma_r\left(\frac{\gamma-1}{\gamma}\hat{\lambda} + \frac{(\gamma-1)^2}{\gamma}\sigma_r b(T-t)\right)\right)\hat{A}_2(\tau)\end{aligned}$$

$$\begin{aligned}\hat{A}_0'(\tau) =& \left[\kappa\bar{r} + (\gamma-1)\sigma_r\left(\frac{1}{\gamma}\hat{\lambda}_1 + \frac{\gamma-1}{\gamma}\sigma_r b(T-t)\right)\right]\hat{A}_1(\tau) + \frac{1-\gamma}{2}\hat{A}_1^2(\tau)\sigma_r^2\\ &+ \frac{1}{2}\hat{A}_2(\tau)\sigma_r^2 + \frac{\gamma-1}{\gamma}\bar{\lambda}_1\sigma_r b(T-t) - \frac{\gamma-1}{\gamma}\sigma_r\hat{\lambda}_1 b(T-t)\\ &- \frac{(\gamma-1)^2}{2\gamma}\sigma_r^2 b(T-t)^2 - \frac{1}{2\gamma}\hat{\boldsymbol{\lambda}}^\top\hat{\boldsymbol{\lambda}} + \frac{1}{\gamma}\hat{\lambda}_1\bar{\lambda}_1 + \frac{1}{\gamma}\hat{\lambda}_2\bar{\lambda}_2,\end{aligned}$$

where we still have the condition $\hat{A}_0(0) = \hat{A}_1(0) = \hat{A}_2(0) = 0$ to satisfy the terminal condition $\hat{g}(r,T) = 1$. Following Appendix C.3 in Munk (2013) a system of ordinary differential equations of the form

$$\begin{aligned}A_2'(\tau) =& a - bA_2(\tau) + cA_2^2(\tau)\\ A_1'(\tau) =& d + fA_2(\tau) - \left(\frac{1}{2}b - cA_2(\tau)\right)A_1(\tau)\end{aligned}$$

with the initial conditions $A_1(0) = A_2(0) = 0$, will have the following solutions

$$\begin{aligned}A_2(\tau) =& \frac{2a(e^{v\tau}-1)}{(v+b)(e^{v\tau}-1)+2v}\\ A_1(\tau) =& \frac{d}{a}A_2(\tau) + \frac{2}{v}(db+2fa)\frac{\left(e^{v\tau/2}-1\right)^2}{(v+b)(e^{v\tau}-1)+2v},\end{aligned}$$

with $v = \sqrt{b^2-4ac}$. From the system of ordinary differential equations it is seen that $a = 0$. For the differential equation $\hat{A}_2'(\tau)$ the result is therefore $\hat{A}_2(\tau) = 0$. With this result, the solution for $\hat{A}_1'(\tau)$ will be simplified. The differential equation for $\hat{A}_1'(\tau)$ will fit into the form $\hat{A}_1'(\tau) = d - \frac{1}{2}b\hat{A}_1(\tau)$ and we can use the solution, which was also used in Section 5.5.2, with

$$A_1(\tau) = \frac{d}{\frac{1}{2}b}\left(1 - e^{-\frac{1}{2}b\tau}\right).$$

From the equations we define

$$d = 1 + \frac{1}{\gamma}\left(\hat{\lambda}_1\tilde{\lambda}_1 + \hat{\lambda}_2\tilde{\lambda}_2\right) + \frac{\gamma-1}{\gamma}\tilde{\lambda}_1\sigma_r b(T-t) \quad \text{and} \quad b = 2\kappa.$$

The solution is thereby

$$\hat{A}_1(\tau) = \left(1 + \frac{1}{\gamma}\left(\hat{\lambda}_1\tilde{\lambda}_1 + \hat{\lambda}_2\tilde{\lambda}_2\right) + \frac{\gamma-1}{\gamma}\tilde{\lambda}_1\sigma_r b(T-t)\right)\frac{(1-e^{-\kappa\tau})}{\kappa}.$$

Before the calculation of the percentage of additional wealth needed, a solution to the differential equation $A_0'(\tau)$ must also be found, as it is part of the loss function. To find the solution we refer to Section 5.5.2, where we defined

$$A_0(\tau) = \int_0^\tau A_0'(s)ds.$$

This is combined with $\hat{A}_0(\tau)$ from above. Before using the integral, $\hat{A}_2(\tau)$ is removed, as it is equal to zero. For the integration we again use the results

$$\int_0^\tau A_1(s)ds = \frac{\tau - A_1(\tau)}{\kappa} \quad \text{and} \quad \int_0^\tau A_1^2(s)ds = \frac{\tau - A_1(\tau)}{\kappa^2} - \frac{A_1^2(\tau)}{2\kappa},$$

such that we can write the equation

$$\begin{aligned}\hat{A}_0(\tau) =& \frac{\gamma-1}{\gamma}\bar{\lambda}_1\sigma_r b(T-t)\tau - \frac{\gamma-1}{\gamma}\sigma_r\hat{\lambda}_1 b(T-t)\tau - \frac{(\gamma-1)^2}{2\gamma}\sigma_r^2 b(T-t)^2\tau \\ &- \frac{1}{2\gamma}\hat{\boldsymbol{\lambda}}^\top\hat{\boldsymbol{\lambda}}\tau + \frac{1}{\gamma}\hat{\lambda}_1\bar{\lambda}_1\tau + \frac{1}{\gamma}\hat{\lambda}_2\bar{\lambda}_2\tau + \frac{1-\gamma}{2}\sigma_r^2\left(\frac{\tau - \hat{A}_1(\tau)}{\kappa^2} - \frac{\hat{A}_1^2(\tau)}{2\kappa}\right) \\ &+ \left[\kappa\bar{r} + (\gamma-1)\sigma_r\left(\frac{1}{\gamma}\hat{\lambda}_1 + \frac{\gamma-1}{\gamma}\sigma_r b(T-t)\right)\right]\frac{\tau - \hat{A}_1(\tau)}{\kappa}.\end{aligned}$$

7.4.2 Calculation of the Loss

We can calculate the loss from following the model in Chapter 5, when the true model is given in Chapter 6, where the market price of risk is non-constant. The three necessary ordinary differential equations for the suboptimal situation are defined above, and we are thereby halfway in having the necessary information to calculate the loss function. With the general definition from Equation (7.1.1), the form of $g(r,t)$, and the result $\hat{A}_2(\tau) = 0$, we can define the loss function as

$$\ell = \exp\left\{A_0(T-t) - \hat{A}_0(T-t) + (A_1(T-t) - \hat{A}_1(T-t))r + \frac{1}{2}A_2(T-t)r^2\right\} - 1.$$

Two of the three ordinary differential equations for the optimal situation are defined in Equation (6.4.3) and Equation (6.4.4). The solution to A_0' is still missing. The integrale of the ordinary differential equation A_0' is complicated and even though Kim and Omberg (1996) show that a solution to a similar system exist, we follow Moos (2011) and choose not to calculate the closed-form solution. The expression for A_0' will not be easy to interpret and we therefore use numerical integration to determine the value of A_0. See Appendix C for justification of this approach.

As there is no closed-form solution to the loss function, we go directly to presentation and interpretation of the estimated loss.

7.4.3 Interpretation and Implications

In this section, we again consider a loss function for an investor who is employing a suboptimal model. In this case, we are considering an investor who uses a model with a stochastic interest rate and a constant market price of risk. The investor's strategy is evaluated under the optimal assumptions of a investment strategy with a stochastic interest rate and a non-constant market price of risk.

As in the previous analysis of a loss function in Section 7.3.3, we expect that an investor who is using a suboptimal investment strategy consistently over a longer investment horizon will have an additional amount wealth needed to obtain the same utility level. The effect of risk aversion is again difficult to back out from the loss function expression. Especially in this case, as we do not have a closed-form solution. However, in Section 6.5.2 it was shown that the amount of wealth invested in stocks and bonds was approximately close to each other between the two models. The difference was due to the amount of hedging an investor required.

In Figure 7.3, we can see the behaviour of the loss function across investment horizon and risk aversion levels. This behaviour confirms that the percentage of additional wealth needed is increasing in the investment horizon, since making a suboptimal choice of investment strategy consistently, naturally builds up a loss as time passes.

The wealth needed is decreasing in risk aversion, and this result is the opposite of the former loss function. This is quite a surprising result, but a decomposition of this result can be useful. As mentioned earlier, the two investment strategies yield almost the same amount of wealth allocated to stocks and bonds, but there was a discrepancy between the strategies in terms of the hedging required. The hedging

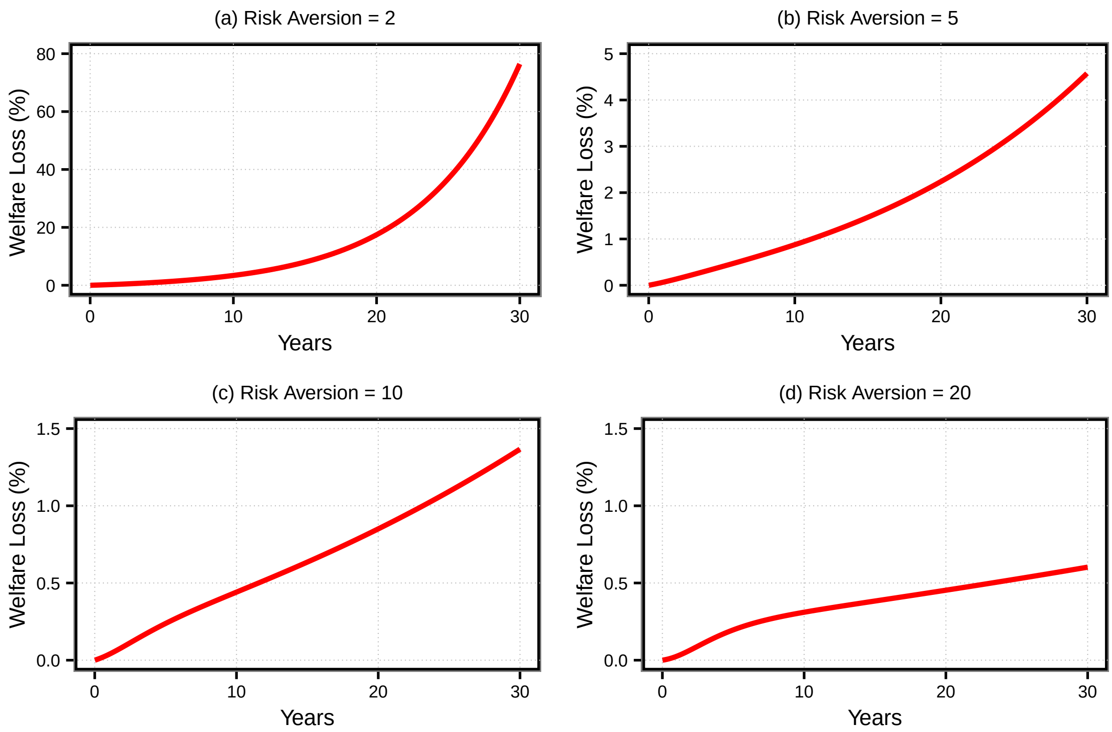

Figure 7.3: Illustrates the additional wealth for investors who are assuming a constant market price of risk, while it is non-constant under the optimal assumptions. The additional wealth is displayed for different levels of risk aversion over an investment horizon of $T = 30$.

term under the model with stochastic interest rate and non-constant market price of risk is slightly more complex than its comparable in the model only extended with a stochastic interest rate. This implies, that the surprising effect of risk aversion could originate from the hedging terms.

In Table 6.1, the hedging term of the optimal investment strategy is decomposed, where it is seen that the differential equations $A_1(\tau)$ and $A_2(\tau)$ are decreasing in risk aversion. For the suboptimal case is $\hat{A}_1(\tau)$ also decreasing in risk aversion. A comparison shows that the decrease in these elements makes them converge towards the same value. As they are converging, the difference in the hedging term will decrease, and the two models are thereby coming closer in their allocation. Therefore is the percentage of additional wealth needed for a highly risk averse investor minimal. This is why the wealth needed is decreasing, when the risk aversion is increasing. The high degree of similarity in the allocation results is a consequence of the calibration of the models. When the model is extended with the market price of risk as a function of the interest rate, the model is calibrated such that the market price of risk on average will match the constant values for λ_1 and λ_2 from the previous model.

Besides the calibration of the model, there is another issue with this calculation of welfare loss. To simplify the problem, we have used a constant interest rate for the example in Figure 7.3. Due to the calibration, this makes the portfolio weights very similar, despite the hedging term. Presenting the loss with a stochastic interest rate would lead to fluctuations in the result, as we saw in Section 6.5. This is because the stochastic interest rate brings the market timing element into the investor's allocation strategy.

Chapter 8

Numerical Example

In this section we will focus on a specific investor. For this investor we will compare the findings in the three dynamic allocation models, and afterwards estimate the loss from choosing the suboptimal solution. We assume one specific institutional investor with an investment horizon of $T = 30$, and a risk aversion of $\gamma = 2$. The rest of the numerical values follows the historical estimates, which we introduced in Section 5.6.3.

Following the three models, we can create Table 8.1 which present the allocation results for the models. The allocation results are similar to those from Section 5.6.3 and 6.5. The similarity in the allocation for the tangency portfolio is not surprising, as already described in Section 5.6.2 where the first two models are compared and in Section 6.5.5, where the two extended models are compared. The common trends

Model	Tangency	Stock	Bond	Hedging	Cash	Exp return	Std
Constant	1.102	0.926	0.176	0.000	-0.102	8.33%	0.191
Stochastic	1.102	0.926	0.679	0.504	-0.606	8.88%	0.211
Affine	1.102	0.926	0.688	0.513	-0.615	8.89%	0.212

Table 8.1: Allocation results and their returns for the three different models from Chapter 4, 5, and 6. $T = 30$ and $\gamma = 2$

are the increased hedging and the decreased cash allocation when going from one model to another. The decrease in the allocation for the locally risk-free asset is similar to the increase in the hedging term, as the tangency portfolio is unchanged. The expected return is increasing, but the relative increase in the standard deviation is even larger. The return per unit of standard deviation is therefore decreasing as the amount of hedging is increasing, when the model becomes more complicated.

This means that there is a lower return from following the extended models, but comparing the result does not make sense. If one model is true, then the other two must be false. Even though the allocation results are similar, the effect of not using the correct model can be large. This is what is reflected in the loss functions, which we defined in Chapter 7. For this specific case the losses are given in Table 8.2. This

Loss: Constant-Stochastic	0.000684%
Loss: Stochastic-Affine	76.44%

Table 8.2: Losses for $T = 30$ and $\gamma = 2$

means that the investor will have to come up with 0.000684% extra initial wealth to match the result from the model extended with a stochastic interest rate. The effect is even larger, when this specific investor does not extend the model with the market price, as an affine function of the interest rate. The investor will have to bring about 76% of extra initial wealth to match the same result. For these results to be true, our assumed parameters and the models will have to be true. One could set the time horizon or the risk aversion to some other level, but we here simply assume $T = 30$ and $\gamma = 2$ to illustrate how the models work.

Chapter 9

Conclusion

This thesis has demonstrated the effect that a stochastic interest rate has on the investor's wealth dynamics, and how the allocation in the different asset types is changed by this modified assumption. The model is further extended with a non-constant market price of risk of which the effects are also analysed. To justify the extensions, we calculate the loss which an investor faces, when she assumes a sub-optimal allocation model.

We consider institutional investors and their focus on maximising utility only as a function of the terminal wealth. The institutional investors are of interest, since they are companies who have significant amounts of assets under management, and they have a large role in the global economy.

The maximisation problem has its intuitive starting point in the mean-variance analysis in Chapter 3, where we have described how the trade-off between return and risk creates a frontier of efficient portfolios. From here the tangency portfolio is part of a two-fund separation result. The idea is the same in Chapter 4, where we introduce the dynamic multi-period model with constant investment opportunities. The underlying assumptions are more realistic, but the allocation result is again two-fund separation between a risk-free asset and risky assets.

We extend the relatively simple model by relaxing the assumption of constant investment opportunities. This is done by introducing a stochastic interest rate, which is assumed to follow an Ornstein-Uhlenbeck process as in Vasicek (1977). The extension takes two different forms; one is only extending the model with the stochastic interest rate, whereas the other extension also makes the market price of risk non-constant. Both models have closed-form solutions to their maximization problem, which we go through by solving a Hamilton-Jacobi-Bellman equation. The allocation results are similar, but they are not identical.

In the model only extended with a stochastic interest rate, the allocation results are different from the model with constant investment opportunities. The portfolio weights for allocation in risky assets have a new component. Besides allocation in the tangency portfolio and the locally risk-free asset, the investor will also hedge the interest rate risk. This new hedging term includes the interest rate volatility and is therefore increasing as the uncertainty about the interest rate is increasing. The allocation in stocks is still constant over time, whereas the allocation in bonds through the hedging term will vary over time.

The second extension of the model is making the market price of risk a function of the stochastic interest rate, which again leads to new allocation results. The amount of wealth allocated in stocks is still independent of time, but due to the new definition of the market price of risk, the allocation becomes dependent on the interest rate level. The hedging term in this quadratic model is complicated. Interpretation of the mathematical result becomes complex, but an increasing volatility for the interest rate will still lead to an increase in hedging.

To determine whether any of these extensions are relevant, we do in Chapter 7 calculate the losses from following suboptimal allocation results. This is done for three cases, where the first situation is based on Munk (2013) and shows the situation where the dynamic model with constant investment opportunities is the true model. The second case is a calculation from assuming the constant investment opportunities, when the interest rate in fact is stochastic. The third case is a numerical calculation of the loss from only extending the model with a stochastic interest rate, when the market price of risk, in fact, should be non-constant.

To investigate the effect of the three dynamic allocation models and the potential loss functions, we do in Chapter 8 consider all models for one specific investor with $T = 30$ and $\gamma = 2$. Under our assumptions, we do not find changes in the tangency portfolio allocation across models. The new hedging term is, however, changed across models, which leads to different returns from the models. A calculation of the loss functions for this specific investor shows that mainly the final extension with non-constant market price of risk is important to consider, when it is the true model.

The overall result from the loss functions shows that it makes sense for the investor to work with the more complicated models, if they are actually true. A simple model cannot describe the market correctly and an investor will therefore face a loss from basing her investments on the simple model.

The relative importance of the extensions is ambiguous. Not considering the extension with a stochastic interest rate leads to a loss, which is increasing as the risk aversion is increasing. The loss from assuming the market price of risk to be constant is high, but it is decreasing, when the risk aversion is increasing.

Our extensions are in the interest of the institutional investors due to the welfare losses from suboptimal allocation. Assuming our model to be perfect would however also lead to suboptimality, but we still expect it to be a step in the right direction. Our findings of suboptimality are good arguments for why one should do further research in creating a more realistic model.

We think, that further research can be within three main categories: models for different types of investors, changes in market assumptions, or relaxing general modelling assumptions.

Considering a different type of investor such as a private investor, would change the optimisation problem, since consumption must be considered in the utility function. In addition, labour income must be a part of the wealth equation. Labour income is, however, uncertain and the questions is whether the future income is to be considered as risky as a stock, or as safe as a bond. Current topics in portfolio optimisation for individuals are for example to include housing decisions, as it for many individuals will be the largest investment in their life. The length of the investor's life is also uncertain, which in most models is assumed to be known.

Considering different investors is not necessarily a choice between institutional investors or individuals. Other investor types could simply be either of the two groups, but with a different objective than wealth maximisation to maximise utility. As an example, an investor could gain utility from investments in green sources of energy. Across investor types, there are still assumptions to relax. Among others, one should consider how to create a model that accounts for issues such as stochastic inflation, stochastic volatility, or other assumptions for the market price of risk. General modelling assumptions is the third category. We also imagine further work can be done in making assumptions about continuous rebalancing, transaction costs, and taxation more realistic.

Several of the ideas within the three categories are old news, but have only been considered in isolation. The next step in research is a combination of these ideas to come a step closer to reality.

The ideas about extensions lead to a trade-off between the amount of extensions and the interpretation of the model. Building further extensions into an allocation model can make it so complex that no closed-form solutions exist. Even if closed-form solutions exist, they may be difficult to interpret, which is already the case for our model with stochastic interest rate and non-constant market price of risk.

Even a very good model may be wrong, if the input is of low quality. It is therefore not only a trade-off between extensions and interpretations, but it is also a question about how good the result will be, when the input is not perfect. Imperfect information can thereby make a good model irrelevant, as it will be wrong anyway. This leads to a whole other research area; model ambiguity and parameter uncertainty, such as Maenhout (2004) does it for the uncertainty about the return process.

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Appendix A

Constant Investment Opportunities

A.1 Distribution for Stock Price and Stock Returns

$$
\begin{aligned}
d \ln S_t &= \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial S_t^2}(dS_t)^2 \\
&= 0dt + \frac{1}{S_t}(\mu S_t dt + \sigma S_t dz_t) - \frac{1}{2}\frac{1}{S_t^2}(\mu S_t dt + \sigma S_t dz_t)^2 \\
&= (\mu dt + \sigma dz_t) - \frac{1}{2}\frac{1}{S_t^2}(\mu S_t dt + \sigma S_t dz_t)^2 \\
&= (\mu dt + \sigma dz_t) - \frac{1}{2}\frac{1}{S_t^2}((\mu S_t dt)^2 + (\sigma S_t dz_t)^2 + 2\mu\sigma S_t^2 dt \cdot dz_t)
\end{aligned}
$$

Following Øksendal (2003): $(dt)^2 = dt \cdot dz_t = 0$ og $(dz_t)^2 = dt$

$$
\begin{aligned}
&= (\mu dt + \sigma dz) - \frac{1}{2}\frac{1}{S_t^2}((\mu^2 S_t^2 \cdot 0) + (\sigma^2 S_t^2 dz^2)) \\
&= (\mu dt + \sigma dz) - \frac{1}{2}\frac{1}{S_t^2}(\sigma^2 S_t^2 dt) \\
&= (\mu dt + \sigma dz) - \frac{1}{2}(\sigma^2 dt) \\
&= (\mu - \frac{1}{2}\sigma^2)dt + \sigma dz
\end{aligned}
$$

From this, the expected return between time t and $t + \Delta t$ is

$$\begin{aligned}
\mathrm{E}[\ln(S_{t+\Delta t}/S_t)] &= \mathrm{E}[(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma \underbrace{(z_{t+\Delta t} - z_t)}_{\sim N(0,t'-t)}] \\
&= \mathrm{E}[(\mu - \frac{1}{2}\sigma^2)\Delta t] + \mathrm{E}[\sigma \underbrace{(z_{t+\Delta t} - z_t)}_{\sim N(0,t'-t)}] \\
&= (\mu - \frac{1}{2}\sigma^2)\Delta t
\end{aligned}$$

where we for the return until the terminal period have

$$\mathrm{E}[\ln(S_T/S_0)] = (\mu - \frac{1}{2}\sigma^2)T.$$

Likewise, we can calculate the variance of the stock returns

$$\begin{aligned}
\mathrm{Var}[\ln(S_{t+\Delta t}/S_t)] =&\mathrm{E}[((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma(z_{t+\Delta t} - z_t))^2] \\
&- (\mathrm{E}[(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma(z_{t+\Delta t} - z_t)])^2 \\
=&\mathrm{E}[((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma(z_{t+\Delta t} - z_t))^2] - (\mathrm{E}[(\mu - \frac{1}{2}\sigma^2)\Delta t])^2 \\
=&\mathrm{E}[((\mu - \frac{1}{2}\sigma^2)\Delta t)^2 + (\sigma(z_{t+\Delta t} - z_t))^2 \\
&+ 2(\mu - \frac{1}{2}\sigma^2)\Delta t \cdot \sigma(z_{t+\Delta t} - z_t)] - (\mathrm{E}[(\mu - \frac{1}{2}\sigma^2)\Delta t])^2 \\
=&\mathrm{E}[(\sigma(z_{t+\Delta t} - z_t))^2 + 2(\mu - \frac{1}{2}\sigma^2)\Delta t \cdot \sigma \underbrace{(z_{t+\Delta t} - z_t)}_{\sim N(0,1)}] \\
=&\mathrm{E}[\sigma^2(z_{t+\Delta t} - z_t)^2] \\
=&\mathrm{E}[\sigma^2(\varepsilon_{t+\Delta t}\sqrt{\Delta t})^2] \\
=&\mathrm{E}[\sigma^2\Delta t] \\
=&\sigma^2\Delta t,
\end{aligned}$$

where it for the terminal period will be

$$\mathrm{Var}[\ln(S_T/S_0)] = \sigma^2 T.$$

A.2 Reducing the General Hamilton-Jacobi-Bellman Equation

From substitution we will have the equation

$$\mathcal{L}^{\pi} J(W,t) = W J_W(W,t) \left(-\frac{J_W(W,t)}{J_{WW}(W,t)W} (\underline{\underline{\sigma}}^{\top})^{-1} \boldsymbol{\lambda} \right) \underline{\underline{\sigma}} \boldsymbol{\lambda}$$
$$+ \frac{1}{2} J_{WW}(W,t) W^2 \left(-\frac{J_W(W,t)}{J_{WW}(W,t)W} (\underline{\underline{\sigma}}^{\top})^{-1} \boldsymbol{\lambda} \right) \underline{\underline{\sigma \sigma}}^{\top} \left(-\frac{J_W(W,t)}{J_{WW}(W,t)W} (\underline{\underline{\sigma}}^{\top})^{-1} \boldsymbol{\lambda} \right).$$

This is reduced as the matrices for the volatility cancel out

$$= W J_W(W,t) \left(-\frac{J_W(W,t)}{J_{WW}(W,t)W} \right) \|\boldsymbol{\lambda}\|^2$$
$$+ \frac{1}{2} J_{WW}(W,t) W^2 \left(-\frac{J_W(W,t)}{J_{WW}(W,t)W} \boldsymbol{\lambda} \right) \left(-\frac{J_W(W,t)}{J_{WW}(W,t)W} \boldsymbol{\lambda} \right)$$
$$= W J_W(W,t) \left(-\frac{J_W(W,t)}{J_{WW}(W,t)W} \right) \|\boldsymbol{\lambda}\|^2 + \frac{1}{2} J_{WW}(W,t) W^2 \left(\frac{J_W(W,t)}{J_{WW}(W,t)W} \boldsymbol{\lambda} \right)^2$$
$$= \left(-\frac{J_W(W,t)^2}{J_{WW}(W,t)} \right) \|\boldsymbol{\lambda}\|^2 + \frac{1}{2} \left(\frac{J_W(W,t)^2}{J_{WW}(W,t)} \|\boldsymbol{\lambda}\|^2 \right)$$
$$= -\frac{1}{2} \left(\frac{J_W(W,t)^2}{J_{WW}(W,t)} \|\boldsymbol{\lambda}\|^2 \right)$$

A.3 Partial Derivatives of Potential Utility Function

$$
\begin{aligned}
J_W(W,t) &= \left(g(t)^\gamma(1-\gamma)W^{1-\gamma-1} + 0\cdot W^{1-\gamma}\right)\cdot(1-\gamma)^{-1} + \left(g(t)^\gamma W^{1-\gamma}\right)\cdot 0 \\
&= g(t)^\gamma W^{-\gamma} \\
J_{WW}(W,t) &= 0\cdot W^{-\gamma} + g(t)^\gamma \cdot -\gamma W^{-\gamma-1} \\
&= -\gamma g(t)^\gamma W^{-\gamma-1} \\
\frac{\partial J}{\partial t}(W,t) &= \left(\gamma g(t)^{\gamma-1} g'(t) W^{1-\gamma}\right)\cdot(1-\gamma)^{-1} \\
&= \frac{\gamma}{1-\gamma} g(t)^{\gamma-1} g'(t) W^{1-\gamma} \\
J_r(W,r,t) &= \frac{\gamma}{1-\gamma} g(r,t)^{\gamma-1} g_r(r,t) W^{1-\gamma} \\
J_{rr}(W,r,t) &= \frac{\gamma W^{1-\gamma}}{1-\gamma}\left((\gamma-1)g(r,t)^{\gamma-2} g_r^2(r,t) + g(r,t)^{\gamma-1} g_{rr}(r,t)\right) \\
J_{rW}(W,r,t) &= \gamma g(r,t)^{\gamma-1} g_r(r,t) W^{-\gamma}
\end{aligned}
$$

Appendix B

Stochastic Interest Rate Model

B.1 Rewriting Vasicek's Result for Bond Price

To ensure consistent notation and to show that the results match, we have rewritten Vasicek's model for an Ornstein-Uhlenbeck process step-by-step into the model, which we work with for the zero-coupon bond price. The starting point is the result from Vasicek (1977), which is rewritten by first changing the order of the terms

$$P(t,s,r) = \exp\left[\frac{1}{\alpha}(1-e^{-\alpha(s-t)})(R(\infty)-r) - (s-t)R(\infty) - \frac{\rho^2}{4\alpha^3}(1-e^{-\alpha(s-t)})^2\right]$$

$$P(t,s,r) = \exp\Bigg[R(\infty)(\frac{1}{\alpha}(1-e^{-\alpha(s-t)}) - (s-t)) - (\frac{1}{\alpha}(1-e^{-\alpha(s-t)})r$$
$$- \frac{\rho^2}{4\alpha^3}(1-e^{-\alpha(s-t)})^2\Bigg]$$

$$P(t,s,r) = \exp\Bigg[R(\infty)(\frac{1}{\alpha}(1-e^{-\alpha(s-t)}) - (s-t)) - \frac{\rho^2}{4\alpha^3}(1-e^{-\alpha(s-t)})^2$$
$$- \frac{1}{\alpha}(1-e^{-\alpha(s-t)})r\Bigg]$$

$$P(t,s,r) = \exp\Bigg[-R(\infty)((s-t) - \frac{1}{\alpha}(1-e^{-\alpha(s-t)})) - \frac{\rho^2}{4\alpha^3}(1-e^{-\alpha(s-t)})^2$$
$$- \frac{1}{\alpha}(1-e^{-\alpha(s-t)})r\Bigg]$$

The order of the terms which e is raised to the power of are now in the order, which we would like. Left is to switch the notation. Here we set

$$(s-t) = \tau, \quad R(\infty) = y_\infty, \quad \alpha = \kappa, \quad \rho = \sigma, \quad \text{and}$$

$$R(\infty)1 = \gamma + \rho q/\alpha - \frac{1}{2}\rho^2/\alpha^2 = y_\infty = \left(\bar{r} + \frac{\lambda_1\sigma_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}\right)$$

After rewriting using this notation the price of the zero coupon bond will be

$$\begin{aligned} B_t^{\bar{T}} &= e^{-y_\infty\left(\tau - \frac{1}{\kappa}(1-e^{-\kappa\tau})\right) - \frac{\sigma_r^2}{4\kappa^3}(1-e^{-\kappa\tau})^2 - \left(\frac{1}{\kappa}(1-e^{-\kappa\tau})\right)r_t} \\ &= e^{-a(\tau)-b(\tau)r_t} \\ &= e^{-a(\bar{T}-t)-b(\bar{T}-t)r_t}. \end{aligned}$$

where

$$\begin{aligned} b(\bar{T}-t) &= \frac{1}{\kappa}(1-e^{-\kappa(\bar{T}-t)}) \\ a(\bar{T}-t) &= y_\infty((\bar{T}-t) - b(\bar{T}-t)) + \frac{\sigma_r^2}{4\kappa}b(\bar{T}-t)^2 \\ y_\infty &= \left(\bar{r} + \frac{\lambda_1\sigma_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}\right). \end{aligned}$$

B.2 Bond Price under an Ornstein-Uhlenbeck Process

The calculation of the bond price under the Ornstein-Uhlenbeck process here is based on method used in Puhle (2008), which we see as the most intuitive method. For three alternative methods see Mamon (2004), which presents the same price calculation as we do her but by using three different methods.

As we assume the markets to be complete and arbitrage-free there will exist a unique stochastic discount factor according to the two fundamental theorems of asset pricing. We can then write

$$P_t = E_t\left[\frac{\zeta_T}{\zeta_t}P_T\right],$$

with ζ being the stochastic discount factor, P_T the return at time T, and P_t thereby being the price at time t. The dynamics of the stochastic discount factor are expected to be

$$\zeta_T = \zeta_t e^{\int_t^T -r_u du + \Sigma_{i=1}^d \int_t^T \lambda_{iu} dz_{iu} - \Sigma_{i=1}^d \frac{1}{2}\int_t^T \lambda_{iu}^2 du},$$

which makes it possible to derive the price of a zero coupon bond as

$$P_t = E_t \left[e^{\int_t^T -f(u,u)du + \Sigma_{i=1}^d \int_t^T \lambda_{iu} dz_{iu} - \Sigma_{i=1}^d \frac{1}{2} \int_t^T \lambda_{iu}^2 du} \right] \tag{B.2.1}$$

when we assume that $P_T = 1$.

We then need to find the movement of the interest rate and information about the market price of interest rate risk, λ. We already know that the market price of interest rate risk is constant. The dynamics of the interest rate is here found by using Itô's lemma, where we use the function $g = r_t e^{\kappa t}$. For a better introduction to Itô's lemma, we refer to Section 5.3.1. Here we simply use

$$dg = \left(\frac{\partial g}{\partial t} + \mu \frac{\partial g}{\partial r} + \sigma^2 \frac{1}{2} \frac{\partial^2 g}{\partial r^2} \right) dt + \sigma \frac{\partial g}{\partial r} dz_t,$$

in which we can substitute the values from the Ornstein-Uhlenbeck process and obtain

$$\begin{aligned} dr_t &= \left(\frac{\partial g}{\partial t} + (\kappa[\bar{r} - r_t]) \frac{\partial g}{\partial r} + \sigma_r^2 \frac{1}{2} \frac{\partial^2 g}{\partial r^2} \right) dt + \sigma_r \frac{\partial g}{\partial r} dz_t \\ &= \left(\kappa e^{\kappa t} r_t + (\kappa[\bar{r} - r_t]) e^{\kappa t} \right) dt + \sigma_r e^{\kappa t} dz_t \\ &= \left(\kappa e^{\kappa t} \bar{r} \right) dt + \sigma_r e^{\kappa t} dz_t. \end{aligned}$$

This dynamic can then be substituted into the expression for the interest rate r_T. We directly use the form of the expression from Puhle (2008)

$$\begin{aligned} y_T &= y_t + \int_t^T \kappa e^{\kappa u} \bar{r} du + \int_t^T \sigma_r e^{\kappa u} dz_u \\ r_T &= r_t e^{-\kappa(T-t)} + \int_t^T \kappa e^{-\kappa(T-u)} \bar{r} du + \int_t^T \sigma_r e^{-\kappa(T-u)} dz_u \\ r_T &= r_t e^{-\kappa(T-t)} + \bar{r} \left(1 - e^{-\kappa(T-t)} \right) + \sigma_r \int_t^T e^{-\kappa(T-u)} dz_u. \end{aligned}$$

Even though method for solving for the price is based on Puhle (2008), the dynamics for r_T are however solved in a different way, as we used Itô's lemma for the Ornstein-Uhlenbeck instead. We do this to follow the same method as when we are solving for bond price dynamics in Section 5.3.1. Substituting the future value of the interest rate into the price formula from equation (B.2.1) will be the next step in to process of determining the zero coupon bond price, when the interest rate is assumed to follow the Ornstein-Uhlenbeck process. After substitution and using the result that the market price of risk is constant, we will have the price function:

$$\begin{aligned}
P_t =& E_t\left[e^{\frac{(e^{-\kappa(T-t)}-1)r_t}{\kappa}+\frac{\bar{r}\left(1-e^{-\kappa(T-t)}-\kappa(T-t)\right)}{\kappa}+\sigma_r\int_t^T\frac{e^{-\kappa(T-t)}-1}{\kappa}dz_s+\sigma_r\int_t^T\frac{\lambda}{\kappa}dz_s-\frac{1}{2}\lambda^2(T-t)}\right] \\
=& \exp\left\{\frac{(e^{-\kappa(T-t)}-1)r_t}{\kappa}+\frac{\bar{r}\left(1-e^{-\kappa(T-t)}-\kappa(T-t)\right)}{\kappa}-\frac{1}{2}\lambda^2(T-t)\right\} \\
&\cdot E_t\left[\exp\left\{\frac{\sigma_r}{\kappa}\int_t^T e^{-\kappa(T-t)}-1+\lambda dz_s\right\}\right].
\end{aligned}$$

The focus is now on the last part of the equation as it is the stochastic part. This is why we are only using the expectation operator for this part of the equation. We then continue by taking the expectation of the stochastic part:

$$\begin{aligned}
=& \exp\left\{\frac{(e^{-\kappa(T-t)}-1)r_t}{\kappa}+\frac{\bar{r}\left(1-e^{-\kappa(T-t)}-\kappa(T-t)\right)}{\kappa}-\frac{1}{2}\lambda^2(T-t)\right\} \\
&\cdot\exp\left\{\frac{1}{2}\text{var}\left[\frac{\sigma_r}{\kappa}\int_t^T e^{-\kappa(T-t)}-1+\lambda dz_s\right]\right\} \\
=& \exp\left\{\frac{(e^{-\kappa(T-t)}-1)r_t}{\kappa}+\frac{\bar{r}\left(1-e^{-\kappa(T-t)}-\kappa(T-t)\right)}{\kappa}-\frac{1}{2}\lambda^2(T-t)\right\} \\
&\cdot\exp\left\{\frac{1}{2}\int_t^T\left(\frac{\sigma_r}{\kappa}e^{-\kappa(T-t)}-1+\lambda\right)^2 ds\right\} \\
=& \exp\left\{\frac{(e^{-\kappa(T-t)}-1)r_t}{\kappa}+\frac{\bar{r}\left(1-e^{-\kappa(T-t)}-\kappa(T-t)\right)}{\kappa}-\frac{1}{2}\lambda^2(T-t)\right\} \\
&\cdot\exp\left\{-\frac{(3+e^{-2\kappa(T-t)}-4e^{-\kappa(T-t)})\sigma_r^2}{4\kappa^3}+\frac{(1-e^{-\kappa(T-t)})\lambda\sigma_r}{\kappa^2}+\frac{(\sigma_r-\kappa\lambda)^2(T-t)}{2\kappa^2}\right\}
\end{aligned}$$

This expression for the price can be simplified in such a way that we will have a more familiar expression.

$$\begin{aligned}
=& \exp\left\{\bar{r}\left(\frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa}-(T-t)\right)-\frac{1}{2}\lambda^2(T-t)+\frac{\lambda\sigma_r}{\kappa}\frac{(1-e^{-\kappa(T-t)})}{\kappa}\right\} \\
&\cdot\exp\left\{-\frac{(3+e^{-2\kappa(T-t)}-4e^{-\kappa(T-t)})\sigma_r^2}{4\kappa^3}+\frac{(\sigma_r^2+\kappa^2\lambda^2-2\sigma_r\kappa\lambda)(T-t)}{2\kappa^2}+\frac{(e^{-\kappa(T-t)}-1)r_t}{\kappa}\right\} \\
=& \exp\left\{\bar{r}\left(\frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa}-(T-t)\right)+\frac{\lambda\sigma_r}{\kappa}\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}-(T-t)\right)\right\} \\
&\cdot\exp\left\{-\frac{(3+e^{-2\kappa(T-t)}-4e^{-\kappa(T-t)})\sigma_r^2}{4\kappa^3}+\frac{\sigma_r^2(T-t)}{2\kappa^2}+\frac{(e^{-\kappa(T-t)}-1)r_t}{\kappa}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left\{\bar{r}\left(\frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa}-(T-t)\right)+\frac{\lambda\sigma_r}{\kappa}\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}-(T-t)\right)\right\} \\
&\quad\cdot\exp\left\{-\frac{\left(2(1-e^{-\kappa(T-t)})+(1-e^{-\kappa(T-t)})^2\right)\sigma_r^2}{4\kappa^3}+\frac{\sigma_r^2(T-t)}{2\kappa^2}+\frac{(e^{-\kappa(T-t)}-1)r_t}{\kappa}\right\} \\
&= \exp\left\{\bar{r}\left(\frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa}-(T-t)\right)+\frac{\lambda\sigma_r}{\kappa}\left(\frac{1-e^{-\kappa(T-t)}}{\kappa}-(T-t)\right)\right\} \\
&\quad\cdot\exp\left\{-\frac{\sigma_r^2}{2\kappa^2}\left(\frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa}-(T-t)\right)\right\} \\
&\quad\cdot\exp\left\{-\frac{\left((1-e^{-\kappa(T-t)})^2\right)\sigma_r^2}{4\kappa^3}+\frac{(e^{-\kappa(T-t)}-1)r_t}{\kappa}\right\} \\
&= \exp\left\{\left(\bar{r}+\frac{\lambda\sigma_r}{\kappa}-\frac{\sigma_r^2}{2\kappa^2}\right)\left(\frac{\left(1-e^{-\kappa(T-t)}\right)}{\kappa}-(T-t)\right)-\frac{\sigma_r^2}{4\kappa}\frac{\left(1-e^{-\kappa(T-t)}\right)^2}{\kappa^2}\right\} \\
&\quad\cdot\exp\left\{\frac{(e^{-\kappa(T-t)}-1)r_t}{\kappa}\right\}
\end{aligned}
$$

If we then remember the notation

$$
\begin{aligned}
b(\bar{T}-t) &= \frac{1}{\kappa}(1-e^{-\kappa(\bar{T}-t)}) \\
a(\bar{T}-t) &= y_\infty((\bar{T}-t)-b(\bar{T}-t))+\frac{\sigma_r^2}{4\kappa}b(\bar{T}-t)^2 \\
y_\infty &= \left(\bar{r}+\frac{\lambda_1\sigma_r}{\kappa}-\frac{\sigma_r^2}{2\kappa^2}\right),
\end{aligned}
$$

which we stated in appendix B.1, then we are able to simplify the price function into the familiar expression

$$
P_t = B_t^{\bar{T}} = e^{-a(\bar{T}-t)-b(\bar{T}-t)r_t}.
$$

B.3 Continuous Coupon Bond Price Dynamics

This appendix shows the same procedure as the previous appendix did for the zero coupon bond. This appendix is with fewer steps showing the use of Itô's lemma to find the dynamics for a bond with continuous coupon.

The process of the interest rate is still the one, which we have defined in equation (5.2.1), but the price of the bond is no longer the defined in equation (5.3.2), but it does however come close. Based on Munk and Sørensen (2004) the price of a bond which pays a continuous coupon $K(t)$ given as $B_t = \int_t^T K(t) B_t^{\bar{T}}$, where $B_t^{\bar{T}}$ is the price of the zero coupon bond given in equation (5.3.2). We go directly to the general result of Itô's lemma for dynamics of bond prices to avoid repeating ourselves. Therefore, recall from Section 5.3.1:

$$dB = \left(\frac{\partial B}{\partial t} + \mu \frac{\partial B}{\partial r} + \sigma^2 \frac{1}{2} \frac{\partial^2 B}{\partial r^2} \right) dt + \sigma \frac{\partial B}{\partial r} dz_t.$$

With values for the Ornstein-Uhlenbeck process:

$$dB_t = \left(\frac{\partial B_t}{\partial t} + \kappa(\bar{r} - r_t) \frac{\partial B_t}{\partial r} + \sigma_r^2 \frac{1}{2} \frac{\partial^2 B_t}{\partial r^2} \right) dt + \sigma_r \frac{\partial B_t}{\partial r} dz_{1t}.$$

As before we simply substitute in the partial derivatives

$$\begin{aligned} &= \left(\frac{\sigma_r^2 (1 - e^{-\kappa(T-t)})}{2\kappa^2 e^{\kappa(T-t)}} - \left(\frac{\lambda_1 \sigma_r}{\kappa} + \bar{r} - \frac{\sigma_r^2}{2\kappa^2} \right) \left(e^{-\kappa(T-t)} - 1 \right) + r e^{-\kappa(T-t)} \right) B_t dt \\ &- \left(\kappa(\bar{r} - r_t) \frac{1 - e^{-\kappa(T-t)}}{\kappa} B_t - \sigma_r^2 \frac{1}{2} \left(\frac{1 - e^{-\kappa(T-t)}}{\kappa} \right)^2 B_t \right) dt \\ &+ \sigma_r \frac{1 - e^{-\kappa(T-t)}}{\kappa} B_t dz_{1t}. \end{aligned}$$

The steps which then are needed are equivalent to the ones in Section 5.3.1. We therefore simply state the result, which is

$$\frac{dB_t}{B_t} = \left(r_t + \lambda_1 \frac{\int_t^T K(s) B_t^s \sigma_r b(s-t) ds}{\int_t^T K(s) B_t^s ds} \right) dt + \frac{\int_t^T K(s) B_t^s \sigma_r b(s-t) ds}{\int_t^T K(s) B_t^s ds} dz_{1t}$$

where we know that $\sigma_r b(s-t) = \sigma_{B\bar{T}_t}$

$$\frac{dB_t}{B_t} = \left(r_t + \lambda_1 \frac{\int_t^T K(s) B_t^s \sigma_{B\bar{T}_t} ds}{\int_t^T K(s) B_t^s ds} \right) dt + \frac{\int_t^T K(s) B_t^s \sigma_{B\bar{T}_t} ds}{\int_t^T K(s) B_t^s ds} dz_{1t}$$
$$\frac{dB_t}{B_t} = \left(r_t + \lambda_1 \frac{B_t \sigma_{B\bar{T}_t} ds}{B_t ds} \right) dt + \frac{B_t \sigma_{B\bar{T}_t} ds}{B_t ds} dz_{1t}$$

where we now will use the notation

$$\sigma_B(r_t, t) = \frac{\int_t^T K(s) B_t^s \sigma_r b(s-t) ds}{\int_t^T K(s) B_t^s ds}$$

to simplify the expression for the bond dynamics, which then can be rewritten as

$$\frac{dB_t}{B_t} = (r_t + \lambda_1 \sigma_B(r_t, t)) \, dt + \sigma_B(r_t, t) dz_{1t}$$

or equivalently

$$dB_t = B_t \left((r_t + \lambda_1 \sigma_B(r_t, t)) \, dt + \sigma_B(r_t, t) dz_{1t} \right).$$

B.4 Stock Price Dynamics

For the dynamics of stock price, we introduce a two-dimensional setup, because in most asset pricing models we have that there are underlying processes which affect each other, such as price processes for a different asset, i.e. a bond's price dynamics. Adding the time-setting adds a second standard Brownian motion. In our system, we consider two assets, a bond and a stock where B_t and S_t is the notation for each, respectively. Let:

$$\frac{dB_t}{B_t} = \mu_{B_t} dt + \sigma_{B,1_t} dz_{1,t} + \sigma_{B,2_t} dz_{2,t}$$
$$\frac{dS_t}{S_t} = \mu_{S_t} dt + \sigma_{S,1_t} dz_{1,t} + \sigma_{S,2_t} dz_{2,t}$$

where $z_1 = (z_{1t})$ and $z_2 = (z_{2t})$ are independent Brownian motions. And $\mu_{S_t} = r + \psi \sigma_S$, where ψ is the Sharpe Ratio. Firstly, we want to investigate the correlation between the two stochastic processes, so we derive the covariance function of these two as follows:

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

Substitute in our processes:

$$\begin{aligned}
Cov_t(dB_t, dS_t) &= E[dB_t \cdot dS_t] - E[dB_t] \cdot E[dS_t] \\
&= E\Big[(\mu_{B_t} dt + \sigma_{B_{1t}} dz_{z1} + \sigma_{B_{2t}} dz_{2t}) \cdot (\mu_{S_t} dt + \sigma_{S_{1t}} dz_{z1} + \sigma_{S_{2t}} dz_{2t})\Big] \\
&- E[\mu_{B_t} dt + \sigma_{B_{1t}} dz_{z1} + \sigma_{B_{2t}} dz_{2t}] \cdot E[\mu_{S_t} dt + \sigma_{S_{1t}} dz_{1z} + \sigma_{S_{2t}} dz_{2t}]
\end{aligned}$$

After multiplying

$$\begin{aligned}
Cov_t[dB_t, dS_t] = E\Big[&\mu_B \mu_S dt^2 + \mu_B dt \sigma_{S_{1t}} dz_{1t} + \mu_B dt \sigma_{B_{2t}} dz_{2t} \\
&+ \sigma_{B_{1t}} dz_{1t} \mu_{S_{1t}} dt + \sigma_{B_{1t}} \sigma_{S_{1t}} dz_{1t}^2 + \sigma_{B_{1t}} dz_{1t} \sigma_{S_{1t}} dz_{2t} \\
&+ \sigma_{B_{2t}} dz_{2t} \mu_{S_t} dt + \sigma_{B_{2t}} dz_{2t} \sigma_{S_{1t}} dz_{1t} + \sigma_{B_{2t}} \sigma_{S_{2t}} dz_{2t}^2\Big] \\
&- \mu_{B_t} \mu_S dt^2
\end{aligned}$$

Imposing the rules of stochastic calculus, where we remember the rules of stochastic calculus:

$$dt^2 = 0 \qquad dz \cdot dt = 0 \quad \text{and} \quad (dz_t)^2 = dt$$

Using the rules below

$$\begin{aligned}
Cov_t[dB_t, dS_t] =& E\Big[\underbrace{\mu_B \mu_S dt^2}_{=0} + \underbrace{\mu_B dt \sigma_{S_{1t}} dz_{1t}}_{=0} + \underbrace{\mu_B dt \sigma_{B_{2t}} dz_{2t}}_{=0} + \underbrace{\sigma_{B_{1t}} dz_{1t} \mu_{S_{1t}} dt}_{=0} \\
&+ \sigma_{B_{1t}} \sigma_{S_{1t}} dz_{1t}^2 + \underbrace{\sigma_{B_{1t}} dz_{1t} \sigma_{S_{1t}} dz_{2t}}_{=0} + \underbrace{\sigma_{B_{2t}} dz_{2t} \mu_{S_t} dt}_{=0} + \underbrace{\sigma_{B_{2t}} dz_{2t} \sigma_{S_{1t}} dz_{1t}}_{=0} \\
&+ \sigma_{B_{2t}} \sigma_{S_{2t}} dz_{2t}^2\Big] - \underbrace{\mu_{B_t} \mu_S dt^2}_{=0}.
\end{aligned}$$

Now we can further reduce the expression to:

$$Cov_t[dB_t, dS_t] = E[\sigma_{B_{1t}} \sigma_{S_{1t}} dz_{1t}^2 + \sigma_{B_{2t}} \sigma_{S_{2t}} dz_{2t}^2]$$

Using the rule $(dz)^2 = dt$ again, and taking expectations, we get:

$$Cov_t[dB_t, dS_t] = \sigma_{B_{1t}} \sigma_{S_{1t}} dt + \sigma_{B_{2t}} \sigma_{S_{2t}} dt.$$

So the covariance function for the two processes is:

$$Cov_t[dB_t, dS_t] = (\sigma_{B_{1t}} \sigma_{S_{1t}} + \sigma_{B_{2t}} \sigma_{S_{2t}}) dt$$

In order to find the correlation function, we must first find the variances of the two processes, dB_t and dS_t. Applying the variance formula to each process:

$$Var[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

For dB_t:

$$\begin{aligned} Var_t[dB_t] &= E[dB_t^2] - (E[dB_t])^2 \\ &= E[\mu_{B_t}^2 dt^2 + \sigma_{B_{1t}}^2 dz_{1,t}^2 + \sigma_{B_{2t}}^2 dz_{2,t}^2] - [E[\mu_{B_t} dt + \sigma_{B,1_t} dz_{1,t} + \sigma_{B,2_t} dz_{2,t}]]^2 \end{aligned}$$

Again taking expectations and using the rules of stochastic calculus:

$$dt^2 = 0 \qquad dz \cdot dt = 0 \quad \text{and} \quad (dz_t)^2 = dt$$

Then we get:

$$\begin{aligned} Var_t[dB_t] &= E[\underbrace{\mu_{B_t}^2 dt^2}_{dt^2=0} + \sigma_{B,1_t}^2 dt + \sigma_{B,2_t}^2 dt] - [E[\mu_{B_t} dt + \underbrace{\sigma_{B,1_t} dz_{1,t}}_{=0} + \underbrace{\sigma_{B,2_t} dz_{2,t}}_{=0}]]^2 \\ &= \sigma_{B,1_t}^2 dt + \sigma_{B,2_t}^2 dt - \underbrace{\mu_{B_t}^2 dt^2}_{=0} \\ &= (\sigma_{B_{2t}}^2 + \sigma_{B_{2t}}^2) dt \end{aligned}$$

This is the variance for for the bond price process. Now we want to carry out the same procedure for the variance of the stock price:

$$\begin{aligned} Var_t[dS_t] &= E[\mu_{S_t}^2 dt^2 + \sigma_{S_{1t}}^2 dz_{1t}^2 + \sigma_{S_{2t}}^2 dz_{2t}^2] - [E[\mu_{S_t} dt + \sigma_{S_{1t}} dz_{1t} + \sigma_{S_{2t}} dz_{2t}]]^2 \\ &= E[\underbrace{\mu_{S_t}^2 dt^2}_{=0} + \sigma_{S_{1t}}^2 dt + \sigma_{S_{2t}}^2 dt] - [E[\mu_{S_t} dt + \underbrace{\sigma_{S_{1t}} dz_{1t}}_{=0} + \underbrace{\sigma_{S_{2t}} dz_{2t}}_{=0}]]^2 \\ &= \sigma_{S_{1t}}^2 dt + \sigma_{S_{2t}}^2 dt - \underbrace{\mu_{S_t}^2 dt^2}_{=0} \\ &= (\sigma_{S_{1t}}^2 + \sigma_{S_{2t}}^2) dt \end{aligned}$$

Now that we have the variances for each processes, and the covariance between them, we are now able to find the correlation function of the two processes. The correlation function is given as:

$$
\begin{aligned}
Corr_t[dB_t, dS_t] &= \frac{Cov_t[dB_t, dS_t]}{\sqrt{Var_t[dB_t] \cdot Var_t[dS_t]}} \\
&= \frac{(\sigma_{B_{1t}}\sigma_{S_{1t}} + \sigma_{B_{2t}}\sigma_{S_{2t}})dt}{\sqrt{[(\sigma_{B_{2t}}^2 + \sigma_{B_{2t}}^2)dt] \cdot [(\sigma_{S_{1t}}^2 + \sigma_{S_{2t}}^2)dt]}} \\
&= \frac{\sigma_{B_{1t}}\sigma_{S_{1t}} + \sigma_{B_{2t}}\sigma_{S_{2t}}}{\sqrt{(\sigma_{B_{2t}}^2 + \sigma_{B_{2t}}^2) \cdot (\sigma_{S_{1t}}^2 + \sigma_{S_{2t}}^2)}}
\end{aligned}
$$

However, from this derivation of the correlation of the two stochastic processes, we can see that when specifying the two-dimensional process $[dB_t, dS_t]$, and characterizing the first-order local and second-order local moments, which are:

- The first-order local moments are μ_{B_t} and μ_{S_t}.
- The second-order local moments are the volatility coefficients (shock) coefficients $\sigma_{B_{1t}}$,$\sigma_{B_{2t}}$, $\sigma_{S_{1t}}$, and $\sigma_{S_{2t}}$, which defines the two instantaneous variances and the instantaneous correlation coefficient.

From the above derivation, it was evident that the four volatility (shocks) coefficient would give rise to the same variances and the same correlation. This implies, that we have one degree of freedom extra by fixing the coefficients, which mean that we choose to fix one of the volatility (shock) coefficients for the process of the bond price, specific $\sigma_{B_{2t}} = 0$, since we want to see how the stock price dynamics are affected the bond price dynamics. So we can simplify our two processes as such:

$$
\frac{dB_t}{B_t} = \mu_{B_t}dt + \sigma_{B_t}dz_{1t} \qquad \frac{dS_t}{S_t} \qquad \mu_{S_t}dt + \sigma_{S_t}(\rho_t dz_{1t} + \sqrt{1-\rho^2}dz_{2t})
$$

where ρ_t is the correlation between the market returns of stock and bond. Now we can again define the new instantaneous variances and the instantaneous covariance of the newly defined stochastic processes. Now using prior definition of $\mu_{S_t} = r + \psi\sigma_S$, we can obtain the same stock price dynamics as in Munk (2013), which are represented by the following process:

$$
\frac{dS_t}{S_t} = (r + \psi\sigma_S)dt + \sigma_{S_t}(\rho_t dz_{1t} + \sqrt{1-\rho^2}dz_{2t})
$$

This shows the dynamics of the stock price, where the drift term is defined as a mean return with addition of the Sharpe ratio with risk adjustment term, and the volatility term is defined as the correlation between stock and bond market returns.

Further, in this appendix we want show the relationship between the processes. Firstly, we consider the instantaneous variance of dB_t is:

$$\begin{aligned} Var_t[dB_t] &= E[(\mu_{B_t}dt + \sigma_{B_t}dz_{1t})^2] - [E[\mu_{B_t}dt + \sigma_{B_t}dz_{1t}]]^2 \\ &= E[\underbrace{\mu_{B_t}^2 dt^2}_{=0} + \sigma_{B_t}^2 dz_{1t}^2] - [E[\mu_{B_t}dt + \underbrace{\sigma_{B_t}dz_{1t}}_{=0}]]^2 \\ &= \sigma_{B_t}^2 dt - \underbrace{\mu_{B_t}^2 dt^2}_{=0} \\ &= \sigma_{B_t}^2 dt \end{aligned}$$

And the same is done for the instantaneous variance of dS_t:

$$\begin{aligned} Var_t[dS_t] &= E[[\mu_{S_t}dt + \sigma_{S_t}(\rho_t dz_{1t} + \sqrt{1-\rho^2}dz_{2t})]^2] \\ &- [E[\mu_{S_t}dt + \sigma_{S_t}(\rho_t dz_{1t} + \sqrt{1-\rho^2}dz_{2t})]]^2 \\ &= E[\underbrace{\mu_{S_t}^2 dt^2}_{=0} + \sigma_{S_t}^2(\rho_t dz_{1t} + \sqrt{1-\rho_t^2}dz_{2t})^2 \\ &- [E[\mu_{S_t}dt + \sigma_{S_t}(\underbrace{\rho_t dz_{1t}}_{=0} + \underbrace{\sqrt{1-\rho^2}dz_{2t})}_{=0}]]^2 \\ &= \sigma_{S_t}^2(\rho_t dz_{1t} + \sqrt{1-\rho_t^2}dz_{2t})^2 - \underbrace{\mu_{S_t}^2 dt^2}_{=0} \\ &= Var_t\Big[\sigma_{S_t}^2(\rho_t dz_{1t} + \sqrt{1-\rho_t^2}dz_{2t})^2\Big] \\ &= \sigma_{S_t}^2 Var_t\Big[(\rho_t dz_{1t} + \sqrt{1-\rho_t^2}dz_{2t})^2\Big] \\ &= \sigma_{S_t}^2\Big[(\rho_t^2 dz_{1t}^2 + (1-\rho_t^2)dz_{2t}^2\Big] \\ &= \sigma_{S_t}^2\Big[(\rho_t^2 dt + (1-\rho_t^2)dt\Big] \\ &= \sigma_{S_t}^2 dt \end{aligned}$$

Next, we can also find the instantaneous covariance under this new specification:

$$
\begin{aligned}
Cov_t[dB_t, dS_t] &= E[dB_t \cdot dS_t] - E[dB_t] \cdot E[ES_t] \\
&= E[(mu_{B_t}dt + \sigma_{B_t}dz_{1t})(\mu_{S_t}dt + \sigma_{S_t}(\rho_t dz_{1t} + \sqrt{1-\rho^2}dz_{2t}))] \\
&- E[mu_{B_t}dt + \underbrace{\sigma_{B_t}dz_{1t}}_{=0}] \cdot E[\mu_{S_t}dt + \sigma_{S_t}(\underbrace{\rho_t dz_{1t}}_{=0} + \sqrt{1-\rho^2}\underbrace{dz_{2t}}_{=0})] \\
&= E\Big[\underbrace{\mu_{B_t}\mu_{S_t}dt^2}_{=0} + \mu_{B_t}\sigma_{S_t}(\rho_t \underbrace{dz_{1t}dt}_{=0} + \sqrt{1-\rho^2}\underbrace{dz_{2t}dt}_{=0}) \\
&+ \sigma_{B_t}\mu_{S_t}\underbrace{dz_{1t}dt}_{=0} + \sigma_{B_t}\sigma_{S_t}(\rho_t dz_{1t}^2 + \sqrt{1-\rho^2}dz_{1t}dz_{2t})\Big] - \underbrace{\mu_{B_t}\mu_{S_t}dt^2}_{=0} \\
&= \sigma_{B_t}\sigma_{S_t}(\rho_t dz_{1t}^2 + \sqrt{1-\rho^2}dz_{1t}dz_{2t}) \\
&= Cov_t\Big[\sigma_{B_t}\sigma_{S_t}(\rho_t dz_{1t}^2 + \sqrt{1-\rho^2}dz_{1t}dz_{2t})\Big] \\
&= \sigma_{B_t}\sigma_{S_t}Cov_t\Big[z_{1t}, \rho_t dz_{1t}^2 + \sqrt{1-\rho^2}dz_{1t}dz_{2t})\Big]
\end{aligned}
$$

This is the instantaneous covariance between the two stochastic processes, dB_t and dS_t. However, we can transform the embedded covariance, which are inside the instantaneous covariance to an embedded correlation in the following way:

$$
\begin{aligned}
Cov_t[dB_t, dS_t] &= \frac{\sigma_{B_t}\sigma_{S_t}Cov_t\Big[z_{1t}, \rho_t dz_{1t}^2 + \sqrt{1-\rho^2}dz_{1t}dz_{2t})\Big]}{\sqrt{\sigma_{B_t}^2 dt \cdot \sigma_{S_t}^2 dt}} \\
&= \sigma_{B_t}\sigma_{S_t}\rho_t dt
\end{aligned}
$$

From this expression of the instantaneous covariance, we can see if σ_{B_t} and σ_{S_t} are both positive or both negative, this implies that the instantaneous correlation is ρ_t between the two processes dB_t and dS_t, while if they have opposite signs, then the correlation will be $-\rho_t$.

Appendix C

Justification for Numerical Integration

In Section 7.4.2 it is chosen to use numerical integration to estimate the loss from choosing a suboptimal allocation. The numerical integration of A_0' is done by the Trapez-formula

$$\int_t^{t'} A_0'(s)ds = (t' - t)\frac{A_0'(t) + A_0'(t')}{2},$$

where $t' > t$. To have a high precision of the estimation of the integral, the size of the subintervals is set to $t' - t = 0.01$. To verify the precision, the two functions $A_1(\tau)$ and $A_2(\tau)$ have been estimated by the Trapez-formula and then compared to the correct, closed-form solution. Estimation of the value at time $t = 3$ has been done for several sizes of the subinterval. As seen from Table C.1 the marginal effect from a smaller size of subintervals i decreasing. At $t = 3$ the formula estimates 98.0511% of the actual value of $A_2(\tau)$. For $t = 30$ the estimation finds 98.0506% of the actual value of $A_2(\tau)$. Even for a long time horizon, the precision is high, and it does not even seem to decrease very much as time increases. For $A_1(\tau)$ for precision at 30 years is 99.5066%.

Subunit	1	0.75	0.5	0.25	0.15	0.1	0.05	0.01
Estimated	31.6%	38.1%	47.9%	65.3%	76.2%	83.0%	90.8%	98.1%

Table C.1: The table illustrates for 8 different subintervals, $t' - t$, how much of the correct value is estimated by use of numerical integration for $t = 3$.

Appendix D

R-code

All R-programs and Excel files used for this thesis can be provided upon request. Below is the full R-code provided. We apologise in advance for the length.

Listing D.1: R-code

```
###########################
# PROGRAM 1   TABLE 2.1#
###########################

#Load packages
require(stargazer)

#Set working directory
getwd()
setwd("C:/Users/ditle/Desktop/")

#Load Excel file  AUM_DK_WW
AUM.df < data.frame(AUM_DK_WW)

#Use stargazer
stargazer(AUM.df, type = "latex", summary = FALSE)

#######################################################
# PROGRAM 2   MEAN VARIANCE ANALYSIS: FIGURE 3.1 #
#######################################################

#CONTENT:
  #Plot: Efficient Frontier of only risky asset
  #Plot: Efficient Frontier of all assets

#Load Excel  Mean_Variance
mean.variance.df < data.frame(mean_variance)
head(mean.variance.df)
tail(mean.variance.df)

#Create plot
mv.plot < ggplot(mean.variance.df) +
  geom_line(aes(x = Std_Dev, y = Exp_Ret, col = "blue"), size = 1) +
```

```
  geom_line(aes(x = SD_2, y = Exp_ret_2, col = "red"), size = 1) +
  geom_point(aes(x = SD_tan, y = Exp_ret_tan),  size = 3) +
  geom_point(aes(x = SD_GMV, y = Exp_ret_GMV), size = 3) +
  geom_point(aes(x = A1_SD, y = A1_ret), size = 3) +
  geom_point(aes(x = A3_SD, y = A3_ret), size = 3) +
  scale_x_continuous("Standard Deviation (%)") +
  scale_y_continuous("Expected Return (%)") +
  scale_color_manual("Legend Title\n",labels = c( "Efficient Frontier of risky assets ", "Efficient Frontier of all assets"),
                     values = c("blue", "red")) +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=9),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=8, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.48,0.85),
        legend.key.size = unit(0.3,"cm"),
        legend.text = element_text(size = 11),
        legend.direction = "vertical",
        legend.background = element_rect(fill=alpha('white', 0.4)))

mv.plot + geom_text(aes(x = SD_tan, y = Exp_ret_tan, label = "Tangency Portfolio"), hjust=1.1, vjust=0, size = 3) +
  geom_text(aes(x = SD_GMV, y = Exp_ret_GMV, label = "GMV Portfolio"), hjust= 0.1, vjust=0, size = 3)

##############################################
# PROGRAM 3   INTEREST RATES: FIGURE 5.1 #
##############################################

#=====================================================================
#'Varies interest rates for the years 1986 to 2016
#=====================================================================
#install.packages("tseries"); install.packages("quantmod"); install.packages("ggplot2")


#'Neded packages
#'
library(tseries); library(quantmod);library(ggplot2)
require(tseries); require(quantmod);require(ggplot2)


#'
#'10 Year Treasury Constant Maturity Rate
#'
getSymbols( c("DGS10"), src="FRED")
adjyear10 < DGS10[6297:nrow(DGS10),]

#'
```

```
#'3 month Treasury interest rate
#'
getSymbols( c("DTB3"), src="FRED")
adjmonth3 < DTB3[8383:nrow(DTB3),]

#'
#'30 years Treasury interest rate
#'
getSymbols( c("DGS30"), src="FRED")
adjyear30 < DGS30[2352:nrow(DGS30),]

#' Due to data issues we will have a time period with interest rate
#' with the value of zero. This is because the 30 year treasury bond
#' was discontined on February 19, 2002 and reintroduced February 9, 2006.

utils::View(adjyear30)
adjyear30[is.na(adjyear30)] < 5 #Just using a random number to avoid ommiting later

#Combine all three interest rates
combined < cbind(adjyear10, adjmonth3, adjyear30)

#Rename the columns
colnames(combined) < c("Tenyear", "Threemonth", "Thirtyyear")

#Inspect data
utils::View(combined)

#We still have a high number of missing data:
sum(is.na(combined$Tenyear))/nrow(combined)*100
#This is due to Federal holdidays.
#The problem is simply solved by removing days with missing values
combined < na.omit(combined)

dataframecombined < data.frame(combined)

p < ggplot() +
  geom_line(data = dataframecombined, aes(x = index(combined), y = Tenyear, color = "Ten year")) +
  geom_line(data = dataframecombined, aes(x = index(combined), y = Threemonth, color = "Three month"))  +
  geom_line(data = dataframecombined[1:4002,], aes(x = index(combined[1:4002,]), y = Thirtyyear, color = "Thirty year")) +
  geom_line(data = dataframecombined[4997:nrow(dataframecombined),], aes(x = index(combined[4997:nrow(combined),]), y = Thirtyyear, color = "Thirty year")) +
  #labs(title="Interest rates from 1986 to 2016", x="Year", y="Interest rate in %", color="Treasury type")
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
       axis.ticks = element_line(color = "black"),
       panel.grid.major = element_line(color = "grey", linetype = "dotted"),
       panel.grid.minor = element_blank(),
       legend.title = element_blank(),
       axis.text.x=element_text(size=9),
       axis.title.x=element_text(size=9),
       axis.text.y=element_text(size=7),
```

```
        axis.title.y=element_text(size=7),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(1,0.75),
        legend.key.size = unit(0.5,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))
p

p+labs(x="Year", y="Interest rate in %", color="Treasury type")

# theme(axis.line = element_line(colour = "black"),
#       #panel.grid.major = element_line(colour = "black"),
#       #panel.grid.minor = element_line(colour = "black"),
#        panel.border = element_blank(),
#        panel.background = element_blank())

#'==========================
#'Interest rate volatility
#'==========================

retyear10 < log(lag(adjyear10))   log(adjyear10)
retyear10 < retyear10[ 1]
plot(retyear10)
retyear10 < na.omit(retyear10)
plot(retyear10)

retmonth3 < log(lag(adjmonth3))   log(adjmonth3)
retmonth3 < retmonth3[ 1]
plot(retmonth3)
ret3 < na.omit(retmonth3)
plot(retmonth3)

retyear30 < log(lag(adjyear30))   log(adjyear30)
retyear30 < retyear30[ 1]
plot(retyear30)
retyear30 < na.omit(retyear30)
plot(retyear30)


#####################################################
## PROGRAM 4   Simulate Sample Paths: FIGURE 5.2 #
#####################################################

#Load packages
require(ggplot2)

#FUNCTION:
## Define model parameters
r0 < 0.03
theta < 0.10
k < 0.3
beta < 0.03

## simulate short rate paths
```

```
n < 10    # MC simulation trials
T < 30    # total time
m < 200   # subintervals
dt < T/m  # difference in time each subinterval

r < matrix(0,m+1,n)  # matrix to hold short rate paths
r[1,] < r0

for(j in 1:n){
  for(i in 2:(m+1)){
    dr < k*(theta r[i 1,j])*dt + beta*sqrt(dt)*rnorm(1,0,1)
    r[i,j] < r[i 1,j] + dr
  }
}

## Standard matplot:
t < seq(0, T, dt)
rT.expected < theta + (r0 theta)*exp( k*t)
rT.stdev < sqrt( beta^2/(2*k)*(1 exp( 2*k*t)))
matplot(t, r[,1:10], type="l", lty=1, ylab="rt")
abline(h=theta, col="red", lty=2)
lines(t, rT.expected, lty=2)
lines(t, rT.expected + 2*rT.stdev, lty=2)
lines(t, rT.expected  2*rT.stdev, lty=2)
points(0,r0)

#Set up for ggplot2 plot
r.df < data.frame(r)
expected.r < rT.expected
up.limit < rT.expected + 2*rT.stdev
down.limit < rT.expected  2*rT.stdev

#ggplot2 plot
ggplot(r.df, aes(x = t, y= X1)) +
  geom_line(aes(col="darkpurple")) +
  geom_line(aes(y=X2, col = "red")) +
  geom_line(aes(y=X3, col = "green")) +
  geom_line(aes(y=X4, col = "blue")) +
  geom_line(aes(y=X5, col = "darkred")) +
  geom_line(aes(y=X6, col = "darkblue")) +
  geom_line(aes(y=X7, col = "orange")) +
  geom_line(aes(y=X8, col = "darkgreen")) +
  geom_line(aes(y=X9, col = "olive")) +
  geom_line(aes(y=X10, col = "purple")) +
  geom_line(aes(y=up.limit), linetype = 2) +
  geom_line(aes(y=down.limit), linetype = 2) +
  geom_line(aes(y=expected.r), linetype = 2) +
  scale_x_continuous("Years") +
  scale_y_continuous("%") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=9),
```

```
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=9, color="black"),
        legend.position= "none")

#################################################################
## PROGRAM 5    CONSTANT AND STOCHASTIC INVESTMENT OPPORTUNITIES #
#################################################################

#CONTENT
#(i) ggplot of Efficient Frontier, Figure 5.3
#(ii) ggplot of allocation results, RA=2 and RA= 10, Figure 5.6
#(iii) ggplot of bond allocation, constant and stochastic model, Figure 5.4
#(iv) ggplot of cash allocation, constant and stochastic model, Figure 5.5

#Packages
require(ggplot2)
require(gridExtra)

#################################################
#Create dataframe from CSV.file
frontier.df < data.frame(Frontier_Rcsv)*100
#View data
head(frontier.df)

#Efficient frontier
frontier.plot < ggplot(frontier.df, aes(x = Con_Stddev, y = Con_ret)) +
    geom_line(aes(col = "black"), size = 1) +
    geom_line(aes(x = Sto_dev2.5, y = Sto_ret2.5, col= "blue"), size = 1) +
    geom_line(aes(x = Sto_dev5, y = Sto_ret5, col= "green"), size = 1) +
    geom_line(aes(x = Sto_dev30, y = Sto_ret30, col = "reed"), size = 1) +
    coord_cartesian(xlim = c(0,20), ylim = c(0,10)) +
    scale_x_continuous("Standard Deviation") +
    scale_y_continuous("Expected Return") +
    scale_color_manual("Legend Title\n",labels = c("Constant", "T=2.5", "T=5", "T=30"),
                       values = c("black", "blue","green", "red")) +
    theme_bw() +
    theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=9),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=8, color="black"),
        legend.justification=c(1,0),
        legend.position= "right",   #c(1.0249,0.845)
        legend.key.size = unit(0.5,"cm"),
        legend.text = element_text(size = 9),
        legend.direction = "vertical",
        legend.background = element_rect(fill=alpha('white', 0.4)),
        legend.margin = unit(0.5, "cm"),
        legend.background = element_rect(fill = "black", size=0.5, linetype="solid",
                                         colour ="black"))
```

```
##### Bond, Cash and Hedge ###
#Load Excel file
weights.df < data.frame(weights_csv)
#View date
head(weights.df)

#Plot of allocation result, RA = 10
plot.risk10 < ggplot(weights.df, aes(x = Time, y = Bond_10)) +
  geom_line(aes(col = "blue"), size = 1) +
  geom_line(aes(y = Cash_10, col="green"), size = 1) +
  geom_line(aes(y = Hedge_10, col="red"), size = 1) +
  coord_cartesian(xlim = c(0,20), ylim = c( 0.75,1.1)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Bonds", "Risk Free", "Hedge" ),↩
       values = c("blue","green", "red")) +
  labs(title="(b) Risk Aversion = 10") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=9),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.345,0.75),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

#Plot of allocation result, RA = 2
plot.risk2 < ggplot(weights.df, aes(x = Time, y = Bond_2)) +
  geom_line(aes(col = "blue"), size = 1) +
  geom_line(aes(y = Cash_2, col="green"), size = 1) +
  geom_line(aes(y = Hedge_2, col="red"), size = 1) +
  coord_cartesian(xlim = c(0,20), ylim = c( 0.75,1.1)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Bonds", "Risk Free", "Hedge" ),↩
       values = c("blue","green", "red")) +
  labs(title="(a) Risk Aversion = 2") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
```

```
        axis.text.y=element_text(size=9),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.345,0.75),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

#Combine plots
grid.arrange(plot.risk2, plot.risk10, ncol=2)


#Bond allocation, Constant and stochastic model, Figure 5.4

#Load excel file
stochas.df < data.frame(Stochastic_analysis)
#View data
head(stochas.df)

#Bond allocation for constant model
bond.constant < ggplot(stochas.df, aes(x = Gamma, y = Con_Bond)) +
    geom_line(aes(col = "blue"), size = 1.5) +
    coord_cartesian(xlim = c(0,20), ylim = c( 0.75,1.1)) +
    scale_x_continuous("Risk Aversion") +
    scale_y_continuous("Fraction of Wealth") +
    scale_color_manual("Legend Title\n",labels = c("Bonds"), values = c("blue")) ↩
        +
    labs(title="(a) Constant Model") +
    theme_bw() +
    theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=9),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.3,0.87),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

#Bond allocation for stochastic model
bond.sto25 < ggplot(stochas.df, aes(x = Gamma, y = Bond_2.5)) +
  geom_line(aes(col = "blue"), size = 1.5) +
  geom_line(aes(y = Bond_5, col = "darkred"), size = 1.5, linetype = 2) +
  coord_cartesian(xlim = c(0,20), ylim = c( 0.75,1.1)) +
  scale_x_continuous("Risk Aversion") +
  scale_y_continuous("Fraction of Wealth") +
```

```
  scale_color_manual("Legend Title\n",labels = c("Bond, T=2.5", "Bond, T=5"), ↩
      values = c("blue","darkred")) +
  labs(title="(b) Stochastic Model") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=9),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.4,0.82),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

#Combine plots
grid.arrange(bond.constant, bond.sto25, ncol=2)

#(iv) ggplot of cash allocation, constant and stochastic model, Figure 5.5

#Cash allocation for constant model
cash.constant <- ggplot(stochas.df, aes(x = Gamma, y = Con_Cash)) +
  geom_line(aes(col = "green"), size = 1.5) +
  coord_cartesian(xlim = c(0,20), ylim = c( 0.75,1.1)) +
  scale_x_continuous("Risk Aversion") +
  scale_y_continuous("Fraction of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Cash"), values = c("green")) +
  labs(title="(a) Constant Model") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=9),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.3,0.85),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

#Cash allocation for stochastic model
cash.sto25 <- ggplot(stochas.df, aes(x = Gamma, y = Cash_2.5)) +
  geom_line(aes(col = "green"), size = 1.5, linetype = 2) +
```

```
  geom_line(aes(y = Cash_5, col = "darkgreen"), size = 1.5) +
  coord_cartesian(xlim = c(0,20), ylim = c( 0.75,1.1)) +
  scale_x_continuous("Risk Aversion") +
  scale_y_continuous("Fraction of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Cash, T=2.5", "Cash, T=5"), ↩
      values = c("green","darkgreen")) +
  labs(title="(b) Stochastic Model") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=9),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.4,0.8),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

#Combine plots
grid.arrange(cash.constant, cash.sto25, ncol=2)

##############################################################################
## PROGRAM 6   MAIN PROGRAM FOR ALLOCATION RESULTS, PLOTS AND LOSS FUNCTION #
##############################################################################

#CONTENT
#(i) PORTFOLIO MODEL FOR BOTH MODEL 1 AND MODEL 2
#(ii) TABLE 6.1   Bond decomposition
#(iii) PLOTS for Figure 6.1, Figure 6.2, cOMPARE MODEL1 AND MODEL 2
#(iv) Loss FUNCTIONS   BETWEEN CONSTANT MODEL AND MODEL 1

#### Required Packages #####
require(ggplot2)
require(stargazer);
require(gridExtra)
##############################################################################

#### Interest Rate Simulation ####

## Define model parameters
r0 < 0.01
r_bar < 0.01   #theta
kappa < 0.4965   #k
sigma_r < 0.05 #beta

## simulate short rate paths
n < 1    # MC simulation trials
T < 30    # total time
m < 30   # subintervals
```

```
dt < T/m  # difference in time each subinterval

r < matrix(0,m+1,n)  # matrix to hold short rate paths
r[1,] < r0

for(j in 1:n){
  for(i in 2:(m+1)){
    dr < kappa*(r_bar r[i 1,j])*dt + sigma_r*sqrt(dt)*rnorm(1,0,1)
    r[i,j] < r[i 1,j] + dr
  }
}

## plot paths
t < seq(0, T, dt)
rT.expected < r_bar + (r0 r_bar)*exp( kappa*t)
rT.stdev < sqrt( sigma_r^2/(2*kappa)*(1 exp( 2*kappa*t)))
matplot(t, r[,1], type="l", lty=1, ylab="rt")
abline(h=r_bar, col="red", lty=2)
lines(t, rT.expected, lty=2)
lines(t, rT.expected + 2*rT.stdev, lty=2)
lines(t, rT.expected  2*rT.stdev, lty=2)
points(0,r0)


#### PORTFOLIO MODEL ####

#Function begin
gammaList = list()
gammaVec = c(0.5, 1, 2, 5, 10, 20)

for(g in 1:length(gammaVec)){
  gamma = gammaVec[g]

  #Parameters
  lambda_bar2 < 0.3650  #0.3650 #0.15
  lambda_tilde2 < 0.06   #0.06   #8.925
  rho < 0.2
  sigma_s < 0.202
  lambda_bar1 < 0.109    #0.109  #0.05
  lambda_tilde1 < 0.067  #0.067 #2.48
  tau < t
  b_tau < (1/kappa)*(1 exp( kappa*t))
  sigma_b < 0.1
  mu_s < 0.087
  mu_b < 0.021
  #r < rep(0.01,31)

  #### #Model with stochastic interest and affine market price of risk ####

  #ODE's
  v < 2*sqrt(( kappa + ((gamma 1)/gamma)*sigma_r*lambda_tilde1)^2 + ((gamma 1)/↩
      gamma)*((lambda_tilde1^2+lambda_tilde2^2)/gamma)*sigma_r^2)

  A2_tau < (2*((1/gamma)*(lambda_tilde1^2+lambda_tilde2^2))*(exp(v*tau) 1))/((v↩
     +(2*(kappa + ((1 gamma)/gamma)*sigma_r*lambda_tilde1)))*(exp(v*tau) 1)+2*v)

  A1_tau < (1+(1/gamma)*(lambda_bar1*lambda_tilde1+lambda_bar2*lambda_tilde2))/↩
```

```
    ((1/gamma)*(lambda_tilde1^2+lambda_tilde2^2))*A2_tau +(2/v)*((1+(1/gamma)*(↩
    lambda_bar1*lambda_tilde1+lambda_bar2*lambda_tilde2))*
(2*(kappa+((1 gamma)/gamma)*sigma_r*lambda_tilde1))+
2*((kappa*r_bar (1 gamma)/gamma)*sigma_r*lambda_bar1)*
((1/gamma)*(lambda_tilde1^2+lambda_tilde2^2)))* ((exp((v*tau)/2) 1)^2)/ ((v+(2*(↩
   kappa+((1 gamma)/gamma)*sigma_r*lambda_tilde1)))*
(exp(v*tau) 1)+2*v)

 A_functions < data.frame(A1_tau, A2_tau, v)

 #Stock Weight
 pi_s2 < (1/gamma)*((lambda_bar2+lambda_tilde2*r)/(sqrt(1 rho^2)*sigma_s))

 #Bond Weight
 pi_b2 < (1/(gamma*sigma_b))*(lambda_bar1+lambda_tilde1*r  rho/(sqrt(1 rho^2))↩
    *(lambda_bar2+lambda_tilde2*r)) +(((gamma 1)/gamma) * (sigma_r/sigma_b))*(↩
    A1_tau +A2_tau*r)

 pi_b2.1term < (1/(gamma*sigma_b))*(lambda_bar1+lambda_tilde1*r)
 mean(pi_b2.1term)

 pi_b2.2term < (1/(gamma*sigma_b))*( rho/(sqrt(1 rho^2))*(lambda_bar2+lambda_↩
    tilde2*r))
 mean(pi_b2.2term)

 #Hedge part
 pi_b2_hedge < (((gamma 1)/gamma) * (sigma_r/sigma_b))*(A1_tau +A2_tau*r)

 #Mean value of pi_s2
 mean_pi_s2 < rep(mean(pi_s2), 31)

 #### Model only stochastic interest rate #####

 #Parameters:
 lambda1 < (lambda_bar1+lambda_tilde1*mean(r)) #0.11
 lambda2 < (lambda_bar2+lambda_tilde2*mean(r)) #0.3666

 #Stock weight
 pi_s1 < (1/gamma)*(lambda2/(sqrt(1 rho^2)*sigma_s))

 stock_weights_old < data.frame(pi_s1)

 #Bond weight
 pi_b1 < (1/gamma)*(((lambda1)/sigma_b)  ((rho*(lambda2))/(sqrt(1 rho^2)*sigma↩
    _b)))+((gamma 1)/gamma)*(sigma_r/sigma_b)*b_tau

 bond_weights_old < data.frame(pi_b1)

 pi_b1_hedge < ((gamma 1)/gamma)*(sigma_r/sigma_b)*b_tau

 #Fill a vector the constant stock weight
 vac < rep(pi_s1,31)

 #Hedging importance
 hedge1 < pi_b1  pi_b1_hedge
 hedge2 < pi_b2  pi_b2_hedge
```

```
  #Stock Bond Ratio
  ratio_1 < pi_b1/pi_s1
  ratio_2 < pi_b2/pi_s2

  ratio.df < data.frame(ratio_1,ratio_2)

  #Cash fraction
  cash_1 < 1  pi_s1  pi_b1
  check_1 < pi_s1 + pi_b1 + cash_1
  cash_2 < 1  pi_s2  pi_b2
  check_2 < pi_s2 + pi_b2 + cash_2

  #Dataframe for stocks
  stock_weights < data.frame(pi_s2)

  #Dataframe for bonds
  bond_weights < data.frame(pi_b2)

  hedge_weights < data.frame(pi_b2_hedge)

  bond_terms < data.frame(pi_b2.1term,pi_b2.2term, pi_b2_hedge, A1_tau,A2_tau)

  ### LOSS FUNCTION ###
  gain_model < (1/gamma  1)*(1 gamma)/2 * sigma_r^2 *
                  ((kappa*tau (1 exp( kappa*tau)))/(kappa^3)
                     (1 exp( kappa*tau))^2/(2*kappa^3))
  loss_model <  exp(gain_model*r0)  1

  #assign(paste0("G", g), stock_weights)
  gammaList[[paste0("G", gammaVec[g])]] < stock_weights
  gammaList[[paste0("A", gammaVec[g])]] < stock_weights_old
  gammaList[[paste0("B", gammaVec[g])]] < bond_weights
  gammaList[[paste0("O", gammaVec[g])]] < bond_weights_old
  gammaList[[paste0("H", gammaVec[g])]] < hedge_weights
  gammaList[[paste0("T", gammaVec[g])]] < bond_terms
  gammaList[[paste0("V", gammaVec[g])]] < A_functions
  gammaList[[paste0("L", gammaVec[g])]] < loss_model
  gammaList[[paste0("C", gammaVec[g])]] < cash_1
}
#############################################
#############################################
#Producing output
#############################################
#############################################
#Setting up dataframes
aversion.df.s2 < data.frame(gammaList$G0.5,gammaList$G1,gammaList$G2,gammaList$↩
    G5,gammaList$G10,gammaList$G20)
aversion.df.s1 < data.frame(gammaList$A0.5,gammaList$A1,gammaList$A2,gammaList$↩
    A5,gammaList$A10,gammaList$A20)
aversion.df.b2 < data.frame(gammaList$B0.5,gammaList$B1,gammaList$B2,gammaList$↩
    B5,gammaList$B10,gammaList$B20)
aversion.df.b1 < data.frame(gammaList$O0.5,gammaList$O1,gammaList$O2,gammaList$↩
    O5,gammaList$O10,gammaList$O20)
aversion.df.hedge < data.frame(gammaList$H0.5,gammaList$H1,gammaList$H2,↩
    gammaList$H5,gammaList$H10,gammaList$H20)
aversion.df.bondterms < data.frame(gammaList$T0.5,gammaList$T1,gammaList$T2, ↩
    gammaList$T5,gammaList$T10,gammaList$T20)
```

```
aversion.df.A_functions < data.frame(gammaList$V0.5,gammaList$V1,gammaList$V2, ↩
    gammaList$V5,gammaList$V10,gammaList$V20)
aversion.df.cash.m1 < data.frame(gammaList$C0.5,gammaList$C1,gammaList$C2, ↩
    gammaList$C5,gammaList$C10,gammaList$C20)
new.df.A_functions_gamma2 < data.frame(aversion.df.A_functions$A1_tau.2,aversion↩
    .df.A_functions$A2_tau.2,aversion.df.A_functions$v.2)
new.df.b2 < data.frame(aversion.df.b2)

# Selecting time periods
new.SW < aversion.df.s2[c(2,30),1:6]
new.BW < aversion.df.b2[c(2,30),1:6]
new.HE < aversion.df.hedge[c(2,30),1:6]
new.BT < aversion.df.bondterms[(c(1,10,20,30)),1:5]

### NEW THINGS  TABLE MODEL 1 ###
aversion.df.b1[c(2,30),1:6]
aversion.df.s1[c(2,30),1:6]
aversion.df.cash.m1[c(2,30),1:6]

# Defining new dataframes
RA.SW < new.SW[1:2,]
RA.BW < new.BW[1:2,]
RA.HE < new.HE[1:2,]
RA.BT < new.BT[1:4,]

#Creating a Bond Table with decomposed terms
BT.05 < aversion.df.bondterms[(c(2,30)),1:5]
BT.1  < aversion.df.bondterms[(c(2,30)),6:10]
BT.2  < aversion.df.bondterms[(c(2,30)),11:15]
BT.5  < aversion.df.bondterms[(c(2,30)),16:20]
BT.10 < aversion.df.bondterms[(c(2,30)),21:25]
BT.20 < aversion.df.bondterms[(c(2,30)),26:30]

colnames(BT.05) < c("first","2nd", "hedge", "A1", "A2")
colnames(BT.1)  < c("first","2nd", "hedge", "A1", "A2")
colnames(BT.2)  < c("first","2nd", "hedge", "A1", "A2")
colnames(BT.5)  < c("first","2nd", "hedge", "A1", "A2")
colnames(BT.10) < c("first","2nd", "hedge", "A1", "A2")
colnames(BT.20) < c("first","2nd", "hedge", "A1", "A2")
##################################################################
#(ii) TABLE 6.1  Bond decomposition
BT.table < rbind(BT.05,BT.1,BT.2,BT.5,BT.10,BT.20)
# Stargazer table output
BT.table.latex < stargazer(BT.table, summary = FALSE)

#################################################################
#Creating a table for comparison between the models

#Setting up the table
dim(RA.SW) = c(length(RA.SW), 1)
dim(RA.BW) = c(length(RA.BW), 1)
dim(RA.HE) = c(length(RA.HE), 1)

SW.df < t(RA.SW)
BW.df < t(RA.BW)
HE.df < t(RA.HE)
SW.t1.df < data.frame(SW.df[,1])
```

```
SW.t30.df < data.frame(SW.df[,2])
colnames(SW.t1.df) < c("Stocks")
colnames(SW.t30.df) < c("Stocks")

BW.t1.df < data.frame(BW.df[,1])
BW.t30.df < data.frame(BW.df[,2])
colnames(BW.t1.df) < c("Bonds")
colnames(BW.t30.df) < c("Bonds")

HE.t1.df < data.frame(HE.df[,1])
HE.t30.df < data.frame(HE.df[,2])
colnames(HE.t1.df) < c("Hedge")
colnames(HE.t30.df) < c("Hedge")

SW.rbind < rbind(SW.t1.df,SW.t30.df)
BW.rbind < rbind(BW.t1.df,BW.t30.df)
HE.rbind < rbind(HE.t1.df,HE.t30.df)
Cash.df < 1  SW.rbind   BW.rbind
colnames(Cash.df) < c("Cash")

check.df < Cash.df + SW.rbind + BW.rbind

new.table.PF < cbind(SW.rbind,BW.rbind,HE.rbind,Cash.df)

##########################################################
#(iii) Table 6.2 for comparison
new.table.PF.df < new.table.PF
#Table Solution
PF.table.latex < stargazer(new.table.PF, summary = FALSE)
######################################################################

#Labeling to print
rownames(BW.df) = colnames(RA.BW)
rownames(SW.df) = colnames(RA.BW)
rownames(HE.df) = colnames(RA.HE)
rownames(SW.df) = c(0.5,1,2,5,10,20)
rownames(BW.df) = c(0.5,1,2,5,10,20)
rownames(HE.df) = c(0.5,1,2,5,10,20)

#Print
SW.df # Stocks at T= 1 and T=30
BW.df # Bonds
HE.df # Hegde term

####################################################################
#(iii) PLOTS for Figure 6.1, Figure 6.2  ###############
####################################################################
head(aversion.df.b1)
head(aversion.df.b2)

# Bond Plot for RA = 1 #
bond.plot.1 < ggplot(aversion.df.b1, aes(x = t, y = gammaList$01, col = "blue"),↩
     size = 1.2) +
  geom_line() +
  geom_line(aes(y = gammaList$B1, col = "green"), size = 1.2) +
  geom_ribbon(aes(x = t, ymax=gammaList$B1, ymin=gammaList$01, linetype = NA), ↩
      fill="pink", alpha=.3) +
```

```
  coord_cartesian(xlim = c(0,30), ylim = c(0.25,0.75)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction Of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Bonds  M1", "Bonds  M2" ), values = c("blue", "green")) +
  labs(title="(a) Risk Aversion = 1") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=8),
        axis.title.x=element_text(size=8),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= "none", # c(0.4,0.8),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))
bond.plot.1


# Bond Plot for RA = 2 #
bond.plot.2 < ggplot(aversion.df.b1, aes(x = t, y = gammaList$02, col = "blue"), size = 1.2) +
  geom_line() +
  geom_line(aes(y = gammaList$B2, col = "green"), size = 1.2) +
  geom_ribbon(aes(x = t, ymax=gammaList$B2, ymin=gammaList$02, linetype = NA), fill="pink", alpha=.3) +
  coord_cartesian(xlim = c(0,30), ylim = c(0.25,0.75)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction Of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Bonds  M1", "Bonds  M2" ), values = c("blue", "green")) +
  labs(title="(b) Risk Aversion = 2") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=8),
        axis.title.x=element_text(size=8),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= "none", # c(0.4,0.8),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
```

```
        legend.background = element_rect(fill="transparent"))
bond.plot.2

# Bond Plot for RA = 5 #
bond.plot.5 < ggplot(aversion.df.b1, aes(x = t, y = gammaList$O5, col = "blue"),↩
     size = 1.2) +
  geom_line() +
  geom_line(aes(y = gammaList$B5, col = "green"), size = 1.2) +
  geom_ribbon(aes(x = t, ymax=gammaList$B5, ymin=gammaList$O5, linetype = NA), ↩
      fill="pink", alpha=.3) +
  coord_cartesian(xlim = c(0,30), ylim = c(0.5,1)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction Of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Bonds  M1", "Bonds  M2" ), ↩
      values = c("blue", "green")) +
  labs(title="(c) Risk Aversion = 5") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=8),
        axis.title.x=element_text(size=8),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position="none",    # c(0.4,0.8),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))
bond.plot.5

# Bond Plot for RA = 10 #
bond.plot.10 < ggplot(aversion.df.b1, aes(x = t, y = gammaList$O10, col = "blue"↩
    ), size = 1.2) +
  geom_line() +
  geom_line(aes(y = gammaList$B10, col = "green"), size = 1.2) +
  geom_ribbon(aes(x = t, ymax=gammaList$B10, ymin=gammaList$O10, linetype = NA), ↩
      fill="pink", alpha=.3) +
  coord_cartesian(xlim = c(0,30), ylim = c(0.5,1)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction Of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Bonds  M1", "Bonds  M2" ), ↩
      values = c("blue", "green")) +
  labs(title="(d) Risk Aversion = 10") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=8),
        axis.title.x=element_text(size=8),
```

```
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= "none", # c(0.4,0.8),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))
bond.plot.10

# Bond Plot for RA = 20 #
bond.plot.20 < ggplot(aversion.df.b1, aes(x = t, y = gammaList$O20, col = "blue"), size = 1.2) +
    geom_line() +
    geom_line(aes(y = gammaList$B20, col = "green"), size = 1.2) +
    geom_ribbon(aes(x = t, ymax=gammaList$B20, ymin=gammaList$O20, linetype = NA), fill="pink", alpha=.5) +
    coord_cartesian(xlim = c(0,30), ylim = c(0.75,1)) +
    scale_x_continuous("Years") +
    scale_y_continuous("Fraction Of Wealth") +
    scale_color_manual("Legend Title\n",labels = c("Bonds  M1", "Bonds  M2" ), values = c("blue", "green")) +
    labs(title="(d) Risk Aversion = 20") +
    theme_bw() +
    theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=8),
        axis.title.x=element_text(size=8),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= "none", #c(0.4,0.8),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

#All plots
grid.arrange(bond.plot.2, bond.plot.5, nrow = 1, ncol = 2)
grid.arrange(bond.plot.10, bond.plot.20, nrow = 1, ncol = 2)
grid.arrange(bond.plot.1, bond.plot.2, bond.plot.5, bond.plot.10, nrow = 2, ncol = 2)

##########################################################
### STOCKS ###############################################
##########################################################
#RA = 2
stock.plot.2 < ggplot(aversion.df.s2, aes(x = t, y = gammaList$G2)) +
  geom_line(aes(col = "red"), size = 1) +
  geom_line(aes(y = gammaList$A2, col = "darkred"), size = 1, linetype = 2) +
```

```
  coord_cartesian(xlim = c(0,30)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Stocks  M1", "Stocks  M2" ), ↩
     values = c("red", "darkred")) +
  labs(title="(a) Risk Aversion = 2") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.4,0.65),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

#RA = 5
stock.plot.5 < ggplot(aversion.df.s2, aes(x = t, y = gammaList$G5)) +
  geom_line(aes(col = "red"), size = 1) +
  geom_line(aes(y = gammaList$A5, col = "darkred"), size = 1, linetype = 2) +
  coord_cartesian(xlim = c(0,30)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Stocks  M1", "Stocks  M2" ), ↩
     values = c("red", "darkred")) +
  labs(title="(b) Risk Aversion = 5") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.4,0.65),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

#RA = 10
stock.plot.10 < ggplot(aversion.df.s2, aes(x = t, y = gammaList$G10)) +
  geom_line(aes(col = "red"), size = 1) +
```

```
  geom_line(aes(y = gammaList$A10, col = "darkred"), size = 1, linetype = 2) +
  coord_cartesian(xlim = c(0,30)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Stocks  M1", "Stocks  M2" ), ↩
      values = c("red", "darkred")) +
  labs(title="(c) Risk Aversion = 10") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.4,0.65),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

## RA = 20
stock.plot.20 <  ggplot(aversion.df.s2, aes(x = t, y = gammaList$G20)) +
  geom_line(aes(col = "red"), size = 1) +
  geom_line(aes(y = gammaList$A20, col = "darkred"), size = 1, linetype = 2) +
  coord_cartesian(xlim = c(0,30)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Stocks  M1", "Stocks  M2" ), ↩
      values = c("red", "darkred")) +
  labs(title="(d) Risk Aversion = 20") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.4,0.65),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

grid.arrange(stock.plot.2, stock.plot.5, stock.plot.10, stock.plot.20, nrow = 2, ↩
    ncol = 2)
```

```
###########################################################################
### (iv) Loss FUNCTIONS  BETWEEN CONSTANT MODEL AND MODEL 1 ##
###########################################################################
loss.function <- data.frame(gammaList$L0.5,gammaList$L1,gammaList$L2, gammaList$↩
    L5,gammaList$L10,gammaList$L20)

### Risk Aversion = 2 ###
plot.loss.rv2 <- ggplot(loss.function, aes(x = t, y = loss.function$gammaList.L2)↩
    ) +
  geom_line(col= "red", size = 1) +
  coord_cartesian(xlim = c(0,30)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Welfare Loss (%)") +
  scale_color_manual("Legend Title\n",labels = c("Loss" ), values = c("red")) +
  labs(title="(a) Risk Aversion = 2") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=7),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=7),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=8, color="black"),
        legend.justification=c(1,0),
        legend.position= c(1.0249,0.845),
        legend.key.size = unit(0.3,"cm"),
        legend.text = element_text(size = 6),
        legend.direction = "horizontal",
        legend.background = element_rect(fill=alpha('white', 0.4)))
plot.loss.rv2

### Risk Aversion = 5 ###
plot.loss.rv5 <- ggplot(loss.function, aes(x = t, y = loss.function$gammaList.L5)↩
    ) +
                geom_line(col= "red", size = 1) +
                coord_cartesian(xlim = c(0,30)) +
                scale_x_continuous("Years") +
                scale_y_continuous("Welfare Loss (%)") +
                scale_color_manual("Legend Title\n",labels = c("Loss" ), values =↩
                     c("red")) +
                labs(title="(b) Risk Aversion = 5") +
                theme_bw() +
                theme(panel.background = element_rect(colour = "black", size=1.5)↩
                    ,
                  axis.ticks = element_line(color = "black"),
                  panel.grid.major = element_line(color = "grey", linetype = "↩
                      dotted"),
                  panel.grid.minor = element_blank(),
                  legend.title = element_blank(),
                  axis.text.x=element_text(size=7),
                  axis.title.x=element_text(size=9),
                  axis.text.y=element_text(size=7),
```

```
                 axis.title.y=element_text(size=9),
                 plot.title=element_text(size=8, color="black"),
                 legend.justification=c(1,0),
                 legend.position= c(1.0249,0.845),
                 legend.key.size = unit(0.3,"cm"),
                 legend.text = element_text(size = 6),
                 legend.direction = "horizontal",
                 legend.background = element_rect(fill=alpha('white', 0.4)))

### Risk Aversion = 10 ###
plot.loss.rv10 < ggplot(loss.function, aes(x = t, y = loss.function$gammaList.↩
    L10)) +
  geom_line(col= "red", size = 1) +
  coord_cartesian(xlim = c(0,30)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Welfare Loss (%)") +
  scale_color_manual("Legend Title\n",labels = c("Loss" ), values = c("red")) +
  labs(title="(c) Risk Aversion = 10") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=7),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=7),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=8, color="black"),
        legend.justification=c(1,0),
        legend.position= c(1.0249,0.845),
        legend.key.size = unit(0.3,"cm"),
        legend.text = element_text(size = 6),
        legend.direction = "horizontal",
        legend.background = element_rect(fill=alpha('white', 0.4)))

### Risk Aversion = 20 ###
plot.loss.rv20 < ggplot(loss.function, aes(x = t, y = loss.function$gammaList.↩
    L20)) +
  geom_line(col= "red", size = 1) +
  coord_cartesian(xlim = c(0,30)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Welfare Loss (%)") +
  scale_color_manual("Legend Title\n",labels = c("Loss" ), values = c("red")) +
  labs(title="(d) Risk Aversion = 20") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=7),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=7),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=8, color="black"),
        legend.justification=c(1,0),
```

```
        legend.position= c(1.0249,0.845),
        legend.key.size = unit(0.3,"cm"),
        legend.text = element_text(size = 6),
        legend.direction = "horizontal",
        legend.background = element_rect(fill=alpha('white', 0.4)))

# All plots
grid.arrange(plot.loss.rv2, plot.loss.rv5,plot.loss.rv10,plot.loss.rv20, nrow = ↩
    2, ncol = 2)


######################################################
#### Program 7   Alternative Specification of MPR ####
######################################################

#Content#
#(i) Simulation of models with a constant interest rate, if r < rep(0.01,31) is ↩
    active.
#(ii) Simulation of models with a stochastic interest rate, if r < rep(0.01,31) ↩
    is deactive.
#(iii) Figure 6.3 and Fiure 6.4


#### Required Packages #####
require(ggplot2)
require(stargazer);
require(gridExtra)
#######################################################

#### Interest Rate Simulation ####

## Define model parameters
r0 < 0.01
r_bar < 0.01   #theta
kappa < 0.4965    #k
sigma_r < 0.05 #beta

## simulate short rate paths
n < 1    # MC simulation trials
T < 30    # total time
m < 30   # subintervals
dt < T/m  # difference in time each subinterval

r < matrix(0,m+1,n)  # matrix to hold short rate paths
r[1,] < r0

for(j in 1:n){
  for(i in 2:(m+1)){
    dr < kappa*(r_bar r[i 1,j])*dt + sigma_r*sqrt(dt)*rnorm(1,0,1)
    r[i,j] < r[i 1,j] + dr
  }
}

## plot paths
t < seq(0, T, dt)
rT.expected < r_bar + (r0 r_bar)*exp( kappa*t)
rT.stdev < sqrt( sigma_r^2/(2*kappa)*(1 exp( 2*kappa*t)))
```

```
matplot(t, r[,1], type="l", lty=1, ylab="rt")
abline(h=r_bar, col="red", lty=2)
lines(t, rT.expected, lty=2)
lines(t, rT.expected + 2*rT.stdev, lty=2)
lines(t, rT.expected  2*rT.stdev, lty=2)
points(0,r0)


#### PORTFOLIO MODEL ####

#Function begin
gammaList = list()
gammaVec = c(0.5, 1, 2, 5, 10, 20)

for(g in 1:length(gammaVec)){
  gamma = gammaVec[g]

  #Parameters
  lambda_bar2 < 0.301  #Base: 0.365  #Alternative: 0.301
  lambda_tilde2 < 6.5  #Base: 0.06   #Alternative: 6.5
  rho < 0.2
  sigma_s < 0.202
  lambda_bar1 < 0.06   #Base: 0.109  #Alternative:0.06
  lambda_tilde1 < 5    #Base: 0.067  #Alternative:5
  tau < t
  b_tau < (1/kappa)*(1 exp( kappa*t))
  sigma_b < 0.1
  mu_s < 0.087
  mu_b < 0.021
  #r < rep(0.01,31)  ######## IMPORTANT: if ACTIVE: Simulation with constant ↩
      interest rate ####


  #### #Model with stochastic interest and affine market price of risk ####

  #ODE's
  v < 2*sqrt(( kappa + ((gamma 1)/gamma)*sigma_r*lambda_tilde1)^2 + ((gamma 1)/↩
      gamma)*((lambda_tilde1^2+lambda_tilde2^2)/gamma)*sigma_r^2)

  A2_tau < (2*((1/gamma)*(lambda_tilde1^2+lambda_tilde2^2))*(exp(v*tau) 1))/((v↩
      +(2*(kappa + ((1 gamma)/gamma)*sigma_r*lambda_tilde1)))*(exp(v*tau) 1)+2*v)

  A1_tau < (1+(1/gamma)*(lambda_bar1*lambda_tilde1+lambda_bar2*lambda_tilde2))/
                ((1/gamma)*(lambda_tilde1^2+lambda_tilde2^2))*A2_tau +(2/v)*
                ((1+(1/gamma)*(lambda_bar1*lambda_tilde1+lambda_bar2*lambda_↩
                    tilde2))*
                (2*(kappa+((1 gamma)/gamma)*sigma_r*lambda_tilde1))+
                2*((kappa*r_bar (1 gamma)/gamma)*sigma_r*lambda_bar1)*
                ((1/gamma)*(lambda_tilde1^2+lambda_tilde2^2)))*
                ((exp((v*tau)/2) 1)^2)/
                ((v+(2*(kappa+((1 gamma)/gamma)*sigma_r*lambda_tilde1)))*(exp(v*↩
                    tau) 1)+2*v)

  A_functions < data.frame(A1_tau, A2_tau, v)

  #Stock Weight
  pi_s2 < (1/gamma)*((lambda_bar2+lambda_tilde2*r)/(sqrt(1 rho^2)*sigma_s))
```

```
#Bond Weight
pi_b2 < (1/(gamma*sigma_b))*(lambda_bar1+lambda_tilde1*r   rho/(sqrt(1 rho^2))↩
    *(lambda_bar2+lambda_tilde2*r)) +(((gamma  1)/gamma) * (sigma_r/sigma_b))*(↩
    A1_tau +A2_tau*r)

pi_b2.1term < (1/(gamma*sigma_b))*(lambda_bar1+lambda_tilde1*r)
mean(pi_b2.1term)

pi_b2.2term < (1/(gamma*sigma_b))*( rho/(sqrt(1 rho^2))*(lambda_bar2+lambda_↩
    tilde2*r))
mean(pi_b2.2term)

#Hedge part
pi_b2_hedge < (((gamma  1)/gamma) * (sigma_r/sigma_b))*(A1_tau +A2_tau*r)

#Mean value of pi_s2
mean_pi_s2 < rep(mean(pi_s2), 31)

#### Model only stochastic interest rate #####

#Parameters:
lambda1 < (lambda_bar1+lambda_tilde1*mean(r)) #0.11
lambda2 < (lambda_bar2+lambda_tilde2*mean(r)) #0.3666

#Stock weight
pi_s1 < (1/gamma)*(lambda2/(sqrt(1 rho^2)*sigma_s))

stock_weights_old < data.frame(pi_s1)

#Bond weight
pi_b1 < (1/gamma)*(((lambda1)/sigma_b)   ((rho*(lambda2))/(sqrt(1 rho^2)*sigma↩
    _b)))+((gamma 1)/gamma)*(sigma_r/sigma_b)*b_tau

bond_weights_old < data.frame(pi_b1)

pi_b1_hedge < ((gamma 1)/gamma)*(sigma_r/sigma_b)*b_tau

#Fill a vector the constant stock weight
vac < rep(pi_s1,31)

#Hedging importance
hedge1 < pi_b1  pi_b1_hedge
hedge2 < pi_b2  pi_b2_hedge

#Stock Bond Ratio
ratio_1 < pi_b1/pi_s1
ratio_2 < pi_b2/pi_s2

ratio.df < data.frame(ratio_1,ratio_2)

#Cash fraction
cash_1 < 1  pi_s1  pi_b1
check_1 < pi_s1 + pi_b1 + cash_1
cash_2 < 1  pi_s2  pi_b2
check_2 < pi_s2 + pi_b2 + cash_2
```

```
  #Dataframe for stocks
  stock_weights < data.frame(pi_s2)

  #Dataframe for bonds
  bond_weights < data.frame(pi_b2)

  hedge_weights < data.frame(pi_b2_hedge)

  bond_terms < data.frame(pi_b2.1term,pi_b2.2term, pi_b2_hedge, A1_tau,A2_tau)

  ### LOSS FUNCTION ###
  gain_model < (1/gamma 1)*(1 gamma)/2 * sigma_r^2 * ((kappa*tau (1 exp( kappa↩
      *tau)))/(kappa^3) (1 exp( kappa*tau))^2/(2*kappa^3))
  loss_model < exp(gain_model*r0) 1

  #assign(paste0("G", g), stock_weights)
  gammaList[[paste0("G", gammaVec[g])]] < stock_weights
  gammaList[[paste0("A", gammaVec[g])]] < stock_weights_old
  gammaList[[paste0("B", gammaVec[g])]] < bond_weights
  gammaList[[paste0("O", gammaVec[g])]] < bond_weights_old
  gammaList[[paste0("H", gammaVec[g])]] < hedge_weights
  gammaList[[paste0("T", gammaVec[g])]] < bond_terms
  gammaList[[paste0("V", gammaVec[g])]] < A_functions
  gammaList[[paste0("L", gammaVec[g])]] < loss_model
}
##############################################################################
##############################################################################
#Producing output
##############################################################################
##############################################################################

#Setting up dataframes
aversion.df.s2 < data.frame(gammaList$G0.5,gammaList$G1,gammaList$G2,gammaList$↩
    G5,gammaList$G10,gammaList$G20)
aversion.df.s1 < data.frame(gammaList$A0.5,gammaList$A1,gammaList$A2,gammaList$↩
    A5,gammaList$A10,gammaList$A20)
aversion.df.b2 < data.frame(gammaList$B0.5,gammaList$B1,gammaList$B2,gammaList$↩
    B5,gammaList$B10,gammaList$B20)
aversion.df.b1 < data.frame(gammaList$O0.5,gammaList$O1,gammaList$O2,gammaList$↩
    O5,gammaList$O10,gammaList$O20)
aversion.df.hedge < data.frame(gammaList$H0.5,gammaList$H1,gammaList$H2,↩
    gammaList$H5,gammaList$H10,gammaList$H20)
aversion.df.bondterms < data.frame(gammaList$T0.5,gammaList$T1,gammaList$T2, ↩
    gammaList$T5,gammaList$T10,gammaList$T20)
aversion.df.A_functions < data.frame(gammaList$V0.5,gammaList$V1,gammaList$V2, ↩
    gammaList$V5,gammaList$V10,gammaList$V20)

new.df.A_functions_gamma2 < data.frame(aversion.df.A_functions$A1_tau.2,aversion↩
    .df.A_functions$A2_tau.2,aversion.df.A_functions$v.2)
new.df.b2 < data.frame(aversion.df.b2)

# Selecting time periods
new.SW < aversion.df.s2[c(2,30),1:6]
new.BW < aversion.df.b2[c(2,30),1:6]
new.HE < aversion.df.hedge[c(2,30),1:6]
new.BT < aversion.df.bondterms[(c(1,10,20,30)),1:5]
```

```
# Defining new dataframes
RA.SW < new.SW[1:2,]
RA.BW < new.BW[1:2,]
RA.HE < new.HE[1:2,]
RA.BT < new.BT[1:4,]

#Creating a Bond Table with decomposed terms
BT.05 < aversion.df.bondterms[(c(2,30)),1:5]
BT.1  < aversion.df.bondterms[(c(2,30)),6:10]
BT.2  < aversion.df.bondterms[(c(2,30)),11:15]
BT.5  < aversion.df.bondterms[(c(2,30)),16:20]
BT.10 < aversion.df.bondterms[(c(2,30)),21:25]
BT.20 < aversion.df.bondterms[(c(2,30)),26:30]

colnames(BT.05) < c("first","2nd", "hedge", "A1", "A2")
colnames(BT.1)  < c("first","2nd", "hedge", "A1", "A2")
colnames(BT.2)  < c("first","2nd", "hedge", "A1", "A2")
colnames(BT.5)  < c("first","2nd", "hedge", "A1", "A2")
colnames(BT.10) < c("first","2nd", "hedge", "A1", "A2")
colnames(BT.20) < c("first","2nd", "hedge", "A1", "A2")

BT.table < rbind(BT.05,BT.1,BT.2,BT.5,BT.10,BT.20)

BT.table.latex < stargazer(BT.table, summary = FALSE)

#Creating a table for comparison between the models

dim(RA.SW) = c(length(RA.SW), 1)
dim(RA.BW) = c(length(RA.BW), 1)
dim(RA.HE) = c(length(RA.HE), 1)

SW.df < t(RA.SW)
BW.df < t(RA.BW)
HE.df < t(RA.HE)
SW.t1.df < data.frame(SW.df[,1])
SW.t30.df < data.frame(SW.df[,2])
colnames(SW.t1.df) < c("Stocks")
colnames(SW.t30.df) < c("Stocks")

BW.t1.df < data.frame(BW.df[,1])
BW.t30.df < data.frame(BW.df[,2])
colnames(BW.t1.df) < c("Bonds")
colnames(BW.t30.df) < c("Bonds")

HE.t1.df < data.frame(HE.df[,1])
HE.t30.df < data.frame(HE.df[,2])
colnames(HE.t1.df) < c("Hedge")
colnames(HE.t30.df) < c("Hedge")

SW.rbind < rbind(SW.t1.df,SW.t30.df)
BW.rbind < rbind(BW.t1.df,BW.t30.df)
HE.rbind < rbind(HE.t1.df,HE.t30.df)
Cash.df < 1  SW.rbind   BW.rbind
colnames(Cash.df) < c("Cash")

check.df < Cash.df + SW.rbind + BW.rbind
```

```
new.table.PF < cbind(SW.rbind,BW.rbind,HE.rbind,Cash.df)
new.table.PF.df < new.table.PF

#Table Solution
PF.table.latex < stargazer(new.table.PF, summary = FALSE)

#Labeling to print
rownames(BW.df) = colnames(RA.BW)
rownames(SW.df) = colnames(RA.BW)
rownames(HE.df) = colnames(RA.HE)
rownames(SW.df) = c(0.5,1,2,5,10,20)
rownames(BW.df) = c(0.5,1,2,5,10,20)
rownames(HE.df) = c(0.5,1,2,5,10,20)

#Print
SW.df # Stocks at T= 1 and T=30
BW.df # Bonds
HE.df # Hegde term

##########################################################
####### (iii) Figure 6.3 and Fiure 6.4 ################
##########################################################
head(aversion.df.b1)
head(aversion.df.b2)

# Bond Plot for RA = 1 #
bond.plot.1 < ggplot(aversion.df.b1, aes(x = t, y = gammaList$01, col = "blue"),↩
    size = 1.2) +
  geom_line() +
  geom_line(aes(y = gammaList$B1, col = "darkblue"), size = 1.2) +
  geom_ribbon(aes(x = t, ymax=gammaList$B1, ymin=gammaList$01, linetype = NA), ↩
      fill="pink", alpha=.3) +
  coord_cartesian(xlim = c(0,30), ylim = c(0.25,0.75)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction Of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Bonds  M1", "Bonds  M2" ), ↩
      values = c("blue", "darkblue")) +
  labs(title="(a) Risk Aversion = 1") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=8),
        axis.title.x=element_text(size=8),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= "none", # c(0.4,0.8),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))
bond.plot.1
```

```
# Bond Plot for RA = 2 #
bond.plot.2 < ggplot(aversion.df.b1, aes(x = t, y = gammaList$O2, col = "blue"),↩
    size = 1.2) +
  geom_line() +
  geom_line(aes(y = gammaList$B2, col = "darkblue"), size = 1.2) +
  geom_ribbon(aes(x = t, ymax=gammaList$B2, ymin=gammaList$O2, linetype = NA), ↩
    fill="pink", alpha=.3) +
  coord_cartesian(xlim = c(0,30), ylim = c( 1.5,4.5)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction Of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Bonds  M1", "Bonds  M2" ), ↩
    values = c("blue", "darkblue")) +
  labs(title="(a) Risk Aversion = 2") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=8),
        axis.title.x=element_text(size=8),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= "none", # c(0.4,0.8),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))
bond.plot.2

# Bond Plot for RA = 5 #
bond.plot.5 < ggplot(aversion.df.b1, aes(x = t, y = gammaList$O5, col = "blue"),↩
    size = 1.2) +
  geom_line() +
  geom_line(aes(y = gammaList$B5, col = "darkblue"), size = 1.2) +
  geom_ribbon(aes(x = t, ymax=gammaList$B5, ymin=gammaList$O5, linetype = NA), ↩
    fill="pink", alpha=.3) +
  coord_cartesian(xlim = c(0,30), ylim = c( 1.5,4)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction Of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Bonds  M1", "Bonds  M2" ), ↩
    values = c("blue", "darkblue")) +
  labs(title="(b) Risk Aversion = 5") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=8),
        axis.title.x=element_text(size=8),
        axis.text.y=element_text(size=8),
```

```
         axis.title.y=element_text(size=8),
         plot.title=element_text(size=9, color="black"),
         legend.justification=c(1,0),
         legend.position="none",      #  c(0.4,0.8),
         legend.key.size = unit(0.35,"cm"),
         legend.text = element_text(size = 7),
         legend.direction = "vertical",
         legend.key.size = unit(2, "cm"),
         legend.background = element_rect(fill="transparent"))
bond.plot.5

# Bond Plot for RA = 10 #
bond.plot.10 < ggplot(aversion.df.b1, aes(x = t, y = gammaList$O10, col = "blue"↩
    ), size = 1.2) +
  geom_line() +
  geom_line(aes(y = gammaList$B10, col = "darkblue"), size = 1.2) +
  geom_ribbon(aes(x = t, ymax=gammaList$B10, ymin=gammaList$O10, linetype = NA), ↩
      fill="pink", alpha=.3) +
  coord_cartesian(xlim = c(0,30), ylim = c( 1.5,4)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction Of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Bonds  M1", "Bonds  M2" ), ↩
      values = c("blue", "darkblue")) +
  labs(title="(c) Risk Aversion = 10") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
         axis.ticks = element_line(color = "black"),
         panel.grid.major = element_line(color = "grey", linetype = "dotted"),
         panel.grid.minor = element_blank(),
         legend.title = element_blank(),
         axis.text.x=element_text(size=8),
         axis.title.x=element_text(size=8),
         axis.text.y=element_text(size=8),
         axis.title.y=element_text(size=8),
         plot.title=element_text(size=9, color="black"),
         legend.justification=c(1,0),
         legend.position= "none", # c(0.4,0.8),
         legend.key.size = unit(0.35,"cm"),
         legend.text = element_text(size = 7),
         legend.direction = "vertical",
         legend.key.size = unit(2, "cm"),
         legend.background = element_rect(fill="transparent"))
bond.plot.10

# Bond Plot for RA = 20 #
bond.plot.20 < ggplot(aversion.df.b1, aes(x = t, y = gammaList$O20, col = "blue"↩
    ), size = 1.2) +
  geom_line() +
  geom_line(aes(y = gammaList$B20, col = "darkblue"), size = 1.2) +
  geom_ribbon(aes(x = t, ymax=gammaList$B20, ymin=gammaList$O20, linetype = NA), ↩
      fill="pink", alpha=.5) +
  coord_cartesian(xlim = c(0,30), ylim = c( 1.5,4)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction Of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Bonds  M1", "Bonds  M2" ), ↩
      values = c("blue", "darkblue")) +
  labs(title="(d) Risk Aversion = 20") +
```

```
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=8),
        axis.title.x=element_text(size=8),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= "none", #c(0.4,0.8),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))
bond.plot.20

#All plots
grid.arrange(bond.plot.2, bond.plot.5, nrow = 1, ncol = 2)
grid.arrange(bond.plot.10, bond.plot.20, nrow = 1, ncol = 2)
##########################################################
#(iii) Figure 6.3 and Fiure 6.4
grid.arrange(bond.plot.2, bond.plot.5, bond.plot.10, bond.plot.20, nrow = 2, ncol↩
     = 2)
##########################################################

##########################################################
### STOCKS ###
##########################################################

#RA = 2
stock.plot.2 < ggplot(aversion.df.s2, aes(x = t, y = gammaList$G2)) +
  geom_line(aes(col = "red"), size = 1) +
  geom_line(aes(y = gammaList$A2, col = "darkred"), size = 1, linetype = 2) +
  coord_cartesian(xlim = c(0,30)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Stocks   M1", "Stocks   M2" ), ↩
      values = c("red", "darkred")) +
  labs(title="(a) Risk Aversion = 2") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.4,0.65),
        legend.key.size = unit(0.35,"cm"),
```

```
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

#RA = 5
stock.plot.5 <  ggplot(aversion.df.s2, aes(x = t, y = gammaList$G5)) +
  geom_line(aes(col = "red"), size = 1) +
  geom_line(aes(y = gammaList$A5, col = "darkred"), size = 1, linetype = 2) +
  coord_cartesian(xlim = c(0,30)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Stocks  M1", "Stocks  M2" ), ↩
      values = c("red", "darkred")) +
  labs(title="(b) Risk Aversion = 5") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.4,0.65),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

#RA = 10
stock.plot.10 <  ggplot(aversion.df.s2, aes(x = t, y = gammaList$G10)) +
  geom_line(aes(col = "red"), size = 1) +
  geom_line(aes(y = gammaList$A10, col = "darkred"), size = 1, linetype = 2) +
  coord_cartesian(xlim = c(0,30)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Stocks  M1", "Stocks  M2" ), ↩
      values = c("red", "darkred")) +
  labs(title="(c) Risk Aversion = 10") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.4,0.65),
```

```
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

## RA = 20
stock.plot.20 < ggplot(aversion.df.s2, aes(x = t, y = gammaList$G20)) +
  geom_line(aes(col = "red"), size = 1) +
  geom_line(aes(y = gammaList$A20, col = "darkred"), size = 1, linetype = 2) +
  coord_cartesian(xlim = c(0,30)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Fraction of Wealth") +
  scale_color_manual("Legend Title\n",labels = c("Stocks  M1", "Stocks  M2" ), ↩
      values = c("red", "darkred")) +
  labs(title="(d) Risk Aversion = 20") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=8),
        axis.title.y=element_text(size=8),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.4,0.65),
        legend.key.size = unit(0.35,"cm"),
        legend.text = element_text(size = 7),
        legend.direction = "vertical",
        legend.key.size = unit(2, "cm"),
        legend.background = element_rect(fill="transparent"))

grid.arrange(stock.plot.2, stock.plot.5, stock.plot.10, stock.plot.20, nrow = 2, ↩
    ncol = 2)
###########################################################

######################################################################
### Program 8  Loss function for constant investment opportunities ##
######################################################################

#Content
#(i) Loss function for risk aversion for Figure 7.1
#(ii) Loss function for time horizon for Figure 7.1

#Packages
require(ggplot2)
####################################################################

#Loss function  Gamma
loss.function.gamma.df < data.frame(loss_function_constant1)
head(loss.function.gamma.df)

loss.con.gamma < ggplot(loss.function.gamma.df) +
  geom_line(aes(x = pi, y = gamma_1, col = "blue"), size = 1) +
```

```
  geom_line(aes(x = pi, y = gamma_2, col = "red"), size = 1) +
  geom_line(aes(x = pi, y = gamma_3, col = "green"), size = 1) +
  geom_line(aes(x = pi, y = gamma_6, col = "purple"), size = 1) +
  scale_x_continuous("Fraction of Risky Assets") +
  scale_y_continuous("Welfare Loss (%)") +
  scale_color_manual("Legend Title\n",labels = c("RRA = 1", "RRA = 2","RRA = 3","RRA = 6"), values = c("blue", "red", "green", "purple")) +
  labs(title="(a) Time = 10") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=9),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.14,0.5),
        legend.key.size = unit(0.3,"cm"),
        legend.text = element_text(size = 9),
        legend.direction = "vertical",
        legend.background = element_rect(fill=alpha('white', 0.4)))

#Loss function  Time
loss.function.time.df < data.frame(loss_function_constant2)
head(loss.function.time.df)

loss.con.time < ggplot(loss.function.time.df) +
  geom_line(aes(x = pi, y = time_1, col = "blue"), size = 1) +
  geom_line(aes(x = pi, y = time_10, col = "red"), size = 1) +
  geom_line(aes(x = pi, y = time_20, col = "green"), size = 1) +
  geom_line(aes(x = pi, y = time_30, col = "purple"), size = 1) +
  coord_cartesian(xlim = c(0,2), ylim = c(0,1)) +
  scale_x_continuous("Fraction of Risky Assets") +
  scale_y_continuous("Welfare Loss (%)") +
  scale_color_manual("Legend Title\n",labels = c("T = 1", "T = 10","T = 20","T = 30"), values = c("blue", "red", "green", "purple")) +
  labs(title="(b) Risk Aversion = 2") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=9),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=9),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=9, color="black"),
        legend.justification=c(1,0),
        legend.position= c(0.13,0.5),
        legend.key.size = unit(0.3,"cm"),
        legend.text = element_text(size = 9),
        legend.direction = "vertical",
```

```
        legend.background = element_rect(fill=alpha('white', 0.4)))

grid.arrange(loss.con.gamma, loss.con.time, nrow = 2, ncol =1)


##########################################################
### Program 9   Loss function between model 1 and model 2##
##########################################################

#Content
#(i) Figure 7.2   Loss function for model and model 2

#Packages
require(ggplot2)
require(gridExtra)

#Load Excel file
loss.function.df <- data.frame(loss_function_2)
head(loss.function.df)
tail(loss.function.df)

#Plot for RA=2
plot.loss.rv2 <- ggplot(loss.function.df, aes(x = time, y = loss_g2)) +
  geom_line(col= "red", size = 1) +
  coord_cartesian(xlim = c(0,30), ylim = c(0,80)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Welfare Loss (%) ") +
  scale_color_manual("Legend Title\n",labels = c("Loss" ), values = c("red")) +
  labs(title="(a) Risk Aversion = 2") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=7),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=7),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=8, color="black"),
        legend.justification=c(1,0),
        legend.position= c(1.0249,0.845),
        legend.key.size = unit(0.3,"cm"),
        legend.text = element_text(size = 6),
        legend.direction = "horizontal",
        legend.background = element_rect(fill=alpha('white', 0.4)))


#Plot for RA=5
plot.loss.rv5 <- ggplot(loss.function.df, aes(x = time, y = loss_g5)) +
  geom_line(col= "red", size = 1) +
  coord_cartesian(xlim = c(0,30), ylim = c(0,5)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Welfare Loss (%)") +
  scale_color_manual("Legend Title\n",labels = c("Loss" ), values = c("red")) +
  labs(title="(b) Risk Aversion = 5") +
  theme_bw() +
```

```
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=7),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=7),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=8, color="black"),
        legend.justification=c(1,0),
        legend.position= c(1.0249,0.845),
        legend.key.size = unit(0.3,"cm"),
        legend.text = element_text(size = 6),
        legend.direction = "horizontal",
        legend.background = element_rect(fill=alpha('white', 0.4)))

#Plot for RA=10
plot.loss.rv10 < ggplot(loss.function.df, aes(x = time, y = loss_g10)) +
  geom_line(col= "red", size = 1) +
  coord_cartesian(xlim = c(0,30), ylim = c(0,1.5)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Welfare Loss (%)") +
  scale_color_manual("Legend Title\n",labels = c("Loss" ), values = c("red")) +
  labs(title="(c) Risk Aversion = 10") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
        axis.text.x=element_text(size=7),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=7),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=8, color="black"),
        legend.justification=c(1,0),
        legend.position= c(1.0249,0.845),
        legend.key.size = unit(0.3,"cm"),
        legend.text = element_text(size = 6),
        legend.direction = "horizontal",
        legend.background = element_rect(fill=alpha('white', 0.4)))

#Plot for RA=20
plot.loss.rv20 < ggplot(loss.function.df, aes(x = time, y = loss_g20)) +
  geom_line(col= "red", size = 1) +
  coord_cartesian(xlim = c(0,30), ylim = c(0,1.5)) +
  scale_x_continuous("Years") +
  scale_y_continuous("Welfare Loss (%)") +
  scale_color_manual("Legend Title\n",labels = c("Loss" ), values = c("red")) +
  labs(title="(d) Risk Aversion = 20") +
  theme_bw() +
  theme(panel.background = element_rect(colour = "black", size=1.5),
        axis.ticks = element_line(color = "black"),
        panel.grid.major = element_line(color = "grey", linetype = "dotted"),
        panel.grid.minor = element_blank(),
        legend.title = element_blank(),
```

```
        axis.text.x=element_text(size=7),
        axis.title.x=element_text(size=9),
        axis.text.y=element_text(size=7),
        axis.title.y=element_text(size=9),
        plot.title=element_text(size=8, color="black"),
        legend.justification=c(1,0),
        legend.position= c(1.0249,0.845),
        legend.key.size = unit(0.3,"cm"),
        legend.text = element_text(size = 6),
        legend.direction = "horizontal",
        legend.background = element_rect(fill=alpha('white', 0.4)))

#Combine plots
grid.arrange(plot.loss.rv2, plot.loss.rv5,plot.loss.rv10,plot.loss.rv20, nrow = ↩
    2, ncol = 2)
```