# A COMPARISON OF THE BLACK-LITTERMAN MODEL AND THE MEAN-VARIANCE APPROACH

Generating investor views through premium prediction models

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# Abstract

The scope of this thesis is to examine asset allocation using Markowitz's Modern Portfolio Theory and the Black-Litterman model. Further, it compares the performance of the allocation models over an out-of-sample period running from 01-01-2000 to 31-12-2018, reflecting a real investment scenario. The analysis applies a simple multi-asset portfolio consisting of equities (SPX Index) and bonds (LUATTRUU Index).

Generating portfolio allocations using historical measures has often shown to be imprecise, which can be seen in the mean-variance optimization process. The implication is finely solved by Black and Litterman (1990), who used equilibrium returns derived from the Capital Asset Pricing Model as a benchmark for the expected excess returns of the portfolio. Further, the model gives the investor the possibility of combining their subjective views of the return movements with quantitative benchmark data, using Bayesian statistics. Equity and bond prediction models are applied to determine the individual beliefs of the investor, and the respective uncertainty regarding the established views.

The thesis investigates the out-of-sample performance of mean-variance, CAPM and Black-Litterman portfolios using rolling window estimates of the expected return vectors and covariance matrices. The models are evaluated by performance measures such as the Sharpe ratio, the certainty equivalent, M-squared and t-statistics, in addition to presenting the cumulative realized portfolio returns. The overall conclusion of this study provides evidence that the Black-Litterman portfolio performs more inferior than the traditional mean-variance portfolio, especially during recessions and crisis, when evaluating the performance measures stated above. All portfolios show significant t-statistics; however, their performance appears to differ substantially over the total out-of-sample period.

# Forewords and Acknowledgements

First and foremost, we would like to thank our supervisor, Kasper Lund-Jensen, for insight, comments and supervision. We are sincerely grateful for the academic help throughout this study. He made it possible for us to explore an unknown model, providing us with intuitive recommendations. This thesis is written on behalf of the interest in applied portfolio management, which we have obtained during our time as students at CBS. The process has been exciting but has included challenging times. Moreover, we want to thank our family and friends who have helped us along our process.

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# 1. Introduction

## 1.1 Introduction to investment

Portfolio managers and traders have had disagreements regarding their preferred choice of asset allocation approaches over the years. It is often the case that they favour either quantitative- or qualitative approaches or passive allocation models as opposed to active allocation methods. Typical quantitative strategies often require historical observations of the asset prices to optimize the portfolios or to generate forecasting models. These are commonly known to require a lot of assumptions and constraints to perform optimally, which can be both expensive and time-consuming, leading to estimation errors. The other central aspect tends to focus on a qualitative approach where the investor, for instance, performs market research and fundamental company valuations to express their beliefs about the future prices of the assets in question.

The ground-breaking work of Markowitz (1952) changed how academics and practitioners within finance looked at portfolio allocation choice. The mean-variance optimization approach is well-known in the academic world, as well as for practitioners within asset management. Markowitz's Modern Portfolio Theory presented an optimization of the risk-return trade-off between assets, which was quite revolutionary at this point. The model made it possible to diversify and allocate assets in the investors' portfolio in a sophisticated manner. Despite the excitement regarding the model at first, the mean-variance approach has, both empirically and practically, shown to have quite a few shortcomings and sparse estimation over time. The framework of Markowitz applies historical return and volatility as a proxy for future expectations in the allocation model. Consequently, it can be an inaccurate assumption that the returns and variances will be the same as they have previously been in the future. It can be a discussion on whether this baseline is an appropriate assumption of the future realizations of the asset returns.

Black and Litterman (1990) delicately solves this issue by applying an equilibrium return, originating from the Capital Asset Pricing Model, as the baseline for defining the expected return vector. A central motivation for the construction of the Black-Litterman approach rests on a desire to combine the contradicting opinions mentioned above, about quantitative and qualitative asset allocation methods, which they do by letting the investor allocate individual views on the assets of their choice in the portfolio. The incorporation of the views in the model is done by the use of Bayesian statistics, so that the investor is able to combine their views with the market equilibrium. As a result, the models should supposedly provide more stable and intuitive weights than those obtained in traditional allocation models.

The application of the Black-Litterman framework is relatively straightforward, although, establishing the investors' subjective views, and defining its uncertainty, can be demanding and challenging since a tremendous number of factors are impacting the financial markets. The market movements and trends have had massive changes over the years. For instance, a remarkable factor in the stock-bond relationship has been the transition of the sign changing from positive to negative. The relationship between equities and bonds is one of the fundamental building blocks of portfolio asset decisions, which could change the whole way investment managers view their allocations, combined with the way they hedge and diversify. Further, many have tried to find clever and efficient ways to determine individual investor views that would not require an extensive use of time, like doing market research, using analysts' recommendations, and fundamental valuations. Finance practitioners have found a generous number of variables that allegedly have the power to predict the future return of an asset or an index. A sub-goal of this analysis will be to exploit the literature on stock- and bond predictability, and thereby apply it as a tool to solve the problem of generating subjective views on the expected return of an asset. This paper will use a quite straight forward premium prediction model for equity index and bond index.

## 1.2 Research question

The project seeks to investigate the construction of mean-variance portfolios and the Black Litterman model. The goal is to find empirical evidence of the relative performance of the Black-Litterman portfolio, compared to an optimized mean-variance portfolio. More specifically, the tangency portfolio. The performance will be tested over a significantly large out-of-sample period, where we observe different kinds of market movements. A second focus is given to the determination and generation of the individual investors market views, used to adjust the expected excess return vector in the Black Litterman model. The main research question will be as follows:

> *How do we generate portfolios using the Black-Litterman model and the mean-variance approach, and how well do they perform out-of-sample?*

This will be investigated by answering the following sub-questions:

- *How do we construct portfolios applying Markowitz's Modern Portfolio Theory and the Black-Litterman approach?*
- *Can equity and bond prediction models provide us with intuitive and well-working investor views in the Black-Litterman model?*
- *How do the portfolio weights differ when comparing the approaches, and how does it affect the risk allocation of the portfolios?*
- *How will the mean-variance, CAPM and Black-Litterman portfolios perform out-of-sample?*
- *How does the asset allocation perform and behave when comparing it over different periods?*

The structure of this study will comprise seven sections, illustrated in Figure 1. The sections are organised the following way; Section 1 introduces the project with a primary research question (and relevant sub-questions) followed by the delimitations of the analysis. Section 2 and 3 overall contains the investigation of this study, which is the theoretical body of the project. It provides an overview of the literature and previous research done on the subject, which is after that followed by the theoretical approaches concerning Modern Portfolio Theory (MPT) and the Black-Litterman model (BL). Further, a description of the data and methodology applied during this paper is present in Section 4. Section 5 gives a detailed description of the implementation with the belonging empirical measures, including computations of the out-of-sample performance measures. Proposals for further research and possible improvements will be discussed in Section 6. Lastly, the final section provides an overall conclusion of the findings arising from the research and analysis.

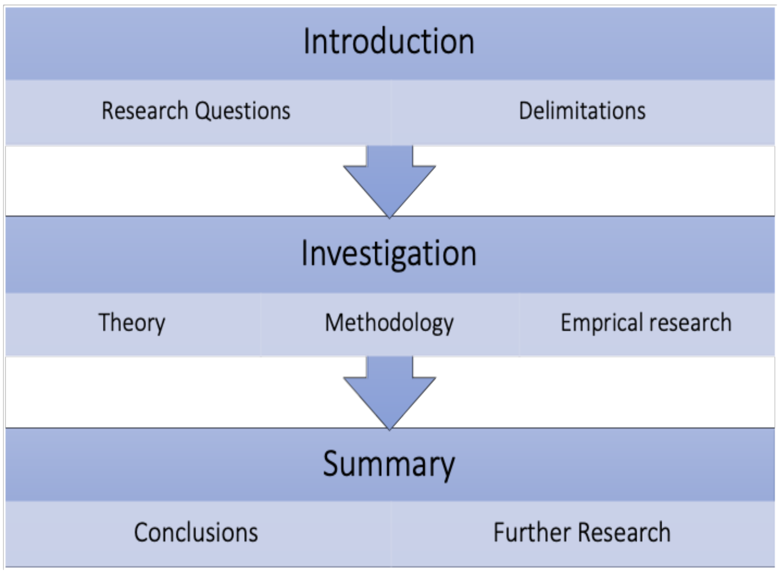


Figure 1: Overview of the structure of the study

## 1.3 Motivation

The traditional mean-variance approach is a well-known portfolio theory, which has gotten a great deal of attention in academic curriculums. While the model gives intuitive understanding, it is not widely applied in practice. Numerous models have been investigated throughout the years, but no models have been acknowledged as a standard procedure in the real-world setting. The part of combining passive and active portfolio allocation is a curious strategy of optimization, thus investigating the Black-Litterman model is, in fact, relevant and exciting research in our opinion. The model gives the ability to implement practical observations of an investor to be thereby applied in asset allocation processes. We find that many have done investigations into the Black-Litterman model to look at the features of the model, which includes the view generating process. To our knowledge, it did not appear to exist much research using premium prediction models to determine the investor views. Due to this fact, it has motivated us to analyse this possibility. Lastly, it is interesting to look at the change in market movements over the total sample period. The correlation in stocks and bonds have had a shift in the sign over the period, and it is an important input in the allocation models. It made us curious to which extent this proposes changes to our portfolio allocation over time, or that perhaps rolling estimation schemes can help to sort this issue.

## 1.4 Delimitations and assumptions

The thesis will evaluate portfolios consisting of a simple multi-asset investment universe based on stocks and bonds, where the Standard & Poor 500 index is representing the equity market. In addition, the 10-year Treasury index will serve as an approximation of the bond market. In order to determine the market portfolio, the portfolio is excluding asset classes like commodities, real estates, derivatives, currencies, etc., since the Black Litterman model requires the approximation of the market portfolio. Therefore, anticipated to limit such asset classes. i.e. a combination of the equity- and the bond index will be considered as the market portfolio. The investment universe is focusing only on the U.S. market, which is mostly due to previous research regarding the return predictability in the U.S.

Due to the simplicity of this study, the finding will not include any results adjusted for transaction costs. Furthermore, the mean-variance optimization and the CAPM assumes no taxes. Hence taxes on gains and deductible taxes will not be taken into account during the thesis and will show homogenous investors. The trading costs are, actuality, relatively low for the treasury index and the S&P 500, because these are only trackers of the market. There is no active management of these indices, like mutual funds, which is why it must be cheaper to trade. Because the inputs are based on historical information, primarily on the tangency portfolio, no portfolio constraints are imposed in the analysis. Historical information is known to generate outliers' weights, or negative positions for the mean-variance optimization, therefore it could be desirable to apply constraints. However, this study allows for short selling.

To meet requirements of statistical modelling analysis, it must be secured to have a large sample size, therefore applying data with a span of 40 years. However, the relation of the data is in danger of changing over time. Using shorter periods, i.e. rolling windows, could improve the sample forecast. Using a rolling window is determined to make the statistical inference more robust in the application of historical data. Also, different economic states, including the bull and bear markets, are present within the data frame. The dataset provided by Goyal and Welsh (2007) only contains data until December 2018, hence the rest of the analysis will be restricted to apply data within the same period, further, to provide consistency.

# 2. Literature review

This section seeks to describe prior literature and empirical evidence found on the relevant subjects. A special focus will be given to subjects such as shortcomings/limitations of Markowitz's modern portfolio theory as an explanation and how the Black Litterman model can be more intuitive to use for portfolio allocation. Furthermore, there will be described the findings on the performance of mean-variance portfolios (and Black-Litterman portfolios) over time. Lastly, the literature on stock- and bond predictability will be presented for the explanation of the generation of views of the Black Litterman portfolios.

## 2.1 Modern portfolio theory

How to allocate assets is one of the most important decisions that investors take at the very beginning of their investment process. Necessarily, investors can allocate their portfolio by reducing risk through diversification. This is presented by the modern portfolio theory, which comprises mean-variance optimization, arguing that the investor can create the optimal portfolio by maximizing the return by the optimal risk. This theory has become one of the most traditional portfolio allocation theories among investors and researchers. Harry Markowitz created and pioneered the theory, published in the essay "Portfolio Selection" in the Journal of Finance in 1952. He argues that the value of additional security added to a portfolio should be measured with the relationship to all other securities in the portfolios. Notably, he showed that the variance of return was an essential measure of the portfolio risk under a given set of assumptions. This so-called mean-variance optimization was the beginning of the concept of diversification and the capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965).

When testing whether the Mean-Variance portfolio is efficient based on a portfolio where all assets are risky, it is equivalent to testing the validity of the Capital Asset Pricing Model. This was especially reviewed by theorists, e.g. Roll (1977) and Ross (1977), whose idea was an unobservable market portfolio, and the creation of the actual market portfolio was impossible (it was impossible to create the actual market portfolio). Numerous empirical examinations, among Gibbons (1981), Gibbons et al. (1989) & MacKinlay and Richardson (1991), provided evidence for the inefficiency of the market portfolio, where the proxy typically is far away from the efficient frontier. Briére et al. (2013) support this conclusion, where they apply the same examinations in their empirical analysis. They find no mean-variance efficiency of the market portfolio for the U.S. equity market.

A variety of literature regarding portfolio selection has taken application in the modern portfolio theory; however, the approach has suffered much criticism during the years applied in the real asset management setting. It is remarkable for the mean-variance to be highly sensitive to small variations in the model input (expected return, variance) since small changes in expected return can lead to drastic changes in the portfolio construction. In other words, the optimal portfolio weights are sensitive to parameter estimates, especially the mean return vectors. Michaud (1989) defines this sensitivity as "error-maximization" of the risk-return estimates (Michaud, 1989 p. 33). This phenomenon indicates extreme portfolio reallocations when the mean-variance estimation overweight in assets with a high expected return, negative correlation, and small variance, oppositely, underweights assets that have a lower expected return, positive correlation, and high variance. These assets are those having high exposure to estimation errors, which often tend to give poor out-of-sample results (Unger, 2015). This statement is supported by Jorion (1985) & Merton (1980), who find difficulties when estimating the expected return under the assumption of the quadratic utility preferences of portfolio theory. They argue, is one of the main reasons why mean-variance efficient portfolios perform poorly out-of-sample.

Several authors have come up with solutions to the shortcomings of the MV, concerning the error-maximization. Jorion, (1985 and 1986) suggests the Bayesian method as input variables for the mean-variance analysis. This Bayesian approach results in decreasing sample returns, shifting towards a minimum variance portfolio, which is known as Bayes-Stein shrinkage estimation. Another method of the Bayesian approach proposed by Pastor (2000) and Pastor and Stambaugh (2000) builds on the prior beliefs of an asset allocation model, where the investor believes can be an essential determination model for decision models.

## 2.2 The Black-Litterman model

The first publication of the Black-Litterman asset allocation model was issued in 1990 by Fisher Black and Robert Litterman. They suggested a model that would solve the problem of the less natural results and weights arising from traditional mean-variance optimization, as stated above, by incorporating investor views with historical return observations. It made it easier for portfolio managers to combine quantitative and qualitative investment strategies, which had been a conflict of interest over the years. This approach allows for the investor to assign views on asset returns, either as a relative measure to another asset or as an absolute view. The investor can select views to a self-chosen number of assets, in which, they have an opinion that deviates from the equilibrium return, also known as the market portfolio. The market portfolio they define as the reference point of the investor views, which is one of the crucial features of the approach. Black and Litterman (1990) apply their model on a global equity portfolio. However, they argue that the model conveniently can be applied to a wide spectre of asset classes such as equities, fixed income, etc.

In 1999, He and Litterman published an article that sought to explain the intuition behind the Black-Litterman model by comparing it with the mean-variance optimization approach. He and Litterman (1999) argue that the mean-variance asset allocation process often suffers from unstable weights, and the empirical results are sensitive to changes in the model input and often appear not to be especially intuitive. Additionally, they present various ways where the Black-Litterman asset allocation model provides improved and more intuitive results showing this with multiple examples. Due to properties like establishing investors' views with the equilibrium excess return vector as a starting point, investors can adjust their views from this CAPM equilibrium given personal interpretation of the future excess return of an asset. Other papers such as Cheung (2009) also strive to explain the workings of the Black-Litterman model, in addition to specifying the assumptions of the model and suggesting methods for coping with large portfolios within the model. Lastly, Izadorek (2004) presents a “Step-By-Step” approach to easily apply the model and understand the workings of it. They can be reviewed for further assessment.

Various academics provide an overview of the Black Litterman model and presents us with examples of the application and generation process of the model, and also, discussion of the estimation parameters. Satchell and Scowcroft (2000) presented a paper called “*A demystification of the Black-Litterman model”* focusing on the quantitative and mathematical approach of the asset allocation model, especially the Bayesian framework used to incorporate the individual investor opinions with quantitative data to form new opinions. This is also known as the “Alternative Reference model” among academics. Satchell and Scowcroft (2000) argue that a comprehensible paper about the model was presented by Lee (1999); however, it still failed to present a legible explanation of the mathematical concepts underlying the model. The economic interpretation of Satchell and Scowcroft (2000) will be further explained throughout the theory section.

Meucci (2010) also discusses the original model pioneered by He and Litterman, where he states the value of scalar, tau, should be set between 0 and 1 instead of applying an extension. This is due to posterior distributions building on two settings, returns and covariance, which is dependent on whether views are extremely confident or if the investor has no views, meaning that the investor views either goes towards infinity or, is zero. Meucci (2010) stated, the posterior model should be the implied model when the confidence in the view is zero and oppositely, when the confidence of the view is high, the posterior model should be the combined model which includes the views.

Even though a variety of literature has discussed the BL model, there has yet been a considerable amount of testing the out-of-sample performance of the Black Litterman approach. However, Wolff, Bessler & Opfer (2012) present multi-asset portfolios and analyses the out-of-sample performance. Here, they use the performance measures Sharpe ratio, Maximum drawdown, and the Portfolio Turnover for each portfolio. First, they find that multi-asset portfolios can be applied in the Black-Litterman model, not only stocks, as often shown in the literature. Secondly, their empirical findings show that the Black-Litterman model is performing better in terms of Sharpe ratio and Maximum drawdown when they test the out-of-sample performance.

From the Black-Litterman, as mentioned earlier model, the subjectivity of the investor's views has been challenging for most practitioners and researchers to obtain. A variety of studies have investigated this to provide an explanation of these, and hence, generate investor views. The model is satisfyingly describing the views. However, the model does not answer the question of how to form these views. There have been examinations of the application of the statistical framework to find the investor views. Both Beach and Orlov (2007) and Duqi, Franci, & Torluccio (2014) suggest the utilization of the statistical approach based on forecasting the volatility of returns to derive towards views. The model of volatility is based on a GARCH where they incorporate the stylized facts, e.g. volatility clustering, kurtosis, mean revision, time-varying volatility, among others. In particular, their investigations show the preference of an EGARCH-M argument that it captures the regularities of stock returns.

## 2.3 Premium predictability

A central part of the following view generating process in this paper relies on the prospect of predictability in stock- and bond premium, which will serve as argumentation for using stock- and bond predictability as an approach to generate views in the Black-Litterman framework.

The story of return predictability originates in the efficient market hypothesis, which ultimately states that markets are efficient, and therefore are correctly priced. After the 1960s and forward, many economists, academics, and practitioners in the finance area have strived to identify variables that can predict the stock- and bond market. This section will, therefore, provide a description and a recap on research done on variables that have shown to have predictable features, combined with some history concerning return predictability.

### 2.3.1 Stock return predictability

The stock market premium has been known to differ extensively over time but has indeed been high on average. For many years, academics thought that the risk premium of stocks followed a random walk, where the best expectation of tomorrow's return is a constant. An extensive amount of regressions has been used to try to explain where the equity return is heading. This is often completed by regressing any indicator or signal today on tomorrow's return, with the desire that it will show predictable features. Lettau & Ludvigson (2001) concludes that it is, nowadays, widely accepted that assets have a time-varying risk premium and can be predicted by various variables. Their findings also suggest that the variables have especially shown a good ability to predict expected returns over longer investment horizons, as opposed to shorter investment horizons.

Many variables, especially valuation ratios and macro variables, have been discovered in the literature over the past years that supposedly have, both statistically and economically, shown the ability to predict the return of stocks and indices, mainly in-sample. Dow (1920), Fama & Schwert (1977), Fama & French (1988), Campbell & Shiller (1988 & 1998) and Kothari & Shanken (1997) investigated the predictive power of valuation ratios such as the dividend/price ratio, the book-to-market ratio and the earnings-price ratio. Many of them found in-sample evidence of predictive power by regressing these variables on tomorrow's stock return. Further, predictability in economic variables such as inflation, term- and default spreads, net equity expansion, consumption-wealth ratio, and stock market variance, etc., has also been popular to exploit. This exploit has for instance been investigated by Nelson (1976), Fama & Schwert (1977), Baker & Wurgler (2000), Campbell (1987), and Lettau & Ludvigson (2001) among many others.

The paper of Goyal & Welsh (2007) presents a review of the performance of many of the previously mentioned financial ratios and macroeconomic variables briefly mentioned above. First, they provide a complete data set of fourteen economic variables to analyse, making their findings easy to replicate. Further, they re-examine the models and evaluate them using four criteria: (1) in-sample significance, (2) out-of-sample performance, (3) relation to outliers, and (4) the long-term performance which should hold over a minimum period of three decades. Their overall conclusion states that the individual models perform poorly when evaluated both in-sample and out-of-sample. It is also mentioned that some of the models fail to pass standard diagnostic tests used in statistics.

The paper presented by Goyal & Welsh (2007), with the belonging data set, has been applied by many academics with an attempt to design equity premium prediction models that outperform the benchmark or historical average. Another method was proposed by Campbell and Thompson (2007), where they place restrictions on the coefficient signs, which according to their findings, improves the return forecast and provide useful information to mean-variance investors. However, the empirical analysis still failed to present consistent out-of-sample significance over time, and the performance was still highly uneven over time. Later on, Rapach et al. (2007) showed a method to predict the equity premium of indices, out-of-sample, by using combination forecasts and covariate estimation. Their findings, motivated by Goyal & Welsh (2007), suggest that none of the fourteenth economic variables can beat the historical average individually measured by MSPE. However, by combining the individual regression models in this manner, Rapach et al. (2007) were able to present a model that improved the out-of-sample forecast and consistently outperformed the historical average. The results are presented for multiple lengths of the out-of-sample periods, and the conclusions are similar for the various out-of-sample periods.

### 2.3.2 Bond return predictability

There is a wide harmony among financial experts that returns on nominal U.S. Treasury bonds can be predicted at different investment horizons or, equivalently, evidence for the existence of time-varying expected excess return of the government bonds.

It is, however, reasonable that the expectations hypothesis, that the investor was expected to gain zero of a constant excess return on bonds based on the predictability of the short-term interest rate built on the long-term rate, has been rejected through studies. Empirical findings have shown to have statistically and empirically significance to predict bond returns. This has been supported by economists such as Fama and Bliss (1987), Cochrane and Piazzesi (2005), Campbell and Shiller (1991), and Ludvigson and Ng (2009). They carry out variables using forward rates, yield spread, and macroeconomic variables. Furthermore, it was found that the expected term premium is related through a business cycle, as the term premium gets positive in booming economy and negative during recessions.

Engsted & Møller (2013) investigate the predictability in US bond returns in expansion and recessions, applying univariate regressions and forecasting techniques. Their study rejects unpredictability in-sample and out-of-sample in both expansions and recession. Here, they take into account the utility for a mean-variance investor which includes predictability patterns when investing capital. The economic significance is found to be positive during expansions consequently, negative in recessions.

In recent times, newer methods are pioneered to predict the term structure. This is determined from the movements on long-term rates which consist of two parts; the first consisting of the expected return from the short-rate, and an additional component, also known as term premium, which compensates investors in long-term bonds for interest rate risk. It is often known that the term premium is calculated as the difference between model-implied fitted yield and the model implied average expected return on the short rate. Economist Adrian, Crump, and Moench (2014) from the New York Fed examine this Treasury term premium, which is the compensation for bearing risk associated with a long-term bond. Older methods mostly applied infrequent data, e.g. inflation or forward rate, contrary to traditional methods, however, ACM used available nominal yield data. Their research shows how to price the term structure of interest rates using linear regressions. In their study, they apply pricing factors and thus estimate the term premium. Apart from four-factor models from Cochrane and Piazzesi (2008), they present a five-factor model from coupon-bearing yields that essentially outperforms Cochrane and Piazzesi models in an out-of-sample estimation, making their specification of term premium applicable. Nevertheless, they conclude that the term premium tends to move with measures of uncertainty of disagreement about the future level of yields. The accuracy of the yield shows superior performance for the ACM five-factor model compared to three-factor models. Furthermore, they compare their implied ten-year yield, found from the five-factor model, with the ten-year yield from the GSW zero-coupon yield, where their findings show their implied yield fits the yield from Gurkaynak, Sack and Wright (2006) quite well when going back to June 14, 1961.

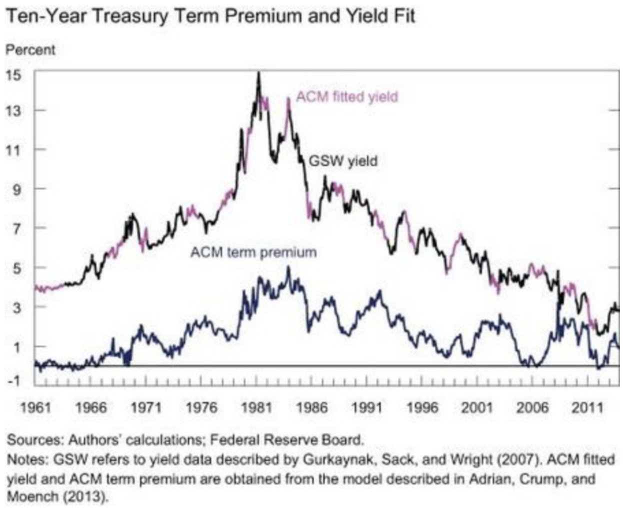


Figure 2: Ten-Year Treasury yield and term premium (Adrian, Crump, Mills, and Moench, 2014)

However, practitioners often employ a much simpler method to obtain the term premium rate, by assuming that the term premium is generated of the difference of the long-end bond yield and short-rate (Tang, Li & Tandon, 2019; BIS Quarterly Review, 2007). If this method applies, the underlying assumption is based on a random walk, where the expectation of the short rate is equivalent for an infinite period. This means that the long-term yield prices where it expects the short-term yield front-end to be in the future etc. The correlation of stocks and bonds is a driver of the long-term bond prices and the corresponding term premia.

Contrarily to prediction in bond returns, literature has shown to provide little evidence for predictability in bond returns to improve investor's utility. Thorntorn and Valente (2012) and investigate the predictability of bond returns out-of-sample. Thorntorn and Vante (2012) find that forward rates do not add higher economic value compared to a non-predictable benchmark. Gargano, Pettenuzzo, and Timmerman (2017) find both economic and statistical significance of out-of-sample predictability in US Treasury bond excess returns, applying variables as the forward spread, forward rates, and macro factors.

# 3. Theoretical framework

This section seeks to describe the modern portfolio theory presented by Markowitz's (MPT), the Capital Asset Pricing Model (CAPM), and the Black-Litterman model, in addition to present basic risk and return computations. Furthermore, the section includes mathematical explanations of these theories to get familiarized with the approaches. It's crucial when applying the Black Litterman framework that modern portfolio theory and CAPM is understood, as the approach takes practice in these. These models are therefore carefully explained throughout this section.

## 3.1 Basic risk and return calculations

To calculate the risk and returns of the portfolios in question, the formulas used to assess these measures are defined. The standard arithmetic average, $\bar{r}_p$ and sample standard deviation $\sigma_p$ is generated as following (Munk, 2018)

$$\bar{r}_p = \frac{1}{T}\sum_{t=1}^{T} r_{pt}$$

$$\sigma_P = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(r_{pd} - \bar{r}_p\right)^2}$$

The expected return, variance and standard deviation of a mean-variance portfolio (and other portfolios) is estimated using historical observations of the return process, and is computed as follows:

$$\mu(\pi) = \pi \cdot \mu = \sum_{1i=1}^{N} \pi_i \, \mu_i \qquad \text{(Equation 3.1.1)}$$

$$\sigma^2 (\pi) = \pi \cdot \underline{\Sigma}\pi = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i \, \pi_j \Sigma_{ij} \qquad \text{(Equation 3.1.2)}$$

$$\sigma(\pi) = \sqrt{\pi \cdot \underline{\Sigma}\pi} = \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i \, \pi_j \Sigma_{ij}\right)^{1/2} \qquad \text{(Equation 3.1.3)}$$

where $\mu(\pi)$ and $\sigma(\pi)$ is, respectively, the weighted mean and standard deviation of the portfolio given the weights invested. Another rule stated by Markowitz implied that the investor should diversify and that he should maximize expected return. The investors should diversify it in the way that he invests in all securities that give the highest expected return (Markovitz, 1952).

## 3.2 Mean-Variance analysis

Markowitz (1952) developed a theory in his paper "Portfolio Selection", that investigated trade-offs to identify the optimal portfolio over a certain period, allowing the investor to observe the maximum expected return given the lowest amount of portfolio risk. The mean-variance optimization is built as a theoretical foundation of Modern Portfolio Theory (MPT), which assumes that the investor makes rational decisions based on complete information. Sharpe (1964) interprets the theory where "*the process of investment choice can be broken down into two phases: first, the choice of a unique optimum combination of risky assets; and second, a separate choice concerning the allocation of funds between such a combination and a single riskless asset*".

### 3.2.1 Mean-variance portfolio

A mean-variance analysis is applied to make decisions about which securities to invest in given the level of risk and expected return. The mean-variance portfolio choice is based on several important assumptions (Markowitz, 1952), all listed below:

1. When investors choose among portfolios, they consider only the expected return and the return variance of the portfolios over a fixed period of time
2. Investors like high expected returns
3. Investors dislike high return variances, which indicates risk aversion.

Investors who invest like this, are known to be mean-variance optimizers.

First, the combination of risky assets will be explained. Second, the allocation between risky assets and risk-free assets will be explained further below in Section 3.2.5.

#### 3.2.3.1 Mean-variance efficient portfolios

A portfolio is mean-variance efficient, between risky assets, if the portfolios contain the minimum variance among all portfolios with the same mean return or a portfolio that maximizes the expected return for a given amount of risk. Following the methodology of Markowitz (1952), he assumes that two constraints exist when minimizing the objective function. The first constraint is that the investor must be fully invested, meaning all of the capital is invested at this moment and the second that the expected return is fixed since we are minimizing the risk given this return. The fixed return is given by $\mu_p$. The $\pi_i$ is explained as the fraction of the total portfolio value invested in asset *i*, and the portfolio vector must therefore satisfy

$$\pi \cdot 1 = \pi_1 + \pi_2 + \ldots + \pi_N = 1$$

The sum of the vector should sum up to 1.

The lowest variance for the quadratic given mean, $\bar{\mu}$, is found by solving the minimum variance, since we want to minimize the risk;

$$min\, \pi \cdot \sum \pi$$
$$s.t. \pi \cdot \mu = \bar{\mu}$$

The variance-covariance matrix is derived as following applying a two-asset case, which is denoted by

$$\sum = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

And its inverse is

$$\sum^{-1} = \frac{1}{(1-\rho^2)\sigma_1^2\sigma_2^2}\begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}$$

Where $\sigma_1$ and $\sigma_2$ are the standard deviation of the two assets and $\rho$ is the correlation between the assets.

The auxiliary constants are defined as,

$$A = \mu^T \sum^{-1} \mu = \mu \cdot \sum^{-1} \mu$$
$$B = \mu^{\mathrm{T}} \sum^{-1} 1 = \mu \cdot \sum^{-1} 1 = 1^T \sum^{-1} = 1 \cdot \sum^{-1} \mu$$
$$C = 1^T \sum^{-1} 1 = 1 \cdot \sum^{-1} 1 = \frac{1}{(1-\rho^2)\sigma_1^2\sigma_2^2}(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)$$

which is applied to the computations of the mean-variance optimization.

This means that the expression for the variance becomes,

$$\sigma^2(\bar{\mu}) = \pi(\bar{\mu}) \cdot \sum \pi(\bar{\mu}) = \frac{C\bar{\mu}^2 - 2B\bar{\mu} + A}{D}$$

Which is the formula for the variance on the portfolio for the efficient frontier. Of all the portfolio with the expected return on $\bar{\mu}$, this equation provides the one with the lowest variance. Taking the square root, we obtain the standard deviation that can be plotted into a (standard-dev, mean)-diagram.

The mean-variance efficient portfolio weight vector, variance of the portfolio and standard deviation, with the expected return is given as

$$\pi(\bar{\mu}) = \frac{C\bar{\mu} - B}{D}\Sigma^{-1}\mu + \frac{A - B\bar{\mu}}{D}\Sigma^{-1}1$$

$$\sigma^2(\bar{\mu}) = \pi(\bar{\mu}) \cdot \underline{\Sigma}\pi(\bar{\mu}) = \frac{C\bar{\mu}^2 - 2B\bar{\mu} + A}{D}$$

$$\sigma(\bar{\mu}) = \sqrt{\frac{C\bar{\mu}^2 - 2B\bar{\mu} + A}{D}}$$

If investing in three or more assets, many portfolios obtain an equally expected rate of return. Hence, the optimal portfolio with expected return $\bar{\mu}$ is the portfolio with the lowest portfolio variance. The different optimal combinations of standard deviation and mean form a hyperbola in a (standard deviation, mean diagram). This is also known as mean-variance frontier or efficient frontier of risky assets (Munk, 2018).

### 3.2.2 The minimum-variance portfolio

The minimum-variance portfolio is defined as the portfolio that has the minimum variance among all portfolios. The portfolio is also called the global minimum-variance portfolio. The investor wants to invest in a portfolio where he does not care about the expected return but only cares about the lowest amount of risk. Since the investors always invest in an efficient portfolio, he chooses the portfolio on the efficient frontier with the minimum standard deviation.

The minimum-variance is defined by solving the mean, that gives the smallest possible variance

$$min\, \pi \cdot \Sigma\pi$$
$$s.t. \pi \cdot 1 = 1$$

With no constraint on expected return.

The minimum-variance portfolio is given by:

$$\pi_{\min} = \frac{1}{C}\underline{\Sigma}^{-1}1 = \frac{1}{1 \cdot \underline{\Sigma}^{-1}1}\underline{\Sigma}^{-1}1$$

With expected return, variance and standard deviation

$$\mu_{min} = \frac{B}{C}$$

$$\sigma_{min}^2 = \sigma^2(\bar{\mu}_{min}) = \frac{1}{C} \qquad \sigma_{min} = \frac{1}{\sqrt{C}}$$

As the minimum-variance lays on the efficient frontier, the variance and the expected return are therefore related. By minimizing the variance of the portfolio of the mean-variance, the minimum-variance portfolio can be identified. In the minimum-variance portfolio, it is expected that assets with low standard deviation have large weights. However, this method focuses on the importance of the correlation structure of the assets. It is quite useful for diversifying away risk, since the minimum-variance portfolio might have a significant overweight on assets with large standard deviation and that asset might have low correlation with some low-variance asset.

We would look after the slope at the front of the frontier or the right of the frontier. It has to be at the point where it just touches the efficient frontier, which gives the maximum portfolio.

### 3.2.3 The maximum-slope portfolio

A maximum-slope portfolio is known as the portfolio on the efficient frontier with the maximum slope. The Sharpe ratio is the ratio of the expected excess return of the portfolio relative to its volatility. A portfolio more intuitively knows it of risky assets that lie to a point in a (standard deviation, mean)-diagram. Therefore, any point corresponds to the mean-variance frontier. By connection, any point with the origin with a straight line, the slope of the line becomes $\mu/\sigma$. One desires to find the portfolio that maximizes the slope of this line.

The portfolio that gives the maximum slope:

$$\pi_{slope} = \frac{1}{B}\Sigma^{-1}\mu = \frac{1}{1 \cdot \Sigma^{-1}\mu}\Sigma^{-1}\mu$$

Has expected return, variance and standard deviation of

$$\mu_{slope} = \frac{A}{B}$$

$$\sigma_{slope}^2 = \frac{A}{B^2}, \qquad \sigma_{slope} = \frac{\sqrt{A}}{|B|}$$

The length of the expected rate of return along the frontier which would provide the maximum return is already known. Hence, the relationship between the variance and the expected return can be exploited.

The maximum-slope portfolio corresponds to a point on the upward sloping branch of the curved frontier. Note: If $B < 0$, the maximum-slope portfolio is located on the *downward*-sloping branch of the curved frontier and is the portfolio giving the most *negative* slope of all lines considered (Munk, 2019).

### 3.2.4 The efficient frontier

The efficient frontier is a curve that provides all efficient portfolios in a risk-return approach. An investor always invests in an efficient portfolio, since they would always aim for the highest possible expected return. This comes because the investor is risk averse.

From the derived results of the minimum-variance and the maximum-slope portfolio, any mean-variance efficient portfolio is a combination of the maximum-slope and the minimum-variance portfolio.

A mean-variance efficient frontier consisting of risky assets can be stated as a combined portfolio between the maximum slope portfolio and the minimum variance

$$\pi(\bar{\mu}) = \frac{(C\mu - B)B}{D}\pi_{slope} + \frac{(A - B\bar{\mu})C}{D}\pi_{\min}$$

The two portfolios will have a zero covariance, and every frontier portfolio has a zero covariance on the frontier. The two coefficients will sum up to one. So, that $\pi(\bar{\mu})$ is a weighted average of the maximum-slope portfolio and the minimum-variance portfolio.

There are two ways to generate the mean-variance efficient frontier:

1. Use a range of $\bar{\mu}$ values with the corresponding portfolio standard deviations by applying the constants A, B, C, and D.
2. Compute the expected return and variance of the minimum-variance and maximum-slope portfolio.
    a. Then a combination of these two portfolios should be considered
    b. Each combination should contain the expected return and standard deviation
    c. The combination of the portfolio can be described as $\mu(w) = w\bar{\mu}_{min} + (1-w)\bar{\mu}_{slope}$, giving the expected return of the portfolio. The portfolio variance is calculated as $\sigma^2(w) = w^2\sigma_{min}^2 + (1-w)^2\sigma_{slope}^2 + 2w(1-w)\sigma_{min}^2$

This efficient frontier of risky is also known as a two-fund separation, meaning, that the investors can form a portfolio of N risky assets. Furthermore, a mean-variance investor seeks an optimized portfolio being a combination of the two portfolios, the minimum-variance portfolio and the maximum-slope portfolio (Munk, 2019).

### 3.2.5 Tangency portfolio

The investors can create a mean-variance analysis containing both risky assets and a risk-free asset. Investors that have these preferences can invest in the portfolio with the maximum Sharpe ratio. Applying a combination of the straight line from (0,$r_f$) and the point $(\sigma, \mu)$ corresponds to a portfolio of risky assets. The slope of this portfolio is equivalent to the Sharpe ratio of the risky portfolio. The investors prefer a high expected return and a low standard deviation, and therefore, the $\mu > r_f$ should be met as the maximum Sharpe ratio is then fulfilled.

The maximum Sharpe Ratio is defined as the relationship between the return-risk trade-off, which is a measure of the risk premium relative to the total risk of the portfolio, as explained

$$Sharpe = \frac{(\mu - r_f)}{\sigma}$$

In a $(\sigma, \mu)$-diagram the tangency portfolio is the point where the straight line starting at $(0,r_f)$ is tangent to the mean-variance frontier of risky assets. Now the mean-variance efficient portfolio of all assets is a combination of the risk-free rate and a tangency portfolio of risky assets. An expectation of the individual asset with high Sharpe ratios is a more significant allocation in the tangency portfolio. Nevertheless, the correlations are essential, to diversify the risk so that the tangency portfolio might give considerable weight to an asset with a low Sharpe ratio (Munk, 2019).

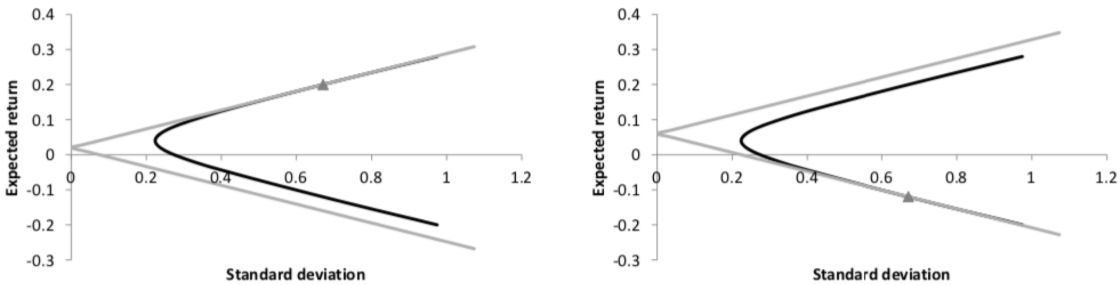


FIGURE 3: THE BLACK IS THE RISKY ASSETS, WHILE THE GREY IS THE RISK-FREE ASSET (MUNK, 2018).

In general, investors prefer to be in the north-west in the standard deviation-mean diagram so that the tangency portfolio can be obtained. The two figures show why it is vital that the risk-free rate is smaller or higher than the minimum-variance portfolio's expected return. The first graph shows that when the risk-free rate is smaller than the expected return, the tangency lies on the upward slope of the efficient frontier of the risky assets. In contrast, the right graph shows the opposite, precisely that the risk-free rate higher than the expected return means a tangency portfolio on the downward sloping efficient frontier. Both of the graphs represent the efficient frontier of all asset, although, no one would never choose a point corresponding to the downward line of the efficient frontier (Munk, 2019).

The tangency portfolio of the risky assets is generated as follows:

$$\pi_{\tan} = \frac{1}{B - Cr_f} \underline{\underline{\Sigma}}^{-1}(\mu - r_f 1) \qquad \text{(Equation 3.2.1)}$$

$$\mu_{tan} = \frac{\mu \cdot \underline{\Sigma}^{-1}(\mu - r_f 1)}{1 \cdot \underline{\Sigma}^{-1}(\mu - r_f 1)} = \frac{A - Br_f}{B - Cr_f} \qquad \text{(Equation 3.2.2)}$$

$$\sigma_{\tan}^2 = \frac{(\mu - r_f 1) \cdot \underline{\Sigma}^{-1}(\mu - r_f 1)}{\left(1 \cdot \underline{\Sigma}^{-1}(\mu - r_f 1)\right)^2} = \frac{A - 2Br_f + Cr_f^2}{\left(B - C_{rf}\right)^2} \qquad \text{(Equation 3.2.3)}$$

$$\sigma_{tan} = \frac{\sqrt{A - 2Br_f + Cr_f^2}}{|B - C_{rf}|} \qquad \text{(Equation 3.2.4)}$$

$$|SR_{tan}| = \sqrt{A - 2Br_f + Cr_f^2} \qquad \text{(Equation 3.2.5)}$$

The mean-variance efficient frontier of the risk-free asset and risky assets is a combined portfolio of the risk-free asset and the tangency portfolio of risky assets, which can be viewed in Figure 4. Denoting, *w,* as the weight in the tangency portfolio and (1-w) and the weight on the risk-free asset, the expected return and standard deviation of the portfolio is generated as follows;

$$\mu(w) = w\mu_{\tan} + (1 - w)r_f$$

$$\sigma(w) = |w|\sigma_{tan}$$

If investors agree on the risk-free rate and expected return and risk of the risky assets, they agree on the construction of the tangency portfolio where every investor would hold the same portfolio of risky assets and risk-free asset.

The mathematical explanation of the tangency portfolio is given as follows:

$$\Sigma^{-1}(\mu - r_f 1) = \frac{1}{(1 - \rho^2)\sigma_1^2\sigma_2^2} \begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix} \begin{pmatrix} \mu_1 - r_f \\ \mu_2 - r_f \end{pmatrix}$$

$$1 \cdot \Sigma^{-1}(\mu - r_f 1) = \frac{1}{(1 - \rho^2)\sigma_1^2\sigma_2^2} \left(\sigma_1^2(\mu_2 - r_f) + \sigma_2^2(\mu_1 - r_f) - \rho\sigma_1\sigma_2(\mu_1 + \mu_2 - 2r_f)\right)$$

$$\pi_{tan} = \frac{\underline{\Sigma}^{-1}(\mu - r_f 1)}{1 \cdot \underline{\Sigma}^{-1}(\mu - r_f 1)} = \begin{pmatrix} \dfrac{\sigma_2^2(\mu_1 - r_f) - \rho\sigma_1\sigma_2(\mu_2 - r_f)}{\sigma_1^2(\mu_2 - r_f) + \sigma_2^2(\mu_1 - r_f) - \rho\sigma_1\sigma_2(\mu_1 + \mu_2 - 2_{rf})} \\ \dfrac{\sigma_1^2(\mu_2 - r_f) - \rho\sigma_1\sigma_2(\mu_1 - r_f)}{\sigma_1^2(\mu_2 - r_f) + \sigma_2^2(\mu_1 - r_f) - \rho\sigma_1\sigma_2(\mu_1 + \mu_2 - 2_{rf})} \end{pmatrix}$$

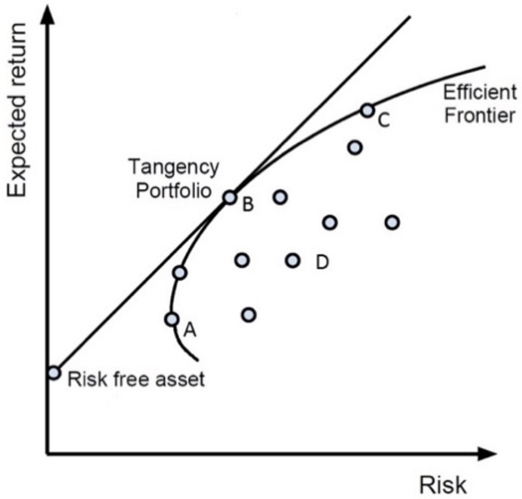


FIGURE 4: THE EFFICIENT FRONTIER INCLUDING RISK-FREE ASSET AND RISKY ASSETS, AND THE TANGENCY PORTFOLIO

### 3.2.6 The optimal portfolio

In general, any mean-variance optimizer chooses a combination of the risk-free asset and the tangency portfolio of risky assets. Therefore, it is desirable to find the optimal $w$ depending on the mean-variance trade-off of the investor. This fraction should be invested in the tangency portfolio, whereas 1 - $w$ of wealth should be invested in the risk-free asset (Munk, 2019).

To find the equation for the optimal value of $w$, we want to set the objective to maximize the investor's expected return minus a constant time the variance.

$$max\left(E[r] - \frac{1}{2}\gamma Var[r]\right) \qquad \text{(Equation 3.2.5)}$$

Where $\gamma$ a is a positive constant. $\gamma$ corresponds to the investor's risk aversion. The excess returns are denoted by the (N x 1) vector $\mu$ and the covariance matrix of returns is denoted by, $\sum$.

The mean-variance optimal vector of the risky assets, w* (Nx1) vector, is computed from the following equation:

$$w^* = arg\ max\ \left\{w'\mu - y * \frac{1}{2} * w'\Sigma w\right\}$$

$$w^* = \frac{1}{\gamma} \cdot \sum^{-1} \mu \qquad \text{(Equation 3.2.6)}$$

Where $\mu$ is a *N x 1* vector of the expected rates of excess return and $\Sigma$ is a *N x N* variance-covariance matrix. *w,* is the fraction of the total portfolio value, which is invested in risky assets, where (1-w) in the risk-free asset.

If $w$ is a mean-variance efficient portfolio concerning a universe of assets with a known return vector $u$ and covariance matrix ∑, then there exists a linear correlation between u and ∑w. Furthermore, covariance is known to be more accurate to estimate rather than expected returns. Thus, if a mean-variance weight vector is known and the covariance is accurately estimated, the linear relation between u and ∑w can be exploited to create implied expected returns. Pointed out by Munk (2019), reducing the risk involved by investing in stocks and bonds, predictability through momentum is best exploited by allocating long positions in assets with recent positive excess returns and short positions in recent negative excess returns.

### 3.2.7 Utility function

One key driver of the mean-variance analysis must be that the investor decides their investment based on the expected return and the risk. In particular, the decision of an investor is often represented as a utility function. The mean-variance objective can be justified, meaning that an optimal solution can be derived for the optimal portfolio.

The investor's wealth is denoted as $W_0$ being the start of the period, and if assuming the investor where to invest all of his wealth, it would end up as W given as:

$$W = W_0(1 + r)$$

Where the *r* is given as the rate of return. The overall wealth depends on the portfolio choice. The utility function is then defined as the function that is attached to each wealth function, for a given portfolio, at the end-period *u(W)*, were the objective function can be stated as the maximum of expected utility, on all possible portfolios E[u(W)]. The utility is also reflected by the risk-averse of an investor, as an increasing utility function means that the investor wants as much wealth as possible, and a low utility is assumed to mean decreasing in wealth. An investor being risk-averse also means, rejecting risky investment when expected profit is negative. This is shown by the first derivative of the utility function (Munk, 2019).

#### 3.2.7.1 Quadratic utility function

Assuming a quadratic utility function,

$$u(W) \ = \ a + bW - cW^2$$

where a, b and c are given as constants. The expected utility is as follows:

$$E[u(W)] = a + b\ E[W] - c\ E[W^2] = a + b\ E[W]\ - c(Var[W] + (E[W])^2)$$

The expected utility only replies on the expectation of wealth and the variance, and therefore, the mean-variance is a reasonable fit for quadratic utility investors, even though the returns are non-gaussian (Munk, 2019).

### 3.2.8 Critique of the mean-variance analysis

There are several assumptions about investors and markets which point towards a lack of eligibility. Despite the importance of the theory, there are critical drawbacks of the underlying framework regarding the accurateness of MPT conclusions in the real world.

If the distribution of returns is non-gaussian, there are limitations of the predictability. However, the return of financial returns is assumed to be normally distributed. It is crucial since it supports the assumption that investors only care about the expected return and risk of their portfolio. This is because investors only look at the first two moments of the return distribution (Hull, 2012). It is known that the return, risk, and correlation from MPT is based on the use of expected values. Investors have to predict the return and volatility based on historical data, meaning that they are subject to be changed by variables that are currently not known or considered. Although MPT is not concerned with estimating variables, it is usually estimated by quantitatively analysing historical data (Fabozzio, et al., 2002). One of the issues with estimating the variables is to choose a representative subset of data, as the data should represent the period predicted. Often the historical data is not enough to say how the future returns should evolve. Also mentioned in the literature review, the model is quite sensitive towards inputs causing error-maximization. A solution proposed by Munk (2019) is to apply asset classes instead of individual assets, since the inputs of the mean-variance increases in step with the number of assets. A low number of inputs is limited and may quickly provide a realizable forecast of the expected returns of the asset's classes rather than on individual assets. Asset classes are more robust against individual equities turbulence, such as. M&A, patents, CEO, etc, which might be more representative against the future returns.

In practical terms, the framework is not really applied since it does not fit the real world. Investment managers prefer to focus on small segments in their investment universe and find assets they feel are the right pick. Although the MPT takes into account the expected return, it is required that they are specified for every component of a relevant universe but in reality, they are defined by a benchmark (Black & Litterman, 1992). Furthermore, the portfolio weights can contain constraints, such as short sales. Black and Litterman (1992) state that excluding short sales, which investment managers often find necessary, the portfolio construction will give quite a large position for a few assets. If involving short positions, the optimal portfolio can easily contain large negative weights in certain assets. The fundamentals of the mean-variance portfolio should hold when including constraints, however, the interpretation can be very complicated.

## 3.3 Capital Asset Pricing Model

The Capital Asset Pricing model (CAPM) was derived from Treynor (1961), Sharpe (1964), Lintner (1965) and Mossin (1966) twelve years after Harry Markowitz (1959) introduced his mean-variance portfolio theory (Bodie et al, 2014). Markowitz's modern portfolio theory laid the groundwork for the Capital Asset Pricing Model. Sharpe and Lintner applied this to an economy-wide setting where an assumption is that the portfolios of investors are held mean-variance efficient, and their views are homogeneous in a frictionless market (Campbell et al., 1996). The model provides a relationship between expected return and riskiness of the asset, which can serve as a benchmark for future investments or help estimate the expected return of assets that are not yet publicly traded (Bodie et al., 2014). The model is ultimately based on the fact that the market is in equilibrium, where assets are priced correctly when the assumption of market equilibrium is held. In other words, the definition of the market equilibrium is the adjustment of the prices, influenced by the beliefs and expectation of the investor, until the expected returns are in equilibrium where the demand matches the supply (He & Litterman, 1992). The model also allows us to use various risk measures for different kinds of assets, and also get the relationship of efficient and inefficient assets. Because the market is in equilibrium, the prices of assets are such that the tangency portfolio is the market portfolio, which is composed of all risky assets in proportion to their market capitalization.

The CAPM model is based on several assumptions which have to be fulfilled. The assumptions are quite similar to the once introduced in modern portfolio theory, and are given as follows (Jensen, 1967) (Bodie et al., 2014):

- The wealth of individual investors is small compared to the overall wealth in the total market
- Investors have the same holding horizon and homogeneous market views, i.e. the same underlying distribution of future expected returns
- The risk-free rate is the same for all investors, and they can lend and borrow at this rate
- The investor is risk averse, however, he still seeks to maximize his/her wealth
- The decision-making is determined from the risk-return perspective
- The market is in equilibrium
- There are no market frictions, taxes, etc.

Some of the assumptions mentioned above are a simplification of the real world which does not necessarily hold under true market conditions. Regardless, they are essential tools to be able to explain the market equilibrium. The assumption that all investors seek to hold or replicate the market portfolio, which in theory contains all publicly traded assets, is in reality hard to establish.

The basic CAPM-equation is defined as

$$E(r_i) = r_f + \beta_i(E(r_m) - r_f) \qquad \text{(Equation 3.3.1)}$$

where $E(r_i)$ is the individual asset return, $E(r_m)$is the return on the market portfolio, $r_f$ is the risk-free rate and $\beta$ is defined as a measure of the asset risk, given as

$$\beta_i = \frac{cov_{r_i,r_m}}{{\sigma_m}^2} \qquad \text{(Equation 3.3.1)}$$

where ${\sigma_m}^2$is the variance of the market portfolio, and $cov_{r_i,r_m}$is the covariance between the asset excess return and the excess return on the market portfolio. The equation shows that it is far from straightforward to estimate the market beta. One approach is to use probabilities of different outcomes of the asset return, $r_i$ and of the return of the market portfolio, $r_m$. This method builds on the probability distribution of the stochastic variables. It is then used to calculate the covariance between the market return and the asset returns and the variance of the generated returns. The uncertainty of the assumptions that is done to estimate the beta is, therefore, significant (Munk, 2019). Estimating beta through observations on historical returns of the asset and the market is a more conventional method. This makes it possible to build a regression model where the historical data helps us obtain a relationship between the market return and the asset return.

CAPM makes some assumptions of varying degrees of plausibility. For use in the reverse optimization of equilibrium excess returns performed in Section 3.3. Given a vector of specified market-clearing asset prices, agents must agree on the joint distribution of asset returns from this period to the next. This assumption entails that any market portfolio must be on the minimum-variance frontier if the market is to clear all positions (Fama and French, 2004). Additionally, CAPM assumes that investors are only concerned with the asset returns and variances, the first two moments.

### 3.3.1 Capital allocation line

The Capital Allocation Line (CAL) is a graphical lie that illustrates the relationship of the risk-and-reward combinations of assets and is often associated with its application to find the optimal portfolio. The slope of CAL is the increase in the expected return of the portfolio per unit of additional risk, also referred to as the Sharpe ratio. The line is mathematically expressed as follows:

$$E(r_p) = r_f + Sharpe_p \sigma_p \qquad \text{(Equation 3.3.2)}$$

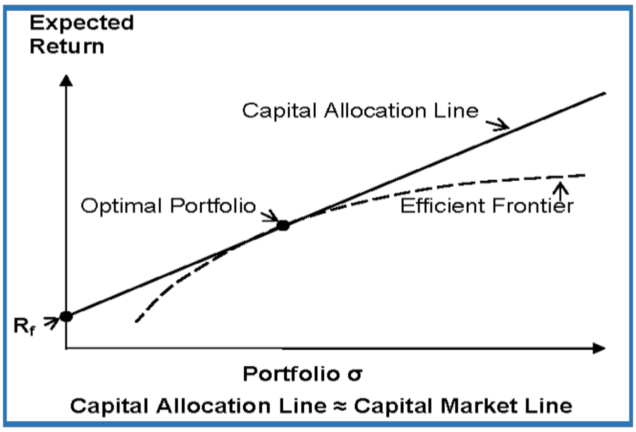


FIGURE 5: GRAPHICAL ILLUSTRATION OF THE CAL

The different allocation options also mean that one optimal portfolio is present but does depend on the different level of risk aversions. The different levels of allocation depend on how much we want to hold in risk-free assets and correspondingly, in risky assets. The optimal portfolio is found, when the CAL is tangent to the efficient frontier, illustrated by the graph. The point at $r_f$, means 100% investment in risk-free assets, whereas the point (Optimal Portfolio) shows 100% investment in the portfolio. Between the risk-free rate and the optimal portfolio, investors that lie there hold positions in both risk-free assets and the portfolio and represent an investor lending a part of their portfolio, as investors are not 100% invested in the portfolio. The point after the optimal portfolio, show a leveraged position, being more than 100% invested in the portfolio, and therefore borrowing capital to buy more portfolio (Bodie et al. , 2014).

### 3.3.2 Single index models

The market portfolio is in practice a mean-variance efficient portfolio consisting of all risky assets. Testing the efficiency of the market portfolio requires construction of a value-weighted portfolio of significant size, which can be demanding and often not feasible. Hence, correcting for this issue requires additional assumptions (Bodie et al., 2014).

Sharpe (1963) developed the well-known Single-Index Model where, as opposed to regular factor models, the return on the market portfolio or a stock market index is used as a factor to explain the excess return of an individual asset (Munk, 2019). The model illustrates a relationship between the expected asset return and its respective beta, which is usually formulated as:

$$E(r_i) - r_f = \alpha_i + ß_i[E(r_m) - r_f]$$

where the notation can be recalled under Section 3.2 and $\alpha$ is the abnormal return or the difference between the expected return and the realized return. The traditional CAPM is just one example of the Single Index model, but instead, the single index applies an economic variable to explain the excess return. The theoretical CAPM implicitly predicts that, for all assets, the alpha should yield zero. If the asset has an alpha that deviates from zero, it is not correctly priced according to the theoretical CAPM. Under CAPM, taking additional risks may be reduced through diversification, which is compensated from beta by taking additional systematic risks. This is not compensated when taking risks associated with alpha.

## 3.4 The Black-Litterman model

The Black-Litterman model is an asset allocation model created in 1990 by Fisher Black and Robert Litterman from Goldman-Sachs but has since then been further developed, see Black and Litterman (1990, 1991, 1992 & 1999). They found that the traditional MPT, where the return is maximized for a given level of risk, was known to produce unrealistic results, and therefore they tried to solve the problems using their method. In 1998 a paper published by Goldman-Sachs discussed how to implement the Black-Litterman approach: "*Investors should take risks where they have views, and correspondingly, they should take the most risk where they have the strongest views"* (Bevan and Winkelmann, 1998). In other words, the model intended to incorporate the investors' subjective views with an asset allocation model. By doing so, they accomplished to create a more intuitive portfolio by providing a better estimate for the expected return. This is based on the Bayesian methodology that seeks to combine current opinions with new opinions (Satchell & Scowcroft, 2000). A simplified illustration of the procedure is presented in Figure 6 and will be explained in detail throughout this section.

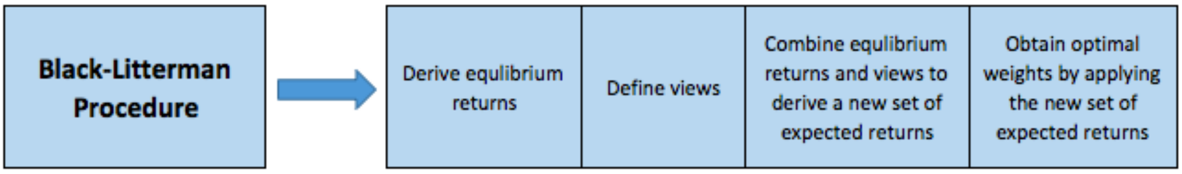


FIGURE 6: SIMPLIFIED BLACK-LITTERMAN PROCEDURE

### 3.4.1 Bayesian approach

The Black-Litterman model applies a Bayesian approach to combine the subjective views of an investor with historical quantitative data. The Bayesian approach is a statistical theory named after the British statistician Thomas Baye (Stigler, 1982). The Bayesian approach allows the forecaster to account for information or events that occur unexpectedly and simultaneously be able to reflect it in the existing objective model by mixing their respective probability distributions. Subjective interpretations of the future or new information, such as prior knowledge or individual beliefs, denoted as the "prior distribution", will be incorporated with relevant existing data in the analysis forming a "posterior distribution" (Insua et al., 2012). The expected return vector in the Black-Litterman model is suggested to be an outcome of two separate normal distributions merged (Satchell & Scowcroft, 2000). As mentioned, Satchell & Scowcroft (2000) found that there were not yet provided sufficient readings on the mathematical approach of the Black-Litterman model, which seems to be a strong motivation for their paper describing the underlying mathematics. This section will discuss the general features of Bayes' Theorem, which is the groundwork/baseline for the mathematical approach when deriving the Black-Litterman model suggested by Satchell & Scowcroft (2000).

A primary goal in statistical analysis is to obtain knowledge and information about various parameters. Using P(A) as a parameter of a stochastic process to investigate further, as in the Black-Litterman world will equal the expected returns. Further, we have a sample of observations denoted P(B), which will be used to calculate the updated distribution denoted as P(A|B) (Insua et al., 2012). The reasoning of the approach builds on Bayes' Theorem, which is given as

Bayes' Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \qquad \text{(Equation 3.4.1)}$$

The "posterior" distribution is the output of the distribution mix, denoted P(A|B). The notation P(B|A) is often called the "sampling" distribution, which can be read as the conditional probability of B given A. P(A) and P(B) is the probability of respectively A and B, where P(A) is commonly known as the "prior" distribution as stated above, and P(B) is a normalizing constant that is different from zero (Agresti, Franklin & Klingenberg, 2017). When these rules are applied to the Black-Litterman model, the equation should yield in this manner:

$$pdf(E(r)|\pi) = \frac{pdf\big(\pi\big|E(r)\big)pdf(E(r))}{pdf(\pi)} \qquad \text{(Equation 3.4.2)}$$

As explained by Satchell and Scowcroft (2000), pdf(.) are the probability density functions of the parameters in the equation above, and the information and formula above can be directly used to apply this to a Black-Litterman setting.

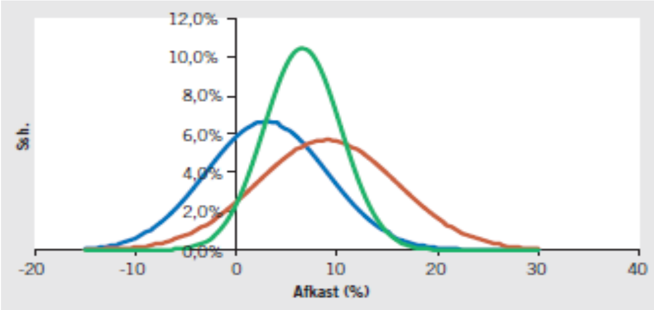


FIGURE 7: FINANS/INVEST NO. 4 (2016)

Further, Figure 7 shows how the two probability distributions of the prior and the views will be combined to a posterior distribution using the approach as explained above. The blue line illustrates the prior distribution, while the red line presents the view distribution. These are combined to a posterior distribution, viewed by the green line. In general, including more observations in the conditional distribution, the posterior distribution will converge towards the investor views, as we will introduce further below. If the investor is a 100% confident in mean estimates of the view distribution, making the standard deviation close to zero, then the Bayesian approach will not be significantly reflected in the posterior distribution (Plesner, 2016).

## 3.4.2 The original Black-Litterman model

Black-Litterman has been researched and investigated by academics ever since the Goldman Sachs article was released in 1990. Therefore, there are a variety of models known as "Black-Litterman", however, this project applies mostly the same approach as the Black and Litterman original "canonical" model described in He and Litterman (1999). Black-Litterman is based on the theoretical framework of CAPM because the starting point in Black Litterman is given by the equilibrium expected returns. These weights are a neutral reference point since the prices adjust after all assets are in equilibrium. Furthermore, the model applies the views held by the investor since these investors might have different opinions about the expected return on an asset than those that hold in an equilibrium. Accordingly, the framework is combining the information from the market equilibrium with investor's views about the market, so the optimal portfolio drifts away from the market portfolio towards the investor's views and creates a diversified portfolio (Plesner, 2016). This creates a new set of vectors of expected returns which can then be applied for portfolio optimization.

### 3.4.2.1 Equilibrium return vector

The equilibrium return vector is defined as the reference point of the Black Litterman model, and it is, therefore, important that the capitalized weights are represented through these. The CAPM weights can be computed in several ways. However, literature shows that either derivation from CAPM or reverse optimization is the most applied method. In the Bayesian framework, the equilibrium is defined as the "prior distribution", which is the market portfolio that includes a set of weights that all sum up to one and is more significant than zero. Eventually, when adding the investor's view, a "posterior distribution" is obtained as explained further in Section 3.3.4. The model retrieves the weights that are then included in the portfolio in accordance with the covariance between the view and the equilibrium (He and Litterman, 1999).

He and Litterman (1999(b)) found that the weight on a view is likely to increase when an investor is more bullish on the view. In contrast, it is reflected in the magnitude of the weight, which increases when the investor becomes less uncertain about the view (He and Litterman, 1999(b)). By deriving the equilibrium excess return $\pi$ from the market portfolio, an assumption regarding the asset returns must be that the returns are normally distributed with μ being the expected return and the covariance matrix $\Sigma$:

$$r \sim N(\mu, \sum)$$

where $r$ is the return (He and Litterman, 1999). The mean μ is also considered as a random variable which means that the investor needs to estimate μ. This is referred to as the "prior distribution", since the mean μ at this point is unknown, $\mu = \pi + \epsilon$. The following way of deriving the equilibrium return to obtain the mean μ thereby, is explained further.

**Reverse optimization**

The equilibrium return can be backed-out from the market portfolio by applying reverse optimization. Accordingly, reverse optimization can be defined as a contradiction to the mean-variance framework, as it uses portfolio weights to retrieve excess return vectors. The method provides a better alternative of forecasting equilibrium return since the input of vectors easier can be predicted. Secondly, the method gives a more realistic expected return rather than those in the mean-variance approach (Scowcroft, 2003).

The equilibrium excess return acquired from CAPM portfolio applies the unconstrained objective *quadratic utility function,* which is seen in the mean-variance optimization (Plesner, 2013) (He and Litterman, 1999(b)):

$$U = w^T \Pi - \frac{\lambda}{2} w^T \sum w_m$$

where $w_m$ is a vector with equilibrium weights, $\sum$ is a covariance matrix, U is the utility for an investor, $\Pi$ represents the equilibrium excess returns for each asset, and $\lambda$ as the risk aversion for an investor. In Markowitz's framework, the optimal portfolio weights are derived by differencing the utility function for the weights and setting it equal to zero:

$$\frac{dU}{dW} = \Pi - \lambda \sum w_m = 0$$

$$\Leftrightarrow$$

$$w_m = (\lambda \sum)^{-1} \Pi$$

It is assumed that the market is in equilibrium and therefore, the implicit equilibrium return is calculated from the capitalization weights. The model applies "equilibrium excess returns" as a neutral starting point given by:

$$\Pi = \lambda \sum w_m \qquad \text{(Equation 3.4.3)}$$

The equilibrium vector is an *n x 1* matrix with the excess return for the assets that are included in the CAPM market portfolio. $\sum$ is an *n x n* covariance matrix that is estimated from historical returns from the market portfolio. $\lambda$ is a positive constant stated by $\lambda = \frac{\mu_m - r_f}{\sigma_m^2}$, and is defined as the average risk tolerance which represents the inputs of the investment managers.

The expected returns are different from the original CAPM equation. The market portfolio only contains risky assets, since the investors, by definition, are only rewarded for taking on systematic risk. Therefore, all risk-free assets (hence $\beta = 0$) are not taking into account the market portfolio of BL. The problem using this equilibrium return, must be the estimation of covariance, $\sum$. It must often be assessed that the covariance comes from historical returns.

The Bayesian approach assumes that the expected returns $\mu$ is centred at the equilibrium values, which is normally distributed with a mean of $\Pi$ denoted by $\mu = \Pi + \epsilon^{(e)}$ where, $\epsilon^{(e)}$ is a vector that is normally distributed with a mean of zero and covariance of $\tau\Sigma$. $\tau$is representing the uncertainty of the CAPM prior. Thus, the expected return can be defined as a random variable, with a mean of $\Pi$ (He and Litteman, 1999).

This approach is the most applicable method and thus, also the most practical one. This method seeks to use a benchmark as a proxy for the market portfolio since the real estimation of the market portfolio is difficult to obtain. In the real world, applying an index or a benchmark portfolio is the most employed method to measure an investment manager's performance. Their performance is often compared to a benchmark, for example, to observe whether the manager outperformed the S&P Global 1200.

**Risk aversion**

Risk is usually associated with the standard deviation; however, some investors are more risk averse than others, i.e. being willing to risk more capital than others, and the behaviour of investors is indicated by the risk aversion coefficient as mentioned in Section 3.2.7. A higher risk aversion means a higher risk premium is required. Depending on the level of risk aversion, a higher risk aversion coefficient indicates a more risk averse investment, since the risk aversion is the definition on how much expected return, we are willing to miss out on, in order to reduce the risk. In other words, the risk aversion coefficient is thought of as a description of human investment behaviour. Since the risk-aversion coefficient requires the parameters of expected return and variance known, it may be more complicated to calculate. Although a majority of literature is provided on how to measure risk aversion, there has yet been a commonly accepted estimate. Probably the most referred estimates of the relative risk aversion coefficient lie between 1 and 3, however, a vast range of the coefficients in the literature show everything between 0.2 to 10. He and Litterman (1999) use a risk-aversion coefficient for the portfolios on 2.5, since it corresponds to the world level of risk-aversion.

### 3.4.2.2 Investor views

The next step in the model is to combine the views. The model blends the prior distribution with the conditional distribution, which is given by the investor views. Incorporating this into the model, the theoretical formula for the posterior distribution change, which gives different calculations for the returns and variance.

The investor's views are based on the fact that an investor can, and often do, have strong market beliefs that the Black-Litterman model manages to account for since they are staying away from the essential market neutral view. When incorporating the views, the portfolio drifts away from the market portfolio and thereby giving the ability to form it however one wants to. In the model, a view is defined as a statement about the expected excess return for any portfolio. Investment managers tend to focus on a small segment of an investment universe, where they are choosing assets, they feel are undervalued, assets with momentum or finding relative value trades. Additionally, investment managers often think of weights in a portfolio, as opposed to balancing expected returns relative to risk (He and Litterman, 1992). If the investor does not have any views on the market according to a BL model, the expected return is equivalent to the market equilibrium, which is the unconstrained optimal portfolio. Relative to the MPT, the BL expected returns is adjusted from their starting values by stating that the expected return is raising or dropping from an implied value. Including these views comes with a degree of uncertainty which has to be incorporated (He and Litterman, 1992).

The investor has $K$ views regarding the portfolio that is expressed by three components, Q, P and Ω. The prerequisites from the following components are leading to the following specification,

$$P\mu = Q + \varepsilon^{(v)} \qquad \text{(Equation 3.4.4)}$$

where $\varepsilon^{(v)}$ is a normal distributed random variable with mean equal zero, represented as an investor who has an uncertainty of views, and covariance matrix Ω, which expresses the uncertainty in each of the views. Since the views are mutually uncorrelated, the covariance matrix Ω is applied diagonal. If the variance is zero, it means that the investor is sure about the view. The vector $\mu$ is the unknown expected return, hence, needs to be estimated (Satchell and Scowcroft, 2000).

- *Q is a [k x 1] vector that expresses the relative change returns which contains either absolute or relative returns, i.e. the expected return of particular asset yields (absolute returns) or the expected difference in return between assets (relative view).*
- *P is a [K x N] matrix that shows portfolio weights in Q to the N assets given by the investment universe.*
- *Ω represents the uncertainty of the views given by a [k x k] variance matrix.*

This is an example of how views can be expressed in a matrix, where the vector Q are matched to specific assets by matrix P.

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} ; \quad Q = \begin{bmatrix} X\% \\ X\% \end{bmatrix}; \quad \Omega = \begin{bmatrix} \omega_{1,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_{k,k} \end{bmatrix}$$

This shows how the matrices are generated by examining six different assets where:

- *An absolute view where the investor believe that Asset 2 will create a return at X% with confidence $\omega_{1,1}$*
- *A relative view where the investor believe that Asset 3 will outperform Asset 4 by X%*

- *The rest of the asset does not hold any views from the investor and the return should not be adjusted*

Black and Litterman (1992) states for most of the view to be relative, saying one asset will outperform another. Another approach is to say the investor looks at the market to be bullish or bearish, explained as, above natural or below natural, of the market. Black and Litterman do not specify how views should be formulated; thus, it gives complete liberty for the investor to characterise these views. It should be noted that the P matrix indicates the subjective views of the investor and does not provide any indication whether these, in fact, are true or if these are unbiased. We assume that the investor believes that his predictions are unbiased.

**3.4.2.2.1 Covariance Ω**

If the investors are insecure about the views, thereby indicating low confidence, the views will have a relatively large standard deviation. Oppositely, if the investor is secure about his views indicating high confidence, the views have a lower standard deviation. If the standard deviation is low, it will not necessarily impact the expected return a lot. However, if the standard deviation is high, it will have much more influence on the expected return. This means that the variance of the views is proportional to the variance of the asset returns (Plesner, 2016.).

Black and Litterman did not specify a mathematical explanation of the **Ω** in their paper. Thus, the question of how to specify the diagonal parameters of **Ω** is yet not answered. Therefore, the covariance can be computed in several ways.

**Proportional uncertainty,** Ω

He and Litterman (1999) applied a computational method that used proportionality to the variance of the prior, ∑. This assumes that the variance is uncorrelated by each other. The ratio of $\Omega/\tau$is equal to the variance of the view denoted by the following expression (Walters, 2007):

$$\Omega = diag(P(\tau\Sigma)P', where \qquad \text{(Equation 3.4.5)}$$

$$w_{u,j} = P(\tau\Sigma)P' \ \Lambda i \ = \ j$$

$$w_{i,j} = 0 \ \ \Lambda i \neq j$$

For each of the $m$ expressed views, Ω is an $m \times m$ diagonal covariance matrix of the error that represents the uncertainty in each view. As described in e.g. Idzorek (2005) (and many other places):

$$\Omega = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{bmatrix} = \begin{bmatrix} \tau(p_1\Sigma p_1^T) & 0 & 0 \\ 0 & \tau(p_2\Sigma p_2^T) & 0 \\ 0 & 0 & \tau(p_3\Sigma p_3^T) \end{bmatrix}$$

He and Litterman describe the diagonal element of **Ω** as a function of (tau).

Meucci (2010) applies Ω differently and ignores the diagonal of the matrix. This is computed as

$$\Omega = \frac{1}{c} P \sum P^{\mathrm{T}}$$

Where c represents overall confidence in the views.

**Confidence interval,** Ω

One intuitive way of calculating the covariance is to specify a confidence interval of the estimate of the expected return. This could, for example, be stated as: "*It is expected, with a 95% probability, that the stock lies between a confidence interval of [0%;2%]."* Since the Black Litterman framework assumes normality, the confidence interval can be translated into standard deviation, as the covariance is the uncertainty of the estimate of the mean. As the probability can be translated into standard deviation, in this example, the standard deviation is 2%.

**3.4.2.2.2 Tau**

The $\tau$ is originally introduced in He and Litterman as a constant, where they mention it should be close to 0. Tau is associated with the uncertainty of view hence also the uncertainty of the investor's prior estimation of equilibrium returns, which would say the uncertainty of CAPM. $\tau$ is as described previously, the measure of the investor's confidence in the prior estimates. It, therefore, shows the constant of proportion between $\sum(\mu)$ and $\sum$.

> *"Because the uncertainty in the mean is much smaller than the uncertainty in the return itself, $\tau$ will be close to zero. The equilibrium risk premiums together with $\tau\Sigma$ determine the equilibrium distribution for expected excess returns. We assume this information is known to all; it is not a function of the circumstances of any individual investor". (He and Litterman, 1992)*

He and Litterman (1999) propose considering $\tau$ as the ratio of the sampling variance to the distribution variance, and thus it is 1/t. They use $\tau$= 0.05 since it corresponds to the confidence level of the prior CAPM if they applied 20-year of historical data.

"*...corresponds to using 20 years of data to estimate the CAPM equilibrium returns.*"

The value should be close to 0 because the uncertainty of the mean should be lower than the uncertainty of the variables argued by, He and Litterman. Following the work by Satchell and Scowcroft (2000) and Meucci (2010), the alternative reference model, they apply a setting of $\tau$equivalent to 1[1] and therefore it does not appear in the final model (Walters, 2013). Walter (2010) mention three different approaches to obtain $\tau$.

### 3.4.2.3 Posterior distribution

When combining the prior distribution following the framework of Black-Litterman, the posterior distribution, containing the new combined return distribution, is generated and denoted as a normally distributed variable with a mean $\bar{\Pi}$ and a covariance $M^{-1}$. This is also widely known as the Black Litterman asset allocation model. The investor's views and expected returns from the Bayesian prior both contain normally distributed random vectors and combining these, assuming that $\epsilon^{(e)}$and $\varepsilon^{(v)}$ are independent, the following is obtained (He and Litterman, 1999(b)):

$$\begin{pmatrix} \epsilon^e \\ \epsilon^v \end{pmatrix} \sim N\left(0, \begin{bmatrix} \tau\Sigma & 0 \\ 0 & \Omega \end{bmatrix}\right)$$

This gives conditional distribution where the expected returns are normally distributed $N(\bar{\Pi}, M^{-1})$, where the mean $\bar{\Pi}$ and the covariance matrix $\bar{\sum}$, denoted as $\bar{\Sigma} = \Sigma + M^{-1}$ , is given as the following

$$\bar{\Pi} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\pi + P'\Omega^{-1}Q] \qquad \text{(Equation 3.4.6)}$$

Equation 3.4.3 mainly consists of two elements. The first element of the equation functions as a risk-scaled weighted average of the excess equilibrium return and view returns. $(\tau\sum)^{-1}$is the scaled inverse matrix, $(P'\Omega^{-1})$ by the confidence of the views (He and Litterman, 1999(b)). The equilibrium can be written as (derivation can be seen in Appendix 1)

$$\bar{\Pi} = \Pi + \tau\sum P'[(P\tau\sum P') + \Omega]^{-1}[Q - P\Pi] \qquad \text{(Equation 3.4.7)}$$

And for the covariance matrix of the returns

[1] According to Walter (2014), Satchell and Scowcroft (2000) proposes "point estimates" as opposed to regular distributions, which is why the parameter tau is allowed to be set to 1.

$$M^{-1} = ((\tau\Sigma)^{-1} + P'\Omega^{-1}P)^{-1} \qquad \text{(Equation 3.4.8)}$$

And the posterior covariance is given the covariance matrix of return as well as the prior covariance

$$\bar{\sum} = \sum + M^{-1} = \sum + ((\tau\Sigma)^{-1} + P'\Omega^{-1}P)^{-1} \qquad \text{(Equation 3.4.9)}$$

The derivations of these can be seen in Satchell and Scowcroft (1997). The mean equation takes both volatility and correlations into consideration when computing the expected returns, and it is clear that the mean is estimated with uncertainty. This expected return is the return suggested to obtain the most optimized portfolio.

$(\tau\sum)^{-1}$= confidence on CAPM

$\Omega^{-1}$= investors confidence on the view

The mean equation, Equation 3.4.6, looks fairly complicated, however, the formula comes with specifications regarding the investor. If the investor does not hold any view on the market, the equilibrium is used, which then leads to the market portfolio as the implied equilibrium return, given by the formula $\bar{\mu}$. When switching the approach towards Black-Litterman, the methodology is the same, just instead assuming that Equation 3.4.6 contains P = 0 (showing that the investor does not have specific views about the market). This leads to

$$[(\tau\Sigma)^{-1}]^{-1}[(\tau\Sigma)^{-1}\pi] = \pi$$

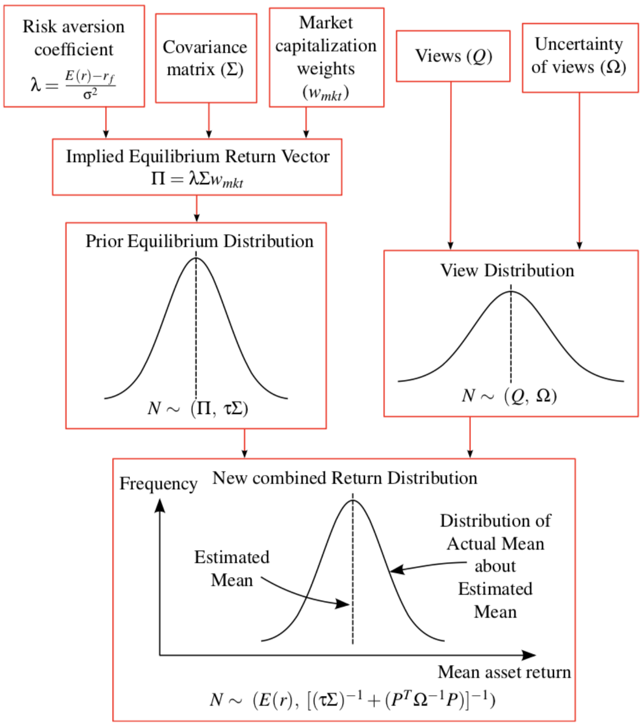


FIGURE 8: THE BLACK-LITTERMAN MODEL (SATCHELL AND SCOWCROFT, 2000)

Figure 8 simply illustrates the Black-Litterman approach, and the inputs, suggested by Satchell and Scowcroft (2000). The prior distribution, generated from the implied market equilibrium, will be combined with the view distribution arising from the premium prediction models. This finally provides us with the new combined distribution used in the optimization process.

### 3.4.2.4 Unconstrained optimal portfolio

In equilibrium, all investors hold the market portfolio $w_m$, and the equilibrium return is optimal in the sense that the demand for these assets equals the supply (Black, 1989) (He and Litterman, 1999). However, it can be quite challenging trying to understand the mathematical intuitions looking at the equations. In simplicity, the optimization process, in similarity with other traditional optimization cases, seek to account for the relationship, the return and volatility of the assets. The portfolio expected returns can be modified by applying relative volatility and correlations, since that simply would create a more sophisticated portfolio. Thus, looking at the optimal portfolio weight, instead of directly the expected returns, we can get a better understanding of the transition from view to the optimal portfolio. If an investor had unconstrained optimal portfolio, it could be represented as the risk aversion equivalent to $\lambda$ (He and Litterman, 1999(b)).

The unconstrained optimal portfolio, that was derived in Section 3.2.6, the expected returns were random variables that had to be estimated. Therefore, the expected returns do not follow the initial distribution, $N(\bar{\mu}, \Sigma\ )$, meanwhile the return distribution is now given as follows:

$$r \sim N(\bar{\mu}, \bar{\Sigma})$$

Where the covariance is denoted as $\bar{\Sigma} = \Sigma + \bar{M}^{-1}$.

After the posterior mean of the expected returns and the posterior covariance is estimated, they are applied in the application of the optimal portfolio weights. If returning to the optimal portfolio weights where the objective was to maximize the utility (He and Litterman, 1999):

$$w^* = (\lambda\sum)^{-1}\bar{\mu} = \frac{1}{\lambda}\Sigma^{-1}\bar{u}$$

Where $\Pi = \bar{\mu}$ if the investor is certain about his views. An alternative way of deriving this can be seen in Appendix 2 (also He and Litterman, 1999(b)).

The unconstrained optimal portfolio in Black Litterman is the market equilibrium portfolio $w_m$ and including a weighted sum of portfolios that comes from the investor's views which are scaled by the factor $\tau$, to adjust for uncertainty in the CAPM equilibrium. In other words, the sum of the portfolio weights will not summarize to 100%, like the equilibrium weights, due to the scaling factor $\frac{1}{1+\tau}$. Because of the Bayesian approach of being uncertain in the prior, they do not want to invest 100% of the wealth in risky assets.

The interpretation of Formula 3.4.9. can be stated as the more robust the view of an investor, the more allocation is towards the optimal portfolio. This can either be in terms of a higher expected return or a lower degree of uncertainty. Furthermore, the formula also shows the weights of the covariance of the investor's view and the market portfolio, which is being taken into account. This is because the covariance of the investor's view of a portfolio combined with the market portfolio indicates that the view has less new information (He and Litterman, 1999(b)).

**Comparison of Mean-Variance and Black Litterman**

The differences of the models are illustrated in TABLE 1:

| | Mean-variance optimization | Black-Litterman approach |
|---|---|---|
| **Asset mean** | Mean of historical asset returns | Blended asset return from prediction models and equilibrium returns |
| **Asset Covariance** | Covariance of historical returns | Covariance of historical asset returns and estimation of uncertainty of the asset returns |

TABLE 1: DIFFERENCES OF THE MODELS

Overall, the MV applies historical returns for the mean and covariance. The BL model combines the equilibrium returns with the views for the asset mean, and for the covariance, the approach blends the covariance of historical assets with the uncertainty of the asset returns. This will be further elaborated in the analysis.

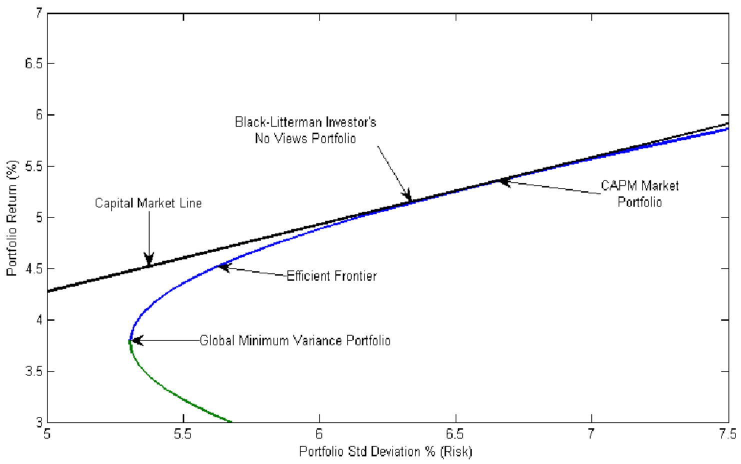


FIGURE 9: COMPARISON OF BLACK LITTERMAN, CAPM AND MPT (WALTERS, 2014)

Figure 9 illustrates the position of the respective portfolios used in the analysis. The point showing the Black-Litterman investor with no views can be interpreted as the CAPM equilibrium. When views are introduced, i.e. when the investor have opinions about the future asset movements, the Black-Litterman portfolio will deviate from this point. Including the uncertainty parameter, $\tau$, the efficient frontier will move slightly to the right as we are investing less than 100% in risky assets. This is adjusted later on by scaling upwards to 100% (Walters, 2014).

# 4. Data and methodology

## 4.1 Data description

The data description reports how data is screened and which criteria that has been chosen for the data sample throughout this study. Besides, it also presents the methodology of the data of indices, bond-and stock predictability, benchmark and the risk-free rate.

### 4.1.1 Data on portfolio allocation

#### 4.1.1.1 Data sample

The performance of MV-model, CAPM and BL-model will be compared in a simple investment universe. Several methods can be applied to available data to construct a well-diversified portfolio. However, this project is restricted to apply a simple U.S. multi-asset portfolio, consisting of U.S. Equity Index and U.S. Treasury bonds. The asset classes in the data are represented by an equity index and a bond index. It has been decided to use American-domiciled indices as a data reference in this study. Initially, this is also discussed in Section 2.3, where literature has shown that many examinations have concerned the US. market. The decision regarding the selection of the investment universe has been made on the basis of one factor; literature found on prediction variables. Literature has shown that the prediction of U.S. equity and bonds is relatively robust evidence. It was suggested from Munk (2019) that the application of asset classes was more robust in terms of predicting future returns, since these where less sensitive towards unsystematic risk. Since the aforementioned Black Litterman model requires view generation of assets, the prediction of the multi-asset portfolio seems to provide the most statistical evidence.

#### 4.1.1.2 Database

The data of the equity and bond index are extracted from the Bloomberg database. Bloomberg is a database that provides financial news, analysis, and real pricing data of financial data, furthermore, the database is highly applied by practitioners in the financial industry. It offers possibilities to obtain high-quality data on practically any financial asset traded. Using data from this database has some advantages that ultimately enhances the reliability of the conclusions drawn in this paper. The main advantage is that Bloomberg is one of the leading data providers within asset management. Also, Bloomberg is a trusted data source that seeks to deliver high-quality data. Thus, the data extracted is representing the accurate data on portfolio allocation desired.

### 4.1.1.3 Sample period

The relationship between stock and bonds is quite relevant and influences the asset allocation of the portfolios. Since the late 1900s, it has been widely known that the correlation between stocks and long-term bonds has mostly been positive, i.e. increasing stock prices resulted in rising bond yields (Shiller and Beltratti, 1992; Campbell and Ammer, 1993). Recent studies show variation in the stock-bond correlations is negative in some periods, also pointed out by Ilmanen (2003), Cappiello, Engle, and Sheppard (2006), and Andersson, Krylova, and Vähämaa (2008). Periods with financial uncertainty, causes an opposite shift in the prices of stock-bond correlation, due to investors changing from stocks to bonds.

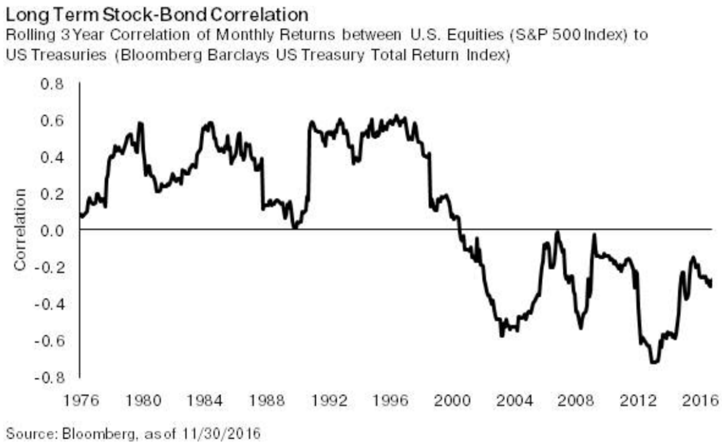


FIGURE 10: VARIATION IN THE STOCK-BOND CORRELATIONS

FIGURE 10 represents the correlation from the year 1976-2016. The correlation is based on a three-year rolling window of the monthly returns on the S&P 500 and the U.S. Treasury. In general, the stocks-bonds have had a positive correlation from 1980 to 2000, while negative correlation had been observed from 2000. A positive correlation means for the stocks-bonds relationship, that when stock prices reduce, oppositely, the bond prices increase. In the last decade, accordingly, low positive or/and negative correlation appeared during stock market turbulence, indicating that stock and bond diversification of each other is better during a recession than over expansionary economy. Aiming the study within this specific period makes it possible to compare different sub-periods and distinguish between opposite correlations in stock-bonds. Since many asset management algorithms are based on the fact that stocks and bonds were positively correlated back then, this project analyses both perspectives. In this specific study, the sample period follows from 1980 - 2019. This period includes economic cycles in terms of strong bull and bear trends, high/low volatility periods, and stagnating periods, which should, to a great extent, give the ability to capture the short-term market circumstances. Since portfolio allocation depends on these states of the market, it is essential to include these different economic cycles. Furthermore, the period is split into different sub-periods from the years 1980 - 2000 to model in-sample estimates and the years 2000 - 2019 to predict pseudo-out-of-sample. This gives the ability to generate prediction by applying the first sub-period as an estimation window and then the second sub-period to estimate out-of-sample to observe the allocation in 2000 based on the historical data.

Time series analysis requires available historical information over time. It is often troublesome to select the best-fitted frequency of the data set because it highly depends on the hypothesis of the analysis and what you want to investigate. Various obstacles could interrupt the results of the analysis during the period of the data. Common disturbance in daily data is often related to time zone delay, different currency, structural breaks, etc. To eliminate some of the disturbance, monthly data will be applied adjusted for closed trading days, etc, since these noises will even be evened out in a monthly scheme over a daily. Monthly observations are often sufficient enough to capture information, according to literature. Primarily, the benefit of using monthly returns instead of weekly returns is that the returns are approximately normally distributed. Furthermore, long-horizon returns tend to approximate closer to normal distribution compared to short-term returns (Campbell et al., 1997). However, daily observation has the benefit of giving more accurate and precise covariance matrix. Nevertheless, due to the lack of available daily and weekly data in the 10-yr treasury index, monthly frequency is used.

### 4.1.2 Portfolio allocation

#### 4.1.2.1 S&P 500

For this study, the equity index is simply covered by the S&P 500. This index tracks the 500 most prominent companies in the U.S. and indicates the stock market's performance by applying the return and risk from the companies in the index. The index has approximately 80% coverage of the available market capitalization, thus, gives a broad representation if used as a proxy of the total market (Bloomberg). The index applies a market capitalization weight which gives a higher allocation for those companies with a larger market capitalization. The S&P 500 index is probably the most commonly referenced U.S. equity benchmark, and many regards it as the single best way to track the overall performance of the largest and most dominant American companies. The data extracted covers the period 01/01/1980 to 31/12/2018 and is provided on weekly prices but resampled into a monthly scheme instead.

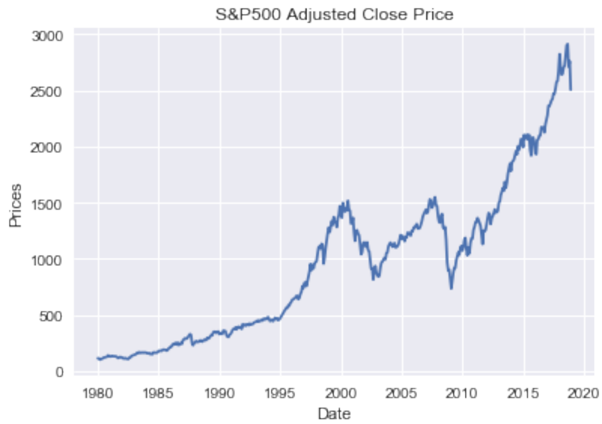


FIGURE 11: S&P 500 PRICES

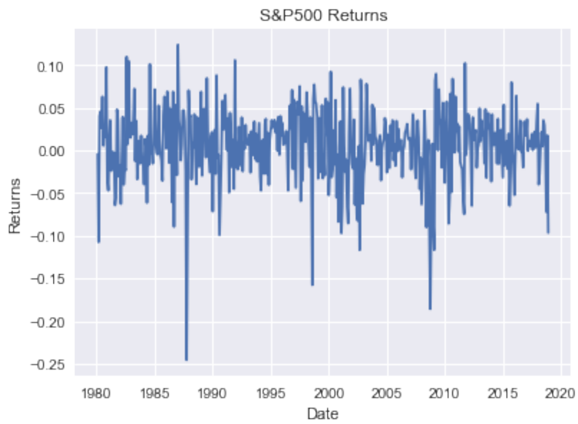


FIGURE 12: S&P 500 RETURN

For S&P500, the level data exhibit rising prices of the whole sample period. From 1980 to 2000 the index appears to have a substantial rise from approximately 1996 - 2000. Nevertheless, this is followed by a period with declining prices in 2000 and 2009 due to respectively, the dot com bubble and financial crisis. The stock index, however, appeared to have reversed again to a bear market. Looking apart from this, the stocks, generally, have risen during the years. The stocks have performed superior, especially, when looking at 2010 to 2018. The overall boost in the market seems to be a result of fuelling from the tech companies and uncertainties. In the past years, since 2013, the index S&P 500 has led to its best performance besides looking at the decrease in 2018. This decrease was a result of fear of economic slowdown along with Brexit and a slowdown in the Chinese economy. From the beginning of 2018, the stock index experienced a decrease due to market uncertainties, which had not been seen for many years (since the financial crisis). FIGURE 12 shows the return on the S&P 500, which indicates higher returns during the beginning of the 1980s. During the 1990s the returns appeared to be small and up towards the 2100th century, the returns began to show a higher degree of volatility clustering.

#### 4.1.2.3 10-yr U.S Treasury

To represent the 10-year U.S. Treasury, the investment-grade bond - Bloomberg Barclays U.S. Treasury Index - is applied. The Treasury index is an index based on the action of the U.S. Treasury bills and is commonly used as a benchmark when determining interest rates. The index measures U.S. dollar-denominated, fixed-rate, nominal debt issued by the U.S. Treasury. It is often common for the 10-year U.S. Treasury to be used as a benchmark. Since the Treasury securities are considered to be a riskless investment that is guaranteed and credited to the U.S. government, the securities are evaluated as a risk-free return. The data extracted cover the period 01/01/1980 to

31/12/2018 based on a weekly horizon. However, from 1980 to 1994, the index is only provided monthly, and therefore the data is, as a whole, resampled into monthly data.


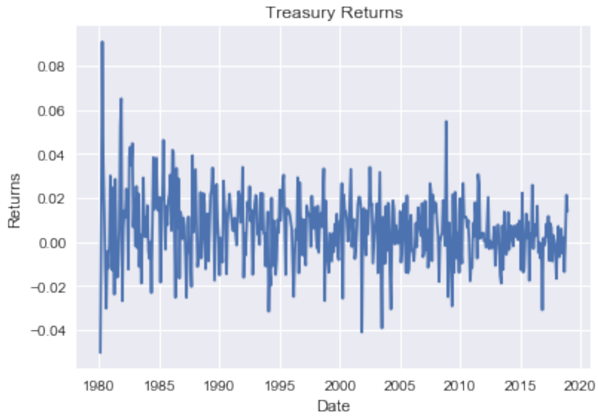


FIGURE 13: 10-YEAR TREASURY PRICES

FIGURE 14: 10-YEAR TREASURY RETURNS

Figure 13 shows the price level of 10-year Treasury over a horizon of 1980 - 2019. The prices of the Treasury have risen during the years, due to the fallen bond yields. The Treasury index appears to have been in a bear market, with a trend of rising prices and the bull market as a result of decreasing bond yield. The downward trend of bond yields has been observed since the early 1980s, where the U.S. Federal Reserve raised interest rates with the commitment to beat the inflation as the inflation peaked in the early 1980s. The steadily decrease in Treasury yields is one of the most lasting effects in finance. Also, in 2008 during the financial crisis, the Federal Reserve brought back bonds which increased prices and decreased the yields. The returns on 10-year Treasury shows that the returns have been quite high during the 1980s, and with a more stable outcome of returns.

### 4.1.2.2 Descriptive statistics

The analysis aims to identify the portfolio returns, and therefore the prices are transformed into returns. The returns are given as the change simple return of each market index price:

$$r_{i,t} = ln\,(P_{i,t}/P_{i,t-1}) - 1$$

Where $Pi(t)$ is the monthly adjusted closing price of stock $i$ at day $t$.

The descriptive statistics will be provided in a table to compare the indices.

| Descriptive Statistics | Mean | Std.dev | Observations | Sharpe ratio | Kurtosis | Skewness |
|---|---|---|---|---|---|---|
| S&P 500 | 7.9368% | 14.9549% | 467 | 53.0715% | 3.3596 | -0.9234 |
| 10-year Treasury | 7.0008% | 5.4331% | 467 | 128.8546% | 2.3717 | 0.3713 |

TABLE 2: DESCRIPTIVE STATISTICS OF ASSET ANNUAL RETURNS (FULL SAMPLE)

Table 2 provides an overview of the descriptive of the S&P 500, and 10-yr U.S Treasury provided on an annual basis, giving the overview of the full data sample presented in this paper. For the stocks, the sample mean is on average 7.9369 % compared to bonds where the average return has been 7%. In general, the stocks do not contain a much larger return compared to the bonds during this sample period. The standard deviation of 14.96% on stocks indicates high volatility, compared to bonds, where the standard deviation is 5.43%. Stocks tend to follow trends of volatility clustering, and therefore, it is natural for the stocks to have a higher standard deviation compared to bonds, having lower exposure towards risk. The Sharpe ratio on stocks is 53 % which have been considerably lower than bonds with a Sharpe ratio of 128.85 %. Often, many financial data are now to exhibit non-normal distribution, which is not the case for this specific sample. Assumptions on the mean-variance and Black-Litterman requires returns to be normally distributed, therefore describing the kurtosis and the skewness. Kurtosis describes the tails in a probability distribution and with positive values of 3.3596 on stock and 2.3717 on bonds, compared to a kurtosis on 3 of the normal distributions, indicating for the distributions to be a nearby normal distribution. For the skewness of the returns, it is normally observed for them to deviate from 0, although, they are close to being symmetrical. A rule of thumb for an acceptable skewness is a range within [-1,1], and anything outside this, is considered as a highly skewed distribution (George & Mallery, 2010; Ryu, E, 2011; Bachman, 2004). The returns of S&P 500 are displaying negative skewness with a value of - 0.17 and indicates for most of the returns tends to be negative. The Treasury is showing positive skewness, which represents for the bonds to be positive mostly. If very well-known that returns rarely display normality and therefore, well accepted.

Stock volatility varies over time also shown by the volatility of the descriptive statistics and is an anomaly defined by Mandelbrot (1963) as: "*large changes tend to be followed by large changes, of either sign or small changes tend to be followed by small changes*". In other words, volatility clustering indicates that large or small returns often come together in periods, meaning that volatility is non-constant over time. This means significant autocorrelation for all lags in returns. When observing fairly short horizons, stock returns tend to have positive autocorrelation, also known as short-term momentum. Usual, negative autocorrelation appears on longer horizons, which is called long-term reversal or mean-reversion. DeBondt and Thaler (1985), Fama and French (1988), Jegadeesh and Titman (1993, 2001), Campbell, Lo, and MacKinlay (1997), and Cochrane (2005) all discuss prediction in returns. Overall, it can be said that past positive returns predict returns in the near future, whereas negative returns predict returns later into the future (Munk, 2019).

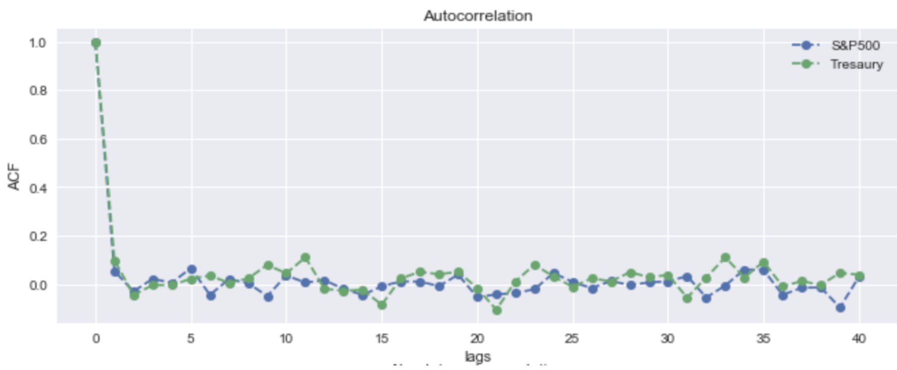


FIGURE 15: AUTOCORRELATION FOR S&P500 AND 10-YR TREASURY

FIGURE 15 displays the autocorrelation for the S&P 500 and 10-yr Treasury. For the absence of autocorrelation, it appears to be within a range of -0.2 and 0.2 hence, the autocorrelation is not said to be strong for the indices. This suggests no significant autocorrelation (>0.2) present at time t and up to 40 months behind. Black (1976) observed for stock volatility to have a negative correlation with the return, meaning that high volatility is present in periods with low returns and the other way around. The autocorrelation appears to be decreasing towards negative, mostly for stock. The returns exhibit oscillating movement between positive and negative autocorrelation. However, it cannot be said whether these have momentum or mean-reversion since none of them has persistently positive or negative autocorrelation. Though, stock returns appear to have more negative autocorrelation, which might indicate mean reversion during longer horizons. Times-series momentum has been documented for stocks, bonds, among other securities, where returns over 12 months have positively predicted the returns for the next month (Moskowitz, Ooi, and Pedersen, 2012). The methodology on predictability on stocks and bonds will be mentioned further in Section 4.2.2.

### 4.1.4 Data on risk-free rate

The risk-free rate is often known as the rate at which the investor, as a minimum, expects to receive on an investment that carries zero risks. The rate is often proxied by the short-term interest rate, which is the rate that is short-term borrowed or short-term issued/traded. Short-term interest rate is often based on three-month rates (OECD, 2020). Accordingly, for the US-market the 3-month Treasury bill is often used as a benchmark for the risk-free rate.

For this reason, the 3-month Treasury Bill: Secondary Market Rate (TB3MS), extracted from the Federal Reserve Bank of St. Louis (FRED), is used as the benchmark for the risk-free rate, also applied by Goyal & Welch (2007). The treasury bills are associated with low exposure towards financial risk since these are issued by the U.S. Government, with no incentives to default its bonds. The risk-free rate applied in the sample and spans from 01-01/1980 to 31-12/2018 containing 468 observations.

The rate is given in yearly observation and therefore, reformulated to monthly log-rate consistently aligned with the comparable data's

$$r_{f,m} = \frac{ln(1 + r_f)}{12}$$

where $r_{f,m}$is the monthly risk-free rate.

The risk-free rate is required in order to calculate the excess return on the assets in the portfolio.

### 4.1.5 Data on premium prediction

#### 4.1.5.1 Data on equity premium prediction

Goyal & Welch (2007) provides an updated data set, which presents data on the fourteen different regressors. This will be used to explain the monthly changes of the excess returns of the equity index S&P 500 in the time period 01/1980 to 12/2018. The excess returns are constructed as the total rate of return minus the short-term interest rate. This empirical analysis requires the use of continuously compounded excess returns of the S&P 500, combined with a data frame consisting of the predictive variables from 01-01/1980 to 01-12/2018.

The data frame collected from Goyal consists of the following 14 variables: (1) the sum of dividends on the S&P 500 index, (2) risk-free rate reflected by the treasury bill, (3) earnings, (4) stock return variance, (5) cross-sectional premium, (6) book value, (7) corporate issue activity, (8) Treasury bills, (9) long-term yield, (10) returns on corporate bonds, (11) inflation, (12) investment to capital ratio, (13) all variables mentioned above and (14) consumption/wealth ratio (income ratio). The dividend, earnings and net equity expansions are computed as the 12-month moving sum, while the stock return variance is calculated as the sum of (daily) squared returns on the S&P 500 index. Further description and origin of the data frame can be further explored in Goyal and Welch (2007). This section of our paper seeks to replicate the study of Rapach et al. (2007). However, it required some adjustments to make it applicable in our analysis. Accurately, the variables applied in our analysis from Goyal's dataset is presented as follows (Description of these in Appendix 3): [2]

- Dividend Price ratio ($DP_{log}$)*
- Dividend Yield ($DY_{log}$)*
- Dividend Payout ratio ($DE_{log}$) *
- Earnings Price ratio ($EP_{log}$)*
- Book to market ratio (b/m)
- Stock return variance (svar)
- Yield spread (yieldspread)*
- T-bill rate (tbl)
- Long term yield (lty)
- Net equity expansion (ntis)
- Inflation (infl)
- Long term return (ltr)*
- Term Spread (tms)*
- Default return spread (drs)*

The graphs of single variables are shown in Appendix 4. Due to a lack of observations in the dataset, the variable Cross-Sectional Premium (csp) had to be excluded.

[2] Variables containing * are calculated variables described in Appendix 2

### 4.1.5.2 Data on bond premium prediction

The data applied in the bond premium prediction is given by The New York Fed economists Adrian, Crump, and Moench (*ACM-model*). They present term premia estimates for maturities from one to ten years, with corresponding fitted yields and expected short-term rates (New York Fed). The data is updated daily and re-estimated every month to ensure that the data is reflected on the most recent information. The data is employed in a monthly setting and spans the period 01-01/1980 to 31-12/2018, i.e. a total number of 475 observations. For each date, the yield is provided per annum in percent.

In this study, only the term premia estimate for the 10-year maturity point is applied to investigate the prediction power of the 10-year term premium. Although this data is given for a longer horizon, this project only seeks to apply data from 1980 since earlier historical data should not be longer than our sample period and be a possible explanation of the view generation of the BL-model.

| **Names** | **Description** | **Transformation** | **Source** | **Frequency** |
|---|---|---|---|---|
| ACTP10 | Term premia: t10yr-t3m | Level | ACM | Percent |
| LUATTRUU | 10-year Treasury | Returns | Bloomberg | Percent |
| rf | 3-month T-bill | Level | FRED | Percent |

TABLE 3: DATA ON BOND PREMIUM

To apply to bond premium prediction, the excess return is computed as

$$r_t^{\text{bond}} - r_f = log\left(\frac{y_t}{y_{t-1}}\right),$$

where $r_t$ - $r_f$ is the excess return on the 10-year bond while $y_t$ is the 10-year treasury index.

The term premium is applied on level data, see Appendix 5. The difference between the risk neutral yield and the model-implied fitted yield is the definition of the term premia according to ACM, where their model-implied yield is obtained from their five-factor-model, also the no-arbitrage term structure model.

## 4.1.6 Market portfolio

A market cap allocation is widely applied in the real world among portfolio managers, to obtain the optimal return through the exposure of a diversified portfolio consisting of equity and fixed income. This allocation is composed of 60% equities and 40% fixed income (Chaves et al., 2011). It has been confirmed that the equity weights in the U.S vary over time, but the positive correlation has seen to be present throughout the earlier year. Therefore, the portfolio of 60:40 has shown to provide a favourable diversification. Roll (1977) states, the market portfolio definitions in the theoretical CAPM, is an index of multi-asset classes, for example, equities, bonds, etc. Since the CAPM is a capitalized weighted index; hence, the CAPM allocation should use the weights of the market cap between stocks and bonds.

Theoretically, the definition of the market portfolio is widely understood. However, the computation of the market portfolio is very restricted, due to the challenging way to obtain all the comprehensive data required. To characterize the market portfolio in the U.S. market by the application of the stock/bond setting, we define the market capitalization of equity by S&P500 and the market capitalization of bonds by U.S. Treasury.

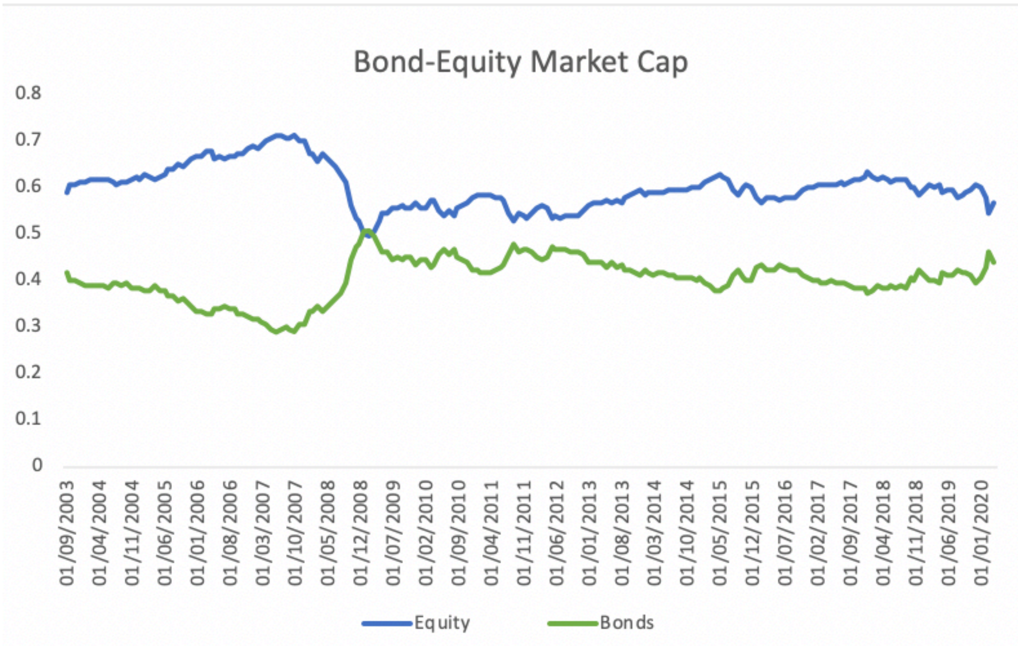


FIGURE 16: MARKET ALLOCATION OF BONDS AND EQUITIES FROM 2003 – 2019

The graphs in Figure 16 illustrate the global market allocation and is proxied by Bloomberg World Exchange Market Capitalization (WCAUWRLD Index) and Bloomberg Barclays Global multi-currency benchmark (LEGATRUU Index). In general, the allocation between equity and bonds fluctuates around 60:40, like the standard allocation of bonds and equities. There appear to be periods with a higher market capitalization in stocks and periods with less, correspondingly the same is observed for bonds. This indicates that it is valid to use the standard allocation to proxy the market capitalization in order to obtain equilibrium weights of the Black-Litterman model. Even though this market capitalization is shown globally, this approximation is also used for the U.S. market. Due to the lack of data for the stock index before 2003, the allocation of 60:40, in respectively stocks and bonds, will be applied constantly throughout the analysis as a proxy.

## 4.2 Methodology

The methodology of this study is based on a quantitative approach, which is based on the times series of financial data. The main content of the methodology concerns the specification of the regression models, combined with the approach of the portfolio generation and the evaluation of the performance. To evaluate the regression models, error measures will be presented. The conclusion, on how to create the most established portfolio allocation, can be drawn from the performance of these portfolio. Finally, the theory presented above, is considered crucial to construct the portfolios.

### 4.2.1 Portfolio estimation

In order to investigate the workings and the performance of the mean variance and Black-Litterman model, the portfolios will be evaluated using two different methods. The first two allocation models are based on the in-sample period, providing us with one set of optimal weights to invest our wealth in at the beginning of the out-of-sample period. Furthermore, this will be used to investigate the workings of the different models. However, there is a possible weakness of the approach, based on constructing the model parameters over a long period, as it may lead the covariance matrix to estimation errors. If there is an indication of time-varying covariance, a change in the correlation matrix or other parameters during our in-sample period, these estimation errors might appear. Due to the fear that the relation in the indices' returns may vary over time, a rolling estimate will be applied.

| Model | Description | Estimation period | Window (H) | Abbreviatio n |
|---|---|---|---|---|
| i | Mean-variance, unconstrained portfolio | 01-01-1980 to 01-12-1999 | In-sample period | $MV_s$ |
| ii | Black-Litterman, unconstrained portfolio | 01-01-1980 to 01-12-1999 | In-sample period | $BL_s$ |
| iii | Mean-variance, unconstrained rolling portfolio | 01-01-1980 to 01-12-2018 | 72 | MV |
| iv | Black-Litterman, unconstrained rolling portfolio | 01-01-1980 to 01-12-2018 | 72 | BL |

TABLE 4: DESCRIPTION OF THE PORTFOLIOS AND ESTIMATION PERIODS

A central part of the following analysis is the application of rolling estimates to capture the change in movements of the indices. This is done by splitting the data frame into multiple sub-samples/periods, hopefully providing a more accurate presentation on how the variables behave within these periods since the approach only takes into account the most recent information. It is, furthermore, possible that this method will give a more precise allocation, due to a shorter estimation period. The model will, possibly, have better performance because it is estimated on more recent data, which could eliminate some estimation errors, especially if the relations of the index returns vary over time.

In this paper, we use monthly observations of the return processes of the assets, however, it is entirely possible to use more frequent data observations to obtain a more precise estimate of the expected return vector and covariance matrix. The procedure of rolling estimation follows the approach applied by DeMiguel & Uppal (2009), and will be carefully explained:

1. We select a window with the length H=72, reflecting a window of six years using monthly data.
2. The initial starting point will be T= H+1
3. For each observation t, the expected return vectors and covariance matrix will be estimated on the last H observations as illustrated in Figure 17.

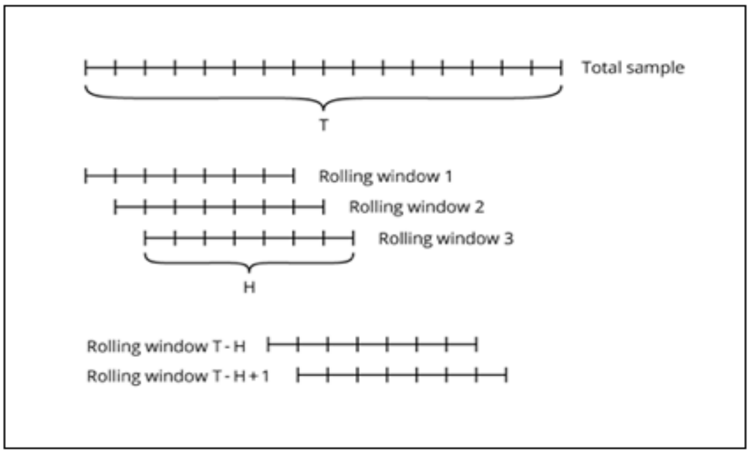


FIGURE 17: ILLUSTRATION OF THE ROLLING WINDOW ESTIMATE

4. This provides 227 total estimated observations to further apply in the optimization processes to implement the given allocation strategies.

### 4.2.2 Testing the out-of-sample performance

The portfolios' out-of-sample performance will be evaluated using multiple performance measures. The out-of-sample models will also be split into different sub-periods, to further investigate the models performs better in some particular periods. In addition to the assessment of the cumulative portfolio return in the out-of-sample period, we will evaluate the performance of our models, $k$, based on the Sharpe Ratio (SR), M-squared and the Certainty Equivalent (CE) explained by DeMiguel and Uppal (2009) and Bodie et al (2014). Lastly, the t-statistics of the portfolio will be computed.

The out-of-sample performance will mainly be evaluated by the portfolios respective Sharpe Ratios. The Sharpe Ratio provides us with a measure of the risk-return relationship for each of the portfolio allocation approaches, denoted as k (Munk, 2019). It allows to compare if one portfolio is better than another and evaluate if some periods provide better performance in contrast to other periods. The formula is defined as the out-of-sample excess return, $\hat{\boldsymbol{\mu}}_k$, divided by the out-of-sample standard deviation $\hat{\sigma}_k$. The formula is presented followingly:

$$Sharpe\ Ratio\ (SR)_k = \frac{\hat{\mu}_k - r_f}{\hat{\sigma}_k} \qquad \text{(Equation 4.2.1)}$$

The statistical significance of the out-of-sample Sharpe Ratio is tested using the approach of Jobson & Korkie (1981). In order to test the Sharpe ratios in one sub-period relative to another, we propose the following approximate Sharpe ratio estimator to adjust for any observation bias:

$$\widehat{SR}_{I*} = \frac{(\hat{\boldsymbol{\mu}}_k - r_f)}{\hat{\sigma}} * \left(\frac{1}{1+\left(\frac{0,75}{T}\right)}\right) \qquad \text{(Equation 4.2.2)}$$

The approach is applied on both allocation models, and sub-periods, to establish their individual performance assessment (out-of-sample). Further, the statistical performance of the Sharpe ratios will be tested using the procedure suggested by Jobson & Korkie (1981). The following hypothesis tests the portfolios:

$$H_0: SR_{k_1} - SR_{k_2} = 0,$$

where it is assumed that $SR_k \sim N\left(\frac{(\hat{\boldsymbol{\mu}}_k - r_f)}{\hat{\sigma}}, \left(\frac{1}{T}\right)\left(1 + \left(\frac{\hat{\mu}_k^2}{2\sigma_k^2}\right)\right)\right)$.

The transformed difference for Sharpe measure

$$\widehat{Sh}_{in} = s_n \bar{r}_i - s_i \bar{r}_n$$

The Sharpe statistics, the variance is given as:

$$\theta = \frac{1}{T}[2\sigma_i^2\sigma_n^2 - 2\sigma_i\sigma_n\sigma_i + \frac{1}{2}\mu_i^2\sigma_n^2 + \frac{1}{2}\mu_n^2\sigma_i^2 - \frac{\mu_i, \mu_n}{2\sigma_i\sigma_n}(\sigma_i^2 + \sigma_i^2\sigma_n^2$$

The Certainty equivalent will serve as a measurement for the utility score of an investor (Bodie et al., 2014), used to investigate the rank of the portfolio return accounted for the risk, in addition to the risk aversion. According to DeMiguel and Uppal (2009), the investor is willing to take on a risk-free rate of this measure (CE) instead of the given risky portfolio allocation. The measure is defined as

$$Certainty\ Equivalent\ (CE_i)_k = \boldsymbol{\mu}_k - 0{,}5\lambda\ \sigma_k^2 \qquad \text{(Equation 4.2.3)}$$

where $\boldsymbol{\mu}_k$ is the excess return on each portfolio k, $\lambda$is the risk aversion and $\sigma_k$ is the standard deviation of the return process for each portfolio.

As a final performance measure, we will look at the M-squared measure. The measure was originally developed by Modigliani and Modigliani (1997) seeking to find a better measure for performance interpretation than what can be assumed and seen from the Sharpe Ratio. The M-Squared is given as follows;

$$M - Squared_k = \boldsymbol{\mu}_k \left(\frac{\sigma_m}{\sigma_k}\right) + r_f \left(1 - \left(\frac{\sigma_m}{\sigma_k}\right)\right) \quad \text{(Equation 4.2.4)}$$

This computation uses the market portfolio volatility so that the portfolios and a risk-free asset are weighted with the proportion $w = \frac{\sigma_m}{\sigma_k}$ and $1 - w = 1 - \frac{\sigma_m}{\sigma_k}$ respectively. This makes it possible to assess the performance with the risk taken (Munk, 2019).

The statistical significance of the portfolios has to be tested, because it is not sufficiently reflected by the measures described above. This is done by calculating the respective t-statistics of the assets and portfolios the following way:

$$\text{t} - \text{statistic} = \boldsymbol{\mu}_k \cdot \sqrt{T}/\sigma_k$$

## 4.2.2 Regression models

OLS regressions are computed in order to specify investor's views on the Black Litterman portfolios. The predictive power of the variables is explored in order to obtain the premiums. Regression analysis is an important statistical tool to investigate and establish if a chosen variable, or multiple variables, have an impact on the future movement of another variable (Enders, 2014). The regression models are based on historical data by applying time-series modelling. The models will be tested to detect out-of-sample prediction of the views, meaning that from the period 1980 to 2000 will be used as an estimation window to get a model that can explain out-of-sample for 2000 and further.

### 4.2.2.1 Equity premium prediction

Multiple regression models will be used to explain the change in tomorrow's return on the S&P 500 index. A general prediction model for equity excess return is given as

$$r_{t+1} = \alpha + \beta x_{i,t} + \epsilon_{t+1} \quad \text{(Equation 4.2.5)}$$

where $r_{t+1}$ is the premium of an equity index, $x_{i,t}$ are the fourteen different variables, presented by Goyal and Welch (2007), that supposedly have predictable features over the index in question and $\varepsilon_{t+1}$ is the error term of the model.

The forecast analysis' will follow a large part of the methodology of Rapach et al. (2007) to predict the fourteen variables, and weighted them together in a combination forecast, using an average of the predictor variables. Goyal and Welch (2007) and Rapach et al. (2007) use a recursive estimation window. However, we will apply linear OLS regressions based on the in-sample forecast to predict out-of-sample forecast for the S&P 500. This means, that constants model coefficients, alpha and betas are carried out as estimates to forecast.

After performing the described procedure on all mentioned variables above, the combination forecast suggested by Rapach (2007) is applied to produce a forecast of the premium on the S&P 500 index using the fourteenth predictor variables. Rapach et al. (2007) present multiple suggestions on how to calculate a premium forecast that consistently outperforms the historical average. However, we chose to use the following simple averaging method presented in their paper:

$$\hat{r}_{c,t+1} = \sum_{i=1}^{N} \omega_{i,t}\hat{r}_{i,t+1} \qquad \text{(Equation 4.2.6)}$$

The equation is based on the results arising from the N individual prediction models estimated in Equation 4.2.6 above and the equation is based on the forecasted returns of the SP 500 index rather than the actual realized returns. This approach uses the mean average approach providing us with $w_{i,t} = 1/N$ for i = 1,.., N for each monthly observation t. Further, we use the estimated expected excess return, $r_{i,t+1}$, for all monthly observations t in our out-of-sample period.

### 4.2.2.1 Bond premium prediction

The methodology of the bond premium prediction is based on the economists from New York Fed (ACM) where the bond yield prediction is explained by OLS linear regression. Following three-step regressions for the parameters of the model:

1. The bond premium is estimated applying ordinary least squares decomposed into the in-sample period.
2. The excess returns are regressed on a constant and a lagged pricing factor, i.e. term premia according to the regression model

$$r_t^{\text{bond}} - r_f = \alpha + \beta TP_{t-1} + \varepsilon_t \qquad \text{(Equation 4.2.7)}$$

Where the $r_t - r_f$ is the excess bond premium, and TP is the term premia given from ACM.

The model applies the coefficient of the OLS-regressions

$$[\hat{\alpha} \ \ \hat{\beta}]$$

and store the model coefficients, alpha and beta.

3. The regression model is now used to the application of the out-of-sample dataset, to predict the excess return based on the in-sample a linear forecast.

In the real estimation of the term structure, ACM applies a five-factor model as their baseline specification, in which they compare the four-factor model known from Cochrane and Piazzesi. They notice that the risk-free short-term rate, including other pricing factors, provides them with the estimates of the zero-coupon yield without observing the curves. Generally, the term premium is said to reflect compensation for holding long-term bonds, but in reality, several factors influence the bond yield being the expectations and term premium components. One should note, that the methodology of ACM estimates the bond-yield, while we in fact, tries to estimate the bond price.

The regressions require that the term premium will have some information about the excess return in the long-term bonds. However, this is already found to be accurate, since the term premia is based on their the five-factor model of ACM, which fitted the data of zero-coupon yield provided by Gurkaynak, Sack, and Wright (2007) exceptionally well. As shown in APPENDIX 6, the term premium has been positive for many years, and based on the fact that this study is estimating the term premia from 1980 – 2000, is expected to have positive regression of the bond premium mostly of the out-of-sample.

## 4.2.3 Further empirical procedure

To further assess the validity of our prediction models, measurements have to be applied. Regularly, prediction models can either be validated through information criteria like AIC and BIC, or else $R^2$. We apply error metrics to assess the models with the error.

The most commonly applied residuals statistics to evaluate the prediction are MAE, RMSE or MSE. These are used to measure the magnitude of the error in the prediction models. MSE shows the mean squared error and computes how close the predicted values are to the true values. Many works of literature propose to use the same measures MAE, RMSE or MSE. MAE has been cited for being the primary measure for comparing forecasting out-of-sample (Chen, Twycross & Garibaldi, 2017). The measure is the average over the out-of-sample of the absolute differences between prediction and actual values. MSE is defined as

$$MSE \ = \frac{1}{N}\sum(y - \hat{y})^2$$

where y is the true value and $\hat{y}$ is the predicted value.

RMSE (Root Mean Square Error) is the standard deviation of the residuals. RMSE measures how the prediction errors are spread out, and how the data is fitted, also defined as the measure of the difference values between the predicted values and actual values observed. An RMSE value of 0 indicates a perfect fit to data, and it will always be an achievement to aim for the lowest prediction error (Chen, Twycross & Garibaldi, 2017).

$$RMSE = \sqrt{\sum_{t=1}^{T}(\hat{y}_t - y_t)^2}$$

Eventually, the models with the lowest implied MSE and RMSE are preferred, since they provide the lowest errors of predicted values relative to the true values. The method takes into account the estimated bias but also takes in the estimate variance (Chen, Twycross & Garibaldi, 2017) (Enders, 2014).

Another model implied statistics measure of error is the R-squared, which measures how close the data are to the regression. In other words, it is defined as the proportion of the variation in the dependent variable that is predictable from the independent variable. The measure is also known as the coefficient of determination:

$$R^2 = \frac{explained\ variation}{total\ variation} = 1 - \frac{\sum(y_i - \hat{y})^2}{\sum(y_i - \bar{y})^2}$$

The R-squared is always in between the value of 0% and 100%, where 0% means that the model explains no variability around the model mean. In comparison, 100% indicates that the model explains all the variability around the mean. In general, a higher R-squared proposes for a higher fitting model (Tsay, 2002).

The regression model is estimated on the in-sample dataset, which is then used to forecast the out-of-sample period. The coefficients of the regression models will subsequently be provided in addition to the test-statistics, $t = \frac{coef}{std\ error}$ and the $R^2$. Similar to the approach of Rapach et al. (2007), the historical average is applied as a benchmark $\bar{y} = \frac{1}{N} \cdot \sum Y$.

# 5. Empirical analysis

In this section, the empirical findings will be presented, and outcomes of all relevant models will be revealed. The results will be conferred separately and in comparison, with one another, in addition to computing rolling results as opposed to a one-period estimation window, given as the in-sample period, to illustrate the aspects of the various models. The computation of the models requires historical data as presented in Section 4.1.2, and the estimates are presented as monthly annualized figures.

## 5.1 Initial description of the portfolios

The empirical analysis applies a mean-variance portfolio, given as the tangency portfolio, and a Black-Litterman portfolio, as explained in the theoretical framework. Both allocation models (the mean-variance model and the Black-Litterman model) consist of a simple investment universe represented by two assets; the S&P 500 index and the 10-Year Treasury index.

A common feature of the portfolios is the estimates of the covariance matrices, while the expected return vectors, $\mu$, varies across the allocation models. This means that the optimal active allocation is based on the investors' estimate of the expected return and covariance. The covariance matrices will only be presented in Section 5.2. However, the same matrices will be continuously applied throughout the three different models; Mean-Variance, CAPM and Black-Litterman.

The results will be presented using two different estimation methods. Firstly, the portfolio allocation will be constructed using the in-sample period running from 1980 - 2000 for model parameter estimation. Throughout this first part of the analysis, we aim to present the reader with examples of the generation of the models. Secondly, an investigation of a rolling estimation scenario for each observation $t$ is applied based on an estimation window of six years (H = 72 months). A rolling approach gives the possibility to compare a fixed rolling scheme throughout the sample, hence compare different periods. For asset allocation, it is desirable to use out-of-sample back-testing to evaluate the performance of the portfolio. The allocation is based on historical information, and forecasts, of the asset returns. We do this to investigate what an investment today (year 2000) would be worth in the future, i.e. throughout our investment horizon. It is also worth noticing that we allow short selling in our optimization process; thus, negative weights can appear in the solution.

## 5.2 Model estimation based on the full in-sample period

This section focuses on getting familiarized with the model building approach and possible aspects to be aware of going forward with the rolling estimation. The optimization methods are all estimated on the in-sample subset, and thereby a brief discussion of the out-of-sample performance. The risk allocation of the two models will also be presented to illustrate the probable difference between the two models.

### 5.2.1 Mean-Variance

In order to optimize the risk-return relationship, it is desired for a mean-variance analysis to apply the tangency portfolio to create the optimal allocation. The mean estimates of the in-sample optimization are based on the application of historical data. This procedure requires mean/averaged estimates generated from the past realizations of the assets.

| | **Return** | **Excess Return** | **Standard deviation** |
|---|---|---|---|
| SPX Index | 12.83% | 6.27% | 15.20% |
| LUATTRUU Index | 9.45% | 2.90% | 6.07% |

TABLE 5: MEAN ESTIMATES BASED ON THE IN-SAMPLE PERIOD (1980 - 2000)

TABLE 5 provides an overview of the annualized return, excess return and volatility for the indexes. The annualized average return for the stock index is 12.83%, and for bonds, the return is 9.45%. Stocks have in the past been known to generate higher returns than bonds, but they have also been associated with higher risk as observed in the table. The average excess return on the stock index is estimated to 6.27%, suggesting that we might want to allocate more in the stock index when assuming that the investor wants to maximize his/hers return and wealth. This indication will quickly change when we introduce the investor's risk aversion. When we account for the risk we take on, the volatility on the bond index is estimated to 6.07%. This is actually providing us with a higher Sharpe ratio, due to a lower volatility in the past realizations of the asset. Bonds experienced a strong performance due to inflation influence, which lead to a rough/rash repricing in the 1970s to the 1980s. This phenomenon stabilized during the 1990's, however bond performance has been relatively good since. This will be further discussed when looking at the rolling window estimation below. As mentioned, this allocation model assumes that the future returns will look like the historical returns on average, and that may not be a very intuitive and accurate assumption in reality.

The expected Sharpe ratio is a quite important parameter in the tangency portfolio optimization process, as it seeks to maximize this measure to scale the returns with the respective risk on every asset.

| | **SPX Index** | **LUATTRUU Index** |
|---|---|---|
| Sharpe ratio | 41.5790% | 47.5133% |

TABLE 6: MONTHLY ANNUALIZED SHARPE RATIO

It appears from TABLE 6 that the Sharpe ratio is 41.58% and 47.51%, respectively, for the stock- and the bond index. The Sharpe ratio for the bond index is higher than the equivalent for the stock index, which might indicate that the mean-variance investor will allocate a greater amount in bonds relative to stocks. When risk-adjusting the asset returns, there is more attractiveness in owning bonds than equities, when bonds have a higher Sharpe ratio.

The next estimate we need for the optimization process is the correlation. As mentioned, the correlation among the assets does also play a central role in diversifying the risk of the portfolio, and therefore it could be that the allocation preference changes due to the relation of the assets.

| **Correlation** | **SPX Index** | **LUATTRUU Index** |
|---|---|---|
| SPX Index | 1 | 0.2559857 |
| LUATTRUU Index | 0.2559857 | 1 |

TABLE 7: CORRELATION (1980 - 2000)

TABLE 7 presents the correlation between the two assets. As a quick reminder, a correlation coefficient of zero means that there is no linear dependence between the assets, while the sign of the coefficient indicates in which direction they move against one another. The table illustrating the correlation of the assets shows that there is a positive relation between stocks and bonds over the in-sample period. The positive correlation coefficient indicates that stocks and bonds to some extent move in the same direction. This means that if we have an increase in the return of one asset, it would lead to an increase in the other asset. However, the correlation coefficient is not especially high, and therefore, the assets are said to have a weak positive relationship.

Moving on from the correlation matrix, we estimate the covariance matrix based on the in-sample period, which also obviously suggests that there is a positive covariation among the stock- and bond indices. The covariance is estimated as a 2 x 2 matrix, due to the fact that our portfolios only contain two assets.

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\,\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

| Covariance | SPX Index | LUATTRUU Index |
|---|---|---|
| SPX Index | 0.001896 | 0.000196 |
| LUATTRUU Index | 0.000196 | 0.000310 |

| Inverse Covariance | SPX Index | LUATTRUU Index |
|---|---|---|
| SPX Index | 564.1353346 | -356.630847 |
| LUATTRUU Index | -356.630847 | 3453.6916992 |

TABLE 8: COVARIANCE- AND INVERSE COVARIANCE MATRIX

TABLE 8 displays the covariance of the historical returns of the stocks and bonds, which show that there is a positive relationship of 0.2%. This fits well with what we observed in the data section (Section 4.1.1.1), where it was mentioned that the two assets had a positive correlation in the past century. To obtain the optimal portfolio, this requires an inverse covariance matrix (mean-variance optimization).

To create the mean-variance efficient portfolio, auxiliary constants are computed for simplicity. The mathematical computations of these were mentioned in Section 3.1.3.1. These calculations are applied further to obtain the portfolio weights, returns and standard deviation of the tangency. However, the constants are not required in order to calculate the tangency portfolio. The measures are viewed in Table 9.

| | Constants |
|---|---|
| A | 0.2187244 |
| B | 26.61548796 |
| C | 3304.56533944 |

TABLE 9: AUXILIARY CONSTANTS

The execution of the mathematical procedure described provides the following weight allocation towards the SPX index and the LUATTRUU index:

| | **SPX Index** | **LUATTRUU Index** |
|---|---|---|
| Weights | 24.54% | 76.10% |

TABLE 10: WEIGHTS

TABLE 10 shows the notional optimal weights of the portfolio which is dominated by 76.10% to bonds, followed by 24.548% left in stocks. An overweight in the allocation of the bond index was to some extent anticipated when we looked at the model input and the Sharpe ratio, as bond in general did observingly well in the period from 1980 to 2000, without being too volatile. It is assumed that we invest all our wealth in risky assets, and therefore the weights sum up to 1. From the perspective of investing our wealth in the year of 2000, this portfolio allocation is preferred in a mean-variance setting based on the monthly observation scheme.

The observation arising from the Sharpe ratio calculations suggests an allocation of the assets close to 50:50, which is not what the weight allocation proposes. To correct the portfolio weights for the risk associated with each asset and get a sense of where the risk originates from, we compute the risk allocation. The risk allocation is calculated as $\omega_i * \sigma_i$, and thereafter scaled to derive the respective risk weights. TABLE 11 shows that the risk allocation is actually close to 50:50 during this period, due to lower volatility in the bond index.

| | **SPX Index** | **LUATTRUU Index** |
|---|---|---|
| Notional risk allocation | 3.7295% | 4.6191% |
| Risk allocation (weighted) | 44.67% | 55.33% |

TABLE 11: RISK ALLOCATION IN THE ASSETS

The risk allocation of the portfolio weights shows how the risk is justified in the portfolio. Since a majority of the notional weight allocation is observed in bonds, the risk allocation of the equity- and bond index does not deviate a lot. Holding 76.10% in bonds provides a risk of 4.61%, while putting 24.54% in equities gives, respectively, a risk of 3.73%. The risk allocation provides an intuitive sense of how risky our portfolio allocation is towards the assets; therefore, the portfolio risk will be dominated by bonds by a small margin.

| | Annualized |
|---|---|
| Returns | 10.1592% |
| Volatility | 6.5984% |

TABLE 12: EXPECTED EXCESS RETURN AND VOLATILITY OF THE TANGENCY PORTFOLIO

TABLE 12 shows that the annual expected portfolio return obtained from the allocation strategy is 10.16% combined with the risk of 6.6%, which reflects the best trade-off between risk and return of the tangency portfolio. The portfolio returns and risk is a combination which is providing the most efficient portfolio based on the in-sample empirical estimates. As all of the wealth is invested in risky assets, it is fair to state that the tangency portfolio is the optimal portfolio of all risky assets, which was also clarified in the theoretical framework.

The tangency portfolio is the points where the Capital Market Line is tangents to the mean-efficient frontier of risky asset as seen in Figure 18.

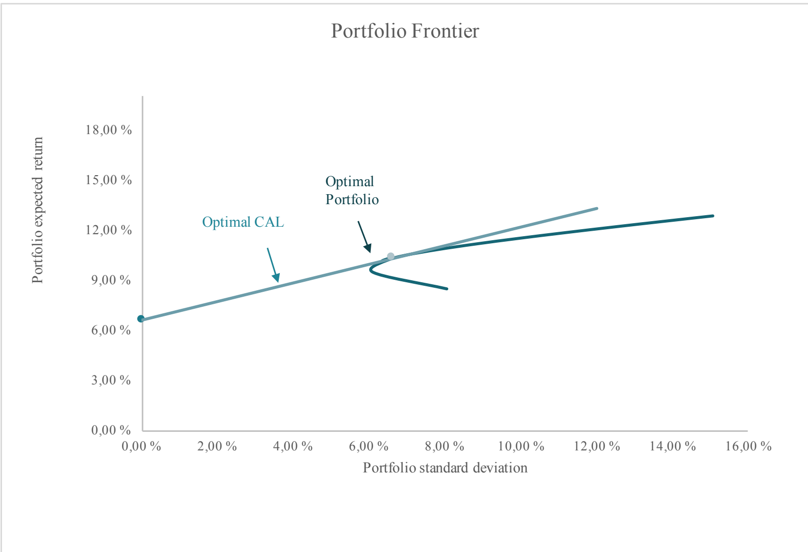


FIGURE 18: EFFICIENT FRONTIER OF RISKY ASSETS (ANNUALIZED)

The efficient frontier exhibits the combinations of the risk-return approaches of all the mean-variance efficient portfolios. Since an investor aims for the highest expected return, the investor would always be investing in the efficient frontier. The mean efficient portfolio including risk-free and risky assets, is a combination of the tangency portfolio, the portfolio with the maximum Sharpe ratio, and a risk-free asset which is illustrated above in Figure 18. The capital allocation line is calculated following equation 3.3.2.

The slope of the portfolio is equivalent to the expected Sharpe ratio

Sharpe ratio (portfolio) = (10.1592% - 6.5761%)/6.59% = 54.37%

The annually in-sample Sharpe ratio of the portfolio is given as 54.37%. Since the expected return is higher than the risk-free rate, the requirement of maximum Sharpe ratio is fulfilled. The Sharpe ratio of the portfolio is larger than the Sharpe ratio of bonds, in addition to being higher than the Sharpe ratio of equity. The portfolio allocation is much better, than just investing in stock or bonds only, providing a more diversified allocation.

To measure the performance of the one-period model out-of-sample, the weights are applied consistently with no rebalancing, indicating that we invest $1 at each t with the weights held constant.


FIGURE 19: CUMULATIVE PORTFOLIO RETURNS USING CONSTANT PORTFOLIO ALLOCATION

The realized performance of the one-period model, gives a portfolio return of 130.97%, based on the asset allocation of the mean-variance investigated over an out-of-sample period. The graph showed overall a steady increase in the cumulative return, with a small dip in 2009 being the small allocation we have in equity. Allowing rebalancing, which will be shown later, one would expect for the mean-variance to provide better portfolio returns at the end of 2018.

## 5.2.2 Black-Litterman

The Black-Litterman approach is applied in contrast to the mean-variance optimization to see how the model is constructed based on a one period model. This simple method will demonstrate how to construct the Black-Litterman portfolio for one finite period.

### 5.2.2.1 Implied excess returns

It was previously shown that the Black-Litterman equilibrium returns was derived from the CAPM relation. This was specifically done by backing-out the equilibrium returns in a reverse optimization, using capitalized weights of 60:40 in respectively stocks and bonds. The derivation of the market portfolio can be reviewed in Section 4.1.6.1.

The computation of the equilibrium excess return applied the risk-aversion, the capitalization weights and the covariance of historical observations. The covariance matrix is equivalent to the one applied in the mean-variance optimization, shown and described in Section 5.2.1.

| | **SPX Index** | **LUATTRUU Index** |
|---|---|---|
| Equilibrium excess returns | 2.9718% | 0.5951% |

TABLE 13: EXPECTED EQUILIBRIUM EXCESS RETURNS (1980 - 2020)

The expected equilibrium return, given a risk aversion of 2, provides an expected return on the stock index of 2.9718% and similarly a return on the bonds equalling 0.5951%, based on the historical performance. It was stated in Section 3.4.2.1 that the risk aversion coefficient usually lies between 1 and 3, which is why we pursuit with the average of these throughout the paper.

**Tau**

The parameter $\tau$ often influences the variance described by $\Omega$ in the diagonal elements in Equation 3.4.5, but the parameter also appears in the posterior return distribution and the posterior covariance matrix like the proportional factor, which will be shown later. It was previously mentioned that He and Litterman (1999) applied a tau = 0.05 and the same assumption will be followed for this study. When using a tau of 0.05 it corresponds to obtaining portfolio weights of 95.23% corresponding to the allocation investing in risky assets. The parameter tau is created as a constant proportionality and will therefore be applied unchanged throughout the analysis.

### 5.2.2.2. View distribution

When generating the implied returns, the reference starting point is the CAPM expected returns as previously shown. The second step is to incorporate how the investor's expectation about the future excess returns can be included in the model. As the Black-Litterman model includes these views using the Bayesian approach, they can be translated into a distribution and subsequently be merged with distribution of the implied returns. We translate the view applying simple steps to demonstrate the context of the model.

To determine the views of the view matrices, we propose a relative view and apply input data using only 1 view. One view will give vectors/matrix estimates shown as follows:

$$P = [N \times K] = 2 \times 1$$
$$Q = [1 \times K] = 1 \times 1$$
$$\Omega = [K \times K] = 1 \times 1$$

We further want to specify our views using risk-return relationship measures in order to develop the views. Since this section investigates the model based on a finite period, the views will be assessed from historical risk-return relationship. Later, it will be showed how prediction models can be used to generate the beliefs of an investor instead of applying historical data. To determine the view, we observe the performance of bonds relative to stocks, which is based on the risk-adjusted return (Sharpe ratio). Furthermore, the relationship will be defined through a relative view showing how the assets perform in comparison to each other. By doing so, the P matrix being the weighted views should sum up to 100% in order to obtain portfolio weights equivalent to investing 100% in risky assets. An indirect objective is to incorporate the view of an investor, which is made at the time where we expect to invest in the asset allocation. The views are determined from the risk-adjusted return, followingly shown in Table 14.

| | **SPX Index** | **LUATTRUU Index** |
|---|---|---|
| Sharpe ratio | 41.5790% | 47.5133% |

TABLE 14: ANNUAL SHARPE RATIO

The Sharpe ratio shows a risk-adjusted return where the bond index has outperformed the stock index in terms of the historical observations. Due to this observation, we generate the views so that we allocate more towards the bond index compared to the stock index. The views-inputs are presented in Table 15-17.

| P | | |
|---|---|---|
| | **SPX Index** | **LUATTRUU Index** |
| Bond > stocks | -1 | 1 |

TABLE 15: P VECTOR

| | Q |
|---|---|
| Target return | 2.43% |

TABLE 16: Q

| | Ω |
|---|---|
| Uncertainty | 6.11% |

TABLE 17: Ω

The P vector, shown in Table 15, simply illustrates the direction we want to (relatively) weigh our views towards. As mentioned above, the annual average Sharpe ratio for the bond index have been higher than the equivalent for the stock index, which leads to underweighting the equities by 100% and overweighting bonds by 100%. The effect of the view on stocks, is the same as multiplying our target return viewed in Table 16, Q, by -1 and correspondingly multiplying Q by 1 for the bond index. The P vector just exhibits the views, and this does not mean that the stock index should have negative weights, but hopefully, that the weight of the stock index should decline from the capitalization weights.

The Q matrix is the target return of the view and will show the relative performance of how bonds will outperform the stocks. The Q-vector is quantified as the monthly returns. Since bonds have had higher risk-return relationship, we use the expectation that the difference of the two Sharpe ratios to obtain a target return where the view defines that bonds have outperformed equity. This is calculated as: $E[R_A] - E[R_B]$, where E[R] corresponds to the risk-adjusted return on either equity or bonds depending on which one outperforming, in this specific case with $Q = E[R_{bond}] - E[R_{equity}]$. In other words, the risk adjusted return difference can be stated by $Q = risk(w_{equity}) \cdot r_{equity} - risk(w_{bond}) \cdot r_{bond}$, where the risk weight = $\sigma_{LR}^{-1}$. The $\sigma_{LR}$ is explained as the long-run volatility[3].

[3] The long-run volatility is approximated as the full-sample volatility. Normally, the long-run volatility can be estimated through GARCH models, however, this is not the focus of the study, therefore assuming volatility as the sample volatility.

Lastly, Ω reflects the uncertainty of the views illustrated in Table 17. Normally, the covariance matrix is often applied in a diagonal direction, but as the view only contains one parameter, the matrix is only a [1 x 1]. In other words, the value will be constant. Therefore, we want to risk weight each variance in relation to the long run variance of each asset. The variance is therefore calculated as

$$\Omega = \sigma_{LR}^{-1}(equity) \cdot \ var\ equity\ +\ \sigma_{LR}^{-1}(bond) \cdot \ var\ bonds$$

### 5.2.2.3. Combined distribution

After both the view distribution and market portfolio are described, the two information inputs will now have to be combined to get the future excess returns. The two distributions are described as two probability distributions, in which there still is uncertainty.

If we are looking at a case where the investor is uncertain about his views, the following returns are obtained below:

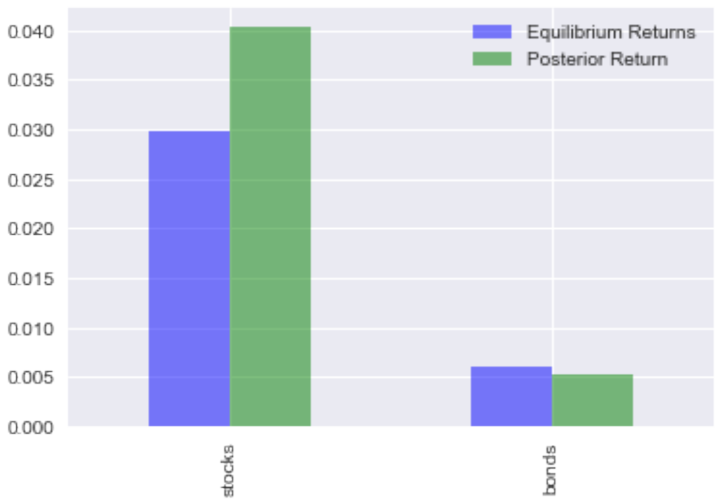


FIGURE 20: EQUILIBRIUM RETURNS VS POSTERIOR RETURNS

| | Prior | Posterior | Difference |
|---|---|---|---|
| SPX Index | 2.971812 % | 4.03239% | -1.06059 % |
| LUATTRUU Index | 0.595104 % | 0.5342747 % | -0.608293% |

TABLE 18: DIFFERENCE EQUILIBRIUM RETURNS VS POSTERIOR RETURNS

The posterior return of stocks has moved slightly both for stocks and bonds. Actually, only a minor difference in bonds is observed when calculating the change in returns based on annual terms, on 0.6%. For stocks a slightly higher difference of the prior returns is observed over bonds, of 1.06%. These differences are affected by the correlation, since it has an impact on how the assets allocate, which is also shown in the equation posterior return. These minor changes come as a result of the small changes in the P-link matrix combined with the higher uncertainty on the view. The returns do not have a large impact, and actually, changing these views does not seem to provide a significant change. This is why the model does not directly fit as a passive model applied based on historical data. The model will appear in a rolling-setting, where we will investigate the difference prospect of the view matrices to see which impact the largest values of the view have for the posterior returns. Normally, it is known that the application of a relative view comes in small proportions, which is verified from above.

**Sigma Posterior, $\bar{\sum}$**

| | SPX Index | LUATTRUU Index |
|---|---|---|
| SPX Index | 0.001925 | 0.000209 |
| LUATTRUU Index | 0.000209 | 0.000307 |

HISTORICAL COVARIANCE MATRIX

+

| | SPX Index | LUATTRUU Index |
|---|---|---|
| SPX Index | 0.000096 | 0.00001 |
| LUATTRUU Index | 0.00001 | 0.000015 |

POSTERIOR COVARIANCE MATRIX

$$\bar{\sum} = \sum + M$$

| | SPX Index | LUATTRUU Index |
|---|---|---|
| SPX Index | 0.002121 | 0.000219 |
| LUATTRUU Index | 0.000219 | 0.000322 |

TABLE 19: COMBINED (POSTERIOR) COVARIANCE MATRIX

The posterior covariance matrix is a combination of the historical covariance and the covariance matrix of the returns. Looking at the posterior covariance matrix of the excess return it becomes clear that the covariance matrix does not provide a large impact on the combined covariance. The combined covariance matrix will always be larger than the historical covariance matrix, due to formula 3.4.7, which states that the posterior covariance of excess return is added on top of the historical covariance. Additionally, a larger variance and covariance means increasing uncertainty in proportion with the $\tau$. In other words, larger values of $\tau$ will increase the variance of the view and therefore also the combined covariance matrix. Furthermore, larger value of the scalar, $\tau$ would mean higher volatility and thereby also higher return, if following the CAPM relationship of risk-return. Since $\tau$ is a scalar factor, it also affects the sum of the portfolio weights, as we become more uncertain in the risky assets. The expectation is a lower value in the sum of the portfolio weights.

| Weights | Posterior | Equilibrium |
|---|---|---|
| SPX Index | 58.93% | 60% |
| LUATTRUU Index | 41.07% | 40% |

TABLE 20: COMBINED (POSTERIOR) COVARIANCE MATRIX

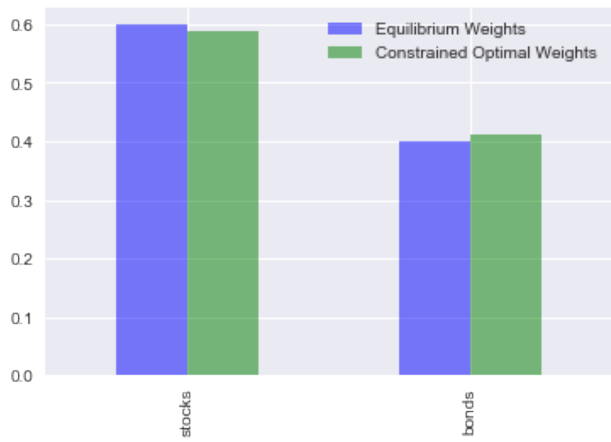


FIGURE 21: EQUILIBRIUM WEIGHTS VS. POSTERIOR WEIGHTS

The plot shown in FIGURE 21 is displaying an adjustment of the weight allocations, both for stocks and bonds. The stocks have changed the weights towards 58.93% while the bonds have increased to 41.07%. The view stated a relative view, where the view showed a bullish view on bonds and a bearish view on stocks. The changes were quite intuitive and led to a decrease in the stock index of -1.07% and correspondingly, an increase in the weights on 1.07%. The change in the weights does not appear to be of significant nature, and this might indicate the view distribution is not prominent enough to have a huge impact. However, since this allocation is based on a monthly setting, it seems fairly reasonable. The weights of Black Litterman changed in the same direction of the view, although the magnitude was small. This may be also due to the uncertainty of the view, since it was said to be around 6%, indicating to some extent a quite noticeable uncertainty.

The weights are obtained from the same optimization as seen in the mean-variance setting, but since taking into account the risk-aversion, the weights do not allocate 100% in risky assets. The weights before scaling was given as 56.12% on the stocks and 39.11% towards the bonds only summing up to 95.24% equivalent to the scale factor $1/(1+\tau)$. When multiplying the weights against $(1+\tau)$, the asset allocation is in a position in which, 100% is invested in risky assets.

We calculate the risk allocation of the assets as shown in Section 5.2.1. The risk allocation for the Black-Litterman in Figure 21 shows to have a large overweight in stocks. Since stock naturally carry more risk than bonds, it also means that allocating higher weights in stock results in more risk. Even tough the views stated relative underperformance towards stocks, the views were not prominent enough to make a noteworthy difference. This illustrates that the risk allocation is more uneven in the Black-Litterman model, compared to the mean-variance scenario.

| | **SPX Index** | **LUATTRUU Index** |
|---|---|---|
| Risk allocation | 2.5663% | 0.7228% |
| Risk allocation (weighted) | 78.02% | 21.98% |

TABLE 21: RISK ALLOCATION IN THE ASSETS


FIGURE 22: CUMULATIVE PORTFOLIO RETURNS

The out-of-sample of the one period performance of the cumulative portfolio returns show a worse performance when comparing it to the mean-variance one-period analysis, which makes good sense. The Black-Litterman model is an active allocation model, hence it makes minimal sense to estimate a weight allocation and further assume that our views will not change over an 18-year period.

Overall, the Black Litterman model blends the equilibrium excess return- and the view distribution of the model, affected by the scalar tau and risk aversion. This is leading to changed expected returns, which again influences the weight allocation of the model. When allocating a large part of the portfolio towards equity, as observed in this model, the portfolio returns will also follow a larger part of the market turbulences.

### 5.2.3 Sub-conclusion

The main intention of the first section is to present the workings of our respective models, making it easier to cope with the rolling estimate scenario in the next section. The one-period estimation of the mean-variance analysis provides weights dominated by 76.10% towards bonds and 25.54% in equities. The portfolio allocation using the mean-variance approach is optimized by the Sharpe ratios, which is shows higher values of bonds over stocks. Therefore, the allocation is expected to be higher for bonds in contrast to stocks when observing the historical estimates. In the two-asset case between stocks and bonds, the intuitive interpretation has been quite meaningful when looking at the historical observations. In reality, bonds have, in terms of risk-adjusted return, performed fairly well in the period running from 1980 to 2000. It was lastly displayed that the mean-variance portfolio almost unveiled a 50:50 allocation of the risk between the indexes.

The 60:40 allocation approximated the market capitalization used in the Black-Litterman analysis. The annual expected return on stocks and bonds proposed by CAPM was quite lower than the mean-variance portfolios expected returns. This means that actually placing 22% in stock compared to 60% and 76% in bonds instead of 40% gives a higher expected return for the portfolio. Since the market portfolio is a reference point of the Black Litterman, this might indicate a higher return for the MV-optimization over the BL-optimization, unless the views are having a significant large impact on the equilibrium portfolio, also affected by the risk aversion and a scaling factor. The investor's view of the assets was generated from the historical observations, where it was observed for the bonds to have higher risk-adjusted return over stock, indicating to have a positive view of bonds and relative underperformance of stocks. The investor view was implied towards bonds outperforming stocks by 2.43%, with an uncertainty factor of 6.11%. The BL model compared to the MV, over weighted relative more stocks than bonds, and indicate that BL performs better in expansions while MV in recession. Overall, the mean-variance analysis seems to provide superior performance in terms of expected returns outperforming both CAPM and BL based on a one-period model.

## 5.3 Rolling model estimation

This section will focus on a rolling estimation scenario. The procedure is explained in the methodology section, where we define the rolling window, H, to measure six years. The purpose of this section is to find empirical evidence, and explanations in the data, of which model that provides the best results when re-estimating the expected returns and the covariances every month using the previous window H as the estimation period.

A construction of historical returns based on rolling sample windows is used to estimate the expected return, $\mu_t$, and covariance matrix, $\sum$, at each time t. At each time month, t+1, new information is incorporated and updated for the next periods allocation. Based on the estimate, the optimization problem is updated each month and the portfolio weights are computed for every first day of each month. Thereby, a monthly rebalancing is applied to construct a portfolio that takes into account the changes in the risk-return trade-off continuously through the investment horizon. The idea is pretty straightforward, and can be thought of as: if our investment starts today, how should the portfolios be rebalanced at each month t, in order to follow the newest contained information for the allocation? This out-of-sample period spans from 2000 to 2018. This specific time-period is applied in order to obtain allocation estimates from the start of 2000, since the rolling window requires six years of historical data for the first allocation estimate. This sample period contains periods with both booming and recession cycles of the economy, which is an important reason to choose the shorter window for the continuous re-allocation.

### 5.3.1 Mean-variance portfolio allocation

The mean estimates of the tangency portfolio are generated as described in Section 5.2.1; however, they are re-estimated at each t observation, using information contained from the last six years.

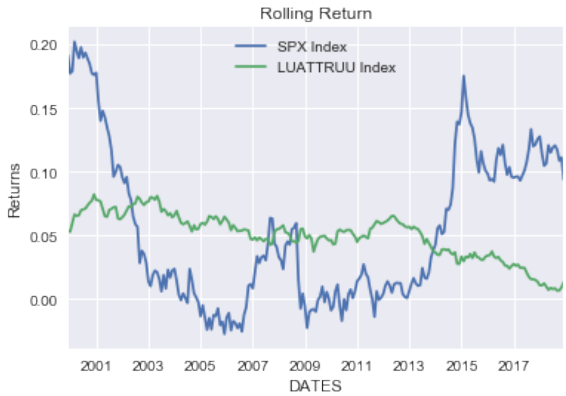


FIGURE 23: RETURNS

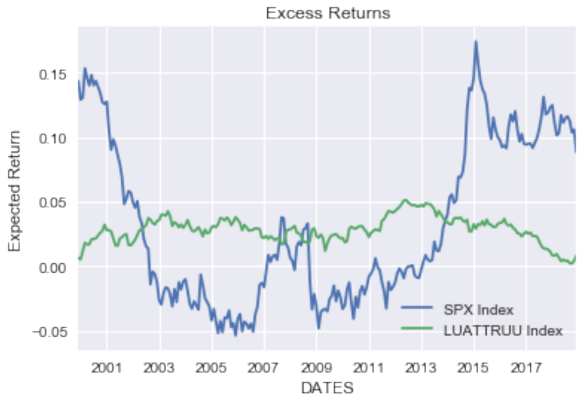


FIGURE 24: EXCESS RETURNS

FIGURE 23 provides an overview of the monthly annualized returns of the multi-asset portfolio. The returns of the stock index show up and down movement of the returns, additionally, indicating periods with both positive and negative returns. It is important to keep in mind that the estimate at each observation t is generated based on the historical information from the past six years when interpreting the graph. The expected returns of stocks have following indicated a bear market from the start of the out-of-sample estimation period, where the financial turbulence from the dot com bubble turned the market upside down followed by the economic effects of 9/11. A bear market appeared around 2008, which means that the past six years of information experienced decreasing returns. However, since 2009 the expected returns appear to have increased until the beginning of 2015, indicating a bull market. The returns of bonds are presented to be more stable over time, however, with a declining trend the last years of the horizon. The excess returns of stocks and bonds are exhibited in FIGURE 24. The two plots are almost equal despite the calculation of the risk-free rate. As can be observed, no returns are constant over time and these are quite fluctuating. The excess return indicates, for stocks to be overweight in the model from 2000 until 2002 and again from 2014 to 2018. The model implies higher allocation in bonds, based on the excess return remaining years. However, these considerations might disappear when adjusting for the risk.

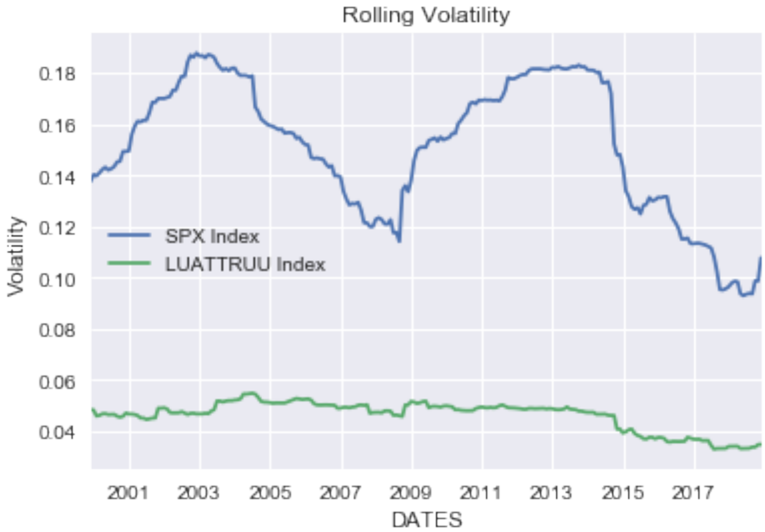


FIGURE 25: ROLLING STANDARD DEVIATION OF EXCESS RETURNS

The time-varying variance, illustrated in FIGURE 25, shows a quite different scenario for the two assets classes. For bonds, it indicates a pretty persistent volatility over time, which is quite low and measures approximately 5% on average. From 2015 and further, the volatility has been decreasing to a risk below 4%, which indicates that the six years before 2015 had a declining volatility. In reality, the interest rate has been declining since 2015, which in fact, means for the bonds to normally react stronger and thereby be more volatile for changes in the environment. Although, the rolling method does not take this into account, therefore, showing lower volatility on bonds. For low-volatility stocks, they tend to benefit from the declining interest rate. Observing the graph, the volatility of the stock decreases in 2015, where high-volatility stocks often have a higher exposure for changes in the interest rate. The elevated risk is not surprising, as the volatility tends to increase during market turndowns. Increasing stock allocation at these times, will also increase the exposure towards risk.

The rolling Sharpe ratio, computed from the mean and standard deviation above, are illustrated in FIGURE 26. As mentioned, the Sharpe Ratio serves as a fine indicator of how our assets, preferably, should be allocated when the objective is to maximize this measure.

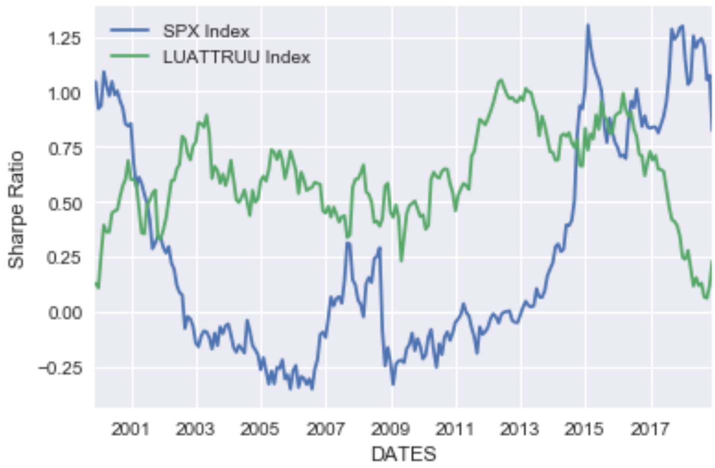


FIGURE 26: ANNUALIZED MONTHLY SHARPE RATIOS (2000 - 2018)

The stocks appear to have quite moving Sharpe ratios, which have similar patterns associated with the excess return, even after adjusting for the risk. The volatility of the stock index appears to be considerably large, particularly between 2001 to 2005 and again from 2009 to 2015. The same period registered decreasing Sharpe ratios for the treasury. From 2014 until 2018 the Sharpe ratio for the stock index has been increasing and peaked in 2015 and 2018 around 125%. For bonds, the patterns of the Sharpe ratio are to some extent showing similar patterns related to the excess returns. Overall, the Sharpe ratio of bonds has been somewhat steady. The expectation of the Sharpe ratio must be that there will be more allocation in bonds relative to stocks in periods where the Sharpe ratio of bonds is over the Sharpe ratio of stocks. Consequently, the allocation is indicating an overweight towards bonds from 2002 to 2014 and overweight in equities from 2000 to 2001 and again from 2014 to the end of 2018.

As previously mentioned in Section 5.2, the two assets have correlated positively towards the 2100th century. According to the information prior to the year 2001, it appears that the correlation exhibits a negative relation. Around times of crisis, the correlation between stocks and bonds have been fluctuating, consequently, consistent in terms of negative sign. Thus negative correlation implies that bonds can be applied for hedging towards stocks when the economy is in a recession. This negative correlation is extremely beneficial as opportunities to diversify are effective, making portfolio loss minimal.

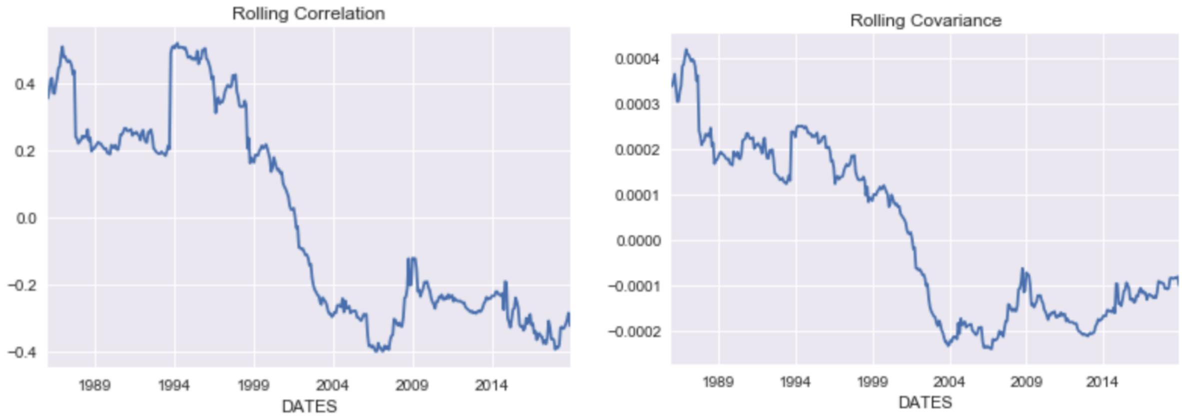


FIGURE 27: ROLLING CORRELATION AND COVARIANCE BETWEEN SPX & LUATTRUU

The purpose of FIGURE 27 is to illustrate how the equity- and the bond index covary over time. It is quite clear that both the rolling correlation and the rolling covariance is time-varying. The in-sample period estimate confirmed a weak positive covariance between the indices, while the out-of-sample period indicates signs of negative covariance. Overall, this illustrates the fact that the covariation has switched from a positive to a negative sign around the year 2000 (a bit earlier due to the fact that it is estimated on the past six years from this point). This was also described in Section 4.1.1.1, where the bonds and equity showed negative correlation for the past 20 years.

The significant shift from positive to negative correlation during this sample could change the whole way our portfolio should be optimally constructed. It is typical issues like this we try to avoid using rolling estimation windows so that we have a more recent estimate of the relationship between our indexes. Hopefully, this gives a stable representation of the current relationships among the assets. If the calculated weights of our model were based on the one-period model and not rolling estimates, it could lead to misleading or spurious investing, due to the transformation in the relationships among our assets. Doing this, we could be in danger of making a lousy allocation, since we assume that the relation in the future is going to be the same as the historical relation.

Moving forward, the mean-variance setting is applied in a closed-form solution. Therefore, auxiliary constants are created for each observation t using the rolling window to obtain the A, B and C coefficients over time to generate the mean-variance efficient portfolio.

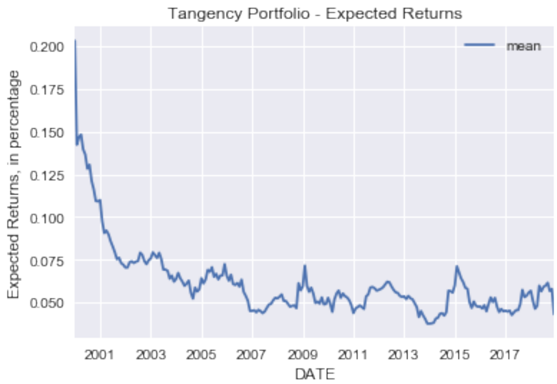


FIGURE 28: EXPECTED RETURN-TANGENCY

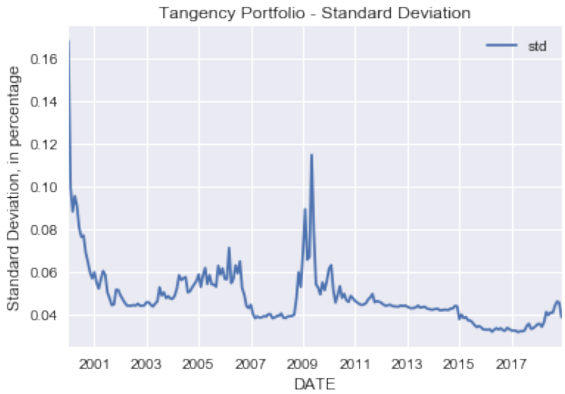


FIGURE 29: VOLATILITY-TANGENCY

The monthly annualized expected returns are displayed in FIGURE 28, while FIGURE 29 presents the expected risk of the tangency portfolio. This shows, on expectation, how the portfolio returns and risk are developing. The portfolio allocation indicates expected return on the tangency portfolio to produce a high return at the beginning of 2000, actually as high as 20%. This is also linked to high volatility measuring to approximately 16% at the same point in time. This indicates that turbulence might have happened right before the century shift. The expected return on the tangency portfolio is followingly decreasing and varies around a more constant/consistent level further on. The volatility of the portfolio is also exponentially decreasing towards a steadier state, then what we observed around the starting point. However, the volatility still illustrates some periods with higher risk (for instance, around the oil- and financial crisis). The expectation of the volatility is lower for the latter year compared to the beginning of the investment. Another factor is also a previously positive correlation between bonds and equity compared to our out-of-sample period, which appeared to change to a negative relationship from the start of the out-of-sample period to the end.

It was earlier described the calculation regarding how to optimally generate balanced portfolio weights using Markowitz. These are created manually; therefore, short selling has been allowed. This means that negative weights will be present in different periods, and the negative weights will indicate a short sale where we gain profit from actual negative returns in the indices. This means for every rolling estimation window; we obtain the optimal weights of the tangency portfolio.

The plot below displays the weights that are rebalanced every month until 2020. Besides looking at the time horizon and financial goals, asset allocation is a crucial decision when constructing a portfolio. The way assets are allocated is the primary determinant of the risk-return trade-off for a portfolio. When investing over time, the portfolio construction will generate investments that contain different returns and thereby move further away from the initial asset allocation. Thus, the risk and returns may likely be inconsistent with the investor's goals and preferences. Therefore, portfolio rebalancing is especially essential because of the ability of the investors to maintain their asset allocation target. Using the monthly rebalancing, the investors can eliminate the tendency of portfolio drift, and reduce their exposure to risk relative to the portfolio allocation.

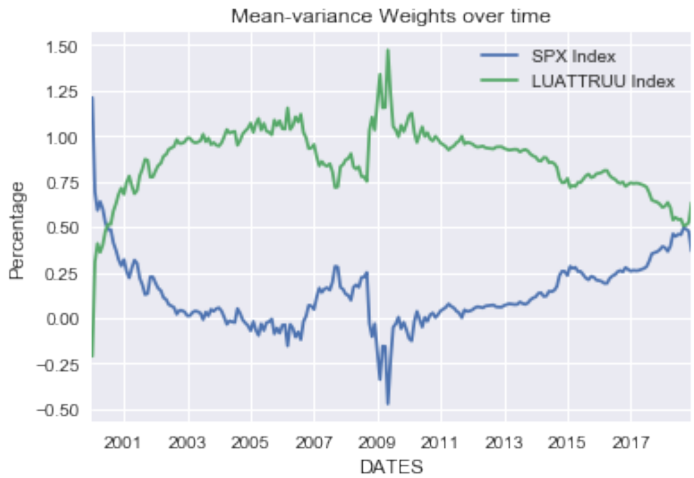


FIGURE 30: MEAN VARIANCE WEIGHT ALLOCATION OUT-OF-SAMPLE

In general, there is overweight in bonds compared to stocks, besides in the beginning of 2000, where the allocation briefly preferred short positions in the bond index and a long position in the stock index. This may be because of a positive correlation in the beginning, combined with a high Sharpe ratio of the stock index before the year 2000. It was previously shown that bonds have had the highest Sharpe ratios over the investment horizon. This indicates that the volatility of stock has had a large impact on the weights concerning the correlation as well as the Sharpe ratio. Regardless of the economic cycle, almost every period shows a higher preference for allocation towards bonds. Periods with shorted stock are present, especially right around the financial crisis, meaning that the bond allocation is in overweight. As a result, a period experiencing a recession, such as the financial crisis, does not affect the portfolio that much. The mean-variance optimization seems to capture the post-effect of the financial crisis, which indicates, for the portfolio to provide substantially good results when stocks are affected by the economic crisis'. Due to higher volatility in stocks, increasing variance means that there is more attractiveness investing in bonds.

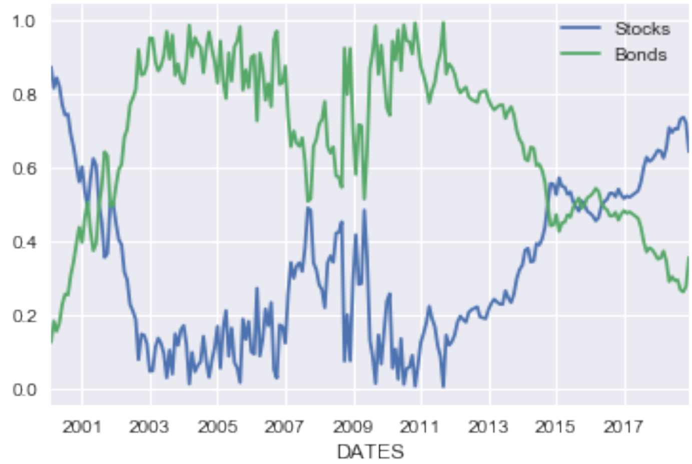


FIGURE 31: RISK-ALLOCATION OF STOCKS-BOND

Table 31 illustrates the risk allocation of the portfolio over time. In risk terms, it appears that the equity risk is dominating the portfolio during the first year of the investment horizon, due to a significant allocation towards stocks. However, from year 2002 to 2014, the risk is heavily allocated towards bonds, which is also linked to an overweight in bonds during this period. It can be observed that there is a few points in the graph that indicates that the portfolio risk is allocated close to 50:50. In periods where the Sharpe ratio of bonds and equities are approximately equivalent, around 2001 to 2002 and 2014, the risk allocation seems to be crossing according to the plot of Sharpe ratios seen in Figure 26. When the Sharpe ratio of stocks is higher than it is for bonds, it becomes clear that most of the volatility is coming from equities, and therefore, the notional allocation will still allocate a large part in bonds due to the risk. In periods where the bonds have a higher Sharpe ratio than equities, even a larger allocation in notional weights of bonds is observed. These are then becoming more desired, as the risk of the portfolio will mainly follow the risk of bonds. The maximization of the Sharpe ratio, combined with wanting a minimal risk exposure and superior returns, leads the notional allocation to a majority in bonds.

To show how the evolution of the weight would have performed in reality, the cumulative portfolio returns are being calculated, according to Equation 3.1.1. The cumulative returns will be provided for comparison in relation to the Black Litterman approach and CAPM during the comparison of the models.


FIGURE 32: CUMULATIVE REALIZED RETURN

Observing the graph above, it provides us with the cumulative return based on a monthly rebalancing scheme. Using a fixed window of six years, we allow our portfolio to update based on the newest information. The portfolio returns show how the performance of the re-optimization will be, only applying a shorter period of historical data to optimize the portfolio in the future. If rebalanced every month, we increase the return over 100% of the initial, if looking at an investment horizon over 20 years based on stocks and bonds. The cumulative return for the rebalanced portfolio is 110.58% in 2019. The portfolio allocation during the financial crisis seemed to be robust, probably due to an overweight in bonds, which showed to perform well during the recession in the economy. The portfolio is fairly satisfactory.

## 5.3.2 Black-Litterman asset allocation

### 5.3.2.1 Implied equilibrium returns

The CAPM market implied return is treated in the Black Litterman model, hence the CAPM allocation model will be described (also for the purpose of using it as a benchmark when estimating performance measures). The CAPM strategy uses another approximation of the expected excess return apart from the traditional CAPM equation (explained in theory), instead applying the expected capitalized excess return of the BL-framework, providing us with a reference point for the views of investors.

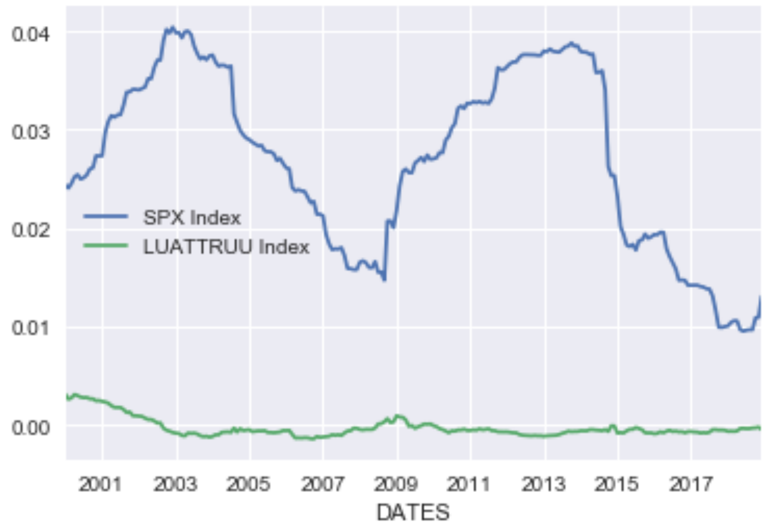


FIGURE 33: ANNUALIZED EXPECTED RETURNS

The expected equilibrium excess return arising from CAPM, provided by BL model, is plotted in FIGURE 33 for the rolling estimation scenario. We observe a surprisingly small amount from the bond index even though placing 40% allocation. From 2010 to 2015 negative expected returns for bonds are observable, however these are not especially large. Overall, the expected return of the bonds is quite consistently fluctuating around zero. Stock performance shows higher expected returns compared to the bonds. There are high returns in the beginning of the year 2000, which declines from 2003 towards the financial crisis. The stock return is rising again after the financial crisis between 2009 to 2013, and in 2014 the equilibrium returns are decreasing. The expected returns of the market portfolio show similar observations in relation to the historical standard deviation. This implies, that the variance-covariance matrix has a quite large impact on the expected return over the capitalized market weights, although the expected return computation also takes into account the risk aversion. Overall, the expected return is mainly dominated by the stock.

The allocation consisting of 60:40, is approximated as the market portfolio. The weights are used to construct the cumulative returns of CAPM, to observe what the realization has been during in reality. This will serve as the market portfolio and will be used to compare how the Black-Litterman portfolio is deviating from the market portfolio.


FIGURE 34: CUMULATIVE CAPM

The portfolio return for CAPM gives a cumulative realized return of 125.74% at the end of 2018. Holding 60% equity during the times with recession shows quite a downward movement in the realized return of the CAPM, which is implied by the dot com bubble as well as the financial crisis in 2008. Since equity is mostly dominated in the market portfolio, this will also have a larger impact of changes in return compared to the effects of the changes in bond return. From 2009 going forwards, it appears that the market portfolio regularly has been performing quite properly, and stock returns have been rapidly increased after the crisis. An increased allocation towards bonds would have smoothen the drops in the downward market cycles.

### 5.3.2.2 View distribution

To generate the view of the Black Litterman, as previously mentioned, prediction models are implemented. Looking at the in-sample estimation of the Black Litterman model, we described the views. This will be equivalent to the rolling period as well. For each time t, we will only apply one relative view. Although this means we have 228 specific views at time t since rolling gives 228 periods. All periods use a relative view, but it will be different whether stock or bonds will be weighted negative or positive. Due to this assumption, there will only be applied one view as input to the view, illustrated in Section 5.2.3.2. The approach and format of the view-distribution estimates are generated in the same manner as previously.

Subsequently, the prediction models will be described and analysed, giving input variables in the view distributions and following how the views have been determined.

5.3.2.2.1 Prediction models

In this section the statistical evidence for out-of-sample measures will be provided of the equity- and bond premium at a one month forecast horizon. It should be noticed that the realized return in a period often will diverge considerably from the prediction at the start of the prediction period.

**Equity premium**

Based on the approach and methodology of Rapach et al. (2007), described in Section 4.2.3.1, the equity premium forecast is estimated using fourteen single predictor variables, which all have shown, according to literature, to some extent having predictive power related to the equity premium. The purpose is to find out whether these in reality do have predictive power out-of-sample, and if the combination forecast of all these variables is able to verify this.

| In-sample: 1980 - 2000 | | | | |
|---|---|---|---|---|
| ***Predictor models:*** | | Coefficient | t-stat | $R^2$ |
| Dividend-Price ratio | alpha | -0.018 | -0.712 | 0.004 |
| | beta | -0.0066 | -0.92 | |
| Dividend-yield ratio | alpha | 0.0056 | 1.382 | 0.000 |
| | beta | -0.0106 | -0.161 | |
| Earnings-price ratio: | alpha | -0.0066 | -0.326 | 0.001 |
| | beta | -0.0043 | -0.584 | |
| Dividend-pay-out ratio | alpha | -0.0032 | -0.274 | 0.002 |
| | beta | -0.0114 | -0.734 | |
| Stock-variance ratio | alpha | 0.0069 | 2.215** | 0.009 |
| | beta | -0.8608 | -1.433 | |
| Book-to-market | alpha | 0.0107 | 1.798* | 0.005 |
| | beta | -0.0111 | -1.072 | |
| Net equity expansion | alpha | 0.0073 | 2.238** | 0.008 |
| | beta | -0.2011 | -1.383 | |
| T-bill | alpha | 0.0198 | 2.77*** | 0.021 |

| | | | | |
|---|---|---|---|---|
| | beta | -0.2162 | -2.241** | |
| Long-term yield | alpha | 0.027 | 2.443** | 0.018 |
| | beta | -2.464 | -2.051** | |
| Long-term return | alpha | 0.0042 | 1.402 | 0.006 |
| | beta | 0.0991 | 1.148 | |
| Term spread | alpha | 0.0003 | 0.058 | 0.006 |
| | beta | 0.231 | 1.1789 | |
| Default yield | alpha | 0.0062 | 0.861 | 0.000 |
| | beta | -0.0937 | -0.162 | |
| Default return spread | alpha | 0.0052 | 1.853 | 0.024 |
| | beta | 0.629 | 2.409** | |
| Inflation | alpha | 0.0141 | 3.337*** | 0.033 |
| | beta | -2.7654 | -2.854*** | |

TABLE 22: IN-SAMPLE EQUITY PREMIUM RESULTS[4]

Table 22 shows the regressions model output and provides the coefficients for each single predictor variable. Overall, the single predictor variables display insignificant coefficient estimates, consequently giving poor regression results. Same observations are found by Goyal and Welch, which explains the individual predictor variables perform poorly, both in-sample and out-of-sample. However, it can be observed that both the inflation, the long-term yield and the risk-free rate indicate significant coefficient estimates.

The $R^2$ of the prediction models indicates that some of the variables to have better explanatory power than others, i.e. does not manage to explain a lot of the variation in the model. The default yield and dividend-yield ratio indicate no explanatory element. Therefore, these could, perhaps, be considered to be removed from the model, as none of them contributed satisfactory, when observing the performance reflected by the test-statistics. Inflation is the predictor variable containing the highest explanatory model, followed by the default return spread. In addition, inflation also show statistical significance at a 1% level on both the alpha and beta coefficients. Overall, it would often be the case that the $R^2$ measures of the magnitude we observe in our

[4] The number of stars *, **, *** corresponds to significance levels respectively equalling 10%, 5% or 1%

results, are considered low within statistics. This is, however, not necessarily the case in finance, where lower values actually are considered quite satisfactory.

Recalling the findings from our literature review, we mentioned that some papers found predictive power in some valuation ratios such as the price-dividend, the book-to-market and the earnings-price, at least in-sample. According to our finding, this does not apply to our sample period, where we find that at least some of these predictors are not statistically significant, even in-sample.

We want to observe the out-of-sample mean squared prediction errors to evaluate if the predictor models in fact does provide linear prediction. For such models, the evaluation of the parsimonious possibility that the predicted y is true to the real value of y has to be assessed. The out-of-sample error prediction measures, illustrated in Table 23, provides an overview of the calculated measures. In general, a lower error measure is favourable.

| Error measures | MSE | RMSE |
| --- | --- | --- |
| ***Predictor models:*** | | |
| Dividend-Price ratio | 0.001872 | 0.043273 |
| Dividend-yield ratio | 0.001823 | 0.042705 |
| Earnings-price ratio | 0.001843 | 0.042931 |
| Dividend-pay-out ratio | 0.001866 | 0.033220 |
| Stock-variance ratio | 0.001756 | 0.041907 |
| Book-to-market | 0.001851 | 0.043032 |
| Net equity expansion | 0.001888 | 0.043454 |
| T-bill | 0.002000 | 0.044720 |
| Long-term yield | 0.002017 | 0.044907 |
| Long-term return | 0.001811 | 0.042551 |
| Term spread | 0.001826 | 0.042730 |
| Default yield | 0.001816 | 0.042612 |
| Default return spread | 0.001881 | 0.043369 |
| Inflation | 0.002025 | 0.045003 |
| ***Combination forecast:*** | | |
| ***Mean*** | 0.001837 | 0.042862 |

TABLE 23: FORECAST ERRORS

The MSE for the single predictor model and the combination forecast of mean, indicate close to similar errors being around 0.17 %- 0.21%. The highest forecast error is coming from the inflation, long-term yield and the t-bill, which was the individual predictors coefficient that provided most explanatory power. The combination forecast also yields a MSE of 0.18%, which is close to the values of the single predictor models. The RMSE shows more persistent values for all of the variables, and the combination forecast yields a value of 4.29%. Overall, it appears that the level of the error measures is fairly low, where the highest error of 4.5% is coming from the inflation regression.

When averaging the fourteenth single predictor model into a combined forecast and comparing it against the benchmark, we find that the combination forecast is persistently outperforming the historical average. Interestingly, the combination forecast is able to consistently beat the historical average, which is also supported by the similar empirical findings of Rapach, Strauss & Zhou (2010). The forecast performs poorly for the individual predictor; however, the combination forecast is providing significant outperformance of the constant mean.

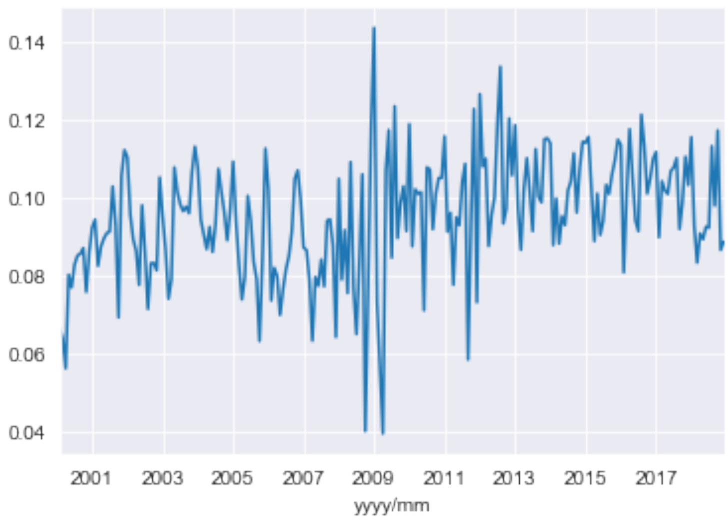


FIGURE 35: EQUITY PREMIUM PREDICTIONS OUT-OF-SAMPLE

The predictions arising from the combination forecast mean over time is shown in Figure 35. The forecast seems to show moderate prediction. In Section 4.1.2.2, we described that mean-reversion often appears investigating longer horizons. The stock appeared to have a slightly more negative autocorrelation, which could indicate some sort of mean-reversion. However, it seems to fail predicting a lot of the variation/volatility that is observed in the realized return of S&P 500.

**Bond premium**

The term premium computed by ACM is said to have a strong predictive power of the bond premium, simply because the term premia is given as the difference in the long-term yield and expected short-term rate. Based on the regression model, we can find out whether the term premium is able to forecast the bond premium.

The obtained regression model estimates are given as following:

| | | In-sample: 1980 - 2000 | | |
|---|---|---|---|---|
| **Predictor models** | | Coefficient | t-stat | $R^2$ |
| Term premia: | alpha | -0.0002 | -0.03245 | 0.000 |
| | beta | 0. 0004 | 0.07998 | |

TABLE 24: REGRESSION MODEL OUTPUT[5]

Table 24 above provides an overview of the regression model. The coefficient estimate, $\hat{\alpha}$, gives a value of -0.02%, and the $\hat{\beta}$-estimate a value of 0.04%, in which none appears to be statically significance. The regression model output shows that the linear relationship is positive, represented by the beta coefficient. This is not aligned with findings of ACM, as they find highly significant predictive power of the term premium. ACM find that the monthly term premium explains over 75% of the yield looking at short horizons, and more than 90% for longer horizons. Our regression model implies an $R^2$ of 0.0%, indicating that the model fails to explain the variation in the bond return.

As in equity premium, we want to evaluate the prediction based on the error metrics described in Section 4.2.3, to assess whether the prediction models lie close to the true observations. These are shown in TABLE 25.

| Error measures | MSE | RMSE |
|---|---|---|
| Term premium | 0.00017 | 0.1302 |

TABLE 25: FORECAST ERROR

[5] The number of stars *, **, *** corresponds to significance levels respectively equalling 10%, 5% or 1%

The mean squared error for the term premium is given by 0.017% which is seems to be quite low compared to those presented in equity premium. Intuitively, an error less than 1% seems to be relatively small. In terms of RMSE, the error computed is 1.30%, also indicating a small error term. Thus, the error seems satisfactory, since the small error terms means that the bond premium lies close to the true values of the bond premium. Even though the regression model did not provide any explanatory power, the linear regression seems to capture the true values of the bond premium.

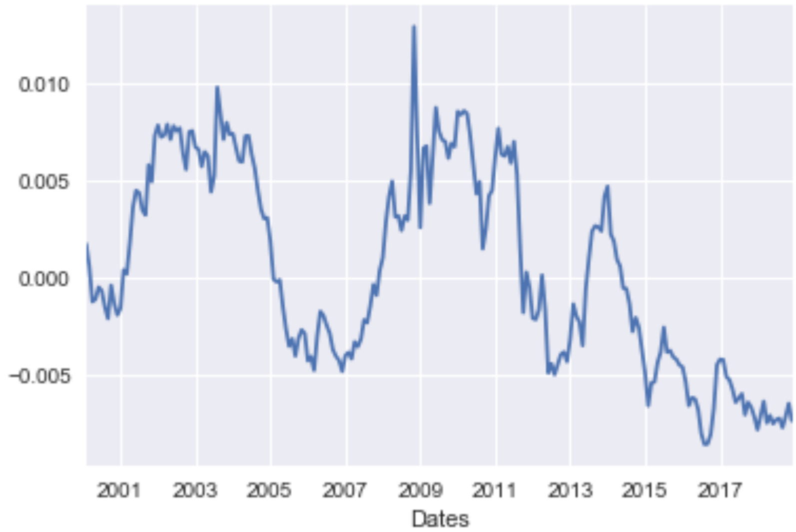


FIGURE 36: PREDICTION OF THE BOND PREMIUM

The out-of-sample prediction of the bond premium is exhibited in FIGURE 36. Based on the regression model, the bond premium is following the fluctuations of the term premia as shown in Appendix 5, which is mostly expected as this variable is the only one used to predict the excess returns of bonds. The bond premium is showing a downward trend, which is reflecting a decrease in the term premium, since the regression model implied a positive beta coefficient. In reality, the 10-year yield is represented by the 10-year forward term premia, thus should also be able to forecast the yield. In this setting, the term premium is expected to forecast the bond prices, and therefore, it is natural for the bond premium to follow the term premium. From the graph above, the returns in bond premia seems like there is an indication of mean-reversion, as it is shifting between periods of low negative or high positive returns, but this cannot be verified.

The bond premium overall seems to reflect a larger part of the magnitude, compared to the equity premium. These spikes may appear large when these are employed into the prediction models of the investor. In this paper it is argued that bond premiums are predictable thus the point is to apply the bond premium out-of-sample forecast to generate the investor views required in the Black Litterman model. Hence, it will be used to compare the performance of stocks relative to bonds.

**P matrix**

The prediction models of the return premiums shown above has to be further computed into relative views. Since stocks in reality do have a higher return than bonds, we want to look at the values when adjusting for risk. If we do not adjust for risk, then the equity premium just outperforms the bond premium for many horizons. After accounting for the risk, we get a clearer picture of the actual risk-adjusted performance of the respective assets. The P-matrix will describe the views of the model based on a monthly rebalancing scheme, meaning the P-matrix is adjusted every beginning of the month. Further, it will illustrate in which direction we want to modify the assets returns, i.e. when the risk-adjusted measure of bonds is higher than stocks, we want to adjust a higher allocation towards bonds.

As described in methodology, the premium for stocks and bonds is forecasted for the out-of-sample period based on our monthly dataset. Standing in 2000, regression models were built from the in-sample period to predict the out-of-sample forecast. FIGURE 37 below shows the forecasted premium for bonds and equity accounted for the risk. The predicted premium of bonds is illustrated in the green graph, while equities are illustrated by the blue graph. It is observable, for many periods, that bond and stock cross each other, which is the shift where either under- or overperformance of, respectively, stocks or bonds is taking place. When stocks are below bonds, it is desired to allocate the view positively in bonds and negatively in stocks. Contrary, during periods where stocks are above bonds, the view should be weighted positively in stocks and negatively in bonds. In general, stocks indicate a somewhat persistent risk-scaled return, whereas actually bonds forecast quite volatile risk-scaled returns. The spikes in the graph illustrate how much that is over- or undervalued in relation to each other. Normally, equities are associated with having more risk, which is not the same for our forecast models. It can be the case that much of the magnitude of the equity returns are removed, since the combination forecast uses an averaged mean, which smoothens the equity more than the bonds. Another way of illustrating this fact is by calculating the difference between the predicted values of equity and bonds, which is shown in Figure 37. When the difference process goes below zero, the bond premium outperforms the equity premium, which will indicate that these are points in time where we want to adjust our view-vector.

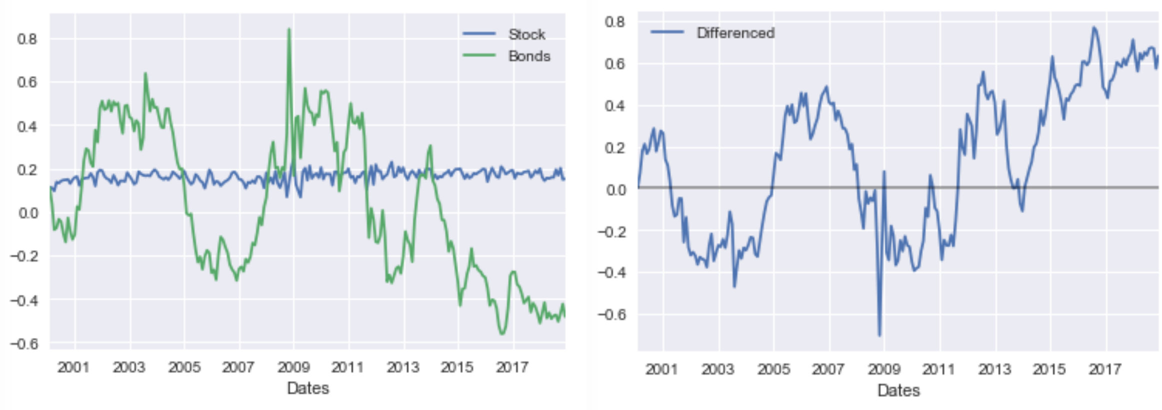


FIGURE 37: PREMIUM PREDICTIONS FOR SP500 AND TREASURY + DIFFERENCE PROCESS

**Q vector**

The Q-vector expresses the relative change in the performance of the view, i.e. Q shows how stocks outperforms bonds by X%, oppositely the same thing when bonds outperforms stock at each month. As expressed earlier, Q contains the value of return in relative performance. The approach is the same as previously.

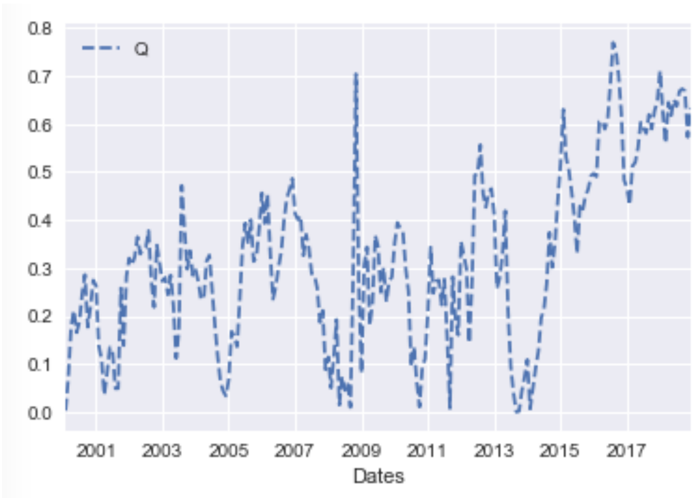


FIGURE 38: TARGET RETURN

For relative performance the Q represents the value that one asset is outperforming the other value with. Figure 38 exhibits the target returns of the views, calculated from the difference of the risk-scaled stock and bonds. Looking at the graph, it cannot be observed which asset is outperforming the other since it only tells us about the return of the (over/under) performance, i.e. the P-matrix is important for the interpretation. Depending on the P-values, the negative weight of P x Q gives the performance of the asset negatively weighted while the positive weights of P x Q give the performance of the positively weighted asset. This is also represented through

Equation 3.4.4. Furthermore, the target returns seem to vary a lot over time and indicating that there are not always significant target returns. If Q is close to zero, none of the are performing much better than the other. The highest target returns appear to be around the end of 2008 and 2016 measuring to approximately 70%. It is expected that the target returns at these periods will have a large impact on the expected returns.

**Ω vector**

The Ω-vector represents the uncertainty of each view or explained as the variance of the covariance-matrix of the view. The view distribution only contains one specific view; hence it cannot be applied diagonal. Therefore, uncertainty will be risk-weighted in regard to the variance likewise shown earlier. The Ω were computed as the sum of 1/LR vol · risk premium for both assets.

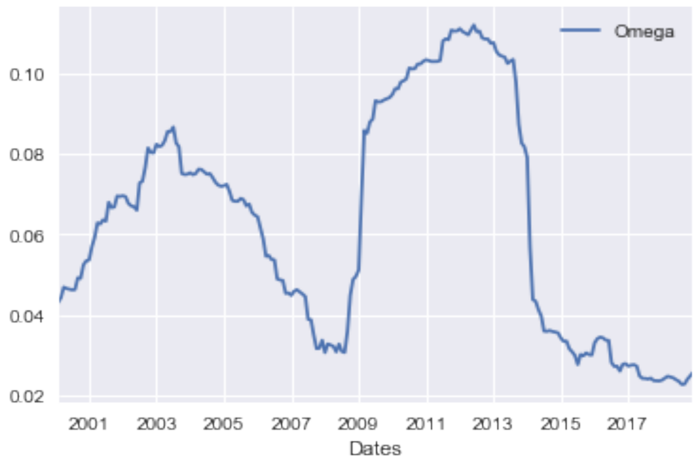


FIGURE 39: OMEGA

The uncertainty of the view is very similar to the distribution expected return of CAPM and also very closely related like the mean-variance standard deviation, as it applies the long-run volatility (also approximated as the full sample-volatility). When risk weighting the uncertainty of the view, it is closely following the stock rather than the bond volatility, since bonds did not show any significant volatility, most of the risk is weighted from the stock. The volatility of the bonds was relatively low, around 5%, and seemed to be very persistent over time, the uncertainty of the view has just decreased with the proportionality of bond risk. Since the uncertainty is very similar to the expected return, it may imply that we are very uncertain when high volatility is present and becomes more confident when the expected return is low. Each view is independent, which is why normally the variance-covariance is applied diagonal. The confidence level of the investor view is represented through the inverse of omega $\Omega^{-1}$.

The matrix Ω is employed with the scalar tau (which was discussed earlier), which is a very controversial part of the BL model. Since Ω is the uncertainty, the scalar tau will measure it to be smaller as tau scales the view distribution lower as we are less confident in the view. This is employed in the posterior distribution of returns.

### 5.3.2.3 Posterior distribution

For the posterior distribution both the combined expected returns and covariance matrices are input variables to the Black-Litterman optimization. These are therefore calculated, and further applied in the optimal portfolio to look at how the asset allocation are going forward in time.

**Posterior returns**

The posterior returns are generated using Equation 3.4.7. The posterior returns when rebalancing monthly is displayed where the blue and green line represent the returns of stocks and bond, respectively.

PRIOR POSTERIOR

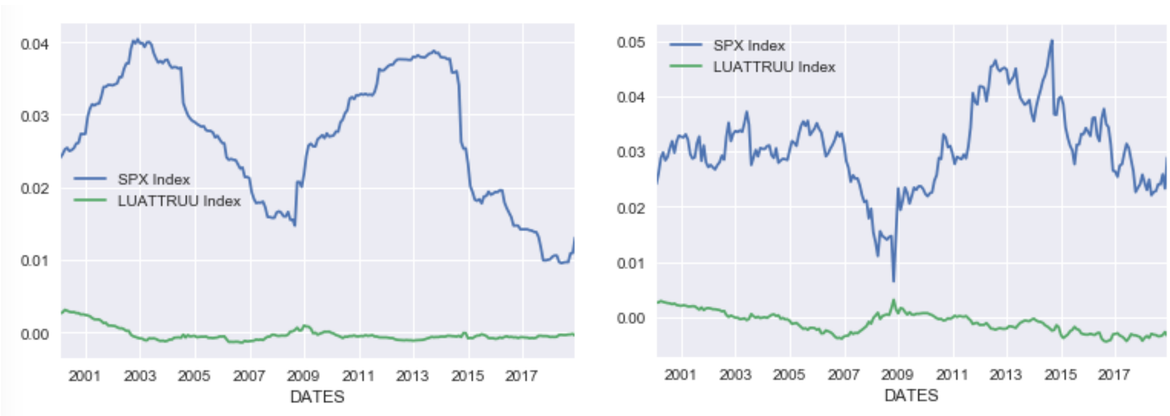


FIGURE 40: EXPECTED EXCESS RETURNS

The prior and the posterior returns are showing to have very similar patterns, perhaps, because we are more confident in the prior distribution compared to the view distribution. Some periods illustrate a bit lower return on stocks, than observed from the prior. This is, however, not always the case. In general, the posterior stock returns have more variation compared to the implied returns. The posterior bonds return seems to approximately, fluctuate around the same level, but also view some minor deviations.

**Combined Posterior Covariance**

The simple Black-Litterman example in Section 5.2.3, showed how the combined covariance was computed. This was done by combining the historical covariance and the covariance of the posterior returns. The covariance of the rolling approach (as seen in the MV analysis) and the covariance matrix of the expected returns from the Black-Litterman model is applied in the same formula. This is done to obtain the covariance matrix for the posterior distribution. We know for a fact, that the posterior covariance is higher than the covariance of the expected returns, since the two covariance were merged together. However, there is not a tremendous deviation as the changes are minimal. The posterior covariance contains the covariance of the excess expected return and furthermore, does also contain the variances for each asset shown in the diagonal elements of the matrix.

**Optimal portfolio weights**

From the theory section previously described, the optimal unconstrained portfolio, derived from mean-variance optimization, requires the inputs of the covariance matrices and the expected excess return vectors. The weights are the last part which follow the same step of the optimization portfolio, to obtain the portfolio allocation.

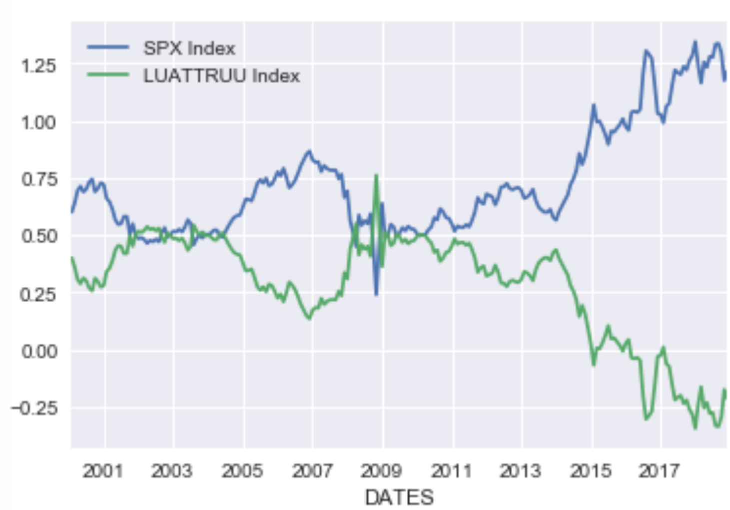


FIGURE 41: PORTFOLIO WEIGHTS

FIGURE 41 plots the monthly allocation weights of the Black-Litterman optimization. The allocation of the Black-Litterman portfolio indicates fluctuating portfolio weights, mainly having a higher allocation in stocks over bonds. The weights appear to deviate from the reference point, particular after 2014, showing considerably higher allocation towards stocks. In 2015 and further, the weights allocate a short position in bonds and long position in stocks. On expectation, the portfolio returns should be high if having long position in stocks when the market cycle shows expansions. The allocation weights show small crossovers around 2002 and 2008. In addition, it is also observed that the allocation, on some points, are close to 50:50. These weights appear to quickly revert, probably due to changes in the views. It was previously seen the mean-variance weighted more towards bonds as opposed to the Black-Litterman model, which is allocating more in stocks.

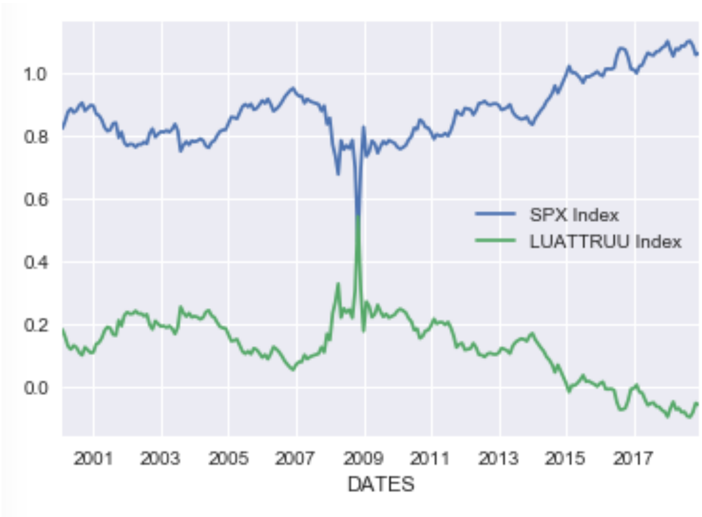


FIGURE 42: RISK ALLOCATION

The risk allocation, appearing in Figure 42, show that the equity risk is overrepresented in the portfolio throughout the whole period. From 2000 to 2008, it can be proxied that around 80% of the risk of the portfolio is resented by equities and respectively 20% by bonds. In 2008 the notional allocation weight in bonds is 75%, and 25% in equity, which seems to provide a 50:50 allocation in terms of risk. The risk allocation for bonds is declining consistently from 2008, which is naturally because of the notional weight allocation is lower. Correspondingly, the opposite appears for equities which is followed by an increasing risk. This risk allocation is also strongly related to the views of the model, as the periods where stocks is outperforming bonds is reflected by a heavier risk allocation in stocks. It appeared that bonds were underweighted relative to stocks in 2014, looking at the notional weight allocation. This corresponds to the same period experiencing a steady decrease in the observed risk-allocation towards bonds. As opposed to the mean-variance optimization, seeking to maximize the Sharpe ratio, the risk-allocation in Black-Litterman is influenced by the views. Overall, the two models show quite contradicting risk-allocations.

Going further, we want to look at the cumulative performance of the generated portfolio if we invest our wealth in the year 2000.

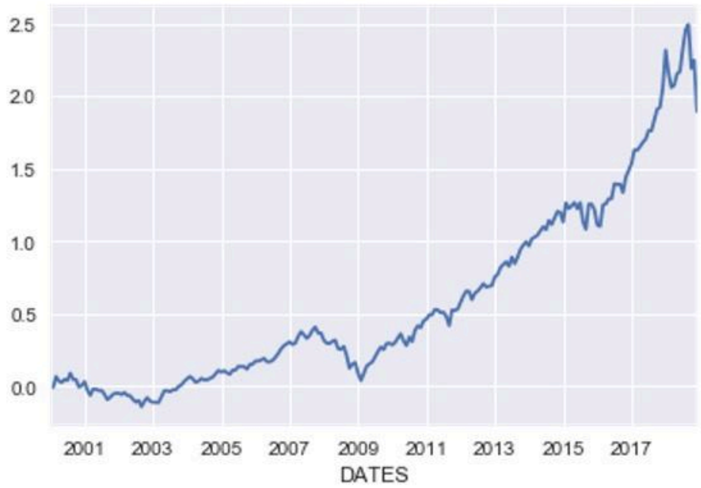


FIGURE 43: CUMULATIVE BLACK-LITTERMAN RETURNS

The cumulative returns in FIGURE 43 shows the performance of the asset allocation, which have shown similar patterns as the cumulative returns of CAPM. An explanation could be that the Black-Litterman weights does not diverge that much from the CAPM allocation. This is at least the case until 2015, where the cumulative returns have almost shown an exponential effect. The returns will in the ending period be 189.3%, based on an investment today. The investor views provide effective results, in periods where the change is large enough. Since 2016 the returns are doubled, perhaps because the views are heavily weighted with a lower uncertainty towards the view. This is clear for the optimal portfolio weights, where both long and short positions are observed. The posterior returns also show quite a deviation from 2016, again, as a result of the lower uncertainty of the views.

## 5.4 Out-of-sample comparison of the allocation models

After presenting and discussing the rolling estimates of our two allocation models, the models are compared and evaluated for their out-of-sample performance and statistical significance. The out-of-sample Sharpe ratio will be the main measurement, while the M-squared, the Certainty Equivalent and the T-statistics will be assessed to support the findings arising from the Sharpe ratios.

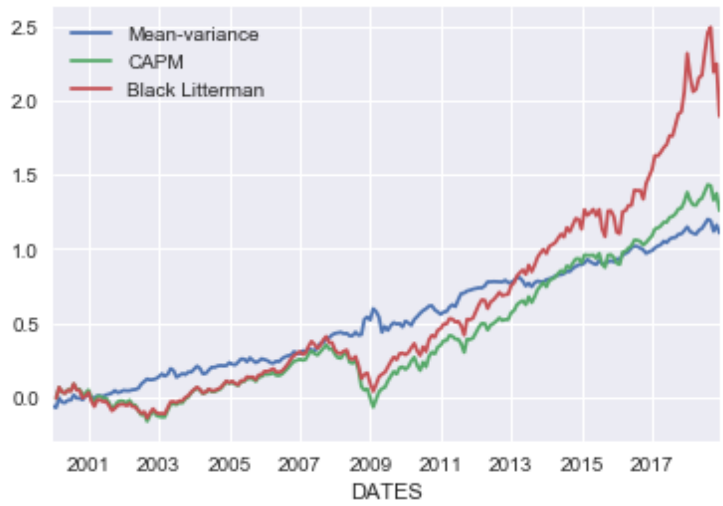


FIGURE 44: CUMULATIVE LOG-RETURNS ON THE PORTFOLIOS

Figure 44 illustrates the out-of-sample cumulative realized return performance of the mean-variance-, the CAPM- and the Black-Litterman portfolio. Our mean-variance portfolio appears to perform quite well throughout the out-of-sample period and does not have as significant negative drops around periods such as the financial crisis etc., as it appears to have on the Black-Litterman- and CAPM portfolio. Actually, the BL and MV portfolios are almost equivalent in 2013, looking at the cumulative returns for the respective portfolios. The mean-variance is consistently increasing in terms of cumulative returns, without any major spikes. Further, an observation of the Black-Litterman portfolio indicates to follow the fluctuations of CAPM portfolio quite narrowly, at least until 2013. The Black-Litterman portfolio significantly outperforms the two other portfolios after 2013 in terms of cumulative realized returns based on the allocation to the respective portfolios.

The BL portfolios deviation from the CAPM portfolio observed after this can be traced back to the weights presented in Figure 41. Up until year 2009, we observed that the BL portfolio only has small deviation from the 60:40 market capitalization allocation. After this, the estimated weights are changing more significantly for each t, with an increasing allocation towards stocks. In combination with high realized returns on the stock index, this give us a good performance on the BL portfolio the latter years. The high allocation towards stocks in both the BL-model and the CAPM also explains why we previously observe the negative spikes at some points in the graph when the stock index has periods of negative returns. This is especially the case in times of crisis and recession. At the end of 2018, the summarized return for the mean-variance is 110.58%, and the corresponding values for CAPM and BL is, respectively, 125.74% and 200.51%. Actually, the BL model has almost doubled the return compared to MV in terms of cumulative returns at the end of the horizon. This indicates that changing views over time can possibly provide effective results, especially when stocks perform well.

| | **Asset Statistics** | | **Portfolio Statistics** | | |
|---|---|---|---|---|---|
| | **S&P 500** | **Treasury 10-yr** | **MV** | **CAPM** | **BL** |
| Arithmetic mean | 3.101 % | 4.395 % | 4.065 % | 4.658 % | 6.096 % |
| Excess return | 1.522 % | 2.816 % | 2.486 % | 3.079 % | 4.517 % |
| Volatility | 14.683 % | 4.540 % | 5.285 % | 8.316 % | 9.660 % |
| Sharpe Ratio | 10.365 % | 62.036 % | 47.035 % | 37.026 % | 46.757 % |
| Adjusted SR | 10.331 % | 61.831 % | 46.880 % | 36.905 % | 46.603 % |
| M-squared | 2.454 % | 6.820 % | 5.553 % | 4.707 % | 5.529 % |
| Certainty equivalent | -0.634 % | 2.610 % | 2.207 % | 2.388 % | 3.584 % |
| t-statistics | 1.562* | 9.347*** | 7.087*** | 5.579*** | 7.045*** |
| Observations | 227 | 227 | 227 | 227 | 227 |

TABLE 26: ANNUALIZED DESCRIPTIVE SUMMARY STATISTICS OF THE RETURN PROCESS AND THE PORTFOLIOS (OUT-OF-SAMPLE)[6]

TABLE 26 presents the descriptive statistics of S&P 500 index and treasury index in the out-of-sample period, combined with the portfolios respective out-of-sample statistics and measurements. The parameters will be used to compare the overall performance of the models over the out-of-sample period.

When just observing the statistics of the individual assets, it appears for bonds, illustrated by the 10-yr treasury, to have performed surprisingly well compared to the stock index. This is in fact ultimately well-described by looking at the out-of-sample Sharpe ratios of the respective assets. This comes from the fact that bonds have performed well in terms of excess returns in this period, combined with the fact that the volatility has been relatively low. Looking at the certainty equivalent, it appears that an investor, with a risk aversion of 2, would rather invest in a negative risk-free asset than the S&P 500 index during this out-of-sample period.

[6] The number of stars *, **, *** corresponds to significance levels respectively equalling 10%, 5% or 1%.

For further assessment, we can observe the M-squared measure, which in relation with the Sharpe ratio tells us something about the risk-return relationship of the assets. Lastly, the computed t-statistics (with its respective significance level) tells us that the excess return on the bond index is statistically significant on a 1% level, while the S&P 500 index shows to be statistically significant on a 10% level. This is probably an important note to why the mean-variance portfolio shows a nice and steady increase in the portfolio return over time, due to a general overweight in the allocation of bonds compared to stocks. The rolling mean-variance weights in Figure 30 shows that the model assigns a quite significant weight to bonds, probably due to the Sharpe ratio performance relative to stocks (in combination with other things such as correlation etc.).

Further, we want to look at the overall out-of-sample performance of the three portfolios generated. Even though it appears from Figure 43 that the mean-variance portfolio performs better during the (approximately) first 13 years of the investment horizon, we see that the Black-Litterman portfolio had a higher arithmetic mean and excess return overall. It can also be mentioned that the CAPM portfolio also had a slightly higher arithmetic mean and excess return over the total period than the mean-variance portfolio. However, when scaling for the respective volatilities, we see that the CAPM portfolio provides a lower score in terms of Sharpe ratio than the two remaining portfolios. The Black-Litterman portfolio provides us with the second highest Sharpe ratio over the out-of-sample period, nonetheless a significant amount of the returns is accumulated at the end of the investment horizon. This is lastly confirmed by the certainty equivalent, which claims that an investor would be willing to invest at a risk-free rate at 3.58%, where the accepted rate for the MV and CAPM portfolio is a bit lower. The bond index and the remaining portfolios all show positive measures, although the MV portfolio outrival the CAPM- and BL portfolio by a small margin overall.

That being said, we know that the mean measures calculated over the total out-of-sample period suffers from a lot of variation in the returns and risk over time. This fact compromises the preciseness of the mean results, and as mentioned, the portfolios appear to perform better in some sub-periods than others. As pointed out by Munk (2019), this is ultimately the problem when working with averaged values over a given time period. When working with arithmetic means, it is usually the case that having a long period of observations is better than having too few. Having a minimal number of return observations can therefore result in less informative estimates. Further, when regressing the market portfolio on a return process of a given strategy or allocation model, it is rarely the case that the alpha obtained from the regression is statistically significant.

As pointed out when computing the weight allocations for the mean-variance model, we noticed that the bonds provided a higher expected Sharpe ratio in the beginning of the out-of-sample period. This led the objective to maximize the portfolio allocation towards an overweight in bonds, which by looking at the realized returns gave us a nice and steady gain over the first years of the investment horizon. The first six years also showed us that the CAPM capitalized weights, which have an overweight towards stocks, performed poorly. The deviation from these weights were not substantial in the Black-Litterman model, which also led this portfolio with a bad performance during the first years. To illustrate the split in the series, we will compute the equivalent measures as above by dividing the total out-of-sample into two subsets. This will be done by splitting the sample in the changing point of 2009 and 2010. Lastly, we will have a look at the performance for the six last years (approximately) of the out-of-sample period where we observe an exceptional performance of the Black-Litterman portfolio.

| | **Annualized descriptive portfolio statistics** | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 2000-01-01 to 2009-12-01 | | | 2010-01-01 to 2018-12-01 | | | 2013-01-01 to 2018-12-01 | | |
| | MV | CAPM | BL | MV | CAPM | BL | MV | CAPM | BL |
| Arithmetic mean | 4.080 % | 2.295 % | 3.004 % | 4.048 % | 7.261 % | 9.502 % | 2.958 % | 6.771 % | 9.598 % |
| Excess return | 1.438 % | -0.346 % | 0.363 % | 3.650 % | 6.863 % | 9.104 % | 2.407 % | 6.220 % | 9.047 % |
| Volatility | 6.461 % | 9.253 % | 8.805 % | 3.583 % | 7.110 % | 10.475 % | 3.376 % | 6.130 % | 11.267 % |
| Sharpe Ratio | 22.264 % | -3.739 % | 4.121 % | 101.876 % | 96.524 % | 86.906 % | 71.289 % | 101.469 % | 80.302 % |
| Adjusted Sharpe Ratio | 22.126 % | -3.716 % | 4.095 % | 101.173 % | 95.859 % | 86.306 % | 70.554 % | 100.423 % | 79.474 % |
| M-squared | 4.523 % | 2.325 % | 2.990 % | 9.006 % | 8.554 % | 7.741 % | 6.575 % | 9.125 % | 7.336 % |
| Certainty equivalent | 1.021 % | -1.202 % | -0.413 % | 3.521 % | 6.357 % | 8.006 % | 2.293 % | 5.844 % | 7.778 % |
| t-statistics | 2.439** | -0.410 | 0.451 | 10.587*** | 10.031*** | 9.031*** | 6.049*** | 8.610*** | 6.814*** |
| Observations | 120 | 120 | 120 | 108 | 108 | 108 | 72 | 72 | 72 |

TABLE 27: DESCRIPTIVE STATISTICS FOR VARIOUS OUT-OF-SAMPLE PERIODS[7]

[7] The number of stars *, **, *** corresponds to significance levels respectively equalling 10%, 5% or 1%.

It appears from TABLE 27 that the portfolios indeed have performed quite unstable over different time periods of the out-of-sample period. As explained, we adjust the measures to account for observation bias when comparing different time periods. For the first split, running from 2000-01-01 to 2009-12-01, we observe that the mean-variance portfolio outperforms the CAPM- and BL portfolio in terms of Sharpe ratio and M-squared. It can also be observed that the utility measure suggests that an investor would rather take on negative risk-free assets than invest in the CAPM- and BL portfolio. In fact, the CAPM portfolio appears to have a negative Sharpe ratio due to negative excess returns in this specific sub-period. The BL portfolio has a positive, however, very low Sharpe ratio. Further, it can be observed that both the CAPM- and BL portfolio suffers from insignificant t-statistics based on the levels we test on. On the other hand, the MV portfolio provides a statistical significance at a 5% level. We see that the portfolio returns have been relatively low on average during this period, which can be partially explained by negative returns on the stock index, which have been assigned the most weight in the CAPM- and BL portfolio allocation in this period.

The story is quite different for the CAPM and BL portfolio in the next sub-period running from 2010-01-01 to 2018-12-01. This period provides us with considerably higher returns on the CAPM and BL portfolio. All portfolios view high Sharpe ratios, yet the CAPM and MV outperforms the BL portfolio in this period as well, due to high volatility in the BL portfolio. Even though the BL portfolio has the highest averaged return over the period, it only delivers a Sharpe ratio of 86.31%, compared to values closer to 100% in the two other portfolios. The volatility of the MV portfolio has contrary approximately been halved. The return on the MV portfolio has not changed significantly when comparing the two first periods. This leads the portfolio to obtain a Sharpe ratio over hundred, outperforming both the BL and CAPM portfolio. The same conclusions as seen in the Sharpe ratio is backed up by the M-square measure, which shows that all portfolios seemed to perform well, but in this period, the MV portfolio also marginally outperformed the others. Finally, it is observed that all the portfolios deliver significant t-statistics in this sub-period on a 1% level. The certainty equivalent is the only measure that argues in favour of investing in the BL portfolio.

Lastly, we will have a look at the six ending years of the investment horizon, running from 2016-01-01 to 2018-12-01, where the BL portfolio specifically appeared to perform very well. Firstly, it appears to be quite some deviations in the arithmetic means of the portfolios, where the average return of the BL portfolio is approximately three times as high as the equivalent measure of the MV portfolio. In exchange for this, we observe a much higher volatility in the BL portfolio compared to both the CAPM- and MV portfolio. Comparing the risk-reward relationship illustrated in the Sharpe ratio, we see that the BL portfolio outperforms the MV portfolio, but the CAPM portfolio lastly provides the highest ratio. In addition, the CAPM portfolio beats the MV and BL portfolio in terms of M-squared, and secondly gives us the highest t-statistic. Anyhow, all portfolios show significant t-statistics on a 5% level. As pointed out, the sub periods contain different numbers of observations, but according to Jobson & Korkie (1981), the Sharpe ratio adjustment accounting for the observation effect works for sample sizes down to 12 observations. It appears that the certainty equivalent still favours the BL portfolio, which has been the case for all periods.

# 6. Discussion

## 6.1 Evaluating results

From the empirical results, the Black Litterman rolling portfolio outperformed the market portfolio, CAPM, and the mean-variance approach in terms of cumulative returns. However, it is clear that the outperformance is not consistent for all horizons.

The mean-variance returns showed to have consistently increasing returns over different time-periods. As aforementioned, mean-variance optimization is often leading to error-maximization (Michaud, 1989; Fabozzi, Markowitz, Kolm, Gupta, 2012). Due to fixed investment horizon and historical observations of assets, this could lead to problematic outputs, as the asset returns often appear to be a quite imprecise estimate of future asset returns. Since the MV is quite sensitive to the input variables, this could mean for the analysis to provide substantial wrong expectations and weights for the portfolio. This uncertainty of the MV optimization is pioneered by Roll (1979), Jobson and Korkie (1980, 1981), Shanken (1985), Jorion (1986, 1992). The rebalanced portfolio weights in our analysis do genuinely appear very satisfactory, thus does not provide sustainable miscalculated weights. This might be to the choice of applying rolling estimation, giving only the most recent information within a given time period, instead of using a large amount of historical data. The portfolio allocation seems to hedge itself against a recessionary economy but does not give substantially good returns in terms of an expansionary economy. Intentionally, the allocation of the portfolio from 2013 should have been allocated differently in order for the MV-optimization to beat the BL and CAPM. An examination by Merton (1980) found that the expected returns of the MV-analysis are difficult to estimate. This is also the case for our findings, as the tangency portfolio shows to have a significant large expected return in the out-of-sample estimation during the beginning of the period, however, this is not actually a realization in the true portfolio returns.

Moreover, a two-asset case also gives quite an intuitive interpretation of the result, but for instance, selecting a larger investment universe would also have affected the expectations and results of the mean-variance analysis. Simply, including more (low correlated) assets would result in decreasing portfolio variance because of the diversification (also known as unsystematic risk), which is even more diversified, if the assets have a negative correlation. Generally, an important key factor is to strive after portfolio diversification, due to maximization in the protection of the investments (Fabizzo, Markowitz, Kolm & Gupta, 2012). Increasing securities in the portfolio result in lower risk if the risk-free rate is maintained the same. Yet, the S&P 500 is already well-diversified and Meccui (2009) specifies a well-diversified portfolio as not heavily exposed to an individual stock. The chosen portfolio allocation can, therefore, be argued to be well-diversified, but does hedge us in terms of diversification if it is going bad in stocks, not many correlated assets are present in order to account for a possible decrease in stock and the other way around.

Bessler and Wolf (2015) investigate the effects of adding commodities into a stock-bond portfolio. They examine different portfolios among MV-optimization as well as BL-optimization on out-of-sample data. They conclude that the BL optimized portfolio performs better than the comparable portfolios measured by the Sharpe ratio and among other measurements. The same is found in our study, and especially from 2016 - 2018, the Sharpe ratio is significantly high for the BL portfolio compared to the MV and CAPM benchmark. Additionally, research implied by Bessler and Wolf (2015) suggests developing and estimate return prediction models. This is carried out for our analysis, where the forecast of stocks and bonds is used in relation to the views of the Black Litterman model. This approach indicated superior performance in terms of the Sharpe ratio for all periods and sub-period besides 2000 - 2009. Since the market portfolio performs poorly in recessions, observed in the cumulative portfolio returns, the BL must deviate from the market portfolio significantly in order to have better performance measures during this sub-period. Harries et al. (2017) examine the out-of-sample performance of BL compared to a benchmark strategy and native portfolio. They conclude, under different performance measures, that the BL portfolios outperform the benchmark and the native portfolios. For our overall out-of-sample period, their conclusion is similar to our findings.

It can be argued whether the effect of transaction costs, which under real-world scenarios is included, would have resulted in different conclusions. Transaction costs would overall have an impact on the profitability, but dependent on the exchanges as these fees vary. Throughout our research, the transaction fees are excluded. Though, including transaction cost would give a more realistic picture of the profitability, since many of these transaction costs are based on the percentage of the amount traded. In fact, the benefits of rebalancing may be smaller than the inclusion of costs, in terms of turnover, see Davis and Norman (1990) for a mathematical explanation, and Acharya & Pedersen (2005) for empirical study. There have been discussions on the role of the transaction cost in optimization. A vast majority of the literature uses the transaction costs ex-post, e.g. Bollerslev (2016) and Hautsch et. al (2015), analyzing how a portfolio strategy would have performance including transaction costs of a given size. But the transaction costs, in practice, are often included ex-ante so it becomes a part of the optimization problem. Hautsch and Voigt (2017) point out, including transaction costs is quite important for the reallocation of the portfolios but reduces the benefit of predictive models. Since our model is applying a predictive model in terms of views generation, we do not wish for the model to reduce the explanation of the forecast.

Furthermore, the discussion of investor views can be important and is rather complicated determining the view being the main component influencing the views. Many investors just rely on financial analysts' reviews to determine their views. A framework that is more validated is the application of GARCH-derived views proposed by Beach and Orlov (2007). Conclusion on their analysis, based on a risk-adjusted decision, shows to have beneficial moments applying EGARCH-M, giving the highest return among the compared portfolios. Another ambiguous decision related to the views is the application of absolute views instead of relative views. A substantial part of our study concentrates around the stock-bond relationship, hence prevalent to implement relative considerations. They apply absolute views, which also make sense since GARCH takes into account volatility clustering which occurs when large changes in returns are followed by another large change (in absolute terms) and oppositely, small changes followed by other small returns. Walter (2013) refers, the absolute view is giving larger improvements in the precision of the estimate. It does not appear for literature to provide much information about predictions and any empirical evidence in terms of using absolute returns. Mostly, many have applied a relative view in comparing assets, just like our empirical analysis. The P-matrix has also been introduced differently. A proposal by Satchell and Scowcroft (2000) is an equal-weighted scheme to express the weights of the asset related to an indirect relative view, while He and Litterman (1999) apply a market capitalization weighting scheme.

Allaj (2019) and Fabozzi et al (2006) use momentum-strategies to generate views and compare this out-of-sample with respect to different portfolios. Allaj (2019) suggests that the views are derived by maximizing the expected value of a quadratic utility function of the portfolio excess returns, where views are generated by using reverse optimization. In particular, this means that the investor defines its own portfolio weights, allowing the investors to directly to express views. They find the superior performance of the BL model, consequently, public information combined with private gives benefits that have to be incorporated in the model. It cannot be said which proposal of the view provides the best and most accurate results, as no literature has shown the comparison of these different approaches, though it can be said that the forecast estimation provides a more proper estimate rather than just assuming or applying analyst recommendation.

## 6.2 Suggestions for further research

Further research suggested the application of rolling data in the prediction models, to obtain more consistent data throughout the analysis, just as the rest of the analysis. The use of rolling data in the prediction models should give a more precise forecast of the premiums. Since the equity premium model consistently is outperforming the historical average, according to Rapach et. al (2008) and as well as the analysis, the equity premium is doing superior compared to how the returns in reality perform, and since the investor view does depend on these regression models, a more accurate picture of the stock return predictability is desired. The bond premium seemed to capture more of the magnitude behind the returns, hence it should be provided with a more precise forecast in terms of the rolling window.

Clark and McCracken (2008) from Federal Reserve in St. Louis investigates the terms of improving forecast accuracy by combining recursive and rolling forecasts. Often, the terms either apply to the rolling window, because financial data are known to have long observations (Fama MacBeth, 1973), or recursive estimation. Their findings show beneficial by combining the recursive and rolling forecast, evaluated on a 2.5% significance level, in order to provide better individual prediction models that consistently improve the forecast accuracy. The bias of forecasting accuracy is measured in terms of MSE and RMSE. Here, in line with Clark and McCracken (2008), one would expect for the equity premium prediction models in our analysis to have been more accurate in terms of forecast, thus giving different investor's views with the thought that the views should have more adjusted of the forecast period, which would result in better portfolio returns. Since the BL model was not able to hedge itself against recession, it might have been able to improve this ...

Practically, implementing recursive and rolling forecasts would imply to calculate combinational weights. A linear combination of the recursive and rolling forecast is equivalent to the corresponding estimates. The linear estimation combination is given as, $\beta_{w,t} = \alpha_t \beta_{R,t} + (1 - \alpha_t)\beta_{L,t}$. This gives both the ability to derive the optimal window but also the optimal combining weights, $\alpha_t$, in the presence of a single structural break. The optimal strategy found by Clark and McCracken is to combine a rolling forecast using post-break observations with a recursive forecast that uses all observations. Overall, this would be interesting to implement and could have led to more precise estimations of the view generation, which might have resulted in improved portfolio returns of the Black-Litterman model.

A large part of applying prediction models was the ability to provide a model which actually focused on real financial data, instead of assuming data, hence these investor views. The $R^2$ of the regression models was used as a determinant measure in order to grasp the robustness of the OLS-regression on the sample set. Some of the results gave a poor coefficient in terms of $R^2$ but was still applied in the combination forecast as the results of the equity premium prediction, aware that the $R^2$ was zero for some of the single predictor variables and could lead our conclusion of prediction to be questioned for validity. Accordingly, an improvement of the analysis, a suggestion would be to remove the predictor variables with zero explanation in the equity premium. However, if these single predictor models were removed from the model, the conclusion would still likely be very similar, that the forecast combination outperforms the historical average.

Investigations of the risk-adjusted coefficient, $\lambda$, or the scalar, $\tau$ can be analysed in terms of sensitivity. Many researchers have sought to establish these constants, but no final statements have been made. Practically, most people have different risk aversion depending on the world-economy and static risk aversion is not a real-world implementation. Further research would imply for a setup where an investor is very risk-averse, compared to the opposite, less risk aversion. A situation where the investor is more risk-averse would imply a large divergence between the prior and the posterior. Various discussions regarding calibration of $\tau$ have been made. Black and Litterman (1992) describe the uncertainty in the mean to be smaller than the uncertainty of the returns, hence it will be close to zero. They propose in their paper from 1999, the as a ratio related to the distribution variance, therefore calculated as 1/t. Walter (2013) investigates three methods of selecting, by estimating tau from standard error of the equilibrium covariance matrix, using confidence intervals or examining the investor's uncertainty as expressed in their portfolio. This study uses the third examination, where the $\tau$ is considered from the view of a Bayesian investor, where their fraction of the wealth is invested in $1/(1+\tau)$ risky asset and the fraction $\tau/(1+\tau)$ in risk-free assets. Bevan and Winkelman (1998) estimate the factor to be around 0.5 - 0.7, while Satchell and Scowcroft (2000) use a tau equal to 1.

# 7. Conclusion

The purpose of this thesis was to investigate the construction of portfolios using Markowitz's modern portfolio theory and the Black-Litterman model, and thereafter evaluate their performance out-of-sample. By estimating equity- and bond prediction models, we were able to generate our investor views regarding the multi-asset portfolio used in the Black-Litterman portfolio.

The generation of the optimal portfolios followed two different construction methods. The process required historical return observations of the assets to generate the expected return vectors in the mean-variance model. In contrast, the Black-Litterman model applied the market equilibrium returns arising from the CAPM relation. This led to two quite deviating foundations for the optimization process in the allocation models. The common feature of the models was the computation of the covariance matrix, indicating the relationship among the assets.

Furthermore, the Black-Litterman model required the subjective views of the investor to deviate from the market equilibrium. This was done by estimating premium prediction models for the stock index and the bond index. It appeared from our results that the prediction models provided us with views that led to some deviation from the CAPM allocation. However, it did not lead to significant outperformance compared to the other portfolios, accounted for risk. Overall, the allocation overweight was in general observed in the stock index, mainly fluctuating around the 60:40 CAPM allocation, which probably contained an optimistically high allocation towards stocks, to begin with.

The optimal weight allocation in the rolling mean-variance model was overall dominated by the bond index, which provided a steady portfolio return and a lower risk. This allocation made sense, considering that the Sharpe ratio was the objective of the optimization. On the other hand, the Black-Litterman portfolio provided an overweight towards the stock index, especially at the last years of the investment horizon. This led to a total opposite risk allocation in the two models, where the main risk in the Black-Litterman portfolio originated from stocks, while the equivalent in the mean-variance portfolio suggested a risk overweight in bonds. The one period estimation period indicated that the risk allocation in the Black-Litterman model was more unbalanced than what we observed from the mean-variance model, where the risk allocation was close to 50:50.

When testing the performance of the portfolios out-of-sample, we found that the mean-variance allocation overall provided us with the superior portfolio performance, when testing by their respective Sharpe ratios and M-squared; however closely followed by the Black-Litterman portfolio. The Black-Litterman portfolio generally followed the CAPM market portfolio allocation quite closely during the first years of the investment horizon without any significant deviations. This lead the Black-Litterman portfolio to drop in accordance with the CAPM portfolio, due to a high allocation towards stocks in bear markets. On the other hand, the mean-variance portfolio seemed to provide a steady increase in portfolio return without any extreme negative spikes in periods of recession or uncertainty. In terms of cumulative realized returns, the Black-Litterman model was the favoured model; however, it was also the riskiest portfolio.

When splitting the out-of-sample period into sub-samples, we found that the mean-variance portfolio significantly outperformed the Black-Litterman- and CAPM portfolio for many years. The mean-variance portfolio was the only portfolio with a satisfactory Sharpe ratio during the first ten years of the out-of-sample period, where Black-Litterman and CAPM obtained low (and even negative) Sharpe ratios. The outstanding performance from the BL model did not occur until, roughly, year 2016. In this period, the BL and CAPM portfolio both outperformed the MV portfolio by all measures.

The mean-variance portfolios appeared to have performed surprisingly well, considering the mentioned shortcomings of the model. It awarded a high allocation towards bonds, which made it perform well during recessions, such as we seen in our sample. Oppositely, the Black-Litterman portfolio assigned higher weightings in stocks, which were penalized during economic slowdowns. However, it did remarkably well during the last observed expansion. Finally, our results do not necessarily mean that this is consistently accurate. It is highly dependent on the asset composition and the selected parameters. It is fair to assume that a different use of parameters would have changed the results significantly.

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# 9. Appendices

## Appendix 1 - Master formula of Black Litterman

A derivation for than alternative formula of the Black Litterman for the posterior expected return can be shown below (Chen et. al, 2015):

$$\mu^* = ((\tau\sum)^{-1} + P^T\Omega^{-1})^{-1}((\tau\sum)^{-1}\Pi + P^T\Omega^{-1}Q)$$

$$= ((\tau\sum)^{-1} + P^T\Omega^{-1})^{-1}(\tau\sum)^{-1}\Pi) + ((\tau\sum)^{-1} + P^T\Omega^{-1}P)^{-1}P^T\Omega^{-1}Q$$

$$= (\tau\sum - \tau\sum P^T(P\tau\sum P^T + \Omega)^{-1}P\tau\sum))(\tau\sum)^{-1}\Pi + ((\tau\sum)^{-1} + P^T\Omega^{-1}P)^{-1}P^T\Omega^{-1}Q$$

$$= (\Pi - \tau\sum P^T(P\tau\sum P^T + \Omega)^{-1}P\Pi) + ((\tau\sum)^{-1} + P^T\Omega^{-1}P)^{-1}P^T\Omega^{-1}Q$$

$$= (\Pi - \tau\sum P^T(P\tau\sum P^T + \Omega)^{-1}P\Pi) + (\tau\sum)(\tau\sum)^{-1}((\tau\sum)^{-1} + P^T\Omega^{-1}P)^{-1}P^T\Omega^{-1}Q$$

$$= (\Pi - \tau\sum P^T(P\tau\sum P^T + \Omega)^{-1}P\Pi) + (\tau\sum P^T(P^T)^{-1})(\Omega(P^T)^{-1} + P\tau\sum)^{-1}Q$$

$$= (\Pi - \tau\sum P^T((P\tau\sum P^T + \Omega)^{-1}P\Pi) + (\tau\sum P^T)(\Omega + P\tau\sum P^T)^{-1}Q$$

$$\Pi + (\tau\sum P^T((P\tau\sum P^T + \Omega)^{-1})(Q - P\Pi)$$

## Appendix 2 – Optimal portfolio alternative

An alternative way of writing this, after incorporating the new combined return distributed

$$w^* = \frac{1}{\lambda}\bar{\Sigma}^{-1}\bar{M}^{-1}[(\tau\Sigma)^{-1}\,\Pi + P'\Omega^{-1}\,Q]$$

Sorting for covariance

$$\bar{\sum}^{-1} = (\Sigma + \bar{M}^{-1})^{-1} = \bar{M} - \bar{M}\,(\bar{M} + \Sigma^{-1})^{-1}\bar{M}$$

$\bar{\sum}^{-1}\,\bar{M}^{-1}$ is simplified

$$\bar{\sum}^{-1}\,\bar{M}^{-1} = \frac{\tau}{(1+\tau)}\left(I - P'^{A^{-1}}P\,\frac{\Sigma}{(1+\tau)}\right)$$

Where $A = \frac{\Omega}{\tau} + \frac{P\Sigma}{(1+\tau)P}'$ (He and Litterman, 1999(b)). Applying this rule, the optimal weights can also be denoted as

$$w^* = \frac{1}{(1+\tau)}(w_m + P' \, x \, A)$$

Where $w_m = (\lambda\Sigma)^{-1}\Pi$ is the market equilibrium portfolio and the weights of each portfolio is given by the vector A defined as:

$$A = \frac{\tau\Omega^{-1}\,Q}{\lambda} - \left[\frac{\Omega}{\tau} + P\Sigma P'\right]^{-1} P\Sigma w_m - \left[\frac{\Omega}{\tau} + P\Sigma P'\right]' P\Sigma P'\tau\Omega^{-1}\,\frac{Q}{\lambda}$$

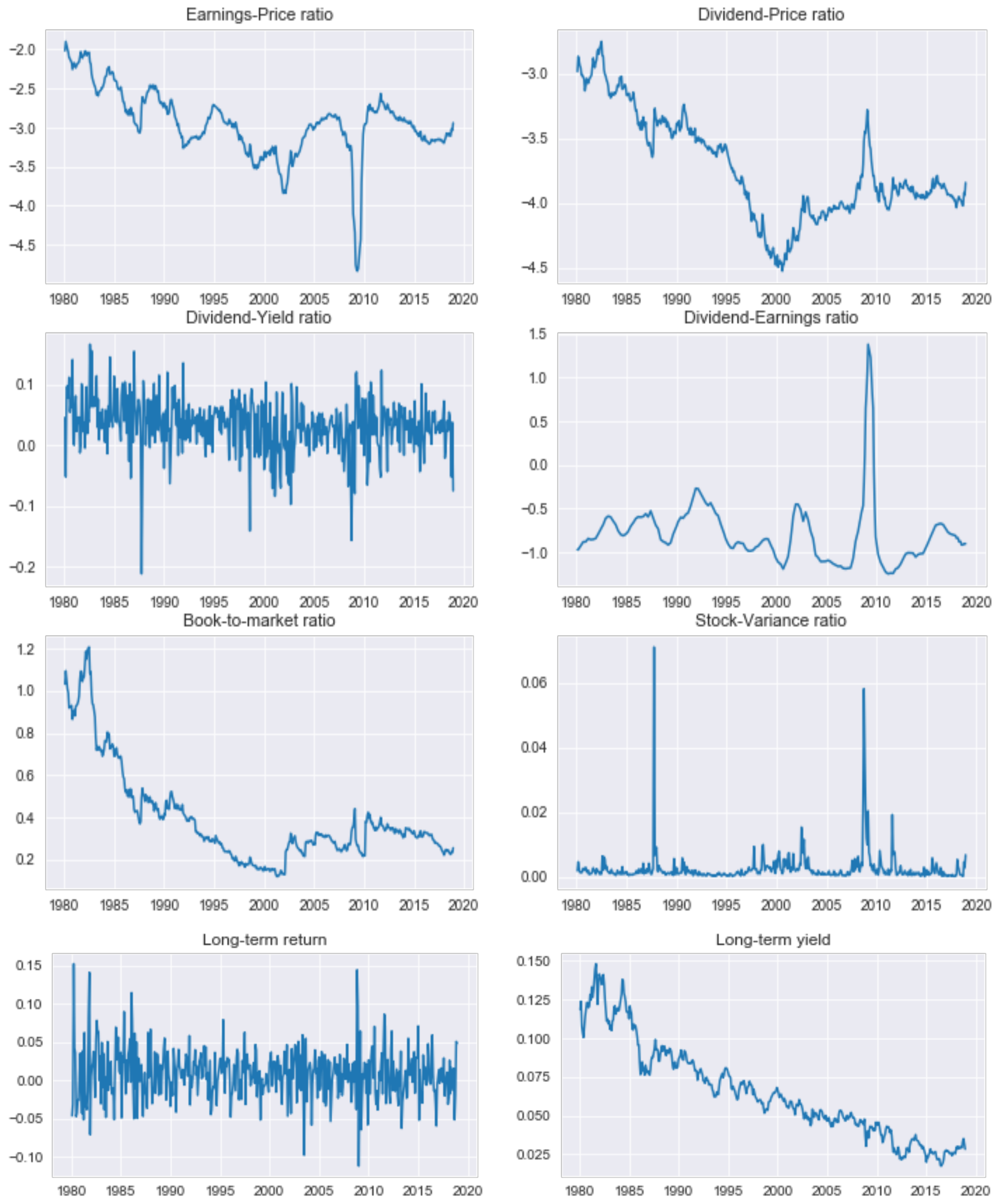
# Appendix 3 – Equity premium predictor

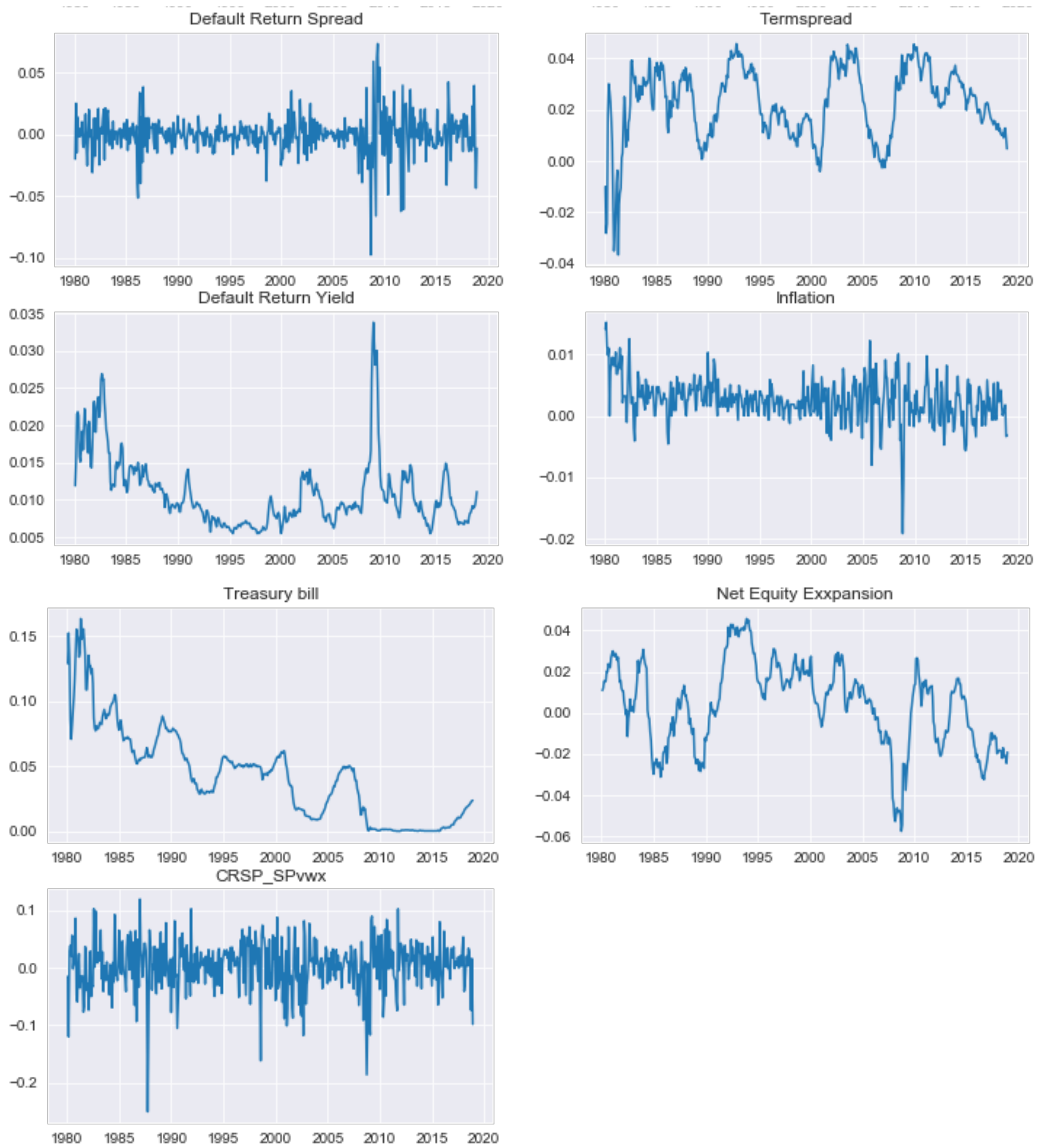
All data are provided by Goyal and Welch (2008). The data are available at www.bus.emory.edu/AGoyal/Research.html.

| Names | Description | Source |
|---|---|---|
| CRSP_wv | The total return of the stock minus the risk-free rate (T-Bill). The CRSP value-weighted index without dividends is applied. The continuously compounded returns are applied for this r=ln(Pt/Pt-1) | CRSP |
| Dividend-Price ratio (D/P) | Difference between log of dividends and log of prices | NBER |
| Earnings-Price ratio (E/P) | Difference between log of earnings and log of prices | Robert Shiller |
| Book-to-Market ratio (B/M) | The ratio of book value to market value for the Dow Jones Industrial Average. | Value Line & Dow Jones Industrial Average |
| Dividend-Payout ratio (DE) | Difference between log of dividends and log of earnings | |
| Dividend-yield (D/Y) | Difference between log of dividends and log of lagged prices | |
| Stock Variance (svar) | The sum of squared daily returns on S&P 500 | CRSP |
| Net Equity Expansion (ntis) | The 12-month moving sums of net from NYSE listed stock divided by total end-of-year market capitalization of NYSE stocks | NYSE |
| Treasury Bills (tbl) | 3-month Treasury bill Secondary market | FRED |
| Long-term yield (lty) | Long-term government yield | Ibbotson |

| | | |
|---|---|---|
| Long-term return (ltr) | Long-term government returns | Ibbotson |
| Term Spread (tms) | Difference between long-term yields and treasury bill | |
| Default Return Yield (dfy) | Difference between BAA rated bonds and AAA rated corporate bonds bonds | FRED |
| Default Return spread (drs) | Difference between the long-term corporate bonds and long-term government bonds | FRED |
| Inflation (infl) | Consumer Price index | Labour Statistics |

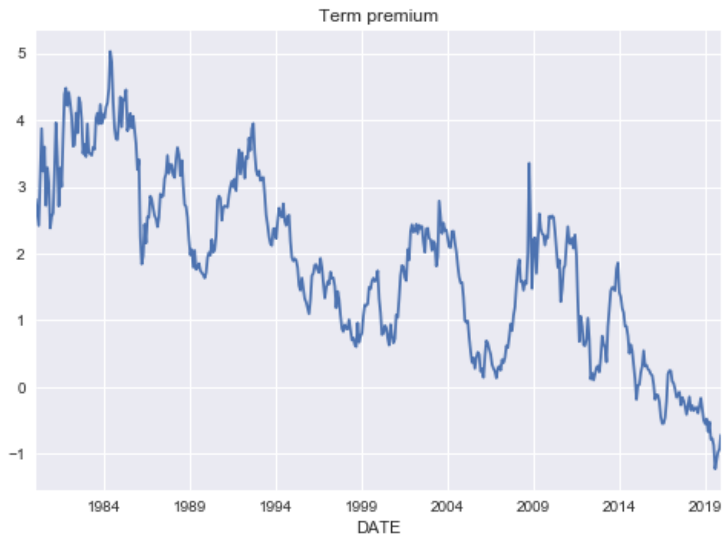
# Appendix 4 – Graphs of single equity premium predictors



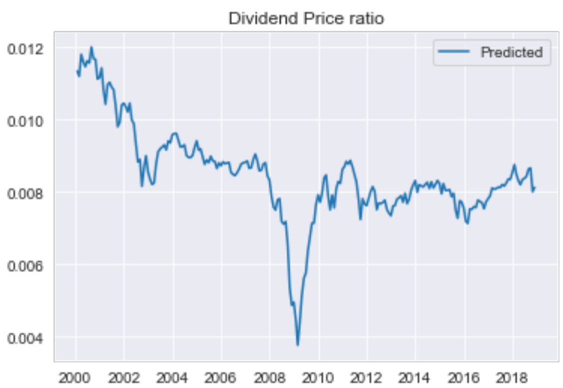

Default Return Spread
0.05
0.00
−0.05
−0.10
1980 1985 1990 1995 2000 2005 2010 2015 2020
Termspread
0.04
0.02
0.00
−0.02
−0.04
1980 1985 1990 1995 2000 2005 2010 2015 2020
Default Return Yield
0.035
0.030
0.025
0.020
0.015
0.010
0.005
1980 1985 1990 1995 2000 2005 2010 2015 2020
Inflation
0.01
0.00
−0.01
−0.02
1980 1985 1990 1995 2000 2005 2010 2015 2020
Treasury bill
0.15
0.10
0.05
0.00
1980 1985 1990 1995 2000 2005 2010 2015 2020
Net Equity Exxpansion
0.04
0.02
0.00
−0.02
−0.04
−0.06
1980 1985 1990 1995 2000 2005 2010 2015 2020
CRSP_SPvwx
0.1
0.0
−0.1
−0.2
1980 1985 1990 1995 2000 2005 2010 2015 2020

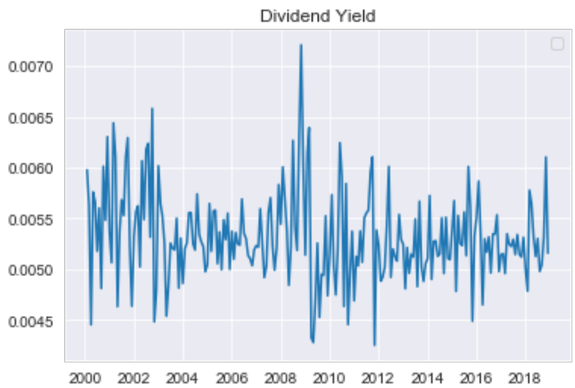
# Appendix 5 – Graph of term premium

Term premium from 1980 to 2020.

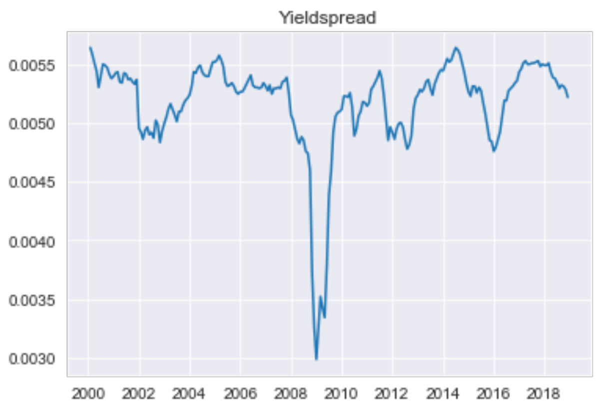


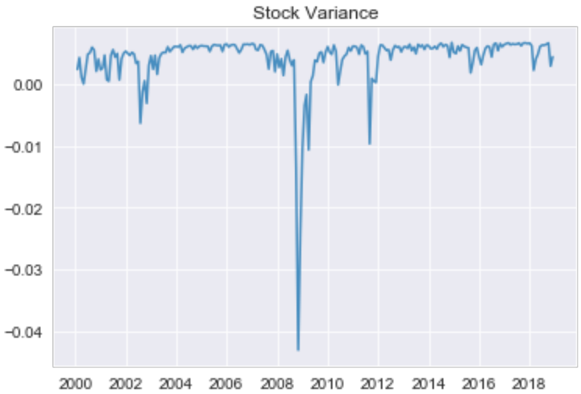

# Appendix 6 – Graphs of equity premium forecast

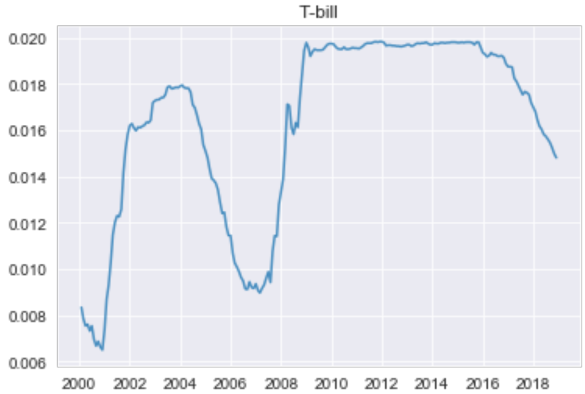


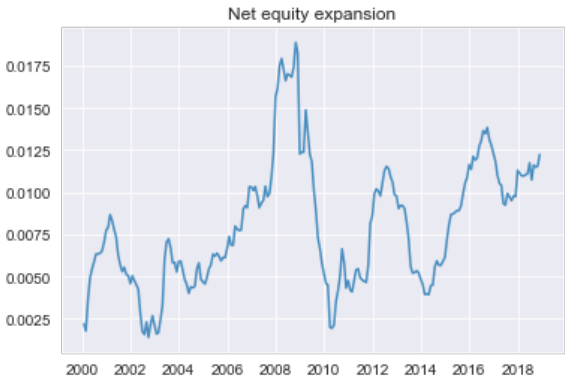



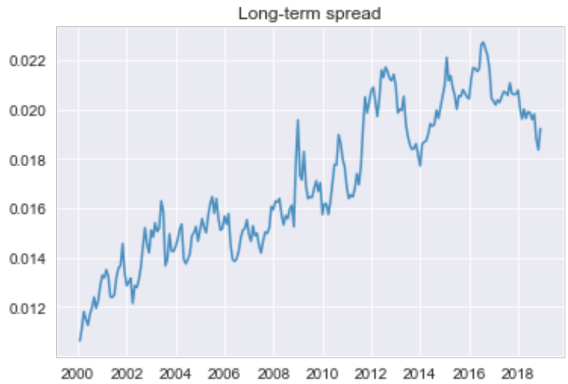


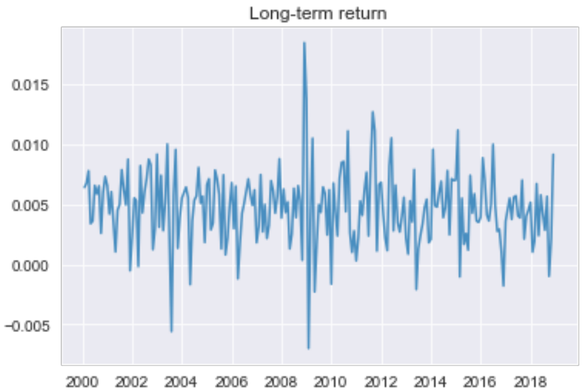


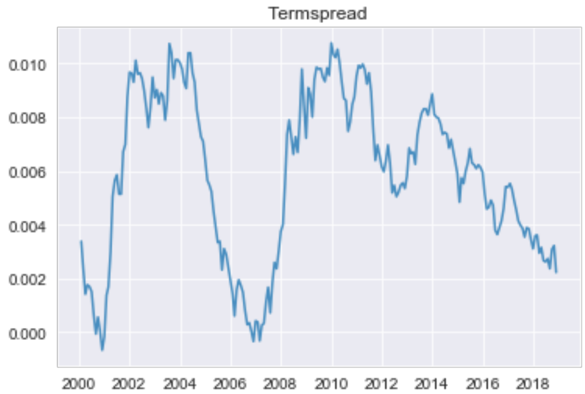

Stock Variance
0.00
−0.01
−0.02
−0.03
−0.04
2000 2002 2004 2006 2008 2010 2012 2014 2016 2018

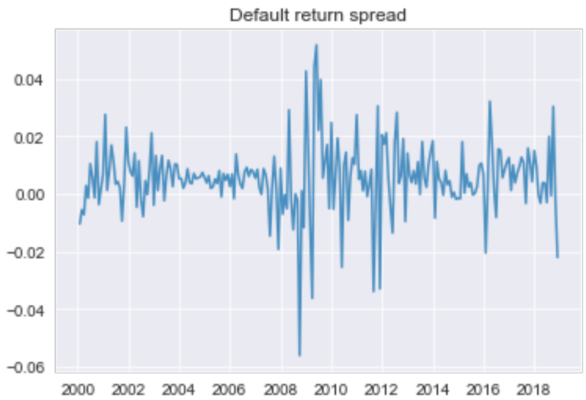

T-bill
0.020
0.018
0.016
0.014
0.012
0.010
0.008
0.006
2000 2002 2004 2006 2008 2010 2012 2014 2016 2018

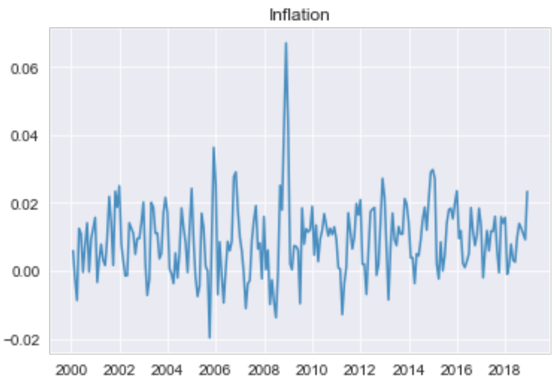

Net equity expansion
0.0175
0.0150
0.0125
0.0100
0.0075
0.0050
0.0025
2000 2002 2004 2006 2008 2010 2012 2014 2016 2018

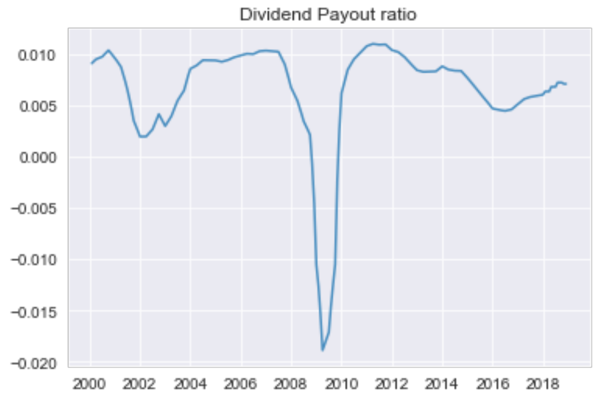

Long-term spread
0.022
0.020
0.018
0.016
0.014
0.012
2000 2002 2004 2006 2008 2010 2012 2014 2016 2018

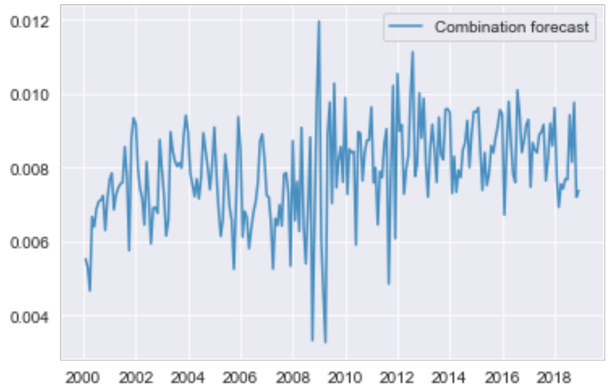

Long-term return
0.015
0.010
0.005
0.000
−0.005
2000 2002 2004 2006 2008 2010 2012 2014 2016 2018


Termspread
0.010
0.008
0.006
0.004
0.002
0.000
2000 2002 2004 2006 2008 2010 2012 2014 2016 2018

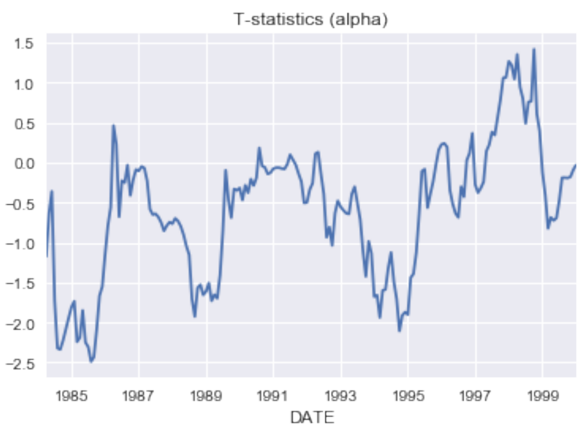

Default return spread
0.04
0.02
0.00
−0.02
−0.04
−0.06
2000 2002 2004 2006 2008 2010 2012 2014 2016 2018

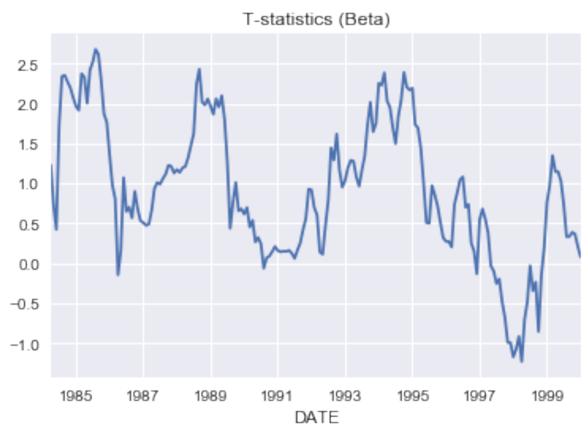

Inflation
0.06
0.04
0.02
0.00
−0.02
2000 2002 2004 2006 2008 2010 2012 2014 2016 2018


# Appendix 7 – Regression statistics on bond premium



# Appendix 8 – Black-Litterman weights before scaling