Empirical analysis of GARCH option pricing models

Master Thesis

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Abstract

Extensive literature on financial time series returns find evidence that return exhibit statistical characteristics making the Black-Scholes-Merton model for option pricing unable to model observed data with great accuracy. In a response to the stylized fact that financial return volatility is changing over time and show sign of clustering, a new class of option pricing models have been developed. The models builds on the General-Autoregressive-Conditional-Heteroskedacity (GARCH) volatility modeling framework of Bollerslev (1986). This thesis conduct an empirical comparison of several option pricing models where the return follows a discrete GARCH process. We employ the HN-GARCH model of Heston & Nandi (2000), the NGARCH model of Engle & Ng (1993), the EGARCH model of Nelson (1990) and the GJR-GARCH model of Glosten et al. (1993). Using option data written on the S&P 500 Index from 2014 to 2016 we test the range of GARCH in terms of pricing errors, ability to replicate the volatility smile and significant biases through regression. The models are set up against the Black-Scholes-Merton model, serving as a benchmark. We obtain the model parameters through maximum likelihood on historical time series of return and through minimizing the mean-squared-error (MSE) on a cross sectional data-set of option prices. Our analysis find evidence that the cross sectional approach manages to produce substantially less pricing errors across all models.

Measuring the relative performance of the models through MAE, MAPE, RMSE, %RMSE and IVRMSE we find evidence that the HN-GARCH display the least pricing errors in both an in-sample and an out-of-sample analysis. The NGARCH and EGARCH display slightly higher absolute pricing errors overall and show similar characteristics both in- and out-of sample. The GJR-GARCH is outperformed over both moneyness and maturity, and demonstrate larger residuals as well as variability in the pricing, especially out-of-sample. While all models manage to replicate the volatility smile characteristics, we present evidence implying that the HN-GARCH is the preferred model to replicate the observed smile. Furthermore, we confirm the relation between pricing errors, time to maturity and moneyness though a regression over MAPE, on a 1% significance level. In accordance with the voluminous existing literature this analysis confirms the GARCH-models relevance within the option pricing theory.

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1 Introduction

The global financial environment has seen immense growth over the last decades with increased globalization, faster flow of information and less limited access to the financial markets. In the wake of this globalization, derivatives and options have risen from purely speculative instruments to one of the pillars of modern finance. Following the increased interest in options as instruments for various purposes the literature develop and employ innovative ways of correctly pricing these claims within applicable frameworks. Following the famous Black-Scholes-Merton for valuating European contingent claims the literature have expanded trying to capture stylized facts observed in financial return data better than its predecessors. Observing that volatility of underlying assets return are changing throughout options maturity encouraged a new class of option pricing models. These models known as General Autoregressive Conditional Heteroskedasticity (GARCH) have gained popularity in modern finance due to their relatively ease of implementation and ability to capture stylized facts.

1.1 Motivation and Research Question

As the global financial markets make themselves more and more dependent on the active trading of derivatives and options in particular, the need for correct pricing is equally important. Throughout our years as students of finance we have caught interest in options due to their versatile use and importance in the modern financial world. Furthermore, the cavities and unrealistic assumptions of the well renowned Black-Scholes-Merton model have been put under scrutiny and empirically challenged. This motivates us to analyze an alternative class of models for option valuation that have gained attention in the academic literature over the last decades. We see this as an opportunity to gain some more knowledge on a topic within finance that have not been explicitly covered by our syllabus, as well as explore an important topic within volatility modeling and option valuation.

We will put the focus of this thesis on the valuation of options within the GARCH framework. We also narrow our analysis to GARCH models as the theory on these models are rich, but few have put all the different models directly to test against each other. We limit our thesis to test the asymmetric models that allow for a leverage effect in returns, as these empirically have produced the best performance in terms of option prices. It is also of particular interest to investigate GARCH-models as they tend to provide better fit than stochastic models.

Research Question:

Which asymmetric GARCH(1,1) option pricing model generates the lowest pricing errors when pricing options written on the S&P 500 Equity Index.

1.2 Delimitation

As this thesis aim to empirically test and analyze the relative performance of already established GARCH option pricing models, the scope of the thesis does include the derivation of the relevant models. Furthermore we do not explore, or challenge the validity of the underlying assumptions of the models beyond comments.

We limit the study to only cover European call options, as similar conclusions can be derived for European puts through the put-call-parity explained in Section 4.1. American options are excluded due to their more complex structure.

1.3 Structure

We have aimed to structure this thesis in a way that it naturally builds up from the presentation of theory and models, the implementation on sampled and filtered data, the discussion around the theory and models and in the end our final results/discoveries. First we will swiftly introduce the history of option pricing followed by a short note on the main applications for options as financial instruments. From there we will present the academic literature supporting our topic and models together with the latest literature concerning discrete model option pricing. We will continue with the presentation of our models in the methodology section, where we present the option pricing models, the performance measures as well as analytical tools we utilize. The empirical analysis itself will concentrate around the relative performance of the models compared to each other with basis in existing literature.

Before presenting our conclusion we will discuss the results of our study in regards to existing biases and shortcomings in order to establish the validity of this empirical study.

2 On Options

In this chapter we will present a brief introduction to options. First we will summarize the development of options throughout history followed with some more recent statistics. The intention is to shine a light on the history and development of options, its various uses and its recent popularity. The presentation of the background of modern option pricing further aims to catch the interest of those not familiar with options as well as further strengthen our rationale.

2.1 Brief history of Options

Contrary to the belief of many, the use options within finance is not a new phenomenon developed in the recent decades. According to Poitras (2009), the first recorded use of an option like contract for the purpose of making a profit is mentioned in Aristotle's *Politics* written in the ancient Greece. He writes the tale of Thales of Miletus who forecasted a good harvest for olives in the coming summer, and bought the right but not the obligation to use all the olive presses in Miletus. When the good harvest later indeed realized, he sold his rights to use the olive presses making a good fortune.

Throughout history, there have been many examples of contracts or gambles analogous to the contractual form of modern day options. The first known marketplace where speculators actively traded "to arrive" contracts on ships with commodities, was at the bourse of Antwerp during the second half of the 16th century. During the 17th and 18th centuries, trading of option and future contracts on the Amsterdam exchange advertised many similarities to the modern trade of derivatives. Here the writing of both puts and calls with regular expiration dates yields evidence of the first instance of regular exchange trading in financial derivatives (Gelderblom & Jonker 2005).

Adapting many of Amsterdam's speculative techniques for stock and derivatives trading, the London bourse was active from the 1690's. To centralize the trade London Stock Exchange was formally established in 1773. The speculative use of options has not always considered reputable, and in the early 1800's the Barnard's Act banned the trading of options. The ban did not last long, however, as members of the exchange saw the options transactions as important sources of profit and unbanned the trading in the 1820's (Poitras 2012). Later trade of option like contracts (prämiengeschäfte) was also traded on the 19th century Berlin and Paris exchanges (Courtadon 1982). And while the London Stock Exchange brought the structure and regulation needed to trade derivatives, it is the development in the US derivatives market that have had the most impact on option trading as we know it today.

US 19th - 20th century

US stocks and obligations began to be actively traded on the New York Stock and Exchange Board from 1817, but the exchange did not offer trading in options or derivatives until much later. The main reason for this was that US investors did not hold derivatives trading especially high and thus there was little interest in developing a derivatives market. The little trading that were conducted was brokered by a small group of OTC dealers with a niche clientele located mainly in New York. The trading was quite illiquid with little interest from larger investors until 1972. In 1972 the Option Clearing Corporation was founded as a subsidiary of the Chicago Board Options Exchange (CBOE). Trading commenced on CBOE in April 1973 and marked the beginning of modern option trade. By 1977 the interest had grown substantially, and the introduction of the put option marked the next milestone for the exchange.

In the following years several other exchanges in the US followed and started to include option trading. Maturities and strike prices were standardized to advocate liquidity, and the costs associated with transaction and trading became lower. In 1982 options on commodities such as gold and sugar commenced together with options on stock indices and US treasury bonds. This marks the last milestones for modern options trading as we know it, and the interest for such instruments have increased rapidly over the last decades.

At the opening in 1973 CBOE wrote options on 16 traded stocks. At the end of 2014 the *open interest* [1] of equity options at CBOE was close to 210 million contracts

[1] Open interest - The total number of open or outstanding (not closed or delivered) options contracts that exists on a given day.

(CBOE 2014). Later on in 2016 the SPX options average traded daily volume were over 1 million contracts CBOE (2017). SPX being the traded fund tracking the S&P 500 Index - the 500 biggest companies in the US.

Technology

The introduction of information technology during the early 1990's was probably the main driver behind the popularity of options as we know it today. While trading over telephone and telegraph was disruptive in their own sense, the use of computer technology has introduced a whole new aspect to options trading in terms of liquidity, interest and range. Markets are available instantly at any time of the day to anyone with access to a computer. More market makers and lower entry barriers for brokers have driven the transaction costs to all time lows. The entry of new technology also blurred the difference between OTC [2] and exchange traded securities as market makers could more easily connect investors.

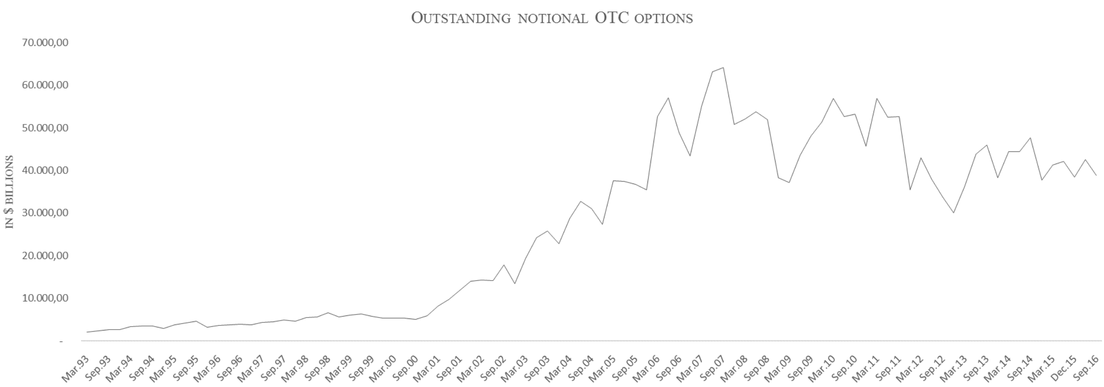


Figure 2.1: The semi-annual growth in the OTC options market 1993 - 2016. ***Source: Bank of International Settlements http://stats.bis.org/statx/srs/table/d1***

Starting as speculative bets in unregulated marketplaces, derivatives have developed to be a substantial part of the global economy. Figure 2.1 displays the development and growth in the OTC options market as valued by The Bank of International Settlements (BIS). Peaking in 2007 before the financial crisis the outstanding notional amount[3] of options have seemed to stabilized somewhat in the following years on a level around $40.000 billion on average. Even though that is a tremendous amount, options is only a part of the total global outstanding notional amount of the derivatives market that is measured to exceed $544,000 billion as of 2016 (Bank of International Settlements 2016).

[3]Over the Counter (OTC) refers to bilateral trading between two counterparties without the supervision of a exchange or clearing house.

[3]The notional amount refers to the gross nominal value of all options concluded and not yet settled on the date of reporting.

Thus with the increasing use of options in the financial markets, the need for correct pricing is of utmost importance. We therefore find further motivation in testing established option pricing models in order to see what model should be preferred to price contingent claims.

2.2 The use of Options

Throughout history the use of options has limited itself to the art of speculation on market movements and the delivery of commodities or goods. Only with the development of more sophisticated evaluation techniques and standardization of exchanges and trade, have the use of options extended that of speculation. In newer time the use of options as instruments for hedging purposes and management compensation have emerged and gained attention in both academia and practice.

Speculation

As mentioned in Section 2.1 it was the speculators that first took interest in the use of financial derivatives when making bets on market movements. The peculiar property of options is that bets can be taken in any direction, depending of the type or combination of options used. Speculators are able to place bets on positive, negative and sideways market movements with limited effort, and often little initial cost. Selling options without owning enough of the underlying asset to cover the position, also known as a naked option is also a way of utilizing the option market. While the markets have embraced and adapted the use of derivatives beyond speculation, not all share the enthusiasm. Warren Buffet, one of the greatest investors who have ever lived still see the derivative market as pure speculation. In Bershire Hathaway Inc. (2002) he refers to derivatives as ticking time bombs, and that there is only a matter of time before an event will make their toxicity clear.

Hedging

Options are in hedging used as protection against fluctuations in the value of the underlying asset and can be seen as insurance for the hedger. Investors might use options to take an offsetting position in the stock market in order to lower the volatility on their portfolio. Producers of goods might want to hedge the fluctuations in the costs of raw materials, and the producers of raw materials might hedge the price they receive for their raw materials. Thus the use of options for hedging purposes does not limit itself to the financial world. Following Figure 2.2 we can see how a hedged position on the underlying asset with a put-option aid in reducing the loss in the case of a decline in the price of the underlying compared to an un-hedged position. Further discussion on options as instruments for hedging can be found in Section 4.1.1.

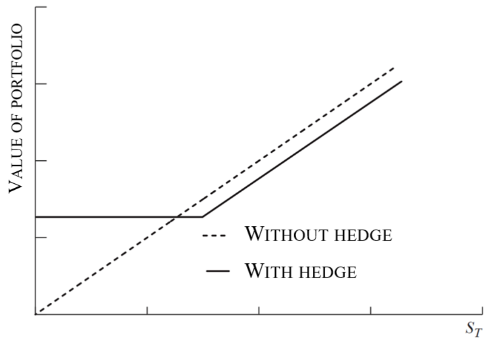


Figure 2.2: Hedging effect

EMPLOYEE COMPENSATION

Options have also found their way into managerial and employee incentive schemes. Being contingent claims, the use of options on stocks can create incentives for employees to work harder in order to increase firm value and thus their potential future reward. The actual benefit of using options as compensation have been discussed in academia for decades, but no clear answer has arised as to their effect on employee performance. The use is however, especially common in the US as a way of senior management compensation until this day.

3 Literature Review

"...every option pricing model has to make three basic assumptions: the underlying price process (the distributional assumption), the interest rate process, and the market price of factor risks. For each of the assumptions, there are many possible choices." – (Bakshi & Chen 1997, pp. 2003).

The literature regarding the pricing of contingent claims have become quite voluminous over the decades. This section is dedicated to outline the theoretical progression from the early 1900's until the latest proposed models we know of, and to make record of the literature that have helped research get where it is today.

The theory of option pricing can be traced back to Bachelier (1900) and his doctoral dissertation *Théorie de la spéculation*. As the first to mathematically approach option pricing and recognize the particular movement of asset prices he proposed the theory of a Brownian motion and the use of stochastic calculus. The concept of the Brownian motion with zero drift are considered one of the most influencing discoveries of the 21th century finance. His application of the theory to the mathematical modelling of price movements and the evaluation of options became the predecessor for what is today known as the topic of option pricing.

Early Option Pricing

Inspired by the work of Sprenkle (1961), Ayres (1963), Samuelson (1965) and others, Myron Black and Fischer Scholes proposed the Black-Scholes model for the pricing of European options. The BS model for pricing contingent claims where the first model to offer a closed form solution to the pricing problem. Previous researches had made the same assumptions underlying the model but had difficulties modelling the correct discount rate for the expected payoffs. Utilizing the capital asset pricing model (CAPM) to explain the relationship between the market return and required return of the underlying stock gave birth to the Black & Scholes (1973)[4] model.

[4] For more on the Capital Asset Pricing Model see (Sharpe 1964)

Merton (1973) extended the model to include the dividend rate, and made the model more sophisticated by further explaining the discount rate. Merton showed that the dynamic trading strategy in the BS-model used to offset the risk exposure of an option, would provide a perfect hedge in the limit of continuous trading - this without relying on the assumptions of the CAPM. That is, following the BS dynamic trading strategy, using the riskless asset and the underlying asset would yield an exact replication of European option payoffs.

As a recognition of their contribution to the option pricing theory Black, Scholes and Merton received the Nobel Price of Economics in 1997. Their model have been renowned as the most influential development in the history of finance practice (Merton 1998).

Several extensions to the BSM-model have been developed to price different types of contingent claims. Garman & Kohlhagen (1983) extended BSM to price foreign currency options, Black (1976) extended the BSM framework to price commodity options, while Margrabe (1978), Duffie (1988), Boyle (1988) amongst others have extended BSM to price exotic types of contingent claims where the underlying differs from stocks.

Even though the Black-Scholes-Merton model have been celebrated for its simple procedures and numerical computational ease, the model has its deficiencies. The BSM model rely heavily on assumptions such as a constant measurable volatility, that asset prices follow a random walk, returns on assets are normally distributed, there is no transaction costs in the market and that the markets are perfectly liquid. These assumptions have proven themselves to be unreasonable and therefore demonstrates systematic bias (Rubinstein 1985).

Empirical studies on the Black-Scholes-Merton model report that the BSM prices deviates from the market prices. A study performed by MacBeth & Merville (1979) found that BSM prices on average are less (greater) than market prices for OTM (ITM) options. While Black (1975*a*) report that the model overprices deep ITM options and undervalues deep OTM options. In another study, Cohen et al. (1972) documents that the model underprices calls written on low volatility stocks and calls with short maturities.

The studies of the BSM model show that the model are prone to substantial mispricing. The mispricing is attributed to its many unreasonable simplified assumptions, as asset returns empirically does not fit these assumptions. While the BSM model assumes log-normal returns, financial asset return tend to be fat-tailed in their distribution (Hull 2015). This cannot be replicated through Gaussian modelling (Boyarchenko & Levendorskii 2002). The assumption that volatility is constant have also been deemed poor in relation to observed data. When volatility is backed out or implied from observed option prices and plotted against maturity and moneyness, the result displays a smile/smirk. The smirk deviates from the flat surface predicted by the model. Trying to improve the BMS-model, the ad hoc Black–Scholes model of Dumas et al. (1998) uses a separate implied volatility for each option to fit to the smirk/smile in implied volatility.

Another empirical observations found in financial time series is that times of market uncertainty tends to be followed by more uncertainty. The phenomenon known as volatility clustering challenges the assumption of the BSM model and there is now substantial evidence supporting that variance change over time (Rosenberg et al. 1972, Blattberg & Gonedes 1974, Black 1975*b*, Latane & Rendleman 1976, Schmalensee & Trippi 1978).

Basic Numerical Procedures

Cox et al. (1979) developed a discrete time model of option pricing based on binomial movements in the underlying. The method is practical for American options, and options with some alteration to the stock diffusion (e.g. one time dividends) as it is easily altered throughout any specific period of time. It relies heavily on non-arbitrage arguments. As we are not using this method we refer to Hull (2015) for more applications and alterations.

Schwartz (1977) developed a Finite Difference method using partial differential equation (PDE) based on BSM. This is to solve issues when dividends are paid discrete, or simply to find the optimal exercise strategy for American options. I.e. the same issues as the binomial tree from Cox et al. (1979) handles. The finite difference method can be split up to Implicit Finite Difference and Explicit Finite Difference - which according to Hull (2015) is equivalent to the lattice based trinomial tree that was developed by Boyle (1986).

Changing Volatility

The insecurity in the stock market has always been present, but it wasn't before mid 20th-century that solid theories on how to measure this insecurity came clear. Economists as Markowitz (1952) and Sharpe (1966) started to quantify the risk in terms of standard deviation, and combining it with expected returns for performance analyses. Time series models for forecasting stochastic processes were also under development at the same period, as Whitle (1951) introduced the ARMA-process.

One of the most simple time series models is the Moving Average (MA)-model, which combines several white noise processes in order to model a stochastic process with lagged error terms. Another basic model is the Autoregressive Process (AR) process that lags its previous values (Brooks 2014). Combining the models yields the Autoregressive-Moving-Average (ARMA)-process in which creates a weakly stationary model that incorporates lags. The ARMA model were further developed to the ARIMA model which integrated the change of previous realizations in an alternative way. Box (1970) generalized and implemented parameter estimation methods to the models. All of which were improvements in order to fit, and better predict time series.

The increased interest in volatility modeling continued when Engle (1982) proposed the Autoregressive Conditional Heteroskedasticy (ARCH) model. The ARCH-model has similarities to a MA-process where volatility depends on previous squared innovations. Generalizing upon his results Bollerslev (1986) later proposed the Generalized Autoregressive Conditional Heteroskedasticy (GARCH) model. The model has become the most popular econometric model for forecasting conditional variance and has been extended numerous times over the last decades. Engle & Ng (1993) proposes the NGARCH that account for the leverage effect or asymmetry of positive and negative shocks to the conditional volatility. Nelson (1991) develops the exponential GARCH or EGARCH to ensure that the conditional volatility stay positive without restrictions on the parameters. His model also accounts for the leverage effect, but have met criticism as the logarithmic transformation could lead to biases in forecasts of the level of future variance. Engle & Bollerslev (1986) constructs the integrated GARCH or IGARCH where the persistent parameters sum to one - making the volatility of today affect the volatility into the indefinite future. In an attempt to capture the asymmetry of the leverage effect Glosten et al. (1993) models the GJR-GARCH by implementing a threshold making negative shocks yield greater impact on the conditional variance. Bearing resemblance to the GJR-GARCH, another threshold GARCH model was implemented by Zakoian (1994). Nelson & Cao (1992) introduced some new restrictions to the basic GARCH to make sure that the volatility stays positive. Klüppelberg et al. (2004) developed a continuous-time generalization in COGARCH as opposed to Bollerslev's discrete time model. The Family GARCH (FGARCH) set up by Hentschel (1995) is a compilation model of many of the various popular symmetric and asymmetric GARCH models.

Awartani & Corradi (2005) evaluates the relative out-of-sample predictive ability of different GARCH models at different horizons. Using daily data of the S&P 500 from January 1990 to September 2001, they use pairwise comparisons of the GJR-GARCH, EGARCH, NGARCH amongst others against the GARCH(1,1) model. For the one-step-ahead comparison against the GARCH(1,1) model using squared returns as a proxy, all the asymmetric models provide the best fit. In the case of multiple comparison, the GARCH(1,1) are again outperformed by the asymmetric models, but not by other GARCH models that do not allow for asymmetry. Overall the EGARCH show the best performance with lowest mean-squared-error (MSE) in-sample. The authors argue that their findings points towards that asymmetry play a crucial role in prediction volatility.

Using data from October 1992 to May 20115 of the Tel-Aviv index TA25, Alberg et al. (2008) found that asymmetric models improve the forecasting performance relative to their symmetric counterparts. Furthermore, the EGARCH model with a student-t skewed distribution outperformed the GARCH and GJR-GARCH in the Tel-Aviv market.

Peters (2001) find that noticeable improvements can be made when using an asymmetric GARCH model for the conditional variance. Using data from January 1986 to January 2001 on the FTSE 100 the authors results show evidence that GJR-GARCH seem to outperform the EGARCH model using MSE, MAE and Adjusted MAPE among other tests to compare performance.

Option valuation under changing volatility

Following the criticism of the Black-Scholes-Merton model, the option pricing theory divided itself into two main strains in order to capture the empirical fact of changing volatility according to Cox et al. (1981). One class utilizes the continuous stochastic theory where volatility follows a diffusion process, and the other follow a discrete time GARCH modelling of volatility. Both classes take advantage of the concept of risk neutrality developed by Cox & Ross (1976), stating that the discounted price process is a martingale[5] and extend upon the theory.

The theory of stochastic volatility option pricing can be traced back to Hull & White (1988) who suggest that stochastic return volatility may be causing the BS biases. Their model uses stochastic volatility where the price process follows a Wiener diffusion process. Another important model is the stochastic volatility closed

[5] A martingale process is a process where past events cannot help determine future events, only current events matter. See Hall & Heyde (2014) for a more thorough explanation.

form affine model of Heston (1993). The model is the first closed-form option pricing model with stochastic volatility, greatly improving the computational burden from earlier models. The model manages to capture both volatility clustering and the leverage effect and have been proved effective in many empirical studies. The model does however have some drawbacks as its affine square root structure fail to capture some stylized facts according to Christoffersen et al. (2007). Testing the Heston (1993) model against a class of non-affine stochastic models Christoffersen et al. (2007) find that non-affine models in general are superior in terms of likelihood values from a long time series of returns and a root-mean-squared-error (RMSE) comparison.

Further extensions of stochastic option pricing models are utilized by Melino & Turnbull (1990) who develop a model for pricing spot foreign currency options. Letting the interest rate be stochastic provides good fit for observed option prices. Further Bates (1996) develops a method for pricing American options on stochastic volatility/jump [6]-diffusion processes under systematic jump and volatility risk. The model fails to replicate the volatility smile in a satisfactory manner.

Christoffersen et al. (2009) extend the Heston (1993) model by incorporating a new factor into the conditional volatility hoping that a more richly parameterized model will help to better explain the observed data. The authors find that their extension greatly improves the models ability to capture the volatility term structure and the level and slope of the volatility smile outperforming the Ad Hoc Black-Scholes-Merton model. Heston & Nandi (2000) further report that stochastic continuous time models can be hard to implement but effective for option pricing. It is very difficult to filter a continuous variable from discrete observations such as discrete returns of the S&P 500.

Taking stochastic volatility into accordance in the search of improving the BSM formula is of first-order importance according to (Bakshi & Chen 1997) that develops a stochastic model and test it through hedging, out-of-sample pricing and internal consistency of parameters with observed data.

The other class of models explore the discrete time GARCH framework as the process path of the volatility. The GARCH models have become successful in the literature due to their relatively few parameters and computational ease together with the fact that the volatility can be observed directly from discrete asset price data.

[6]Jumps in terms of volatility modelling refers to the use of conditional distributions of the innovations that allow for sudden large movements (Eraker et al. 2003). Allowing for the replicating large conditional movements in the volatility such as financial crises.

As one of the first to explore the connection between option prices and the GARCH framework Engle & Mustafa (1992) try to estimate the implied stochastic process of the volatility of an asset from prices of options written on the same asset. The first attempt to price European contingent claims using a GARCH approach can be traced to Amin & Ng (1993), proposing a GARCH based option pricing model that did not work due the the violations of the risk neutral valuation relationship. Building on their findings, Duan (1995) managed to generalize a locally risk-neutral valuation relationship (LRNVR) needed for the correct pricing of European options following the conditions of (Cox & Ross 1976). Successfully transforming the physical GARCH process into a risk-neutral equal process he uses the asymmetric NGARCH model of Engle & Ng (1993) to estimate GARCH parameters under the risk neutral measure using a single day of option prices and try to calculate prices.

Duan (1995) argues that the one period ahead conditional variance does not change with changes in the risk-neutralized pricing measure under LRNVR. He further demonstrates that this is important as in GARCH processes any conditional variance beyond the one-period ahead does change with changes in measures caused by risk neutralization. Using this finding he see that the process changes from a chi-squared innovation to a non-central chi-square random variable in the conditional variance. Comparing his GARCH models with the conventional Black-Scholes, Duan (1995) finds that in a low-variance state, the GARCH delta hedging calls for smaller portions of options relative to the BS. The models also capture the fat tails and skewness of the conditional returns. Many later models build on Duan's framework, as this model captures both current and lagged innovations, while the BMS-framework is based on current spot prices and constant volatility.

Defining a special case of affine equations for the conditional volatility and return process for GARCH(1,1), Heston & Nandi (2000) develops a closed form model for valuating European call options under the discrete GARCH framework. Substantially easing the computational burden from earlier GARCH option valuation models that require Monte Carlo simulation in order to obtain the prices. Their HN-GARCH model also allow for a leverage effect and volatility clustering.

Duan et al. (2002) extend the NGARCH of Duan (1995) to include jumps in returns and the conditional volatility. They find that the jumps created by a Poisson distributions help to generate fat tails and thus have a better data fit than the regular NGARCH model for all moneyness and maturity.

In an attempt to create an approximation for a closed-form solution Duan et al. (2006) provide the risk neutralization of the price process for the EGARCH and GJR-GARCH models. Constructing analytical approximations to the models by considering limiting models of the GARCH-jump process, the authors find that their models perform good in terms of price fit for short maturities, but that for longer maturities the fit is poorer.

Ritchken & Trevor (1999) construct a binomial lattice approach for pricing American options under GARCH and establishes that a range of GARCH models converge to a rich family of stochastic volatility bivariate diffusions. By constructing the NGARCH by Engle & Ng (1993) in the lattice framework they find that the NGARCH has a limiting bivariate diffusion that include the model of Hull & White (1988).

Härdle & Hafner (2000) observe that out-of-the-money options strongly depend on the volatility specification. OTM calls may be severely overpriced by assuming symmetric shocks and thus empirically test their TGARCH model that incorporate asymmetry. Using option data on the German stock index (DAX) maturing in 1992, they find that their TGARCH prices outperform both the BS and NGARCH. The authors estimate the GARCH parameters using time series of returns under the physical distribution and use these parameters in the valuation measure.

Expanding the model of Engle & Ng (1993), Christoffersen & Jacobs (2004*b*) find evidence that richer parameterization of the NGARCH model does not improve the models fit in regard to option valuation out-of-sample. In-sample comparison makes the more richly parameterized models perform slightly better using a dollar MSE as performance measure.

Christoffersen et al. (2005) find that NGARCH outperform HN-GARCH by 7% in sample and 9% in sample. The NGARCH outperform Heston (1993) by 10% in-sample and 6% out-of-sample. The difference between discrete and stochastic models is not large and the models are quite similar in pricing terms. The main differences in fit can be found in affine vs non-affine models. The article argues that the price of having a closed form model is poorer fit to the data. They also find that differences in fit are mainly dependent on time to maturity. The article utilizes RMSE to estimate the parameters using in- and out-of-sample Wednesday data.

Using a Value-at-risk, relative performance error and absolute relative performance error measures Lehar et al. (2002) compare the performance of the BS model, the stochastic volatility model of Hull & White (1988) and the NGARCH option pricing model of Duan (1995). The authors concludes that the NGARCH model outperform the stochastic volatility and the Black-Scholes model in terms of pricing errors. No measurable differences were found in the VaR measure between the models as all displayed a poor fit to the realized profit and losses. Furthermore, the NGARCH model fails to forecast the tails of the return distribution.

Investigating the relative performance of the NGARCH and HN-GARCH option pricing models, Hsieh & Ritchken (2005) find that both models manage to explain significant amounts of the maturity and strike bias produced by the BS model. Even when not re-estimated for periods over one year, both models display satisfactory fit compared to observed data. Their concluding remarks state that the non-affine NGARCH model exhibit better fit than the affine HN-GARCH, especially for deep OTM contracts.

Studying option data from the TXO (Taiwan) option market on a sample from 2003 - 2008 Huang et al. (2011) find that GARCH outperforms BS and a stochastic model using Absolute Relative Pricing Error (ARPE) as a performance measure. Their results using a regression on ARPE find that the pricing error is decreasing in time-to-maturity and moneyness.

Su et al. (2010) study the performance of the HN-GARCH against the ad hoc Black-Scholes. Their findings suggest that the HN-GARCH display lower pricing errors for both the updated and non-updated model compared to the ad hoc Black-Scholes using RMSE, MAPE, MAE and RMIVE as measures. The results hold for both in-sample and out-of-sample comparisons.

The recent focus within option pricing literature seem to be to a large extent revolving around the discrete GARCH framework. The NGARCH and HN-GARCH have been the topic of several empirical studies, whereas most of the remaining GARCH models have yet to endure voluminous testing. This thesis aim to fill that gap in the existing literature.

4 Theory of Options

This sections purpose is to look at the underlying theory behind options, how they work as well as how their behaviour relates to, or complements, the underlying asset. In order to fully comprehend the analytical part of this thesis we believe that an introduction to options theory is fundamental. We will fist present pure option theory before we look at some stylized facts found in financial time series.

4.1 Fundamentals of Options

In the derivatives world options are fundamentally different from futures, forwards and swaps. Where the latter instruments commits buyer and seller of a contract to a future obligation, the options gives the buyer a right but not the obligation to do something. Contrary to futures, forwards and swaps that have zero up-front payments ignoring possible margin, or collateral requirements, options have an initial cost (Hull 2015).

There are two main types of basic options; call and puts. A call gives the buyer an option to buy an asset for a certain price, while a put the right to sell an asset for this price. The price is determined at purchase and is referred to as the exercise price, or strike price - commonly denoted as K. The most common types within these options are European and American. For European options the predetermined maturity date is the only time for exercising the option, while for American options the holder can choose to exercise at any time before maturity. There is many alterations, or hybrids, to these plain vanilla options[7]. Bermudan-, Binary-, Weeklys- and Doom-options are examples, but as we only look at European call options we will not go deeper into these exotic options[8]. Similarly, are options written on many types of underlying assets, as foreign currency, futures etc. - but we will focus primarily on Index and Stock options.

[7]Plain vanilla - The most basic, or standard version of a financial instrument.

[8]Exotic - A financial instrument more complex than the plain vanilla versions.

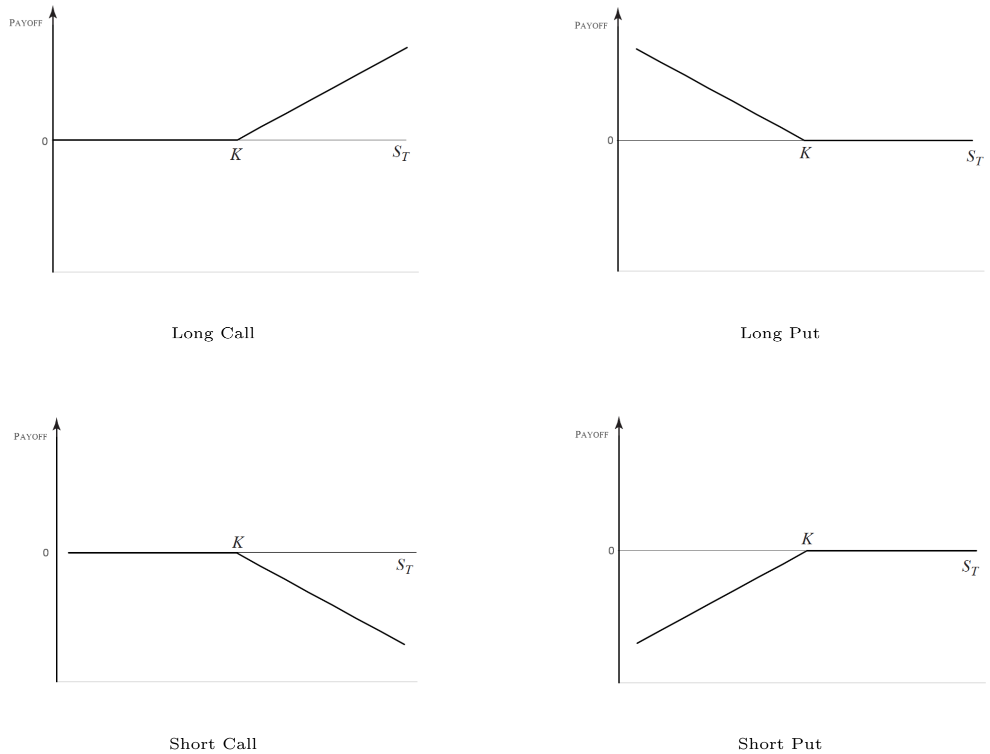


Figure 4.1: Illustration of various option payoffs

It is common to differentiate between 4 different positions you can hold in regards of options. At time of purchase the buyer of a long position has to pay the option price, that the seller with a short position receive. At maturity the seller will have to payout in case of exercise by the buyer. Table 4.1 and Figure 4.1 display the payoff structure of plain vanilla European options on any underlying linear asset. Here C and P is the option prices of a call and a put respectively. While S_T is the stock price at time of expiration, T.

If gambling on movements in the underlying, a long call (short put) would represent a belief in upwards moving stock price. While gambling on a decrease in prices would similarly be achieved with a long put (short call). It is worth noticing that written or sold positions will never have a higher payoff than the initial price of the option. Going long in a call option will potentially have a infinite up-side at maturity, similarly the short call options will have infinite down-side. Put positions have a maximum payoff equal to the strike price when the underlying asset price fall to zero.

The options at maturity will either yield a payoff equal to the increase in the spot above the strike, or zero. Common terminology for the payoff of the options at any given time is referred to as in-the-money (ITM), at-the-money (ATM) and out-of-the-money (OTM). ITM options are the options that have a positive payoff where $S > K$, ATM options are option with zero payoff as $S = K$ and OTM options where $S < K$. The different states are often represented by a measure referred to as moneyness. Moneyness is defined as S/K where values below 1 is OTM, above 1 is ITM and at 1 is ATM. It is also worth noticing that the terminology can be used at any point in time, not only at maturity. So for instance if the stock price is above the strike, a call option could still be referred to as ITM - even if the expiration date is not close.

	Payoff	
	t=0	*t=T*
(a) Long Call	$-C$	$Max(S_T - K, 0)$
(b) Short Call	C	$-Max(S_T - K, 0)$
(c) Long Put	$-P$	$Max(K - S_T, 0)$
(d) Short Put	P	$-Max(K - S_T, 0)$

Table 4.1: Payoff structure of regular stock options

Arbitrage arguments to the price

When pricing European options the main sources of insecurity is what the stock price will be at maturity. Option pricing models have different approaches to conquer this, of which many are based on some kinds of no-arbitrage arguments, often accompanied with some risk-neutral valuation relationship similar to the one that are explained further in Section 5.2.5. Some theoretical principles are, however, independent on which model is used. One of which are the lower bound of the option price:

$$S_0 - Ke^{-rfT} \tag{4.1}$$

Where rf denotes the risk-free interest rate. Equation 4.1 defines the lower bound for the price on a European call option and the theory behind is quite straight forward. At the point of maturity an investment in a call and a risk-less investment providing a payoff of K at time of expiration must be greater, or equal to the stock price (Hull 2015).

$$max(S_T - K, 0) + K \geq S_T \tag{4.2}$$

Similarly, this will have to be true today in the absence of arbitrage:

$$C + Ke^{-rfT} \geq S_0 \tag{4.3}$$

or

$$C \geq S_0 - Ke^{-rfT} \tag{4.4}$$

If $C < S_0 - Ke^{-rT}$ arbitrage can be made through forming a portfolio where one short one share, buy one call and purchase Ke^{-rT} risk-free bonds maturing on the expiry date. At maturity the portfolio will earn zero if $S_T < K$ or a positive value $K - S_T$ if $K > S_T$. Similar arguments can be made to derive the theoretical lower bound of a put option:

$$P \geq Ke^{-rfT} - S_0 \tag{4.5}$$

Breaking Down Option Prices

The price paid for an option is often referred to as the option premium. The option premium paid again consists of two factors known as intrinsic value and time value. The intrinsic value is directly speaking the payoff above the strike price if we look at a European Call option, and represent the fundamental true value of the option. The intrinsic value is therefore dependent on the price of the underlying and the strike price of the option at any given time. However, the price tends to be a bit higher than the intrinsic value itself, this difference is referred to as the time value. This value reflects the potential for change in the underlying and are higher the longer time to maturity. This is intuitive as the option have longer time to end up ITM at maturity. The time value is dependent on the volatility of the underlying, the risk free rate, and the time left to maturity.

$$C_{Observed} - C_{Intrinsic} = time\ value\ T \tag{4.6}$$

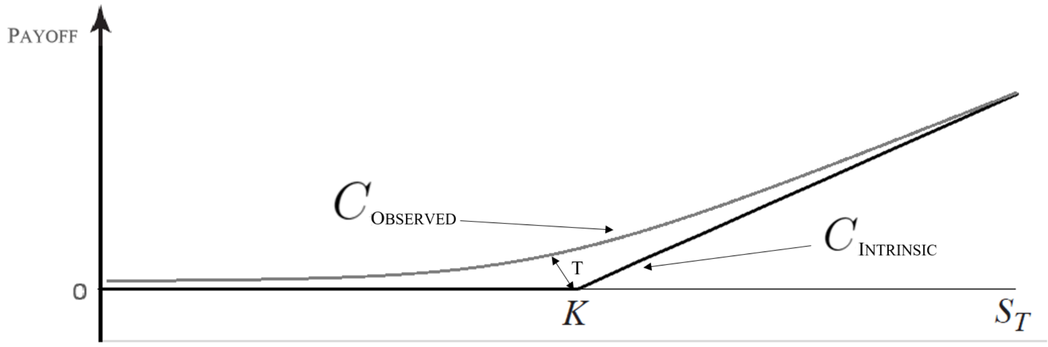


Figure 4.2: Intrinsic Value & Time Value

The put-call parity

Another useful property of options is the put-call parity:

$$P + S_0 = C + Ke^{-rfT} \tag{4.7}$$

The parity defines the relation between a call option price and put option price of instruments written on the same underlying with the same strike price K. The derivation is based on quite simple arbitrage-free arguments. We can consider left hand side and right hand side of the equation as two portfolios. Portfolio 1 has a put (P) and the underlying stock (S_0), while portfolio 2 has a call (C) and a risk-free investment providing (K) at time of maturity. At the time of maturity, we will then have following scenarios:

Payoff under scenarios of the stock price		$S_T > K$	$S_T < K$
Portfolio 1:	Put and stock	$0 + S_T$	$(K - S_T) + S_T$
		$= S_T$	$= K$
Portfolio 2:	Call and risk-free	$(S_T - K) + K$	$0 + K$
		$= S_T$	$= K$

Table 4.2: Payoff structure of the put-call parity

Without any opportunities for arbitrage the value of the portfolios will have to be equal today as well. The parity requires no assumption about the probability distribution of the underlying and can be applied to any model (Hull 2015).

4.1.1 Hedging

As briefly mentioned in Section 2.2 is the use of options not only for gambling on stock price movements. When the seller of an option does not possess the underlying at the time of sale, it is called a naked position. There are some regulations to this as the potential down-side can be impairing, and without any profound belief, a hedging strategy will be a far safer option. It is common to hedge all kind of positions, and even with the plain vanilla options described earlier there is endless combinations that can yield a satisfactory payoff structure. The greatest limit to those is that you only can hedge positions involving a traded asset in which there also is written an option upon. Or at least assets that are highly correlated to the underlying of the option. The put-call parity in equation 4.7 offers greater insight when constructing different structures on options. This relation is useful not only to find put (call) prices when having the call (put) price, but also constructing portfolios. For instance can a regular put be mimicked by buying a call, shorting the underlying and placing Ke^{-rT} amount of cash in a fund offering the risk-free rate, rf.

Several structures such as different kinds of spreads and combinations as straddles, strips, straps and strangles creates creative payoff structures from simple plain vanilla options. Key to many of these is to make combinations of different option types, strikes and even maturities (Hull 2015). One of the most basic hedges where the investor owns a covered put option with the underlying is illustrated in Figure 2.2.

4.2 Stylized Facts About Financial Time Series

Financial time series are known to empirically display some characteristics that are not in accordance with many option valuation models assumptions. That return time series are not Gaussian when observed is one example of characteristic that make many models struggle to replicate the observed prices with great accuracy. In the following section we will outline some of these characteristics.

4.2.1 Kurtosis

Kurtosis is the measure of a distributions tails, and is referred to as the fourth standardized central moment of a distribution. The kurtosis[9]of a normal distribution equals three, so in terms of standardization, excess kurtosis is measured and three is withdrawn in order to make comparison easier. Leptokurtic kurtosis is when the distribution is more peaked around its mean and display fatter tails than a normal distribution. The platykurtic is much "lower" and display thinner tails as shown in Figure 4.3 below.

Empirically financial time series returns are known to display leptokurtic kurtosis thus having fat tails. Fat tails means that even though the majority of returns lie around the mean, the probability for more extreme outliers is higher than in the

[9]Kurtosis for a random variable X is calculated as Kurt$[X] = E\left[\frac{(X-E[X])^4}{Std[X]^4}\right] - 3 = \frac{E[(X-E[X]^3])}{Std[X])^4} - 3$

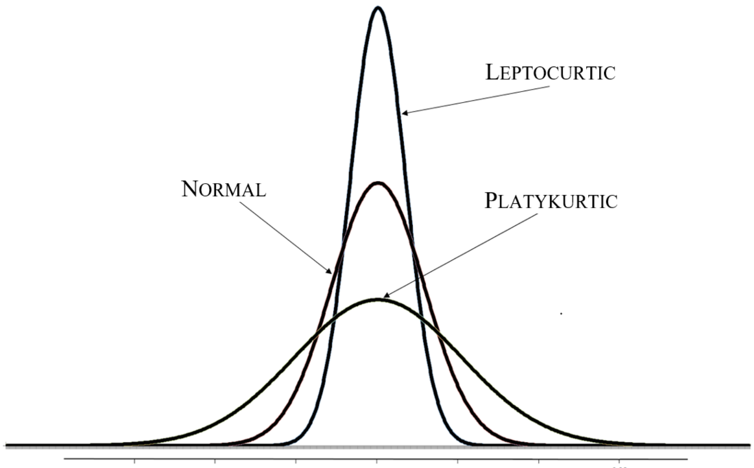


Figure 4.3: Kurtosis Types

normal distribution. In a financial setting these outliers will be synonymous to large gains or losses.

4.2.2 Skewness

The skewness of a distribution is defined as the asymmetric property of the distribution of a random variable X and is referred to as the third standardized central moment[10]. For the negative skewed distributions, most of the probability lies left of the mode and vice versa for positive skewed distributions. Examples of positive skewed distributions is the log-normal distribution whereas for normal distribution, or other symmetric distributions the skewness is zero. This even though the distributions have the same mean and standard deviation.

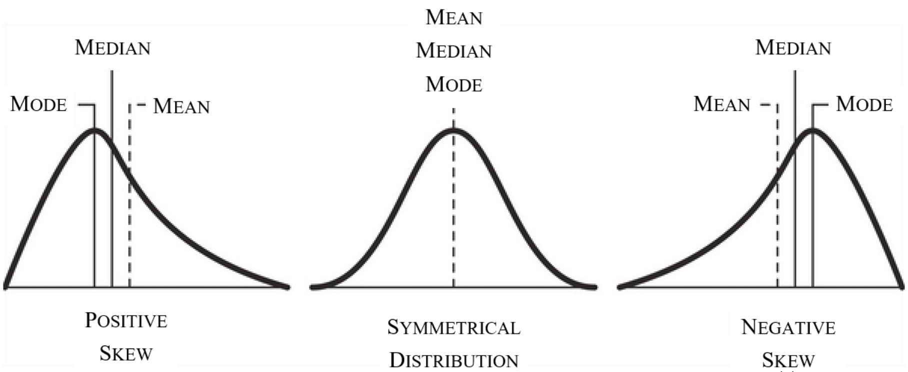


Figure 4.4: The three possible types of skew [11]

Empirically, asset log-returns tend to be positive skew, but one should be careful to

[10]Skewness for a random variable X is calculated as Skew$[X] = E\left[\frac{(X-E[X])^3}{(Std[X])^3}\right] = \frac{E[(X-E[X]^3])}{(Std[X])^3}$

[11]Source: https://www.safaribooksonline.com/library/view/clojure-for-data/9781784397180/ch01s13.html

generalize on this matter. Before the financial crisis of 2008, returns was positively skewed after long periods of positive returns, but in the subsequent periods return was highly negative skewed. However in most cases of financial time series, the return is rarely perfectly normal in its distribution.

4.2.3 Volatility Clustering

Volatility clustering is a stylized fact in asset returns, and are referred to as the positive autocorrelation of volatility over longer periods. The fact that volatile periods tents to be followed by volatile periods, and calm periods followed by calm periods is often observed in the data. From Figure 4.5 one can identify the clustering of high returns in several periods. The period of return stretches over the financial crisis of 2008, as we can see from the clustering.

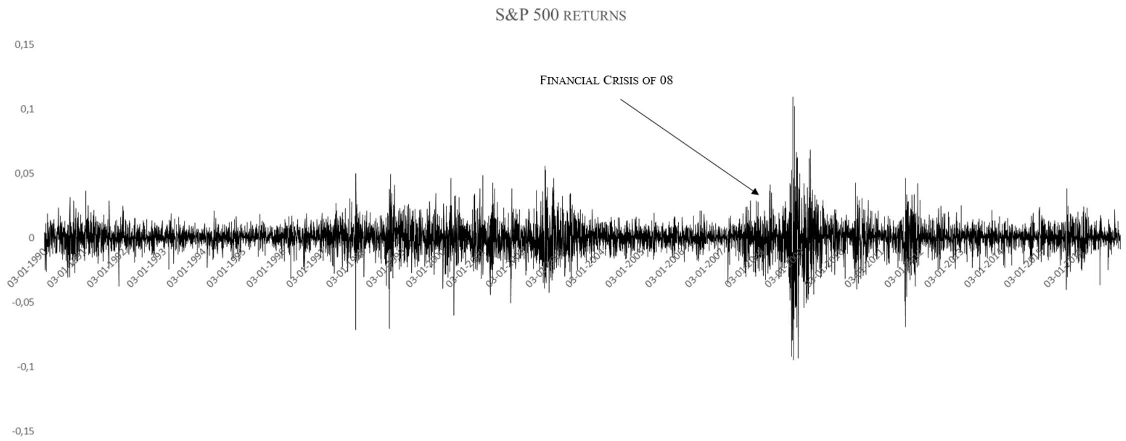


Figure 4.5: S&P 500 returns 1990 - 2017, Source: Datastream

4.2.4 Leverage Effect

The leverage effect is an asymmetric effect that captures the empirical fact that negative returns have a greater influence on future volatility than positive returns of equal magnitude. In other words, the asymmetry represents the negative correlation between an asset return and its changes in volatility. The effect was first discovered by Black (1976) and have later been confirmed in several empirical studies according to Engle & Ng (1993). Empirically increasing asset prices are followed by lower volatility and vice versa will falling prices lead to more insecurity, thus more volatility. The explanation behind the "leverage" arises from the debt equity relation in firms, as firms value of debt increases relative to the equity when the asset prices decline, the stock become more volatile. This is still only a hypothesis, but the term has been adopted by academics to explain the statistical regularity in question according to Ait-Sahalia et al. (2013).

4.2.5 Volatility Smile

When pricing options a big challenge is to find the correct volatility to use. As the volatility is not observable, the implied volatility has become a popular measure. The implied volatility is the volatility backed out from the Black-Scholes-Merton model to arrive at a given observed empirical price. This measure will be equal for both put and calls as long as same spot, strike and underlying asset is used. If plotting Implied Volatility as a function of the strike price of an option at a certain point in time, we get what is commonly known as the volatility smile. The volatility smile will look the same for both call and puts for that given strike price (Hull 2015). Empirical findings does however point to different shapes of volatility smiles depending on the class of underlying asset the option is written on.

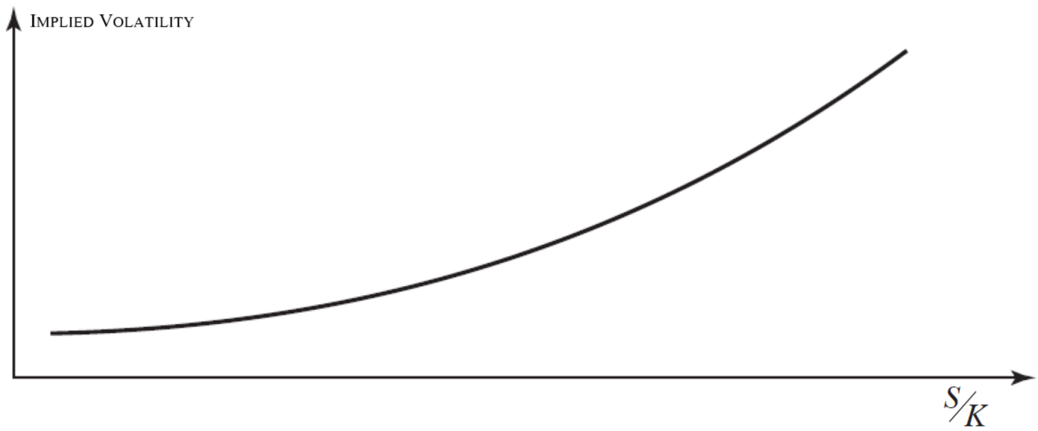


Figure 4.6: Historical Volatility Smirk of Equity Indexes

Since the Stock Market Crash in 1987 the volatility smile has been used by traders to price stock options. The form differs from for instance Foreign Currency Options which actually looks more like a "smile" and Figure 4.6 shows why it for stocks sometimes are referred to as the *volatility smirk*. While it is common to plot the smile on different strikes, please note that we use moneyness, S/K on our x-axis. This is to be consistent with later results when implied volatilities have different levels of stock prices. To interpret the plot here we will then have to consider it as from one point in time, where S is constant while K is the changing variable. Main take-away is anyway that ITM calls tend to have a higher implied volatility than OTM calls Hull (2015).

The implications of using higher volatility to price low strike options compared to high strike options are not as benign as it might sound. If we consider Figure 4.7 we see that it has implication to the probability distribution of underlying returns. Here a log-normal distribution with the same mean and standard deviation is compared to the implied distribution. Let us consider a OTM call with high strike price, K_2. We see from Figure 4.7 that the stock price has a lower probability of exceeding this strike under the implied distribution than the log-normal, thus we expect a low observed price on the call. A low implied price would be equivalent with low implied volatility, which is confirmed by Figure 4.6 for high strikes. Same argumentation for a put with strike price K_1 will show why low strike options have a high implied volatility. It is, however, important to remember as we mentioned earlier, that the skewness can be different over time.

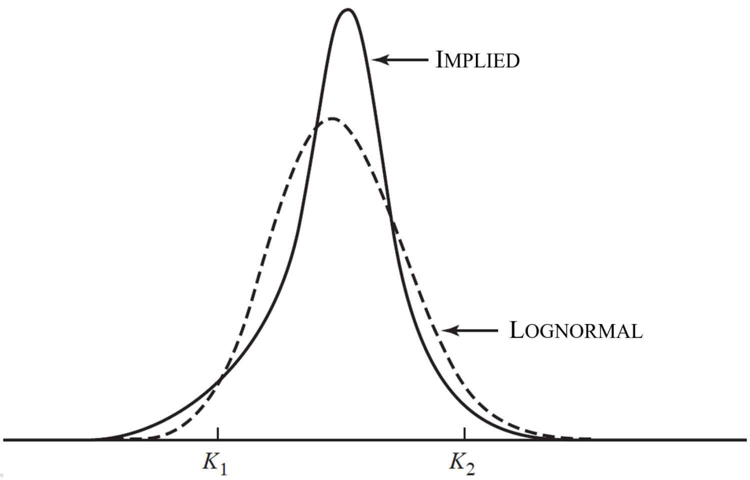


Figure 4.7: Implied return distribution of stocks - need Ks and S on axis

A possible explanation to this observation can be the leverage effect explained in the previous section. Figure 4.7 is also consistent with a negative relationship between the stock price and volatility. More equity in a firm-setting yields lower leverage, which in turn is less risky in terms of volatility. Another theory is *Crashophobia*, which suggests investors being worried about market drops and will affect prices accordingly. This theory is partly supported with evidence that before Black Monday in 1987 the pattern were different and now the smirk is more steep in times of a low stock price (Hull 2015). Many models, including Black-Scholes assume a log-normal distribution, but as we see is this not necessarily realistic. The implied volatility indicates a heavier left tail while accompanied with a smaller right tail - a negative skewness.

4.2.6 The VIX Index

We saw in the previous section that the implied volatility is a measure with a lot of information. Besides information about return distributions and option prices we see from Figure 4.8 that implied volatility is particularly high in times of financial crises. The VIX Index is a Volatility Index (VIX) offered by CBOE tracking the implied volatility on S&P 500 Index Options between 23 and 37 days to maturity. We see that the implied volatility is high around the Dot-Com burst around year 2000, the Financial Crisis of 2008 and the European Debt crisis after 2010. For this reason VIX is sometimes referred to as the "Fear Index".

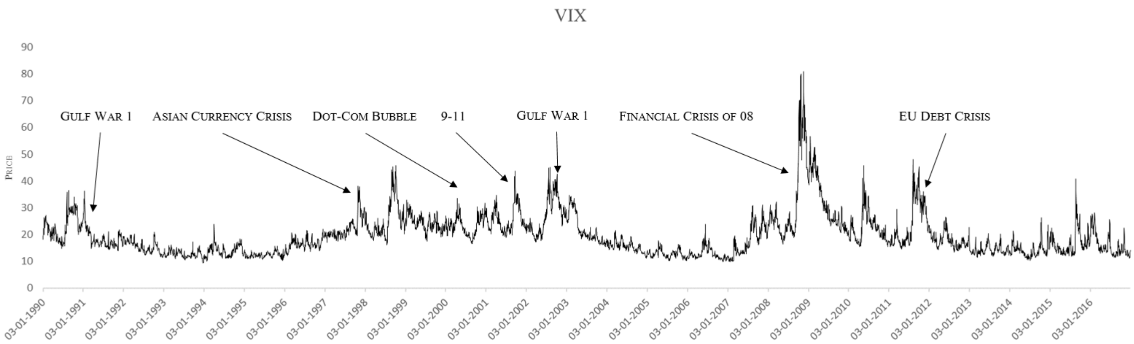


Figure 4.8: CBOE VIX Index with causes to high volatility, Source: Datastream

Notice that this is not based on historical volatility, which also might be high in times of distress. The implied volatility is rather a measure of market temperature. A high implied volatility points to high volatility expectations. A way to see it is that when the economy show signs of weakness, or uncertainty, demand for put options increase as they can hedge the stock portfolio. Furthermore will this higher demand increase prices and yield higher implied volatility.

As for stock price movements there are several trading strategies proposed based on trends in the VIX Index. For instance "When VIX is high, it's time to buy. When VIX is low, it's time to go." This justified with high implied volatility indicating a pessimistic market, that are likely to turn. Having a portfolio with a positive dependence on the stock price ($\Delta > 0$) and a negative dependence on volatility ($\nu < 0$) are then argued to be favourable as stock prices will increase and volatility decrease (Fisher 2016)[12]. But as any trading strategy there is still much insecurity, and furthermore, theories about dominant strategies being extinguished when becoming common knowledge. The focus in our paper will anyway not be to analyze such trading strategies.

[12]For more information about measures of portfolio sensitivities look at the Greeks Letters in Hull (2015)

5 Theory of the Models

In this chapter we present the theory of the option pricing models along with the theory behind and performance measurement procedures. First we present the theory behind the pricing models of constant and changing volatility, before we look at the estimation procedures and numerical procedures for pricing. Conclusively we will present the performance evaluation models that we will use to compare the pricing models. The section will only present the overall theory needed to perform an empirical analysis, and not the mathematical derivations, or proof of validity. For this we refer to the original authors cited.

5.1 Black-Scholes-Merton

The Black-Scholes-Merton (BSM) model is the option pricing model that have gained most accreditation for making option pricing theory applicable for practicioners as well as students of finance. Their model nest from the theory where the stock price process following a geometric Brownian motion [13] and is presented as a convenient closed form solution for the analytical pricing of European call options.

We use the BSM model as a benchmark for mainly two reasons. Even though the model require some assumptions listed below that have been empirically invalidated, the convenience of implementing the model along with the models ability to replicate observed prices when fed with implied volatilities makes it a suitable benchmark. The model can easily obtain European call prices as all variables except the volatility are directly observable in the market. Thus easing the data filtering and processing power needed.

[13] A geometric Brownian motion is a continuous stochastic process in where the logarithm of the random varying quantity follows a Brownian motion or Wiener process with drift.

5.1.1 Black-Scholes-Merton Option Pricing Model

The classic BSM model for pricing European call options is defined as.

$$C_t = S_t N(d_1) - Ke^{-rfT} N(d_2) \tag{5.1}$$

where:

$$d_1 = \frac{Ln(\frac{S_t}{K}) + (rf + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{5.2}$$

and

$$d_2 = d_1 - \sigma\sqrt{T} \tag{5.3}$$

Variables

S_t – Underlying stock price at time t.
K – Strike price.
rf – Continuously compounded risk-free interest rate.
T – Time to maturity.
σ – Volatility, or standard deviation of the underlying.
$N(*)$ – Denotes the cumulative standard normal distribution.

5.1.2 Implied Volatility

One of the controversies about the BSM model surrounds its volatility. Of all the parameters, the volatility is the only one that cannot be directly observed, Furthermore, there is no method to inverse the Black-Scholes-Merton formula in 5.1 to give the volatility parameter as an output. Implied volatilty is often used in practice as a measure of the true volatility of an underlying asset, this is because while historical volatility are backwards looking, implied volatility is forward looking. It captures the markets expectations for the future explained by the option price. The implied volatility is also used to check whether any of our models are able to replicate the volatility smile observed in market data.

The most efficient way to obtain the implied volatility is to use a iterative bisection method, where one utilizes that $\frac{\partial C}{\partial \sigma} > 0$ – i.e. the price is an increasing function of the underlying volatility. The algorithm will calculate the BSM prices until it matches the observed price with a predetermined accuracy and return the volatility

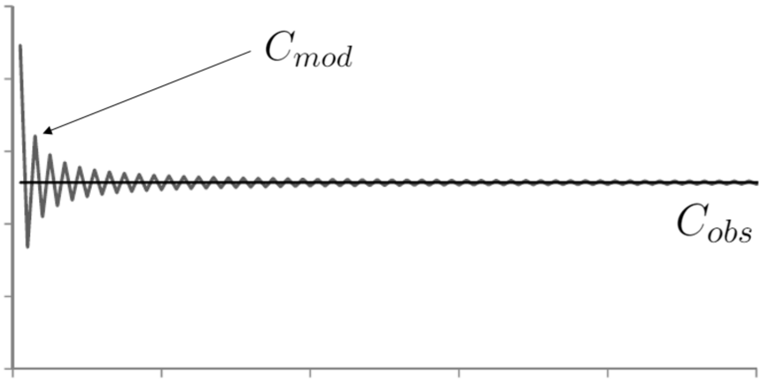


Figure 5.1: Implied Volatility Convergence

used to model that price. The optimal convergence is shown in Figure 5.1, where C_{mod} represents the calculated BSM price for the given volatility input.

5.2 GARCH Framework

The GARCH-models is the basis of this thesis, and in this Section we aim to present all information about the models needed for the modelling of GARCH processes and the following option pricing. We will also give some introductory notes for the supporting theory needed to understand some of the fundamental underlying principles of GARCH modeling. It is worth noticing that we have changed some notation from original authors to maintain a consistency throughout our thesis.

5.2.1 ARCH(p)

Empirical observations of asset returns show that high returns often tend to be followed by more high returns. This is true for both signs (positive and negative), implying that returns are serially correlated and show signs of clustering (Campbell et al. 1996).

As one of the pioneers in the field of volatility modeling, Engle (1982) proposed the Autoregressive Conditionally Heteroskedacity also known as the ARCH(p) model in order to capture the observed autocorrelation. By recognizing the difference between the conditional and unconditional variance, it allows the conditional variance to change as a linear function of past errors, or return residuals. The model is defined as:

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-1}^2, \tag{5.4}$$

Here ω and $\alpha_1, ..., \alpha_p$ are constant parameters, and the impact of the error term or return shock i periods ago is driven by α_i. News that arrived more than p lags ago have no effect on the current volatility at time t. To ensure that the conditional variance stays positive, both ω and the coefficients α_i must stay non-negative. The ARCH(p) model manages to capture the volatility clustering effect where extreme values of h_t tend to be followed by other extreme values of any sign.

5.2.2 GARCH(1,1)

Bollerslev (1986) proposes a generalized version of the ARCH(p) model known as the GARCH(p,q) model, and is defined as:

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-1}^2 + \sum_{i=1}^{q} \beta_i h_{t-i}, \tag{5.5}$$

The most common variant of this model is the GARCH(1,1) process, with only one lag in both the past errors and conditional realizations:

$$h_t = \omega + \alpha \varepsilon_{t-1}^2 + \beta h_{t-i}, \tag{5.6}$$

Here $\omega = \gamma V_L$ where V_L refers to the long term variance, and γ is the weight assigned to V_L. The constant parameters γ, α and β sums up to 1, so after ω, α and β have been estimated one can calculate γ as $1 - \alpha - \beta$. For a stable GARCH(1,1) process we demand that the first order condition $\alpha + \beta < 1$, or else the weight of the long term variance become negative. β can further be interpreted as the "decay rate" or "GARCH-effect" that defines the relative importance of the lags in determining the current variance, thus the effect of the error term declines exponentially at rate β. α is seen as the mean reversion factor or "ARCH-effect" that makes the process revolve around its mean at rate α (Hull 2015).

The GARCH(1,1) model have become a successfully estimator of variance due to its easy implementation and understandable process. The parameters are in most cases estimated using maximum likelihood methods explained in 5.5.1 that is fitted on sampled historical data. Critique of the GARCH(1,1) is that it does not consider the leverage effect or asymmetric impact of news on returns, and asymmetric GARCH almost always perform better than symmetric models to fit data for equity, equity indexes and commodities for any frequency (Alexander 2008).

Following the success of the GARCH(1,1) model, a whole family of GARCH-models have been designed to capture different effects (See Section 3) and some of them have been extended into the field of option pricing. We will outline some of these models that have been extended to option pricing below.

5.2.3 Short note on stationarity

The stability the $\alpha + \beta < 1$ condition give to the GARCH(1,1) is related to the stationarity of the process. As a non-stationary process will "wander off," it is important that all of our GARCH-models stay stationary. In order to keep the conditional volatilities from ending up with extreme values, we require our models to be stationary by setting these bounds on the parameter inputs. The requirements for (weakly) stationarity is that the population mean, variance and auto-covariance is finite, constant and time-invariant (Cuthbertson & Nitzsche 2005).

5.2.4 The Distribution of the Error Term

One of the majorly discussed subjects in relation to the modelling of GARCH-models is the distribution of the error terms, random innovations or shocks ε_t. In GARCH modelling the error term jointly decides the distribution of the returns, as the only noise factor impacting the return not explained by the model itself. Empirically most researchers assume that the error term is Gaussian, but non-normal distributions such as the Student's t-distribution (Bollerslev et al. 1988) Generalized Error Distribution (GED) (Nelson 1991) or standard Cauchy distributions may also be considered.

The rationale behind choice of distribution is what represents the best fit for the underlying data. Empirically financial return data display fat tails and are skewed, thus a non-normal or skewed distribution could arguably provide a better fit on the return data. The choice of distribution impacts the modelling of the innovations and hence the definition of the maximum likelihood estimator. In this thesis we follow the majority of GARCH modelling literature and assume that the error term follow a normal distribution. This is further motivated by the fact that assuming a normal distribution is more robust against misrepresentation than wrongly assume any other distribution. Furthermore, our main focus is to compare models on same basis, rather than exploring which distribution which may fit the best.

The density function of the normal distribution with a given mean μ and standard deviation σ is expressed as:

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^{"}}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{5.7}$$

5.2.5 Risk-neutrality and Duan's LRNVR

In this section we will touch upon the principle behind the concept of risk neutralization for option pricing in general, and the local-risk-neutral-valuation-relationship (LRNVR) that is the foundation of GARCH option valuation.

The concept of risk neutral valuation is of paramount importance within the field of derivatives valuation theory. The concept states that when pricing a derivative like options, it is possible to make assumption that investors are risk-neutral. Since we are pricing options in terms of the price of the underlying, the investors individual risk preference is deemed irrelevant. The implications to option pricing is that: 1. we can assume that the expected return of the underlying asset equals the risk free rate and 2. that we can discount the expected payoffs from an option with the risk free rate. It is useful that assuming a risk neutral preference and computing option prices yields the same option price in the real world, simplifying the potential complexity of choosing a correct discount factor. Another way of looking at the concept is that the price of an option today equals the expected future payoffs discounted at the risk-free rate.

The first to explain the concept of the risk-neutral valuation principle is Cox & Ross (1976) who specify a risk-neutral set of preferences in order to price options under stochastic volatility. Extending the concept, Rubinstein (1976) specify a representative agent economy and impose a set of conditions on the agents preferences in order to obtain a risk-neutralization framework for European option valuation. A representative agent with constant relative, or absolute risk aversion and normally distributed relative changes in aggregate consumption maximizes expected utility using LRNVR (Härdle & Hafner 2000).

Duan (1995) was the first to successfully apply the risk-neutral measure to a GARCH framework by demanding that the conditional return distribution remained normal and that

$$h^P(r|\mathscr{F}_{t-1}) = h^Q(r|\mathscr{F}_{t-1}) \tag{5.8}$$

where $\mathscr{F}$ denotes the set of all information prior to and including time t. This mean that moving from the physical (P) to the risk-neutral (Q) measure does not change the local conditional variance, nor the distributional class of the one-period ahead continuously compounded conditional return. Therefore the expected one-period ahead return, both conditional and under Q, equals the risk-free rate and are often referred to as the Equivalent Martingale Measure (EMM). Furthermore the local risk-neutral valuation relationship is satisfied when the stochastic discount factor and the one-period return are jointly conditionally lognormal.

The framework of Duan (1995) is only one of several ways to change the numéraire. As an alternative to specifying a agent economy to obtain the option prices, some specify a class of Radon-Nikodym (RN) derivatives and derive restrictions that yields an equivalent martingale measure that will in order make the discounted stock price process a martingale. We will not venture further into the risk-neutralization process as this is beyond the scope of this thesis. We will rather refer to Christoffersen et al. (2013) for more information on Radon-Nikodym derivatives.

5.3 HN-GARCH

Heston & Nandi (2000) propose a class of discrete affine GARCH-models that allow for the closed form solution for the price of a European call option. The model allow for a leverage effect and volatility clustering, but have been criticized for imposing a volatility process that are too restrictive, thus obtaining a poorer fit to historical data than its numerical peers.

P-Measure

The price follow the HN-GARCH(1,1)-process that are a set of affine equations designed to yield a closed form solution to option valuation:

$$Ln\left(\frac{S_{t+1}}{S_t}\right) = r_{t+1} = rf + \lambda h_{t+1} + \sqrt{h_{t+1}}\varepsilon_{t+1} \tag{5.9}$$

$$h_{t+1} = \omega + \beta h_t + \alpha(\varepsilon_t - \gamma\sqrt{h_t})^2 \tag{5.10}$$

Where:

Variables

rf – Discrete risk-free interest rate.

h_t – Conditional variance.

ε_t – Error (shock term) term, I.I.D. $N(0,1)$.

Model Parameters

λ – Equity risk premium.

ω – Is similarly as for GARCH(1,1), a combination of the long-term variance (V_L) and the weight assigned to it (γ).

β – Weight assigned to the lagged variance, thus determining the relative importance of the lag in the current variance. The higher the value of β the stronger clustering of volatility can be observed.

α – Determines kurtosis, and the higher the value of α the deeper the volatility smile.

γ – Represents the leverage effect. The higher the value of γ the higher is the impact of negative shocks on the conditional variance.

The conditional variance h_{t+1} is known at the end of day t. To value options at each date t one need an estimate of the conditional volatility h_t on that particular date, that is often referred to as the filtering problem. Discrete time GARCH-models offer a simple way of dealing with the filtering problem, which is one reason to their popularity (Christoffersen et al. 2005).

Unconditional Variance

The unconditional variance is a combination of all parameters of the process and represents the long-term variance independent of time. This is analogous to the long-term variance, V_L, we explained under 5.2.2.

$$\mathbb{E}[h_t] = \sigma^2 = \frac{\omega + \alpha}{1 - \alpha\gamma^2 - \beta} \tag{5.11}$$

Furthermore is the process stationary with a finite mean and variance if the following condition is met (Heston & Nandi 2000):

$$\beta + \alpha\gamma^2 < 1 \tag{5.12}$$

Q-Measure

Heston & Nandi further propose the risk-neutral process of the price of the GARCH(1,1) in (5.9) and (5.10) through their **Proposition 1** as

$$Ln\left(\frac{S_{t+1}}{S_t}\right) = r_{t+1} = rf - \frac{1}{2}h_{t+1} + \sqrt{h}_{t+1}\varepsilon^*_{t+1} \tag{5.13}$$

$$h_{t+1} = \omega + \beta h_t + \alpha(\varepsilon^*_t - \gamma^*\sqrt{h_t})^2 \tag{5.14}$$

Where $\gamma^* = \gamma + \lambda + 0.5$ and $\varepsilon^*_t \sim N(0,1)$. This makes the expected one period return from investing in the asset equal to the risk free rate.

Option Pricing

The following formula is derived to yield the value at time t of an European call option with strike price K and maturity T:

$$C = e^{-rf(T-t)} E_t^*[Max(S(T) - K, 0)]$$

$$= \frac{1}{2}S(t) + \frac{e^{-rf(T-t)}}{\pi} \int_0^\infty Re\left[\frac{K^{-i\phi} f^*(i\phi - 1)}{i\phi}\right] d\phi$$
$$- Ke^{-rf(T-t)} \left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left[\frac{K^{-i\phi} f^*(i\phi)}{i\phi}\right] d\phi\right) \tag{5.15}$$

In order to obtain the option prices one must integrate over the real part $Re[\,]$ of a complex number where the imaginary number is i and represents $\sqrt{-1}$. $f^*(i\phi)$ is the conditional moment generating formula of the log asset price under the risk neutral measure. Some argue that the Heston & Nandi model only yield a quasi closed-form solution as one must use an univariate numerical integration to obtain the prices. The integration are however easily conducted and does not require much time, or computational power (Christoffersen et al. 2013). To obtain the conditional moment generating formula one recursively solve these set of equations backwards from time of maturity using the terminal conditions $A_T = B_T = 0$ (Rouah & Vainberg 2012).

$$f(\phi) = S_t^\phi \exp(A_t + B_t h_{t+1}^2) \tag{5.16}$$

$$A_t = A_{t+1} + \phi r + B_{t+1}\omega - \frac{1}{2}log(1 - 2\alpha\beta_{t+1}) \tag{5.17}$$

$$B_t = \phi(\lambda + \gamma) - \frac{1}{2}\gamma^2 + \beta B_{t+1} + \frac{\frac{1}{2}(\phi - \gamma)^2}{1 - 2\alpha B_{t+1}} \tag{5.18}$$

Put prices can easily be obtained through the put-call parity, but will not be a part of the analysis conducted in this thesis.

5.4 Numerical GARCH-models

The other GARCH-models we use are various alterations over the conditional variance process. All has in common that they cannot yield closed-form option pricing models and share the same proposed stock price process. This means that they require a Monte Carlo simulation in order to obtain the price estimates. Another common feature is that all include the leverage effect in some way, and that parameters in each model are estimated by using Maximum Likelihood. Each model has its own implications from this. Furthermore, we will only apply the GARCH(1,1) case for each model.

5.4.1 NGARCH

The NGARCH model, also known as the asymmetric GARCH or A-GARCH adds a leverage parameter to the original GARCH(1,1) to capture the leverage effect, thus it is known as the simplest GARCH-model that contains volatility clustering and a leverage effect. The model was initially proposed by Engle & Ng (1993) and extended through LRNVR by Duan (1995) in order to price options.

P-Measure
The physical price process is given by the return and conditional volatility equations:

$$Ln\left(\frac{S_{t+1}}{S_t}\right) = r_{t+1} = rf + \lambda\sqrt{h_{t+1}} - 0.5h_{t+1} + \sqrt{h_{t+1}}\varepsilon_{t+1} \tag{5.19}$$

$$h_{t+1} = \omega + \beta h_t + \alpha h_t(\varepsilon_t - \gamma)^2 \tag{5.20}$$

Where:

Variables
rf – Dicrete risk-free interest rate.
h_t – Conditional variance.
ε_t – Error (shock term) term, I.I.D. $N(0,1)$.

Model Parameters
λ – Equity risk premium.

ω – Is similarly as for GARCH(1,1), a combination of the long-term variance (V_L) and the weight assigned to it (γ).

β – Weight assigned to the lagged variance, thus determining the relative importance of the lags in the current variance.

α – Determines kurtosis.

γ – Determines the leverage effect.

Unconditional Variance

The unconditional variance can be written as:

$$\mathbb{E}[h_t] = \sigma^2 = \frac{\omega}{(1 - \beta - \alpha(1 + \gamma^2))} \tag{5.21}$$

In order to ensure stationarity of the process we demand the condition

$$\beta + \alpha(1 + \gamma^2) < 1 \tag{5.22}$$

The non-affine nature of the NGARCH process cannot provide a closed form solution for the European call option prices. In this way the GARCH-model differs from the one in Heston & Nandi as it is designed to fit underlying return rather than yielding a closed-form solution.

Q-Measure

Using the local risk-neutral valuation relationship, the risk-neutral price process from (5.19) and (5.20) is obtained.

$$Ln\left(\frac{S_{t+1}}{S_t}\right) = r_{t+1} = rf - 0.5h_{t+1} + \sqrt{h_{t+1}}\varepsilon^*_{t+1} \tag{5.23}$$

$$h_{t+1} = \omega + \beta h_t + ah_t(\varepsilon^*_t - \gamma^*)^2 \tag{5.24}$$

The changes to the equations when changing to the risk-neutral measure is minor, with $\gamma^* = \gamma + \lambda$ and $\varepsilon^*_t \sim N(0, 1)$. The risk neutralization establishes that the discounted price process is a martingale by nature. The risk-neutralization is an operation that exclusively addresses the valuation problem.

Put prices can easily be obtained through the put-call parity.

5.4.2 EGARCH

The exponential or EGARCH model was developed by Nelson (1990) as a more complex GARCH framework in order to capture the leverage effect that GARCH fails to capture. The EGARCH allow for asymmetries between return and volatility through the *gamma* parameter in the conditional variance function. Furthermore, is an advantage of the EGARCH that when modelling $Log(h_t)$, or the logged conditional variance, h_t will stay non-negative regardless of the values of the parameters.

P-Measure

The price process of the EGARCH(1,1) may be formulated in several ways, we follow the formulation used by Duan et al. (2006) due to its simplistic formulation and the provision of the risk-neutral process for option pricing. The EGARCH models state that return follow the GARCH process:

$$Ln\left(\frac{S_{t+1}}{S_t}\right) = r_{t+1} = rf + \lambda\sqrt{h_{t+1}} - 0.5h_{t+1} + \sqrt{h_{t+1}}\varepsilon_{t+1} \tag{5.25}$$

$$ln(h_{t+1}) = \omega + \beta ln(h_t) + \alpha(|\varepsilon_t| + \gamma\varepsilon_t) \tag{5.26}$$

Where:

Variables

rf – Discrete risk-free interest rate.

h_t – Conditional variance.

ε_t – Error (shock term) term, I.I.D. $N(0,1)$.

Model Parameters

λ – Equity risk premium.

ω – Is similarly as for GARCH(1,1), a combination of the long-term variance (V_L) and the weight assigned to it (γ).

β – Represents the persistence in conditional volatility irrelevant of the of what happens in the market. The higher the value of β the stronger clustering of volatility can be observed.

α – Determines the kurtosis of the distribution. The higher the α the deeper the volatility smile.

γ – Represents the leverage effect, where the size of γ determines the impact of negative shocks to the variance.

The existence of the γ parameter determines the leverage effect. If $\gamma = 0$, model is symmetric as the GARCH(1,1) model. When $\gamma < 0$ negative shocks have greater impact than positive shocks and vice versa for $\gamma > 0$.

The conditional variance is a function of the past standardized shocks ε_t rather than a function of the past innovations of $\sqrt{h_t}\varepsilon_t$. Some argue that this is a reason why the EGARCH empirically show less volatility clustering than the normal GARCH(1,1) (Mikosch et al. 2012).

Unconditional Variance

The long term or unconditional variance can according to Alexander (2008) be calculated as :

$$\mathbb{E}[h_t] = \sigma^2 = exp\left(\frac{\omega}{1-\beta}\right) \tag{5.27}$$

thus bearing some resemblance to the GARCH(1,1) $\alpha + \beta < 1$ condition.

To ensure stationarity we demand the condition:

$$|\beta| < 1 \tag{5.28}$$

Q-Measure

Using the local risk-neutral valuation relationship, the risk-neutral price process from 5.25 and 5.26 is obtained.

$$Ln\left(\frac{S_{t+1}}{S_t}\right) = r_{t+1} = rf - 0.5h_{t+1} + \sqrt{h_{t+1}}\varepsilon^*_{t+1} \tag{5.29}$$

$$ln(h_{t+1}) = \omega + \beta ln(h_t) + \alpha\left[(|\varepsilon^*_t - \lambda|) + \gamma(\varepsilon^*_t - \lambda)\right] \tag{5.30}$$

Where $\varepsilon^*_t = \varepsilon_t + \lambda$ is the random term becoming a standard normal variable under Q. The condition $\beta < 1$ ensures the conditional variance does not explode with time. Furthermore is the stationarity condition equal under both P and Q, unlike for instance GJR-GARCH. This is because we in EGARCH have a additive mechanism of volatility shocks entering the time series (Duan 1997).

Put prices can easily be obtained through the put-call parity.

5.4.3 GJR-GARCH

The Glosten, Jagannathan, Runkle GARCH or GJR-GARCH was presented by Glosten et al. (1993) and are included in the family of threshold-GARCH and is closely related to the TGARCH model of Zakoian (1994). The GJR-GARCH model is also modelled to capture asymmetry in the shocks. The model consists of one extra leverage parameter γ that weights the negative impact of the shock in the $max(*)$ function. The $max(*)$ function makes sure that only negative shocks are considered as any positive shocks will be filtered out as their value becomes negative in the function. Furthermore we follow Duan et al. (2006)'s formulation of the processes.

P-Measure
The return process under the physical measure are defined as

$$Ln\left(\frac{S_{t+1}}{S_t}\right) = r_{t+1} = rf + \lambda\sqrt{h_{t+1}} - 0.5h_{t+1} + \sqrt{h_{t+1}}\varepsilon_{t+1} \tag{5.31}$$

$$h_{t+1} = \omega + h_t\left[\beta + \alpha\varepsilon_t^2 + \gamma max(0, -\varepsilon_t)^2\right] \tag{5.32}$$

or equivalently:

$$h_{t+1} = \omega + \alpha X_t^2 + \gamma max(0, -X_t)^2 + \beta h_t, \quad X_t = \sqrt{h_t}\varepsilon_t \tag{5.33}$$

Where:

Variables

rf – Discrete risk-free interest rate.
h_t – Conditional variance.
ε_t – Error (shock term) term, I.I.D. $N(0,1)$.

Model Parameters

λ – Equity risk premium.
ω – Is similarly as for GARCH(1,1), a combination of the long-term variance (V_L) and the weight assigned to it (γ).
β – Weight assigned to the lagged variance, thus determining the relative importance of the lags in the current variance.
α – Determines kurtosis, or sensitivity to shocks in the market.

γ – Represents the leverage effect. The higher the value of γ the higher is the impact of negative shocks on the conditional variance.

Making sure to generate a positive conditional variance we impose that $\omega \geq 0$, $\alpha \geq 0$, $\beta \geq 0$ and $\gamma \geq 0$. It is also natural as γ is typically found to be positive so that volatility increases proportionally more for negative shocks (Bollerslev 2008).

Unconditional Variance

The unconditional variance can according to Alexander (2008) be written as:

$$\mathbb{E}[h_t] = \sigma^2 = \frac{\omega}{1 - \alpha - \frac{\gamma}{2} - \beta} \tag{5.34}$$

where $\frac{\gamma}{2}$ comes from the normality assumption of ε_t.

To ensure stationarity under the physical measure we demand the condition:

$$\alpha + \frac{\gamma}{2} + \beta \ < \ 1 \tag{5.35}$$

Q-Measure

Using the local risk-neutral valuation relationship, the risk-neutral price process from 5.31 and 5.32 is obtained:

$$Ln\left(\frac{S_{t+1}}{S_t}\right) = r_{t+1} = rf - 0.5h_{t+1} + \sqrt{h_{t+1}}\varepsilon^*_{t+1} \tag{5.36}$$

$$h_{t+1} = \omega + h_t\left[\beta + \alpha(\varepsilon^*_t - \lambda)^2 + \gamma max(0, -\varepsilon^*_t + \lambda)^2\right] \tag{5.37}$$

Where $\varepsilon^*_t = \varepsilon_t + \lambda$ is the random term becoming a standard normal variable under Q. Due to the multiplicative nature of volatility shocks in GJR-GARCH the condition for stationarity is different under P and Q. Under Q the variance is found to be weakly stationary if:

$$\beta + (\alpha + \gamma N(\lambda))(1 + \lambda^2) + \gamma\lambda n(\lambda) < 1 \tag{5.38}$$

Where $N(*)$ and $n(*)$ is respectively the cumulative standard normal distribution and the standard normal probability density function (Duan et al. 2006).

Put prices can easily be obtained through the put-call parity.

5.4.4 Numerical Pricing With the GARCH-models

When we have estimated the model parameters we use the risk-neutral price processes for the NGARCH, GJR-GARCH and EGARCH, to numerically calculate the option prices. All the models share the same price process, and a similar procedure can therefore be applied to all models. We use Monte Carlo to produce a large amount of price-paths for a correct pricing. The methodology will further be explained in Section 5.6.

We will obtain the option prices through averaging the expected value of the payoff for each simulated risk neutral price-path on the given option. The price is a function of the conditional variance given by each distinct model.

$$C(h_{t+1}) = \frac{1}{N} \sum_{t}^{T} e^{-rf(T-t)} E_t^*[Max(S_T - K, 0)] \tag{5.39}$$

here t denotes time of pricing and T maturity date of the option priced.

5.5 Parameter Estimation

In order to fit our GARCH option pricing models to real data we need to find the correct set of parameters. As with any GARCH-model the conventional theory is to fit our model to observed data, while when applying them to option pricing we risk-neutralize the parameters. As described more thorough under each model we use their physical measure to fit models to observed return data, and similarly to the process described in Heston & Nandi (2000) we make these estimations risk-neutral in order to price options. Following several empirical studies such as Christoffersen & Jacobs (2004*b*) we also employ another process in order to obtain less pricing errors. With the terminology used in Chorro et al. (2014) we first estimate parameters by mapping them in physical measure optimizing a maximum likelihood function. Secondly we calibrate a new set of parameters by minimizing the mean squared error (MSE).

5.5.1 Maximum Likelihood - Estimation

In order to estimate the parameters we use in our models, we use Maximum Likelihood Estimation (MLE). The principle of the method is quite intuitive where you calculate the conditional probability of each observation.

$$Probability(x_i|\theta) = p(x_i|\theta) \tag{5.40}$$

Where x_i is the observation estimated with our parameters, or input, θ. Furthermore we sum all these probabilities to find what parameters are most likely to be true.

$$p(x_1, x_2, \cdots, x_n|\theta) = p(x_1|\theta) \times p(x_2|\theta) \times \cdots \times p(x_n|\theta) = \prod_{i=1}^{n} p(x_i|\theta) \tag{5.41}$$

This is now the probability of our observations with given parameters. As the product usually becomes infinitely big, it is common to use a handy property of logarithm functions.

$$ln\left[p(x_1, x_2, \cdots, x_n|\theta)\right] = ln\left[p(x_1|\theta)\right] + ln\left[p(x_2|\theta)\right] + \cdots + ln\left[p(x_n|\theta)\right] = \sum_{i=1}^{n} ln\left[p(x_i|\theta)\right] \tag{5.42}$$

We then use an optimization tool to maximize this probability over the parameters.

$$\max_{\theta} \sum_{i=1}^{n} ln\left[p(x_i|\theta)\right] \tag{5.43}$$

To estimate this, one sum each probability of finding the observed returns with the modeled volatility, or summing the probability of finding implied error-terms (ε) given that they are standard normally distributed. We use the former approach and find the probability of realizing observed returns given that they are normally distributed with their theoretical mean and modelled standard deviation.

$$\max_{\theta} \sum_{i=1}^{n} ln\left[f(r_i)\right], \quad where \quad f = n(\mu, h_i|\theta) \tag{5.44}$$

Here f is the normal density distribution and r_i the observed returns. There is many different ways of solving the maximum likelihood optimization, and we have chosen the method found in Chorro et al. (2014). The reason for this is that it is not only quite basic, but also easy to implement throughout all of our models in the same manner without significant derivations. In this way we can for each model use a theoretical mean for returns based on the stock price processes expected return and h as the modeled standard deviation from our simulation.

If we denote the normal density function as normal density function with notation $n(\mu, \sigma)$, we then optimize the following for HN-GARCH:

$$\max_{\theta} \sum_{i=1}^{n} ln\, p\left[r_i|r_i \sim n(rf_i + \lambda h_i\,, h_i)\right] \tag{5.45}$$

And as the NGARCH, GJR-GARCH and EGARCH are assumed to follow the same price process we optimize:

$$\max_{\theta} \sum_{i=1}^{n} ln\, p\left[r_i|r_i \sim n(rf_i + \lambda\sqrt{h_i} - 0.5h_i\,, h_i)\right] \tag{5.46}$$

for each distinctively modelled h_i.

Quasi Maximum Likelihood

Wedderburn (1974) introduced the theory of Quasi-Maximum-Likelihood (QML).

As opposed to Maximum Likelihood in which we have to know the distribution of the observations, QML only need to know a relation of the mean and variance. The estimate will still be consistent and asymptotically normal, and even if we do not believe that normal distribution is indeed the true distribution it produces a more robust estimate than wrongly applying any other distribution (Francq et al. 2004).

Following this we then use QML with the framework previously outlined. Even if we know that empirical return data present fat-tails we will estimate it based on an assumption of normality. This is in order to not misspecify the error function with for instance a t-distribution while a Generalized Error Distribution would fit better.

5.5.2 Calibration

A quite applied method that is arguably easier to grasp is simply to start pricing the options in-sample and then calibrating the parameters by reducing the pricing errors. For analytical models this is relatively straightforward, whereas for the numerical models where we use Monte Carlo methods to reach the observed price the procedure is more computationally demanding. The advantage of using this method compared to maximum likelihood is that it incorporate forward looking information from the options themselves, that the historical maximum likelihood does not. An effective way of doing this is by minimizing the *mean squared error* (MSE):

$$MSE(\$) = \frac{1}{N}\sum_{j=1}^{N}(C_j^{market} - C_j^{model})^2 \tag{5.47}$$

When squaring the errors we then ensure that any error, positive or negative carries weight. It is, however, put exponentially more weight on outliers. For that reason, it can be preferable to either use root mean squared error (RMSE), or mean absolute deviation (MAD). It simply depends on whether large errors should be punished more, or weighting of the error should be proportional to the error itself (Meindl & Chopra 2015). Christoffersen & Jacobs (2004*a*) argue that minimizing MSE functions is a good general-purpose loss function for option pricing. They also emphasize the importance of using the same loss function for all models in order to obtain a fair comparison. Different loss functions will be explained later in Section 5.7.

Optimization

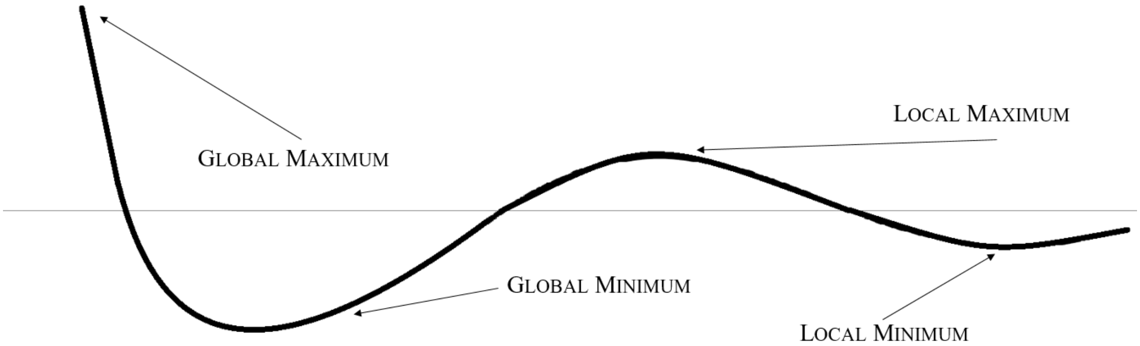


Figure 5.2: Local- and Global Extreme Points

Both the ML estimation and the MSE calibration is done by optimizing the function value by changing the parameters. The optimal solution of this exercise is extremely complex and there exists many local optimums. The difficult task is to find the true global optimum for the models, and avoid the local optimums. From Figure 5.2 we can see the difference of a local and global optimum in a two dimensional setting, for our models the dimensions are equal to the parameters we try to find, making the function a "multi dimensional surface" rather than a simple line.

5.6 Monte Carlo Simulation

Monte Carlo simulation are in this thesis used to obtain the option prices following the GARCH processes that cannot be solved analytically. Under the assumption of no arbitrage the price of a generic derivative security equals its discounted expected payoff at maturity. The expectation is taken in relationship of the equivalent martingale measure or in GARCH-models the local risk-neutral valuation relationship. We facilitate Monte Carlo as it is widely implemented in the option pricing literature, is easy to implement and yields flexibility.

In option pricing the biggest issue is to predict the price of the underlying at maturity. As explained in 4.1 we have that the payoff of a plain vanilla European call options is:

$$Payoff\ at\ maturity = (S-K)^{+} = max(S-K,0) \tag{5.48}$$

We know the time to maturity (T), the strike price (K), the risk-free rate (rf) and the current spot price (S_0). What we cannot know with certainty is the spot price at maturity. By constructing a Monte Carlo to simulate the spot price until maturity we can obtain a "best guess" of what value the spot can take at maturity.

An important notion about the Monte Carlo simulations are the random aspect. As all other variables and parameters are hold constant, the changing factor over time is the shock. Under the risk-neutral measure, the shock ε_t^* is standard normal distributed, thus the shock represents new market information that impacts the price in either direction. The more simulations, the less is the isolated effect of the shocks, and the more precise results.

The steps in the Monte Carlo simulations we use are:

1. Simulate N paths for the price process following the relevant risk-neutral GARCH process over the horizon to maturity.
2. Obtain the discounted cash flows yielded by the process given the GARCH process.
3. Average the discounted cash flows over the number of sample paths.

The Monte Carlo method is in effect computing a multidimensional integral, as the expected value of the discounted payoffs over the space of sample paths (Boyle et al. 1997).

The drawback of using Monte Carlo simulation is that as the complexity of the problem grow, huge amounts of simulations is required in order to obtain satisfactory results. The confidence of a Monte Carlo is also non-linear; it is actually proportional to the square root of the number of simulated paths. So to half the insecurity you would have to quadruple the number of trials (Hull 2015).

5.6.1 Variance Reduction Techniques

For enormous amounts of data it is relevant to consider computational power. In that regard there is some computational techniques that concentrate estimates in use of Monte Carlo. They are in different natures reducing the uncertainty of the results, and in practice reduces the needed amount of simulations.

Antithetic variables

One of the techniques we are using is simply reducing the amount of random numbers needed. By calculating two different prices based on the same number, the computational power is reduced and the variance of the estimates are significantly lowered. As all the random numbers are standard normal distributed with a mean of zero, their value distributions will be symmetric and one realization which is positive could equally possible be negative by the same value. Thus is the theory simply that if one price estimate (C_1) is estimated using ε, another price (C_2) can be estimated by using $-\varepsilon$. The sample value will then be the average of the two, and follows that if one of the estimates are above the true value, the other tends to be below and vice versa (Hull 2015).

We then have that:

$$\bar{C} = \frac{C_1 + C_2}{2} \tag{5.49}$$

The final Monte Carlo estimate will be the average of all $\bar{C}$'s. If we denote $\bar{\omega}$ as the standard deviation of the $\bar{C}$'s and n as the number of simulated pairs, the standard error of the estimate will be $\bar{\omega}/\sqrt{n}$. This is usually substantially less than the error achieved when calculating $2n$ random trials without using antithetic variables (Boyle et al. 1997).

Moment matching

Another techniques we use in combination with antithetic variables is Moment Matching, or also referred to as quadratic resampling as presented by Barraquand (1995). The theory is to adjust each moment in a sample of random realizations to generate a mean and standard deviation equal to their true value.

$$\bar{\varepsilon}_i^* = \frac{\varepsilon_i - m}{s} \tag{5.50}$$

Where ε_i^* is the adjusted shocks when the random variable ε_i is adjusted with the sample mean m and standard deviation s.

Antithetic variable and moment matching are often used together in Monte Carlo simulation. The methods will reduce computational power as less simulations are needed, but particularly the latter might lead to memory problems as two sets of random numbers will have to be stored (Hull 2015). If you choose to generate the samples outside of loops and calculations, this will, however, be a problem of inferior importance.

5.7 Performance Measures

To be able to compare the models fairly it is important to use different performance measures to create a full picture of the pricing errors. Different loss functions will tell a different story, and using various target functions for optimization would have different effects. We will outline the functions we focus on in this paper with some short interpretation and indication following each function. We have chosen each function based on its traits to describe pricing errors, as well as what we have found to be common in previous research.

5.7.1 Loss Functions

In general we present various methods to represent the pricing errors so that both positive and negative deviations are captured. One of the most conventional ways is to square the error ensuring that positive and negative errors does not cancel each other out. The most basic of which is the *Mean Squared Error (MSE)* which measures the average of these squared pricing errors.

$$MSE(\$) = \frac{1}{N}\sum_{j=1}^{N}(C_j^{market} - C_j^{model})^2 \tag{5.51}$$

As the interpretation of a squared error is somewhat vague it is common to take the root of this average. The *Root Mean Squared Error (RMSE)* displays a figure that is easier to grasp.

$$RMSE(\$) = \sqrt{\frac{1}{N}\sum_{j=1}^{N}(C_j^{market} - C_j^{model})^2} \tag{5.52}$$

These measures does, however, put a exponential weight on big errors as we square each deviation. And as we are using MSE when calibrating our models, we deliberately punish high pricing errors more than prioritizing model accuracy. This might be important to keep in mind when analyzing the results. Nevertheless the method seem to be the convention, and Christoffersen & Jacobs (2004*a*) also confirm that it also performs best compared to others. However, when presenting the data we want to compliment with other figures to describe results better.

While RMSE($) measures the errors in currency units ($), we sometimes prefer to look at percentage errors. *Root Mean Squared Error (RMSE(%))* is similar to RMSE($), but uses relative errors. In this way it can be interpreted as a percentage pricing error. Note that we do not multiply the error with 100, so the measure is not quoted in percentage, but rather in decimals.

$$RMSE(\%) = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(\frac{C_j^{market} - C_j^{model}}{C_j^{market}}\right)^2} \tag{5.53}$$

The advantage if this measure is that while RMSE($) will not differentiate between $1 error on a $5, or a $50 option - RMSE(%) will. When the figures are looked at together we will get a more nuanced picture of how the pricing errors are distributed. For instance can a low RMSE($) coupled with a high RMSE(%) indicate high pricing errors on low priced options (OTM).

Another way of capturing both positive and negative errors in sum is to use the absolute errors. *Mean Absolute Error (MAE)* is similar to RMSE($), but will have a more linear interpretation as the errors are not squared.

$$MAE = \frac{1}{N}\sum_{j=1}^{N}|C_j^{market} - C_j^{model}| \tag{5.54}$$

As this measure will be dependent on the average price level and we also utilize the *Mean Absolute Percentage Error (MAPE)*. Which has an even simpler interpretation than RMSE(%) as it really is the average relative pricing error, and will not depend on the magnitude of the relative error.

$$MAPE(\%) = \frac{1}{N}\sum_{j=1}^{N}\frac{|C_j^{market} - C_j^{model}|}{C_j^{market}} \tag{5.55}$$

Implied Volatility root mean squared error (IVRMSE) look at the squared difference in the implied volatility from the models, and the implied volatility from the observed option prices. The loss function is convenient, as implied volatilities are not very different across moneyness and maturities.

$$IVRMSE = \sqrt{\frac{1}{N}\sum_{j=1}^{N}(\sigma_j^{market} - \sigma_j^{model})^2} \tag{5.56}$$

All the measures above have their own advantages, and will complement each other in the analysis.

5.7.2 Regression Analysis

By running regressions on variables we see if some particular independent variables are affecting some particular dependent variable. Setting up a regression equation we can determine the linear dependence and strength of the relationship between any of the variables. Following a similar procedure as Madan et al. (1998) we use the Equation 5.57.

$$PE_i = \beta_0 + \beta_1 MNY_i + \beta_2 MNY_i^2 + \beta_3 TTM_i + \beta_4 rf_i + \varepsilon_{i,t} \tag{5.57}$$

Variables:

PE_i – Pricing error of option i
MNY_i – Moneyness, S/K
TTM_i – Time to maturity
rf_i – Risk free interest rate
β_j – Coefficient of its respective variable, $j = 1, 2, 3, 4$

By utilizing the method of Ordinary Least Squares (OLS), we can determine the coefficients (β's) for each variable. And when the coefficients are determined we can run some different tests to determine the strength of the relationship. For our regression analysis we choose to focus on a few key test statistics. While we use Microsoft Excel's data analysis package for regression, we base our tests on Murray (2005).

To help the analysis it is helpful to realize that Total Sum of Squares (TSS) which is the variance in the dependent variable will consist of Explained Sum of Squares (ESS) and Sum of Squared Residuals (SSR). ESS will be the variance in which the OLS regression can explain. While SSR is what cannot be explained by the regression, but our dependent variable nevertheless possesses.

To start off we will observe the coefficient of determination, commonly referred to as the R^2. Interpreted as a percentage this figure tells us how much of the variance in the dependent variable can be explained by the variance of the independent variables. I.e. $R^2 = ESS/TSS$. We will report the adjusted R^2 which adjusts for extra explanatory power imposed by extra variables.

To check for significance of the relationship between any independent and the dependent variable we run a t-test. This is done by running a test on the coefficient (β) of the respective variable, to check if the variable itself has some significant explanatory power. The null-hypothesis (H_0) will then be that the coefficient is zero, which is tested against the alternative hypothesis (H_A) that it is different from zero. Given that our estimated coefficients ($\hat{\beta}_j's$) are normally distributed by the true mean and standard error (σ_{β_j}) of the coefficient we will under H_0 have that:

$$\frac{\hat{\beta}_j}{\sigma_{\beta_j}} \quad \sim N(0,1) \tag{5.58}$$

But as we do not know the true variance we will have to replace the true variance with a unbiased estimated standard error, ($\hat{\sigma}$), and the fraction will be t-distributed with (n-m) degrees of freedom. Where n is number of observations and m is the number of independent variables.

$$H_0: \ \beta_j = 0 \qquad H_A: \ \beta_j \neq 0 \tag{5.59}$$

$$\frac{\hat{\beta}_j - \beta_j}{\hat{\sigma}_{\beta_j}} \ \sim \ t(n-4) \tag{5.60}$$

The relevant output of the test is the p-value in which tells us how likely H_0 is true. If the p-value is 1% we say that the variable is significant on a 1% significance level. I.e. with 99% certainty can the respective variable explain variation in the dependent variable.

Another key figure we will look at is a joint test. We want to see if all of the β's can be said to be zero at the same time. In other words, are we equally well-off by modelling the dependent variable with some stochastic process instead of our selected variables. We also want to test if moneyness can really be said to have no significance. Without going too far into details this requires testing several the parameters at once. To do this we see if a restricted regression without the relevant variables have significantly less explanatory power than the unconstrained regression. Check if there is a significant difference in R^2 if you like. As they both have the same TSS, this will mainly depend on SSR. To test for moneyness we will then have to test following hypotheses:

$$H_0: \ \beta_1 = \beta_2 = 0 \qquad H_A: \ H_0 \ is \ not \ true \tag{5.61}$$

And we will have a restricted regression based on:

$$PE_i = \beta_0 + \beta_3 TTM_i + \beta_4 rf_i + \varepsilon_{i,t} \tag{5.62}$$

We then find the SSR for both equations by taking the difference of what the regression can explain and what is observed.

$$SSR^u = (PE_i - \beta_0 - \beta_1 MNY_i - \beta_2 MNY_i^2 - \beta_3 TTM_i - \beta_4 rf_i)^2 \tag{5.63}$$

And if testing moneyness we will have the restricted version:

$$SSR^r = (PE_i - \beta_0 - \beta_3 TTM_i - \beta_4 rf_i)^2 \tag{5.64}$$

We then run a F-test on the difference of these figures.

$$\frac{\frac{SSR^c - SSR^u}{r}}{\frac{SSR^u}{n-m-1}} \sim F(r, n-k-1) \tag{5.65}$$

Where r is the number of restrictions imposed to the regression; two for testing moneyness and four for all variables. The exact same framework can be applied when testing all variables.

6 Empirical

In this section we employ the GARCH models earlier described and present the results of our empirical analysis. First off we present as we as filter the return and option data following peer reviewed literature. This is in order to remove potential biases found in for instance illiquid and very low priced options. After filtering the data, we continue to extract the model parameters from return data and a cross sectional option data sample. We continue to price options on an in-sample and an out-of-sample data set with the obtained parameters and analyze the models in relation to each other through different procedures and present our final findings at the end.

6.1 Data Sampling and Description

To test our range of GARCH option pricing models we have chosen European call options written on the Standard & Poor's 500 Index with expiration date from 01.01.2014 to 31.12.2016 as the foundation. The S&P 500 is the leading US large-cap equity index consisting of 500 value weighted stocks chosen to represent the US economy. To ensure a sufficient amount of data on each maturity we choose to include options with up to 3 months' maturity. The periods of option data represented in our sample is presented in Table 6.1.

Table 6.1: Time periods of option prices

Year	Price quotes	Expiration dates
2014	23.10.2013 - 10.12.2014	18.01.2014 - 20.12.2014
2015	22.10.2014 - 09.12.2015	17.01.2015 - 19.12.2015
2016	21.10.2015 - 07.12.2016	15.01.2016 - 16.12.2016

To download the time series data, Thompson Reuters Datastream is used as it is free to use on Copenhagen Business School, and is known as a reliable source of financial time series data. Here we also obtain the Adjusted Close prices, meaning that dividend payments are accounted for in the price quote. From Datastream the data is filtered in Microsoft Excel before R-Studio is used to conduct the analysis itself. The rationale behind using R-Studio as the main software for the analysis is due to its computing power compared to Excel. We believe that R-Studio will prove sufficient enough in handling the amount of calculations we require in this thesis.

The S&P 500 Index

The Standard & Poor's Index is an equity index consisting of 500 large-cap companies specifically chosen by the S&P Index committee that consists of expert economists and analysts working at Standard & Poor's. The index itself is considered the best gauge on the US economy as the majority of equity value is considered. To qualify for the index, companies must have a market capitalization of over \$5.3 billion from 2014 (\$6.1 billion after March 20. 2017), most of its shares in public hands and headquarters in the US among other criteria. The index is value-weighted, and the market capitalization of each included company is calculated using available public shares relevant to the index. The implication of this is that that the most valuable companies will have a greater impact on the index than the small-cap ones. It is also prone to what we call the survivorship bias, as only the 500 biggest firms at each point in time is included. The index began back in 1923 and was known as "The Composite Index" which only tracked a small number of stocks at the time, and it was not until 1957 that the index began to track the full 500 companies as we know today.

Figure 6.1: S&P 500 daily prices 1990 - 2017, Source: Datastream

We have chosen call options written on the S&P 500, as this market is the most active options market in the United States, with the most open interest[14]. The underlying is a cash asset and the options are European, which is suitable for our

[14]Source: http://www.cboe.com/data/current-market-statistics#openInterestIndex

models. The Index have no features that makes pricing complicated and according to Rubinstein (1994), the S&P 500 is one of the best markets for testing option pricing models.

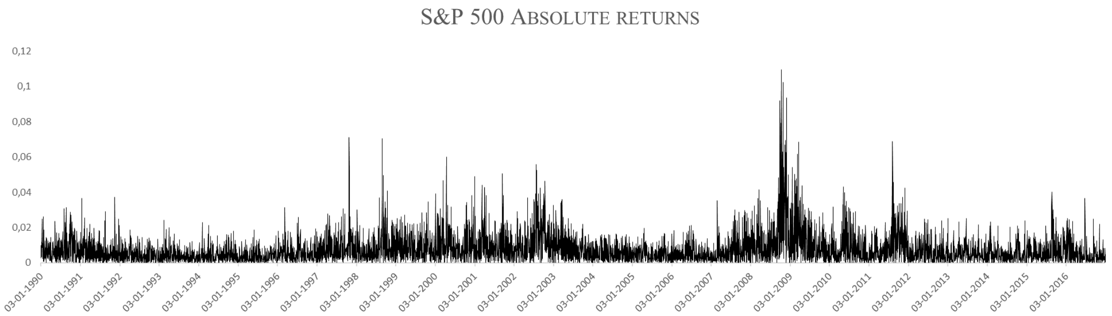


Figure 6.2: S&P 500 daily absolute returns 1990 - 2017, Source: Datastream

Pictured in Figure 6.1 we see that the S&P 500 Index have overall had an increasing trend, commonly referred to as a bull-market over the course of our analysis. The clearest exceptions are the Dot-Com burst in 2001 and the Financial Crisis of 2008, which are indicated by the major price declines, or bear-market if you may. We see that in the end of 2008 it has the most dramatic drop in the Index value, which also is confirmed with the largest absolute change in Figure 6.2. There is also some smaller declines in the latest years which could be explained with some effects of the European Debt Crisis, or the later increase in the Federal Funds Rate.

Options Data Filtering

We obtain the prices from the Options Price Reporting Authority (OPRA), that sources its information from a range of market data vendors such as Chicago Board Options Exchange (CBOE), NYSE Amex and more, through Datastream. Due to ease of obtaining and filtering the prices we assume that the prices will hold as true market prices for the sake of our thesis. The prices are not timestamped.

Before running the models on the observed data we need to filter and sort the data. This is done in order to remove the option prices that are considered to be too illiquid, far out-of-money or far in-the-money to be correctly quoted. Filtering of the data should leave us with the best sample for a fair comparison of the models, and a number of options that we can handle computationally.

First, we follow Christoffersen et al. (2005) and Heston & Nandi (2000) using only Wednesday data in our sample. The reason for this is that Wednesdays are the day in the week that are least likely to be a holiday, thus giving us a better time series. Wednesdays are also less prone to day-of-the-week effects such as Mondays and Fridays[15]. In the cases where Wednesdays are holidays or non-trading days, Thursdays have been used. It also worth mentioning that the computational burden is lessened with picking fewer weekdays in our sample. The option data is further filtered in order to get rid of biases that may impact the models in different ways. Following previously literature we apply filters according to Bakshi & Chen (1997) and Heston & Nandi (2000).

1. Options with less than 6 days to maturity are excluded as short term options have substantial time decay and liquidity related bias.

2. Options with moneyness below 0.95 and exceeding 1.05 are excluded from the sample. This will exclude the most OTM and ITM options that are rarely traded. We continue to define moneyness as S_t/K.

3. Options below the no-arbitrage boundary from Equation 4.4, further explained in Cox & Rubinstein (1985), are excluded.

4. Option prices below \$0.5 are excluded to remove the options where the price are too low for the models to handle.

After the filtering is done we are left with 15,780 European call options written on the S&P 500 Index. In Table 6.2 we have sorted the data after days to maturity and moneyness in order to see if there is any categories that are over- or underrepresented in our sample. From the looks of it, the data seem to be stable over our categories which makes the sample fit for model testing. Most options are in the middle maturity bracket and in the ATM bracket, this is where we would expect the options to be most liquid and thus correctly priced.

For a more detailed description we also present the yearly descriptive in Appendix 10.5 -10.7. We find our data to be reasonable distributed, with a bit heavier representation of middle maturity ATM options. It is slightly less of short-term OTM options, but we find the sample to be representative to capture the options we might expect have the least bias. Observing the monthly average prices from Appendix 10.1 we cannot detect any particular seasonality trends, so we feel safe to split the year into parts.

When looking at the average implied volatility we find reasonable figures in comparison with VIX. We can also see this in Appendix 10.10, where the implied volatility is plotted against the VIX. Whereas VIX uses options to predict the change in S&P

[15] Day-of-the-week effect is the empirically supported theory that Mondays have more volatility than the other weekdays. We will refer to Gibbons & Hess (1981) for a more thorough explanation.

500 over the next 23-37 days it has a higher average volatility than our sample, but the trends follow the same pattern. We also see that our sample has an increasing implied volatility in regards to moneyness.

Table 6.2: S&P 500 Index Option data, 2014-2016

	9>DTM<28	29>DTM<48	49>DTM<68	69>DTM<88	**Total**
			Panel A. Number of call options		
0.95>S/K<0.97	429	889	927	755	**3000**
0.97>S/K<0.99	836	897	891	733	**3357**
0.99>S/K<1.01	874	871	854	712	**3311**
1.01>S/K<1.03	829	831	828	679	**3167**
1.03>S/K<1.05	703	799	786	657	**2945**
Total	**3671**	**4287**	**4286**	**3536**	**15780**
			Panel B. Average Option Price		
0.95>S/K<0.97	$1.917	$4.088	$7.714	$12.681	**$7.060**
0.97>S/K<0.99	$5.538	$13.085	$20.200	$27.314	**$16.201**
0.99>S/K<1.01	$20.932	$32.439	$40.844	$48.521	**$35.028**
1.01>S/K<1.03	$48.897	$59.033	$66.724	$73.865	**$61.571**
1.03>S/K<1.05	$81.606	$89.260	$95.714	$101.932	**$91.982**
Total	**$33.139**	**$38.255**	**$44.449**	**$51.263**	**$41.662**
			Panel C. Average Implied Volatility		
0.95>S/K<0.97	9.632%	8.438%	8.507%	8.970%	**8.764%**
0.97>S/K<0.99	8.893%	9.118%	9.405%	9.818%	**9.291%**
0.99>S/K<1.01	10.172%	10.399%	10.506%	10.790%	**10.451%**
1.01>S/K<1.03	11.689%	11.558%	11.474%	11.646%	**11.589%**
1.03>S/K<1.05	13.196%	12.495%	12.301%	12.397%	**12.589%**
Total	**10.739%**	**10.340%**	**10.361%**	**10.663%**	**10.511%**

Note: The sample data consists of European call options on the S&P 500 Index in the period 15.10.2013 - 15.12.2016. We only use Wednesday data. The implied volatility is calculated numerically trough Black-Scholes and annualized by being multiplying with $\sqrt{252}$.

Risk-Free Rate

The Federal Funds rate is the US domestic interest rate that a depository institution, or bank can borrow unsecured funds from other depository institutions over night. It is also known as the "overnight rate". This interest rate can be said to have zero risk, as it is the lowest interest rate available to institutions. This implies that it is only available to the most creditworthy institutions. For the purpose of option pricing, the rate serves as a good proxy of the risk-free rate as it characterizes much of the state of the US economy that is also incorporated in option prices.

The rate itself is calculated as a volume-weighted median of overnight federal funds transactions and is quoted annually. The rate is annualized using a 360-day year rather than using 252 trading days[16]. For our models we then utilize the discrete daily rate computed from the annual rate.

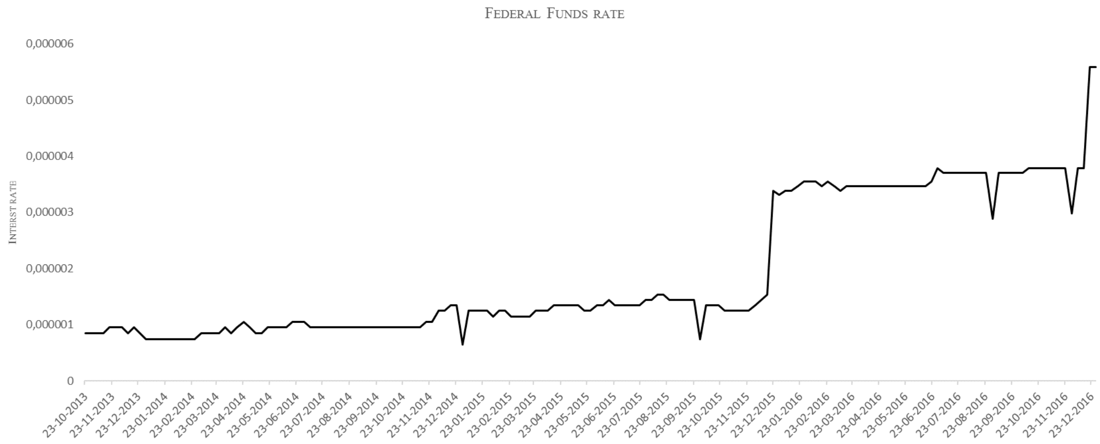


Figure 6.3: Federal Funds rate 10-2013 - 12-2016, Source: Datastream

As we can see from the plot in Figure 6.3, the rate was kept stable and low in 2014 and most of 2015. In late 2015, the Federal Reserve in the US decided to increase the rate by 25 basis points, and again in late 2016. For the sake of our analysis, more options would possibly be filtered by the boundary condition in filter number 3. This is however not confirmed with a higher option count in our filtered sample the latest years. We find what effect the interest rate changes might have to our models to be of interest.

Black-Scholes Volatility

In order to find a volatility to use for input in the BSM-model there is several approaches including the VIX Index, historical volatility on returns, and various approaches using implied volatility. While it is possible to help the BSM performance substantially with implied volatilities for different levels of moneyness and time to maturity - we choose to do it relatively simple. We average the implied volatility for each sample when we do the pricing. Mainly because BSM only works as a benchmark, but also to see how important volatility updating is, or how other models capture smiles better by nature.

In Table 6.3 we have averaged the implied volatility which is calculated by the method described under Section 5.1.2. Furthermore is the average done over the first period as described later in Table 6.7, or the whole year as described in Table 6.1. The volatility is then annualized by multiplying it with $\sqrt{252}$. This represents the yearly volatility as implied by the BSM-model. From Figure 10.9 in Appendix

[16] https://www.federalreserve.gov/releases/h15/#fn1

Table 6.3: Average implied volatility

	Time Periods	
Year	**First half**	**Whole year**
2014	9.0935%	9.0916%
2015	10.7052%	11.2370%
2016	12.1325%	11.0454%

10.10 we see how the averages look in comparison to the whole series, and to the VIX. As the price of a call option is increasing in volatility it is interesting to see if our averages lead to either over, or underpricing the options. If we for instance find that BSM underprice more than our GARCH-models, an average of VIX could have been a better estimate.

Random Numbers

As we need to run Monte Carlo simulation on some of the models we have chosen to generate a big set of standard normal random numbers. Looking at previous research we find that a Monte Carlo simulating 1000 price paths is sufficient (Christoffersen et al. 2005). The set is also generated with methods of *antithetic variables* and *moment matching* to increase confidence in our findings as we explained in Section 5.6.1. Furthermore is the length set to the longest maturity in our option data. This set will then be used for the execution of parameter estimation and option pricing. This will ensure a higher degree of comparability across models, and as it is generated outside of any computation it will save us for a lot of computational power.

6.2 Parameter Estimation

In the parameter extraction there is five parameters to be obtained - the model dependent variables $\theta = (\omega, \alpha, \beta, \lambda, \gamma)$. As the parameters of the GARCH models determines the fit of the models with respect to the observed data, the extraction of correct and unbiased parameters is of utmost importance. In the literature two different methods of parameter filtering are commonly used in order to obtain as true values as possible. The first method is maximum likelihood, that uses the return of the S&P 500 index in order to calculate the maximum likelihood estimates for all parameters in the GARCH(1,1) models. This is the conventional method within volatility modeling and used in the early work on GARCH option pricing as seen in Duan (1995). The parameters are fitted under the physical GARCH process on historical return data, before being risk neutralized to price options under the Q-measure. Later research is however, introducing a second method as a standard of fine-adjusting the parameters (Christoffersen & Jacobs 2004*a*, Chorro et al. 2014). Referred to as calibration the second method take advantage of the observed cross sectional option prices in the market, and minimize the pricing error from the models directly. We will be referring to the maximum likelihood parameter extraction as estimation and the pricing error minimization extraction as calibration.

When pricing after both methods we can see if historical prices of the underlying are indeed able to explain the option prices. Furthermore, we can compare whether the choice of parameter fitting greatly affect the models ability to price options. It is natural to assume that the calibration of the parameters through minimizing the pricing error will produce parameters that will better fit the observed option prices as maximum likelihood estimation tends to ignore information that complement the time series of underlying option prices (Hsieh & Ritchken 2005). The prices contains forward looking information that historical returns have difficulties explaining. Such information can refer to future expectations represented by the intrinsic value and time value discussed in Section 4.1.

In- and Out-sample

To test the different pricing models we want to see how the models perform in pricing the sample it is fitted on, and how well that fitting can explain prices outside of the fitting-sample. We will use a separate definition for in-sample and out-of-sample depending on the estimation method. For the estimated parameters we define in-sample as the last respective year in which the parameters are fitted on. For the calibration we will refer to the first half of the respective year as in-sample and the second half as out-of-sample.

6.2.1 Estimation

For the maximum likelihood (ML) estimation we use daily return data of the S&P 500 Index spanning over 3 periods depending on the in-sample period we want to price. For 2014 we have the period 02.01.1997 - 09.12.2014, for 2015 we use the period 02.01.1998 - 08.12.2015 and for 2016 we use the period 04.01.1999 - 06.12.2016.

Following arguments from Christoffersen & Jacobs (2004*b*) it is important to estimate GARCH models using long time series of data in order to obtain a satisfactory parameter fit. Using 17 years, we believe that the parameters have enough time to converge towards their true value, following Chorro et al. (2014) we believe that over 4000 daily return observations is satisfactory to obtain a fair maximum likelihood.

Table 6.4: Summary Statistics of the S&P 500 daily returns.

	02.01.97 - 09.12.14	02.01.98 - 08.12.15	04.01.97 - 06.12.16
# of observations	4517	4516	4516
Maximum	10.95%	10.95%	10.95%
Minimum	-9.46%	-9.47%	-9.47%
Mean	0.02%	0.01%	0.01%
Median	0.06%	0.05%	0.04%
SD	1.26%	1.26%	1.24%
Skewness	-0.22	-0.20	-0.18
Excess Kurtosis	7.53	7.54	7.82
JB	10715 (2.2e-16)	10752 (2.2e-16)	11555 (2.2e-16)
Ljung-Box (SQ-ret)	195.31*	192.22*	195.97*

Note: (*) denotes significance at the 1% level.

For our GARCH models to correctly be able to model the observed return we check for stylized facts in the underlying time series data. We look for GARCH effects, normality, kurtosis and skewness. The descriptions of the observed data will give us indications of what degree we should expect the GARCH models to capture the observed data characteristics.

The descriptive statistics of the return data used in the MLE is presented in Table 6.4. We can see that the excess kurtosis is positive for all years, implying that the true distribution of the returns are leptokurtic and display thicker tails than the normal distribution. The excess kurtosis is also implied from the density-plot of the logarithmic returns in Appendix 10.1. Here we clearly see a leptokurtic form of the plotted distribution curve compared to that of the normal bell curve. The skewness is slightly negative, telling us that the returns are negatively or left skewed, which is in alignment with a median higher than the mean. This is in accordance with the stylized facts earlier discussed and raise indications of the leverage effect.

We also perform the Jarque-Bera[17] test for normality to check whether the returns are normally distributed. The JB test is asymptotically distributed as chi-square with 2 degrees of freedom, and we test whether the sample data comes from a normal distribution. The null hypothesis is a joint hypothesis that the excess kurtosis is zero and that the skewness is zero. Any deviation from the null increases the JB

[17]The Jarque-Bera test is calculated as $JB = \frac{T}{6}\left(\widehat{Skew}^2 + \frac{(\widehat{kurt}-3)^2}{4}\right)$

statistic, and in our samples we reject the null in all three cases with significance at the 1% level.

We also check the returns for GARCH-effects or volatility clustering through examining for autocorrelation in the squared and/or absolute log returns. From the correlograms in Appendix 10.3 we see that the logarithmic returns does not display any autocorrelations. The squared and absolute log returns do however display positive autocorrelation. The rates seem to decay slow, implying possible long memory behaviour. We test the significance of the autocorrelation with the Ljung-Box test. The significant value for the Ljung-Box test confirms the presence of autocorrelation in the squared log return, thus confirming GARCH effects.

The statistics tell us that the distributions of the returns are not normal, displaying both excess kurtosis and negative skewness. Possible explanations for this could be the large outliers in the absolute returns occurring in the years '02-'03, '08-'09, '11-'12 and '15-'16 seen in Figure 6.2. The incidents are among others the dot-com-bubble and the financial crisis as explained in Section 4.2.6. Establishing that GARCH effects does indeed exist we believe that the returns can be modeled with a GARCH framework, in order to obtain the parameters and option prices that are in accordance with the observed values. We further commence the parameter extraction through ML estimation.

Obtaining The Parameters

To obtain the parameters we model the log-returns with each respective GARCH model and fit the modeled data to the observed data. We set the initial volatility equal to the long-term unconditional variance in accordance with Christoffersen et al. (2005). To maintain comparability we have used the same random numbers for the models, so that any bias created by random shocks will affect all models simultaneously. Even more importantly we use the same procedure and formula to compute the maximum likelihood as described in Section 5.5.1 across all models.

As the optimization of the parameters are a quite complex non-linear problem, the estimation is not straightforward. Systematically the models are extraordinary sensitive to the initial parameter guesses and the slightest change in decimals will make the models unstable and send the conditional volatility to either positive infinity, or negative infinity. We try to obtain the optimal parameters with initial guesses inserted into the model through trial and error. Furthermore, we set restrictions on the parameters to all models so that the unconditional variance cannot be negative. We also find that Microsoft Excel's Solver-function as a optimizing tool exhibits very poor ability to search for global extremes, and frequently ends up with no solution. R-Studio's .nlm solver does a better job in searching, but are equally sensitive to initial guesses close to the local optimum. The difficulty of obtaining the optimums can be explained by the multidimensional nature of the optimization. As all the parameters are inter-dependable, there will exist several local optimum in the parameter surface.

After using trial and error with regards to initial guesses in both Excel and R-Studio we finally came up with the parameter estimates and the respective maximum likelihood as presented in Table 6.5. We found that the GenSA solver-package of Yang Xiang et al. (2013) gave a satisfactory convergence for the maximum likelihood global optimum. The GenSA optimizer use a simulated annealing approach to the optimization problem. The method is a stochastic heuristic method, and starts at a randomized point x in the parameters space and evaluate nearby points. We believe that the converged solutions are the true global optimum as the GenSA optimizer outperforms most global optimizers in R-Studio in terms of finding the correct optimum according to Mullen et al. (2014).

Table 6.5: Results of Maximum Likelihood Estimation

	1997-2014					
	ω	α	β	λ	γ	ML
HN-GARCH	2.872211e-12	3.995971e-06	0.7719184	2.999846e+00	2.208513e+02	14281.29
NGARCH	2.568076e-06	8.109654e-02	0.8034266	2.065644e-02	1.128217e+00	14347.92
EGARCH	-0.420222243	0.133449535	0.965149228	0.006937182	-1.150400096	14303.12
GJR-GARCH	1.978983e-06	6.309820e-07	0.9039489	2.875400e-02	1.580757e-01	14322.57
	1998-2015					
	ω	α	β	λ	γ	ML
HN-GARCH	4.571968e-14	4.299157e-06	0.7638649	2.999957e+00	2.155321e+02	14339.84
NGARCH	2.577466e-06	8.109670e-02	0.7918907	1.362413e-02	1.192762e+00	14410
EGARCH	-3.575447e-01	1.098150e-01	0.9700947	3.828254e-07	-1.473789e+00	14362.42
GJR-GARCH	1.968960e-06	8.058618e-08	0.9026514	1.958492e-02	1.612309e-01	14381.9
	1999-2016					
	ω	α	β	λ	γ	ML
HN-GARCH	2.780723e-12	5.575514e-06	0.7855257	2.999964e+00	1.733111e+02	14415.29
NGARCH	2.538075e-06	8.109601e-02	0.7793669	4.555432e-03	1.266156e+00	14501.78
EGARCH	-3.101335e-01	6.723237e-02	0.9718247	3.153842e-07	-2.423817e+00	14413.63
GJR-GARCH	2.213562e-06	3.813912e-08	0.8951764	1.525685e-02	1.721303e-01	14463.45

Note: All parameters are obtained by fitting the observed data with the same ML function for comparability. The results are satisfactory in terms of low variability across the maximum likelihood estimates, further supporting that the global optimum are found for each model.

THE PARAMETERS

From the estimation of the models we see that the ML of the models are not differing a whole much - and model performance seems to be consistent over the tree years, something that is to be expected. The results are interesting in several ways; first we see that the best performing model in terms of the highest maximum likelihood is the NGARCH. The GJR-GARCH is not surprisingly also performing well as it is especially designed to fit market data. Not surprisingly is HN-GARCH performing worst, but this could be explained with the cost of the models affine structure for closed for option pricing. EGARCH also have a relatively low ML, which can be explained by its criticism for not being able to model stylized facts such as volatility clustering especially well.

Comparing to other research such as Christoffersen et al. (2005) and Hsieh & Ritchken (2005), we can see that our parameters for the NGARCH and the HN-GARCH is reasonably concurring. While ω and γ under HN is quite low, β is relatively high. Respectively representing weight assigned to long-term variance, the leverage effect and lagged variance this could point to an overall lower, yet more clustered variance in our data. That the leverage parameter γ is substantially positive for both models indicating that shocks to returns and volatility indeed are negatively correlated. Our findings of the NGARCH parameters is more similar to their research. Worth noticing is it that they have a higher variability in the parameters, but this could be explained by our bigger data with the majority of years being present in each sample. Furthermore, the others used annual and semi-annual calibration approaches. Looking at Alexander (2008), fitting on CAC 40, DAX 30, FTSE-100 and Eurostoxx 50 from 1991-2005 we find the GJR-GARCH parameters to be quite similar, except for a quite low α value indicating weak mean reversion. Determining the kurtosis, it might point to less fat tails and extreme observations in our data. Knowing about the later turmoil in the markets, it could possibly be explained by lower sensitivity to market shocks compared to the relative quiet 90s. EGARCH seem to be having relatively coinciding results despite some minor differences in the modeling. The minor differences in results can point us to some trends - but if the reason actually is different data, or poorer fitting, is hard to tell without further investigation. We would expect some differences from the early 90s compared to our data and find our estimates overall reasonable.

To analyze the parameters further, we calculate the unconditional variance our fitted parameters represent. As this is numbers based on our parameter estimations we call this the implied unconditional variance. This will represent the variance our parameters would imply without any market information. It can be interpreted as the expected long term variance, and while the models are fitted on a data set of 17 years, it should be comparable to VIX over the same period and the implied volatility from our samples. We are reporting the annualized standard deviation for interpretational purposes in Table 6.6. With the VIX-Index fluctuating between 10% and 25% in the 17 year data set, the figures seem more or less reasonable for the HN-GARCH, NGARCH and GJR-GARCH. While EGARCH is relatively low. Even for our implied volatility which is between 9% and 12%. Interesting is it to notice that it does not seem to be a pattern with better fit and more reasonable implied volatility. With the sensitivity of the parameter estimation and without other tests we do not, however, want to draw some further conclusions from this.

Table 6.6: Implied Unconditional Standard Deviation from MLE

Year	HN	NGARCH	EGARCH	GJR
2014	17.42%	22.98%	3.83%	17.12%
2015	17.25%	23.62%	4.02%	17.22%
2016	17.29%	25.91%	6.46%	17.24%

6.2.2 Calibration

Following existing literature we calibrate our models by minimizing the pricing errors represented by mean squared errors (MSE). To save computational power and calibration time we divide the annual samples of option prices into in-sample and out-of-sample semi-annual datasets. We will refer to the first semi-annual dataset as in-sample and the second semi-annual dataset as out-of-sample. The reason for doing this is so we can calibrate our data on the in-sample dataset, and extend our analysis by pricing the out-of-sample dataset with the same parameters. The in-sample calibration will tell us to what extent the models are able to fit the option data, and are important in terms of model theory. The out-of-sample pricing are important in order to analyze the models ability to correctly price options based on historical data, and can be seen in a practitioners perspective as the parameters persistence over time.

Ensuring no overlap in the underlying stock path, we divide each year into options maturing before mid-July and the options we have price quotes on from mid-July. This procedure eliminates a few hundred options for each year and lay the foundation for a more thorough analysis. More details can be found in Table 6.7.

While the estimation only simulates the GARCH process and try to fit this, the calibration will have to calculate each option price and minimize this accordingly. In the case of the HN-GARCH that presents a closed form solution, the procedure is quite straight forward. Problems arise when minimizing the GARCH models that require numerical pricing. The computational power required to price each option with Monte Carlo under the optimizing procedure have shown itself as extremely demanding. For that reason, we will have to extract a representative sub-set of data to save computational power for each optimization.

Table 6.7: Semi-Annual Data Set

		Number of options		
		Quoted after	**Expiring before**	**Number of options**
2014	In-sample	23-10-2013	19-07-2014	2742
	Out-of-sample	19-07-2014	20-12-2014	1678
2015	In-sample	22-10-2014	17-07-2015	3109
	Out-of-sample	17-07-2015	19-12-2015	1821
2016	In-sample	21-10-2015	15-07-2016	3182
	Out-of-sample	15-07-2016	16-12-2016	1855

When making the representative sub-set when we calibrate, we want to capture the option price characteristics over both moneyness and maturity of the in-sample dataset. With our cutoffs in maturity there is only 12 different days to maturity for each option. We choose to take 3 different maturities in this specter, where we define 87 days as long maturity, 45 days as middle- and 10 days as short maturity. This filtering yields around 1000 option prices, a subset that we still see as too large to effectively run our calibration on. For that reason we filter out more options by extracting far OTM and ITM as well as ATM options. Using a cutoff of options with $S/K \leq 0.952$, $0.998 \leq S/K \leq 1.002$ or $S/K \geq 1.048$ we are left with around 50 representative prices for each sample seen in Appendix 10.5. We observe that the lowest maturity options deeply OTM are not well represented in the subset. This is mainly due to the strictly imposed criteria for being accepted into the sub-set, and we believe that an overweight of ATM option will help us price more correctly on average. Furthermore, the low-maturity, low-moneyness bracket exhibits the lowest option prices, and the many of these are already excluded by the lower-boundary and lower price filtering. The sub-set should represent all characteristics of the option prices, and are hopefully a good replication of the in-sample datasets. We find the average prices from the sub-set reasonable, and the count fair considering the spread of the options.

When the calibration sub-set data is ready filtered, we then need to price the options and minimize the sum of MSE from observed prices in order to calibrate the parameters.

Trying several optimizers in R-Studio as Optim and nlm based on Nelder–Mead, Newton-type and conjugate-gradient algorithms we encounter difficulties in obtaining satisfying solutions. As the functions have numerous local minimums the choice of initial parameters are particularly essential, especially as validating any global minimums is impossible. We also find HN-GARCH not to be very apt to such optimizers, as the integral often yield no answer coinciding with particular sets of parameters. Even when setting reasonable intervals for the solution for each parameter, most optimizers will stop after reaching a "no answer" realization. Furthermore, is the process of running a optimizer vastly time consuming, and the chance of fruitless results big. Manual guessing is, however, possible - but can lead to local minimum that is hard to get out of without shocking any of the parameters. After some rounds of trial and error we have again found the GenSA optimizer to be of most help as it converges to better estimates and do not depend as much on initial guesses. As the optimizer does not stop when reaching a local minimum, we can also be more sure that the solution indeed is a global minimum. Nevertheless, the process on calibrating the parameters with only 50 prices do take between 10-23 hours for each model, each year. The procedure for pricing will be further explained under Section 6.3.

The Parameters

After running the optimizer to minimize pricing errors we find the parameters presented in Table 6.8. The first thing to notice is that the models seem to be more aligned in terms of optimal solution than from the estimation. Comparing the parameters to those of the estimation we see some substantial differences, but overall the trends of the parameters can be said to be somewhat similar. Naturally we also get more parameter variation over the in-sample years due to the difference in the underlying market conditions of the data on which it is fitted. While the estimation were fitted on a 17 year long stock price path which did not allow for much variance in parameters, the calibrated parameters will fit more specific to each year. How much the parameters change can also indicate how well the models are to price out-of-sample. It is no point in comparing the models, but the fact that they change quite substantial over the years shows that frequent calibration might be of importance.

Interesting is it to see that while the HN-GARCH had most trouble fitting the data under MLE, it obtains the lowest MSE in almost all three years when calibrated, only beaten by the EGARCH in 2014. The EGARCH is close to hit its upper bound on the β parameter in all years, but as it presents low MSE and does not hit the boundary we find it reasonable. NGARCH is still performing better than GJR-GARCH, which now actually is the worst model at pricing correctly in terms of the highest MSE. EGARCH is clearly better than NGARCH in 2014 and 2015, but in 2016 NGARCH shows slightly less pricing errors on the small sub-sample. Looking at the HN-GARCH one might start to suspect that despite its bad fit to time series data, the model can present completely different attributes when utilized directly for option pricing without employing the physical measure.

Table 6.8: Results from the calibration

	ω	α	β	λ	γ	MSE
			2014			
HN-GARCH	2.911876e-06	7.066537e-06	4.600001e-01	1.200000e+00	1.912600e+02	7.564329
NGARCH	6.705188e-08	5.549488e-02	9.420739e-01	7.843371e-01	4.088499e-05	14.07908
EGARCH	-0.07016999	0.0603161	0.9935334	1.062813e-07	-0.7662512	6.454334
GJR-GARCH	9.677251e-06	3.832354e-01	9.672779e-07	1.523714e-02	1.085095e+00	22.02949
			2015			
HN-GARCH	1.022033e-06	5.502867e-06	3.716308e-01	2.421528e+00	2.998210e+02	17.09186
NGARCH	1.642126e-06	8.915774e-03	8.728269e-01	3.999881e+00	2.895139e-08	29.38044
EGARCH	-1.073249e-01	6.704799e-02	9.899286e-01	4.851942e-08	-9.383851e-01	25.76237
GJR-GARCH	1.342236e-05	4.926594e-01	8.175669e-12	1.411061e-02	8.941153e-01	34.74352
			2016			
HN-GARCH	1.602767e-06	5.616600e-06	3.716308e-01	2.421528e+00	2.998210e+02	46.92612
NGARCH	5.537153e-08	2.661920e-02	9.693977e-01	8.864790e-01	3.152471e-01	48.47109
EGARCH	-8.247109e-05	0.03944052	0.9999918	3.24247e-07	-1.52052	48.48613
GJR-GARCH	1.296679e-05	5.754054e-01	2.115665e-07	2.167333e-02	7.373130e-01	65.82331

Note: Results from calibrating parameters on in-sample data minimizing MSE.

When comparing with the estimated parameters it is noticeable that the change in parameters is not necessarily in the same direction for each model. For the HN-GARCH it is quite interesting that the first year seem to be different from the rest in terms of change. All parameters except for β changed in the opposite direction the first year compared to the last two. This might however be spurious results as we also experienced the worst fit on the HN-GARCH under its physical measure.

While the HN-GARCH and particularly GJR-GARCH shows signs for less volatility clustering represented by β it is interesting that the NGARCH and EGARCH shows the exact opposite. When comparing VIX over the 17 year sample with the 3 years in question this is more or less incomparable simply because of the vast difference in sample size. When looking at the HN-GARCH, NGARCH and EGARCH there is signs of lower γ in earlier periods as opposed to higher in 2016. However, GJR-GARCH has a substantially higher γ in all years, and again it is hard to draw any good explanation from the results. Similarly do the models have conflicting pointers with regards to α and λ. All of this can also come from model mechanisms to find better fit, as we find that if one parameter tend to go up, another tend go down and vice versa.

Table 6.9: Implied Unconditional Standard Deviation from calibrated parameters

Year	HN	NGARCH	EGARCH	GJR
2014	9.45%	8.34%	6.99%	18.17%
2015	11.09%	5.92%	7.70%	24.95%
2016	12.14%	10.21%	10.39%	24.33%

Interestingly we find a trend in ω to be smaller in all models, whereas it is bigger (less negative) for EGARCH. This is a bit contradicting as with all else equal, a lower ω will imply a lower unconditional variance for all models. To analyze this more closely we calculate the implied unconditional standard deviation from these parameters as well. The results can be seen in Table 6.9.

Indeed we find that all model parameters except for EGARCH indicate a higher unconditional variance than the one we saw from the estimated parameters. As we have seen that our sample of option prices has a lower implied volatility than VIX, we find it reasonable to only compare with the implied volatility here when the parameters are derived solely from these data. The results seem to have stabilized more, and are way more similar to the implied volatilities between 9% and 12%. EGARCH has increased to more reasonable figures, while HN-GARCH and NGARCH has declined to better indications. GJR-GARCH seems to still be quite off, but if this is due to bad fit, or its more complex nature around the P- and Q-measure is too early to say. It might also not be anything crucial with regards to the option pricing if the figures are not coinciding.

We still cannot detect any clear pattern between fit and the unconditional volatility. While HN-GARCH has the lowest pricing errors, we do not find the second best model to have the most similar unconditional volatility.

6.3 Model Implementation/Option Pricing

After obtaining the parameters through estimation and calibration, we use those parameters to price the options. First we price options using the estimated parameters before we conduct a more thorough analysis of the calibrated parameter prices. The benchmark Black-Scholes prices will be calculated with the volatility and risk-free rate previously discussed, and will only serve as a proxy for the GARCH models. In terms of pricing, the BSM and the HN-GARCH is quite straightforward due to their their analytical form, while NGARCH, EGARCH and GJR-GARCH needs a Monte-Carlo simulation as explained earlier. To save computational power and maintain a higher degree of comparability we still use the same big set of random generated numbers.

Pricing with Estimated Parameters
With the estimated parameters we price the whole respective year that is included in the GARCH fitting return sample outlined in Section 6.2.1, and will refer to this price sample as in-sample. For all the options in the respective year we then calculate the MAE and MAPE for each model and report the results in Table 6.10. We will only price options with estimated parameters in-sample as the small variability in parameters over the years in our sample indicates that pricing an out-of-sample data-set would yield marginal differences in the prices given that there is no radical changes in the market situation.

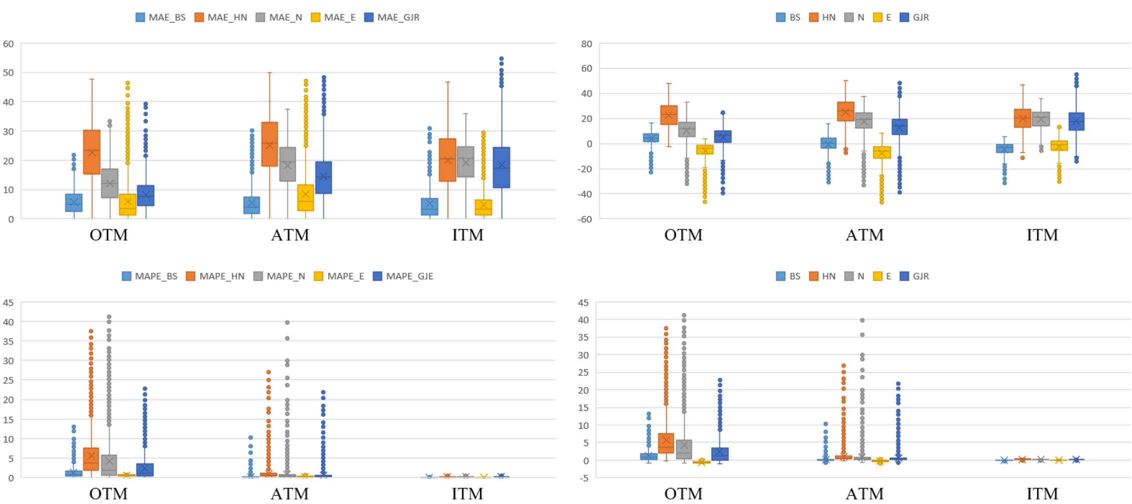
Table 6.10: Pricing Errors with parameters obtained through ML

		BSM	HN-GARCH	NGARCH	EGARCH	GJR-GARCH
2014	MAE	3.4105	27.0877	17.0266	3.8455	19.2587
	MAPE	0.3858	2.8463	1.9601	0.2596	1.5929
2015	MAE	6.2138	21.8148	14.7155	9.2693	11.3470
	MAPE	0.5632	1.8989	1.4738	0.4129	0.8994
2016	MAE	6.4507	21.6977	18.3797	7.5818	10.1697
	MAPE	0.6078	1.9798	1.9218	0.3121	0.9071

Note: We only present MAE and MAPE as the same conclusion is drawn from all loss functions.

Using the parameters from the maximum likelihood estimation, EGARCH is the model displaying the lowest pricing errors of the GARCH models. The BSM model have the lowest pricing errors overall, but a direct comparison between the benchmark model and the GARCH-models is rather unfair toward both parts. The benchmark model benefits from being fed with the average implied volatility for the respective in-sample year while the GARCH models are estimated using historical returns. As mentioned earlier, it the maximum likelihood method unfavored due to using backward-looking information when pricing forward looking derivatives.

Ignoring the BSM model for a while we see that the EGARCH is by far the GARCH model displaying the fewest pricing errors in terms of dollar MAE and in relative terms through MAPE. The EGARCH's superiority over the other models is somewhat surprising, as the model previously are criticized for not being able to fit observed data satisfactory. An MAE of only 3.8455 tells us that the model misprices the options of \$3.8 on average for the whole set. The worst performing model is the HN-GARCH, displaying an average error of \$27.08 on every option. This is a substantial mispricing of the options on average, and may be caused by the affine structure of the GARCH. The HN-GARCH did also have the lowest log-likelihood values for all the three years, which may help explain why the model perform so bad relative to the other models. That the NGARCH outperforms the HN-GARCH in absolute terms are in accordance with the findings of Christoffersen et al. (2005) and Hsieh & Ritchken (2005), but not to such an extreme extent.



The plots to the left display the MAE and MAPE averaged over all years. The right plots display the over- and underpricing in actual and relative terms in relation to the absolute measures. All plots are sorted after moneyness and incorporate all maturities. The relative and actual pricing errors are calculated with the following loss functions $(C^{model} - C^{market})/C^{market}$, $C^{model} - C^{market}$.

Figure 6.4: ML Pricing Errors for All Years

Implied from the ML values we would believe that the NGARCH would outperform the other models due to having the highest log-likelihood and thus the best fit to observed data for all three years. The explanation for this may be found in the theory stating that even though maximum likelihood is in most cases the preferred method of estimating GARCH parameters, it may have a hard time explaining option prices.

From Figure 6.4 the EGARCH is still outperforming the other GARCH models over all moneynesses in terms of both relative and absolute pricing errors. Where the EGARCH price options on average a little too low, the rest is exhibiting substantial overpricing and cannot be said to be anywhere near the observed prices on average. Interestingly enough are the pricing errors appearing to be worst for ATM options in absolute terms, especially when looking at variability. The results are somewhat disappointing and might back up indications of ML estimation not being a particular good way of obtaining the parameters for most of the models.

It is, however important to bear in mind that the errors are presented for all maturities. The models could benefit or be punished for differences in their ability to price certain maturities. If we were to analyze the pricing errors more substantially we would go deeper into the different maturities and moneyness to see whether the spectrum of pricing errors is more nuanced. Nevertheless, as the errors from the estimation are so high overall we will not venture further into the results of the estimation. We will however, perform a deeper analysis on the basis of the results from the calibration. As we do not want to investigate the theoretical-, but rather the practical applicability of the GARCH option pricing models we expect the calibration to yield more relevant results.

Pricing with Calibrated Parameters

For the calibration we divided each year into two different periods as described in Table 6.7. We price in-sample with the whole data-set under the first half of each year. I.e. the whole in-sample period, not the sub-set the parameters were calibrated on. Out-of-sample we price the second half. When pricing with the parameters obtained by calibration we get the results presented in Table 6.11.

Overall it is interesting to see that despite its good fitting on the calibration sample, the EGARCH is now falling a bit behind while the HN-GARCH seems to overall price the options best particularly in-sample. EGARCH and NGARCH are not much worse, and while it is no clear pattern it might seem as EGARCH is best in-sample whereas NGARCH out-of-sample. It is pleasing to see that BSM throughout the data presents high pricing errors, thus in accordance with previous research the model performs worse than most of the GARCH option pricing models. The results are nevertheless not much worse than GJR-GARCH which is by far the worst performance of the GARCH models. And calculated with average implied volatility, BSM actually prices better than GJR-GARCH in 2014 in-sample. BSM can still not be said to be significantly far behind the other GARCH models.

We find the relative pricing errors to be surprisingly high. With figures between 15-55% for MAPE in-sample the models do not impress, but it is reassuring to see that BSM seems to be equally struggling. Furthermore, we suspect that much of the error is due to mispricing low priced options, likely far OTM options. When looking closer into in-sample performance we see that the HN-GARCH is close to superior in all performance measures, including IVRMSE. With minor differences the only exception is in 2014, where NGARCH performs better in the relative measures. A possible explanation could be for instance that HN-GARCH being worse to price low price options, thus having a low RMSE in terms of dollar - but a higher relative measure. A further investigation of this will follow when we look closer into pricing performance of different types of options. GJR-GARCH is in most cases outperforming our benchmark BSM model, and are not far behind EGARCH and NGARCH - but the performance will nevertheless have to be called mediocre at best.

While we observed that the HN-GARCH and NGARCH had most problems performing with estimated parameters, they have a strong come-back after the calibrating. EGARCH seems to only be slightly improved, as it can be said to have displayed quite strong relative performance from the estimation this is somewhat expected. Nevertheless, we cannot be laying to much weight on the change in the performance measures from estimated to calibrated parameters as they are based on different data and extraction procedures. It is, however, interesting to observe that even if the EGARCH outperformed HN-GARCH when estimating parameters, the HN-GARCH is superior when being calibrated. While it could be an indication of the HN-GARCH to have some loss in fit in the trade of a closed-form solution under ML, we find the results from calibrated parameters to be challenging this view. It could imply that some of the models have a more robust theoretical framework in the transition from modeling returns as a GARCH-model to pricing options. This could be the first pointer to the HN-GARCH loosing some fit in order to have a closed form solution, but being much more precise when calibrated.

For the out-of-sample pricing we see that the pricing errors increase substantially. While it is natural and expected, it presents itself in both absolute as well as relative terms. For the earliest years it seems like the increase in absolute terms is around

Table 6.11: Option Pricing Errors From Calibration

	Panel A. 2014 In-Sample					**Panel B. 2014 Out-of-Sample**				
	BSM	HN-GARCH	NGARCH	EGARCH	GJR-GARCH	BSM	HN-GARCH	NGARCH	EGARCH	GJR-GARCH
$RMSE	3.399	2.667	3.078	2.690	4.065	5.839	5.190	5.161	5.131	5.519
%RMSE	0.644	0.393	0.313	0.398	0.487	0.884	0.574	0.436	0.593	0.606
IVRMSE	9.84e-04	7.75e-04	7.98e-04	8.09-04	1.27e-03	1.56e-03	1.32e-03	1.34e-03	1.46e-03	1.48e-03
MAE	2.677	2.007	2.197	2.117	3.276	4.258	3.705	3.535	3.700	4.269
MAPE	0.298	0.188	0.150	0.199	0.235	0.436	0.312	0.242	0.312	0.317

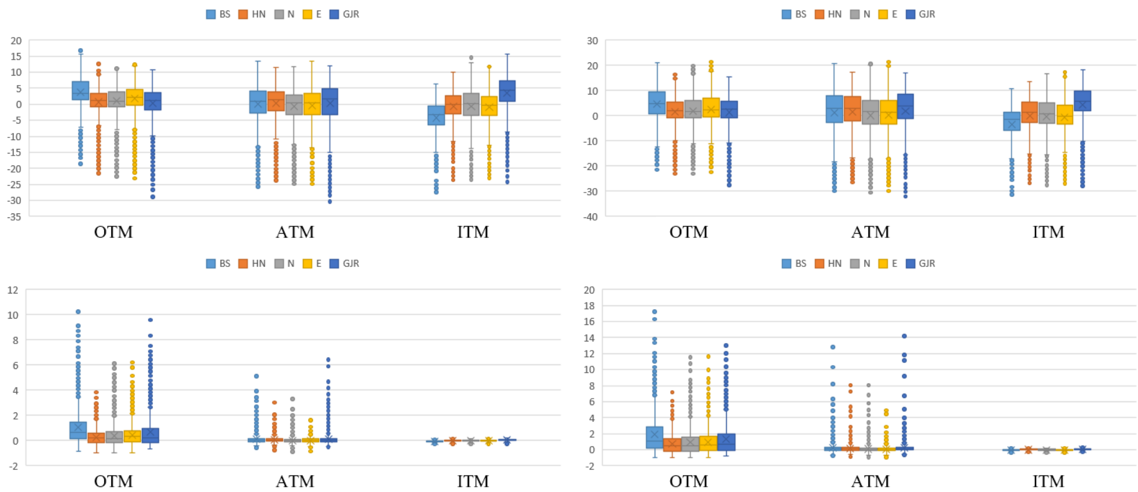
	Panel C. 2015 In-Sample					**Panel D. 2015 Out-of-Sample**				
	BSM	HN-GARCH	NGARCH	EGARCH	GJR-GARCH	BSM	HN-GARCH	NGARCH	EGARCH	GJR-GARCH
$RMSE	5.978	4.556	5.257	5.215	5.510	11.1527	10.3247	10.7440	10.576	10.5378
%RMSE	1.014	0.355	0.569	0.567	0.947	1.0848	0.5069	0.6780	0.680	1.0485
IVRMSE	1.49e-03	1.07e-03	1.53e-03	1.41e-03	1.43e-03	2.68e-03	2.48e-03	2.96e-03	2.80e-03	2.35e-03
MAE	4.804	3.712	4.265	4.226	4.611	8.2338	7.5774	8.0501	7.905	8.1732
MAPE	0.468	0.220	0.297	0.297	0.424	0.5073	0.3204	0.3789	0.375	0.4952

	Panel E. 2016 In-Sample					**Panel B. 2016 Out-of-Sample**				
	BSM	HN-GARCH	NGARCH	EGARCH	GJR-GARCH	BSM	HN-GARCH	NGARCH	EGARCH	GJR-GARCH
$RMSE	8.516	7.592	7.859	7.836	8.090	9.892	8.622	8.435	8.766	9.711
%RMSE	1.186	0.484	0.668	0.695	1.000	2.702	1.270	1.688	1.750	2.106
IVRMSE	2.02E-03	1.81E-03	1.92E-03	1.91E-03	1.93E-03	2.26e-03	1.96e-03	1.81e-03	1.87e-03	2.49e-03
MAE	6.824	5.940	5.980	6.067	6.479	8.363	7.617	7.125	7.454	8.880
MAPE	0.535	0.292	0.341	0.353	0.438	1.359	0.725	0.869	0.951	1.058

Note: The pricing errors are calculated In-Sample and Out-of-Sample from the definition in Section 6.2.2. The in-sample errors are based on 2742 prices in 2014, 3109 prices in 2015 and 3182 prices in 2016 while the out-of-sample errors are based on 1678 prices in 2014, 1821 prices in 2015 and 1855 prices in 2016. It is important to bear this in mind if comparison between the two is to be made.

a factor of two. We even find relative pricing errors above 100% for some models in the later years. Despite not being as clear, the trends are similar, however. The HN-GARCH is having the overall lowest pricing errors in 2015, while NGARCH seem to be best in 2014 and 2016. EGARCH seem to be outperforming NGARCH in 2015 and HN-GARCH in 2014. GJR-GARCH tend to outperform BSM, but there are several observations of measures where BSM has lower pricing error. IVRSME tells a slightly different story than the other pricing error measures. HN-GARCH outperforms the other models in 2014, while GJR-GARCH and NGARCH is best in 2015 and 2016 respectively. The measure is overall very low, and it is hard to draw some clear conclusion when it changes independently over the years.

Where Table 6.11 presents the pricing errors in relative and absolute number, it display the errors in terms of total average in-sample and out-sample. The results say that on average the HN-GARCH, NGARCH and EGARCH is the best performing models over all samples. We must dig deeper into the numbers to see the nuance of the models pricing errors. Option models are empirically better at some maturities than others, the same goes for moneyness. We will further analyze the pricing errors to see whether the models are punished on average because of poor performance in given moneynesses or maturities.



The plots at the top describe the actual pricing error for each moneyness, while the bottom plot describe the relative errors. The In-Sample errors is to the left while the Out-of-Sample errors is placed to the right. Note that the negative residuals in the relative measure represents underpricing due to the formulation of the actual relative loss function $(C^{model} - C^{market})/C^{market}$ and the actual loss function $C^{model} - C^{market}$.

Figure 6.5: Box and Whiskers Plot for All Years

Figure 6.5 give a introductory view of the option price errors, and exhibits the actual and relative pricing errors for all modelled prices in the samples. The plot indicates that the OTM options, who's pricing error is best pictured by the relative measure seems to be severely variable with large positive outliers for all models. The BSM model have residuals up to 10 in-sample, representing a overpricing of 1000%, this could occur if the model prices a $0.6 option to $6.

Figure 10.6 and 10.7 from Appendix 10.9 display the box and whiskers plot over moneyness for the pricing errors in relative absolute and absolute as well as relative and actual pricing errors respectively. The box and whiskers plots captures the variability within the pricing errors and while the whiskers represents the range of the errors, the box contains the upper and lower quartiles as the limits and the dots above and below the whiskers represent extreme outliers. This type of plot makes it easier to interpret the data sorted after moneyness in Appendix 10.11, as it visualizes the errors relative to each other through both absolute average (MAE) and relative average (MAPE) errors. The Figures using actual and relative measures help us observe the over- and underpricing of the models. We have sorted the data with smaller bins for moneyness, and given the plot three levels of moneyness to save space. The moneynesses are divided into ITM, ATM and OTM, and represents 0.95-0.98, 0.98-1-03, 1.03-1.05 respectively in moneyness, the plots does however paint the same picture as the more fragmented Tables in Appendix 10.11. Furthermore, the maturity sorted plots use Short, Medium and Long maturities that include the options maturing in 9-24, 30-66, and 72-87 days respectively. Also here note that the negative residuals in the measures represents undervaluation as we set up the relative loss function to be $(C^{model} - C^{market})/C^{market}$ and the actual loss function as $C^{model} - C^{market}$.

The first noticeable trend is that all models are exhibiting quite low absolute pricing errors in relative terms for ITM options. This is due to that the relative errors are so small for high prices options that the effect is marginalized. Therefore, it makes more sense to keep the focus on the dollar terms when analyzing the models performance on ITM options. The relative pricing errors is a much more interpretable statistic for the OTM options that have low prices. Here will the relative pricing error paint a more correct picture of the models pricing ability.

In-Sample

Further we will split the analysis into the in-sample part and look at those numbers separately. The reason for this is that the in-sample performance of the models can be attractive to see what model can provide the best fit for already observed data while out-of-sample performance is more attractive for practitioners in term of the obtained parameters persistence and ability to model future data.

The first thing to notice is that all GARCH models outperform the benchmark BSM model in relative terms for OTM options with the HN-GARCH displaying the best performance. The performance of the BSM model is highly dependent on its volatility that is averaged over the in-sample period. Thus the model might having a hard time capturing extremes in terms of moneyness. We will further analyze each model separately to look at its relative performance to the others.

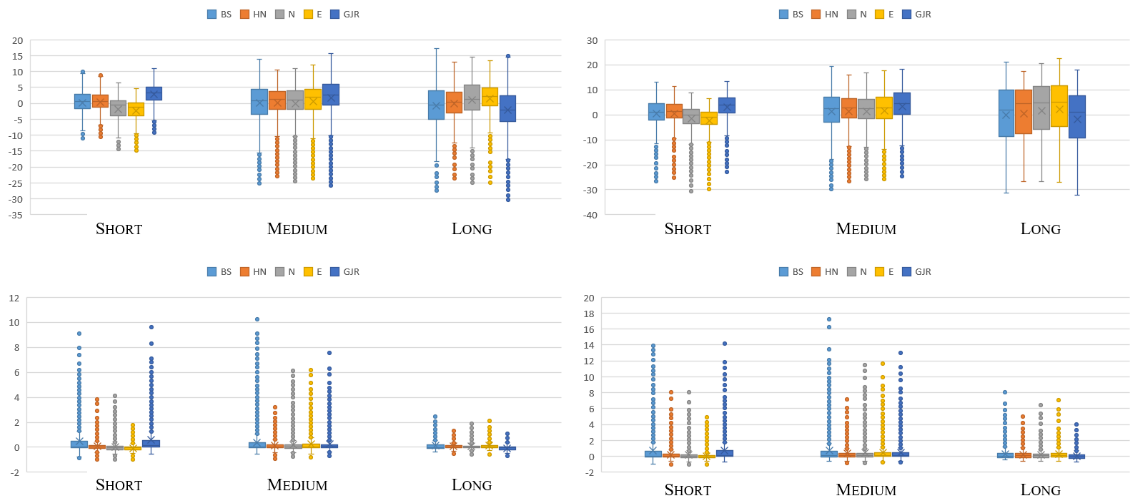
The HN-GARCH model show promising results overall in-sample, fitting the data quite well compared to the other models. With lowest relative pricing errors for OTM options in 2015 and 2016 displaying some overpricing with small variability. If we look at the Box-Whisker plot in Figure 10.8 in Appendix 10.9, we can investigate why NGARCH outperformed HN-GARCH with total relative figures in 2014 as we noticed earlier. In Figure (a) on the left hand side we indeed see that HN has higher errors OTM in relation to NGARCH, but for ITM options the Table is turned. The HN-GARCH is the best model ATM in 2015 in both relative and dollar terms, but are slightly beaten by the NGARCH model in 2014 and 2016. For ITM options the HN-GARCH model dominates and display the lowest pricing errors in both dollar and relative terms.

Exploring the performance of the HN-GARCH over maturity reinforces the relative superior performance of the model. The model displays absolute average errors slightly increasing in maturity, implying that the model struggle more with longer maturities. Even if outperforming other models for all maturities we do, however, find tendencies for the model to overprice with some extreme outliers.

The NGARCH model is also displaying overall good performance in-sample relative to the other models. It presents lowest absolute pricing errors in 2014 OTM, and is second to the HN-GARCH in 2015 and 2016. The model show tendencies to overvaluate OTM options in-sample. Furthermore, the model seem to be overall the best in pricing ATM options, with some slight underpricing in 2015 and 2016. The model displays the second best performance after the HN-GARCH in pricing ITM options. Where it is overpricing in 2014 and underpricing in 2015 and 2016. Looking over the maturity specter, the model displays increasing errors in maturity - pricing short-term options best.

With the lowest MSE off all models in 2014, the EGARCH seem to perform well overall. For all ranges of moneyness the model outperform the benchmark, and display overall the best performance for ITM options. The model seem to be slightly weaker than the HN-GARCH and NGARCH models, and show a tendency to overprice OTM options. It is quite even for ATM options, and underprice ITM options. The pattern across maturity is underpricing for short maturities, while it slightly overprices medium- and long-term maturities.

The GJR-GARCH does not perform as good as the HN-GARCH, EGARCH and NGARCH and show tendencies of overpricing. It is the only model that display similar patterns as the benchmark BSM model in terms of large residuals for short maturities. The tends to have a smaller spread of pricing errors for higher moneynesses, on contrary to other models. But the errors are over all higher. Over maturities the model tend to overprice short-term options and underprice long-term options.



Notes are the same as in Figure 6.5.

Figure 6.6: Box and Whiskers Plot Divided into Maturities

The overall performance of the models in-sample are not especially satisfactory as the models display quite substantial pricing errors in both relative and absolute measures. HN-GARCH and NGARCH models are performing best in-sample outperforming the benchmark BSM model over both maturity and moneyness. EGARCH manages to price the options fairly, but have larger errors then the NGARCH and HN-GARCH. GJR-GARCH is not as bad when considering relative errors, but struggle to fit the option data satisfactory and over-valuate the contracts on average, showing similar results as the benchmark.

Out-of-Sample

With out-of-sample pricing we measure the models ability to price options outside of their fitted sample. This can be particularly attractive for practitioners as they are interested in how the models can price options with a forward looking perspective. We interpret the pricing errors as before. Earlier we also commented on the persistence of the parameters over time. As they require frequent calibration we would naturally expect a worse performance, but to see what model that handles this transition best is of interest. It will describe the qualities of the models in a more applied perspective.

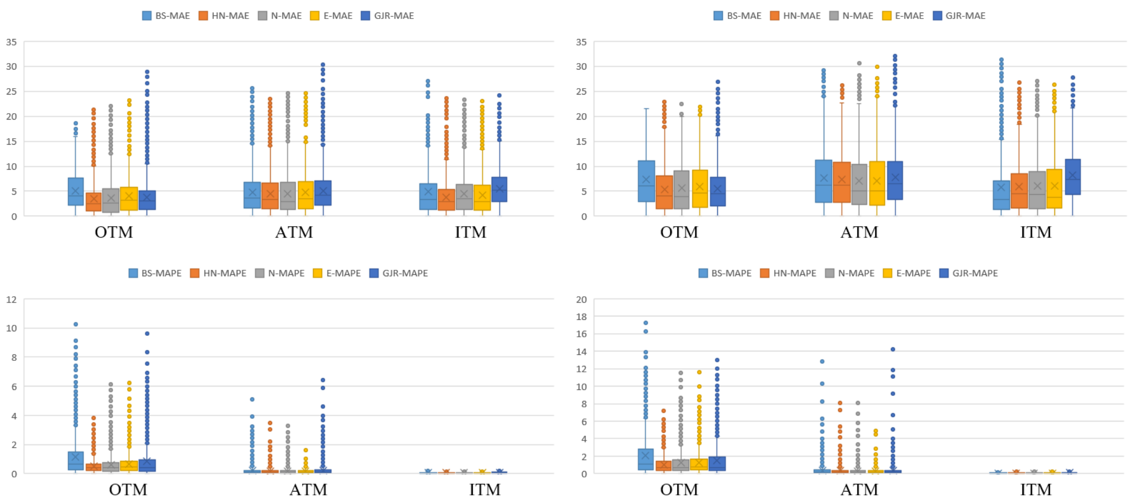
Indeed, we find higher pricing errors out-of-sample, but it is also interesting to notice that there is some trends applying to all models. All models seem to be underpricing options in 2015 and overpricing in 2016. This seem to be particularily drastic for long-term options. Furthermore it seems like the models have higher absolute pricing errors for ATM options in 2015 and 2016 which is slightly surprising. The ATM options were well represented in our calibration sample, and one would expect that ITM options had higher errors in absolute terms. Nevertheless, is the relative pricing error decreasing for higher moneyness, which is expected.

Out-of-sample the HN-GARCH model is still performing very good. Throughout it seems to have highest absolute pricing errors on ATM options, while in relative terms it shows a falling trend for moneyness. In 2014 it has higher absolute pricing errors for the middle maturities, whereas in the last year the pricing error increases with higher time to maturity. The relative pricing error is quite flat, with minor indications of lower errors for longer maturities especially in 2014.

The NGARCH model is also performing satisfactory relatively to the HN-GARCH, and in 2014 it is overall presenting the lowest pricing errors. It shows a slightly stronger trend for higher pricing errors for ITM options in 2014 than the HN-GARCH, but is also overall presenting higher absolute errors for ATM options. NGARCH has a particular good relative fit to long maturity options in 2014, but the relative error is quite flat in the time to maturity specter. The absolute error is increased with longer maturities.

The EGARCH model perform well overall out-of-sample despite having some variability in the errors. The variability imply that the model has a large spread in the pricing errors, and have occasionally high errors. Out-of-sample the model seems to be dominated more clearly by NGARCH, even if it has the lowest RMSE$ in 2014. Showing similar trends as the NGARCH, EGARCH tend to overprice OTM options and underprice ITM options, while they all are on average overpricing in 2016. Along with HN-GARCH it also has the clearest trends of highest absolute pricing error coupled with lower relative error for higher moneyness. In terms of pricing different maturities it does not seem to be differentiating a lot from other models. If anything it seem to have a clearer trend of higher pricing errors for longer maturities. We still find the model to disappoint after it seemed to calibrate very good on our sub-sample.

The GJR-GARCH cannot be said to perform especially good relative to the HN-GARCH, NGARCH and EGRACH. The model tends to overprice more than the other models, and it has the most distinctive overpricing errors for ITM options with the least spread. For OTM options it exhibit more variability than other models in 2014 and 2015, but ATM it can be said to price options as good as the other models.



Total MAE and MAPE over the sample years. In-sample to the left, and Out-of-Sample to the right .

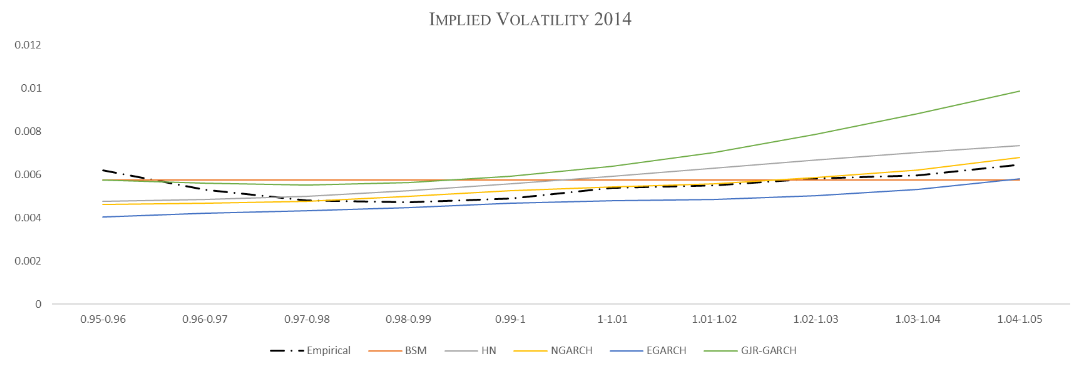
Figure 6.7: Box and Whiskers Plot Divided into Total Moneyness

Broadly is the performance out-of-sample of the models worse, but not completely off compared to in-sample. Compared to the BSM, the models perform satisfactory, and for OTM options they outperform BSM quite clear in absolute terms. This is relatively natural as BSM is calculated with the average implied volatility, but we are a bit surprised that it is not more clear for ITM options. The HN-GARCH performs best, while EGARCH has taken more ground from NGARCH. We find that pricing errors is relatively stable in absolute terms, and decreasing in relative terms for options with a higher moneyness. When pricing options with longer maturities we find decreasing relative pricing errors coupled with slightly higher absolute errors.

6.4 Volatility Smile

To further analyze the pricing errors, we look at each models ability to replicate the observed volatility smile/smirk. The observed implied volatilities in Appendix 10.4 tell us that the volatility indeed is changing over moneyness. To analyze the models, we back out the implied volatilities from the model-generated option prices with the Black-Scholes-Merton model and plot them accordingly. The models should be able to replicate the smile/smirk observed in the option data to perform satisfactory. For given points in moneyness where the implied volatility exceeds that of the observed data we say that it overprices, and in the cases where the models display lower implied volatility the model underprices. We plot only the implied volatility over moneyness for the observed maturity that exhibit the most characteristic smile. This to save space and time, furthermore, we believe that we can draw conclusions based on the models ability to replicate the smile for a given maturity and generalize upon this.

First of all, we are not surprised to see that the smile produced by the observed option prices are in accordance with the upward facing smirk that index options empirically display according to Hull (2015). We see the smile over all our sampled maturities to be increasing in moneyness and also at far OTM options.



Note: The maturity 17 days to maturity exhibited the most characteristic smile of all maturities in 2014 in-sample. We therefore want to see what model can best fit that smile.

Figure 6.8: Volatility Smile for options with 17 days to maturity in-sample

For the in-sample 2014 plotted in Figure 6.8 we first notice that the BSM model exhibit a flat volatility over all moneyness. This because we gave it the average implied volatility from the in-sample observed prices in order to not give it favouring abilities. It is interesting to see that the smile is more present at low moneyness, and if the models are able to capture the slight increase in implied volatility at lower moneyness it would be considered favorable. This is not the case, however. The GARCH-model that best captures the smirk in the 2014 in-sample plot is NGARCH and HN-GARCH. NGARCH obtain a visible best fit, especially above the 0.97 moneyness mark. This is a bit peculiar as we found that HN-GARCH outperformed NGARCH in-sample 2014 based on pricing errors, but it is important to remember that this is for only one specific maturity. The HN-GARCH is also pretty close to the observed, but is a little below for low moneyness and a little above for high moneyness. This imply that the model overprices options for high moneynesses and underprice at low moneynesses as option prices is an increasing function of volatility.

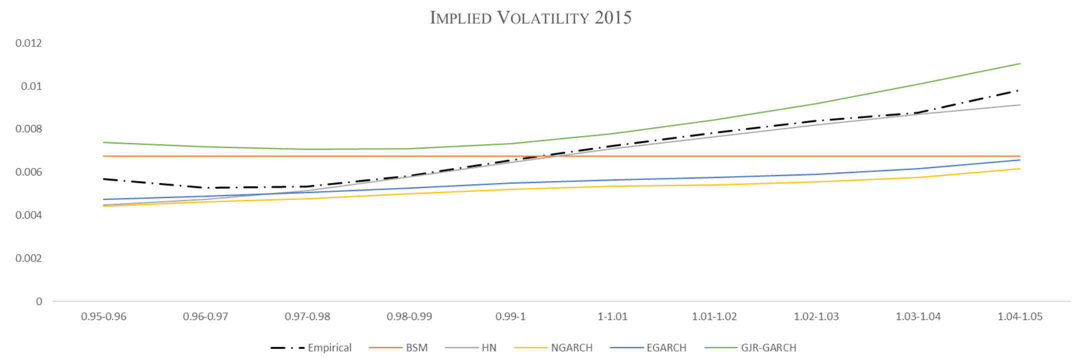


Figure 6.9: Average Volatility Smile for options with 16 and 17 days to maturity in-sample

In 2015 the volatility for the similar maturity is a bit higher, ending above 0.008 for deep ITM options. Following the results from 2014, the HN-GARCH display the best fit to the smile for moneynesses over 0.96. The NGARCH seem to underprice the options more substantially and display a flatter term structure of volatility, not being able to capture the smile satisfactory. GJR-GARCH is actually the model that fits observed implied volatility second best for ITM options. EGARCH underprice options for all moneynesses, but manages to follow the increase in volatility for higher moneyness.

The volatility is further increasing in 2016, something that can be seen in relation to the increase in return volatility in Figure 6.2. The reason for this would require further economic interpretation that will not be covered by this thesis. In this year there is no model that manages to perfectly replicate the smile for the sample maturity. The GJR-GARCH that we saw displayed tendencies to overprice options manages to follow the increase in volatility for ITM options, but are still overpricing the options. The best models in this year is the HN-GARCH and NGARCH, whereas the latter seem to be more linear. HN-GARCH manages to describe the

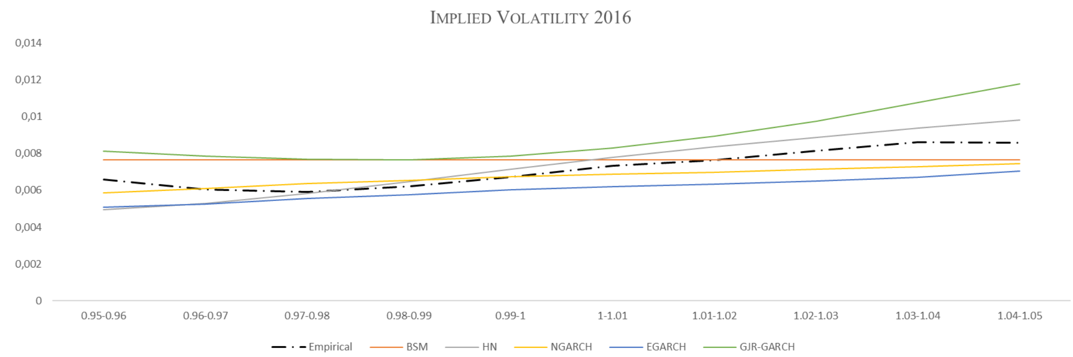


Figure 6.10: Volatility Smile for options with 16 days to maturity in-sample

characteristics of the smile better, despite that it underprices far OTM and overprice ITM.

The conclusion of this analysis regarding the models capability to replicate the empirically seen volatility smile must be said to be straightforward. The HN-GARCH display overall the best fit to the observed volatilities and the other models display similar patterns found in the previous analysis. Considering the models being fitted on a rather small sub-sample from the original in-sample data, the models must be said to capture the smile satisfactory with the volatilities changing across moneyness.

6.5 Regression

In order to analyze the performance of the option pricing models further, we run some regressions on the pricing error. In that way we can also perform statistical tests to evaluate any biases the model may possess. Based on the framework outlined under Section 5.7.2 we start off by trying different regression equations to see what makes the most sense to investigate further. We find that including the squared moneyness as a independent variable increase explanatory power to the regression in terms of R^2, and that the variable is significant when running on the same pricing error. Furthermore, as we want the regression to be as interpretable as possible, we choose MAPE as our dependent variable. This is because we find the linear nature of MAPE to be easier to grasp than any of the squared measures. After trying several options we also find that the choice of pricing error as dependent variable does not alter the results significantly for the other variables. Our choice of regression equation falls on Equation 6.1 for all i options at time t.

$$MAPE_{i,t} = \beta_0 + \beta_1 \frac{S_t}{K_i} + \beta_2 \left(\frac{S_t}{K_i}\right)^2 + \beta_3 TTM_i + \beta_4 rf_t + \epsilon_{i,t} \tag{6.1}$$

We find the regression to be of interest as it tries to capture any biases we are particularly curious to see if the models possess. Earlier we mentioned that BSM has been criticized for not being able to capture the proper volatility smirk. We will investigate if the GARCH-models to a greater extent manages this by seeing if moneyness has some effect on their pricing errors. Furthermore we want to see if the models have some maturity bias - if they have a higher pricing error on options with a longer maturity (TTM). Lastly we find the interest rate increase in end of 2015 and 2016 to be a potential disturbance, and as all the models rely on risk-free arbitrage arguments heavily dependent on the risk free rate - it will be interesting to study further. Even if such big changes in the risk-free rate is arguably extraordinary, any effect to models viability will be of importance.

Table 6.12: Regression Results

	In-sample							**Out-of-sample**						
	2014							**2014**						
	$R^2_{adjusted}$	**F-stat**	β_0	β_1	β_2	β_3	β_4	$R^2_{adjusted}$	**F-stat**	β_0	β_1	β_2	β_3	β_4
BS	0.3769	416	180	-350	170	-0.0062	3043	0.3960	276	298	-578	281	-1.8314	-2423
HN	0.3119	312	70	-135	65	-0.0033	1913	0.3859	264	153	-293	141	-0.8838	-1483
N	0.2227	197	48	-91	44	-0.0025	1003	0.3405	217	98	-188	90	-0.8340	-1152
E	0.4279	514	137	-267	130	-0.0018	1152	0.4559	352	239	-466	227	-0.4516	-889
GJR	0.2371	214	40	-74	34	-0.0048	2021	0.3057	186	100	-188	89	-1.7441	-2259
	2015							**2015**						
	$R^2_{adjusted}$	**F-stat**	β_0	β_1	β_2	β_3	β_4	$R^2_{adjusted}$	**F-stat**	β_0	β_1	β_2	β_3	β_4
BS	0.4430	619	430	-842	412	-0.0075	1086	0.2824	180	313	-611	297	-0.0050	820
HN	0.4747	703	94	-181	87	-0.0014	405	0.3749	274	104	-200	96	-0.0012	265*
N	0.4337	596	218	-426	208	-0.0025	433	0.3322	227	183	-356	173	-0.0020	520
E	0.4462	627	224	-437	213	-0.0019	507	0.3269	222	187	-363	176	-0.0016	555
GJR	0.3645	447	292	-569	277	-0.0108	722	0.2584	160	237	-458	222	-0.0069	649*
	2016							**2016**						
	$R^2_{adjusted}$	**F-stat**	β_0	β_1	β_2	β_3	β_4	$R^2_{adjusted}$	**F-stat**	β_0	β_1	β_2	β_3	β_4
BS	0.3561	441	414	-809	395	-0.0080	174	0.5641	601	1201	-2341	1143	-0.0225	-909*
HN	0.4268	593	116	-225	108	-0.0022	183	0.4685	410	372	-716	346	-0.0059	-544*
N	0.3661	460	213	-415	202	-0.0032	179	0.5122	488	700	-1362	665	-0.0087	-771
E	0.3716	471	208	-406	198	-0.0017	154	0.5150	493	680	-1327	648	-0.0020	-642*
GJR	0.2941	332	258	-502	244	-0.0095	253	0.4588	394	727	-1410	686	-0.0254	-885*

Note: The asterisk * denotes significance at the 5 % level. Parameters without asterisk are all significant at the 1% level.

The regression results are presented in Table 6.12. Here we have simply used the moneyness, squared moneyness, days to maturity and yearly risk-free interest rate on the day of the option price quote as independent variables. Most notably is the strength in all of the regressions and parameters. All of the F-statistics rejects any null-hypothesis that the *all* parameter coefficients are jointly zero. In other words are all of the regressions meaningful to interpret further. The R^2 is however a bit harder to interpret correctly, observing relatively low R^2 values does not necessarily mean that the independent variables are not explanatory to pricing error. It could indicate that we lack some important explanatory variables, which we find good as we do not expect our independent variables to be the only determinants to option pricing. It is also good in regards to the models we are testing; that they are not exclusively dependent on these variables.

For single variables we find that all variables are significant if we accept a significance level of 5%. Furthermore, we see that the risk-free rate is not accepted as a explanatory variable on a 1% significance level for several of the models out-of-sample in 2015 and 2016. They are still significant at a 5%-level, but it is interesting as the increase in interest rates came within these exact data sets. It is also peculiar that the coefficients for the interest rate is positive in-sample, while they are negative out-of-sample in 2014 and 2016. We also find the coefficients for yearly interest rate to take high values in positive, or negative direction. We suspect that we have found some spurious effect from the interest rate with our regression on this particular data. The change in interest rates for the in-sample is minuscule, and when pricing particularly 2015 and 2016 out-of-sample we expected that the interest rate would either have none or positive effect on the pricing errors. With conflicting results we do not want to draw any clear conclusion of the relationship between interest rate changes and pricing errors.

When looking at days to maturity we do find some clearer patterns, as its coefficient is exclusively negative for all models throughout the years. This would imply that all models are better at pricing long-term options in terms of MAPE, which backs up what we found in the pricing error tables in Appendix 10.9.

When looking at moneyness we find a significant negative coefficient for all models, while squared moneyness has a positive coefficient. The marginal effect of one unit of moneyness is $\frac{\partial Y}{\partial S/K} = \beta_1 + 2\beta_2$. So when observing that the first coefficient is exclusively over twice as big as the second, we find further evidence for the models to be better at pricing ITM options in relative terms. The relationship is, however, decreasing as β_2 is negative (Murray 2005). To further test if moneyness really can be said to have a significant effect on the pricing error, we also run a F-test with the null-hypothesis that $\beta_1 = \beta_2 = 0$. Results are presented in Table 6.13.

All F-values are significant and we find further evidence that there exists a negative relation between pricing error and moneyness.

When looking at the results in conjunction with our previous indications we find

Table 6.13: F-test for Moneyness

		In-sample					Out-of-sample				
		BSM	**HN**	**N**	**E**	**GJR**	**BSM**	**HN**	**N**	**E**	**GJR**
2014	**F**	669.0	485.0	311.1	992.0	278.5	532.6	520.3	416.3	702.2	311.3
2015	**F**	1167.4	1371.2	1173.0	1240.3	730.2	347.8	544.2	448.0	438.5	293.1
2016	**F**	815.2	1082.8	863.6	916.4	515.3	1165.9	812.6	968.9	980.3	688.3

Note: All F-values are significant on a 1% significance level.

that there is a trend for the models to have a positive relationship between time to maturity and relative accuracy. Furthermore, we find a positive relationship between moneyness and accuracy. This backs up most of previous findings of pricing error as well as the Box-Whisker Plots in Appendix 10.9. Disappointingly is any effect of the interest rate strongly conflicting, and we do not want to point to any effects as we cannot be sure that we isolate any effects in our setup.

7 Discussion and Limitations

In this chapter we aim to further analyze, concentrate and critically examine any results and the methodology used to obtain them. Furthermore, we will assess our process in obtaining the pricing parameters and pinpoint limitations to our work. Of particular interest is it to determine any potential fallacies our results exhibit due to our methodology, data or presentations.

Results

We found that the HN-GARCH performed overall best of all our models, both in-sample as well as out-of-sample in both absolute and relative measures. The EGARCH impressed in terms of low pricing errors and proved to be the best model in terms of MAE and MAPE when estimated on historical return data. It was also the only model that did not overprice options across all moneyness. The NGARCH model did also perform well, and were arguably better than EGARCH in the in-sample calibration. The GJR-GARCH were by far the worst performing GARCH-model from the calibration, and were actually outperformed by BSM in some samples. The models outperformed BSM overall, but as BSM had average implied volatility as input this was to some degree expected. When looking closer at the results we find all models to price shorter maturities relatively worse, with a statistically significant effect on a 1% significance-level under the calibration. All models also have pricing errors that are statistically significantly dependent on moneyness and interest rates. While the models shows to be pricing low moneyness worse than high, it is no clear pattern on how the interest rates affect the model accuracy. That the GARCH-models have decreasing pricing errors for time-to-maturity and moneyness is in accordance with for instance Huang et al. (2011).

Parameter Estimation

Before the empirical comparison of the models we had some predetermined expectations of how the models would perform relative to each other. Based on previous research we expected the non-affine GARCH models to outperform HN-GARCH from

(Christoffersen et al. 2005, Hsieh & Ritchken 2005), but EGARCH, NGARCH and GJR-GARCH had conflicting and limited research from before. While EGARCH were supposed to be easy to work with, it received criticism for creating some biases due to its logarithmic transformation. These presumptions have to some extent been challenged, as our results point in a somewhat contrary direction.

As all models nest from the same underlying GARCH structure and are equally parameterized it is to a large extent expected that the models performance are somewhat equal. We see that the models obtain a maximum likelihood with small variability, but we cannot draw any conclusions on whether the distribution of the error terms are causing the models to under- or over-fit the return data. The estimation did however confirm to a large degree that the non-affine models manage to fit the return data time series better than the affine counterpart in terms of ML values. Thus we believe to have found confirming results that the HN-GARCH does indeed loose some fitting attributes due to its structure allowing for a closed-form analytical option pricing model. The overall results can hardly be said to be satisfactory as the models display quite large pricing errors in the sample. The EGARCH is however, superior in terms of both MAE and MAPE, and is the only model that produce ballpark satisfactory results. The model outperforms the BSM benchmark model in all measures and display a slight underpricing overall. We did not find the applicable nature of EGARCH to be of particular significance, but it was quite dominant in the transition from modeling returns to pricing options.

The results from the estimation are interesting, and provide a basis for the analysis for each models ability to fit data relative to each other. It is interesting that the NGARCH model that provide the highest ML value is not the best model in terms of pricing errors. As the model is outperformed by both the EGARCH and the GJR-GARCH it could point towards that the model loose fit in the transition from the P- to the Q-measure, or that the other models benefit from the transition. This is quite interesting as the tables turn under the calibration where the HN-GARCH and NGARCH are the best performing models. When analyzing results we wanted to have more focus on their performance from more of a practitioners view - how they priced options independently of theoretical framework. This requires that they capture stylized facts of empirical option prices, and price correctly out-of-sample. For that reason we did not investigate the theoretical framework for transitioning GARCH processes to a option pricing model any further.

Arguably it could be that the models require differing horizons for obtaining the true parameters through ML, and that our 4000+ observations is not sufficient for the HN-GARCH, NGARCH and GJR-GARCH to converge properly. Further, our sample period of 17 years can be argued to contain some quite volatile market periods such as the dot-com bubble and the financial crisis of 2008 as explained in Section 4.2.6. Whereas our pricing sample display some calmer characteristics with a more smooth bull market seen in Figure 6.1. It would be interesting to analyze further if the EGARCH happens to smooth the volatility clustering in the volatile estimation period and thus obtain better prices in a less volatile period as per its criticism. This could prove a topic for further investigation for sure.

We found that the estimation of the parameters did not require much computational power and the procedure itself is both applicable and straightforward.

Parameter Calibration

Without analyzing it any further we notice how much the calibration reduces the pricing errors. In the estimation we use a total of 17 years of historical data to fit the GARCH-models properly using QML. With this setup the parameters change minimally from year to year and while the fit overall is poor, it does not seem to be a better proxy for any particular year. Even if we compare in-sample pricing using estimated parameters with out-of-sample pricing using calibrated parameters the pricing errors reduces significantly. Thus we view this as a confirmation that the calibration should be the preferred way of obtaining the parameters.

From being worst in class under the estimation, the HN-GARCH outperform all GARCH models in terms of MAE and MAPE for all maturities in-sample. The superiority is however not as distinct as of the EGARCH under the estimation.

When fine-adjusting the models through calibration the tables turn from the estimation, and the models started to distinguish more from each other in terms of usability. HN-GARCH which were supposed to be the easiest one applying with its closed-form solution presented big weaknesses in terms of fitting. The integral in the function is very sensitive to parameter inputs and tend to crash the optimizer in cases was the combination of parameters did not cohere with the integral. While the computation were not particularly demanding, we found the model function to be very unstable. The other GARCH-models were on the other hand computationally harder to fit as they require Monte Carlo simulation within the minimization procedure itself. This in its own term cause a problem for optimization procedures as the complexity of the calculations is vastly extended. All models showed complexity in their multi-dimensional nature of parameter optimization and here we find several pitfalls in terms of computational tools which could have affected our results, or at least affected our computational efficiency to a great extent.

The prices obtained from the calibrated parameters have its own characteristics. We are a bit surprised that the models do not dominate BSM more for the low- and high moneyness. The relatively high mispricing for OTM could be explained with liquidity bias in the observed prices. The performance in regards to ATM and ITM options have some more interesting results, however. A low relative pricing error on ITM options is natural, but in absolute terms one could expect higher - or at least equal errors to ATM options. Our data set for calibration on option prices had highest representation of ATM options which we also would expect to be of the most liquid options.

In regard to this, the choice of loss-function could be a source of disturbance. By minimizing MSE when calibrating parameters, we know that this method punishes large outliers more than ensuring average fit. We believe that using some relative measure as MAPE could lead to higher absolute pricing errors for ITM than the other option classes. A possible negative effect of this might however be an overfitting on OTM options despite their potential liquidity biases. Having used IVRMSE could have made the results more impartial as the measure does not differ as much over maturity and moneyness.

A final comment on our usage of loss functions is also that by optimizing on MSE when calibrating the parameters, it is not completely fair to evaluate the models on other pricing error measures. As we have a slightly higher focus on the MAE and MAPE through our Box-Whisker plots for interpretational purposes we might judge on sub-optimal parameters (Christoffersen & Jacobs 2004*a*).

Data & Theoretical Framework

Observing empirical return data characteristics, we found evidence for stylized facts such as fat-tails, skewness and autocorrelation/GARCH effects in squared returns. This would imply that modelling returns with asymmetric GARCH-models is reasonable to capture these characteristics. We did however not investigate options to model the fat-tails observed in empirical return data. We know that a Gaussian distribution for the error terms will never be able to model fat-tails for the GARCH-models, and may cause a deterioration of the overall return-fit. The fat-tails might imply that skewed distribution such as the student-t distribution could prove better to model returns in our ML estimation. The normal distribution is, however more robust to specifications in data distributions, so we chose to use Quasi Maximum Likelihood. Another solution could be to use a jump-diffusion process for the returns similar to Duan et al. (2002). Either method could potentially improve our results, but without any comparative analysis we cannot validate this. Furthermore, as all models were based and evaluated on the same framework and with equal assumptions we find our comparison of the models to still be valid.

As we wanted to use liquid and unbiased big data, we ended up using options written on the S&P 500 Index. With close to 16,000 options the amount of options should arguably be big enough for a robust analysis, but our filtering could still be biased. We believe that the choice of filters outlined in Section 6.1 does not create any bias in regards to real market data as they are not particularly strict. It could nevertheless create some biases for our analysis if they include too many biased option prices from illiquid options and other abnormalities. The choice of analyzing only 3 years were based on the ease of needing few different parameters, and assuring a representative sample with relatively few filters. When calibrating the parameters, we could, however, have used more years to smooth any potential annual anomalies. Nevertheless, we believe our period represent normal market conditions with no extreme attributes and should be representative for longer periods of time. Furthermore, would further calibration in terms of more periods require more of our scarce computational power.

The fact that we do not use timestamped data for our option price quotes or the S&P 500 spot price creates some possibilities that our data contains synchronous bias. As we cannot say that the observed option price quotes directly correspond to our observed spot at the same date, there might be some marginal errors depending on what time of the day the prices were collected. Although this might be of minor importance in our context, we acknowledge its presence and that it may impact our errors.

Another potential weakness is that we for each model use its unconditional variance as a proxy for the initial variance. While it might be argued that it is a good guess, and we find that this is a common way to initialize GARCH models - it might nevertheless be sub-optimal. Observing the implied unconditional variance from each model in Table 6.6 and 6.9 we see that the unconditional variances is varied and will put the models in quite different start positions. According to Duan (1995) is the initial conditional variance unlikely to be equal to the unconditional variance. If this is used, we would have local risk neutralization implying that the process under the risk-neutral process would revert to a new and higher unconditional variance. One alternative would be to parameterize the initial variance and solve for it in our parameter optimization, similarly to Ritchken & Trevor (1999). This would however require yet another parameter to estimate, and furthermore make our optimization even more complex and demanding. As all models are evaluated using the same procedure, we find our method to still be valid and in accordance with other earlier research (Christoffersen et al. 2005, Hsing 2003).

Computational Tools

We chose R-Studio as our preferred tool for computational analysis as it is a free open-source language with a large community for troubleshooting. While R-Studio is a way more efficient mathematical programming language than for instance Microsoft Excel, we believe that even more efficient languages such as Mathlab or C++ could have helped us reaching more robust results even faster. With faster executing and confidently finding global optimal solutions we would be able to execute more realistic fitting procedures based on larger data set. Even when calibrating the parameters on a relatively small sub-sample the procedure was extremely time consuming, taking up to 23 hours for one calibration. It is obvious that this constraint has an impact on the final results. Arguably is this bottle-neck with a small sub-set of option prices to calibrate on, one of the biggest potential risks to our calibration results.

With more computational power several interesting applications come to mind. For estimating through ML we could have tried different methods such as a rolling window optimization for bigger data and averaged the parameters to analyze their variability over time and smooth any abnormalities in the underlying return data. The calibration would have benefited from being able to fit on the whole in-sample data set, and increased the Monte Carlo for each price reducing the pricing bias from the fitting itself by ensuring better convergence to the true option price. Even if all models are calibrated on the same sub-sample data, it is reasonable to believe that any bias the calibration sub-set may possess could fit one model better than another. It might also be that this relatively small set of options is sufficient to fit one model, while another need more data to perform adequately. Without drawing any conclusion we see that the HN-GARCH is the model that seem to be performing best overall, and as the only GARCH model not being dependent on a numerical pricing procedure it could demonstrate a slight advantage over our relatively small Monte Carlo.

With more computational power we could also enrich our analysis with some different setups for 1-3 day ahead forecasting with updated numerical calibration. This could be a better approach to replicate a practitioner applicability and capturing more model characteristics. This might mainly affect the comparison between HN and the other GARCH models, but possibly also individual fitting of the models.

External Validity

We believe that our results are representative given our fundamental theories. Furthermore we think that we have applied methods in a theoretical correct manner. But there is still some pitfalls and biases that might be present due to either wrong choice of method, or some aberration that we do not account for. Our results are also victim of being neglected due to our poor performing computational tools. As we find small differences within the models it might for that reason be hard to generalize too much about their relative performance. Nevertheless, we do believe that we can point to some general characteristics of GARCH option pricing models, and discuss their different applicableness.

Through our research we have tried to analyze results from different approaches and methods. Most of which have had concurring indications which would be strengthening its viability. By going through the whole method of gathering data, estimating parameters and pricing options we can also say quite much about how the models are to apply in relation to each other.

8 Conclusion

The purpose of this thesis was to empirically test the performance of the HN-GARCH model of Heston & Nandi (2000), the NGARCH model of Engle & Ng (1993), the EGARCH model of Nelson (1990) and the GJR-GARCH of Glosten et al. (1993) as option pricing models against the Black-Scholes-Merton model of Black & Scholes (1973) as a benchmark.

In the parameter extraction procedure we find evidence that obtaining the model parameters from a price-error minimizing calibration yield substantially better option prices that the conventional maximum likelihood estimation on historical return data. Our results imply that historical data lack information needed to correctly price forward looking claims and see this as a interesting topic for further investigation in its own right.

Applying the calibration procedure on a sub-set of cross-extracted option prices in order to maintain sample characteristics the overall pricing errors drop, but are still substantial from a practitioners perspective. In-sample the pricing errors are naturally less than the out-of-sample dataset on average, implying poor persistence in the obtained parameters, especially in 2014 and 2015.

Our results find evidence that the model of Heston & Nandi (2000) display overall superiority, especially in-sample. The NGARCH and EGARCH display similar patterns in pricing errors in-sample in terms of MAE and MAPE. Where the NGARCH are arguably better on OTM options, the EGARCH are slightly better on ITM options. The GJR-GARCH exhibit higher pricing errors in most instances and is the only model not capable to beat the benchmark model overall. Out-of-sample the models display higher pricing errors across moneyness and the patterns are less clear. The HN-GARCH exhibit superiority for OTM options aligned with overall less variability in the prices, while the EGARCH and NGARCH manage to lie in the same range both pricing ATM and ITM options equally well. The GJR-GARCH overprices ITM options, especially with short and medium maturities.

The results are strengthen by the models relative ability to replicate the observed volatility smile. On our sample of short maturity options, it is the HN-GARCH that manages to replicate the smile with the least error. We do however, find that all models manages to create a volatility smile confirming that the GARCH models captures what the Black-Scholes-Merton cannot. Regressing MAPE confirms that the pricing errors are less for long maturities and ITM options for all models significant at the 1% level. The regression further confirm the relationship between the option prices and the time to maturity and moneyness.

8.1 Further Research

Going forward, we want to suggest some further research that we feel is missing within our limited scope of GARCH option pricing, or that can further enrich our findings. While we found asymmetric GARCH models to be dealing with the observed skewness in return data, we find fitting using other distributions to strengthen Maximum Likelihood optimization to be an area worth investigating further. While some jump diffusion models have been suggested and applied we find few empirical comparisons of different models to capture the fat-tails that characterizes historical returns.

Even if earlier research has used many different loss functions to calibrate GARCH option pricing models, there is few comparing studies. While Christoffersen & Jacobs (2004*a*) has a quite extensive paper, we find investigation into different attributes optimizing with different error functions to be of interest and value. For instance could a comparative analysis of functions through specters of maturity and moneyness be of particular interest to practitioners, but also academics.

As we failed to draw some conclusive results from the effect of interest rates on option pricing, it would be interesting to see some research focusing on this effect and see the isolated effect on the models.

In accordance with recent studies we have reviewed on GARCH option pricing we conclude consistently; that the GARCH models are superior to BSM and other conventional option pricing models. We miss some applied research where some simple application, code or program are offered to simply price options with different GARCH models. We find the amount of work, resources and practicality of the models to be very extensive.

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10 Appendix

10.1 Option Statistics

Table 10.1: Average option price pr. month in sample

	Average Price pr Month in $											
Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2014	33.96	34.36	36.34	35.49	34.21	35.99	35.64	34.79	35.53	36.81	38.44	40.74
2015	41.85	43.40	46.09	43.48	41.47	42.68	41.19	41.90	43.36	47.14	49.55	45.22
2016	44.06	47.72	51.23	50.19	42.50	51.84	42.76	42.06	42.38	40.95	42.67	41.73

10.2 Returns Histogram

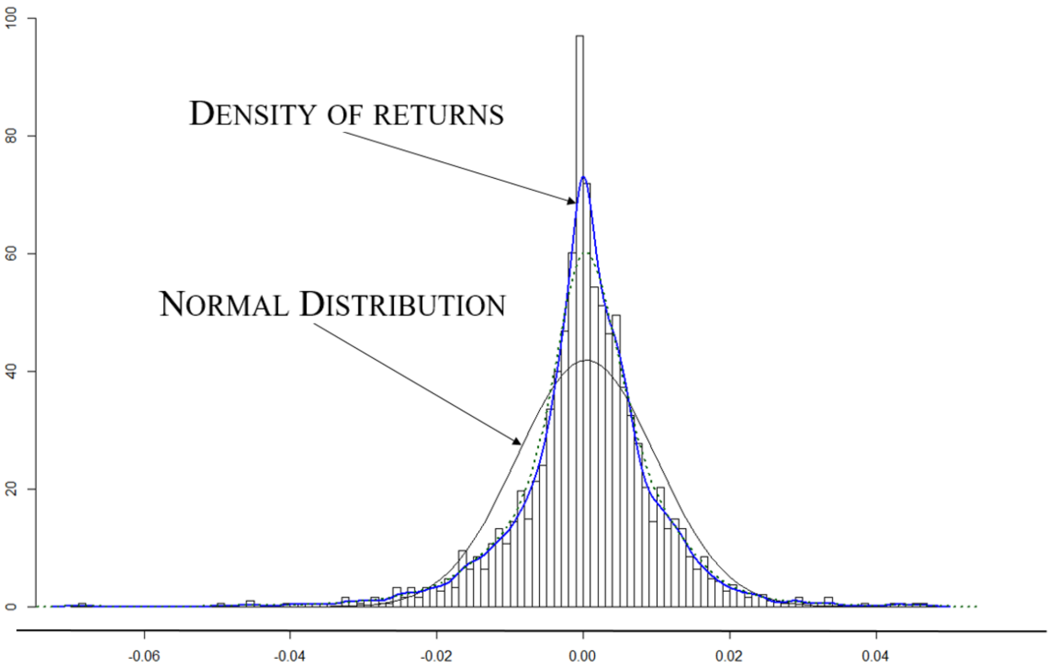
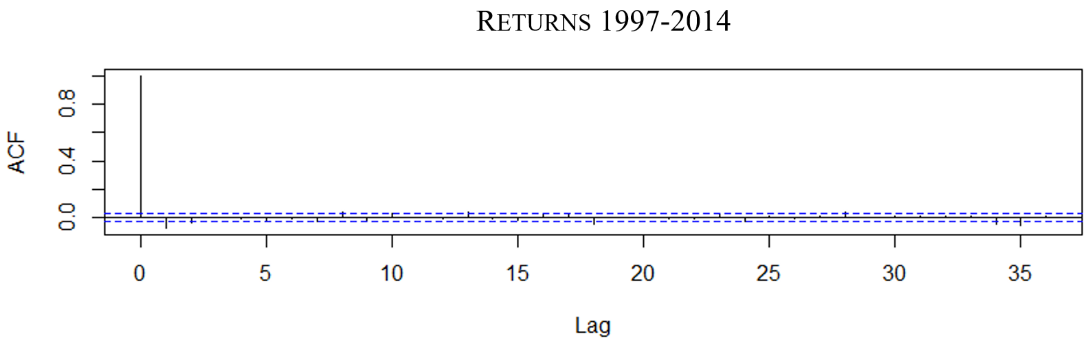
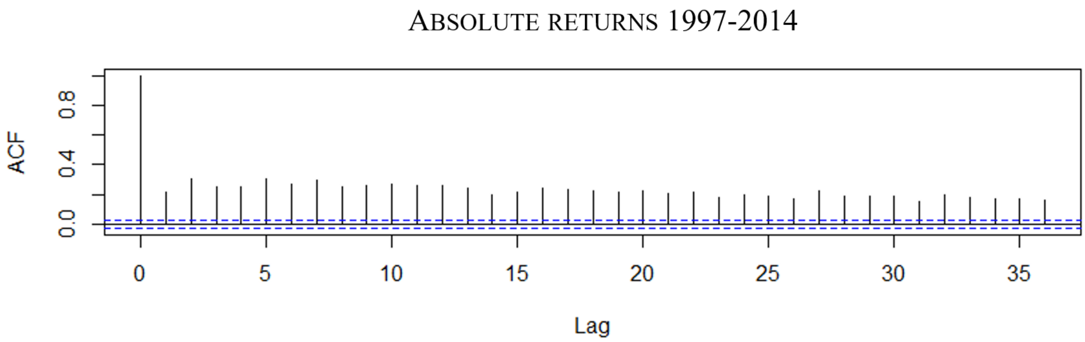


Figure 10.1: S&P 500 returns 1990 - 2017 density,

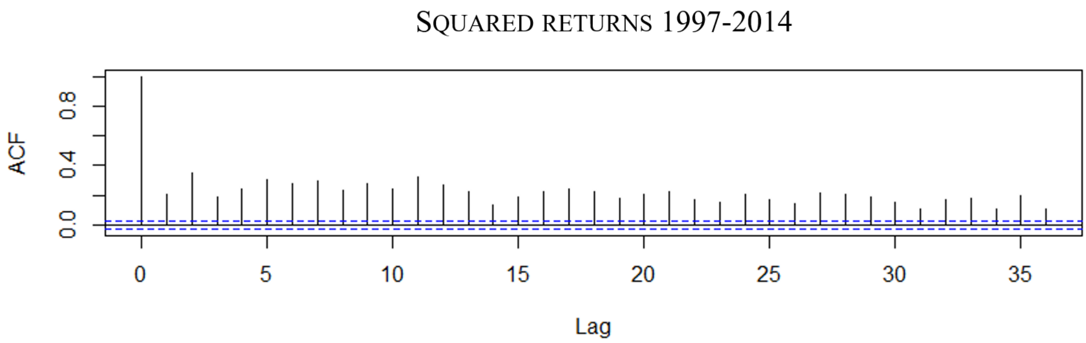
10.3 Autocorrelation



Autocorrelation of S&P 500 log returns

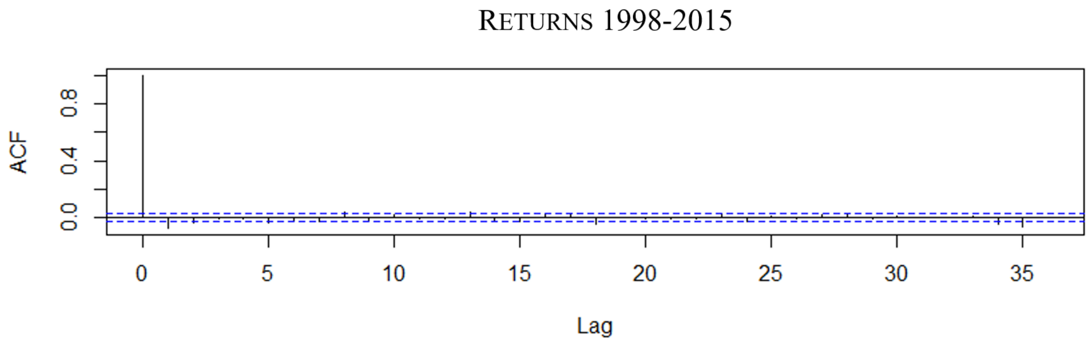


Autocorrelation of S&P 500 absolute returns

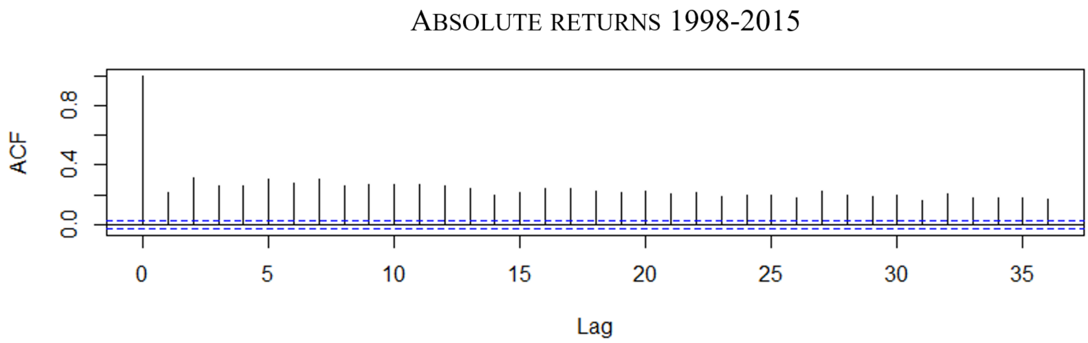


Autocorrelation of S&P 500 squared returns

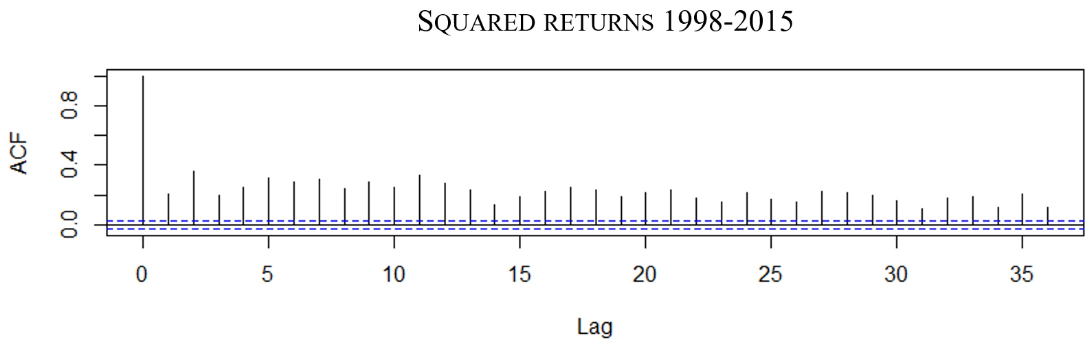
Figure 10.2: Autocorrelation/Heteroskedacity in S&P 500 returns



Autocorrelation of S&P 500 log returns

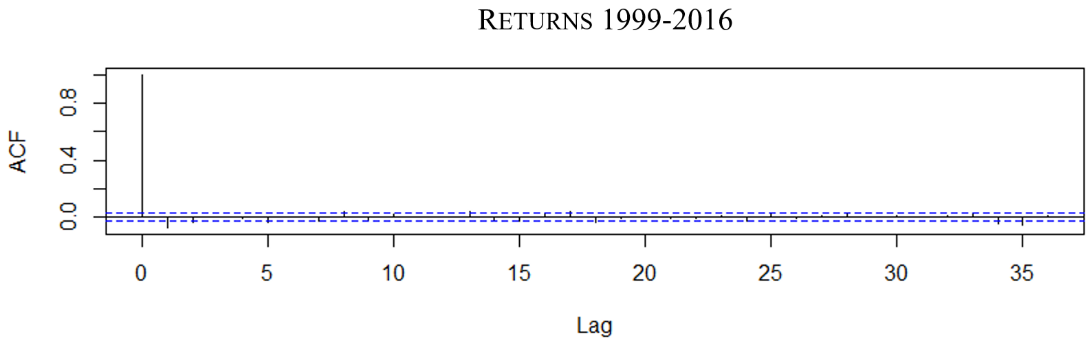


Autocorrelation of S&P 500 absolute returns

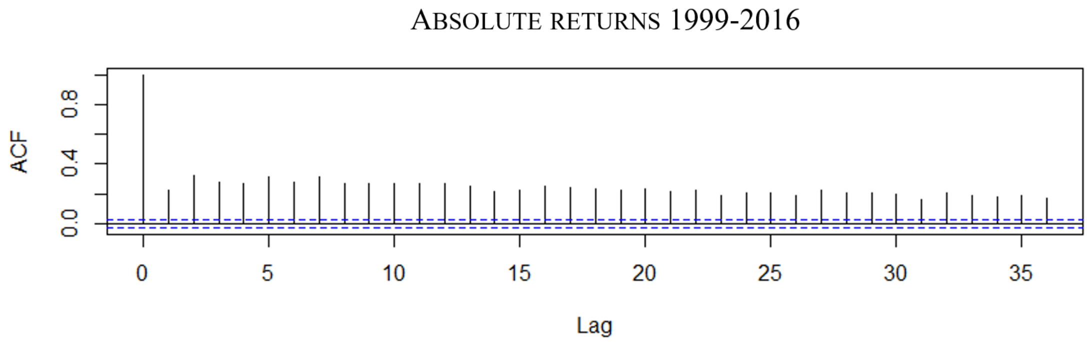


Autocorrelation of S&P 500 squared returns

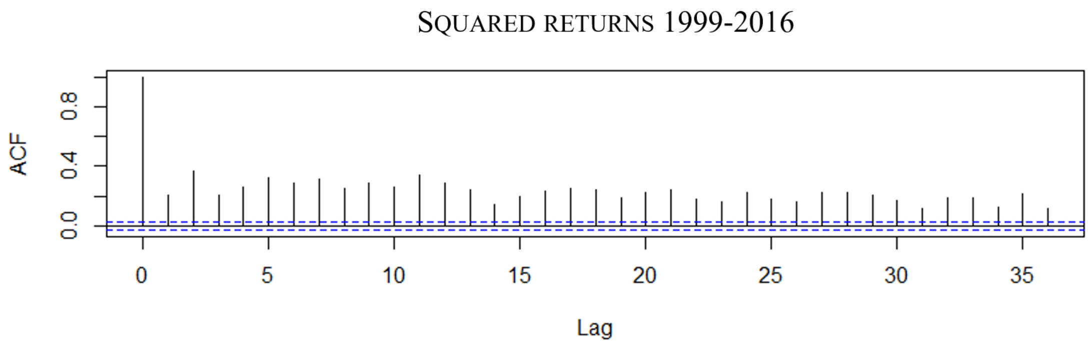
Figure 10.3: Autocorrelation/Heteroskedacity in S&P 500 returns



Autocorrelation of S&P 500 log returns



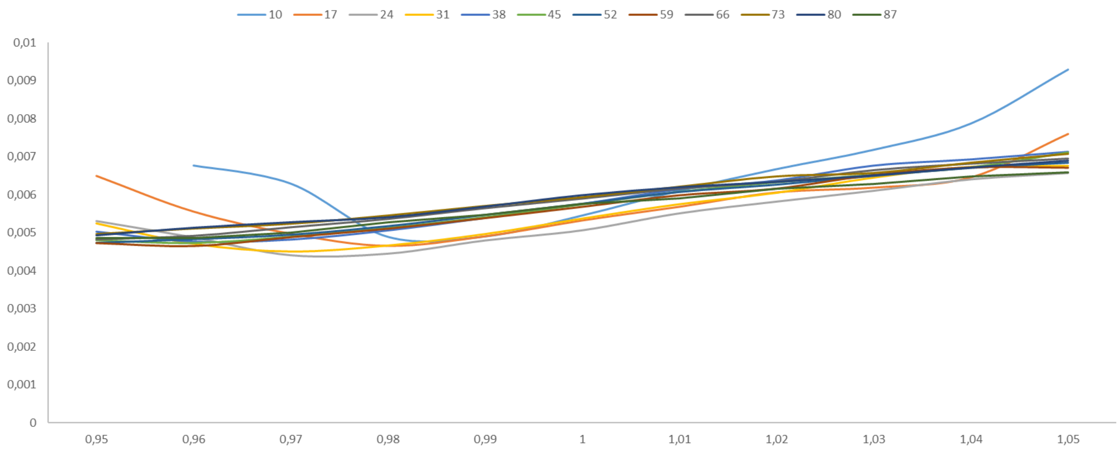
Autocorrelation of S&P 500 absolute returns



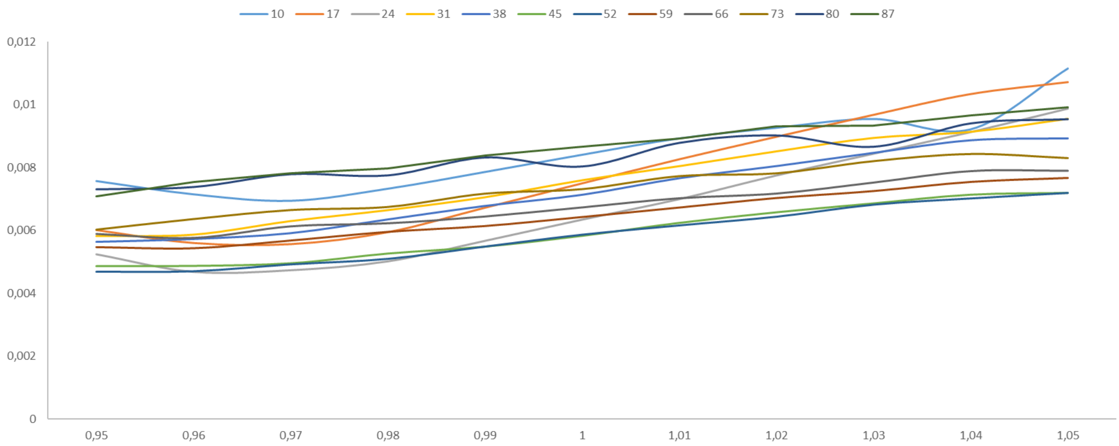
Autocorrelation of S&P 500 squared returns

Figure 10.4: Autocorrelation/Heteroskedacity in S&P 500 returns

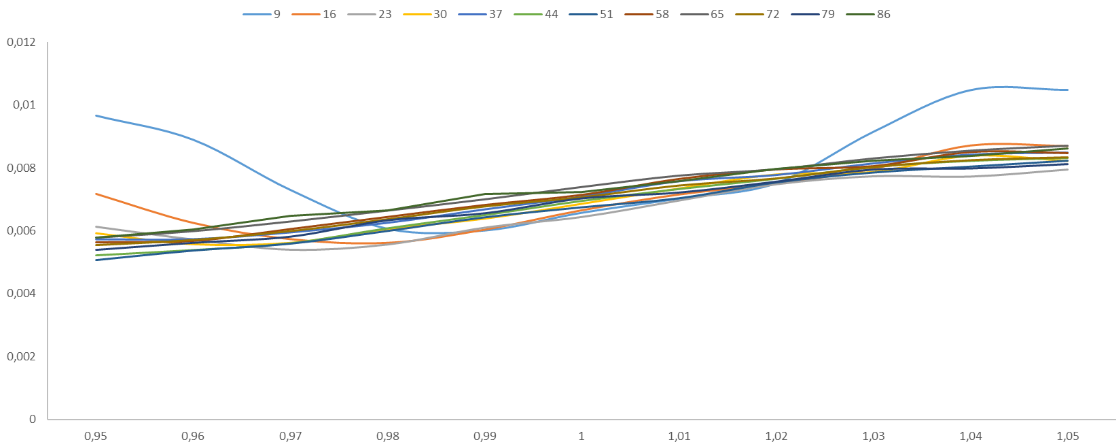
10.4 Volatility Smiles



Average Implied volatility 2014 option data



Average Implied volatility 2015 option data



Average Implied volatility 2016 option data

Figure 10.5: Average Implied Volatility from Sample Data

10.5 Data Statistics 2014

Table 10.2: S&P 500 Index Option data, 2014

	9>DTM<28	29>DTM<48	49>DTM<68	69>DTM<88	**Total**
			Panel A. Number of call options		
0.95>S/K<0.97	96	255	288	222	**861**
0.97>S/K<0.99	254	283	278	228	**1043**
0.99>S/K<1.01	270	271	264	221	**1026**
1.01>S/K<1.03	268	261	259	213	**1001**
1.03>S/K<1.05	222	248	244	204	**918**
Total	**1110**	**1318**	**1333**	**1088**	**4849**
			Panel B. Average Option Price		
0.95>S/K<0.97	$1.026	$2.690	$5.270	$8.729	**$4.924**
0.97>S/K<0.99	$3.643	$9.458	$15.298	$20.778	**$12.073**
0.99>S/K<1.01	$16.862	$26.685	$33.720	$39.495	**$28.669**
1.01>S/K<1.03	$43.999	$51.997	$57.799	$62.797	**$53.655**
1.03>S/K<1.05	$75.697	$81.128	$85.723	$89.477	**$82.891**
Total	**$30.786**	**$33.600**	**$37.928**	**$43.228**	**$36.306**
			Panel C. Average Implied Volatility		
0.95>S/K<0.97	8.101%	7.601%	7.658%	7.981%	**7.774%**
0.97>S/K<0.99	7.441%	7.872%	8.277%	8.561%	**8.026%**
0.99>S/K<1.01	8.387%	8.932%	9.182%	9.330%	**8.939%**
1.01>S/K<1.03	9.765%	9.954%	9.970%	10.012%	**9.920%**
1.03>S/K<1.05	11.092%	10.821%	10.700%	10.606%	**10.807%**
Total	**9.019%**	**9.004%**	**9.094%**	**9.266%**	**9.092%**

Note: The sample data consists of European call options on the S&P 500 Index in the period 15/10/2013 - 10/12/2014. We only use Wednesday data. The implied volatility is calculated numerically trough Black-Scholes and annualized by multiplying by $\sqrt{252}$.

10.6 Data Statistics 2015

Table 10.3: S&P 500 Index Option data, 2015

	9>DTM<28	29>DTM<48	49>DTM<68	69>DTM<88	**Total**
			Panel A. Number of call options		
0.95>S/K<0.97	171	321	313	258	**1063**
0.97>S/K<0.99	297	303	302	240	**1142**
0.99>S/K<1.01	296	301	292	232	**1121**
1.01>S/K<1.03	285	284	282	225	**1076**
1.03>S/K<1.05	254	275	270	219	**1018**
Total	**1303**	**1484**	**1459**	**1174**	**5420**
			Panel B. Average Option Price		
0.95>S/K<0.97	$2.292	$4.466	$8.352	$14.906	**$7.794**
0.97>S/K<0.99	$6.731	$14.778	$21.762	$30.822	**$17.904**
0.99>S/K<1.01	$23.711	$35.006	$43.056	$52.814	**$37.806**
1.01>S/K<1.03	$51.725	$62.366	$69.808	$78.793	**$64.933**
1.03>S/K<1.05	$84.064	$92.864	$99.139	$107.656	**$95.515**
Total	**$34.922**	**$40.227**	**$46.752**	**$55.197**	**$43.951**
			Panel C. Average Implied Volatility		
0.95>S/K<0.97	10.133%	8.681%	8.752%	9.578%	**9.153%**
0.97>S/K<0.99	9.830%	9.681%	9.744%	10.573%	**9.924%**
0.99>S/K<1.01	11.580%	11.055%	10.880%	11.609%	**11.263%**
1.01>S/K<1.03	13.213%	12.355%	11.976%	12.507%	**12.515%**
1.03>S/K<1.05	14.584%	13.316%	12.792%	13.379%	**13.507%**
Total	**11.934%**	**10.929%**	**10.754%**	**11.453%**	**11.237%**

Note: The sample data consists of European call options on the S&P 500 Index in the period 22/10/2014 - 09/12/2015. We only use Wednesday data. The implied volatility is calculated numerically trough Black-Scholes and annualized by multiplying by $\sqrt{252}$.

10.7 Data Statistics 2016

Table 10.4: S&P 500 Index Option data, 2014-2016

	9>DTM<28	29>DTM<48	49>DTM<68	69>DTM<88	**Total**
			Panel A. Number of call options		
0.95>S/K<0.97	162	313	326	275	**1076**
0.97>S/K<0.99	285	311	311	265	**1172**
0.99>S/K<1.01	308	299	298	259	**1164**
1.01>S/K<1.03	276	286	287	241	**1090**
1.03>S/K<1.05	227	276	272	234	**1009**
Total	**1258**	**1485**	**1494**	**1274**	**5511**
			Panel B. Average Option Price		
0.95>S/K<0.97	$2.048	$4.840	$9.260	$13.784	**$8.045**
0.97>S/K<0.99	$5.985	$14.737	$23.066	$29.760	**$18.216**
0.99>S/K<1.01	$21.830	$35.069	$44.988	$52.376	**$37.956**
1.01>S/K<1.03	$50.733	$62.144	$71.749	$79.047	**$65.521**
1.03>S/K<1.05	$84.634	$92.976	$101.278	$107.433	$96.690
Total	**33.367**	**$40.416**	**$48.017**	**$54.499**	**$44.123**
			Panel C. Average Implied Volatility		
0.95>S/K<0.97	10.010%	8.871%	9.020%	9.197%	**9.171%**
0.97>S/K<0.99	9.211%	9.704%	10.085%	10.214%	**9.800%**
0.99>S/K<1.01	10.383%	11.069%	11.311%	11.301%	**11.001%**
1.01>S/K<1.03	11.984%	12.230%	12.338%	12.285%	**12.208%**
1.03>S/K<1.05	13.702%	13.183%	13.251%	13.040%	**13.285%**
Total	**11.020%**	**10.936%**	**11.106%**	**11.127%**	**11.045%**

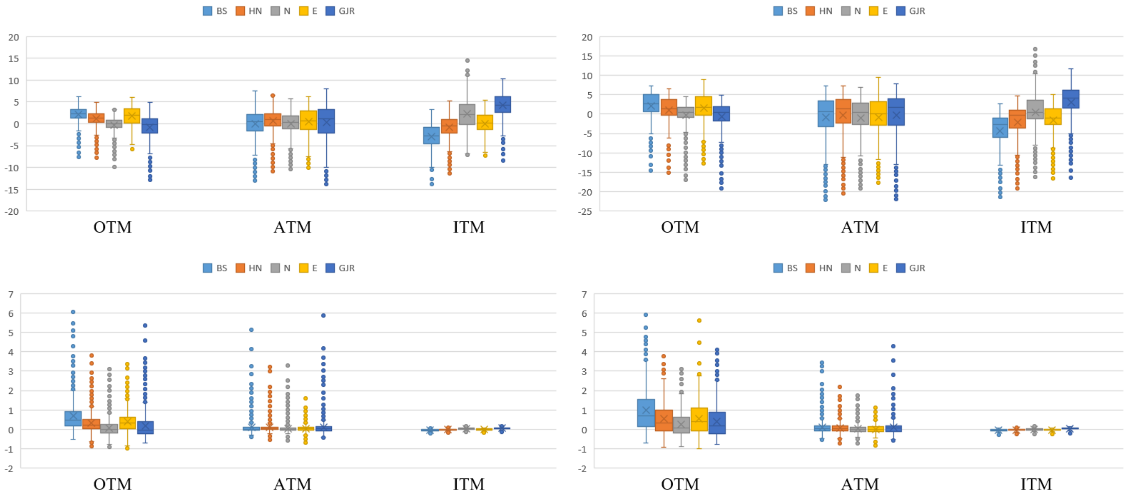
Note: The sample data consists of European call options on the S&P 500 Index in the period 21/10/2015 - 07/12/2016. We only use Wednesday data. The implied volatility is calculated numerically trough Black-Scholes and annualized by multiplying by $\sqrt{252}$.

10.8 Calibration Sample

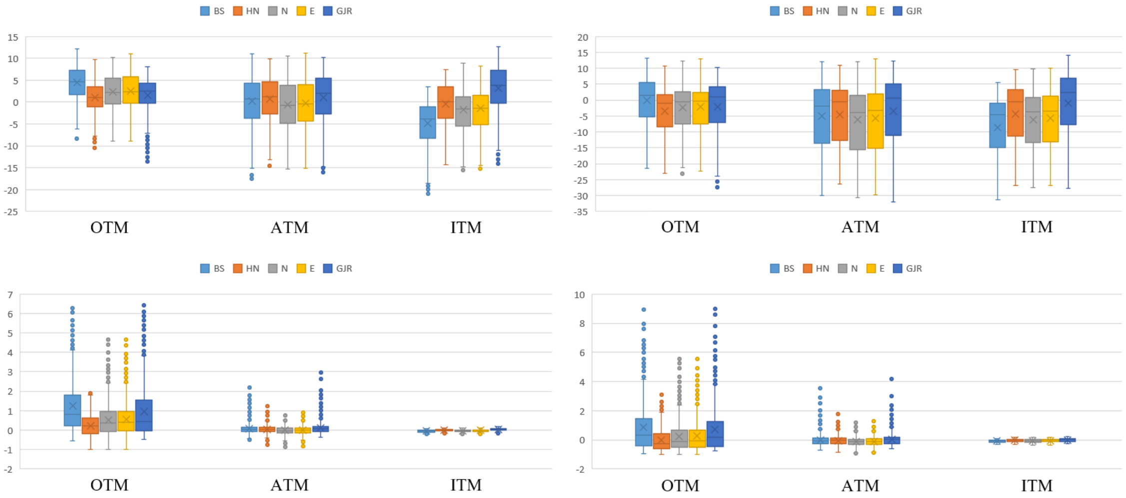
Table 10.5: Sub-Sample of options on which we run our calibration

			2014								
Avg $	DTM	10-29	30-49	70-89	**Total**	Count	DTM	10-29	30-49	70-89	**Total**
	0.95-0.97		2,278	6,513	**4,160**		0.95-0.97		5	4	**9**
	0.99-1.01	11,691	29,558	40,981	**26,053**		0.99-1.01	11	11	8	**30**
	1.03-1.05	89,113	93,540	100,900	**93,268**		1.03-1.05	4	5	2	**11**
	Total	**32,337**	**38,297**	**39,693**	**36,900**		**Total**	**15**	**21**	**14**	**50**
			2015								
Avg $	DTM	9-28	29-48	69-88	**Grand Total**	Count	DTM	9-28	29-48	69-88	**Total**
	0.95-0.97		2,212	9,290	**5,429**		0.95-0.97		6	5	**11**
	0.99-1.01	18,903	36,380	52,611	**36,502**		0.99-1.01	10	10	11	**31**
	1.03-1.05	**99,420**	**109,460**	**120,870**	**109,917**		1.03-1.05	5	5	5	**15**
	Total	45,742	44,018	58,549	**49,825**		**Total**	**15**	**21**	**21**	**57**
			2016								
Avg $	DTM	9-28	29-48	69-88	**Total**	Count	DTM	9-28	29-48	69-88	**Total**
	0.95-0.97		4,358	11,880	**7,777**		0.95-0.97		6	5	**11**
	0.99-1.01	18,348	40,339	58,117	**38,342**		0.99-1.01	10	11	9	**30**
	1.03-1.05	96,363	106,988	124,813	**109,388**		1.03-1.05	4	4	4	**12**
	Total	**40,638**	**42,754**	**60,094**	**48,084**		**Total**	**14**	**21**	**18**	**53**

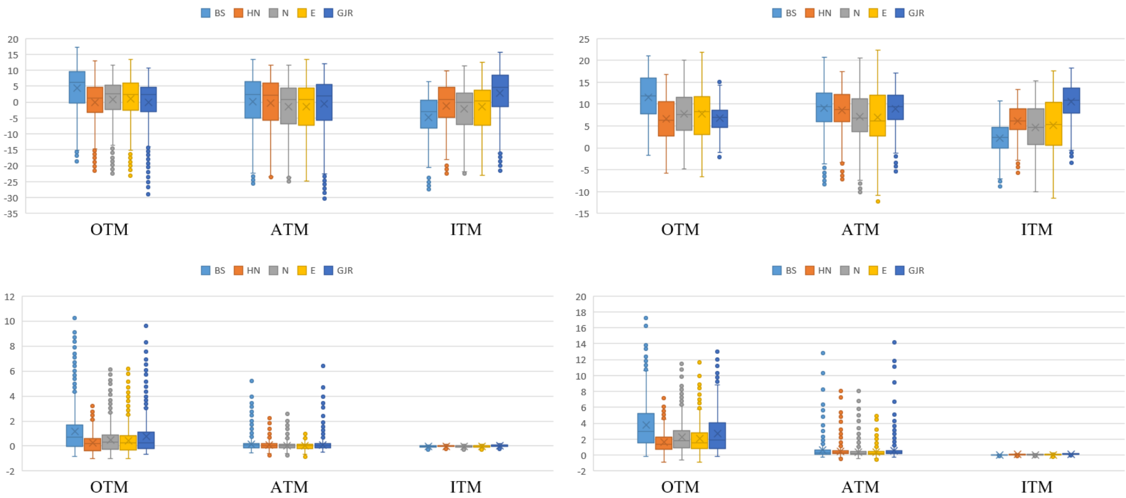
10.9 Box-Whisker Plots of Pricing Errors



2014; In-Sample to the left, Out-of-Sample to the right.

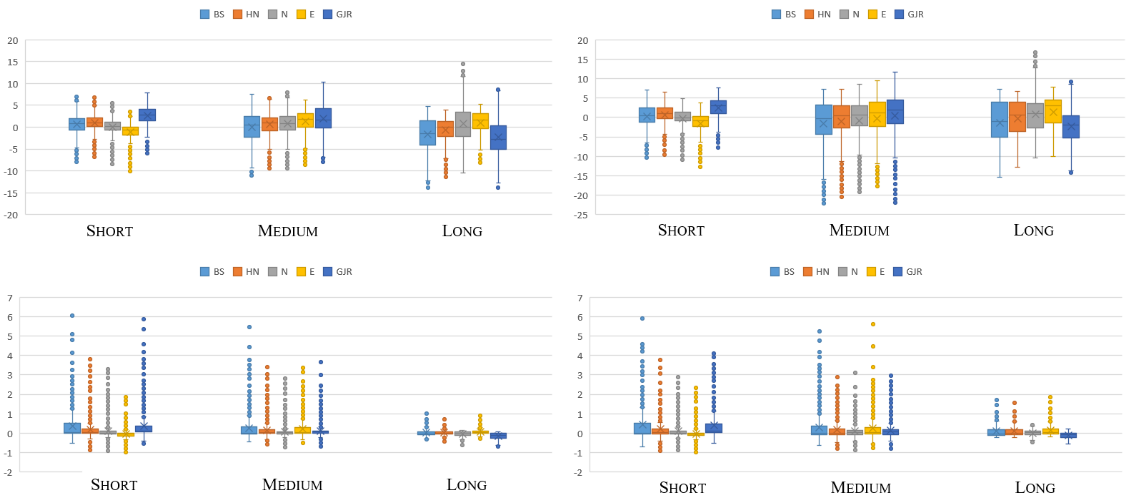


2015; In-Sample to the left, Out-of-Sample to the right.

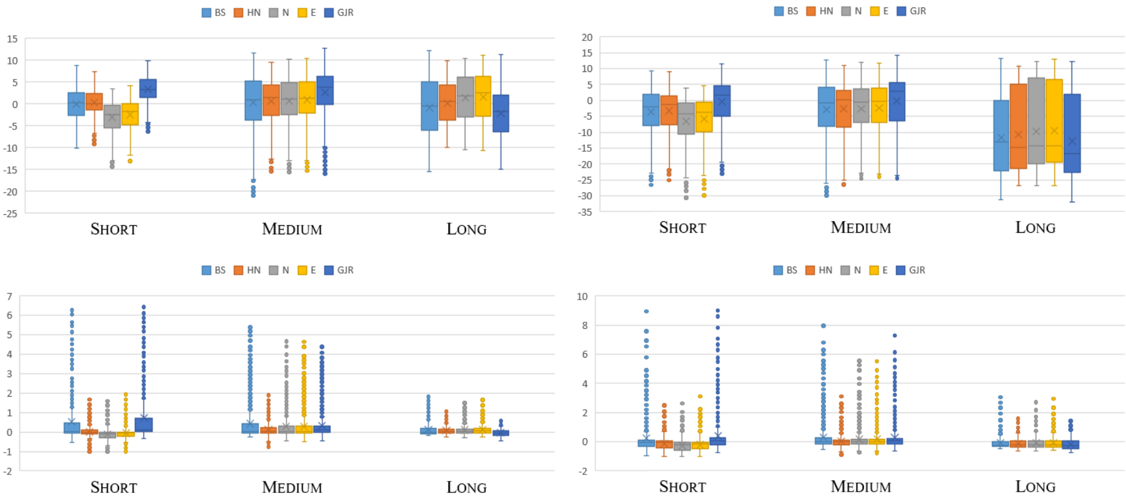


2016; In-Sample to the left, Out-of-Sample to the right.

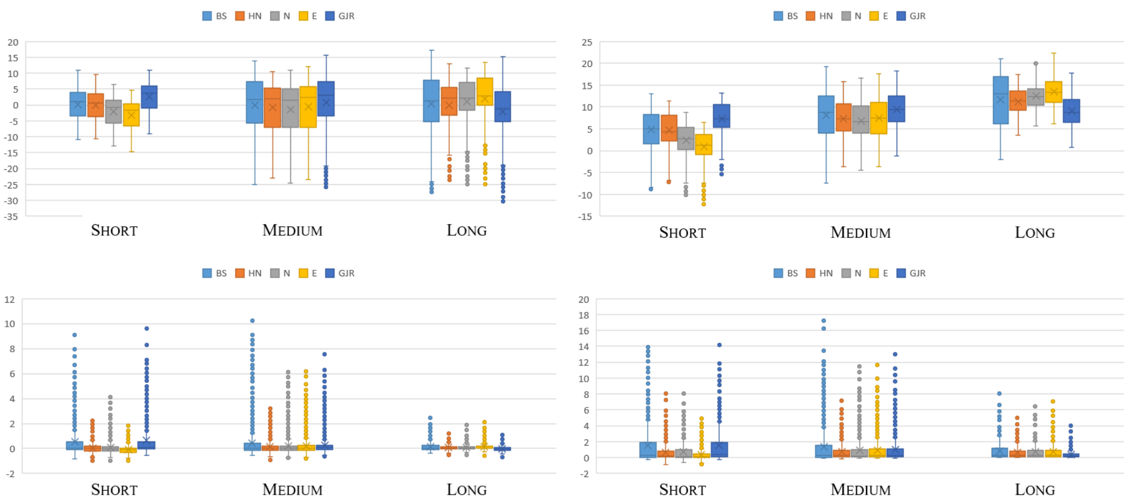
Figure 10.6: Box-Whisker Plots of Actual Errors



2014; In-Sample to the left, Out-of-Sample to the right.

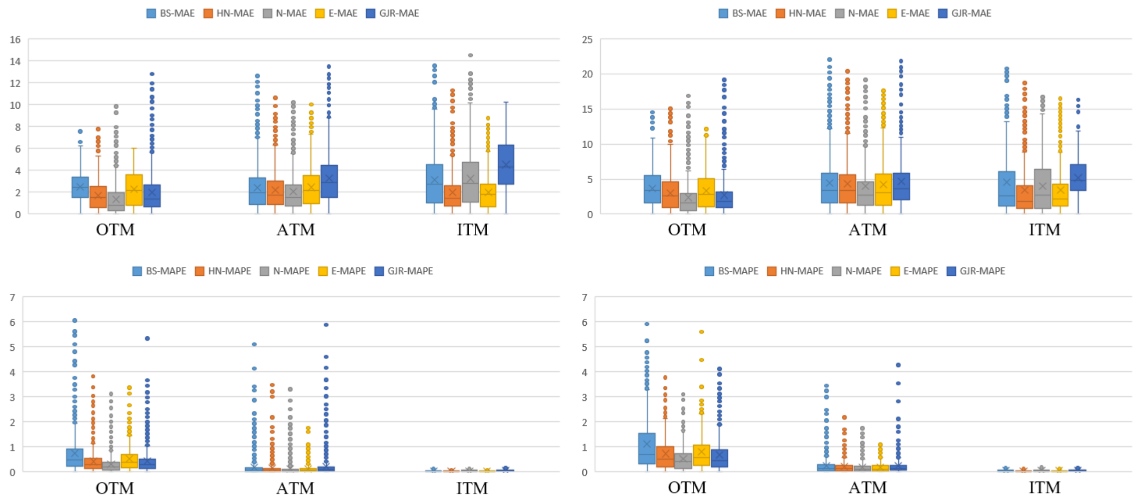


2015; In-Sample to the left, Out-of-Sample to the right.

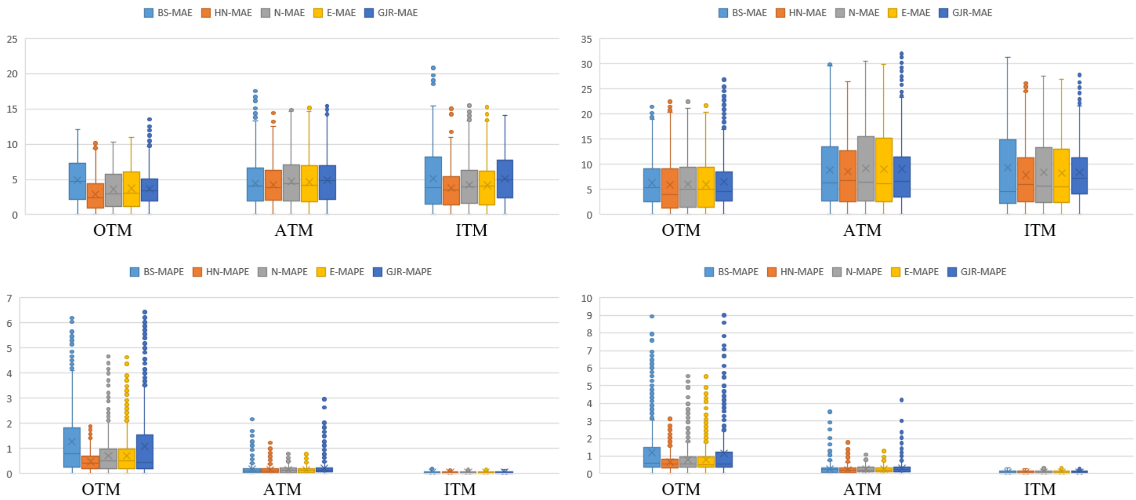


2016; In-Sample to the left, Out-of-Sample to the right.

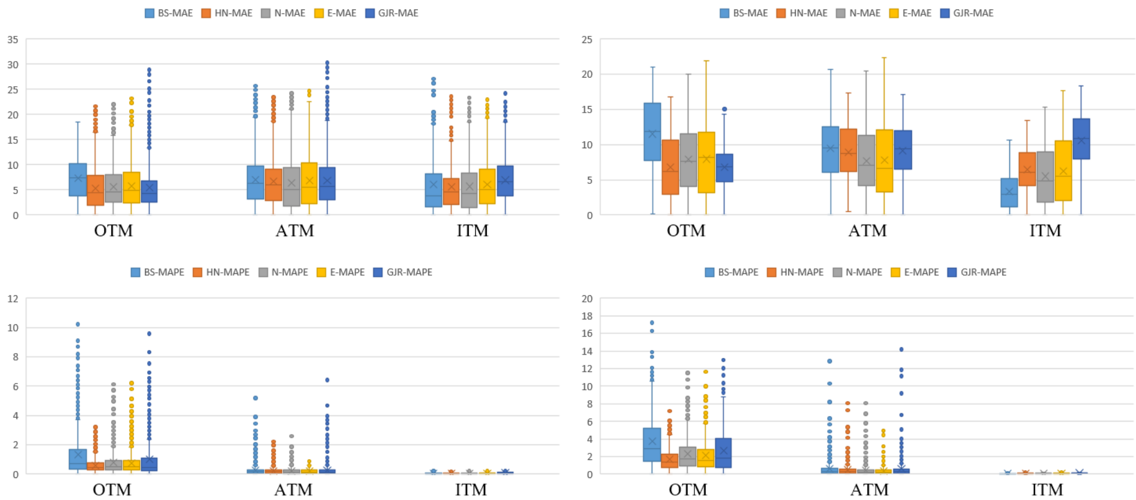
Figure 10.7: Box-Whisker Plots of Actual Errors after Maturity



2014; In-Sample to the left, Out-of-Sample to the right.



2015; In-Sample to the left, Out-of-Sample to the right.



2016; In-Sample to the left, Out-of-Sample to the right.

Figure 10.8: Box-Whisker Plots of Absolute Errors after Moneyness

10.10 Vix vs. Implied Volatility Plot

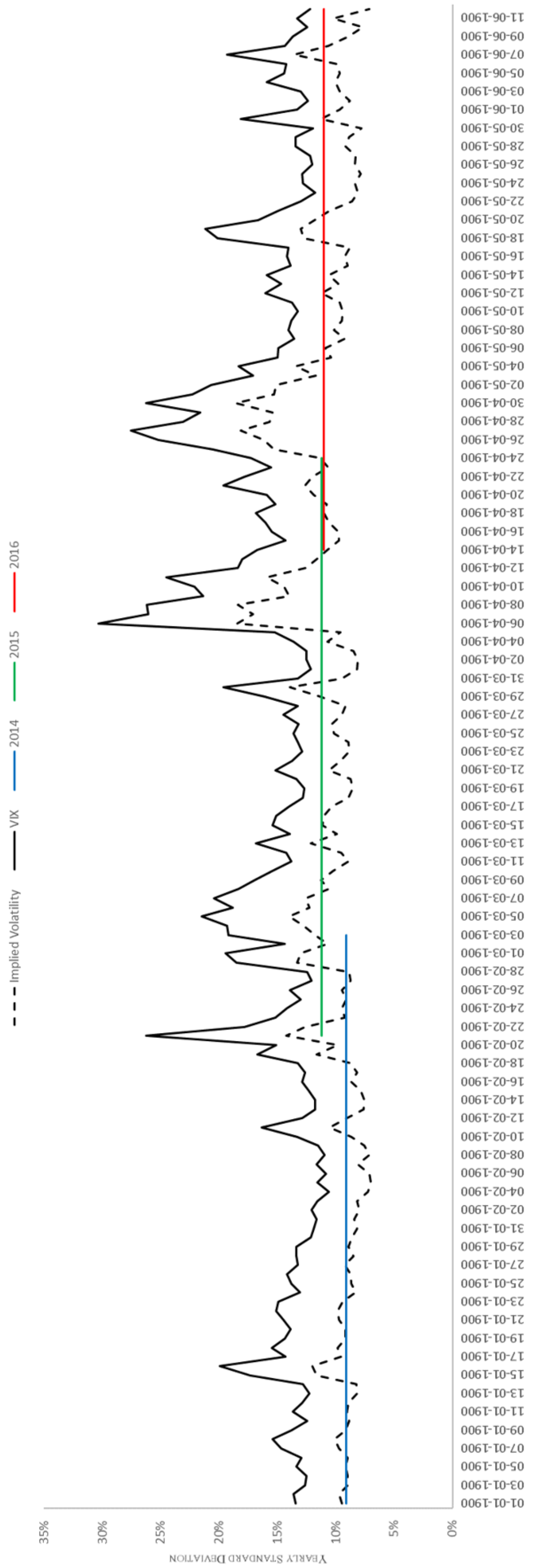


Figure 10.9: Vix vs. Implied Volatility 2013 - 2017.VIX Source: Datastream

The flat lines represent the average implied volatility for each year used for the Black-Scholes model when used as a benchmark for the estimation section.

10.11 Pricing Errors by Maturity and Moneyness

Table 10.6: IVRMSE In-Sample All Year

		2014					2015					2015				
		IVRMSE					IVRMSE					IVRMSE				
	DTM	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**
	0.95-0.97	0.0008	0.0011	0.0009	0.0006	**0.0009**	0.0017	0.0017	0.0017	0.0012	**0.0016**	0.0019	0.0023	0.0022	0.0017	**0.0021**
	0.97-0.99	0.0014	0.0010	0.0006	0.0005	**0.0009**	0.0016	0.0013	0.0013	0.0009	**0.0013**	0.0022	0.0020	0.0019	0.0014	**0.0019**
BS	0.99-1.01	0.0011	0.0007	0.0005	0.0005	**0.0007**	0.0013	0.0010	0.0010	0.0008	**0.0010**	0.0022	0.0019	0.0017	0.0012	**0.0018**
	1.01-1.03	0.0010	0.0008	0.0007	0.0008	**0.0009**	0.0017	0.0014	0.0012	0.0011	**0.0014**	0.0024	0.0021	0.0018	0.0014	**0.0020**
	1.03-1.05	0.0018	0.0013	0.0012	0.0012	**0.0014**	0.0027	0.0019	0.0016	0.0016	**0.0020**	0.0029	0.0025	0.0021	0.0016	**0.0023**
	Total	**0.0013**	**0.0010**	**0.0008**	**0.0008**	**9.84E-04**	**0.0019**	**0.0015**	**0.0014**	**0.0011**	**1.49E-03**	**0.0023**	**0.0022**	**0.0020**	**0.0015**	**2.02E-03**
	0.95-0.97	0.0008	0.0007	0.0006	0.0005	**0.0006**	0.0013	0.0008	0.0010	0.0008	**0.0009**	0.0023	0.0019	0.0017	0.0013	**0.0018**
	0.97-0.99	0.0010	0.0008	0.0006	0.0005	**0.0007**	0.0012	0.0009	0.0011	0.0007	**0.0010**	0.0020	0.0019	0.0017	0.0012	**0.0018**
HN	0.99-1.01	0.0010	0.0008	0.0005	0.0004	**0.0007**	0.0012	0.0010	0.0010	0.0007	**0.0010**	0.0021	0.0019	0.0017	0.0012	**0.0018**
	1.01-1.03	0.0011	0.0007	0.0005	0.0006	**0.0008**	0.0013	0.0011	0.0010	0.0008	**0.0011**	0.0023	0.0019	0.0017	0.0012	**0.0018**
	1.03-1.05	0.0013	0.0009	0.0006	0.0008	**0.0009**	0.0017	0.0012	0.0011	0.0009	**0.0013**	0.0022	0.0020	0.0018	0.0012	**0.0018**
	Total	**0.0011**	**0.0008**	**0.0006**	**0.0006**	**7.75E-04**	**0.0014**	**0.0010**	**0.0010**	**0.0008**	**1.07E-03**	**0.0022**	**0.0019**	**0.0017**	**0.0012**	**1.81E-03**
	0.95-0.97	0.0008	0.0006	0.0004	0.0006	**0.0006**	0.0018	0.0012	0.0014	0.0009	**0.0013**	0.0019	0.0019	0.0018	0.0013	**0.0017**
	0.97-0.99	0.0009	0.0007	0.0004	0.0006	**0.0007**	0.0015	0.0010	0.0012	0.0008	**0.0012**	0.0020	0.0020	0.0018	0.0012	**0.0018**
N	0.99-1.01	0.0009	0.0007	0.0005	0.0005	**0.0006**	0.0021	0.0011	0.0010	0.0008	**0.0014**	0.0023	0.0021	0.0018	0.0012	**0.0019**
	1.01-1.03	0.0009	0.0007	0.0006	0.0007	**0.0007**	0.0028	0.0014	0.0010	0.0008	**0.0017**	0.0027	0.0022	0.0018	0.0012	**0.0020**
	1.03-1.05	0.0013	0.0011	0.0010	0.0015	**0.0012**	0.0032	0.0015	0.0011	0.0008	**0.0019**	0.0030	0.0022	0.0018	0.0012	**0.0022**
	Total	**0.0010**	**0.0008**	**0.0006**	**0.0008**	**7.98E-04**	**0.0024**	**0.0012**	**0.0012**	**0.0008**	**1.53E-03**	**0.0024**	**0.0021**	**0.0018**	**0.0012**	**1.92E-03**
	0.95-0.97	0.0010	0.0008	0.0008	0.0006	**0.0008**	0.0012	0.0011	0.0014	0.0010	**0.0012**	0.0018	0.0019	0.0018	0.0013	**0.0018**
	0.97-0.99	0.0009	0.0008	0.0007	0.0005	**0.0007**	0.0013	0.0010	0.0012	0.0009	**0.0011**	0.0020	0.0019	0.0018	0.0012	**0.0018**
E	0.99-1.01	0.0009	0.0007	0.0005	0.0004	**0.0007**	0.0018	0.0011	0.0010	0.0008	**0.0013**	0.0023	0.0021	0.0017	0.0012	**0.0019**
	1.01-1.03	0.0012	0.0007	0.0005	0.0005	**0.0008**	0.0025	0.0013	0.0010	0.0008	**0.0016**	0.0026	0.0022	0.0017	0.0012	**0.0020**
	1.03-1.05	0.0017	0.0009	0.0006	0.0005	**0.0010**	0.0029	0.0014	0.0011	0.0008	**0.0018**	0.0029	0.0022	0.0018	0.0012	**0.0021**
	Total	**0.0012**	**0.0008**	**0.0006**	**0.0005**	**8.09E-04**	**0.0021**	**0.0012**	**0.0012**	**0.0009**	**1.41E-03**	**0.0024**	**0.0021**	**0.0018**	**0.0012**	**1.91E-03**
	0.95-0.97	0.0006	0.0007	0.0004	0.0010	**0.0007**	0.0019	0.0014	0.0011	0.0007	**0.0013**	0.0018	0.0019	0.0018	0.0013	**0.0017**
	0.97-0.99	0.0013	0.0008	0.0005	0.0010	**0.0009**	0.0019	0.0012	0.0011	0.0009	**0.0014**	0.0023	0.0019	0.0018	0.0014	**0.0019**
GJR	0.99-1.01	0.0014	0.0009	0.0005	0.0008	**0.0010**	0.0015	0.0011	0.0011	0.0010	**0.0012**	0.0023	0.0019	0.0018	0.0014	**0.0019**
	1.01-1.03	0.0020	0.0013	0.0007	0.0005	**0.0013**	0.0018	0.0012	0.0011	0.0009	**0.0013**	0.0025	0.0019	0.0017	0.0012	**0.0019**
	1.03-1.05	0.0030	0.0022	0.0015	0.0010	**0.0020**	0.0028	0.0017	0.0015	0.0009	**0.0019**	0.0030	0.0022	0.0019	0.0014	**0.0022**
	Total	**0.0019**	**0.0013**	**0.0008**	**0.0009**	**1.27E-03**	**0.0020**	**0.0013**	**0.0012**	**0.0009**	**1.43E-03**	**0.0024**	**0.0020**	**0.0018**	**0.0014**	**1.93E-03**

Table 10.7: 2014 In-Sample Pricing Error

		$RMSE					%RMSE					MAE					MAPE				
	DTM	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**
	0.95-0.97	1.0574	2.4426	3.0539	2.9160	**2.6768**	1.1847	1.6191	0.8452	0.3925	**1.1122**	0.8704	2.2195	2.8555	2.6710	**2.3905**	0.9252	1.2784	0.6838	0.3327	**0.8062**
	0.97-0.99	2.9589	3.7328	3.0459	2.7322	**3.1677**	1.6770	0.7749	0.2700	0.1347	**0.9252**	2.6260	3.2918	2.6290	2.0945	**2.6937**	1.2285	0.5385	0.2108	0.1060	**0.5195**
BS	0.99-1.01	3.1934	3.3418	2.5338	3.2195	**3.0859**	0.3441	0.1636	0.0782	0.0733	**0.2013**	2.5705	2.5352	1.9122	2.1468	**2.2988**	0.2321	0.1133	0.0600	0.0516	**0.1168**
	1.01-1.03	2.3221	3.2459	3.6561	4.9787	**3.6033**	0.0563	0.0628	0.0624	0.0771	**0.0645**	1.5159	2.4139	2.8377	4.1882	**2.6689**	0.0368	0.0482	0.0490	0.0662	**0.0493**
	1.03-1.05	2.0275	3.5736	4.5057	6.0598	**4.2572**	0.0277	0.0455	0.0538	0.0684	**0.0507**	1.4186	2.7218	3.9454	5.4866	**3.3627**	0.0196	0.0347	0.0472	0.0623	**0.0407**
	Total	**2.5836**	**3.2992**	**3.3972**	**4.1552**	**3.3987**	**0.8860**	**0.8135**	**0.4162**	**0.1934**	**0.6436**	**1.9436**	**2.6435**	**2.8193**	**3.2785**	**2.6770**	**0.4277**	**0.4106**	**0.2227**	**0.1233**	**0.2980**
	0.95-0.97	0.6924	1.5745	2.1040	2.3332	**1.9066**	0.5508	0.8705	0.5506	0.2822	**0.6221**	0.4958	1.2100	1.8283	1.9388	**1.5156**	0.4441	0.6271	0.4239	0.2318	**0.4409**
	0.97-0.99	2.2528	3.0898	2.6913	2.6474	**2.7033**	1.0424	0.5711	0.2257	0.1248	**0.6038**	1.7046	2.5285	2.2374	2.0220	**2.1438**	0.6900	0.3902	0.1740	0.1000	**0.3397**
HN	0.99-1.01	3.1640	3.4012	2.4889	2.6765	**2.9685**	0.2987	0.1649	0.0815	0.0629	**0.1813**	2.5444	2.6749	1.9744	1.7624	**2.2595**	0.2097	0.1184	0.0638	0.0442	**0.1117**
	1.01-1.03	2.6000	2.8113	2.4969	3.3152	**2.7986**	0.0680	0.0580	0.0442	0.0508	**0.0562**	2.0082	2.2057	1.8416	2.1832	**2.0537**	0.0509	0.0453	0.0330	0.0342	**0.0412**
	1.03-1.05	1.7151	2.5458	2.6020	3.8044	**2.7422**	0.0242	0.0327	0.0310	0.0426	**0.0331**	1.2321	2.0583	1.9109	2.7832	**1.9866**	0.0173	0.0265	0.0229	0.0313	**0.0245**
	Total	**2.4184**	**2.7640**	**2.4782**	**2.9896**	**2.6671**	**0.5415**	**0.4776**	**0.2801**	**0.1437**	**0.3934**	**1.7799**	**2.1417**	**1.9600**	**2.1276**	**2.0068**	**0.2614**	**0.2470**	**0.1517**	**0.0883**	**0.1882**
	0.95-0.97	0.7197	1.3932	1.3686	2.8271	**1.8032**	0.5060	0.6935	0.3127	0.2414	**0.4723**	0.5091	1.0276	0.9252	2.0087	**1.1769**	0.4313	0.4993	0.2064	0.1885	**0.3170**
	0.97-0.99	2.0150	2.5942	1.9707	3.4051	**2.5252**	0.8747	0.4412	0.1389	0.1402	**0.4945**	1.3932	1.9978	1.5139	2.3154	**1.7915**	0.5627	0.2938	0.1071	0.1011	**0.2651**
N	0.99-1.01	2.6794	3.0249	2.4134	2.8968	**2.7603**	0.2433	0.1305	0.0734	0.0702	**0.1494**	1.8915	2.3607	1.9140	1.9895	**2.0423**	0.1542	0.0988	0.0594	0.0498	**0.0922**
	1.01-1.03	2.2436	2.8065	2.9023	3.8656	**2.9648**	0.0542	0.0569	0.0521	0.0606	**0.0558**	1.4188	2.3552	2.4176	3.3560	**2.3389**	0.0345	0.0478	0.0436	0.0533	**0.0444**
	1.03-1.05	1.6257	2.9189	4.3585	7.7904	**4.6549**	0.0227	0.0373	0.0522	0.0875	**0.0543**	1.1174	2.5173	3.9603	7.2649	**3.6401**	0.0155	0.0322	0.0476	0.0824	**0.0437**
	Total	**2.1108**	**2.6113**	**2.7348**	**4.4943**	**3.0781**	**0.4575**	**0.3774**	**0.1655**	**0.1370**	**0.3130**	**1.3861**	**2.0437**	**2.0883**	**3.3326**	**2.1974**	**0.2139**	**0.1983**	**0.0961**	**0.0947**	**0.1503**
	0.95-0.97	0.7693	1.9556	2.8020	2.7488	**2.3926**	0.5370	1.0784	0.7377	0.3733	**0.7774**	0.5484	1.5763	2.5652	2.3798	**1.9885**	0.4615	0.8200	0.5988	0.3013	**0.5774**
	0.97-0.99	1.9310	2.9842	3.3877	2.9980	**2.8971**	0.6686	0.5367	0.2767	0.1587	**0.4597**	1.2106	2.4966	3.0259	2.5060	**2.3371**	0.4130	0.3751	0.2296	0.1288	**0.2911**
E	0.99-1.01	2.6891	3.0766	3.0162	2.7874	**2.9010**	0.1875	0.1347	0.0999	0.0711	**0.1325**	1.8150	2.5190	2.5513	2.2138	**2.2770**	0.1261	0.1060	0.0827	0.0576	**0.0945**
	1.01-1.03	2.6888	2.8316	2.5713	2.8027	**2.7217**	0.0632	0.0568	0.0471	0.0443	**0.0538**	1.8845	2.2531	2.0597	2.0726	**2.0659**	0.0455	0.0458	0.0376	0.0334	**0.0409**
	1.03-1.05	1.9841	2.5740	2.3458	2.7163	**2.4205**	0.0276	0.0332	0.0286	0.0309	**0.0302**	1.3617	2.0804	1.8953	2.1063	**1.8667**	0.0190	0.0268	0.0230	0.0240	**0.0233**
	Total	**2.2785**	**2.7195**	**2.8602**	**2.8141**	**2.6895**	**0.3683**	**0.5475**	**0.3701**	**0.1854**	**0.3977**	**1.4919**	**2.1902**	**2.4376**	**2.2581**	**2.1165**	**0.1796**	**0.2802**	**0.2059**	**0.1093**	**0.1986**
	0.95-0.97	0.7147	1.5330	1.4513	4.2568	**2.4403**	0.7534	0.9387	0.3089	0.3865	**0.6344**	0.5679	1.1938	1.0269	3.6180	**1.6658**	0.5643	0.6638	0.2164	0.3601	**0.4281**
	0.97-0.99	2.6925	2.9693	2.3582	6.1887	**3.7354**	1.5900	0.5533	0.1495	0.2750	**0.8407**	2.3395	2.3479	1.9195	5.5698	**2.9281**	1.1462	0.3642	0.1266	0.2583	**0.4624**
GJR	0.99-1.01	4.0254	3.8992	2.5337	5.4250	**4.0457**	0.4085	0.1847	0.0778	0.1393	**0.2411**	3.6205	3.1809	2.0054	4.5706	**3.3020**	0.3058	0.1384	0.0625	0.1189	**0.1580**
	1.01-1.03	4.2351	4.8526	3.5093	3.1706	**4.0276**	0.1097	0.1000	0.0626	0.0507	**0.0858**	3.9682	4.2808	3.0542	2.3068	**3.4524**	0.0992	0.0868	0.0548	0.0371	**0.0711**
	1.03-1.05	3.9598	6.2849	6.1507	4.9020	**5.4716**	0.0560	0.0804	0.0737	0.0550	**0.0681**	3.6656	5.7868	5.7393	4.4474	**4.9710**	0.0512	0.0741	0.0690	0.0503	**0.0620**
	Total	**3.6192**	**4.1778**	**3.4936**	**4.9134**	**4.0649**	**0.8126**	**0.5024**	**0.1685**	**0.2241**	**0.4868**	**3.1512**	**3.3082**	**2.6626**	**4.1146**	**3.2755**	**0.4161**	**0.2699**	**0.1091**	**0.1659**	**0.2350**

Table 10.8: 2015 In-Sample Pricing Error

	DTM	$RMSE 10-29	$RMSE 30-49	$RMSE 50-69	$RMSE 70-89	$RMSE Total	%RMSE 10-29	%RMSE 30-49	%RMSE 50-69	%RMSE 70-89	%RMSE Total	MAE 10-29	MAE 30-49	MAE 50-69	MAE 70-89	MAE Total	MAPE 10-29	MAPE 30-49	MAPE 50-69	MAPE 70-89	MAPE Total
	0.95-0.97	2.5149	4.5073	6.7783	6.8407	5.6657	3.0905	2.4752	1.7414	0.7715	2.1020	2.0649	4.0143	6.0858	5.9507	4.7970	2.4506	1.9485	1.3425	0.6047	1.5281
	0.97-0.99	4.1173	5.4710	7.3185	5.8453	5.7941	1.4857	0.7395	0.5239	0.2665	0.9060	3.4501	4.7685	6.5327	4.9242	4.9154	1.0923	0.5279	0.4098	0.2035	0.5818
BS	0.99-1.01	3.9785	4.8309	6.2885	5.5369	5.1961	0.2255	0.1644	0.1627	0.1110	0.1743	3.1904	4.0765	5.5157	4.6996	4.3377	0.1673	0.1326	0.1396	0.0941	0.1362
	1.01-1.03	3.9701	5.9200	7.0479	7.4747	6.1527	0.0799	0.0907	0.0937	0.0920	0.0890	3.2472	4.4419	5.3343	6.1584	4.6969	0.0652	0.0696	0.0745	0.0773	0.0713
	1.03-1.05	3.6853	6.5214	7.9642	9.2311	7.0102	0.0438	0.0688	0.0760	0.0851	0.0692	2.9276	5.1636	5.8929	7.8301	5.3117	0.0350	0.0548	0.0574	0.0727	0.0539
	Total	3.8079	5.4594	7.0860	7.0629	5.9776	1.2821	1.2035	0.8416	0.3869	1.0143	3.0773	4.4738	5.8802	5.8808	4.8038	0.5990	0.5805	0.4224	0.2206	0.4683
	0.95-0.97	1.0764	2.0590	3.9247	4.2600	3.2212	0.7083	0.6189	0.7933	0.4398	0.6561	0.7600	1.6690	3.4494	3.7073	2.5434	0.6251	0.5319	0.6384	0.3498	0.5353
	0.97-0.99	2.7255	3.9558	6.0265	5.1020	4.5826	0.5628	0.3883	0.3779	0.2158	0.4146	2.0687	3.4985	5.4328	4.3967	3.8092	0.4617	0.3184	0.3133	0.1733	0.3261
HN	0.99-1.01	3.8116	4.7096	6.3307	5.2661	5.0894	0.2000	0.1540	0.1655	0.1117	0.1639	3.0849	4.0855	5.7197	4.4989	4.3261	0.1558	0.1300	0.1452	0.0930	0.1337
	1.01-1.03	3.4832	4.6439	6.1004	5.4190	4.9708	0.0745	0.0763	0.0888	0.0706	0.0783	2.8145	3.8684	5.2768	4.6880	4.1195	0.0585	0.0640	0.0776	0.0612	0.0655
	1.03-1.05	2.9533	4.0905	5.6129	5.4305	4.5992	0.0369	0.0443	0.0555	0.0509	0.0472	2.3379	3.2926	4.6547	4.6535	3.6810	0.0286	0.0359	0.0470	0.0438	0.0385
	Total	3.1015	3.9800	5.6489	5.0928	4.5558	0.3774	0.3465	0.4130	0.2358	0.3549	2.3670	3.2543	4.8937	4.3708	3.7119	0.2340	0.2255	0.2529	0.1501	0.2195
	0.95-0.97	1.0723	3.2819	5.5087	5.0010	4.3051	0.7439	1.4314	1.4213	0.5582	1.1882	0.7762	2.6608	4.9057	4.1553	3.4021	0.6553	1.1151	1.0729	0.4172	0.8710
	0.97-0.99	2.9193	4.5257	6.8151	5.6541	5.1439	0.5172	0.4687	0.4757	0.2419	0.4493	2.1463	3.9159	6.0746	4.7611	4.1850	0.4532	0.3670	0.3744	0.1900	0.3565
N	0.99-1.01	5.7986	5.2268	6.3898	5.5542	5.7654	0.2733	0.1528	0.1721	0.1167	0.1920	4.7540	3.9388	5.6166	4.9306	4.7908	0.2280	0.1173	0.1448	0.1017	0.1512
	1.01-1.03	5.9586	5.8906	5.9727	5.3850	5.8345	0.1234	0.0922	0.0878	0.0717	0.0971	5.0010	4.3035	5.1986	4.9398	4.8535	0.1022	0.0685	0.0767	0.0654	0.0791
	1.03-1.05	4.1879	5.2332	5.4633	5.1167	5.0239	0.0505	0.0559	0.0548	0.0495	0.0530	3.3130	3.7123	4.6714	4.5218	4.0270	0.0398	0.0398	0.0474	0.0434	0.0426
	Total	4.6046	4.8759	6.0582	5.3464	5.2574	0.3840	0.7044	0.6953	0.2899	0.5685	3.4505	3.6856	5.3018	4.6531	4.2651	0.2640	0.3609	0.3575	0.1707	0.2974
	0.95-0.97	1.0560	3.1603	5.6825	5.4968	4.4809	0.8155	1.3614	1.4498	0.6215	1.1868	0.7842	2.5635	5.0561	4.6260	3.5274	0.6999	1.0650	1.1043	0.4693	0.8834
	0.97-0.99	2.7022	4.4829	6.9474	5.9735	5.2206	0.5106	0.4632	0.4852	0.2609	0.4506	2.0108	3.8877	6.1920	4.9924	4.2187	0.4297	0.3637	0.3829	0.2024	0.3540
E	0.99-1.01	5.0482	5.1911	6.4740	5.7128	5.6203	0.2311	0.1522	0.1755	0.1216	0.1779	4.0672	3.9134	5.7308	5.0220	4.6474	0.1925	0.1170	0.1483	0.1043	0.1429
	1.01-1.03	5.3446	5.7601	6.0025	5.5172	5.6699	0.1097	0.0903	0.0891	0.0734	0.0927	4.4083	4.1922	5.2265	5.0813	4.6997	0.0896	0.0668	0.0775	0.0672	0.0758
	1.03-1.05	3.8932	5.0204	5.5014	5.1908	4.9286	0.0469	0.0536	0.0556	0.0501	0.0518	3.0897	3.5658	4.7226	4.6833	3.9751	0.0371	0.0382	0.0481	0.0449	0.0419
	Total	4.1275	4.7704	6.1525	5.5906	5.2155	0.3917	0.6727	0.7092	0.3202	0.5671	3.0867	3.6052	5.3957	4.8786	4.2257	0.2524	0.3487	0.3669	0.1858	0.2968
	0.95-0.97	2.6730	3.5275	4.4033	4.0035	3.8176	3.4034	1.9749	1.0724	0.2912	1.8253	2.3559	3.0585	3.9673	3.2393	3.2687	2.8838	1.5229	0.8113	0.2504	1.2156
	0.97-0.99	4.5530	5.0513	6.1629	6.4766	5.5518	1.9525	0.6545	0.3694	0.1983	1.0828	4.0968	4.4027	5.3227	5.0985	4.7036	1.4360	0.4737	0.2995	0.1654	0.6217
GJR	0.99-1.01	4.8314	5.3709	6.5999	7.1941	5.9750	0.2935	0.1927	0.1655	0.1341	0.2097	4.0051	4.7547	5.5957	5.7705	4.9724	0.2222	0.1583	0.1394	0.1097	0.1610
	1.01-1.03	4.6792	5.4170	6.6703	5.9477	5.7042	0.1012	0.0931	0.0997	0.0767	0.0942	3.8487	4.8246	5.7737	5.1392	4.8753	0.0811	0.0812	0.0857	0.0665	0.0794
	1.03-1.05	4.8842	6.0956	7.4872	5.6426	6.1380	0.0609	0.0674	0.0777	0.0546	0.0664	4.0364	5.3409	6.4721	4.7770	5.1904	0.0496	0.0586	0.0667	0.0460	0.0559
	Total	4.5409	5.1218	6.3081	5.9237	5.5100	1.5036	0.9713	0.5299	0.1783	0.9473	3.8039	4.4363	5.3920	4.7746	4.6109	0.7479	0.4853	0.2909	0.1316	0.4239

Table 10.9: 2016 In-Sample Pricing Error

		$RMSE					%RMSE					MAE					MAPE				
	DTM	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**
	0.95-0.97	3.0149	6.7589	9.7648	10.0270	**8.1826**	2.8812	3.1142	1.9022	0.9173	**2.3475**	2.5029	6.0884	9.2676	9.1857	**7.1245**	2.0282	2.1914	1.4190	0.7273	**1.5787**
	0.97-0.99	5.3817	8.5963	10.5201	9.2946	**8.6772**	2.0448	0.8404	0.5573	0.3410	**1.1518**	4.8351	7.5259	9.9242	8.3272	**7.6732**	1.4048	0.6063	0.4637	0.2821	**0.6921**
BS	0.99-1.01	6.3724	8.9201	10.0942	8.3073	**8.5236**	0.4234	0.2510	0.2093	0.1475	**0.2821**	5.7292	7.3622	8.3043	6.5297	**6.9869**	0.3124	0.2054	0.1782	0.1192	**0.2074**
	1.01-1.03	5.6772	8.8974	10.0129	8.9094	**8.5225**	0.1088	0.1304	0.1237	0.1011	**0.1171**	4.4840	6.9701	6.8110	6.3576	**6.1523**	0.0873	0.1062	0.0885	0.0756	**0.0899**
	1.03-1.05	4.8207	8.6438	10.4204	9.5898	**8.6722**	0.0568	0.0903	0.0969	0.0843	**0.0840**	3.5550	6.3717	7.1703	7.0155	**6.0529**	0.0423	0.0671	0.0682	0.0631	**0.0605**
	Total	**5.3172**	**8.3739**	**#####**	**9.2516**	**8.5164**	**1.4908**	**1.5009**	**0.9318**	**0.4607**	**1.1862**	**4.3761**	**6.8637**	**8.3591**	**7.5322**	**6.8239**	**0.7148**	**0.6716**	**0.4714**	**0.2651**	**0.5349**
	0.95-0.97	2.5003	5.5218	7.6817	7.3105	**6.3264**	0.7810	1.0900	0.9206	0.5413	**0.8749**	1.7428	3.9650	6.5574	6.3488	**4.9013**	0.6861	0.8094	0.7517	0.4423	**0.6803**
	0.97-0.99	4.5849	8.0217	9.6025	8.2463	**7.8598**	0.8325	0.5255	0.3989	0.2648	**0.5489**	3.6130	6.3867	8.1965	6.9049	**6.2942**	0.6518	0.4142	0.3419	0.2199	**0.4094**
HN	0.99-1.01	6.2333	8.9019	10.0224	8.2783	**8.4634**	0.3600	0.2339	0.2000	0.1448	**0.2514**	5.5045	7.2326	8.0863	6.4751	**6.8263**	0.2805	0.1956	0.1697	0.1171	**0.1939**
	1.01-1.03	5.4904	8.2422	9.4661	8.0967	**7.9578**	0.1132	0.1263	0.1214	0.0947	**0.1152**	4.9426	6.7194	7.2232	5.9784	**6.2279**	0.0997	0.1053	0.0966	0.0727	**0.0943**
	1.03-1.05	4.0805	7.0209	8.7594	7.4674	**7.0810**	0.0494	0.0753	0.0832	0.0666	**0.0703**	3.5930	5.7916	6.4598	5.4216	**5.3539**	0.0433	0.0625	0.0629	0.0496	**0.0550**
	Total	**4.9022**	**7.6132**	**9.1276**	**7.8921**	**7.5917**	**0.5270**	**0.5729**	**0.4799**	**0.2902**	**0.4837**	**4.0452**	**5.9892**	**7.3154**	**6.2509**	**5.9400**	**0.3412**	**0.3312**	**0.2992**	**0.1875**	**0.2920**
	0.95-0.97	2.2696	5.6500	8.0752	7.5989	**6.5640**	1.0910	1.7862	1.3253	0.6220	**1.3139**	1.7170	4.5284	7.2407	6.6425	**5.3271**	0.8432	1.2637	0.9736	0.4886	**0.9136**
	0.97-0.99	4.6040	8.1562	9.7699	8.2852	**7.9614**	1.0130	0.5680	0.4274	0.2742	**0.6350**	3.6890	6.6266	8.0927	6.6159	**6.2820**	0.7565	0.4407	0.3453	0.2154	**0.4421**
N	0.99-1.01	6.5119	9.4050	10.2100	8.2261	**8.7010**	0.3080	0.2291	0.1958	0.1443	**0.2302**	5.0734	7.3952	7.5912	6.0372	**6.5295**	0.2351	0.1916	0.1555	0.1097	**0.1756**
	1.01-1.03	6.3556	9.1475	9.6815	7.9526	**8.4020**	0.1180	0.1339	0.1224	0.0944	**0.1188**	4.7251	6.9349	6.9877	5.8555	**6.1389**	0.0896	0.1049	0.0923	0.0719	**0.0903**
	1.03-1.05	4.9838	7.8535	8.7701	7.3505	**7.4211**	0.0591	0.0830	0.0835	0.0674	**0.0746**	3.6608	6.0015	6.5006	5.9250	**5.5528**	0.0436	0.0638	0.0633	0.0549	**0.0568**
	Total	**5.3309**	**8.1165**	**9.3237**	**7.8999**	**7.8595**	**0.6568**	**0.8771**	**0.6584**	**0.3247**	**0.6683**	**3.9246**	**6.2730**	**7.3057**	**6.2332**	**5.9800**	**0.3755**	**0.4339**	**0.3448**	**0.1958**	**0.3414**
	0.95-0.97	2.2605	5.6890	8.1420	7.6843	**6.6210**	1.1408	1.8649	1.3639	0.6352	**1.3648**	1.7363	4.6101	7.3738	6.7676	**5.4231**	0.8775	1.3192	1.0028	0.5002	**0.9469**
	0.97-0.99	4.5885	8.1460	9.7977	8.3645	**7.9844**	1.0766	0.5879	0.4406	0.2809	**0.6678**	3.7599	6.6743	8.2993	6.7924	**6.4066**	0.7998	0.4525	0.3584	0.2224	**0.4609**
E	0.99-1.01	6.4284	9.3013	10.1543	8.2394	**8.6420**	0.3164	0.2308	0.1986	0.1465	**0.2345**	5.0877	7.4019	7.7631	6.1640	**6.6074**	0.2407	0.1940	0.1609	0.1128	**0.1797**
	1.01-1.03	6.2325	8.9936	9.5947	7.9567	**8.3097**	0.1159	0.1322	0.1223	0.0954	**0.1179**	4.6595	6.8996	7.0977	6.0104	**6.1762**	0.0885	0.1049	0.0943	0.0741	**0.0911**
	1.03-1.05	4.9107	7.7264	8.7129	7.5141	**7.3936**	0.0583	0.0818	0.0833	0.0694	**0.0744**	3.6183	5.9906	6.6628	6.2276	**5.6518**	0.0431	0.0638	0.0650	0.0579	**0.0578**
	Total	**5.2600**	**8.0403**	**9.3032**	**7.9666**	**7.8363**	**0.6918**	**0.9143**	**0.6773**	**0.3316**	**0.6952**	**3.9246**	**6.2930**	**7.4627**	**6.4076**	**6.0673**	**0.3911**	**0.4488**	**0.3557**	**0.2014**	**0.3533**
	0.95-0.97	2.8038	5.6955	7.8888	7.5572	**6.5348**	2.7832	2.2799	1.1264	0.3768	**1.8102**	2.3985	4.9634	6.5244	5.1980	**5.0114**	2.0109	1.6114	0.8231	0.2979	**1.1324**
	0.97-0.99	5.2664	8.0388	9.9308	9.4346	**8.3673**	2.3300	0.6712	0.3839	0.2292	**1.2293**	4.8032	6.6889	7.5250	6.9909	**6.5110**	1.5157	0.4935	0.3024	0.1938	**0.6275**
GJR	0.99-1.01	6.6538	8.8848	10.3679	9.5720	**8.9410**	0.4631	0.2497	0.1998	0.1533	**0.2968**	6.1626	7.3986	7.7415	7.1290	**7.1001**	0.3427	0.2067	0.1586	0.1228	**0.2115**
	1.01-1.03	6.1198	8.3921	9.6786	8.2652	**8.2175**	0.1309	0.1346	0.1276	0.0964	**0.1242**	5.5926	7.1044	8.0436	5.9328	**6.6981**	0.1153	0.1139	0.1088	0.0719	**0.1035**
	1.03-1.05	5.5771	8.1442	9.7488	8.4044	**8.1482**	0.0680	0.0897	0.0971	0.0781	**0.0845**	4.7201	7.1276	9.0627	7.3460	**7.1075**	0.0573	0.0780	0.0901	0.0683	**0.0740**
	Total	**5.5883**	**7.8739**	**9.5318**	**8.6829**	**8.0898**	**1.5561**	**1.1092**	**0.5662**	**0.2215**	**0.9997**	**4.9047**	**6.6207**	**7.7275**	**6.4976**	**6.4787**	**0.7522**	**0.5271**	**0.3117**	**0.1555**	**0.4377**

Table 10.10: IVRMSE Out-of-Sample All Years

		2014					2015					2015				
		IVRMSE					IVRMSE					IVRMSE				
	DTM	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**
	0.95-0.97	0.0011	0.0013	0.0014	0.0010	**0.0013**	0.0022	0.0018	0.0018	0.0019	**0.0019**	0.0027	0.0031	0.0031	0.0031	**0.0030**
	0.97-0.99	0.0015	0.0013	0.0013	0.0007	**0.0013**	0.0026	0.0018	0.0018	0.0023	**0.0022**	0.0031	0.0029	0.0025	0.0024	**0.0028**
BS	0.99-1.01	0.0014	0.0014	0.0014	0.0008	**0.0013**	0.0030	0.0020	0.0020	0.0028	**0.0025**	0.0026	0.0021	0.0017	0.0017	**0.0021**
	1.01-1.03	0.0017	0.0017	0.0016	0.0011	**0.0016**	0.0037	0.0024	0.0024	0.0031	**0.0030**	0.0020	0.0014	0.0011	0.0010	**0.0015**
	1.03-1.05	0.0026	0.0022	0.0020	0.0014	**0.0022**	0.0046	0.0029	0.0028	0.0036	**0.0036**	0.0019	0.0011	0.0007	0.0006	**0.0012**
	Total	**0.0018**	**0.0016**	**0.0016**	**0.0010**	0.0016	**0.0033**	**0.0022**	**0.0022**	**0.0028**	0.0027	**0.0025**	**0.0023**	**0.0021**	**0.0020**	0.0023
	0.95-0.97	0.0009	0.0011	0.0012	0.0008	**0.0011**	0.0035	0.0016	0.0018	0.0022	**0.0023**	0.0012	0.0015	0.0019	0.0022	**0.0018**
	0.97-0.99	0.0012	0.0012	0.0013	0.0007	**0.0012**	0.0030	0.0018	0.0019	0.0024	**0.0024**	0.0020	0.0020	0.0019	0.0020	**0.0020**
HN	0.99-1.01	0.0014	0.0014	0.0014	0.0008	**0.0013**	0.0030	0.0019	0.0020	0.0026	**0.0025**	0.0025	0.0019	0.0016	0.0017	**0.0020**
	1.01-1.03	0.0015	0.0014	0.0014	0.0009	**0.0014**	0.0030	0.0021	0.0021	0.0027	**0.0025**	0.0027	0.0019	0.0015	0.0014	**0.0020**
	1.03-1.05	0.0020	0.0016	0.0016	0.0010	**0.0016**	0.0034	0.0023	0.0021	0.0029	**0.0027**	0.0027	0.0020	0.0014	0.0012	**0.0020**
	Total	**0.0015**	**0.0013**	**0.0014**	**0.0008**	0.0013	**0.0032**	**0.0020**	**0.0020**	**0.0026**	0.0025	**0.0024**	**0.0019**	**0.0017**	**0.0018**	0.0020
	0.95-0.97	0.0008	0.0011	0.0012	0.0006	**0.0010**	0.0043	0.0016	0.0017	0.0022	**0.0026**	0.0014	0.0022	0.0023	0.0024	**0.0022**
	0.97-0.99	0.0011	0.0012	0.0013	0.0007	**0.0011**	0.0034	0.0018	0.0018	0.0024	**0.0025**	0.0020	0.0021	0.0019	0.0021	**0.0020**
N	0.99-1.01	0.0014	0.0014	0.0014	0.0007	**0.0013**	0.0040	0.0021	0.0019	0.0026	**0.0029**	0.0018	0.0016	0.0015	0.0018	**0.0017**
	1.01-1.03	0.0017	0.0014	0.0013	0.0008	**0.0014**	0.0047	0.0024	0.0020	0.0026	**0.0033**	0.0017	0.0013	0.0014	0.0016	**0.0015**
	1.03-1.05	0.0022	0.0014	0.0014	0.0015	**0.0017**	0.0051	0.0026	0.0021	0.0026	**0.0035**	0.0019	0.0014	0.0014	0.0018	**0.0016**
	Total	**0.0016**	**0.0013**	**0.0013**	**0.0009**	0.0013	0.0043	0.0021	0.0019	0.0025	**0.0030**	**0.0018**	**0.0018**	**0.0017**	**0.0020**	0.0018
	0.95-0.97	0.0010	0.0012	0.0013	0.0009	**0.0012**	0.0037	0.0016	0.0017	0.0021	**0.0023**	0.0014	0.0022	0.0023	0.0025	**0.0022**
	0.97-0.99	0.0011	0.0013	0.0013	0.0007	**0.0011**	0.0032	0.0018	0.0018	0.0024	**0.0024**	0.0021	0.0021	0.0020	0.0021	**0.0021**
E	0.99-1.01	0.0016	0.0014	0.0013	0.0007	**0.0014**	0.0038	0.0021	0.0019	0.0026	**0.0028**	0.0019	0.0017	0.0015	0.0018	**0.0017**
	1.01-1.03	0.0021	0.0016	0.0013	0.0007	**0.0016**	0.0043	0.0023	0.0020	0.0026	**0.0031**	0.0017	0.0014	0.0014	0.0017	**0.0016**
	1.03-1.05	0.0026	0.0017	0.0014	0.0007	**0.0019**	0.0048	0.0025	0.0021	0.0026	**0.0033**	0.0019	0.0015	0.0015	0.0019	**0.0017**
	Total	**0.0018**	**0.0014**	**0.0013**	**0.0008**	0.0015	**0.0040**	**0.0021**	**0.0019**	**0.0024**	0.0028	**0.0019**	**0.0018**	**0.0018**	**0.0020**	0.0019
	0.95-0.97	0.0008	0.0011	0.0012	0.0008	**0.0010**	0.0020	0.0017	0.0017	0.0026	**0.0020**	0.0024	0.0024	0.0020	0.0015	**0.0021**
	0.97-0.99	0.0013	0.0012	0.0014	0.0010	**0.0013**	0.0026	0.0018	0.0019	0.0030	**0.0023**	0.0031	0.0024	0.0017	0.0011	**0.0023**
GJR	0.99-1.01	0.0015	0.0013	0.0014	0.0010	**0.0014**	0.0027	0.0019	0.0019	0.0031	**0.0024**	0.0030	0.0022	0.0016	0.0011	**0.0022**
	1.01-1.03	0.0018	0.0014	0.0013	0.0008	**0.0015**	0.0028	0.0020	0.0020	0.0028	**0.0024**	0.0034	0.0024	0.0019	0.0015	**0.0025**
	1.03-1.05	0.0030	0.0018	0.0017	0.0010	**0.0021**	0.0032	0.0025	0.0020	0.0025	**0.0026**	0.0041	0.0032	0.0026	0.0022	**0.0032**
	Total	**0.0019**	**0.0014**	**0.0014**	**0.0009**	0.0015	**0.0027**	**0.0020**	**0.0019**	**0.0028**	0.0023	**0.0033**	**0.0025**	**0.0019**	**0.0015**	0.0025

Table 10.11: 2014 Out-of-Sample Pricing Errors

		$RMSE					%RMSE					MAE					MAPE				
	DTM	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**
	0.95-0.97	1.4468	3.1439	5.1892	4.6991	**4.1221**	2.0144	2.0302	1.6007	0.7815	**1.6833**	1.1775	2.7309	4.6257	4.1143	**3.4320**	1.5427	1.5131	1.2915	0.6210	**1.2483**
	0.97-0.99	3.3822	5.0837	7.1053	4.7344	**5.1725**	1.7782	0.8451	0.5058	0.2550	**1.1251**	2.9323	4.2274	6.1000	4.3710	**4.3016**	1.3374	0.5931	0.4256	0.2198	**0.7242**
BS	0.99-1.01	4.1944	6.2955	8.2881	5.6562	**6.1782**	0.3434	0.2093	0.1933	0.1233	**0.2499**	3.4557	4.5273	6.0776	4.4182	**4.5308**	0.2522	0.1593	0.1582	0.1015	**0.1809**
	1.01-1.03	3.5507	6.8132	8.8391	7.1661	**6.5894**	0.0775	0.1115	0.1235	0.1002	**0.1028**	2.3643	4.8062	5.7956	5.3451	**4.3066**	0.0522	0.0816	0.0855	0.0762	**0.0718**
	1.03-1.05	2.7726	6.6414	8.9289	7.3807	**6.5867**	0.0347	0.0763	0.0922	0.0760	**0.0710**	1.8921	5.2307	6.2877	6.1845	**4.6043**	0.0238	0.0603	0.0664	0.0645	**0.0510**
	Total	**3.4195**	**5.7396**	**7.7348**	**5.9550**	**5.8395**	**1.0515**	**0.9921**	**0.7836**	**0.3939**	**0.8839**	**2.5707**	**4.2908**	**5.7553**	**4.8321**	**4.2585**	**0.5190**	**0.4874**	**0.4255**	**0.2305**	**0.4364**
	0.95-0.97	0.7410	2.6721	4.7127	3.9950	**3.6097**	0.9887	1.1887	1.1663	0.6336	**1.0544**	0.5516	1.8602	3.9174	3.5313	**2.7111**	0.6806	0.8732	0.9657	0.5115	**0.7962**
	0.97-0.99	2.6589	4.8268	6.9831	4.7034	**4.9209**	1.1144	0.6504	0.4570	0.2463	**0.7538**	2.1716	3.7021	5.9029	4.3683	**3.8706**	0.8452	0.4700	0.3949	0.2165	**0.5274**
HN	0.99-1.01	4.1891	6.1035	8.0307	5.2505	**5.9837**	0.3115	0.2059	0.1952	0.1201	**0.2349**	3.5449	4.4343	6.2076	4.4912	**4.5789**	0.2444	0.1580	0.1655	0.1062	**0.1805**
	1.01-1.03	3.3809	5.7545	7.7891	5.7002	**5.6849**	0.0776	0.0964	0.1107	0.0806	**0.0919**	2.5113	3.8889	5.3487	4.2133	**3.8267**	0.0573	0.0679	0.0812	0.0613	**0.0664**
	1.03-1.05	2.5194	5.0475	7.3110	5.2049	**5.1798**	0.0322	0.0579	0.0751	0.0532	**0.0559**	1.7608	3.4639	4.7075	3.7847	**3.2964**	0.0225	0.0400	0.0496	0.0393	**0.0368**
	Total	**3.1583**	**5.0301**	**7.0282**	**4.9698**	**5.1899**	**0.6304**	**0.6184**	**0.5881**	**0.3263**	**0.5743**	**2.3708**	**3.4797**	**5.2109**	**4.0737**	**3.7046**	**0.3320**	**0.3273**	**0.3469**	**0.1991**	**0.3116**
	0.95-0.97	0.6116	2.6950	4.6228	2.7594	**3.3056**	0.7841	0.9828	0.7375	0.2784	**0.7647**	0.4563	1.6741	3.2798	2.0756	**2.1112**	0.5670	0.7176	0.6458	0.2343	**0.5666**
	0.97-0.99	2.1947	4.8341	6.9625	4.2087	**4.7651**	0.8828	0.5251	0.3484	0.1688	**0.5961**	1.6819	3.2473	5.2449	3.1750	**3.2302**	0.6532	0.3695	0.3087	0.1367	**0.4049**
N	0.99-1.01	3.8786	6.3563	8.0094	5.0343	**5.9477**	0.2609	0.1892	0.1852	0.1119	**0.2063**	2.8578	4.2140	5.9658	4.0959	**4.1704**	0.1933	0.1387	0.1539	0.0943	**0.1536**
	1.01-1.03	3.5709	5.7360	7.2013	5.5539	**5.5072**	0.0781	0.0956	0.1068	0.0837	**0.0912**	2.3621	3.6449	5.6089	5.0411	**3.9071**	0.0524	0.0636	0.0875	0.0765	**0.0676**
	1.03-1.05	2.6565	4.5408	6.5422	8.9474	**5.6226**	0.0336	0.0527	0.0714	0.0958	**0.0623**	1.7288	2.8988	5.9957	7.5073	**4.0637**	0.0220	0.0338	0.0659	0.0805	**0.0458**
	Total	**3.0468**	**5.0053**	**6.7393**	**5.5229**	**5.1609**	**0.5019**	**0.5104**	**0.3895**	**0.1694**	**0.4358**	**2.0393**	**3.1481**	**5.1772**	**4.2361**	**3.5352**	**0.2639**	**0.2689**	**0.2623**	**0.1285**	**0.2415**
	0.95-0.97	0.6274	3.0107	4.9560	4.2897	**3.8625**	0.7625	1.4997	1.4052	0.7174	**1.2433**	0.4839	2.3651	4.3880	3.8347	**3.0811**	0.5975	1.1178	1.1621	0.5751	**0.9399**
	0.97-0.99	2.1528	4.9191	7.0031	4.7018	**4.8738**	0.7269	0.6128	0.4991	0.2631	**0.5840**	1.5616	3.6315	6.1955	4.2506	**3.7106**	0.5558	0.4454	0.4298	0.2192	**0.4383**
E	0.99-1.01	4.2119	6.4013	7.7665	4.8180	**5.9294**	0.2384	0.1911	0.1989	0.1168	**0.2011**	2.9467	4.1430	6.4010	4.1752	**4.2996**	0.1779	0.1369	0.1747	0.1013	**0.1542**
	1.01-1.03	4.1332	6.2215	7.3062	4.7402	**5.6869**	0.0904	0.1024	0.1064	0.0692	**0.0948**	2.9639	4.0813	5.3240	3.8959	**3.9732**	0.0656	0.0699	0.0823	0.0583	**0.0696**
	1.03-1.05	2.8183	5.2395	6.4897	3.9283	**4.8061**	0.0356	0.0604	0.0677	0.0412	**0.0531**	1.9533	3.5438	4.5895	3.3064	**3.2711**	0.0247	0.0412	0.0491	0.0350	**0.0370**
	Total	**3.3333**	**5.3023**	**6.7543**	**4.5158**	**5.1310**	**0.4334**	**0.7331**	**0.6970**	**0.3643**	**0.5932**	**2.2242**	**3.5581**	**5.3847**	**3.9085**	**3.6998**	**0.2433**	**0.3669**	**0.3982**	**0.2115**	**0.3120**
	0.95-0.97	0.9137	2.6331	4.7515	3.6645	**3.5443**	1.3070	1.2118	0.7140	0.3102	**0.9430**	0.7303	1.7844	3.2169	2.7100	**2.3045**	0.9648	0.8669	0.6167	0.2651	**0.6666**
	0.97-0.99	2.8547	4.7884	7.4195	6.3578	**5.3935**	1.5447	0.6280	0.3274	0.2506	**0.9474**	2.5172	3.5762	5.1819	5.0797	**3.8892**	1.1773	0.4465	0.2849	0.2128	**0.5984**
GJR	0.99-1.01	4.4895	6.0008	8.3853	6.9343	**6.4014**	0.3736	0.2134	0.1888	0.1528	**0.2664**	3.9852	4.6082	6.0637	5.2899	**4.8622**	0.2839	0.1686	0.1540	0.1196	**0.1958**
	1.01-1.03	4.3363	5.5774	7.4136	4.9490	**5.6237**	0.1019	0.1026	0.1125	0.0717	**0.1006**	4.0238	4.7006	6.1151	3.8877	**4.6819**	0.0914	0.0865	0.0966	0.0570	**0.0859**
	1.03-1.05	4.4894	5.6965	7.7083	5.6093	**5.9393**	0.0589	0.0681	0.0862	0.0597	**0.0693**	4.1601	5.3934	7.2013	4.8956	**5.3844**	0.0541	0.0638	0.0804	0.0522	**0.0631**
	Total	**3.9280**	**5.0801**	**7.2069**	**5.5912**	**5.5193**	**0.8593**	**0.6232**	**0.3774**	**0.2023**	**0.6063**	**3.4189**	**3.9889**	**5.4904**	**4.3322**	**4.2693**	**0.4575**	**0.3312**	**0.2556**	**0.1478**	**0.3171**

Table 10.12: 2015 Out-of-Sample Pricing Errors

		$RMSE					%RMSE					MAE					MAPE				
	DTM	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**	10-29	30-49	50-69	70-89	**Total**
	0.95-0.97	3.7711	5.2550	7.4377	11.1265	**7.0526**	2.1967	2.5264	2.0028	0.9839	**2.0952**	2.8849	4.4986	6.6667	10.1040	**5.7489**	1.4782	1.7698	1.3220	0.6831	**1.3822**
	0.97-0.99	6.9540	7.9801	9.8807	15.4710	**9.7141**	1.7790	0.9088	0.5506	0.4111	**1.1595**	5.5907	6.4275	8.5030	14.2735	**7.8843**	1.1336	0.5699	0.4236	0.3781	**0.6848**
BS	0.99-1.01	9.4649	9.7756	11.5412	18.8859	**11.9617**	0.3336	0.2522	0.2324	0.2805	**0.2812**	7.0858	6.8219	8.9165	17.0016	**8.9839**	0.2667	0.1846	0.1891	0.2659	**0.2249**
	1.01-1.03	9.7684	10.6998	12.9506	20.4145	**12.9683**	0.1616	0.1453	0.1618	0.2204	**0.1681**	7.0914	7.2368	9.7224	17.4280	**9.3736**	0.1245	0.1055	0.1265	0.1932	**0.1304**
	1.03-1.05	8.4532	10.6717	12.8508	21.4037	**13.0643**	0.0940	0.1051	0.1197	0.1806	**0.1215**	6.0896	7.1985	9.6760	17.8206	**9.2280**	0.0697	0.0737	0.0926	0.1518	**0.0900**
	Total	**8.1778**	**9.0213**	**11.0080**	**17.6544**	11.1527	**1.2124**	**1.2497**	**0.9792**	**0.5294**	1.0848	**5.9350**	**6.3897**	**8.6304**	**15.1525**	8.2338	**0.5770**	**0.5692**	**0.4561**	**0.3461**	0.5073
	0.95-0.97	4.5772	5.2314	7.1881	12.7033	**7.5557**	0.9074	0.7981	1.0352	0.7294	**0.8823**	3.0574	3.2510	5.7949	11.6360	**5.4366**	0.8227	0.6131	0.7736	0.6324	**0.7081**
	0.97-0.99	7.6779	8.1424	10.1044	16.2156	**10.1610**	0.8357	0.5626	0.4456	0.4010	**0.6211**	5.3568	5.5846	8.1621	14.9298	**7.5909**	0.6484	0.3772	0.3602	0.3831	**0.4606**
HN	0.99-1.01	9.3902	9.3857	11.4568	17.8973	**11.5930**	0.3172	0.2397	0.2327	0.2708	**0.2704**	6.9993	6.5743	8.9154	16.3296	**8.7810**	0.2566	0.1772	0.1904	0.2589	**0.2187**
	1.01-1.03	8.4358	9.3497	11.2331	17.6490	**11.2409**	0.1428	0.1315	0.1454	0.1930	**0.1495**	6.2312	6.3671	8.6505	15.7458	**8.3319**	0.1115	0.0954	0.1161	0.1780	**0.1185**
	1.03-1.05	6.8904	8.6717	10.1671	17.3161	**10.5342**	0.0774	0.0865	0.0960	0.1465	**0.0989**	5.1330	5.9957	7.7082	15.2687	**7.6897**	0.0596	0.0622	0.0749	0.1317	**0.0760**
	Total	**7.7667**	**8.2470**	**10.0860**	**16.3671**	10.3247	**0.5560**	**0.4674**	**0.5405**	**0.4199**	0.5069	**5.5275**	**5.5088**	**7.8052**	**14.6937**	7.5774	**0.3623**	**0.2752**	**0.3180**	**0.3277**	0.3204
	0.95-0.97	4.5570	4.9196	6.8092	12.4872	**7.3214**	0.9162	1.5431	1.6291	0.8173	**1.3380**	3.0684	3.5364	5.9738	11.3992	**5.5348**	0.8493	1.0530	1.0835	0.6539	**0.9417**
	0.97-0.99	8.3874	8.0127	9.6442	16.2723	**10.2169**	0.7446	0.6261	0.5090	0.4114	**0.6140**	5.6673	5.7103	8.0526	14.9565	**7.7034**	0.6352	0.4115	0.3894	0.3862	**0.4737**
N	0.99-1.01	12.0807	10.1301	10.9462	17.9112	**12.4336**	0.3853	0.2381	0.2263	0.2730	**0.2964**	9.2204	7.1974	8.5485	16.3680	**9.5961**	0.3309	0.1852	0.1844	0.2603	**0.2438**
	1.01-1.03	11.6112	10.4296	10.9766	17.0503	**12.1872**	0.1981	0.1445	0.1420	0.1883	**0.1703**	8.8467	7.4379	8.4061	15.3840	**9.3711**	0.1585	0.1106	0.1127	0.1752	**0.1369**
	1.03-1.05	8.9174	9.5337	9.8539	15.7023	**10.7140**	0.1001	0.0945	0.0933	0.1342	**0.1035**	6.4887	6.4420	7.4498	14.2075	**7.9870**	0.0747	0.0663	0.0725	0.1236	**0.0798**
	Total	**9.7695**	**8.7648**	**9.7033**	**15.9116**	10.7440	**0.5428**	**0.7810**	**0.8063**	**0.4556**	0.6780	**6.9107**	**6.0084**	**7.6542**	**14.3857**	8.0501	**0.3930**	**0.3821**	**0.3896**	**0.3314**	0.3789
	0.95-0.97	4.4737	4.9283	6.8971	12.1443	**7.2276**	0.9428	1.4835	1.6607	0.8629	**1.3366**	3.0214	3.5038	6.1084	11.0903	**5.4938**	0.8550	1.0160	1.1141	0.6623	**0.9412**
	0.97-0.99	8.0876	8.0269	9.6854	15.9067	**10.0613**	0.7885	0.6231	0.5152	0.4141	**0.6320**	5.4265	5.7045	8.1108	14.6622	**7.5930**	0.6176	0.4094	0.3951	0.3839	**0.4685**
E	0.99-1.01	11.4453	10.0999	10.9360	17.7112	**12.1829**	0.3562	0.2380	0.2277	0.2715	**0.2844**	8.5073	7.1511	8.5577	16.2097	**9.3297**	0.2993	0.1842	0.1856	0.2586	**0.2334**
	1.01-1.03	11.0831	10.3201	10.8899	17.0692	**11.9844**	0.1873	0.1429	0.1415	0.1884	**0.1658**	8.2968	7.3043	8.3919	15.3770	**9.1497**	0.1476	0.1086	0.1129	0.1750	**0.1328**
	1.03-1.05	8.6352	9.3682	9.7434	15.8536	**10.6141**	0.0968	0.0929	0.0926	0.1354	**0.1022**	6.2474	6.2981	7.5213	14.3149	**7.9080**	0.0718	0.0648	0.0735	0.1245	**0.0789**
	Total	**9.3469**	**8.7035**	**9.6841**	**15.7649**	10.5759	**0.5580**	**0.7551**	**0.8211**	**0.4742**	0.6796	**6.5263**	**5.9387**	**7.7080**	**14.2451**	7.9049	**0.3799**	**0.3729**	**0.3979**	**0.3326**	0.3745
	0.95-0.97	3.6338	5.0782	6.9003	14.6763	**7.9765**	2.3710	2.2305	1.4105	0.6281	**1.8627**	2.7638	4.1749	6.0106	12.9362	**5.9594**	1.6145	1.5558	0.9694	0.5977	**1.2339**
	0.97-0.99	6.8786	7.9581	10.1081	19.8327	**10.9099**	2.1886	0.8813	0.4680	0.4302	**1.3535**	5.7596	6.4466	8.3052	17.3236	**8.3712**	1.3770	0.5611	0.3713	0.4016	**0.7510**
GJR	0.99-1.01	8.7202	9.3208	11.2314	21.0330	**12.1601**	0.3728	0.2769	0.2337	0.3077	**0.3073**	7.0741	7.4212	9.0175	18.2877	**9.3741**	0.2972	0.2161	0.1943	0.2774	**0.2467**
	1.01-1.03	7.9545	9.0662	10.6925	18.3089	**11.1004**	0.1423	0.1381	0.1441	0.2004	**0.1522**	6.3839	7.3793	8.8586	16.0433	**8.7605**	0.1181	0.1160	0.1217	0.1804	**0.1281**
	1.03-1.05	6.5144	8.9744	9.7613	15.2999	**9.9081**	0.0763	0.0957	0.0975	0.1316	**0.0980**	5.3618	8.0341	8.6157	13.8086	**8.3334**	0.0641	0.0866	0.0865	0.1205	**0.0854**
	Total	**7.1541**	**8.1748**	**9.8030**	**17.9597**	10.5378	**1.4000**	**1.1184**	**0.7052**	**0.3933**	1.0485	**5.6660**	**6.6291**	**8.1082**	**15.6360**	8.1732	**0.6568**	**0.5321**	**0.3671**	**0.3258**	0.4952

Table 10.13: 2016 Out-of-Sample Pricing Errors

		$RMSE					%RMSE					MAE					MAPE				
	DTM	10-29	30-49	50-69	70-89	Total	10-29	30-49	50-69	70-89	Total	10-29	30-49	50-69	70-89	Total	10-29	30-49	50-69	70-89	Total
	0.95-0.97	4.6908	8.9899	13.3534	17.4431	**12.2308**	6.3803	7.0121	4.2472	3.0280	**5.4666**	4.0539	8.5969	13.1722	17.3951	**11.2176**	5.3828	5.9574	3.6852	2.7353	**4.4926**
	0.97-0.99	8.2389	12.3638	14.6624	17.1928	**12.9458**	4.7320	2.3958	1.1437	0.9177	**2.9447**	7.5800	11.6194	14.2923	17.0410	**12.0198**	3.7249	1.8386	1.0102	0.8458	**2.0158**
BSM	0.99-1.01	8.5320	10.5344	11.0823	12.8487	**10.4833**	0.8895	0.4620	0.3319	0.3104	**0.6001**	8.0071	9.4432	10.2977	12.5235	**9.6664**	0.6212	0.3742	0.2899	0.2885	**0.4218**
	1.01-1.03	5.1820	6.5400	6.7348	7.5130	**6.3792**	0.1205	0.1267	0.1116	0.1084	**0.1184**	4.3336	5.6488	5.9523	7.0106	**5.5364**	0.0960	0.1035	0.0938	0.0979	**0.0979**
	1.03-1.05	3.0252	3.8788	3.3594	3.4992	**3.4729**	0.0374	0.0437	0.0355	0.0355	**0.0386**	2.2870	3.0468	2.7180	2.9029	**2.7410**	0.0276	0.0338	0.0281	0.0286	**0.0298**
	Total	**6.7279**	**9.0938**	**10.9074**	**13.1956**	**9.8921**	**3.0580**	**3.3393**	**2.0464**	**1.4698**	**2.7028**	**5.6983**	**7.8045**	**9.5385**	**11.7180**	**8.3639**	**1.5815**	**1.6936**	**1.0882**	**0.8465**	**1.3598**
	0.95-0.97	1.5688	4.0335	7.7774	12.0412	**7.4174**	1.7827	2.8310	2.2375	1.9999	**2.3553**	1.2017	3.3115	7.4238	11.9330	**6.0972**	1.2627	2.2354	1.9910	1.8351	**1.9369**
	0.97-0.99	5.1869	8.5363	10.9917	14.0427	**9.6103**	2.3564	1.5255	0.8322	0.7352	**1.6003**	4.5377	7.6391	10.5139	13.8993	**8.5049**	1.8346	1.1705	0.7360	0.6827	**1.1742**
HN	0.99-1.01	8.0129	9.7519	10.5310	12.7055	**9.9653**	0.7575	0.4110	0.3064	0.3004	**0.5212**	7.5591	8.6979	9.7977	12.4663	**9.1867**	0.5517	0.3367	0.2711	0.2838	**0.3834**
	1.01-1.03	6.9811	8.4911	8.9713	10.0873	**8.4683**	0.1593	0.1597	0.1434	0.1416	**0.1529**	6.3859	7.6318	8.2240	9.7932	**7.7398**	0.1389	0.1378	0.1273	0.1344	**0.1350**
	1.03-1.05	5.1007	6.8235	7.0189	7.8074	**6.6737**	0.0632	0.0777	0.0732	0.0762	**0.0728**	4.3512	6.2009	6.4284	7.5175	**6.0183**	0.0526	0.0692	0.0661	0.0722	**0.0647**
	Total	**6.2746**	**7.8047**	**9.2193**	**11.6537**	**8.6222**	**1.3326**	**1.4662**	**1.1174**	**0.9964**	**1.2709**	**5.4187**	**6.7130**	**8.5176**	**11.2427**	**7.6172**	**0.7417**	**0.8084**	**0.6748**	**0.6334**	**0.7255**
	0.95-0.97	2.0666	5.8564	9.2163	13.3623	**8.7172**	2.6741	4.4874	3.0286	2.3676	**3.4390**	1.7429	5.2049	8.9484	13.1901	**7.5499**	2.0992	3.6435	2.5563	2.0973	**2.7609**
	0.97-0.99	5.0126	8.9478	10.8646	14.7003	**9.8207**	2.6645	1.6835	0.8637	0.8003	**1.7861**	4.5549	7.8618	10.3363	14.2883	**8.5967**	2.0592	1.2585	0.7429	0.7174	**1.2729**
N	0.99-1.01	5.8574	8.4025	9.3119	13.3753	**8.9355**	0.6075	0.3585	0.2757	0.3215	**0.4372**	5.1996	7.1204	8.4771	12.8446	**7.7266**	0.4026	0.2788	0.2377	0.2950	**0.3124**
	1.01-1.03	4.1468	6.2722	8.0334	11.4996	**7.3214**	0.0910	0.1171	0.1282	0.1610	**0.1211**	3.2029	5.1690	7.1968	11.0783	**6.0081**	0.0691	0.0930	0.1114	0.1518	**0.0996**
	1.03-1.05	3.0071	4.9714	7.1606	11.0911	**6.7071**	0.0364	0.0562	0.0739	0.1065	**0.0688**	2.2748	4.0009	6.6047	10.8810	**5.4516**	0.0272	0.0444	0.0674	0.1036	**0.0564**
	Total	**4.5819**	**7.1185**	**9.0666**	**12.9508**	**8.4358**	**1.5675**	**2.1636**	**1.4674**	**1.1671**	**1.6886**	**3.7560**	**5.9312**	**8.3830**	**12.5340**	**7.1252**	**0.8135**	**1.0849**	**0.7887**	**0.7081**	**0.8697**
	0.95-0.97	2.142189	6.073569	9.489517	13.59811	**8.934816**	2.7869	4.6586	3.1084	2.4047	**3.5540**	1.8037	5.4365	9.2248	13.4289	**7.7727**	2.1914	3.7984	2.6289	2.1326	**2.8562**
	0.97-0.99	5.220041	9.277271	11.28563	15.05009	**10.14327**	2.8009	1.7429	0.8931	0.8169	**1.8667**	4.7648	8.1992	10.7773	14.6525	**8.9271**	2.1716	1.3086	0.7716	0.7343	**1.3302**
E	0.99-1.01	6.108029	8.766131	9.812366	13.81806	**9.313003**	0.6358	0.3737	0.2896	0.3313	**0.4568**	5.4895	7.4497	9.0232	13.3108	**8.1157**	0.4252	0.2921	0.2521	0.3053	**0.3285**
	1.01-1.03	4.250946	6.563637	8.520195	12.05421	**7.682033**	0.0943	0.1229	0.1356	0.1684	**0.1270**	3.3307	5.4633	7.7445	11.6652	**6.3600**	0.0722	0.0984	0.1196	0.1596	**0.1054**
	1.03-1.05	3.025134	5.210511	7.622313	11.80014	**7.099603**	0.0367	0.0589	0.0786	0.1132	**0.0727**	2.2773	4.2374	7.1117	11.6113	**5.7856**	0.0273	0.0471	0.0725	0.1105	**0.0598**
	Total	**4.748053**	**7.411509**	**9.485157**	**13.38538**	**8.766122**	**1.6437**	**2.2454**	**1.5073**	**1.1864**	**1.7502**	**3.9182**	**6.2184**	**8.8419**	**13.0013**	**7.4543**	**0.8556**	**1.1311**	**0.8157**	**0.7240**	**0.9045**
	0.95-0.97	3.8402	6.4104	7.6262	7.5207	**6.7751**	5.2749	5.0372	2.5557	1.4043	**3.8663**	3.4331	5.9701	7.2988	7.2354	**6.2880**	4.5817	4.2250	2.1404	1.1943	**2.9915**
	0.97-0.99	7.6114	10.2469	9.5933	7.8199	**8.9803**	4.6800	1.9358	0.7583	0.4406	**2.7670**	7.1574	9.3899	8.8523	7.1338	**8.2241**	3.6707	1.4803	0.6420	0.3696	**1.7238**
GJR	0.99-1.01	9.5226	11.2037	9.9750	8.5121	**9.9729**	0.9555	0.4656	0.2901	0.2008	**0.6212**	9.1105	10.1164	9.0707	7.8374	**9.1750**	0.6870	0.3889	0.2520	0.1790	**0.4210**
	1.01-1.03	8.7842	11.0214	11.0463	10.5350	**10.3081**	0.1982	0.2017	0.1715	0.1441	**0.1852**	8.2606	10.0521	10.3266	10.0929	**9.5733**	0.1787	0.1791	0.1571	0.1366	**0.1668**
	1.03-1.05	8.1976	11.7687	13.1574	13.8470	**11.7733**	0.0990	0.1306	0.1336	0.1311	**0.1242**	7.4914	11.1776	12.7921	13.5924	**11.0781**	0.0894	0.1230	0.1292	0.1286	**0.1170**
	Total	**8.2824**	**10.2867**	**10.3370**	**9.7959**	**9.7116**	**2.8400**	**2.4401**	**1.2468**	**0.6913**	**2.1065**	**7.6377**	**9.3149**	**9.5670**	**9.0465**	**8.8802**	**1.5406**	**1.3032**	**0.7007**	**0.4194**	**1.0582**