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VALUATION OF DANISH CALLABLE MORTGAGE BONDS

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1 Abstract

This thesis develops a pricing model for Danish callable mortgage bonds. Pricing a callable mortgage bond is a complicated process since it contains a Bermudan prepayment option. The borrowers do not exercise the option in a strictly rational manner, which affects the price of the bonds as the prepayments alter the cash flows. Initially, a term structure of zero-rates is created from Danish interest rate swaps. This curve is used as input in a Hull-White trinomial tree to describe the future evolution of the rates. The tree's volatility parameters are then calibrated to the market volatility of several Danish swaptions. The required gain model is used as a prepayment model to estimate the borrowers' future prepayments and is based on empirical data. The model's independent variables are the refinance gain, time to maturity and the pool factor while dividing the debtors into five groups depending on the remaining debt. Finally, the interest rate model and the prepayment model are combined and ten callable mortgage bonds are then priced. The prices deviate somewhat from the market prices. This is a consequence of the prepayment model having limited estimation data and generally tends to overshoot the estimated CPR. The model's prices are expected to become more accurate as the number of bonds in the estimation data is increased along with an extension of the sample period.

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2 Introduction

Danish callable mortgage bonds remain an interesting asset class and take part in creating the largest mortgage market in the world relative to gross domestic product. The Danish credit system is built on a strong mortgage act that highly emphasises the protection of the mortgage bond investors. The mortgage act is the reason why the Danish credit system has survived through several rounds of economic and political turmoil for over two centuries. The level of repossessed dwellings and loans in arrears have been very low, even during periods of declining house prices. In fact, every single issued bond has been fully repaid to the investors. The mortgage bonds are rated ‘AAA’ by Standard & Poor’s and occasionally have better liquidity than the Danish government bonds. Thus, this asset class remains highly attractive for both domestic and foreign investors.

The Copenhagen Fire of 1795 burned approximately 25% of the houses down to the ground. As funding was needed for rebuilding the city, lenders formed a mortgage association to provide loans secured by mortgage on real property. The first Mortgage Bond Act was passed in 1850 and imposed strict limits on the activities and risk taking of the mortgage banks. Since then, the legal framework has undergone several amendments, but the investor protection principles have remained unchallenged. In the late 1950s, the establishment of independent mortgage banks was permitted. In 1989, an EU directive deregulated the market and enabled commercial and saving banks to establish mortgage banks. Consequently, fierce competition arose as new lenders entered the market, which led to consolidation within the sector. In 2000, the competition was intensified further by two large mergers. There are at present seven mortgage banks that issue Danish mortgage bonds.

The purpose of this thesis is to develop a pricing model for Danish callable mortgage bonds with a fixed rate. The callable bond is a complex bond to price as it contains a prepayment option that allows the borrowers to prepay the loan at par[1] and remortgage to a new loan at the current market rate. Borrowers tend to prepay their high coupon loans when market rates are low.

[1] Bonds are trading at par when their price is 100.

However, the borrowers do not use this optionality in a strictly rational manner. As prepayments alter the bond's cash-flow they have an impact on its price. Hence, there is a need to model the semi-rational behaviour exhibited by heterogeneous borrowers to accurately price a callable bond.

The intention is not to recount knowledge or highlight trivia but to build a pricing model based on empirical data that can evaluate Danish callable mortgage bonds. On the above basis, the following research question has been compiled:

How is a pricing model for Danish callable mortgage bonds developed?

Five additional sub-questions have been formulated to specify the research question and to structure and organise the paper:

- *How is the term structure of zero-rates created in the Danish market?*
- *Which interest rate model is the most appropriate for the current economic environment?*
- *How is a prepayment model defined so it captures the heterogeneity within borrowers?*
- *How is the interest rate model and prepayment model combined to price the callable mortgage bonds?*
- *What is the accuracy of the pricing model?*

Deliberate answers to the questions above will step-wise create and examine the accuracy of the pricing model. Due to the complexity and comprehensiveness, the ambition for this thesis is not to develop a perfect pricing model, but to make a judicious and contemplated suggestion.

2.1 Scope

The overall scope of the thesis is defined by the research questions. To create a good flow through the thesis, the scope is further defined on an ongoing basis. Hence, the boundaries are outlined as they are set throughout the sections. However, there are several elements that are applicable for the entire thesis.

The thesis' sole focus is on fixed-rate callable bonds with a maturity of 30 years. Furthermore, the scope only includes remortgaging of an existing loan to a new loan with a lower nominal rate of interest. Borrowers are also able to remortgage to a higher coupon if there is a substantial increase in long-term interest rates but this type of remortgaging is excluded from this thesis.

It is reasonable to believe that the decision of prepaying a loan or not is based on an after-tax basis. It is also of significance whether the borrower is a private household or a business, as the tax deduction rate for the interest paid by the borrower varies due to asymmetric taxation schemes. The tax deduction rate for a private household is reduced by the Spring Package 2.0 from 33% to 25% over a transition period from 2012 to 2019 while it is 33% for a business. However, the taxation and the distinction in the borrower type are both omitted from the scope.

Borrowers must give at least two months' notice before exercising the prepayment option. This feature is ignored throughout the thesis such that the payment decision is taken on the payment date. Finally, the amount of settlement days is set equal to zero.

2.2 Structure

The structure might be considered slightly untraditional compared to a regular thesis as it does not contain a specific section for literature review, methodology, findings or discussion. These are assessed on an ongoing basis, as this provides a better flow throughout the thesis. The thesis consists of six chapters. Chapter 4, 5, 6 and 7 include Excel sheets. These are arranged as described in *Appendix 1*.

Chapter 3 contains a brief introduction to the Danish mortgage model. It provides valuable insights into the unique aspect of the Danish mortgage model, its characteristics and the bond types including the prepayment optionality of the callable bonds.

In chapter 4, the term structure of zero-rates is created. The methods, instruments and models are presented along with the reasoning for selecting these. In the last section, these methods are applied on empirical data and a term structure of zero-rates is created.

Chapter 5 describes the interest rate model utilised in the thesis. Initially, a comparison of interest rate models is performed and the best suitable interest rate model for the current economic environment is selected and described. In the last section, a model is created based on the term structure of zero-rates and calibrated using empirical data.

In chapter 6, a prepayment model is defined. Initially, the chapter presents two different models. Hereafter, the independent variables, the estimation model and the data are then described. Finally, the model is estimated and the results are analysed and discussed.

In chapter 7, the interest rate model and prepayment model are combined and the callable bonds are priced. The theoretical prices from the model are compared to the market prices and the model's accuracy is analysed and discussed.

Finally, chapter 8 summarises the thesis and presents ideas for further work.
The process diagram for the pricing model is presented in *Figure 2.1*. It displays the structure of the pricing model from a data flow perspective.

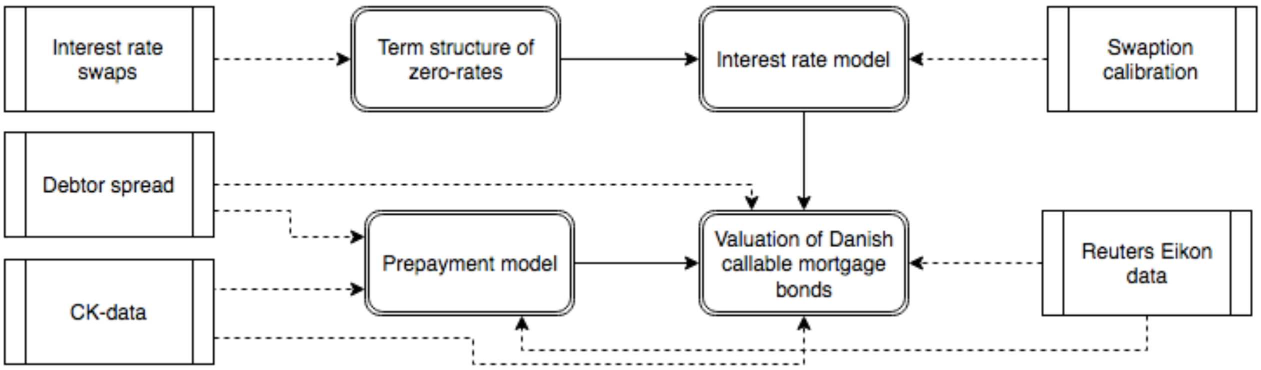

Figure 2.1: Process diagram for the pricing model

3 The Danish mortgage model

Denmark's unique mortgage model ensures a stable market for real estate financing. The more than 200-year-old model, guarantees transparent market-determined prices for the borrowers and a minimal investment risk for the investors. The Danish mortgage model, contrary to other countries' models, has proven its robustness throughout several crises as it kept providing mortgage loans continuously. In its more than 200 years of existence, none of the bondholders have ever suffered losses due to a defaulting mortgage bank (Realkreditrådet, 2016).

Danish mortgage banks provide loans to borrowers, which are secured by mortgages on a real estate property. The mortgage banks act as mediators and fund the loans by selling bonds to investors and are responsible for the borrowers' inability to repay the investors.

Danish mortgage banks apply the match-funding principle, by matching the loan with the bonds that are funding the loan. For instance, if a borrower has been granted a 5% interest rate loan with a 30-year term, the mortgage bank will issue bonds with 5% interest rate and a maturity of 30 years. Since July 2007 mortgage banks have not been legally obliged to follow this principle but have kept maintaining it. This will be explained further in *section 3.1*
(Realkreditrådet, 2016).

The investors charge interest on the bonds, which exactly matches the loan rate payable by the homeowner. As presented in *Figure 3.1*, the borrowers' payments (interest + principal) are passed on directly to the investors. This is defined as the pass-through principal where nothing is added to or deducted from the payments between the investors and borrowers (Realkreditrådet, 2014). This, along with the lack of credit risk minimises the loss risk to the investors and ensures a low bond yield and hence low loan rates. The relationship between the investor and the borrower is anonymised as no investor has a mortgage on a specific real estate, and no borrower has a loan from any specific investor.

Figure 3.1: Pass-through system

Borrowers are not only entitled to pay principal and interest payments to the investor, but are also entitled to pay a margin to the mortgage bank. This margin covers losses, daily operating costs and provides profits. The margin is usually calculated as a percentage of the outstanding debt and is paid throughout the loan term (Realkreditrådet, 2016).

Property rights are defined through a general register of all properties in Denmark through the Danish title number and land registration system. The information from the registration system is publicly available and displays the ownership and encumbrances on each individual property. Furthermore, a compulsory sale procedure is included in case of defaulting borrowers. If a borrower defaults on a payment, the bank has the authority to overtake and sell the house on the real estate market or through a forced sale. This procedure, from default to a sale, may be completed as short as in six months. Both the registration system and the compulsory sale system protect the investor against losses (Danske Bank Markets, 2016).

3.1 Covered bonds

In July 2007, an amendment to the legal framework was enforced. The amendment not only made it possible for Danish universal banks to grant mortgage bonds but also enabled a breakaway from the traditional match funding principle[2]. It further introduced two new bond types in addition to the traditional mortgage bonds (RO); the covered mortgage bond (SDRO) and the covered bond (SDO). Only mortgage banks may issue SDROs and ROs, while both universal and mortgage banks can issue SDOs. The difference between SDROs and SDOs is in what kind of collateral is allowed. While SDROs only may include real estate as collateral, the SDOs can have

[2] This will be further explained in *section 3.1.1.*

real estate, ships, bonds and securities issued by credit institutions (Ministry of Business and Growth, 2015).

Since the legislation came into force mortgage banks have mainly issued SDROs and SDOs. Investors are generally willing to pay a higher price for these bond types compared to the ROs. This is explained by the SDROs and the SDOs have more lenient capital requirements compared to the ROs. It is beneficial for the borrowers as higher prices are equal to lower loan rates (Realkreditrådet, 2016).

Mortgage banks are by law permitted to limit the loan ratio compared to the value of the property. For SDROs and SDOs the loan-to-value (LTV) is set to be 80% for residential properties. Hence, a borrower can maximally loan 80% of the residential property and must finance the remaining 20% by a bank loan unless the borrower possesses the funds. However, a 75% LTV applies if the term is longer than 30 years or the interest-only period exceeds 10 years (Realkreditrådet, 2016). The mortgage banks must provide supplementary security for the bond investors if the value of the mortgaged property decreases and the LTV ratio of the loan exceeds the stipulated LTV limits (Danske Bank Markets, 2016). The mortgage banks are as minimum required to evaluate the value of a residential property every 3rd year and ensure that the LTV ratios are met (Realkreditrådet, 2016). This highlights the disadvantage of the SDROs and the SDOs for the mortgage banks, as they are responsible for the supplementary security, which thereby increases the risk of capital shortage. On the contrary, the LTV ratio requirement on ROs is only necessary to be met when the loans are funded.

3.1.1 Balance principle

The balance principle restricts the market risk exposure for Danish mortgage banks. It imposes numerous of tests and is enforced by the Danish Financial Supervisory Authority. The balance principle must be met at all times as its purpose is to minimise the financial risk exposure such as:

- Interest rate risk, the difference between the loan and the bond rate.
- Exchange rate risk, the difference between the exchange rate on the loan and the bond.

- Option risk, the difference between conditional loan and bond payments (e.g. prepayment options).
- Liquidity risk, the time delay between receiving payments from the borrowers and paying the bondholders.
- Refinancing risk, the difference between the terms of a new bond issue and existing loans.

The above measurements are created as an imbalance can (in a worst-case scenario) lead to bankruptcy. The mortgage banks must choose to adhere to one of the two balance principles; The general balance principle or the specific balance principle. (Nykredit, 2009)

The original balance principle required that the loan to the borrower was exactly matched with the bond issued to the investors. It meant, that the mortgage bank did not take any interest rate, option, exchange rate or liquidity risks. The specific principle is almost identical to the original principle apart from a slight mitigation of the regulations.

The general principle sets similar limitations on financial risk as the specific principle. However, the outline for liquidity and refinancing risk are mitigated as it does not require the mortgage banks to follow the match funding principle. It is thus possible to offer products with an imbalanced risk if this is done within the stimulated limitations. This protects the investor as it minimises the risk.

Even though some mortgage banks have chosen to follow the general principle, all have still chosen to follow the match-funding principle, even though it has not been statutory (Realkreditrådet, 2016). The mortgage banks thereby minimise their financial risks by matching the characteristics (interest rate, currency, repayment profile, options and maturity) of the loans with the bonds they fund them with. This allows the mortgage banks primarily to focus on managing their credit and capital risk. As previously mentioned, the mortgage banks are responsible for borrowers defaulting on their loans and hold the loans on their own balance sheets. It works as an incentive to only provide loans to homeowners with sufficient creditworthiness. This means, that the investors' investments are safer as their risks are being minimised.

3.2 Market

The mortgage banks issue identical bond series (same coupon, repayment model and maturity) on the Copenhagen Stock Exchange. The bonds are rated 'AAA' by Standard & Poor's and have a value of approximately 384 billion EUR as of ultimo June 2016 (Danske Bank Markets, 2016; Nykredit Markets, 2016). There are currently seven mortgage banks in Denmark who issue covered bonds. Their market share is presented in *Figure 3.2*.

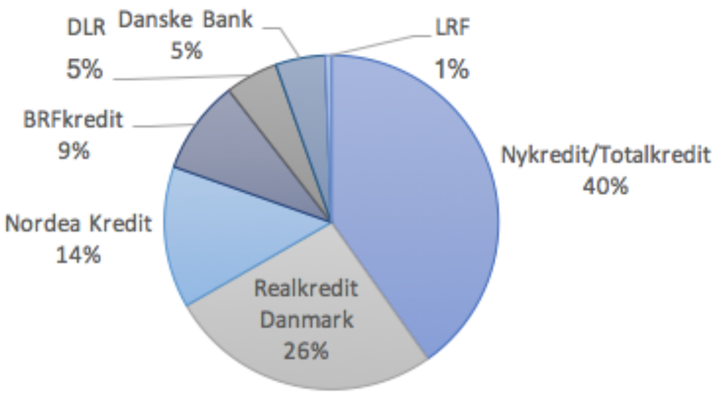

Figure 3.2: Markets share of Danish Mortgage banks. Source: (Danske Bank Markets, 2016)

The series are typically issued in periods of three years to ensure a large circulation size and thereby the liquidity of the bonds. The ambition is to achieve a series with a circulation size of at least 500 million EUR (FinansWatch, 2016). The bonds are issued on tap as required on a daily basis. As a series closes a new series of bonds are issued with new characteristics.

On the contrary, the bond series can close for issues in less than three years. The mortgage banks have agreed on not to offer callable loans[3] based on bonds that are priced above par. This is known as the par rule and is established to avoid arbitrage from borrowers simultaneously disbursing a loan at a price above par and prepaying the loan at par. Thus, the opening period of a bond series is shortened if the bond prices exceed par. The series will, however, be reopened for issues if the price should fall below par (Realkreditrådet, 2016).

[3] Callable bonds are further explained in *section 3.4*

3.3 Loan types

The selection of loan types has (mainly due to legislation) varied throughout the years. Today, the variety consists of three types of loans: fixed-rate loans, adjustable-rate mortgages (ARMs) and floating-rate loans with or without interest rate caps. Furthermore, depending on the loan type, mortgage loans can be raised either as bond loans or cash loans. Fixed-rate loans can be disbursed either as a bond loan or cash loan. ARMs are disbursed as cash loans while floating rate loans are disbursed as bond loans (Nykredit, 2016).

In bond loans, the principal equals the nominal value of the bond issued to fund the loan. The price at which the bonds are traded is usually below par. Hence, the amount disbursed is lower than the loan principal. This capital loss is not tax deductible as it must be repaid as part of the principal payments. The interest rate corresponds to the coupon rate of the bond.

In cash loans, the principal equals the market value of the bonds issued to fund the loan. If the bonds are traded below par a capital loss arises. The capital loss is tax deductible as it is repaid by a higher loan rate. The interest rate corresponds to the yield-to-maturity[4] (YTM) of the bonds adjusted for compound interest. Therefore, the principal is smaller on the cash loan than on the bond loan, while the interest rate is higher on the cash loan. Even though the total payments on each loan are of equal size, the gross payments are not. This is due to the margin, which is calculated based on the outstanding debt, which is different depending on the loan.

As a consequence of a different tax treatment, cash loans generally tend to be prepaid at a slower pace than the bond loans. This is the case, as capital gains on cash loans are typically liable to tax (Nykredit Markets, 2016). Ignoring this, the investor is indifferent towards the borrower's choice of loan type, as both loans are issued on same underlying bonds and thereby provide liquidity.

[4] Yield-to-maturity is the bonds internal rate of return assuming the bond is held until maturity and that all coupon payments are reinvested at the same rate as the bonds current yield.

3.3.1 Fixed-rate loans

This loan has a fixed interest rate until its maturity, which can be up to 30 years. This feature comes, *ceteris paribus,* at the cost of higher interest rate compared to loans with a floating rate. Fixed-rate loans are primarily issued as callable bonds as it sets a cap on the debt. This makes it possible to exploit decreasing rates by prepaying the loan at any time at par and converting it to a new loan with lower interest rate[5]. In the case of increasing interest rates, the price at which the bonds are traded will decrease and thereby lower the outstanding debt. The borrower is then able to prepay the loan and relish tax-free capital gains. This is an attractive feature for borrowers who want to sell their property, as real estate prices often fall when interest rates increase (Realkreditrådet, 2016).

3.3.2 Adjustable-rate mortgages

In this loan type, the rate is fixed for a shorter period. The loan may have a maturity up to 30 years while having an interest rate reset every 1-10 years. At maturity of each period, the borrower's payments are adjusted to a newly achieved interest rate. Hence, a 30-year mortgage loan funded by one-year covered bonds is subject to refinancing 29 times. This loan type is not callable but at maturity of each reset period, the borrower can prepay the loan by buying the bonds back at par.

3.3.3 Floating-rate loans

Floating-rate loans are based on money market interest rates such as CIBOR[6], EURIBOR[7] and CITA[8] with a maturity up to 30 years. The interest rate is commonly based on 3M or 6M CIBOR/EURIBOR rates and 6M CITA rates. Thus, the interest changes terms two to four times a year. Approximately 21% of floating-rate loans have an embedded cap which mostly is set at 5% (Danske Bank Markets, 2016). This serves to protect to the borrower, as the interest rate cannot become larger than its cap. The loans can be either callable or non-callable, but it is always possible for the borrower to buy back the bonds at par at the refinancing dates.

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[5] This will be elaborated on in *section 3.4.*
[6] Copenhagen Interbank Offered Rate. This will be further explained in *section 4.2.*
[7] The Euro Interbank Offered Rate.
[8] Copenhagen Interbank Tomorrow/Next Average.

3.4 Callable bonds

A callable bond gives the borrower the right to prepay the bond at par. Therefore, it can be considered as a portfolio consisting of a non-callable bond and a short call option[9] from an investor's perspective. The option is considered as a Bermudan[10] as the borrower must give two months' notice prior to exercising it (Danske Bank Markets, 2016). The borrower is hereafter obliged to prepay regardless of the development in the interest rate and bond price.

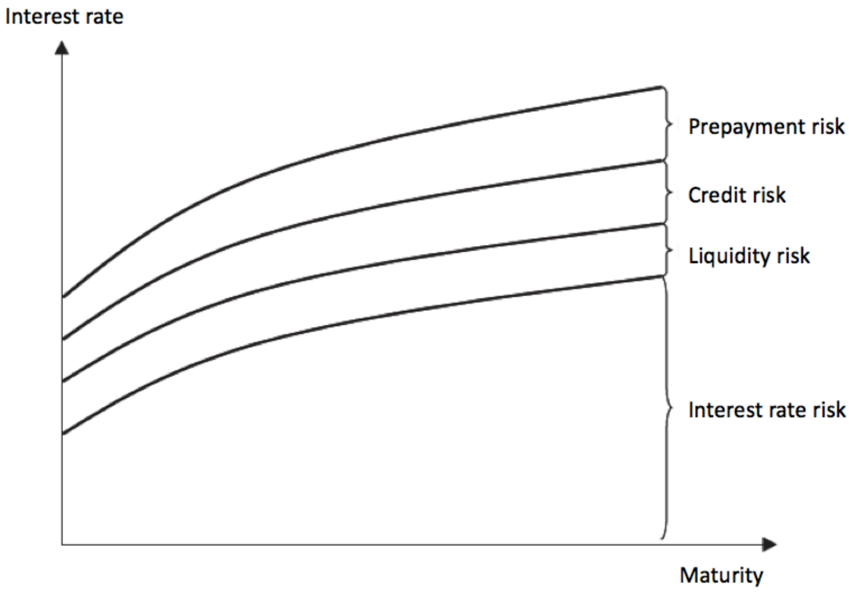


Figure 3.3: Risk compensation. Source: (The National Bank of Denmark, 2003)

In *Figure 3.3*, it is observable that both bond types should be compensated for interest rate risk, credit risk and liquidity risk. However, investors who buy callable-bonds also require a premium for the prepayment risk. The prepayment risk occurs when the interest rate falls and the borrowers prepay their high coupon loans. The investor is therefore forced to reinvest the funds in bonds with a lower coupon rate (assuming the investor desires to invest in the same asset class). Additionally, the investor may suffer a loss if the bond was purchased at a price above par and is being prepaid at par. The prepayment risk causes the prices of callable bonds to be lower compared to the prices of corresponding non-callable bonds.

[9] A call option is an agreement that gives an investor the right, but not the obligation, to buy an underlying asset at specified price within a specific time period.
[10] An option type which can only be exercised on predetermined dates.

From an absence of arbitrage perspective, the price of a callable bond P_C should be equal to the price of a corresponding non-callable bond P_{nc} corrected for the value of the call option C. From the investors view, this is shown in following formula (Christensen, 2014):

$$P_c = P_{nc} - C \qquad \text{eq. 3.1}$$

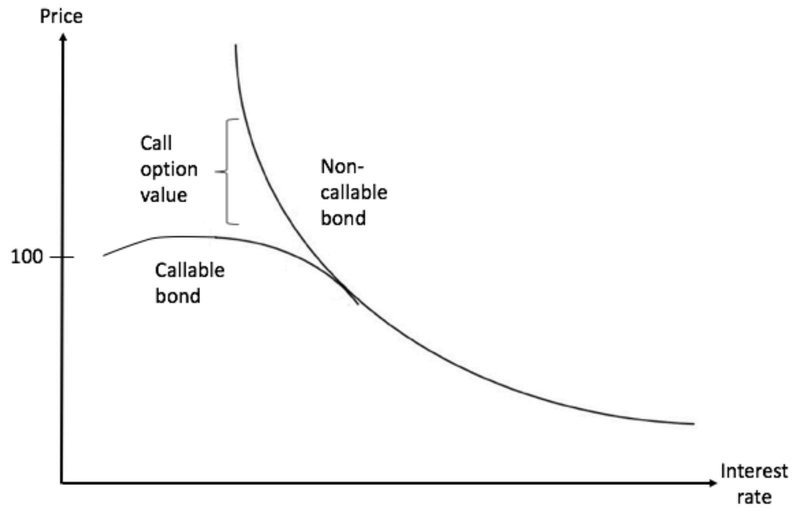


Figure 3.4: Callable bond value. Source: (Christensen, 2014)

In *Figure 3.4*, the relationship between a callable bond and non-callable bond is displayed relative to the interest rate and price. The prices of both bonds converge in a high-interest rate environment. This is explained by the call option being deep-out-of-the-money[11]. The lower the interest rate is, the more in-the-money the call option is and the higher will the option value be. In such setting, the spread between the bonds increases as the probability of prepayment (option exercise) increases. The callable bond can reach a price above par due to opaque market conditions due to heterogeneous and irrational borrowers. However, at very high prices a convergence towards par will commence as the prepayment probability becomes immense. In addition to the interest rate, the value of the call option is *ceteris paribus* positively correlated with volatility and negatively with time to maturity.

The traditional convex relationship between the level of interest rates and the price of the non-callable is evident in *Figure 3.4*. However, the callable bond seems to have an S-shaped curve and

[11] An option which is so far out-of-the-money, that it is unlikely to be exercised prior to expiration.

demonstrates negative convexity[12] at low-interest rate levels. This is explained in the above section – when rates fall the borrowers are likely to prepay their loans. When the callable bond becomes highly exposed to prepayments, the price will decrease as the interest rate further decreases.

3.5 Repayment and prepayment

It is possible to prepay all mortgage loans before maturity, but it can be very expensive if the loan is not callable. If a loan is not callable and prepayment is desired, the borrower may have to buy back the bonds at a price exceeding par. Hence, if the outstanding debt for DKK 1 million and the bonds are trading at 105 the borrower is required to pay DKK 1.05 million for the bonds. If the loan had been callable, the borrower could have bought them back at par.

If the borrower has a callable bond, then there is a choice between immediate prepayment and prepayment on the payment date. The former is the most common option and means that the remaining debt and interest payments are payable to the mortgage bank within three days. As investors are still entitled to receive their coupon payments, the borrowers must pay the coupon until the payment date. However, the borrower is compensated for making the funds available for the mortgage bank until the payment date. This compensation normally consists of a rate close to the current money-market rate (Danske Bank Markets, 2016). The process is visualised in *Figure 3.5*. A borrower who prepays on the payment date does not have to prepay the remaining principal and the coupon due until the prepayment date.

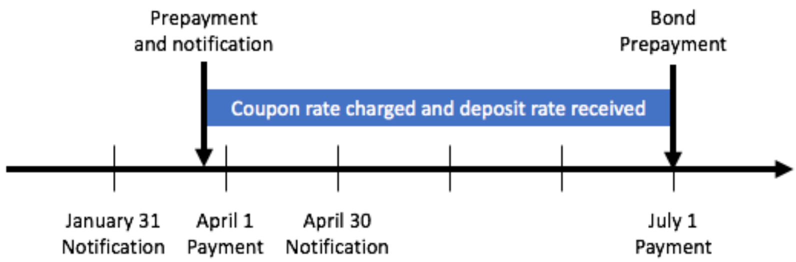


Figure 3.5: Notification and payments in connection with preliminary prepayment. Source: (Danske Bank Markets, 2016)

[12] Convexity expresses the change in duration when yield levels change. The duration is a measure of the interest rate sensitivity.

4 Term structure of zero-rates

The term structure of interest rates describes the relationship between interest rates and different terms or maturities. This structure is generally labelled as the yield curve. A yield curve based on zero-coupon bonds, a zero-coupon yield curve, is used for pricing various assets. The zero-coupon yield curve is a crucial element in practice, as it enables the investors at time t to identify the appropriate yield for discounting future cash flows and thereby pricing an asset.

It is of the utmost importance to the investor to have a good estimate of the term structure of interest rates. A bad estimate would potentially result in a wrong theoretical value of the callable bond, which could lead to a loss for the investor. The term structure of interest rates is fundamental for obtaining future interest rates, which will be used to discount future cash flows from the callable bond to determine its theoretical price today.

The zero-coupon yield is mainly derived in two ways. One way is the non-default Treasury zero-coupon yield curve which can be derived from Treasury bond markets prices. These bonds are normally stripped for coupons and are defined as zero-coupon bonds. A second way is to derive the interbank zero-coupon yield curve. This is derived from a combination (for different types of maturity) of money-market rates, futures contract rates and interest swap rates. The conditions are, that the input assets should be observable, liquid, and have similar credit properties for the asset to be priced (Ron, 2000). As interest rate swaps fulfil all these conditions, they will solely be used to derive the zero-coupon yield curve.

The zero-coupon yield curve is used as an input to the Hull-White model, which will be described in *section 5.1*. The following sections will elaborate on deriving a zero-coupon yield curve based on interest rate swaps.

4.1 Zero-coupon bonds

A *T*-maturity zero-coupon bond guarantees its holder the payment of one unit of currency at time *T* (at maturity) and has no intermediate payments. The value at time $t < T$ is denoted by $P(t,T)$. The value at $P(T,T) = 1\ \forall T$. $P(t,T)$ is thereby a function of two variables; the initiation time t, and the maturity time T. At time t, a zero-coupon bond is a contract that establishes the present value of one unit of currency to be paid at time T. In other words, the investor's yield is equal to the capital gain acquired as the zero-coupon bond is trading below par. Thereby, the discount factor $D(t,T)$ is equal to the yield-to-maturity[13] $Y(t,T)$ for zero-coupon bonds. The discount factor $D(t,T)$ and the price of the zero-coupon bond $P(t,T)$ are linked together and can be viewed as the expectation of the random variable $D(t,T)$ under a particular probability measure. (Martellini, et al., 2003). The cash-flow of a zero-coupon bond is illustrated in *Figure 4.1.*

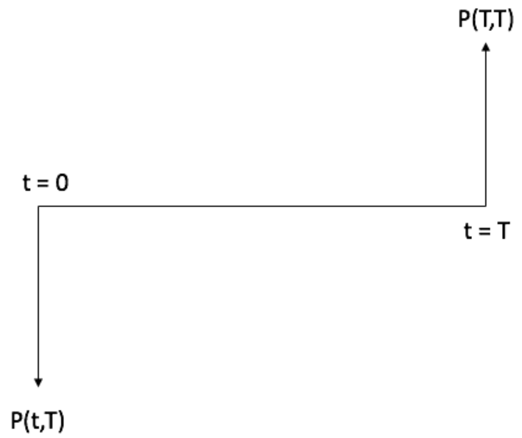


Figure 4.1: The cash-flow of a zero-coupon bond

At present time, $t = 0$, the price of the bond is observable in the market. Thereby, the $Y(0,T)$ of a zero-coupon bond is calculated in the following way:

$$P(0,T) = e^{-Y(0,T)(T-t)} \Leftrightarrow Y(0,T) = -\frac{\text{LN}\, P(0,T)}{T-t} \qquad \text{eq. 4.1}$$

where $P(0,T)$ is the observable market price at time 0 and the bond is paying one unit at date T (Brigo, et al., 2006).

In practice, it is possible to derive the zero-coupon yield curve if the market contains zero-coupon bonds across all maturities needed for the yield curve. The National Bank of Denmark issues zero-coupon bonds but only featuring a maturity of a maximum of 12 months (The National Bank of Denmark, 2017). As callable bonds have a maturity up to 30 years, Danish Treasury bills do not have sufficient maturities for a 30-year yield curve.

[13] YTM

The zero-coupon yield curve cannot directly be extracted as there is no abundance of zero-coupon bonds in the Danish market. As mentioned in *section 4*, the implied zero-coupon rates can be extracted from the interbank market. The yield curve will solely be built on interest swap rates as they are liquid on the Danish market, and contain both liquidity- and credit risk as mortgage bonds do. It is assumed that these risks are of equal magnitude for interest swap rates and the mortgage bonds.

4.2 Interest rate swaps

A general swap contract is a cash-flow transaction with no capital exchange. It enables two parties to swap cash flows that are originated by either loans or investments. The cash-flow from swaps takes place on predetermined dates and are based on a notional principal.

The zero-coupon yield curve will be derived on interest rate swaps, which are defined as plain vanilla swap contracts. Such contracts are formed between two parties and exchange a fixed leg, which payments depend on a fixed rate for a floating leg, which payments depend on a floating rate. If an investor enters a payer (receiver) swap it pays (receives) the fixed rate while receiving (paying) the floating rate as shown in *Figure 4.2*.

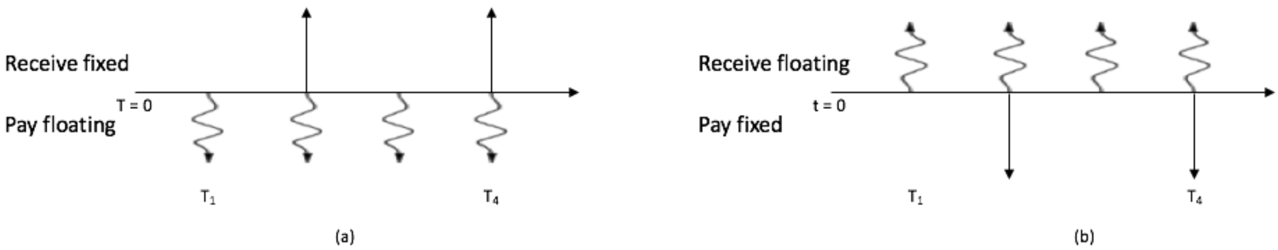


Figure 4.2: Cash-flow of 6M receiver swap (a) and payer swap (b)

The fixed rate is agreed for the entire life of the swap at inception while the floating rate is reset at the beginning of the individual interest periods, usually every 3- or 6-month.

The notional principal remains constant until maturity and is basically fictitious as it is only employed to calculate the actual amount of the different payments on the two legs of the swaps. The principal is a notional principal since only the interest rate payments are exchanged.

Fixed leg			$\boldsymbol{NP*F}$		$\boldsymbol{NP*F}$			$\boldsymbol{NP*F}$
Date	T_0	T_1	T_2	T_3	T_4	…	…	T_n
Floating leg		$-NP*\frac{V_0}{2}$	$-NP*\frac{V_1}{2}$	$-NP*\frac{V_2}{2}$	$-NP*\frac{V_3}{2}$			$-NP*\frac{V_{T_{n-1}}}{2}$

Table 4.1: A 6-month CIBOR swap.

A cash-flow is presented in *Table 4.1* for a 6-month CIBOR receiver swap. It is entered at T_0, matures at T_n and has a notional principal NP. Every 6 months on date T_i ($T_i - T_{i-1} = 6\ months, \forall i\ to\ maturity$) the investor receives the fixed rate F every year on date T_{2i} (until maturity) multiplied by the principal while paying the 6-month CIBOR rate V_{i-1} observed 6 months earlier multiplied by the principal. In a payer swap the cash-flow would be reverse so the investor would be receiving the floating rate and paying the fixed rate (Martellini, et al., 2003).

For Danish interest rate swaps, the CIBOR rate is used in the floating leg (Jensen, 2009). The CIBOR is the Danish reference rate. This rate is the average interbank interest rate at which banks are willing to lend unsecured funds to one another denominated in Danish Kroner for maturities ranging from over-night to 12 months. The floating rate, as shown in above example, is reset[14] at the beginning of the individual interest periods.

In order to ensure that a swap contract is a fair deal, both legs should have the same value at inception. While having an initial value of zero, future prices vary depending on the evolution of the term structure. At date t, the value of a receiver swap is the difference between the value of the floating leg and that of the fixed leg. The value of the fixed leg is easily obtained as it is the present value of future payments discounted to the corresponding fixed spot rate. While only the next payment is known for the floating leg, later payments are unknown at date t as they depend upon the future value of the CIBOR rate. Hence, the fixed rate equals the rate, which makes the value of the swap zero at inception.

[14] Reset dates are where the floating rate is observed.

The swap contract can be priced using the bond method. An interest rate swap is valued by considering the fixed leg[15] and floating leg[16] separately. The two legs do not typically have the same payment frequency. The interest rate swap has its first reset date at time S_0. Assuming now, that the fixed leg makes N fixed payments at time $S_1, S_2, \dots, S_N$, where $S_i <_{S_{i+1}} \forall i$. The floating leg makes n floating payments at times $T_1, T_2, \dots, T_n$. This is based upon the floating rates being reset at time $S_0 = T_0, T_1, T_2, \dots, T_n$, where $t_i < t_{i+1} \ \forall i$. The last payments of both legs are at the date $S_N = T_n$. In the bond method, the interest rate swap is considered as an exchange of a fixed-coupon bond[17] and a floating-rate bond[18] both having a face value equal to the notional principal of the swap. Hence, the notional principle of NP is added to both the fixed and floating leg at time $S_n = T_n$ (Nawalkha, et al., 2007). The coupon-rate on the floating-rate bond is considered as the floating rate while the coupon rate on the fixed-coupon bond is considered as the fixed rate. The price of the floating-rate bond at time $t \leq T_0$ is given as

$$\begin{aligned} P_{FRB}(t) &= P_{flt}(t) + NP * P(t, T_n) \\ P_{FRB}(t) &= NP\big(P(t, T_0) - P(t, T_n)\big) + NP * P(t, T_n) \\ P_{FRB}(t) &= NP * P(t, T_0) \end{aligned} \qquad \text{eq. 4.2}$$

The price of the floating-rate bond always converges to its face value at the reset date T_0.
The price of the fixed-coupon bond at time $t \leq S_0 = T_0$ is given by

$$\begin{aligned} P_{FCB}(t) &= P_{fix}(t) + NP * P(t, S_n) \\ P_{FCB}(t) &= NP * \left(F * \sum_{j=0}^{N-1} \tau_j P\big(t, S_{j+1}\big) + P(t, S_N) \right) \end{aligned} \qquad \text{eq. 4.3}$$

where τ_i is the accrual factor[19] corresponding to the period $S_{j+1} - S_j$ and F is the fixed rate. Recall at $S_N = T_n$ the swap contact between the fixed leg and the floating leg is equivalent to the swap between the fixed-coupon bond and the floating-rate bond, as seen from eq. *4.2* and eq. *4.3*. More formally, the value of a receiver swap at time t ($t \leq S_0 = T_0$) is given as follows

[15] Denoted as fix.
[16] Denoted as flt.
[17] Denoted as FCB.
[18] Denoted as FRB.
[19] Computed using 30/360 day-count conversion.

$$P_{Receiver\ swap}(t) = P_{fix}(t) - P_{flt}(t)$$

$$P_{Receiver\ swap}(t) = NP * F \sum_{j=0}^{N-1} \tau_j\, P\left(t, S_{j+1}\right) - NP\left(P(t, T_0) - P(t, T_n)\right)$$

$$P_{Receiver\ swap}(t) = NP\left(F * \sum_{j=0}^{N-1} \tau_j\, P\left(t, S_{j+1}\right) + P(t, S_N)\right) - NP * P(t, T_0) \qquad \text{eq. 4.4}$$

$$P_{Receiver\ swap}(t) = P_{FCB}(t) - P_{FRB}(t)$$

No knowledge of the future term structure is necessary to valuate a swap contract as all relevant interest rates are known at time t. Recall, the fixed rate makes the value of the swap equal zero at time t which is also referred to as the swap rate. Setting eq. 4.4 equal to zero and rearranging leads the equation for the swap par rate (Brigo, et al., 2006).

$$Swap\ rate(t) = \frac{P(t, T_0) - P(t, S_N)}{\sum_{j=0}^{N-1} P\left(t, S_{j+1}\right)\tau_i} \qquad \text{eq. 4.5}$$

4.2.1 Bootstrapping

The zero-coupon rates can be derived from standard coupon-bearing bonds (Nordea, 2004). The bootstrap method is used to derive the zero-coupon interest rates based on the swap par rates (Ron, 2000).

The bootstrap method constructs synthetic zero-coupon bonds by splitting the coupon-bearing bonds into zero-coupon bonds. Obtaining n coupon-bearing bonds with $1, 2, \dots, n$ periods and respectively having one payment date each period with identical payment dates will allow the construction of zero-coupon bonds for each of these maturities. Each payment from the coupon-bearing bond is transformed into zero-coupon bonds by the law of no arbitrage. Hence, if the same cash-flow could be obtained at a different price there would be an arbitrage opportunity in the market (Munk, 2011).

The prices of the coupon-bearing bonds in the fixed leg of the interest rate swap are known to be priced at par. Additionally, the rate of the coupon-bearing bond is also known as it is the swap rate/the fixed rate in the swap.

The most convenient way of deriving the zero-coupon prices is by using matrix algebra. Denoting the coupon-bearing bond prices as an n-dimensional vector at time t

$$P_t = P_t^1, \dots, P_t^j, \dots, P_t^n \qquad \text{eq. 4.6}$$

The cash flows (coupons and principal) are denoted as a $n * n$ matrix of the corresponding n coupon-bearing bonds

$$CF = \left(CF_{t_i}^{(j)}\right)_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \qquad \text{eq. 4.7}$$

It must be assumed for simplicity that the different bonds have cash flows on the same dates t_j. The cash flows prior to maturity are the yearly coupon payments. At maturity for each bond, the cash-flow is the coupon payment plus the face value of the coupon-bearing bond.

I denote the n-dimensional vector of zero-coupon bond prices at time t, which are to be derived. Note that the relationship between the price and discount factor of a zero-coupon bond was established in *section 4.1*.

$$D_t = D(t, t_1), \dots, D(t, t_n) \qquad \text{eq. 4.8}$$

Absence of arbitrage imposes that the following equation must hold

$$P_t^j = \sum_{i=1}^{n} CF_{t_i}^{(j)} D(t, t_i) \qquad \text{eq. 4.9}$$

Rewriting above equation for n coupon-bearing bonds results in the following matrix equation

$$P_t = CF * D_t$$

$$\begin{pmatrix} P_t^1 \\ P_t^2 \\ \vdots \\ p_t^n \end{pmatrix} = \begin{pmatrix} CF_{t_1}^1 & 0 & \cdots & 0 \\ CF_{t_1}^2 & CF_{t_2}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CF_{t_1}^{(j)} & CF_{t_2}^{(j)} & \cdots & CF_{t_i}^{(j)} \end{pmatrix} \begin{pmatrix} D(t, t_1) \\ D(t, t_2) \\ \vdots \\ D(t, t_n) \end{pmatrix} \qquad \text{eq. 4.10}$$

To determine the value of D_t the linear system must be solved. This is done by multiplying the inversed CF matrix with the P_t vector.

$$D_t = CF^{-1} * P_t$$

$$\begin{pmatrix} D(t,t_1) \\ D(t,t_2) \\ \vdots \\ D(t,t_n) \end{pmatrix} = \begin{pmatrix} CF_{t_1}^1 & 0 & \cdots & 0 \\ CF_{t_1}^2 & CF_{t_2}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CF_{t_1}^{(j)} & CF_{t_2}^{(j)} & \cdots & CF_{t_i}^{(j)} \end{pmatrix}^{-1} \begin{pmatrix} P_t^1 \\ P_t^2 \\ \vdots \\ p_t^n \end{pmatrix} \qquad \text{eq. 4.11}$$

Now that the implied zero-coupon bond prices are derived, the zero coupon rate can easily be extracted. The rate at date $t = 0$ is found by using *eq. 4.1*

$$Y(0,T) = -\frac{\text{LN}\, P(0,T)}{T-0}$$

The Danish swap market features numerous swaps with gradually increasing maturities. However, interpolation is necessary due to missing maturities, as linear dependent bonds are required before bootstrapping can be applied. After bootstrapping the zero-coupon prices, the quarterly rates must be interpolated to obtain rates matching the cash flows of the callable bonds. Splitting the interpolation into two has been necessary, as the cash-flows must have the same coupon dates when bootstrapping the zero-coupon prices.

The bootstrapping approach using matrix algebra will be applied on empirical and interpolated interest rate swap rates in *section 4.3*. The next section will elaborate on interpolation and method selection.

4.2.2 Interpolation

Interpolation is a method of constructing new data points by using a set of discrete known data points. To ensure yield curve validity, the derived yield curve must closely track the observed market data points while being consistent and smooth (Ron, 2000). Numerous curve fitting spline methods have been introduced as they create a smooth curve. In each interval, a spline interpolation uses low-degree polynomials and thereby fits the polynomial pieces smoothly together. Interpolation using quadratic splines and cubic splines were introduced by McCulloch in respectively 1971 (McCulloch, 1971) and 1975 (McCulloch, 1975). Vasicek and Fond used exponential splines in 1982 (Vasicek, et al., 1982) and Shea used B-splines in 1984 (Shea, 1984). In common, all the above methods have been criticised for being 'black box' models and for having undesirable economic properties (Annaert, et al., 2013). Nelson and Siegel introduced in 1987 (Nelson, et al., 1987) a method using parametric curves, which is adaptable enough to describe the various term structure shapes. Hence, the curves are based on parametric equations, which express a set of quantities as explicit functions of several parameters (Stover, et al., 2017). The NS model is parsimonious and consistent with a factor interpretation of the term structure. More notably, it provides a good fit of yields and is a widely-used model amongst financial market practitioners and central banks (Christensen, et al., 2009). Based on the above the NS model is the selected approach for both interpolations.

4.2.2.1 Nelson and Siegel model

The Nelson and Siegel model uses unsmoothed yields to fit a smooth yield curve. Applying this model, it is possible to obtain yields for the remaining maturities while smoothening the yield curve. Nelson and Siegel estimate the term structure of interest rates with the following function (Nelson, et al., 1987).

$$Y_c(0,\theta) = \beta_0 + \beta_1\left(\frac{1-e^{-\frac{\theta}{\lambda}}}{\frac{\theta}{\lambda}}\right) + \beta_2\left(\frac{1-e^{-\frac{\theta}{\lambda}}}{\frac{\theta}{\lambda}} - e^{-\frac{\theta}{\lambda}}\right) \qquad \text{eq. 4.12}$$

where

- $Y_c(0,\theta)$ is the continuously compounded zero-coupon rate at time zero with maturity θ.[20]
- β_0 is the limit of $Y_c(0,\theta)$ as $\theta \to \infty$. In practice, β_0 is regarded as the long-term interest rate.
- β_1 is the limit of $Y_c(0,\theta) - \beta_0$ as $\theta \to \infty$. In practice, β_1 is regarded as the long-to-short-term spread.
- β_2 is the curvature parameter.
- λ is the scale parameter. It measures the rate at which the short-term and medium-term components decay to zero (Martellini, et al., 2003).

[20] Earlier I used $Y(0,T)$ as notation. This is solely changed to match the NS equation.

The NS model applies basic financial economic theory constraints as the corresponding discount curve satisfies $P(T,T) = 1$ and $lim_{T\to\infty}\ P(0,T) = 0$. Thus, the price at maturity is equal to 1 and when time to maturity goes towards infinity then the price of the zero-coupon bond is 0. The NS model further utilises a parsimonious approximation. Parsimony is a desirable feature as it ensures smoothness and has shown to provide an empirically tractable and trustworthy estimation (Martellini, et al., 2003).

The NS model further provides a flexible approximation. Flexibility is essential as a yield curve can take a variety of different shapes across the maturities. Depending on the values of the four parameters $(\beta_0, \beta_1, \beta_2, \lambda)$ the model has the ability to represent the shapes generally associated with yield curves; monotonic, humped and S-shaped. However, it cannot have more than one internal optimum. This constraint is largely nonbinding, as a yield curve commonly only has one internal optimum (Nelson, et al., 1987).

The NS model can either be applied directly to the yield curve or to the coupon-bearing bonds. The NS model will initially be applied to the observable swap rates and then to the zero-coupon rates provided from the bootstrapping procedure. I follow the procedure used by Nelson and Siegel and choose the λ and the corresponding regression estimates of $\beta_0, \beta_1, \beta_2$ that minimise the sum of squared errors (SSE) to maximise the fit of the Nelson Siegel yields (Munk, 2011). The goodness-of-fit measure is only calculated on non-interpolated maturities since it would be impossible to calculate errors on omitted values.

$$SSE = \sum_{i=1}^{n} \left(Y_i(t,T) - Y_{ns,i}(t,T)\right)^2 \qquad \text{eq. 4.13}$$

The minimisation is done by using the GRG non-linear method in the Excel solver.

4.3 Application and results

The interest rate swap data is provided by Nordea and Jyske Bank and is obtained through Thomson Reuters Eikon. The data contains 15 DKK interest rate swaps with maturities ranging from 1 year to 30 years as displayed in *Table 4.2*. The quotes of the interest rate swaps are of end-of-day March 17, 2017.

The interest rate swaps are quoted with both a bid and ask rate. I use the bid rate for further implementation as it infers receiving the quoted fixed rate (receiver swap) (Nordea, 2004). A receiver swap is chosen as the term structure of interest rates will be used to discount the quarterly cash flows from the callable mortgage bonds. The fixed rate received on the interest rate swaps is considered as a good reference rate. The fixed rate is received once a year while the floating rate has two yearly CIBOR fixings.

Interest rate swap (IRS)	**Bid (fixed swap rate)**
DKKIRS 1Y	-0.0731%
DKKIRS 2Y	0.0335%
DKKIRS 3Y	0.1600%
DKKIRS 4Y	0.3010%
DKKIRS 5Y	0.4445%
DKKIRS 6Y	0.5815%
DKKIRS 7Y	0.7180%
DKKIRS 8Y	0.8495%
DKKIRS 9Y	0.9685%
DKKIRS 10Y	1.0780%
DKKIRS 12Y	1.2480%
DKKIRS 15Y	1.4295%
DKKIRS 20Y	1.5810%
DKKIRS 25Y	1.6320%
DKKIRS 30Y	1.6510%

Table 4.2: Interest rate swaps

As mentioned in *section 4.2.1* an interpolation of the remaining annual rates is necessary to obtain linear dependency as required to bootstrap the zero-coupon prices. Therefore, the Nelson Siegel model is applied as described in *section 4.2.2.1*. The result is a well-fitted swap curve containing all annual maturities from 1 year to 30 years. The SSE is 1.32E-06, which must be considered highly acceptable.

The bootstrapping procedure from *section 4.2.1* is then applied to the annual swap rates. The swap rates are considered as coupon-bearings bonds paying annual coupons matching the swap rate. All the bonds are priced at par, which means the zero-coupon bonds can be bootstrapped by using *eq. 4.11*:

$$D_t = CF^{-1} * P_t$$

Multiplying the inversed $30 * 30$ cash-flow matrix with the prices of each coupon-bearing bond, the synthetically zero-coupon prices are derived at each t. The zero-coupon rates at each t are now extracted from the corresponding zero-coupon prices by using *eq. 4.1*:

$$Y(0,T) = -\frac{\text{LN}\, P(0,T)}{T-0}$$

All annual zero-coupon rates are now obtained. As the callable bonds have quarterly cash flows it is necessary to obtain quarterly zero-coupon rates for every quarter up to 30 years. The quarterly rates are obtained by interpolation utilising the Nelson Siegel model. The curve is well-fitted as the SSE only is 2,13E-08. This is perceived as a very satisfactory level. In *Figure 4.3,* the term structure of zero-coupon rates is displayed. The blue points are the annual zero-coupon rates while the orange line is the quarterly interpolated zero-coupon rates. It shows that the interpolation is well-fitted to the annual rates. All yields and corresponding maturities are presented in Excel cf. *Appendix 1*.

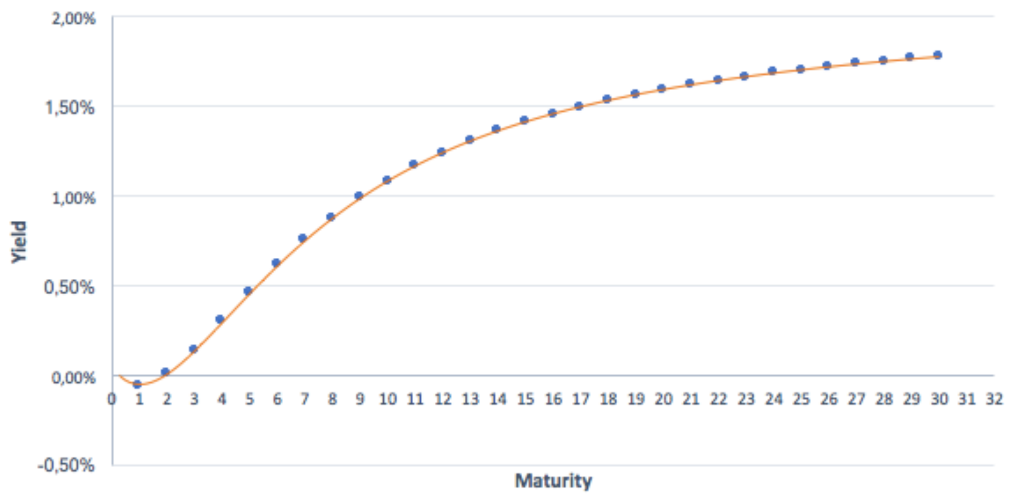


Figure 4.3: Term structure of zero-coupon rates

The zero-coupon yield curve has now been constructed containing quarterly yields ranging from 3 months to 30 years. This yield curve is a fundamental input for the interest rate model to describe how the interest rates evolve over time.

5 Interest rate model

A term structure model combines the zero-coupon yield curve with an estimate of the future interest rate volatility and thereby determines the possible stochastic evolution of future interest rates. The interest rate model is a vital part of the pricing model.

Equilibrium and short-rate models are referred to as endogenous term structure models, where the term structure is an output, rather than an input. Two mentionable short-rate models are the Vasicek model (Vasicek, 1977) and the Cox, Ingersoll, and Ross (CIR) model (Cox, et al., 1985). However, endogenous models cannot reproduce the exact initial yield curve, which is why market practitioners have little confidence in using these to price complex products (Hull, 2015).

On the contrary, no-arbitrage models are constructed to fit the current term structure of interest rates, as it is being used as an input to the models. Future prices evolve in ways which are consistent with the initial price structure and ensure the absence of arbitrage. Five central no-arbitrage models are the Ho-Lee (HL) model (Ho, et al., 1986), Hull-White (HW) model (Hull, et al., 1990), The Black-Derman-Toy (BDT) model (Black, et al., 1990), Heath, Jarrow and Morton (HJM) model (Heath, et al., 1992), and the Black-Karasinski (BK) model (Black, et al., 1991).

The interest rate models can either be considered in continuous-time or in discrete-time. Continuous models feature analytical closed form solutions and are widely used for valuing European derivatives, but they cannot be used to value more exotic derivatives. In a discrete-time model, it is possible to calculate if the Bermuda option should be exercised (bond to be called) or not at each node[21]. In nodes, where exercise is lucrative the option will deliver a cash-flow.

[21] Nodes are described in *section 5.1.*

While the HJM model provides the most accurate description of term structure movement it has the disadvantage of being non-Markov[22]. As a consequence, the model must be implemented by a Monte Carlo simulation or a non-recombining tree, which makes accurate pricing of exotic derivatives almost impossible (Hull, et al., 1994). Moreover, the BDT and BK models assume that the short rate follows a lognormal process, requiring non-negative interest rates. While the HW and HL models both allow non-negative rates the HL model has no incorporation of mean-reversion. When interest rates are high, the investments decline as the economy tends to slow down implying less demand for money resulting in declining interest rates. On the contrary, when interest rates are low, the economy is stimulated as funding is cheap, resulting in rising interest rates. Hence, there are compelling arguments for mean-reversion of interest rates. In the current negative interest rate environment, several challenges would be imposed due to the lognormal process and will result in unsatisfactory pricing. Therefore, on the above basis, the HW model has been selected as the interest rate model used for this thesis.

5.1 The generalised Hull-White model

The short rate r_t is the rate that applies to an infinitesimally short period of time at time t. It is also defined as the instantaneous short rate. Bond and option prices depend on the process followed by r_t in a risk-neutral world (Hull, 2015).

The poor fitting of the implied term structure of interest rates by the Vasicek model was addressed by Hull and White in 1990. The model is a one-factor model, as it only is dependent on one factor viz. the short rate. In the Hull-White model the instantaneous short-rate process evolves under the risk-neutral measure and is modelled through an Itô process[23]. It is described by following stochastic differential equation:

$$dr = [\theta(t) - ar]dt + \sigma\, dz \qquad \text{eq. 5.1}$$

where

- $\theta(t)$ is a function of time chosen to ensure that the model fits the initial yield curve.

[22] The distribution of the interest rates in the next period depends on the current and past rates.
[23] A continuous-time trajectory with a random evolution: Consists of a drift part and a diffusion part.

- a is a constant and the mean-reversion rate parameter. It determines the relative volatility of long and short rates. If a has a high value, it causes short-term rate movement to dwindle so long-term volatility is reduced. $a > 0$.
- r is the zero-coupon rate.
- dt is the time step.
- σ is a constant and the volatility parameter. It determines the overall level of volatility. $\sigma > 0$.
- dz is the Brownian motion[24] driving the process.

The model is built upon a set of assumptions. There are no market frictions, no transaction costs or taxes and all assets are perfectly divisible. Further, trading takes place in discrete time steps and the market is complete i.e. for every time T there exists a bond with respective maturity. The last assumption has been slightly simplified since I had to interpolate missing bond maturities. The approximations created by the interpolation are not assumed to distort the results. Finally, the zero-coupon bonds must satisfy following conditions: $P(T,T) = 1$ and $lim_{T\to\infty}\ P(0,T) = 0$. These conditions are satisfied by the zero-coupon bonds and by the Nelson Siegel model, which is used for interpolation. The Hull-White model assumes that interest rates are subject to mean-revision and that interest rates can become negative. Negative interest rates were traditionally seen as a weakness of the model, but if it is calibrated and used appropriately this should not present any problems (Hunt, et al., 2004). Hull and White address this problem and highlight that in low-interest-rate-environments the model give quite high probabilities of negative rates. Yet, the Hull-White model does fit observed market prices better than the other lognormal models in low interest-rate-environments (Hull, et al., 2015).

As mentioned in *section 5*, a discrete in time model is essential to value an exotic derivative such as a Bermuda option, which is embedded in the callable bonds. This approach is useful in valuing exotic derivatives as it is possible to determine whether to exercise the option early, or to hold it

[24] A continuous-time stochastic process: On the interval $[0,T]$ it is a random variable $W(t)$ that depends continuously on $t \in\ [0,T]$. It satisfies $W(0) = 0$ and $0 \leq s < t \leq T, W(t) - W(s) \sim \sqrt{t-s}N(0,1)$. A brownian motion is a special case of an Itô process with independent increments.

to maturity. It is possible to determine, because the development of the one-period short-rate is shown in each node, and it is thereby possible to determine the value of the bond at each node. Another advantage with the trinomial model is that it ensures the valuation is consistent with the zero-coupon volatility term structure in the market. Finally, the tree guarantees that both the callable bond and the option are valuated concurrently.

A discrete in time model is created by a lattice (tree) model. Lattice models commonly consist of either a binominal tree or a trinomial tree. While a binomial tree assumes that the underlying variable evolves in two states, either up or down, the trinomial tree assumes an extra possible state of 'no-change' as shown in *Figure 5.1*. At each time-step, the short rate evolves in three different states and is recombined. Hence, an up-movement followed by a down-movement resulting in the same rate as a down-movement followed by and up-movement. Instead of having two nodes at next time-step the tree only has one. A non-recombined tree would be very time-consuming and provide an inaccurate valuation (Hull, et al., 1994). Further, in the trinomial tree, the probabilities of each branch are different in different parts of the tree and the branching process is prone to vary from node to node.

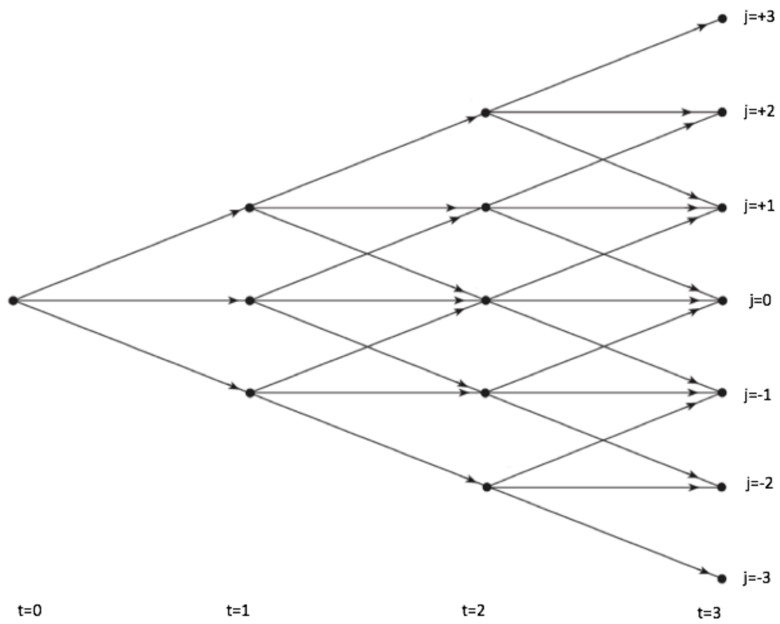


Figure 5.1: Trinomial tree

A binominal model with constant time-steps model makes it impossible to match both the expected drift and instantaneous standard deviation at each node without the number of nodes increasing exponentially with the number of time-steps (Hull, et al., 1993). This is the reason why the Hull-White model uses a trinomial tree, because it is capable of duplicating both the expected drift of the short rate and the instantaneous standard deviation at each node. It has furthermore shown for prices of European derivatives, that are calculated using the tree, converges to the price of the underlying differential equation as the length of the time-step approaches zero (Hull, et al., 1990a). The trinomial tree will be explained further in the following section.

5.2 Construction of the Hull-White lattice

The trinomial tree is constructed using the technique as presented by Hull and White in 1994 (Hull, et al., 1994). Time-steps of length Δt are set to represent movements in r. At the end of each time-step the r-values are considered to be $r_0 + k\Delta r$, where k is a positive or negative integer and r_0 is the initial value of r. The geometry of the tree is arranged such that, the central node always corresponds to the expected value of r. This method has proved to produce more accurate pricing, faster tree construction and more accurate values for hedging parameters (Hull, et al., 1994). The tree branching can take any of the shown forms in *Figure 5.2*. The starting node of the tree is at $(0,0)$, the node at time $i = i\Delta t = 0$ and the $j = 0$.

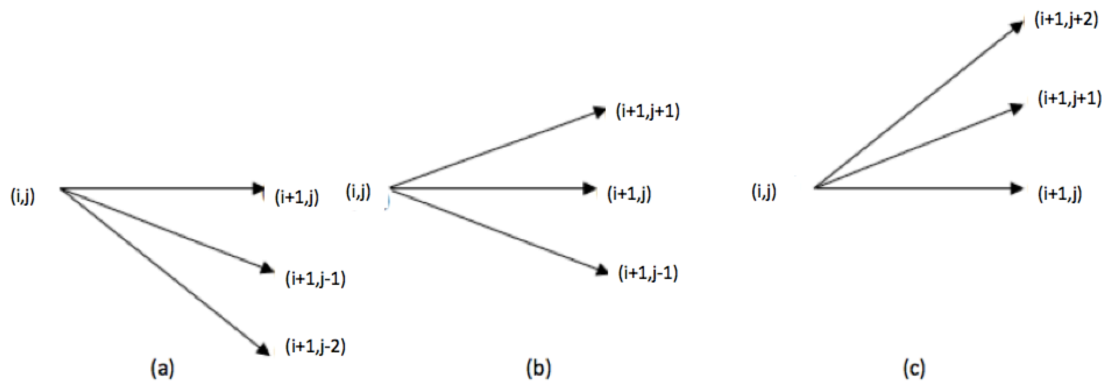


Figure 5.2: Branching methods

The first stage is to build a preliminary stage for r by setting $\theta(t) = 0$ and $r = 0$[25]. The process for r from *eq. 5.1* then changes to

[25] $r(0,0) = 0$

$$dr = -ar\,dt + \sigma\,dz \qquad \text{eq. 5.2}$$

In this process, $r(t + \Delta t) - r(t)$ is normally distributed[26]. For the construction of the tree, r is defined as the continuously compounded Δt-period rate. The expected value of $r(t + \Delta t) - r(t)$ is denoted as M and the variance of $r(t + \Delta t) - r(t)$ as *V*. That is,

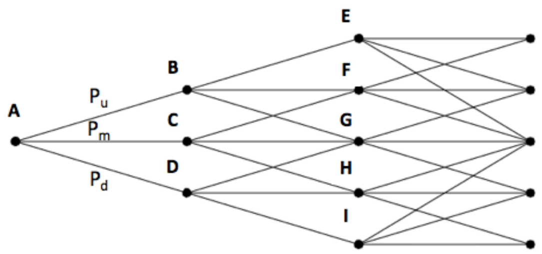
$$\mathbb{E}[r(t + \Delta t) - r(t)] = M \qquad \text{eq. 5.3}$$

$$Var[r(t + \Delta t) - r(t)] = V \qquad \text{eq. 5.4}$$

Now, the size of the time-step Δt is to be determined. The size of the interest rate step in the tree Δr is set as

$$\Delta r = \sqrt{3V} \qquad \text{eq. 5.5}$$

According to Hull and White, the size is set as above because theoretical work in numerical procedures, from an error minimisation perspective, suggests that this serves as a good choice of Δr (Hull, et al., 1994).



Node	A	B	C	D	E	F	G	H	I
r	0.00%	1.73%	0.00%	−1.73%	3.46%	1.73%	0.00%	−1.73%	−3.46%
p_u	0.167	0.122	0.167	0.222	0.887	0.122	0.167	0.222	0.087
p_m	0.666	0.656	0.666	0.656	0.026	0.656	0.666	0.656	0.026
p_d	0.167	0.222	0.167	0.122	0.087	0.222	0.167	0.122	0.887

Figure 5.3: Tree when $\theta = 0, a = 0.1, \sigma = 0.01$ *and* $\Delta t = 1$ *year*. Source: (Hull, et al., 1994)

The initial objective is to construct a tree similar to the tree exhibited in *Figure 5.3*. In the figure, the nodes are evenly spaced in r and t. The branching method from *Figure 5.2* must be determined to construct the tree as this will define its overall shape. Once this is done, the branching probability must be calculated.

Defining (i, j) as the node for which $t = i\Delta t$ and $r = j\Delta r$. At an arbitrary node (i, j), there are three possible branches, respectively p_u, p_m and p_d. These are defined as the probabilities of the

[26] The process is symmetrical when $r = 0$.

highest, middle and lowest branch from an emanating node. The probabilities for the tree are chosen so they reproduce the mean and the variance of *eq. 5.2*. Thus, matching the expected change and variance of r over the next time interval Δt from *eq. 5.3* and *eq. 5.4*. Obviously, the sum of the probabilities must equal unity. The above produces three equations for the three probabilities.

When r is at an arbitrary node (i, j), the expected change during next time step Δt is $j\Delta rM$ and the variance of same change is V. If the branching has the form as shown in *Figure 5.2 (b)*, the p_u, p_m and p_d at node (i, j) must satisfy the following three equations to match the mean and standard deviation (Hull, 2015):

$$\begin{gathered} p_u + p_m + p_d = 1 \\ (\Delta r)p_u + 0 + (-\Delta r)p_d = j\Delta rM \\ (\Delta r)^2 p_u + 0 + (-\Delta r)^2 p_d = (j\Delta rM)^2 + \frac{1}{3}\Delta r^2 \end{gathered} \qquad \text{eq. 5.6}$$

Using $\Delta r = \sqrt{3V}$, the solution of above equations is respectively

$$\begin{gathered} p_u = \frac{1}{6} + \frac{1}{2}(j^2M^2 + jM) \\ p_m = \frac{2}{3} - j^2M^2 \\ P_d = \frac{1}{6} + \frac{1}{2}(j^2M^2 - jM) \end{gathered} \qquad \text{eq. 5.7}$$

Similarly, if the branching has the form shown in *Figure 5.2 (c)*, the equations are

$$\begin{gathered} p_u + p_m + p_d = 1 \\ (2\Delta r)p_u + (\Delta r)p_m + 0 = j\Delta rM \\ (2\Delta r)^2 p_u + (\Delta r)^2 p_m + 0 = (j\Delta rM)^2 + \frac{1}{3}\Delta r^2 \end{gathered} \qquad \text{eq. 5.8}$$

and the probabilities are solved to

$$\begin{gathered} p_u = \frac{1}{6} + \frac{1}{2}(j^2M^2 - jM) \\ p_m = -\frac{1}{3} - j^2M^2 + 2jM \\ p_d = \frac{7}{6} + \frac{1}{2}(j^2M^2 - 3jM) \end{gathered} \qquad \text{eq. 5.9}$$

Finally, if the branching form is as shown in *Figure 5.2 (a)*, the equations are

$$p_u + p_m + p_d = 1$$
$$0 + (-\Delta r)p_m + (-2\Delta r)p_d = j\Delta r M \qquad \text{eq. 5.10}$$
$$0 + (-\Delta r)p_m + (-2\Delta r)p_d = (j\Delta r M)^2 + \frac{1}{3}\Delta r^2$$

while the solved probabilities are

$$p_u = \frac{7}{6} + \frac{1}{2}(j^2M^2 + 3jM)$$
$$p_m = -\frac{1}{3} - j^2M^2 - 2jM \qquad \text{eq. 5.11}$$
$$p_d = \frac{1}{6} + \frac{1}{2}(j^2M^2 + jM)$$

In most cases, the branching presented in *Figure 5.2 (b)* is appropriate. However, when $a > 0$ and *j* is large it is necessary to switch from *(b)* to *(a)*. It assures that the probabilities p_u, p_m and p_d stay positive. Likewise, it is required to switch from *(b)* to *(c)* when *j* is small[27].

Defining j_{max} as the value of *j* where branching (b) is switched to branching (a), and j_{min} where branching (b) is switched to branching (c). The probabilities p_u, p_m and p_d are always positive, when j_{max} is set to be an integer between $-0.184/M$ and $-0.816/M$, and j_{min} is an integer between $0.184/M$ and $0.816/M$.[28] In practice, Hull and White find that setting j_{max} equal to the smallest integer greater than $-0.184/M$ and $j_{min} = -j_{max}$ is most efficient. Furthermore, changing the branching at the first possible node proves to be computationally most efficient (Hull, 2015).

Illustrating the first stage of the tree building is done by constructing the tree presented *Figure 5.3*. The parameters are as presented in Table 5.1

σ	a	t	$Steps$	$\Delta t = t/steps$
0.01	0.1	3	3	1

Table 5.1 Tree parameters

[27] Negative and large in absolute value.

[28] When $a > 0, M < 0$.

Recall M and V from *eq. 5.3* and *eq. 5.4*[29].

1. Step one is to calculate the Δr from Δt. Recall that $\Delta r = \sqrt{3V}$ and that the variance is given by

$$V = \sigma^2 \Delta t = 0{,}01^2 * 1 = 0.0001 \quad \text{eq. 5.12}$$

thereby,

$$\Delta r = \sqrt{3 * 0.0001} = 0.01732 \quad \text{eq. 5.13}$$

2. The next step is to calculate the bounds of j_{max} and j_{min}. The expected return is calculated as

$$M = -a\Delta t = -0.01 \quad \text{eq. 5.14}$$

This means that the bounds for j_{max} are

$$\frac{-0.184}{-0.01} \Leftrightarrow \frac{0.184}{0.01} = 1.84 \text{ and } \frac{-0.186}{-0.01} \Leftrightarrow \frac{0.816}{0.01} = 8.16. \quad \text{eq. 5.15}$$

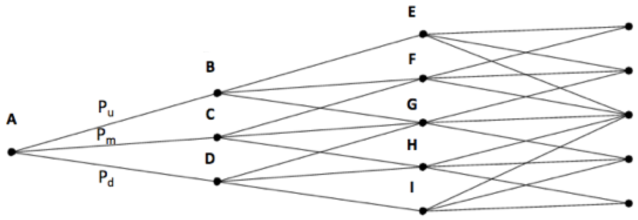
thus,

$$j_{max} = 2 \text{ and } j_{min} = -j_{max} = -2 \quad \text{eq. 5.16}$$

3. In step three, the equations for p_u, p_m and p_d are used to calculate the branches emanating from each node. The probabilities at each node depend only on j. Hence, the probabilities at node B are the same as the probabilities at node F. As it is a symmetrical tree, the probabilities are mirroring each other i.e. the probabilities at node D is a mirror image of the probabilities at node D.

The above steps complete the tree for the simplified process. Now, step four is to introduce the correct time-varying drift. To do so, the nodes at time $i\Delta t$ are shifted by an amount of α_i and thereby producing a new tree. The new and shifted tree is presented in *Figure 5.4*.

[29] Alternatively, M can be set to $e^{-a\Delta t} - 1$ and V to $\sigma^2(1 - e^{-2a\Delta t})/2a$ as it increases the speed of convergence (Hull, et al., 1994).



Node	A	B	C	D	E	F	G	H	I
r	3.82%	6.93%	5.20%	3.47%	9.71%	7.98%	6.25%	4.52%	2.79%
p_u	0.167	0.122	0.167	0.222	0.887	0.122	0.167	0.222	0.087
p_m	0.666	0.656	0.666	0.656	0.026	0.656	0.666	0.656	0.026
p_d	0.167	0.222	0.167	0.122	0.087	0.222	0.167	0.122	0.887

Figure 5.4: Shifted tree

The value of r at an arbitrary node (i, j) in the new tree equals the value of the equivalent node (i, j) in the old tree plus the value of α_i. The probabilities in the new tree are still the same as in the old tree. The α_i values are chosen to equal the observed market price of the zero-coupon bonds, maturing at time Δt consistently with the initial term structure. Thus, α_i is set to equal the initial Δt-period interest rate. In practice, this process starts at time zero and works forward by shifting all nodes at time $i\Delta t$ by α_i to match the zero-coupon prices.

Shifting the tree from *Figure 5.3* to *Figure 5.4* is done by changing the process being modelled from *eq. 5.2*

$$dr = -ar\,dt + \sigma\,dz$$

to

$$dr = [\theta(t) - ar]dt + \sigma\,dz \qquad \text{eq. 5.17}$$

If $\hat{\theta}(t)$ is defined as the estimate of θ given by the tree between times t and Δt, then the drift in r at time $i\Delta t$ at the midpoint of the tree is $\hat{\theta}(t) - a\alpha_i$, so that

$$\left[\hat{\theta}(t) - a\alpha_i\right]\Delta t = \alpha_i - \alpha_{i-1} \qquad \text{eq. 5.18}$$

The above equation relates the $\hat{\theta}$s to the α_is in the limit as $\Delta t \to 0,\ \hat{\theta}(t) \to\ \theta(t)$.

Arrow Debreu[30] prices for all nodes are calculated to facilitate the shift. These are defined as $Q_{i,j}$, the present value of a security that pays off $1 if node (i, j) is reached and is zero otherwise. Both α_i and $Q_{i,j}$ are calculated using forward induction. It is now illustrated how the shift is performed by using the yields presented in *Table 5.2*.

[30] A contract that agrees to pay one unit if a particular state occurs at a particular time in the future, and pays zero units in all the other states. Hence, the price of an Arrow Debreu asset is in a given node depends on which node it reaches.

Maturity	0.5	1.0	1.5	2.0	2.5	3.0
Rate (%)	3.430	3.824	4.183	4.512	4.812	5.086

Table 5.2: Continuously compounded zero-rates used for the example.

The value of $Q_{0,0}$ is 1, while the value of α_i is chosen to give the right price of a zero-coupon bond maturing at time Δt. It is set equal to the initial Δt period interest rate. The α_0 is equal to the one-year yield, 3.82%, since $\Delta t = 1$. Next objective is to calculate the subsequent values of $Q_{1,1}, Q_{1,0}$ and $Q_{1,-1}$. They have probabilities of respectively 0.1667, 0.6667 and 0.1667. The value of $Q_{1,1}$ is calculated by discounting with the discount rate of 3.82% using *eq. 4.1*

$$\begin{aligned} Q_{1,1} &= 0.1667e^{-0.0382*1} = 0.1604 \\ Q_{1,0} &= 0.667e^{-0.0382*1} = 0.6417 \\ Q_{1,-1} &= 0.1667e^{-0.0382*1} = 0.1604 \end{aligned} \qquad \text{eq. 5.19}$$

After calculating above it is now possible to determine α_1. This is set to give the right price of a zero-coupon bond maturing at time $2\Delta t$. Recall $\Delta r = 0.01732$ from *eq. 5.13* and $\Delta t = 1$. The prices of the bonds at following nodes are

$$\begin{aligned} &B: e^{-(\alpha_1 + j\Delta r)} = e^{-(\alpha_1 + 1*0.01732)} \\ &C: e^{-(\alpha_1)} \\ &D: e^{-(\alpha_1 - j\Delta r)} = e^{-(\alpha_1 - 1*0.01732)} \end{aligned} \qquad \text{eq. 5.20}$$

Finally, the price of a zero-coupon bond maturing at $2\Delta t$ is at the initial node A

$$P(0,2) = Q_{1,1}e^{-(\alpha_1+1*0.01732)} + Q_{1,0}e^{-(\alpha_1)} + Q_{1,-1}e^{-(\alpha_1-1*0.01732)} \qquad \text{eq. 5.21}$$

The initial term structure from *Table 5.2* shows that the bond price should be $P(0,2) = e^{-0.04512*2} = 0.9137$. It provides the following equation, when substituting the Qs and then solving for α_1.

$$\begin{aligned} 0.9137 = 0.1604e^{-(\alpha_1+\mathrm{j}*0.01732)} &+ 0.6417e^{-(\alpha_1+j*0.01732)} \\ &+ 0.1604e^{-(\alpha_1+\mathrm{j}*0.01732)} \\ \Leftrightarrow \alpha_1 = \ln\left[\frac{0.1604e^{0.01732} + 0.6427 + 0.1604e^{-0.01732}}{0.9137}\right] &= 0.0520 \end{aligned} \qquad \text{eq. 5.22}$$

To calculate the values of $Q_{2,2}, Q_{2,1}, Q_{2,0}, Q_{2,-1}$ and $Q_{2,-2}$ the process is iterated and done by discounting the value of a single \$1 payment of one of the nodes E-I back through the tree. The calculation for these is shortened by using the previously determined Q-values. Considering $Q_{2,1}$ for example, it is the value of a security that pays off \$1 if node F is reached and is zero otherwise.

The F node can only be reached from node B and C, and then in node B and C the interest rates are, as seen in *Figure 5.4*, 6.937% and 5.205% respectively. The probabilities from node $B \rightarrow F$ and $C \rightarrow F$ are respectively 0.6566 and 0.1667. Thereby, the value at node B of a security that pays off \$1 at node F is $0.6566e^{-0.06937}$ and similarly at node C is $0.1667e^{-0.05205}$. The present value is the sum of each of these weighted by the present value of \$1 received at each corresponding node. Recall, the values of $Q_{1,1}$ and $Q_{1,0}$ from *eq. 5.19*

$$Q_{2,1} = 0.6566e^{-0.06937} * Q_{1,1} + 0.1667e^{-0.05205} * Q_{1,0}$$
$$Q_{2,1} = 0.6566e^{-0.06937} * 0.1604 + 0.1667e^{-0.05205} * 0.6417 \qquad \text{eq. 5.23}$$
$$Q_{2,1} = 0.1998$$

Applying the same procedure, $Q_{2,2} = 0.0182, Q_{2,0} = 0.4736, Q_{2,-1} = 0.2033$ and $Q_{2,-2} = 0.0189$. This above technique is iterated to calculate α_2, then $Q'_{3,j}s$ are computed and so on.

This approach can be described more formally. When $Q_{i,j}$ has been determined for $i \leq m$ ($m \geq 0$), the next step is to determine α_m so that the zero-coupon prices maturing at $(m+1)\Delta t$ are correctly priced. As the interest rate at node (m, j) is $\alpha_m + j\Delta r$, the price of a discount bond maturing at time $(m+1)\Delta t$ is given by

$$P(0, m+1) = \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-(\alpha_m + j\Delta r)\Delta t}$$
$$\Leftrightarrow \alpha_m = \frac{ln \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta r \Delta t} - lnP_{m+1}}{\Delta t} \qquad \text{eq. 5.24}$$

where n_m is the number of nodes on each side of the central node at time $m\Delta t$. Once a_m has been solved and determined, the $Q_{i,j}$ for $i = m+1$ is computed by

$$Q_{m+1,j} = \sum_k Q_{m,k} q_{k,j} e^{-(\alpha_m + k\Delta r)\Delta t} \qquad \text{eq. 5.25}$$

where $q_{k,j}$ is the probability of moving from node (m, k) to node $(m+1, j)$. The is summed for all nonzero values of k.

5.3 Calibration

Until now, the volatility parameters a and σ have been assumed to be known. This assumption no longer applies as these parameters will be determined and used to calibrate the model. Both parameters are constants and not functions of time.

$$a(t) = a$$
$$\sigma(t) = \sigma \qquad \text{eq. 5.26}$$

Alternatively, a and σ can be made into functions of time. This allows the model to be fitted more accurately to the swaptions quoted in the market. However, the disadvantage is that the volatility structure will become nonstationary and no longer be mean reverting. This could lead to quite a different volatility term structure given by the model in the future, compared to the how it is in the present market. Based upon the above, the volatility parameters have been chosen to be constants.

Market data on active traded swaptions[31] are used as calibrating instruments to determine the volatility parameters. The following goodness-of-fit measure is used to determine the volatility parameters, by minimising the sum of squared errors[32].

$$SSE = \sum_{i=1}^{n} (P_i - V_i)^2 \qquad \text{eq. 5.27}$$

where, P_i is the market price of the *I*'th calibrating instrument, V_i is the swaption price estimate provided by the model and n is the number of swaption prices being fit. The calibration is done by choosing the volatility parameters in a manner so the goodness-of-fit measure is minimized. Finally, a threshold of pricing error of approximately 1% is allowed, as it has proven to provide the best results (Hull, et al., 1994).

As both a and σ are constants, there are only two volatility parameters. The number of volatility parameters should not be greater than the number of calibrating instruments and the instruments should be as similar as possible to the instrument being valued (Hull, 2015). Hence, more than two Bermuda swaptions would be optimal as the callable bond has an embedded Bermuda options.

[31] Will be elaborated upon in *section 5.3.1*.
[32] This goodness-of-fit measure is used by Hull and White (Hull, et al., 2000).

However, as quotes for European DKK swaptions are only available at Reuters, these are chosen. In detail, five at-the-money swaptions are chosen aligned with Hull and White's method (Hull, 2015). This will be explained further in *section 5.4*.

Alternatively, the volatility parameters can be estimated by using historical data of the interest rate. The advantage of calibrating to the implied volatility from swaptions observed in the market is that, the model will fit exactly to the volatility of the market. Also, using historical data would require comprehensive data manipulation as the utilised interest rates are not directly observed in the market.

5.3.1 Swaptions

A swap option or swaption is an over-the-counter contract. A European swaption is an option giving the right to enter an interest rate swap, a prespecified underlying swap at a given future time (at maturity of the swaption). There are two main versions of a swaption. A payer European swaption (a call option on an interest rate) gives the holder the right, at expiry of the option, to pay the prespecified fixed rate and receive the floating rate. The equivalent receiver European swaption (a put option on an interest rate) gives the holder the right, at expiry of the option, to receive the prespecified fixed rate and pay the floating rate.

The exercise (or strike) rate is specified in the contract and is determined by a fixed rate at which the buyer can enter the swap. The maturity (or expiration date) is the date when the option can be exercised. The maturity usually ranges between several months to 10 years. The premium of the swaptions, that the buyer pays the seller at entry, is expressed as a percentage of the principal amount of the swap. Finally, the underlying swap is commonly a plain vanilla swap.

A European swaption is quoted as $n * m$, where the option has n-years maturity and the swap has m-years maturity beyond the maturity of the option. Hence, a European DKK6YX4Y swaption with an option having 6-years to maturity that gives the holder the right to enter a 4-years swap if it is lucrative.

Similar to other options, a swaption can either be in-the-money, out-of-the-money or at-the-money. A holder of a payer swaption will exercise the option (and receive a profit) if the CIBOR is above the strike rate (fixed rate) when the option expires. Similarly, a holder of a receiver swaption will exercise the option if the CIBOR is below the strike rate at expiry. In both above cases, the swaption is said to be in-the-money. If the CIBOR rate is equal to the strike rate the option is said to be at-the-money. Finally, if the CIBOR is below (above) the strike rate of a payer (receiver) swaption, the option is said to be out-the-money. In this state, the buyer of the option will not exercise and has thereby lost the premium paid to enter the contract.

In *section 4.3* the swaps used for creating the yield curve were based on receiver swaps. Therefore, only receiver swaptions will be further considered. A European receiver swaption can be considered as a European call option on a coupon bond. Evaluating *eq. 4.4* at time S_0, the terminal payoff of a European receiver swaption is given as

$$\begin{aligned} P_{Receiver\ swaption}(S_0) &= Max\big(P_{Receiver\ Swap}(S_0), 0\big) \\ &= Max(P_{FCB}(S_0) - P_{FRB}(S_0), 0) \\ &= Max(P_{FCB}(S_0) - NP, 0) \\ &= Max\left(\left(NP * F\tau_j \sum_{j=0}^{N-1} P\big(S_0, S_{j+1}\big) + NP * P(S_0, S_N)\right) - NP, 0\right) \end{aligned} \quad \text{eq. 5.28}$$

The last expression in *eq. 5.28* is achieved by noting that $S_N = T_n$ and the price of the floating-rate bond at the reset date $S_0 = T_0$ is equal to its face value NP by applying *eq. 4.2*. Thus, a receiver swaption can be seen as a call option with a strike rate equal to the notional principal NP, that is written on a default-free coupon bond, that is maturing at time S_N with face value NP and having payable coupons of size $NP * F * \tau_j$ at times $S_{j+1}(for\ j = 0,1,2, \dots, N-1)$ (Nawalkha, et al., 2007).

5.3.2 Pricing example of a swaption in a lattice

The calibration will be done by minimising the goodness-of-fit measure presented in *eq. 5.27* by adjusting the two volatility parameters a and σ. This section will show how to price a swaption in a trinomial tree. This example uses the tree constructed in *section 5.2* as presented in *Figure 5.4*.

The swaption has the following characteristics

- It is a plain vanilla receiver DKK2YX3Y swaption.
- The swaption is assumed to have annual payments in the fixed leg and semi-annual payments in the floating leg.
- The strike rate is to 10%.
- The notional principal is set to 100.

In Figure 5.5, the pricing process of the swaption is presented.

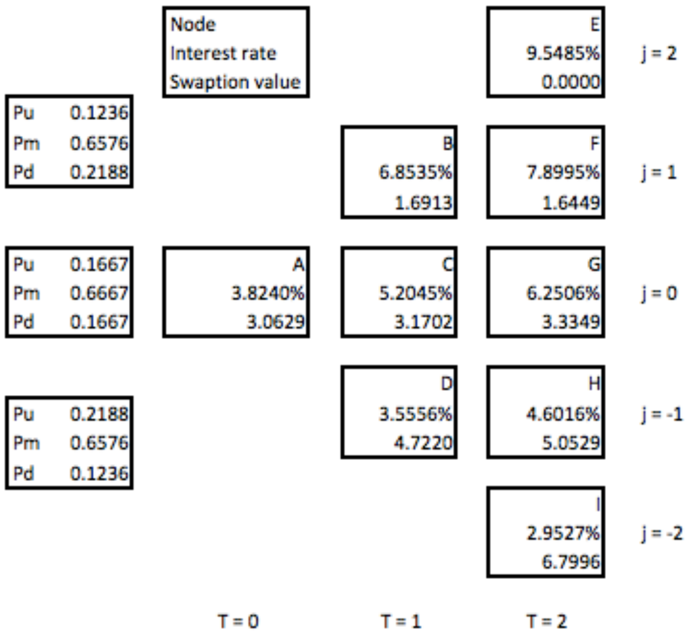


Figure 5.5: Swaption pricing

Each node is displayed in a box, where the upper field equals the node, the middle field shows the interest rates and the lower field presents the swaption value at the specific node. The interest rate and probabilities slightly differ compared to *Figure 5.4*, since the equations for M and V formulas are changed according to *footnote 29* for faster convergence. At all the nodes at $T = 2$, the swaption price is calculated by using *eq. 5.28*. Using backward induction, the prices at $T = 1$ and $T = 0$ are found. Consider the price of the swaption at node B

$$P_{Receiver\ Swaption(1,1)} = (0.1236 * 0.0000 + 0.6576 * 1.6449 + 0.2188 * 3.3349)e^{-1*0.0685}$$
$$= 1.6913$$

Similar procedure is done at node C and D. The price at node A is the price at $T = 0$ and shows that the price of the swaption today with a notional principal of 100 is

$$P_{Receiver\ Swaption(0,0)} = (0.1667 * 1.6913 + 0.667 * 3.1702 + 0.1667 * 4.7220)e^{-1*0.0382}$$
$$= 3.0629$$

This procedure is applied on all the swaptions that are priced in the trinomial tree to calibrate the model.

5.4 Application and results

The Hull-White trinomial tree is constructed by using the term structure of interest rates presented in *section 4.3*. The tree is constructed by following the step-wise procedure presented in *section 5.2* and computing M and V based on *footnote 29.* Δr is calculated using *eq. 5.5* while $j_{min} = -j_{max}$ and j_{max} is set to be an integer between $-0.184/M$ and $-0.816/M$.

After its construction, the two volatility parameters σ and a of the tree are calibrated to five at-the-money plain vanilla DKK European receiver swaptions. By only calibrating to at-the-money swaptions the model will not be calibrated to the entire volatility surface. I continue solely using at-the-money options and acknowledging that it may have an impact when pricing the callable bonds. The pricing may be affected when moving to the tails of the rate distribution. This is caused by the (volatility) smile effect, suggesting that implied volatility is not traded at the same level across the moneyness. Usually, out-of-the-money options are traded at higher volatilities than the corresponding in-the-money options. It must be noted that a perfect calibration for the quoted swaptions prices is not vital, as it may lead to an overfitted model.

The swaption data is obtained from Thomson Reuters Eikon. The swaptions have a 6 month CIBOR swap as underlying while having an annual fixed payment. The swaptions are quoted with both a bid and ask price. I use the bid price for implementation as it infers the right to receive the quoted fixed rate on the underlying swap (Nordea, 2004). The five swaptions are presented in the left side of *Table 5.3.*

#	Swaption	Strike rate (%)	Market Price(bp)	Model Price(bp)	Pricing error (%)
1	DKK2YX2	0.55	124.830	110.298	-13.18
2	DKK3YX2	0.45	90.170	90.170	-2.96
3	DKK3YX3	0.45	75.960	103.721	26.77
4	DKK4YX2	0.46	79.040	77.146	-2.45
5	DKK5YX2	0.40	65.410	65.302	-0.17

Table 5.3: Calibration instruments

The five swaptions have an option period ranging from 2 years to 5 years, while having the right to enter an underlying two to three-year swap. The swaptions quotes are observed at end-of-day March 17, 2017.

Pricing the swaptions has not been done directly in the tree constructed in Excel. Converging five different trees simultaneously, and requiring Excel to find a solution while being dependent on changing cell values is a cumbersome and slow process that has proven to cause errors. Instead, a VBA code has been employed. The VBA code is borrowed from the DerivaGem software which is included in "Options, Futures and other Derivatives" by John C. Hull[33] (Hull, 2015). To ensure identical outcomes, I priced the same swaption in my own constructed tree and through the VBA function. Both methods resulted in the same price. In practise, the optimization is set to minimise the goodness-of-fit factor from *eq. 5.27*, by changing the two volatility parameters σ and a and having a pricing error of absolute 1% if possible. The solution thereby fits the parameters in such a way that the difference between the observed market price and the model's price is minimised. This is with the GRG non-linear solver in Excel.

The post calibration pricing errors (right side of *Table 5.4*), are larger than the proposed 1% by Hull and White. One explanation is that DKK swaptions are illiquid assets. This is present in the bid-ask spreads as illiquid assets usually have higher bid-ask spreads. This is the case as a liquidity

[33] http://www-2.rotman.utoronto.ca/~hull/software/

premium is added to compensate for taking a liquidity risk i.e. not being able to sell the swaption. The lack of liquidity increases the spread and makes it more expensive for an investor to enter or exit trades. The bid-ask spreads on swaption 1-5 are respectively 60.93, 15.87, 1.79, 7.96 and 20.39 (bp). Another approach would have been to use EUR swaptions with an added premium as these are more liquid. An estimation of such liquidity premium requires in-depth market knowledge and is beyond the scope of this thesis.

Another explanation is that the optimisation problem is very complex and the global minimum has not been found. Instead, the local minimum is found as the solution is very sensitive to the initial guess. Hull and White however use the The Levenberg-Marquardt procedure (Hull, 2015) which has shown to be a very stable and a fast minimisation method. This procedure works well in practise and has become the standard method when solving non-linear least-squares problems (Press, et al., 1988). Implementing this procedure requires additional mathematical software and comprehensive work and has not been used, as it is beyond the scope of this thesis.

Swaption (*5)* is within the bounds of 1% and it is roughly assumed that *(2)* and *(4)* are also within satisfying limits despite not being below 1%. This means that three out of the five calibrated swaptions are priced at acceptable levels. The overall calibration is thereby accepted as Hull and White suggest using a greater number of calibrating instruments than the number of volatility parameters.

Based on the initial construction and calibration of the tree the final model is specified with following parameters and presented in *Table 5.4*

σ	a	t	$Steps$	$\Delta t = t/steps$	M	V	j_{max}	j_{min}
0.01298	0.13294	30	120	0.25	-0.03269	0.00004	6	-6

Table 5.4: Calibrated Hull-White trinomial tree parameters

The volatility of the short rate is approximately 0.01298 while the mean-reversion is 0.13294. This is presented in Excel cf. *Appendix 1*. The calibrated tree will further be applied with the prepayment model to price the callable bonds in *section 7*.

6 Prepayment model

The prepayment model is essential when pricing a callable bond as the cash flows are altered when borrowers prepay their loan. The borrowers tend to prepay their high coupon loans when the market rates are low and thereby affecting the investors' return. Modelling prepayments is both interesting and complex as the heterogeneous behaviour of the borrowers are causing them not to remortgage at the same (optimal) time. Initially, a homogeneous behaviour model is described, followed by a model that includes heterogeneity among borrowers by estimating prepayments based on empirical data. It should be noted, that an empirical prepayment model will provide wrong estimations if the prepayment behaviour in the future is different from the prepayment behaviour in the estimation period. If a change in the economic environment is not captured by the independent variables, then the model will generate poor estimates of future prepayment rates (Munk, 2011).

6.1 Optimal prepayment behaviour

A borrower holding a callable mortgage can be seen as the holder of a non-callable bond while having the optionality to buy back the mortgage at the face value on any date until maturity[34]. It therefore seems obvious to model the prepayment behaviour by using standard option theory. This relation is given by *eq. 3.1*.

In the American option model, the callable bond is divided into a portfolio of a non-callable bond and an American call option. At each node (i,j), the borrower must decide if it is ideal to prepay or continue the loan. If the borrower decides to prepay, the value of the loan is denoted as $W(i,j)$. This value includes any interest and repayments from the last period. Transaction costs are assumed to be a fixed percentage γ of $W(i,j)$. The total prepaying cost of the m'th bond will therefore be $W_m(i,j,\gamma) \equiv (1+\gamma)W(i,j)$. If the optionality is not exercised the value of the loan is denoted as $V_m^+(i,j,\gamma)$. Applying the value-minimizing principle, the value of the mortgage at node (i,j) is given by (Jakobsen, 1992).

[34] This is a simplification as the notification period for calling the loan (at the next payment date) expires two months before the next payment date.

$$V_m(i,j,\gamma) \equiv [W_m(i,j,\gamma), V_m^+(i,j,\gamma)]^- \qquad \text{eq. 6.1}$$

where the value in case of prepayment is given by

$$W_m(i,j,\gamma) = f(i,j) + c(i,j) + RD(i,j,\gamma) \qquad \text{eq. 6.2}$$

where $f(i,j)$ is the payment, $c(i,j)$ is the interest payment and $RD(i,j,\gamma)$ is the remaining debt. In case of no prepayment the value of the loan is

$$V_m^+ = f(i,j) + c(i,j) + p(i,j) * (p_u * V_m(i+1,j+1,\gamma) + p_m * V_m(i+1,j,\gamma) + p_d * V_m(i+1,j-1,\gamma) \qquad \text{eq. 6.3}$$

This model assumes that all borrowers have optimal behaviours and act rational. Hence, when conversion happens, the conversion rate jumps from 0% to 100%. The prepayment function thereby has only two outcomes and is given by

$$\lambda(i,j,\gamma) = \begin{cases} 1 if\ W_m(i,j,\gamma) < V_m^+(i,j,\gamma) \\ 0 \qquad otherwise \end{cases} \qquad \text{eq. 6.4}$$

The prepayment function indicates the number of borrowers who prepays their loans. As stated above, this is a binary function due to the rationality assumption in this model. The price of the callable bond is found by backward induction. First the value of W_m and V_m^+ is computed to evaluate the prepayment function. Then, the value of the mortgage at maturity is discounted back through the nodes until time $t = 0$.

In practice, this model is unrealistic as optimal behaviour amongst the borrowers is non-existent. The borrowers are heterogeneous, irrational and do not have full information. Their behaviour is determined by many factors, which need to be included in the prepayment function to describe their actual prepayment behaviour.

6.2 Required gain model

To account for non-rational behaviour and heterogeneous borrowers the prepayment function is modelled as a probit-like model. This model is defined as the required gain model and is created by Jakob Svendsen in 1992 (Jakobsen, 1992). This is the selected model for this thesis as both the rational and non-rational behaviour of the borrowers are being modelled.

Rewriting *eq. 3.1*, the exercise value of the Bermudan option is defined as $C_m \equiv B_m - W_m$ where B_m and W_m denote respectively the value of the m'th non-callable and callable bond. The Bermudan option represents the borrowers' future savings if their loan is prepaid. However, in this model, it is assumed that the individual borrower has an individual required gain g_{mi}, which explains why borrowers call their loans at different times. The required gain is assumed to be normally distributed with the mean of g^* and a standard deviation of σ. Contrary to the prepayment function in American option model, the heterogeneity related to the required gain allows partial prepayment at each decision date in the required gain model.

On every payment date t, a fraction of remaining borrowers who have $g_{mt} > g_{mti}$ will prepay their loan. This fraction is also called the Condition Prepayment Rate (CPR). This means, that the fraction of borrowers who prepay with a gain of g_{mt} is equal to $N(g_{mt}; g^*_{mt}, \sigma_{mt})$. The gain from the prepayment is inserted into the distribution function resulting in a prepayment rate, which can be described as follows

$$\lambda(g_{mt}) = N(g_{mt}; g^*_{mt}, \sigma_{mt}) \qquad \text{eq. 6.5}$$

Consider the decision variable y_{mti} being equal to one if borrower i prepays at time t and equal to zero if borrower i does not prepay at time t. Thus, the probability that borrower i prepays is given by

$$Prob[y_{mti} = 1] = Prob[g_{mt} > g_{mti}] = \Phi\left(\frac{g_{mt} - g^*_{mt}}{\sigma_{mt}}\right) \qquad \text{eq. 6.6}$$

where $\Phi(\cdot)$ is the standard normal distribution function. Assuming the mean of the required gain depends on a vector of the independent variables x_{mt} as well as a vector of the to be estimated β-parameters. Substituting this assumption with g^*_{mt} provides the following (Jakobsen, 1992).

$$\begin{aligned} \lambda_{mt} &= \Phi\left(\frac{g_{mt} - g^*_{mt}}{\sigma_{mt}}\right) \\ &= \Phi\left(\frac{g_{mt} - \sum_{k=1}^{K} \beta_k x_{mtk}}{\sigma_{mt}}\right) \\ &= \Phi\left(\frac{1}{\sigma_{mt}} * g_{mt} - \frac{1}{\sigma_{mt}} \sum_{k=1}^{K} \beta_k x_{mtk}\right) \\ &= \Phi(\beta_1 g_{mt} + \beta_2 x_{mt2} + \ldots + \beta_K x_{mtK}) \end{aligned} \qquad \text{eq. 6.7}$$

Estimation of the β parameters will be elaborated on in *section 6.5.1*. The prepayment function λ is now thereby described by a standard normal function and a set of independent variables and model parameters. This can also be denoted as the expected CPR, $\mathbb{E}[CPR]$. It is important to emphasise, that the value of the $\beta's$ and the x's determine the z-value in a standard normal distribution. Hence, higher (and positive) values of $\beta's$ and x's mean that a prepayment is more likely to happen. A significant and positive β-value will raise the z-score and thereby indirectly affect the estimated CPR.

6.3 Actual prepayment rate

The Conditional Prepayment Rate denotes the fraction of the remaining borrowers who prepay their loan at a given payment date. At each payment date, the total drawing percentage from the series describes the amount being repaid to the investors (Jakobsen, et al., 1999).

$$TD_t = OD_t + PD_t \qquad \text{eq. 6.8}$$

At time t the total prepaid amount consists of the ordinary and the preliminary drawings in percent. The ordinary drawings are the planned repayments from non-calling borrowers while the preliminary drawings are the prepayments. When TD equals 100%, then the entire remaining debt is prepaid to the investors and thereby the TD will equal zero in the succeeding periods.

The CPR can be calculated from the published data. However, it is not as straight forward as it once was, due to an amendment in the legal framework. This will be further explained in *section 6.6*. The total drawing is equal to the fraction of borrowers who prepay their loan prior to maturity plus the remaining non-calling borrowers who repay as scheduled. The OD is calculated as the fraction of borrowers who are not prepaying.

$$TD_t = OD_t + PD_t = OD_t(1 - CPR_t) + 100 * CPR_t \qquad \text{eq. 6.9}$$

The CPR is now isolated in above equation

$$\lambda_t = CPR_t = \frac{OD_t - TD_t}{100 - OD_t} = \frac{PD_t}{100 - OD_t} \qquad \text{eq. 6.10}$$

The actual CPR_t is calculated as shown above. If OD_t is small, then the CPR_t will be close to the value of PD_t. Hence, if there have not been any prepayments, PD_t will be zero and the CPR will consequently be zero.

6.4 The independent variables

The independent variables chosen for this model is not exhaustive. The main interest has been to select variables that significantly can explain the CPR and are easily calculated based on available data. The chosen variables are similar to those used by Danske Bank in their traditional model (Danske Bank Markets, 2016).

6.4.1 Refinancing gain

Recall from *section 6.2*, that a borrower will prepay the loan if the gain from prepaying is larger than the required gain. In 1999, Jacobsen and Svenstrup (Jakobsen, et al., 1999) calculated the refinancing gain as the ratio between the NPV of the old loan and new loan including its costs. Mortgage banks currently focus more on the difference between the first year's payments on the existing loan and the new loan including costs (Danske Bank Markets, 2016). As today's borrowers are currently more focused on their liquidity savings in form of lower payments, I choose to follow the first-year payment approach[35].

The percentage difference is computed by the first-year payment (FYP)[36] of the existing loan and the new loan.

$$g_{mt} = \frac{FYP_{old\ loan,t} - FYP_{new\ loan,t} - costs}{FYP_{old\ loan,t}} \qquad \text{eq. 6.11}$$

The refinancing gain g_{mt} is the FYP gain the borrower achieves by refinancing to a loan with the same time to maturity as the existing loan. Thereby, it is expected that when g_{mt} increases the prepayment rate also increases.

It is assumed that the loans are bond loans as well as having annuity amortizations. Additionally, the remaining debt on the new loan equals the old loan. While this is a relative measure, the

[35] Danske bank uses after tax net payments. Tax is beyond the scope of this thesis and therefore only refinancing costs are included.

[36] The first-year after the payment date.

refinancing costs change as the remaining debt changes. The refinancing gain is therefore calculated for each debtor group by changing the respective costs.

The $FYP_{old\ loan}$ is found by discounting the next year's cash flow with the corresponding zero-coupon curve plus a debtor spread. The $FYP_{new\ loan}$ is found by discounting the new loan's cash flows. The refinancing rate of the new loan is not known and must be determined to calculate its cash flows. It is assumed that the refinancing rate is equal to the zero-coupon curve plus a constant debtor spread, as the borrowers cannot refinance to the zero-rate. Without a debtor spread, the assumed refinancing rate and actual refinancing rate could cause inconsistencies, if the spread between the mortgage bonds and the swap curve widened (Danske Bank Markets, 2016). Finally, it is assumed that the new loan is financed at par.

The debtor spread is the extraordinary spread between the mortgage bonds and the zero-curve (Danske Bank Markets, 2016). The debtor spread is retrieved from *Nordea Analytics* from the first payment date in the estimation period.

First the PMT of the old loan is computed (Jensen, 2009)

$$PMT_{old} = RD \frac{\frac{c}{f}}{1 - \left(1 + \frac{c}{f}\right)^{-t}} \qquad \text{eq. 6.12}$$

where RD is the remaining debt, c is the coupon rate, f is the coupon frequency and t is the number of periods. The PMTs then need to be discounted with the appropriate refinance rate to find the value of the net payments for the first year

$$FYP_{old} = \sum_{t=1}^{T} PMT(1 + r_t)^{-t} \qquad \text{eq. 6.13}$$

r_t is the refinance rate for the given period t. The coupon rate for the new loan is found by adding the debtor spread ds to the zero-coupon rate for the given remaining maturity

$$c_{new} = r_{zcb} + ds \qquad \text{eq. 6.14}$$

then the PMT_{new} is found by using *eq. 6.12* using c_{new} instead of c. FYP_{new} is found by using *eq. 6.13*, but using PMT_{new}. The cost component also needs to be determined before calculating the refinancing gain g_{mt}. The refinancing costs are estimated in *section 6.6.5.*

The refinancing gain g_{mt} will return a positive value when it is profitable to remortgage the loan i.e. when the FYP_{new} including costs is smaller than the FYP_{old} and vice versa. It is to be expected, that the higher the refinancing gain is on the first-year payments, the higher the CPR will be.

6.4.2 Time to maturity

When including the refinancing gain it is also necessary to incorporate the remaining time to maturity of the loans as short-term loans have a lower required gain compared to long-term loans (Jakobsen, et al., 1999). This is explained by simple option theory stating that the longer to maturity, the higher is the probability that the prepayment option will end (deeper) in the money, which increases the required gain. Hence, the borrower with a long-term loan may experience a further potential gain by waiting, as the maturity is in the far future. The short-term borrower will likely not have the time to wait for a better time to remortgage and thereby has a lower required gain.

The time to maturity is given by

$$TTM = \frac{Maturity\ date - Payment\ date}{Maturity\ date - Opening\ date} \qquad \text{eq. 6.15}$$

As time passes TTM will go from 1 to 0. It is expected that when TTM is large the prepayment rate is low and vice versa.

6.4.3 Burnout behaviour/pool factor

The burnout effect explains a low prepayment rate for bond series that have had many historical prepayments. Each series consists of a collection of loans from heterogeneous borrowers. The outcome of the diversity within the borrowers is that they have different incentives for when to call their loans. When rates initially fall, the borrowers with the greatest propensity to call their loans, do so. The remaining borrowers thereby tend to prepay a slower rate than first group. In the end, the borrowers who yet are to call their loan have a very high refinancing threshold. This is when the series is defined as burnt out. Hence, it is a measure for the remaining debt in a series in relation to the expected remaining debt without any called loans (Jakobsen, et al., 1999). A good proxy for the burnout is the pool factor (PF) which measures the current outstanding amount in relation to the maximum outstanding amount.

$$PF = \frac{Current\ outstanding\ amount}{Maximum\ outstanding\ amount} \quad \text{eq. 6.16}$$

A pool factor of 1 indicates that no prepayment has been made in that series. As the loans are prepaid the PF will go from 1 towards 0. Hence, a low PF intuitively implies a low CPR while a high PF implies a high CPR.

6.5 Model estimation

The borrowers are heterogeneous and their readiness to redeem their loans depend on the possible economic gain and their subjective required gain criteria. The economic gain is not directly proportional to the remaining size of the loan, due to the refinancing costs. To capture this effect the prepayment model is estimated for five different debtor groups. The debtor groups can to some extent capture the difference in remaining debt and their subjective criteria for the required gain. The debtor groups are assumed to be homogeneous.

The available data is only published on a series level. This is a hindrance as I estimate a prepayment rate for each debtor group and not for each series. To accommodate this issue, I estimate the prepayment rate by utilizing a Mixture distribution model as suggested by Jakobsen and Svendstrup (Jakobsen, et al., 1999).

6.5.1 Mixture distribution model

This model does not capture the effect of heterogeneity among the borrowers, but models it directly, leading to a prepayment model with a mixture distribution (Jakobsen, 1994). A mixture distribution is a mixture of two or more probability distributions, either made up by different distributions or the same distribution with different parameters.

A prepayment function will be estimated for each debtor group. The prepayment rate will then be weighted by the remaining debt for the corresponding debtor group and summed to a total prepayment rate for the bond series. I choose the β-estimates so that sum of squared errors (SSE) is minimised to maximize the fit. The goodness-of-fit measure is identical with *eq. 4.13*.

$$SSE = \sum_{t=1}^{T} \left(CPR_{act,t} - CPR_{MD,t}\right)^2 \qquad \text{eq. 6.17}$$

where $CPR_{act,t}$ is the actual prepayment rate and CPR_{MD} is the estimated prepayment rate – both for a given bond at time t. $CPR_{act,t}$ is calculated based on the available data, while CPR_{MD} is given by *eq. 6.7* while including the weights from the debtor groups.

$$CPR_{MD,t} = \sum_{i=1}^{I} w_{mt,i}\ \Phi\left(\beta_{1,i} g_{mt,i} + \beta_{2,i} x_{mt2,i} + \ldots + \beta_{K,I} x_{mtK,I}\right) \qquad \text{eq. 6.18}$$

The individual debtor groups are denoted with i $(for\ i = 1, 2, \ldots, I)$, w_i is the remaining debt for the individual debtor group, and $\Phi(\cdot)$ is the standard normal distribution which is a function of the estimated parameters β_k and the independent variables x_k where $k = 1,2, \ldots, K$.

6.6 Data

The Danish mortgage banks are entitled by law to provide data and information about the bonds they issue. The data ensures market transparency and is necessary for investors to price the callable bonds. The data is obtained from Nykredit/Totalkredit, Realkredit Denmark and Nordea Kredit. The mortgage banks publish this data in either text files or Excel format. Nykredit publishes all data (in another format) split by the mortgage series, while the other mortgage banks publish data for all their mortgage series split by the data type. This makes it a comprehensive procedure to manipulate the data.

The published data includes basic bond information, preliminary prepayments, drawings, cash flow information and debtor distributions. The release dates of the data are shown in *Figure 6.1*.

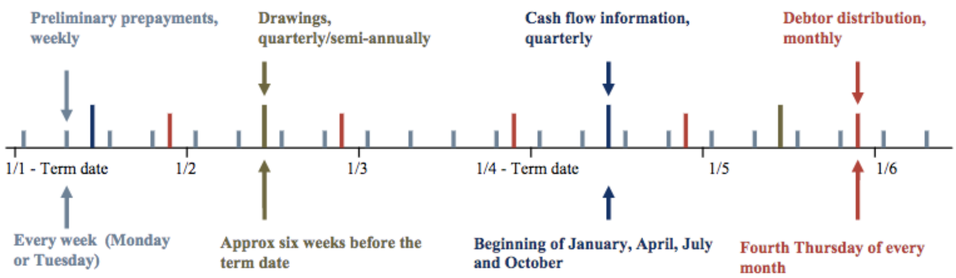


Figure 6.1: Release of bond data. Source: (Nykredit, 2014)

The data is present in different CK-series. The data for the independent variables chosen in *section 6.4* is present in the following; basic bond information, debtor distribution (CK92), preliminary prepayments (CK93), actual prepayments (CK95) and in Nykredit's own format which contains all in one per mortgage series[37].

6.6.1 Basic bond information

Basic bond information is retrieved through Thomson Reuters Eikon. This data includes principal and coupon information, issuance details, identifiers and ratings.

6.6.2 Actual prepayments (CK95)

The individual mortgage banks publish actual prepayments about 40 days prior to the payment date. The format of the CK95-series changed from September 21, 2015, as the Danish law was harmonised to EU-standards[38]. The data published after this date no longer contains the outstanding amount and the preliminary drawings, but only total drawings in percentage and absolute value. The data is therefore in different format to that of the previous years. Since the outstanding amount is disclosed in earlier publications of the data it can be calculated post September 21, 2015 by

$$OA_t = OA_{t-1} - TD_{t-1} \qquad \text{eq. 6.19}$$

[37] Nykredit does publish the CK-series but it is a complicated procedure to obtain it.
[38] See https://newsclient.omxgroup.com/cdsPublic/viewDisclosure.action?disclosureId=660805&lang=da for further information.

Before the actual CPR can be calculated, as described in *section 6.3*, the data on preliminary drawings will be essential.

6.6.3 Preliminary drawings (CK93)

The CK93-series is published on a weekly basis. The data is based on loan terminations for upcoming payment dates. This data allows for estimation of the volume of prepayments for the following payment date. Preliminary payments have a strong exponential increase up to the expiry of the notification period as borrowers tend to decide on calling the loan as late as possible. I therefore use the latest publication prior to the release of the CK95 as it contains the most reliable data.

It is now possible to calculate the ordinary drawings using the relation from *eq. 6.8*

$$TD_t = OD_t + PD_t \Leftrightarrow OD_t = TD_t - PD_t \qquad \text{eq. 6.20}$$

All components for the CPR are available and can now be calculated cf. *section 6.3.*

6.6.4 Debtor distribution (CK92)

Debtor distribution data separates the underlying loans into borrower groups, remaining debt groups and loan types. Concretely, two borrower groups (private, other and commercial), five remaining debt intervals (<200 DKKt, 200-499, 500-999, 1000-2,999 and >3,000) and two loan types (bond and cash loans). The data is published no later than four days before the fourth Thursday of the month.

The separation makes it possible to split the debtors into 20 groups. Many of the groups will however be hard to tell apart. The most important division factor is the size of the remaining debt, as a large remaining debt is called first (Jakobsen, et al., 1999).

Large corporate loans tend to have higher remortgaging than small private loans, as these loans have lower refinancing costs when prepaying, due to having a larger principal. The large principal also adds a psychological effect as a gain of DKK 100,000 is more alluring than a gain of DKK 1,000 (Danske Bank Markets, 2016). Recall from *section 3.3*, that cash loans will be repaid at a slower pace than bond loans, due to the tax liable capital gains. Despite both borrower groups and loan types influencing the prepayment rate, they will not be included in the model. Instead, for simplicity, the five debtor groups implemented based on the five remaining debt intervals.

For each payment date, the CPR is also calculated for each debtor group as

$$TD_{t,i} = \frac{RD_{t-1,i} - RD_{t,i}}{RD_{t-1,i}} \qquad \text{eq. 6.21}$$

The total draw TD for each time t and for each debtor group i is calculated as the ratio between remaining debt from the previous t and the remaining debt from the current t. Since the ordinary drawing for the bond series is relative to the remaining debt, the extraordinary drawing is found by

$$PD_{t,i} = TD_{t,i} - OD_t \qquad \text{eq. 6.22}$$

Finally, the $CPR_{t,i}$ is calculated as

$$CPR_{t,i} = \frac{PD_{t,i}}{100 - OD_t} \qquad \text{eq. 6.23}$$

6.6.5 Refinancing costs

When remortgaging a loan the refinancing costs should be considered. These costs are both variable and fixed. The costs are calculated as an average of the costs of the largest mortgage banks in Denmark: Nykredit/Totalkredit[39], Realkredit Denmark[40] and Nordea Kredit[41] cf. *Figure 3.2*. The costs are obtained from the latest price sheet from each mortgage bank. The costs are estimated solely on private costs. This may cause a slight bias as the groups are not divided by borrower group. This is assumed not to distort the results as the difference is marginal. The average costs are shown in *Table 4.1*.

[39] https://www.nykredit.dk/staticcontent/files/prisblad-laan-dkk.pdf
[40] https://www.rd.dk/PDF/Privat/Prisblad-privat.pdf
[41] https://www.nordea.dk/Images/38-97716/Nordea%20Kredit%20Prisliste%20jan%202017%20A4.pdf and https://www.nordea.dk/Images/38-98139/Priser_serviceydelser_privat_DK_%20-%20februar%202017.pdf

The costs are calculated based on the five remaining debt interval groups. The basis of estimation is the mean of each group. The fixed costs are simply added. The brokerage fee is calculated as a percentage of the spread adjusted price[42]. The spread is calculated as

$$Spread\ cost_i = \frac{P}{P - S_{avg}} * RD_{boe,i} - \frac{P}{P} * RD_{boe,i} \qquad \text{eq. 6.24}$$

where i is the group, P is the market price, RD denotes the remaining debt basis estimate for the individual group, while S is the average spread of the three mortgage banks.

When exercising the prepayment option, a trading fee is normally added and is dependent on the size of the loan. This is beyond the scope of this thesis and it will not be included.

Refinancing costs

Debtor groups	<200,000	200,000 500,000	500,000 1,000,000	1,000,000 3,000,000	>3,000,000
Basis of estimate	100,000	350,000	750,000	2,000,000	5,000,000
Fixed costs					
Mortgage account fee	3,433	3,433	3,433	3,433	3,433
Land registration fee	1,660	1,660	1,660	1,660	1,660
Early repayment charge	833	833	833	833	833
Variable costs					
Spread (0,22)	217	760	1,629	4,343	10,857
Brokerage fee (0,15%)	165	526	1,127	3,007	7,516
Total costs	6,309	7,213	8,683	13,276	24,300
Total costs in %	6.31%	2.06%	1.16%	0.66%	0.49%

Table 6.1: Refinancing costs

Clearly, the remaining debt determines the remortgaging gain as the fixed fees weight more in loans where the remaining debt is smaller.

6.7 Application and results

The CK-data is acquired from the individual mortgage banks' homepage, while the data on bond information is pulled from Thomson Reuters Eikon. The prepayment model is based on historical data ranging from 01-04-2014 to 01-01-2017. The bonds used for this model have been selected based on several requirements. They must be fixed, include callable options and have quarterly

[42] According to Nykredit's customer service.

coupon dates. To ensure a liquid series, they must also have a minimum outstanding amount of DKKm 1,000 on the 01-04-2014 and an annuity amortisation. Finally, the bonds must have a 30-year maturity and be closed, so that new issues will not cause a change in the remaining debt. I select eight bonds over 12 payments dates divided in the five remaining debt intervals. As the estimation sample is limited the characteristics of the bonds are concentrated at 4-5% bonds with maturities ranging from approximately 21 to 27 years. The eight bonds are presented in *Table 6.2*.

ISIN	Coupon (%)	Time to maturity (01/04/2014)
DK0002011386	5	21.01
DK0002023209	4	27.25
DK0009757296	4	21.52
DK0004714458	5	21.01
DK0004715505	4	21.26
DK0009269227	5	21.26
DK0009282329	4	27.52
DK0009753469	5	21.26

Table 6.2: Bonds for model estimation

The borrower must inform the mortgage bank within the notification period to call the loan. The notification period expires two month prior to the payment date. Hence, the refinancing gain should be calculated before expiry of the notification period. I simplify this and calculate it on the payment date.

The zero curves used to calculate the refinancing gain and the debtor spread are directly obtained from *Nordea Analytics.* They are based on interest rates swaps, which have been estimated by bootstrapping and the Nelson Siegel model. The debtor spread is estimated to be 76,01 basis points and is applied as a constant, shifting the curve upwards. The refinancing gain is calculated for each of the five debtor groups.

Time to maturity is directly implemented in the model by using *eq. 6.15*. No adjustments are necessary as the time to maturity does not depend on the remaining debt in the respective groups.

The pool factor requires an adjustment to fit the debtor groups. While *eq. 6.16* is used to calculate the pool factor on a series level, an adjustment is necessary before it can be calculated for each debtor group. One solution would be to calculate the total drawings for each debtor group (based on the CK92 data), and to calculate maximum outstanding amount per debtor group, while accounting for re-openings of the bonds. This is a comprehensive solution and the PF is therefore simplified as the current outstanding amount for debtor group i for series m at time t in relation to the maximum outstanding amount of the series m.

$$PF_{mt,i} = \frac{Current\ outstanding\ amout_{mt,i}}{Maximum\ outstanding\ amount_{m}} \qquad \text{eq. 6.25}$$

6.7.1 Model estimation

The model's dependent variable is the actual CPR for each t while the independent variables include the refinancing gain, time to maturity and the pool factor. The estimation is done by minimising the goodness-of-fit measure from *eq. 6.17* by changing the β-values. The minimisation is done by using the GRG non-linear method in the Excel solver. The minimisation problem becomes increasingly complex when more debtor groups are added to the model as more β-parameters must be estimated. As the model's complexity increases, the more important the initial β-parameters become as they are used as the starting point. Therefore, several estimations have been tested with different initial values. The initial values have been set both as the expected relation between dependentand independent variable but also randomly to test for non-expected relationships. After numerous attempts, there is no guarantee that the outcome is the global minimum, but rather the local minimum. Additionally, models with many parameters are also potentially subject to convergence issues.

Several models are estimated as it is assumed the predicted CPR highly depends on the model's design. Therefore, I estimate models where the input variables are adjusted to account for the debtor groups and models where they are not. This is displayed in Excel cf. *Appendix 1*.

I evaluate the models by comparing their R^2. This coefficient of determination indicates the fraction of the variance in the dependent variable that is predictable from the independent variables. The R^2 is measured between the actual CPR_{act} and estimated CPR_{MD}. The R^2 ranges from 0 to 1, where 1 infers a perfect fit.

In *Table 6.3*, the estimated models are displayed with their corresponding R^2.

Model #	Model	R^2
1	$CPR_{MD} = \Phi(\beta_1 g_{mt} * PF_{mt} + \beta_2 * TTM_{mt} + \beta_3)$	31.70%
2	$CPR_{MD} = \Phi\left(\beta_1 g_{mt} * PF_{mt}^{\beta_0} + \beta_2 * TTM_{mt} + \beta_3\right)$	38.20%
3	$CPR_{MD} = \sum_{i=1}^{I} w_{mt,i}\, \Phi(\beta_{1,i} g_{mt,i} * PF_{mt,i} + \beta_{2,i} * TTM_{mt} + \beta_{3,i})$	47.53%
4	$CPR_{MD} = \sum_{i=1}^{I} w_{mt,i}\, \Phi\left(\beta_{1,i} g_{mt,i} * PF_{mt,i}^{\beta_0} + \beta_{2,i} * TTM_{mt} + \beta_{3,i}\right)$	51.73%

Table 6.3: Estimated models

There are no issues with convergence or finding the global minimum when fitting Model 1 and 2. The models are simple and they only include a few parameters as they are not adjusted for debtor groups. The simplicity is also existent by a low R^2 in both models. These models will not be included for further analysis due to their poor fit.

The β-values determine the weighting of the independent variables and the constant in the prepayment function. The sum of the three parts of the prepayment function is the quantile in the standard normal distribution. The output from the distribution is then adjusted for the weight of remaining debt in the respective debtor group i in relation to the total outstanding amount. Finally, this results in the CPR_{MD} for the respective bond m at time t. As mentioned in *section 6.2*, the β-values do not directly impact the estimated CPR, but raise the z-score and increase the likelihood of the event happening. Only the non-direct impact to the estimated CPR is considered when interpreting the β-values, as a larger (smaller) quantile results in a higher (lower) CPR.

β_1 determines the weight of the product, of the refinance gain and the pool factor. The pool factor is included in this part to adjust for the historical prepayment rate of the bond. Intuitively, the parameter must be positive as a large refinance gain will cause a high CPR. However, even if the refinance gain is large, the CPR might be low if the pool factor is small since the historical prepayments then has been high.

β_2 accounts for the weight of the time to maturity. A large time to maturity results in a small CPR as the borrower would earn a potentially larger gain in the future. Loans with a small time to maturity are thereby more willing to remortgage, as their potential gain by waiting becomes smaller as maturity nears. As the time to maturity goes from 1 to 0 it must be expected that β_2 is negative as the CPR should increase when maturity approaches.

β_3 is the long-term average. It is a constant in the prepayment function.

In *Table 6.4* the β-values are presented for Model 3

Debtor groups	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$
<200,000	9.9588	-2.7160	-3.0408
200,000 – 500,000	10.7259	-6.3743	-39.1176
500,000 – 1,000,000	131.4768	-5.4748	1.9670
1,000,000 – 3,000,000	128.8060	-11.5587	4.0351
>3,000,000	208.5544	-5.6737	2.1594

Table 6.4: β-values for Model 3

The model has an R^2 of 47.53%. The β_1-values are all positive and ranges from 9.9588 to 208.5544, which is a relatively large spread. The values are increasing as the remaining debt increase as expected. The β_2-values fluctuate from -2.7160 to -11.5587 and are negative as expected. The model is both subject to both convergence and local optimum problems despite numerous attempts. Model 3 will not be further analysed as Model 4 has a better fit.

Debtor groups	$\boldsymbol{\beta_0}$	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$
<200,000	2.0574	0.8007	-5.1464	-1.0112
200,000 – 500,000	0.1861	0.5922	-2.4357	-3.8793
500,000 – 1,000,000	0.7342	63.9367	-5.5437	-1.8433
1,000,000 – 3,000,000	0.6806	101.8822	-3.4166	-4.1857
>3,000,000	0.5367	63.8572	-0.3031	-1.6708

Table 6.5: β-values for Model 4

Model 4 explains the CPR_{act} better than Model 3 as it has an R^2 of 51.73%. This is not on a satisfactory level, but will be utilized as the prepayment model for this thesis, since the β-values are almost as expected. Model 4 includes a β_0 parameter that makes the CPR_{MD} proportional to the pool factor raised to a power. Hence, the refinancing gain and pool factor are affected by the values of both β_0 and β_1. Recall, the pool factor goes from 1 to 0, thereby a β_0-value of below (above) 1 will therefore increase (diminish) the pool factor.

All β_0-values are positive for all debtor groups. The highest β_0-value is found in the first debtor group. A value of approximately 2 will lower the pool factor and the CPR. In the remaining groups, the β_0-value in debtor group 2 will increase the CPR most as it has the smallest value. Intuitively, the low β_0 value should be in the debtor groups with the highest remaining debt. However, the β_0 value cannot be analysed isolated as its effect on the CPR is related to the value of β_1.

β_1 is positive for all debtor groups as to be expected, as the higher the gain becomes the more willing borrowers are to remortgage. It infers, that an increasing refinancing gain increases the CPR_{act}. The difference between the high remaining debt groups and the low remaining debt groups is of a considerable size. The largest parameter is found in the fourth group, while group three and five are around the same level and group one and two are both around 1. Intuitively, the parameters should be increasing as the remaining debt increases. However, when the remaining debt is larger, the loan most likely has a higher time to maturity and the borrower is willing to wait for a potentially larger refinancing gain. This is present in the β_2 estimate, which shows to be smallest for the largest remaining debt group. They are affected the least as time to maturity approaches, which means that there is a long time until maturity. The remaining β_2

estimates are all negative as to be expected. β_3 is the long-term average and does not directly relate to any of the independent variables.

In *Figure 6.2,* the CPR_{act} are plotted against the CPR_{MD}. All the points above the $45°$ line are due to a higher estimation of the CPR than the actual observed and vice versa. Hence, if the model had an R^2 of 1 all points would have been on the diagonal line.

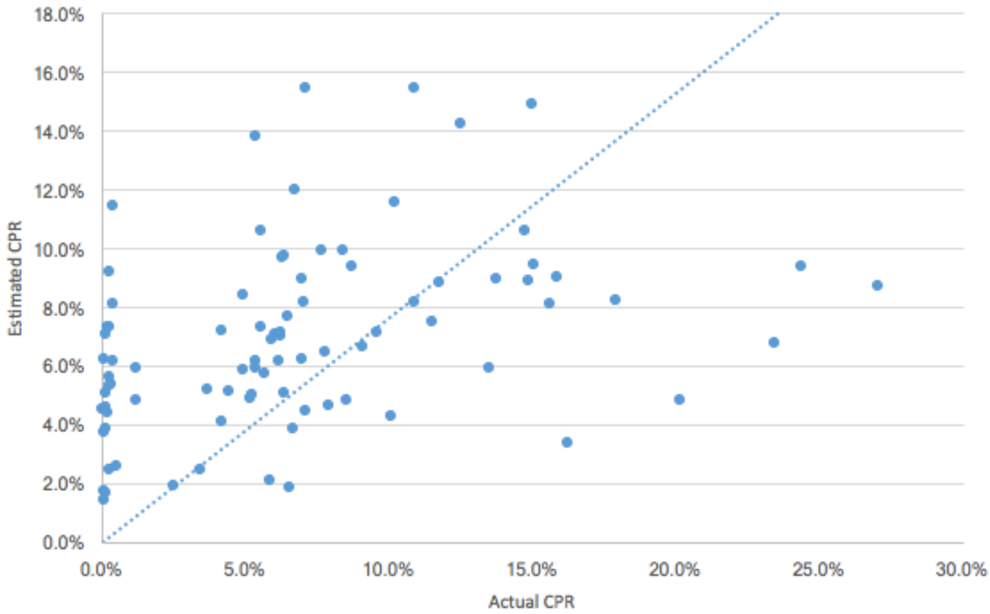


Figure 6.2: Actual CPR versus Estimated CPR

The CPR_{MD} consistently seems to return bigger estimates when CPR_{act} is around 0%. Generally, the model also seems to overshoot the CPR as most points are above the diagonal line. The model tends to estimate the CPR within the interval of 2% and 8% while the CPR_{act} is largely more spread. When the CPR_{act} is high, the model appears to undershoot the estimate.

The discrepancies in the expected values are most likely a consequence of the model being subject to convergence problems and having issues finding the global minimum when being fitted. Convergence problems are likely due to the standard normal distribution as it only takes values from 0 to 1. Actions to prevent this have been implemented[43] and many attempts have been made to rectify the errors. The Excel solver is not the best tool for estimating a model with so

[43] By using constraints and different solvers.

many parameters. The alternative would require an extensive workaround using mathematical software and has therefore not been done.

While considering that the model is based on a very limited dataset the result of getting an R^2 of 51.73% is somewhat impressive. I estimated my model based on a tiny fraction compared to other studies – e.g. Jacobsen use 69 bond series over a period of approximately four years (Jakobsen, 1992), Jacobsen and Svenstrup use 340 bonds over a period of approximately four and a half years (Jakobsen, et al., 1999), while Jacobsen, Svenstrup and Willemann use 288 bonds over five years (Jakobsen, et al., 2004). These studies all get R^2's between approximately 46% to 85%, but cannot be compared to my model as different independent variables are used.

In retrospective, a better approach would be to base the model on a higher number of observations and a greater variety of characteristics. The number of observations used for model estimation has a considerable effect on how well it can predict an outcome close to the actual CPR. It would require extensive work beyond the scope of this thesis to acquire and manipulate the data if 100+ bonds were included. However, it would have been more reasonable to merge the debtor groups, include more bonds and more payment dates. This would have resulted in a model based on more observations while being less complex, as less β-values were to be estimated due to few debtor groups. Finally, the model's R^2 would further increase as the number of independent variables increase since the CPR not only is explained by the three variables I have included in my model.

Nonetheless, Model 4 is used for further pricing of the callable bonds.

7 Valuation

The determination of the evolution of the future interest rates from *section 5* and the estimated prepayment model in *section 6* create the two essential elements to price a callable bond. Pricing bonds are normally done by simply implementing backward induction through the calibrated trinomial tree. The issue with this approach is that it assumes the cash flows at each node to be dependent on only the short-term rate at that note. The recombined tree vastly complicates this procedure as the embedded prepayment option has a path-dependent nature and is thereby no longer Markovian. The consequence of path dependence is that the future cash flows at an arbitrary node (i, j) are a function of both the prevailing interest rates and the path of the interest rates on their way to the node. This implies that there is not a single price at any given node, but multiple prices depending on history of the path to the node. The pool factor is a clear example of how path dependence affects the prepayment behaviour. The most common solution to this is a Monte Carlo simulation, but this will be implemented in the trinomial tree by an alternative method.

7.1 Lattice implementation of the prepayment model

The prepayment model from *section 6* is used to estimate the CPR based on the independent variables while still accounting for the different debtor groups. The CPR is to be determined in all (i, j) nodes of the tree[44]

$$CPR_{i,j,d} = \Phi\left(\beta_{1,d} g_{i,j,d} * PF_{i,j,d}^{\beta_0} + \beta_{2,d} * TTM_i + \beta_{3,d}\right) \qquad \text{eq. 7.1}$$

where (i, j) is a node and d is henceforth denoted as the debtor group. It will now be described how the individual variables are adjusted to fit the trinomial tree.

7.1.1 Refinance gain

The refinance gain is not path-dependent and is determined by forward induction. The computation is similar to *eq. 6.11*, and is calculated for each debtor group at each node

[44] The notation has slightly changed from *section 6* to match the notation from *section 5*.

throughout the tree. The refinance rate is obtained from *Nordea Analytics* from March 17, 2017. This spread is assumed to be constant in the entire period and is applied to the rates in all nodes.

7.1.2 Pool factor

The pool factor is a known path-dependent variable. The pool factor is affected by exposure to remortgaging opportunities in the past. When a series moves into the money, the most reactive borrowers remortgage and exit the series, leaving behind the less reactive borrowers. A series which have been in the money for an extended period in the past is likely to be less reactive than an otherwise similar series which moves into the money for the first time since origination. Hence, the interest rate level at earlier nodes influence the prepayment activity and thereby the pool factor. The method of pricing securities was first described by Hull and White (Hull, et al., 1993; Hull, et al., 1990). The main idea is to augment the state space with additional state variables to represent the past movements. This has shown to be as fast and as accurate as standard Monte Carlo simulation (Svenstrup, 2001). This method considers which nodes the current node has passed and adjusts the pool factor by using the risk-neutral probabilities from the tree. Hence, the procedure is computed by forward induction and for all debtor groups at each node. The pool factor is calculated as

$$PF_{i,j,d} = \begin{cases} PF_{i-1,j,d} * (1 - CPR_{i-1,j,d}) & for\ j = j_{max}\ and\ j = j_{min} \\ PF_{i-1,j,d} * \left(1 - CPR_{i-1,j,d}\right) & for\ 0 < i < j_{max}\ at\ node\ |j| = i \\ PF_{i-1,j+1,d} * p_u * \left(1 - CPR_{i-1,j+1,d}\right) & \\ +PF_{i-1,j,d} * p_m * \left(1 - CPR_{i-1,j,d}\right) & otherwise \\ +PF_{i-1,j+1,d} * p_d * (1 - CPR_{i-1,j+1,d}) & \end{cases} \qquad \text{eq. 7.2}$$

At any node where $t > 0$ the pool factor and CPR is known. This method modifies the pool factor in the current node depending on the pool factor and CPR from earlier connected nodes while adjusting for the branching probabilities. Hence, if the three earlier nodes all had a high CPR then the CPR at the current node will be reduced.

At $t = 0$, the pool factor is given by

$$PF_{0,d} = \frac{current\ outstanding\ amount_{0,d}}{maximum\ outstanding\ amount} \qquad \text{eq. 7.3}$$

7.1.3 Time to maturity

Time to maturity is a relative measure of the time until maturity. It is similar to *eq. 6.15* and only adjusted to acquire input from the tree as seen in *eq. 7.4*.

$$TTM_t = \frac{Maturity\ date - coupon\ date}{30 * 365} \qquad \text{eq. 7.4}$$

No further adjustments are necessary as the time to maturity solely is a function of time and not of the tree state, debtor group or preceding nodes. Hence, it is the same value for all nodes at for each t.

7.2 Valuation

The callable bond is priced by discounting the future cash-flows as part of a backward induction procedure. The cash flows are not known and must be determined as the borrower has the optionality of remortgaging and are thereby affected by the CPR. At each node, the value of the callable bond is thus a function of the value of prepaying the loan, the value of not prepaying the loan and the CPR (Jakobsen, 1992)

$$P_{i,j,d} = CPR_{i,j,d} * W_t + \left(1 - CPR_{i,j,d}\right) * V^{+}_{i,j,d} \qquad \text{eq. 7.5}$$

$$P_T = PMT_T \qquad \text{eq. 7.6}$$

where at node (i, j) for group d, W is denoted as the value of prepaying the loan, while V^{+} is denoted as the value of the loan if no prepayment occurs. Thereby, the price is determined by the fraction of borrowers who prepay their loan the by the fraction of borrowers who continue their current loan. At expiry $t = T$, the value of the callable bond is given by the payment, as there are not future occurring payments. The value of W is given by

$$W_t = RD_t + PMT_t \qquad \text{eq. 7.7}$$

The value of prepaying the loan at time t is the remaining debt RD and the payment PMT. W is neither dependent on path, state nor debtor group, as it is assumed that the remaining debt at time $t = 0$ for all groups is DKK 100. It is thereby only dependent on the coupon rate and the remaining time to maturity. The value of V^{+} is

$$V^{+}_{i,j,d} = PMT_t + e^{-r_{i,j}\Delta t}\left(P_{i+1,j+1,d} * p_u + P_{i+1,j,d} * p_m + P_{i+1,j-1,d} * p_d\right) \qquad \text{eq. 7.8}$$

The value of the loan if no prepayment occurs, is the value of the callable bond from $i + 1$ discounted with the short rate and adding the payment. This procedure is calculated for all debtor groups at each node. It should be noted, that the node at $t = 0$ is the payment date of April 1, 2017. The period length from only $t = 0 + \Delta t$ is adjusted to be $\frac{77}{365} = 0.22$. In the remaining periods $\Delta t = 0.25$.

At time $t = 0$, the callable bond will be priced for each debtor group d. The value of the callable bond P_0 is an average of the value of each debtor group weighted by the relative size of the debtor group at time $t = 0$. This assumes that the relative size remains constant until maturity.

$$P = \sum_{d=1}^{D} w_d * P_d \qquad \text{eq. 7.9}$$

where

$$w_d = \frac{Current\ outstanding\ amount_{0,d}}{Current\ outstanding\ amount_0} \qquad \text{eq. 7.10}$$

As mentioned earlier, this is the price found at the payment date of April 1, 2017 and equals the dirty price. To find the clean price on March 17, 2017, the accrued interest must be subtracted from this price (Hull, 2015)

$$Clean\ price = Dirty\ price - accued\ interest$$

$$Clean\ price = Dirty\ price - \left(RD_0 * coupon * \frac{days\ from\ settlement\ date\ to\ the\ coupon\ date}{days\ in\ coupon\ period}\right) \qquad \text{eq. 7.11}$$

7.3 Price-100 function

Alternative to prepaying the loan, the borrowers are permitted to buy back the bonds at the market price and redeem their loan through the delivery of the bonds to the mortgage banks. Borrowers will use this opportunity when the price is below par as the value of debt is lower than the cost of calling the loan. This should be implemented, as the estimated CPR can be positive even though the price is below par. If this is omitted, the model will assume that the bonds are called despite the price being below par. The price of the bond will then be lower as the model predicts higher prepayments. The pricing formula from *eq. 7.5* is rewritten and adjusted to ensure that the value of prepaying the loan is smaller than the value of not prepaying the loan and otherwise the CPR is set to zero.

$$P_{i,j,d} = V^{+}_{i,j,d} - CPR_{i,j,d} * \left(V^{+}_{i,j,d} - W_t\right) \Leftrightarrow \qquad \text{eq. 7.12}$$
$$= V^{+}_{i,j,d} - CPR_{i,j,d} * \left[V^{+}_{i,j,d} - W_t, 0\right]^{+}$$

The above adjustment assumes that the borrowers only convert their loans when the value of prepaying the loan is smaller than the value of not prepaying their loan.

7.4 Option-Adjusted Spread (OAS)

The theoretical price from the model is in general different from the observed market price. This is the case as the model estimates the value of the option that the borrower has and not model factors like credit- and liquidity premiums or market-psychology. The OAS provides investors with a basis for comparing the value of callable bonds with other investment alternatives (Nykredit Markets, 2016)

The difference between the theoretical price and the market price expresses the market's estimation of the price of additional premiums for a specific bond. Instead of expressing this difference in the price, the premiums are expressed as the spread to the short rate curve. This spread is called the Option-Adjusted Spread. OAS is applied to the rate at each node and solved so the model's price equals the market's price. If the theoretical price is larger (smaller) than the market's price, then the OAS is positive (negative). Additionally, it should be included that besides the premiums the OAS also accounts for errors caused by the pricing model (Jakobsen, et al., 2000).

7.5 Application and results

The valuation is done in Excel[45] in the tree constructed in *section 5.4*. At each node, the independent variables, the CPR and price are estimated for each debtor group and are implemented as stated in previous sections. The CPR is calculated based on the estimated parameters of Model 4. The individual bonds are priced in their own individual tree. Furthermore, an OAS tree is created for each bond and the OAS is solved by using the GRG non-linear solver in Excel.

Ten bonds are priced and selected to represent different coupon rates, maturities mortgage banks, and pool factors. Three of the ten bonds are bonds used for model estimation while the remaining seven are out-of-sample bonds. The data is provided by Reuters Eikon, the CK-series and *Nordea Analytics*. The results are presented in *Table 7.1*.

ISIN	Coupon (%)	Time to maturity (years)	Market price	Model price	Deviation (%)	OAS (bp)
			In-sample bonds			
DK0009753469	5	18.51	113.75	115.75	1.73	199.10
DK0002023209	4	24.52	108.18	103.39	-4.63	-507.14
DK0009282329	4	24.52	109.25	105.77	-3.29	-360.89
			Out-of-sample bonds			
DK0009284028	3	9.51	104.13	97.55	-6.75	-923.46
DK0004715505	4	18.52	108.90	101.36	-7.44	-843.50
DK0002021773	4	24.52	111.10	95.93	-15.82	-1,335.94
DK0009775355	4	24.52	108.8	101.91	-6.76	-736.63
DK0009766289	6	21.52	113.75	130.69	12.96	1,548.01
DK0009763260	5	21.52	112.15	117.69	4.71	541.95
DK0009292047	2	15.51	102.75	93.08	-10.93	-1,218.30

Table 7.1: Valuation of callable mortgage bonds

[45] See *Appendix 1*.

The results are split by in-sample and out-of-sample bonds and the market price is from March 17, 2017. It can be seen, that the in-sample bonds are priced more accurately than the out-of-sample bonds. This is expected, as the in-sample bonds have been used to estimate the model.

Intuitively, there would be a correlation between the deviation and the time to maturity, but this does not seem to remain true. A bond with a longer maturity pays more coupons and its price is thereby based on a higher number of CPR estimates. The bonds with longer maturities do not exhibit the highest deviation. Similarly, a high coupon rate would infer a higher uncertainty in the pricing. This is caused by higher ordinary drawings and thereby a bigger cash-flow deviation, in case of an erroneous CPR. Though, the two of bonds with the highest coupon rates (5%[46] and 5%) do not follow this hypothesis, while the bond with 6% coupon rate has a relatively high deviation compared to the market price. An explanation for the high deviation of the 6% and 2% bond is that there were no bonds with an equivalent coupon rate in the estimation data. The prepayment behaviour of 6% and 2% bonds are not captured by the model and it therefore misprices these substantially. A bond with a coupon of 3% or similar time to maturity is not used in the estimation sample and must therefore be expected to deviate from the market price. Its deviation is however noticeably smaller than the deviation of the 2% bond, which makes the hypothesis plausible in relation to the remaining time to maturity, as it has fewer cash flows and thereby has fewer erroneous estimations of the CPR.

The majority (7/10) of the bonds are estimated below the market price. From *Figure 6.2,* it is visible that the Model 4 tends to overshoot the estimated CPR compared to the actual CPR in the estimation sample. This effect is clearly seen in the in-sample bonds as two of three bonds are estimated below the market price. It is reasonable that this effect applies to the out-of-sample bonds in a similar manner. The relation between the estimated CPR and the price of a callable bond is present in *eq. 7.12*, where a high CPR equals a lower price.

The calibration of the trinomial tree was not within the stipulated errors interval of 1%. This was caused by the illiquid swaptions and local minimum problems. In a low-interest-rate-environment

[46] As this is an in-sample bond it cannot be directly compared with the out-of-sample bond.

this can result in relatively high probabilities for negative short rates. When discounting with negative rates, the value of V^+ will increase and thereby increase the price of the callable bond. However, the calibration could instead have caused a too low probability to negative short rates or a too high probability for positive short rates causing the bonds to be priced below the market price. The direction of the effect caused by the (mis)calibration is hard to determine, but likely plays a role, as the volatility parameters determine the mean reversion and the volatility of the evolution of the short rates.

At first glance, it might seem puzzling that the four 4% coupon bonds with similar time to maturity are priced so differently. However, this is explained by the bonds having different pool factors and the outstanding amount relative to the maximum outstanding amount of each debtor group.

The relationship between the OAS and the deviations in the price are as expected. When the bond prices are below the market prices the OAS is negative and vice versa. The OAS deviate substantially for all bonds, potentially implying that the bonds with a positive (negative) OAS are undervalued (overvalued) due to various premiums and factors which are not modelled. A more likely explanation is that a large part of the OAS is caused by the mispricing of model.

The model's R^2 is 51.73%, which indicates that it cannot explain approximately 50% of the variance of the actual CPR. On one hand, it is impressive, that the model can estimate prices in the region of the market prices. Recall, the model is only estimated by using eight different bonds with coupon rate of 4% and 5% and with remaining maturities of approximately 21 to 27 years[47]. On the other hand, it is expected that there are discrepancies when pricing the bonds due to its poor fit. However, it is expected that increasing the number of bonds and extending the sample period will benefit the prepayment model and the results. Estimating the model on the basis of more data will thus increase the accuracy of the estimated prepayments and thereby the accuracy of the pricing.

[47] As of April 1, 2014.

8 Conclusion and further work

In this thesis, a pricing model for Danish callable mortgage bonds has been produced. The callable bond is a complex bond to price as it contains a prepayment option allowing the borrowers to prepay the loan at par. Borrowers do not exercise the option at the most optimal time due to semi-rationality and heterogeneity, which must be captured by the pricing model.

The model consists of two components; the interest rate model and the prepayment model. It is created on empirical data and its accuracy has been tested on both in-sample and out-of-sample bonds.

The input for the interest rate model, a term structure of zero-rates, has been derived by using 15 Danish interest rate swaps with maturities ranging from 1 to 30 years. Initially, the Nelson Siegel model is used to interpolate annual swap rates as linear dependency is required for further work. The zero-rates are then derived by bootstrapping the annual swap rates through matrix algebra. Finally, to match the cash-flow of the callable bonds, the Nelson Siegel model is again applied to interpolate the quarterly zero-coupon rates and creating a well-fitted curve. Separating the interpolation into two is necessary, as the cash flows must have the same coupon dates when bootstrapping the zero-coupon prices.

The generalised Hull-White model is selected as the interest rate model to describe the evolution of the short rate. This model is most suitable to apply to the current economic environment as it allows for negative interest rates and incorporates mean-reversion. The discrete version of the model is utilised in form of a trinomial tree, as it is essential for pricing the Bermudan option embedded in the callable bond. The Hull-White model implements the current term structure of interest rates then shifts the tree to apply the correct time-varying drift and finally calibrates it to the present volatility structure by fitting the a and σ parameters to market prices of actively traded Danish swaptions. The stipulated pricing errors of a maximum of 1% were not abided due to convergence and local minimum issues. However, the calibration was accepted as three out of the five swaptions were within bearable boundaries of the 1%.

The prepayment model is based on the required gain model, which accounts for the semi-rationality within the borrowers. The estimation is done with a mixture distribution model which is based on the published CK-data from the mortgage banks. The model uses the refinance gain, the remaining time to maturity and the pool factor as independent variables. The economic gain from remortgaging is not directly proportional to the remaining size of the loan, due to the refinancing costs, the heterogeneous borrowers are divided into five homogeneous debtor groups depending on the remaining loan. The refinancing costs are obtained from the price sheets of the mortgage banks and computed for each of the debtor groups. The debtor groups can to some extent capture the difference in the remaining debt and their subjective criteria for the required gain. Several prepayment models are estimated based on eight bonds over a period of 12 payment dates. The best model had a R^2 of 51.73% and featured parameters aligned with the expectations except for few discrepancies but did tend to overshoot the estimated CPRs.

The pricing model is a combination of both the interest rate model and the prepayment model. Initially, time to maturity and the refinancing gain is calculated at each node while the latter is computed for the five debtor groups. The pool factor is path-dependent and is implemented by augmenting the state space with additional state variables to represent past movements. This method considers which nodes the current node has passed and adjusts the pool factor by using the risk-neutral probabilities from the tree. All components are then available in each node allowing the CPR to be estimated. Finally, the clean price of the callable bond is calculated through backward induction while taking into consideration that borrowers do not utilise the prepayment option if the price is below par.

The model has priced a total of ten bonds. Three are in-sample bonds while the remaining are out-of-sample. As expected, the in-sample bonds were more accurately priced than the out-of-sample bonds. The deviation of the in-sample bonds ranges from -3.29% to 1.73% and -15.82% to 4.71% for the out-of-sample bonds. Intuitively, the bonds with the highest time to maturity and highest coupon rates are expected to deviate the most as there are respectively a higher number of CPR estimates and a potentially larger cash-flow deviation arising from an erroneous CPR. The model's results make it hard to accept the above hypothesis. The majority of the bonds are priced below the market price, which harmonises with the prepayment model overshooting the CPR. The relationship between the OAS and the deviations in the price are as expected. As the OAS is relatively high, the most likely explanation is that a considerable part of it is caused by model mispricing.

The deviations in the model's pricing compared to the market price are likely a consequence of the prepayments model R^2 of 51.73%. The estimation data for the model has been limited, and it is therefore not unreasonable to expect that a more accurate model could have been constructed by increasing the sample size. If the sample size is increased by both the amount of bonds and by extending the estimation period, the model is prone to more accurately estimate the CPR and thereby provide a more accurate pricing of the callable bonds. Additionally, more independent variables can be added to the model as factors like level of long-term interest rate or refinancing campaigns very likely can explain some variation in the actual CPR.

To get a greater insight into the prepayment model, it is beneficial to examine the relationship between the independent variables and the estimated CPR. Any discrepancies in the expectations of the variables will be exposed and the sample size and period can on this basis be reconsidered. In practice, callable bond investors assess the risk measures and investment potential through option-adjusted key figures. In addition to the OAS, the option-adjusted duration can be added for calculation of hedge ratios, or the option-adjusted convexity to examine the relationship between the price and the yield of the callable bonds.

Finally, it can be concluded that a prepayment model based on empirical data has been successfully developed. However, as shown by the pricing inaccuracy, the model does leave room for further refinements as suggested in this conclusion.

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10 Appendix

10.1 Appendix 1

The content of the Excel files and their relation to the chapters are explained below:

10.1.1 Chapter 4

Excel file: *"Chapter 4 – NS BS NS"*
Sheet 1, NS: Applying the Nelson Siegel model on the swap rates to create annual rates
Sheet 2, Bootstrapping: Bootstrapping annual zero-coupon rates.
Sheet 3, NS ZCB: Applying the Nelson Siegel model on the annual zero-coupon rates to create quarterly rates.
All the above is related to *section 4.3*.

10.1.2 Chapter 5

Excel file: *"Chapter 5 – Tree"*
Sheet 1, Hull-White: The Hull-White trinomial tree built for *section 5.4*.
Sheet 2, Cali: The pricing of a single swaption in the tree for *section 5.3.2*.

Excel file: *"Chapter 5 – Cali"*
Sheet 1, Swaption: Calibration procedure done with VBA for *section 5.4.*

10.1.3 Chapter 6

Excel file: *"Chapter 6 – Costs"*
Sheet 1, Costs: Costs calculation for *section 6.6.5*.

Excel file: *"Chapter 6 – Model"*
Sheet 1, Data: Compiled CK-92 Data and actual CPR calculations as described in *section 6.3.*
Sheet 2, FYP,total: First-year payments without accounting for debtor groups.
Sheet 3, FYP,i1: First-year payments for debtor group 1.
Sheet 4, FYP,i2: First-year payments for debtor group 2.
Sheet 5, FYP,i3: First-year payments for debtor group 3.
Sheet 6, FYP,i4: First-year payments for debtor group 4.
Sheet 7, FYP,i5: First-year payments for debtor group 5.
Sheet 8, Model 1: Parameter estimation for model 1
Sheet 8, Model 2: Parameter estimation for model 2
Sheet 8, Model 3: Parameter estimation for model 3
Sheet 8, Model 4: Parameter estimation for model 4
All the above is related to *section 6.7*

10.1.4 Chapter 7

Excel file: *"Chapter 7 – Pricing"*
Sheet 1, Bonds priced: Overview of the bonds priced, their price, deviation and OAS.
Sheet 2, 1: Pricing procedure for bond 1
Sheet 3, 1 OAS: OAS calculation for bond 1
(…)
Sheet 20, 10: Pricing procedure for bond 10
Sheet 21, 10 OAS: OAS calculation for bond 10.