Copenhagen Business School
Spring 2020

Portfolio Optimization in the Chilean Pension System

Alternatives and Challenges

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MSc in Economics and Business Administration - Applied Economics and Finance

No. of pages: 80
No. of characters: 24.454

Preface

The submission of this master's thesis concludes my master's degree in Applied Economics and Finance at Copenhagen Business School. I want to express my sincere gratitude to Dr Marcel Fischer, for supervising my work during this period. Without his contribution and ideas, this thesis could not be possible. I also want to utter my endless gratitude to my closest family, especially to my mother, to teach me that critical thinking is the first step to find real knowledge.

Acronyms

AFP Asociacion de Fondos de Pensiones (Association of Pension Funds)

APV Ahorro Previsional Voluntario (Voluntary Retirement Savings)

CMF The Chilean Financial Market Commission

DC Defined Contribution

EW Equally Weighted Portfolio

GMVP Global Minimum Variance Portfolio.

GMVP60 Global Minimum Variance Portfolio optimized under 60% of the sample size.

GMVP70 Global Minimum Variance Portfolio optimized under 70% of the sample size.

GMVP80 Global Minimum Variance Portfolio optimized under 80% of the sample size.

HRP Hierarchical Risk Parity Portfolio

HRP60 Hierarchical Risk Parity Portfolio optimized under 60% of the sample size.

HRP70 Hierarchical Risk Parity Portfolio optimized under 70% of the sample size.

HRP80 Hierarchical Risk Parity Portfolio optimized under 80% of the sample size.

IPSA Chilean stock market index

MPT Modern Portfolio Theory

OCDE The Organisation for Economic Co-operation and Development

PAYG PAY-AS-YOU-GO

TAN Tangency Portfolio

TAN60 Tangency Portfolio optimized under 60% of the sample size.

TAN70 Tangency Portfolio optimized under 70% of the sample size.

TAN80 Tangency Portfolio optimized under 80% of the sample size.

SP Chilean Superintendency of Pension ("Superintendencia de Pensiones" in Spanish)

UF Inflation Index (Unidad de Fomento in Spanish)

VaR VALUE-AT-RISK

Abstract

This thesis aims to describe the problem that a saver belonging to the Chilean pension system faces when allocating asset during the period before to retiring. Through this paper it is illustrated the performance and risk metrics of different portfolio allocations, such as: the naive portfolio allocation (1/n), tangency portfolio, global minimum variance and portfolio choice under quadratic preferences. For this case different degrees of risk aversion has been used . The analysis was extended by allowing short selling, using different sample size to estimate portfolio weights, and comparing different optimization frequencies (static versus monthly re-balancing). The results suggests that portfolio re-balancing generates improvements in terms of performance compared to static optimization. However, some of the main drawbacks of Markowitz portfolio type optimization were detected, these are related to: 1) dramatic portfolio changes when inputs change lightly. 2) Highly concentrated portfolios, in most cases, less than five assets concentrate more than 90% of the portfolio—3) low out-of-sample performance. As a way to mitigate the negative consequences of Markowitz optimization, a novel algorithm was implemented. The Hierarchical Risk Parity method which generate portfolio allocations that are relatively stable through time, high out-of-sample performance (compared to GMVP), and a high level of diversification. These characteristics are desirable in a system as the Chilean one, where portfolio re-balancing is costly, and with low levels of financial literacy, this factor has been link with savers little involvement in investment choices.

Keywords – Portfolio Optimization, Chilean Pension System, Pension Funds, Risk Aversion

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1 Introduction

The Chilean Pensions System has been recognized as one of the best pension systems in the world (Mercer, 2018). Nevertheless, users have expressed their criticism with the pension amounts, and how the system has been set up. The contributors' perception has been confirmed by the data, as the replacement rate reaches a 44% on average (Macías, 2018). This performance is result of the interaction of several variables, some of them exogenous to the system. That is the case of the density of savings. Which roughly reach 53% for current savers. Additionally, during the period between 1985 to 2017, life expectancy in the country has changed from 81 to 85 years for males and from 85 to 90 years for females (Macías, 2018). Furthermore, real interest rates, used to compute the monthly payment have been showing a decreasing trend since 2000, which also have a negative effect in the final pension payout (Macías, 2018).

Nevertheless, the fingers have pointed to the pension fund administrators (AFP for their acronym in Spanish), as the main responsible of the low pensions (Vásquez, 2016). However, the AFP's are responsible for one variable that affects the final pension payout, which is the savings rate of return. When looking at the AFP's track record, this has shown on average a real rate of return of 6.32% for the period between September of 2003 to September of 2017 (Lopez and Otero., 2017), this is comparable with the savings rate of returns, obtained by pension funds in developed countries (Mercer, 2018).

The saver's rising need for achieving a higher rate of return, together with the public discussion about the negative attributes of the Chilean Pension System, have been the triggers for the rise of an industry linked to the private pension system. Financial advisors compose this industry [1] who promote market timing of pension funds with the idea of beating the system. These companies base their asset allocation strategies, arguing the use of market parameters, technical indicators, news, etc. (Cristi Capstick, 2017).

The effect of such recommendations has generated detrimental damages to savers rate of return, the Chilean capital market structure and the AFPs portfolio composition. Firstly, it has been documented that regardless of savers initial fund choices, this combination (no matter how it is designed), perform better than following advice of investment counsellors and follow the "market timing" strategy (Cuevas et al., 2016). Secondly, based on the number of followers that financial advisors have, the effect funds changes have generated pressure in assets price and has increased

[1] The leader investment advisory company called "Felices y Forrados" (Happy and Loaded), provide information to more than 64.000 users (Cristi Capstick, 2017).

the volatility in the Chilean stock market. Additionally, in responding to the large re-balancing flows, fund managers have changed their asset allocations prioritizing more liquid assets (Da et al., 2018).

The problem of assessing the system's information, and understanding the saving options have been intensified, after the reform implemented in 2002. This regulatory change transferred the market risk from the pension funds managers to savers. Before the year 2002, the only option for saving with retirement porpuses was a balanced fund (similar proportion of fixed income and variable income). Currently, there are five different funds, with varying profiles of risk, which gives the possibility for workers to choose based on their risk aversion profile. This set of funds are available under a mandatory pension scheme, where workers get a monthly discount of 10% on their payroll to be transferred into their private accounts. Additionally, a volunteer pension pillar was designed providing tax incentives for saving to retirement. This part of the pension system includes the possibility of investing in more than 268 different funds.

Thus, the increase in the funds' supply adds an additional layer of complexity for the portfolio allocation problem. Based on this scenario, this thesis aims to describe the problem that faces a representative agent that wants to retire in the Chilean Pension System. To do so, a sample of 128 funds have been selected, and the Markowitz portfolio optimization (Markowitz (1952),Markowitz (1959)) has been applied considering several assumptions, such as no transaction costs, assets divisibility, quadratic agent's utility function, no tax benefits or penalties, between others. As a way to deal with the problems that arise when using the mean-variance optimization method, a complementary portfolio algorithm was introduced. This is the Hierarchical Risk Parity (HRP) Method developed by Lopez de Prado (2016). The results suggest that the portfolios optimized under the Markowitz framework, exhibit: low out-of-sample performance, high sensitivity to the window of estimates and low level of diversification. The technique of HRP improves these results by delivering portfolios that compared to the Markowitz optimization, are more stable through time, with a higher level of diversification and exhibit a relatively high out-of-sample performance. Additionally, in all the portfolios under analysis, the mandatory pension funds were included just in the portfolios, optimized under quadratic preferences and assuming high levels of risk aversion. Additionally, out of 128 funds used in the different optimization procedures, less than 5 funds accumulate the largest weights in most of the cases. Last the funds were repeated through time, which provides an indication of a sample of funds that dominate the risk-return relation systematically through time. The contributions of this paper to the existing literature are basically twofold: 1) Provides evidence about the use of optimality criteria to define portfolio allocation in the Chilean Pension System. Most of the literature in the topic of pensions funds selections in Chile has been focused on understanding the determinant of choices and switches between pension funds, and the use of volunteer savings vehicles and tax benefits. However, there is no empirical evidence that integrates the use of both types of funds (mandatory and volunteer saving schemes) under a unique portfolio allocation approach. 2) The use of Markowitz optimization algorithm applied to the Chilean pension funds and the descriptions of their drawbacks when is empirically implemented. Furthermore, the utilization of the Hierarchical Risk Parity (HRP) algorithm to improves the Markowitz optimization output, provides new evidence, of the HRP method performance when it is applied to the portfolio optimization problem using pension funds.

This thesis is structured as follow: in section 2 a description of the Chilean Pension System is presented together with academic literature that links the topic of financial literacy with the use of pension schemes, and the effect of the pension investment limits on the funds' performance. Section 3 provides a description of the portfolio optimization methods use to generate different funds allocation strategies, and the performance measure used to assess the results. In Section 4 the funds' database is listed with the main descriptive statistics of each one of them. In section 5 are shown the results of the empirical implementation of the portfolio strategies is analyzed. In section 6, the results are discussed examining the causes of them, and their applicability in the Chilean context. Finally, a conclusion of this empirical study and its findings will be given in section 7.

1.1 Problem Statement

The main objective of this thesis project is to exhibit the complexity of the problem that savers face when selecting investment vehicles during the accumulating phase under the Chilean pension system scheme. In this regard, the problem of portfolio allocation will be solved, using optimality criteria. The Markowitz portfolio theory has been selected, to solve the optimal portfolio weights. The problem will be solved firstly by employing the mean-variance static optimization problem. Secondly, this case will be extended by using the rolling window sampling method to estimate the portfolios' weights. Afterwards, the savers' preferences will be described as if they follow a quadratic utility function. This allows to solve the problem and review the portfolios' performance for agents with different risk appetite. Finally, the main problems exhibit the empirical implementation of the Markowitz optimization will be partially addressed, introducing the Hierarchical Risk Parity algorithm.

1.2 Motivation

The world's pension systems are facing significant challenges, as a consequence of an increase in life expectancy, decrease in birth rates, low market interest rates, among other factors. In this context, the so-called defined contribution schemes where the savers are responsible for their own pension have been prioritized over the pay-as-you-go schemes, where taxpayers are responsible for the retirement payout of the current pensioners. In a defined contribution plan, one of the most relevant variables that affect the amount saved during the accumulation phase is the rate of return over the saving period. This variable is influenced by the funds selected by savers, which in most cases are assumed to be well informed. In the Chilean case, savers are responsible for choosing the funds to fit them the best, to maximize their savings amount before the retirement date. Nevertheless, this choice is not trivial. Firstly, the pension saving scheme in Chile is based on a mandatory monthly saving. These savings must be allocated in at least one of five funds with differences risk-return profile, and limited to the age of the savers (Closest to the retirement date, only low-risk funds can be chosen). Secondly, an additional component based on voluntary saving was created as a complement to the mandatory savings. This mechanism allows choosing between more than 268 funds. Thirdly, Chile is one of the country members of the OCDE group, with the lowest level of financial literacy (Landerretche and Martinez, 2013). All the factors mentioned before, make very likely that portfolio combination selected by Chilean savers is not efficient (Parraguez, 2017). So far, the academic literature, in this field applied to the Chilean case, has been the variables that determine changes between mandatory pension funds, or the use of volunteer saving instruments. However, the empirical exercise of solving this problem under optimality criteria has not been addressed. Thus, the results of this thesis provide insights at least in the areas of 1) performance measure for portfolio allocation strategies. 2) The applicability of the theoretical tools of the portfolio theory in this context. 3) Analysis of feasible solutions to the Markowitz optimization drawbacks and the portfolio allocation that could be selected by individuals with different risk preferences.

2 Background

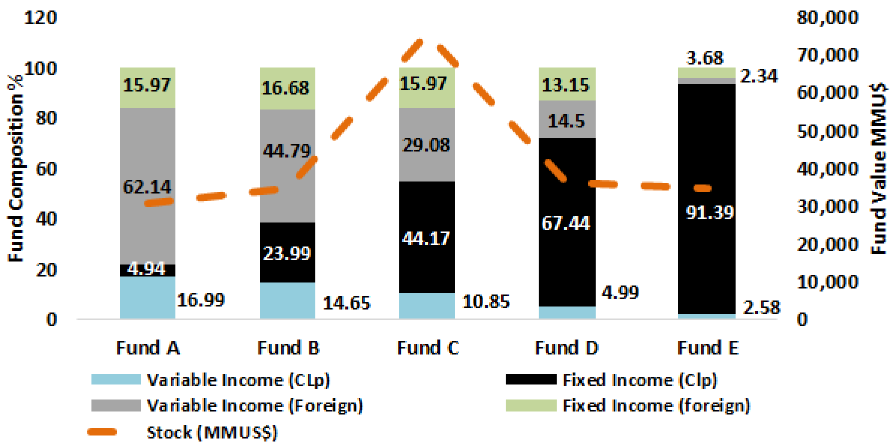
2.1 The Chilean Pension System

During November 1980, the Chilean pension system suffered a dramatic change. It became the first experience in the world to replace a pay-as-you-go system with an individual capitalization scheme (fully funded contribution). The replacement of the pension system has been documented as a driver of macroeconomics improvements in the Chilean economy. The mechanisms that link the pension reform with the macroeconomic development of the country are higher national savings, higher investments rates, improvements in the labour market by increasing contracts formality and labour productivity, and the expansion of the capital market by adding local institutional investors (Corbo and Schmidt-Hebbel, 2003). However, there are several issues that have raised criticism about the pensions system. In this regard, the characteristic of the informal labour market in Chile allows for volunteer participation on the pensions system; thus individuals without formal contracts do not contribute actively in the system. Currently, the unemployment insurance does not cover pension contribution; therefore, savers with high volatile employments have a high probability of not presenting continued saving flows. The life expectancy in the country has increased substantially since the implementation of the system but the retirement age has remained unchanged Lopez and Otero. (2017). Individuals do not participate in the labour market during their young adulthood, which does not allow them to obtain the benefits of compounding saving rates (Ibid). Finally, the real salaries in the formal sector have experienced a yearly growth of 2% during the last decade. All these factors have contributed to generate low replacement rates, which in 2016 reached an average of 40% of the retirees last ten years salaries (OCDE, 2019).

In 2015, a presidential commission called "Bravo" was created –named after the main economist on a charge of the commission, David Bravo. This group was composed by 16 Chilean and 8 international pension experts and its objective was to diagnostic the main drawbacks of the systems and propose improvements. As a result, several measures were implemented, with the objective of poverty relief for elderly people, generating old-age income insurance, and improving consumption smoothing through time (Barr and Diamond, 2016). The commission assessed that the Chilean pension system exhibits the following characteristics 1) low level of coverage. 2) Fund costs were high as a result of the lack of competition between them. 3) High level of pension payout gender inequality (mirror of the labour market). 4) Public opinion hostility towards the pension funds administrators (make political changes impossible to do) and 5) low financial literacy between the users of the system (Ibid). In this context, any improvement of the risk-return relation of the funds' allocation that savers take can potentially raise the replacement rate and with this improve future pensions. Thus, portfolio allocations under optimality criteria, represent a useful tool for improving characteristics of the system.

From the perspective of the fund composition, it replicates the well-known life-cycle investment strategy. In this regard, the Chilean regulator, "Superintendecia de Pensiones" (SP) define investment limits for the kind of fund in which savers can invest, based on their age. Thus, younger savers, are allowed to invest in all the alternatives including the riskier funds, whereas, persons closer to the retirement age, can only consider safer funds. The risk control imposed by the regulators is based on investment limits. Hence, the fund A is allowed to invest in a range of 40% to 80% in variable income whereas the other funds have the following limits: Fund B 25% to 60%, fund C 15% to 40%, fund D 5% to 20% and Fund E 0% to 5% (Pagnoncelli et al., 2017).

Figure 2.1: Pension Funds Composition (June 2019)



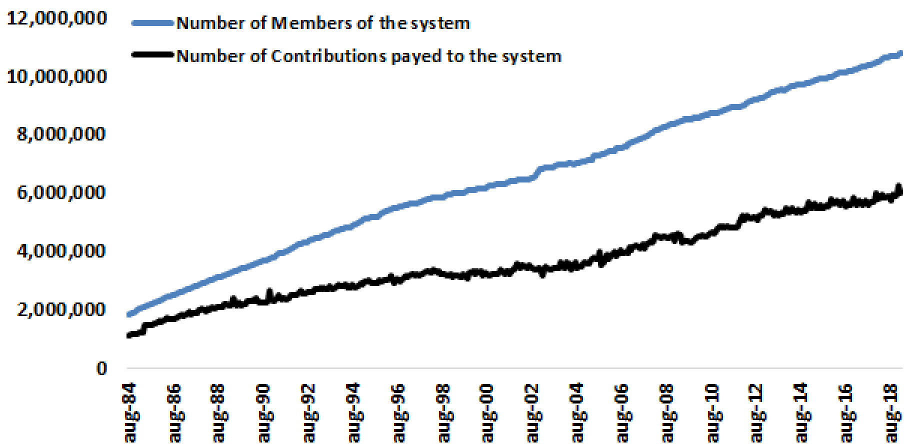
Own elaboration based on data published by Chilean Superintendency of Pensions Fund Administrators

The total stock handle by the pension funds is around US$210 billion, which represent 72% of the GDP (OCDE, 2018) and it is mostly allocated in the fund C. Additionally to the constrain for investments, fund managers must complain with a "minimum yield test". This annual review, compare fund returns for different companies and identify the worst performance for each one of the five options. If the worst fund performance is below the industry average, affiliates must be compensated (Schlechter et al., 2019). Considering this scenario, the fund managers must keep 1% of the fund value, as cash reserve. As a consequence of this kind of regulation, the historical performance of funds has been characterized by herd behaviour, for similar risk-based portfolios (Chant, 2014).

From the perspective of the accumulation phase, the system is organized into a scheme with three pillars:

1) Mandatory payments (Chant (2014)): it refers to the private payments that all employers must deduct (10%) from workers salary plus administration costs and disability insurance. This contribution is subject to a ceiling on monthly earnings of about US$3.000 (indexed inflation). The funds are heritable, non-withdrawable –just at the retirement age and as a pension payment– and managed by a Pension Fund Administrator (AFP). Thus, fund managers are allowed to charge fees without constrains, but this is just collected from active contributors to the funds (half of the members of the system). Savers face the task of choosing between fund managers (new contributors are assign by default to the cheapest fund provider, and are not allowed to change during the first 12 months) and five different portfolios, with different risks profiles (A,B,C,D and E). Additionally, depending on their age, the selection of the riskiest funds is not allowed. The logic of this forced saving is to avoid the opportunistic behaviour of those who prioritize present consumption, assuming future payments from taxpayers.

Figure 2.2: Number of Members in AFP system



Own elaboration based on data published by Chilean Superintendency of Pensions Fund Administrators

2) Solidary component (Chant (2014)): It is based on a contribution from taxes. This public fund is used to cover the pension of those citizens that do not save enough funds so that they can obtain a minimal pension (basic solidary pension). It can be provided as a complement of savings, that allows obtention of this minimal payment or a full pension in cases of persons with no savings. This mechanism targets the 60% with the lowest income, and it also covers the cases of retirement under disability.

3) Voluntary contribution (Chant (2014)): This mechanism includes tax incentives for those

savers that want to extend the mandatory contribution of 10%. The fund allocation process for this additional amount, work slightly different compared to the case of the mandatory 10%. Under this option, savers can choose a more diverse pool of funds, which are offered by banks, AFP, stockbroker houses, among others. The most remarkable characteristic of this scheme is the tax benefits.

2.2 APV funds and state benefits

Voluntary pension savings (APV), is a private and non-mandatory contribution to the individual capitalization accounts. This kind of contribution is based on the idea that when savers have income that exceeds the ceiling mandatory contribution, there is an incentive to increases the monthly contribution to get a higher replacement rate. From the pillars system perspective, this mechanism is under pillar three. Currently, there are three categories of APVs: agreed-upon deposits, voluntary contributions, and collective voluntary pension savings (APVC). The first case (agreed-upon deposits), represents a way of saving that has no limit in the amount that can be saved, but it can be only used at retirement period of time. Companies entirely pay this amount of money and it is agreed between workers and employers. It also represents a tax-deductible expenditure.

Voluntary contributions are periodical money transfers that workers save in private funds, under the only requirement of being an active or passive member of an AFP. The pool of funds that is available for this kind of investments is quite complex. The Chilean Financial Market Commission -(CMF in Spanish) classify the mutual funds based on the portfolio composition (see appendix A, table A2.1) The APV funds replicate the industry of mutual funds with additional fund series. For instance, if there is a mutual fund A, the APV version will be the same fund composition but under other series. It is also possible to save under the APV modality by choosing AFP funds. Furthermore, savers are allowed to make a withdrawal at any period, but they also face penalties for early withdrawals of the funds and taxes over capital gains. Currently, there are two categories of funds under the voluntary contribution scheme:

- APV A: The creation of this modality has its origin in the reform of 2008, under the idea of making pillar three popular in the middle-income sector. It represents a way of support from the state of 15% of the savings up to 270.000 CLP per year. When withdrawal is done, the state subsidy is lost. Thus, as the state benefit does not depend on the tax rate, the subsidy is oriented to those population groups that are exempt from income tax.

- APV B: Represent a tax reduction of up to 15.000.000 CLP per year. The amount of savings are not considered as income tax, but when savers are retired there is a unique tax rate of 15%. There is also a penalty for early retirement that fluctuates between 3% to 5%. The idea of APV B was to complement the mandatory savings for those who have salaries above the top tax rate.

In practical terms, the use of both mechanisms depends not only in the amount intended to save but also in the income tax rate. Thus, the optimal combinations of both mechanisms can be summarized in the following table:

Table 2.1: APVs optimal combination based on Income and Savings per month

Income (Clp thousands)	Savings per month (Clp thousands)						
	100	200	300	400	500	600	700
300	APV A	APV A	APV A	APV A	APV A	APV A	APV A
500	APV A	APV A	APV A	APV A	APV A	APV A	APV A
700	APV A	APV (A+B)	APV (A+B)	APV (A+B)	APV (A+B)	APV (A+B)	APV (A+B)
900	APV A	APV (A+B)	APV (A+B)	APV (A+B)	APV (A+B)	APV (A+B)	APV (A+B)
1,100	APV A	APV (A+B)	APV (A+B)	APV (A+B)	APV (A+B)	APV (A+B)	APV (A+B)
1,300	APV A	APV (A+B)	APV (A+B)	APV (A+B)	APV (A+B)	APV (A+B)	APV (A+B)
1,500	APV A	APV (A+B)	APV (A+B)	APV (A+B)	APV (A+B)	APV B	APV B
1,700	APV A	APV (A+B)	APV (A+B)	APV B	APV B	APV B	APV B
1,900	APV A	APV (A+B)	APV (A+B)	APV B	APV B	APV B	APV B
2,100	APV A	APV (A+B)	APV (A+B)	APV B	APV B	APV B	APV B
2,300	APV A	APV (A+B)	APV (A+B)	APV B	APV B	APV B	APV B
2,500	APV A	APV (A+B)	APV B	APV B	APV B	APV B	APV B
2,700	APV A	APV B	APV B	APV B	APV B	APV B	APV B
2,900	APV A	APV B	APV B	APV B	APV B	APV B	APV B
3,100	APV A	APV B	APV B	APV B	APV B	APV B	APV B

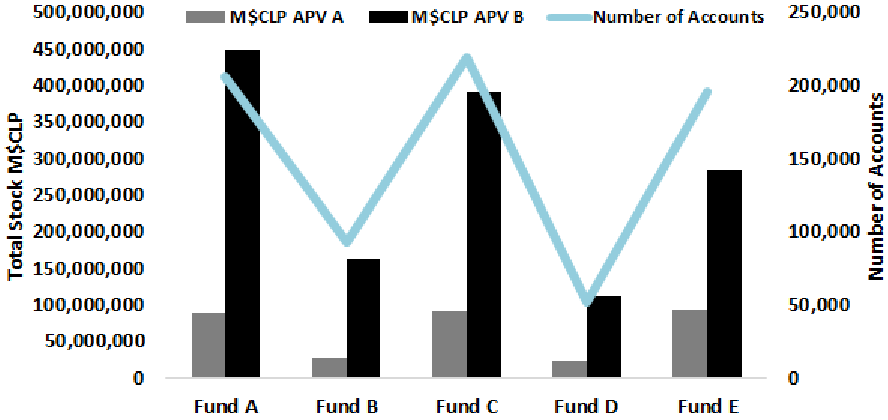
Own elaboration based on data published by Chilean Superintendency of Pensions Fund Administrators

Moreover, for both categories, it is possible to invest in AFP funds. In fact, there is not aggregate information about the endowment invested in the APV funds that replicate mutual funds. However, the information is available for the APV that are invested in the AFP options:

Hence, we can observe that the main vehicle used by APV savers is the Fund A, followed by the fund C. The trend is the same for the number of accounts.

Finally, the collective voluntary pension saving, represents an agreement between employers and workers, who decide an annual or monthly contribution. In this case, both participants are contributors to the worker's savings. The options available are: APV A or APV B.

Figure 2.3: Number of Members in AFP system



Own elaboration based on data published by Chilean Superintendency of Pensions Fund Administrators

2.3 Financial literacy

One of the saver's attributes that academics and policymakers have defined as a variable that explains why households and individuals do not accumulate enough savings to retirements (even-though when they can do it) is the degree of understanding about concepts that surrounds the topics of saving and investments. This pitfall seems to be an inherent component of pensions systems all over the world. Nevertheless, most of the pension systems assume that users are fully informed or that they can at least choose funds based on their assessment of the risk-return relations, compare fees between fund managers and identify optimal pay-out options at the retirement age. The level of individuals financial literacy can have deep consequences in the choice of investment alternatives.Van Rooij et al. (2011). Analyzing the impact of financial education on stock market participation, authors report that those who have lower level of financial literacy are considerably less likely to invest in stocks. Related to the link between financial literacy and portfolio diversification, Abreu and Mendes (2010) reported that for Portuguese retail investors, their education level and financial knowledge have a positive impact in the number of assets included in their portfolio. From a social perspective, several studies have linked belonging to certain population groups with the degree of financial knowledge. Thus, these groups can be defined by variables such as years of scholarship, socioeconomic segment, etc. In this regard, by using the Washington financial literacy survey (EEUU), Lusardi and Mitchell (2007) found that members of groups of low income, low educated, minorities and women were those with worst results in the survey. Those groups had also the lowest expected pension payout. Another research that confirms the high variance between population groups, was conducted by Kalmi and Ruuskanen (2018) in Finland, a country with high levels of education (based on PISA scores) and a developed system of social security. Authors reported that the level of financial literacy was high compared to other countries. Moreover, high levels of financial literacy and retirement planning were most frequent between women, compared to men, which can be attributed to the lower labour market attachment for this group, and their higher disposition to face poverty as retirees. Similar results have been found in research from other developed countries. Bousclair et al. (2017) showed that, in Canada, individuals who understood correctly the concepts of interest compounding, inflation and risk diversification, had a probability of having retirement savings 10% higher to those who did not grasp such concepts. In this research, it was also documented that women, minorities, and those with low years of education answered worse.

Regarding research on this topic in Chile, several studies have analyzed the results of the Social Protection Survey (Encuesta de Prevision Social or EPS), which is a longitudinal survey conducted every two years that includes about 20.000 individuals. One of the early research that connected the topic of financial literacy with the comprehension of savers about the Chilean pensions system was conducted by Arenas et al. (2006). By analyzing the EPS, authors found that 60% of the workers declared to receive quarterly information from the AFP. The 28% of respondents were able to calculate a payroll tax rate, less than 2% of them knew either the variable or the fix AFP fee charge, 20% knew how many funds were part of the system, 38% assessed correctly which fund was the riskiest, and the minority of the AFP users knew their AFP balances (private savings). Related to pension money manager preference (AFP), Mitchell et al. (2007) documented that fund changes are more frequent in individuals who are highly educated, relatively highly paid and posses a higher level of financial literacy. Landerretche and Martinez (2013) investigated the effect of pension system knowledge over "additional pension savings". These authors found that, for every additional right answer in a standardized test of the pension literacy, the probability of pension fund switching increased by 20%, the probability of voluntary affiliation to the pension system increased 30% for independent workers and the chances that individuals will save in at least one of the surveyed periods increases by 50%.

Berstein and Ruiz (2005) analyzed the effect of misinformed consumers in pension fund demand sensitivity. Authors consider variables such as AFPs market share, fees, profitability, savers information level, commissions, and the number of sales agents, as determinants of the number of changes between fund managers, selected by savers over a given period. The results suggest that this market exhibits a low demand sensitivity, which are mostly affected by the number of sales agents that they AFPs use to attract new savers. This effect is intensified after regulatory changes were implemented in 1997. Finally, the authors conclude, that the level of competence in the system can be improved by increasing the level of knowledge that users posses about profitability, fees, etc; and the regulation should also address this issue when designing pension funds legal requirements.

2.4 Chilean Pension Funds and Investment Regulation

From a regulatory perspective, the Chilean pension fund managers face two constraints. One is related to the investment possibilities, for each one of the five type of funds and the other is related to the return of the investments compared to the average of the industry.

In relation to the first case, the investment limit framework was defined for at least, three reasons: 1) To provide protection to uninformed savers and avoid the exposition of them to risky assets when these are not needed. 2) To encourage the expansion of the Chilean productive sector by setting investment limits to choose foreign assets. 3) To avoid principal-agent problem that rises when being the one of the larger shareholder in specific companies (Berstein and Chumacero, 2006). The current investment limits structure is described in appendix ?, table A1.1. As the system itself, the limits set up have been evolved through the time. At the beginning of the system implementation, it was only allowed to invest in fixed income instruments issued in Chile. In 1989, the investment possibilities were extended by including stocks of open Chilean corporations and real state companies. In 1990, the possibility of investment in shares of investment funds, including foreign assets, was introduced. But it was just until 1994 that the limits framework included the possibility of investing directly in foreign fixed and variable income (Schlechter et al., 2019). From the perspective of issuers, there are also limits defined in the participation that AFPs can have in the ownership of individual issuers. For instance, stocks AFPs funds are not allowed to invest more than 7%. However, these limits are lower for interlocked ownership between AFP and controlling shareholders. If that is the case, the fund cannot hold more than 2% of ownrship (Ibid).

The second regulatory constrain is related to the minimum return that is expected for each fund. It considers that the average return over a window of 36 months, must be higher that average return of all funds minus 2% for funds C,D and E; and minus 4% for funds A and B; or fifty percent of the average return of all funds, whichever is lower (Schlechter et al., 2019). This minimum return requirement, the funds administrators must hold 1% of each fund market value as cash reserve that must be invested in the same pension funds. In the scenario that an AFP breached one of the limits above and the cash reserve is insufficient to cover the losses, the authorities will cover it. Furthermore, if this cash reserve cannot be restored, the AFP must be liquidated (Schlechter et al., 2019).

The consequences of this limits structure, have generated an impact on the risk return relation of each fund, and have raised stylized facts that have been addressed by practitioners and academics. These findings can be summarized in the following groups:

1) Herd effect: refers to the fact that fund manager could consider the way of how other portfolio managers invest instead of use their own set of information, even though this behaviour may lead to inefficient asset allocation. This could cause that the same kind of assets composes all funds with a particular risk profile. Vásquez (2004) shows that the minimum return requirement generates distortions in the portfolio composition. Through a static game of incomplete information, the author detects that, when changes to the limits are implemented (those are wider), the herd effect is reduced. In addition, "in order not to deviate from the average profitability, each AFP omits its own information and takes into account for its decision what the other AFP does". Olivares and Sepúlveda (2004), analyzed the link between investment strategies followed by AFPs, authors decomposed the correlation between pension funds and ,as a result, they concluded that most of the correlation is explained by herding behaviour.

Despite the fact that after the reform of 2002 when new funds were introduced, this behaviour remains unchanged. Moreover, the correlation between funds attributed to herding was around 80 per cent. And after the reform of 2002, the correlation increases to 85 per cent. Ruiz and Bravo (2015) quantified stressed scenarios for pension funds, and default probability in the context of herding behaviour. Authors found that the reservation requirement of the 1 percent of the funds value is higher to what can be used under market adverse conditions. In other words, even in the case when pension fund managers deviate from the normal investment strategies, the probability of using the cash reserve is low. By decreasing the requirement of 1 per cent of the fund minimum reserve to 0.5 per cent, the probability of using the reserve increases by 17 percent for the smallest AFP, whereas for the largest continues being zero. This result suggests that the fund manager size should adjust the regulation regarding capital reserve. Stein B. et al. (2011) compares the herd effect observe in the AFPs to similar effects observed in developed economies. The results suggest that the case of Chile the herd effect is more accented that in developed countries. Additionally, they detect asymmetry en the effect, meaning that, in periods of economic contraction, the effect seems to be stronger whereas, for economic expansion, it seem to disappear.

2) Effect of limits in the risk-return relation: The division of funds based on the risk profile of each one of them, which is meant to reduce the exposure of savers to different levels of risks. For instance, savers close to the retirement age are forced to take the less risky fund. In this regard,

one can expect that the risk-return relation for which the five different funds was created should be respected for different period and market conditions. Schlechter et al. (2019) compared the effect of regulatory regimes of Mexico and Chile in pension funds performance.

The case of Mexico was selected based on the fact that the regulation includes quantitative risk-based metrics (such as VaR) to control the portfolios risk profile. On the opposite side, in Chile, the portfolios risk control are asset-class limits based. The researchers concluded that, in the case of Mexico, the funds "delivered returns according to their intended risk profile, and they are consistently ranked correctly in terms of absolute risk, risk-adjusted returns, and cumulative returns". On the contrary, in the case of Chilean funds, in some periods, the most conservative funds outperformed the riskiest funds in term of cumulative returns. This evidence, strongly supports the idea that asset-class limits represent a limited tool from the risk management perspective. Berstein and Chumacero (2006) quantified the cost of the current investment limits by defining the potential portfolio allocation that fund managers could choose in the case of not having asset-class limits, but instead optimizing the portfolios under Value-at-Risk specific targets (i.e. higher VaR for riskier portfolios). Authors found that, the costs of having limits are relevant. In the absence of the asset-class limit constrains, the total assets under administration of the AFP could be at least 10% larger, the affiliates might have faced higher volatility, and the investment asset-class limits could have been breached in 90% of the time. However, they do not consider the potential endogeneity of AFP on the asset prices in the Chilean capital market. Opazo et al. (2009) compared US mutual fund and Chilean institutional investors (pension funds and mutual funds) and their willingness to include in their portfolio allocation, maturity-specific assets. The results showed that Chilean fund managers are more tilted toward the short term than US funds, even after adjusting for assets availability. The explanation that authors attribute to this result is the constrain relate to minimum return requirement, they considered it as an incentive to undertake investment based on short term. Short-term assets are less risky and, as a consequence, fund managers reduce the probability of deviating from their peers. Castañeda and Heinz compared pension funds with a group of index-based funds, with comparable investment profile. Authors, showed that in the case of pension funds constrained to the minimum return requirement, the most optimal way to limit financial deterioration could be using the index-based system. By doing so, manager asset choices could be driven by optimality criteria instead of based on relative performance assessments.

2.5 Risk Aversion and Pension Investments

Investing in financial assets, including retirement funds involve a risk-taking process, the economic concept link to the agents' tolerance to risk has been described as "risk aversion". A high level of risk aversion implies that agents will prefer safety investment alternatives. The opposite can be said for low risk-averse individuals, thus individuals risk aversion could determine investment performance. One research stream in the topic of risk aversion has focused on how sociodemographic characteristics are linked with the willingness to undertake risks. In this regard, there is a broad consensus that women are more risk-averse than men (Borghans et al., 2009).

The explanations for this result has been linked to biological characteristics such as testosterone level and the effect of this hormone in the decision making process (Sapienza et al., 2009), and with cultural roles (Booth et al., 2014). Related to the measure of risk aversion, gender and pension investments, (Bajtelsmit et al., 1999), using the 1989 Survey of Consumer Finances (US), reported evidence that women showed higher relative risk aversion in their allocation of wealth into defined contribution pension assets. Related to other sociodemographic attributes, the evidence suggests that risk aversion decreases with education (Outreville (2015) and (Jung, 2015)) and income or wealth level (Hartog et al., 2002). On the opposite side, risk aversion has been positively related with age (Wang and Hanna, 1997), which implies that elderly people are less willing to take risks than younger people. The empirical evidence has also documented differences between groups with social differences. In this sense, there is evidence that entrepreneurs are less risk-averse than employees and civil servants are more risk-averse than private-sector employees (Hartog et al., 2002).

(Yao and Hanna, 2005) reported differences in risk tolerance for individuals with different marital status. Furthermore, authors describe a hierarchy for risk tolerance; thus, risk appetite is highest for single males, followed by married males, unmarried females and married females, respectively. The literature that analyzes the link risk aversion and pensions plans has also covered the gender differences. Watson and McNaughton (2007) analyzed gender risk aversion differences and expected retirement benefits in Australia, authors found that woman exhibit a higher level of risk aversion when choosing investment strategies, which can be partly attributed to the lower-income received during their working life.

For the case of Chile, using 2009 survey of Chilean pension participants, Kristjanpoller and Olson (2015) detected that ,for the year of analysis, there were no significant differences in the percentage of men and women choosing "default funds" (if users do not choose between 5 AFP funds, the pension managers allocate the savings in a default fund, base on savers age). However, when desegregating the results for demographic characteristics, younger people and men with less education and less income were more likely to choose the default funds and only the age factor was significant for women. Additionally, Ruiz-Tagle and Tapia (2011) reported a positive relationship between risk aversion and early retirement; the mechanism that interacts though these two variables attributed to the assessment of life expectancy. Therefore, individuals who are impatience to take early retirement, are those with higher risk aversion, which reflect uncertainty about the future quality of life.

3 Methodology

3.1 Modern Portfolio Theory

The pioneer research in the analysis of portfolio allocation was developed by Harry Markowitz (Markowitz, 1952). He states in a novel approach the portfolio selection process developed by investors who seek to obtain the highest expected return per unit of risk, but accounting for the movements of the simultaneous asset, it is said, by their correlation. Thus, investing in several uncorrelated assets, portfolio volatility can be reduced. In 1958, James Tobin expanded the idea of Markowitz, by including a risk-free asset to the analysis (Tobin, 1958), the effect of this change in the theory, allowed to generate leverage or deleverage portfolios. Tobin's approach state that market investors independent of their risk tolerance, will choose the same portfolios as long as their expectation about the future is the same. As a result, Tobin concludes that, the main difference between their choices will be based on the proportion of stocks and bonds that they select. In this context, Tobin derivated the "Efficient Frontier" and "Capital Market Line" ideas, based on the previous work of Markowitz (Mangram, 2013). Relate to the assumptions of the Modern Portfolio theory these are: 1) efficiency in the markets, 2) no transaction costs, 3) no taxes, 4) assets are perfectly divisible, 5) agents optimize a cuadratic utility function, 6) agents are rational and risk adverse, 7) agents preferences are a function of the assets returns and variance, 8) Agents prefer the highest portfolio return per unit of risk, 9) returns follow a normal distribution, between others (Markowitz, 1952). In 1959, Harry Markowitz continues their previous work and attached the concept of "Efficient Diversification" (Markowitz, 1959). This concept is link to the mathematical result of computing one of the two following specifications (Bailey and López, 2013):(i) Minimizing the portfolio's standard deviation (or variance) subject to a targeted excess return or (ii) Maximize the portfolio's excess return subject to a targeted standard deviation (or variance).

3.1.1 Mean Variance Analysis

In this section will be shown the main characteristics of the portfolio theory. Let's define the rate of return of an asset as (Neumann, 2015):

$$r = \frac{X(1) - X(0)}{X(0)} \tag{3.1}$$

Where $X(1)$ represent the amount obtain at the end of the investment period, and $X(0)$ represent the capital at the beginning of the investment. Normally, returns are assumed to follow a log-normal distribution, then are computed as follow:

$$log(R) = log(1 + r) = log(\frac{X(1)}{X(0)}) = log(X(1)) - log(X(0)) \tag{3.2}$$

The notation will be extended for i different assets in t periods of time (t):

$$log(R_i(t)) = log(1 + r) = log(X_i(t)) - log(X_i(t-1)), for\ i = 1, ...N\ and\ t = 2, ..., T \tag{3.3}$$

Note that for t=1, the result of the previous equation lead to zero. A common practice of practitioners and academics, is to assume that the return follow a normal distribution. However, there are several stylized facts, that have been documented (Thompson, 2013). Firstly, stock returns in practice follow a most heavy tailed distribution, compared with the normal distribution. The returns volatility mostly exhibit clustering effects, it is said, high volatility periods are follow by low volatility periods. The auto-correlation of returns seems to be dependent on the asset's liquidity. Thus, liquid stocks do not exhibit significant linear auto correlation, the opposite is found for returns categorized as liquid. Finally, the volatility does not follow the same behaviour for price increases than for price decreases. In the first case seem to be higher. This link between price changes and volatility has been described as "leverage effect" (Thompson, 2013).
For the asset allocation process, the proportion invested in each asset will be represented by the portfolio weights: $w = (w_1, ..., w_N)^T$, thus the expected return can be described as follow:

$$r_p(w) = w^T * r \tag{3.4}$$

Additionally, it is assumed the constrain that all the weights must add 1 (all the resources are invested):

$$1^T * w = 1 \tag{3.5}$$

Furthermore, the variance, $\sigma_p^2(w)$, of the return rate of a portfolio can be expressed as follow:

$$\begin{aligned} \sigma_p^2(w) &= E[(r_p(w) - u_p(w))^2] \\ &= E[(w^T(r-u)(r-u)^T w] \\ &= w^T \Sigma w \end{aligned} \tag{3.6}$$

Where, Σ is the matrix of variance and covariance. The standard deviation can be then expressed as:

$$\sigma_p(w) = \sqrt{w^T \Sigma w} \tag{3.7}$$

Finally, the traditional Markowitz optimization approach, seek to obtain the highest return by unit of risk (volatility), thus the problem can be state as follow (derivation based on, Lando and Poulsen (2001)):

$$min_w \ \frac{1}{2} w^T \Sigma W \tag{3.8}$$

$$subject\ to\ w^T u = u\ and\ 1^T w = 1 \tag{3.9}$$

The solution to the unconstrained problem (full investment and target portfolio mean), can be found by minimizing the Lagrange function with respect to the vector of weights (w), and the multipliers (λ_1 and λ_2):

$$L(w, \lambda_1, \lambda_2) = w'\Sigma w + \lambda_1(w'u - r_p) + \lambda_2(w'1 - 1). \tag{3.10}$$

The first order conditions for optimality are as follow:

$$\frac{\delta L}{\delta w} = \Sigma w - \lambda_1 u + -\lambda_2 u = 0. \tag{3.11}$$

$$\frac{\delta L}{\delta \lambda_1} = w^T u - r_p = 0. \tag{3.12}$$

$$\frac{\delta L}{\delta \lambda_2} = w^T 1 - 1 = 0. \tag{3.13}$$

By transforming the previous equations in algebraic elements, it is obtained (4.11) can be expressed as follow (assuming invertibility):

$$w = \Sigma^{-}1[u \quad 1] \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}. \tag{3.14}$$

The part of the system (4.12) and (4.13) gives:

$$[u \quad 1]^T w = \begin{bmatrix} r_p \\ 1 \end{bmatrix}. \tag{3.15}$$

Multyplying both sides of (4.14) by $[u \quad 1]$ and plugging (4.15), thus:

$$\begin{bmatrix} r_p \\ 1 \end{bmatrix} = [u \quad 1]^T \, \Sigma^{-1} 1 \, [u \quad 1] \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}. \tag{3.16}$$

It is defined the matrix A $=[u \quad 1]^T \, \Sigma^{-1} 1 \, [u \quad 1]$. Using matrices multiplication it is obtained:

$$A = \begin{bmatrix} u^T \Sigma^{-1} u & u^T \Sigma^{-1} 1 \\ u^T \Sigma^{-1} 1 & 1^T \Sigma^{-1} 1 \end{bmatrix} := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{3.17}$$

Now the challenge is to show the conditions which make A, positive definite, particularly invertible. If this conditions can be found, the solution to the weights of the optimal portfolio, can be found in a closed form. Let's consider $Z^T = (Z_1, Z_2) \neq 0$, be an arbitrary non-zero vector in $\mathbb{R}^2$. Thus,

$$y = [u \quad 1] \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = [Z_1 u \quad Z_2 1] \neq 0 \tag{3.18}$$

Because the elements on u are not all equal. Finally, from the definition of A it is obtained:

$$\forall Z \neq 0 : Z^T A Z = y^T \Sigma^{-1} y > 0, \tag{3.19}$$

Based on the fact that, Σ^{-1} is positive definite (Σ is). Thus, it has been shown that, A is positive definite and then is possible to solve (4.16) for λ:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = A^{-1} \begin{bmatrix} r_p \\ 1 \end{bmatrix} \tag{3.20}$$

Plugging this result in (4.14), it is obtained the following expression:

$$\hat{w} = \Sigma^{-1} \, [u \quad 1] \, A^{-1} \begin{bmatrix} r_p \\ 1 \end{bmatrix} \tag{3.21}$$

The vector $\hat{w}$, is the "minimum variance portfolio", then based on this result the variance of this optimal portfolio can be computed:

$$\hat{\sigma_p}^2 = \hat{w}^T \Sigma \hat{w} \tag{3.22}$$

$$= [r_p \quad 1] \, A^{-1} \, [u \quad 1]^T \, \Sigma^{-1} \Sigma \Sigma^{-1} \, [u \quad 1] \, A^{-1} \, [r_p \quad 1]^T \tag{3.23}$$

$$= [r_p \quad 1] \, A^{-1} ([u \quad 1]^T \, \Sigma^{-1} \, [u \quad 1]) A^{-1} \, [r_p \quad 1]^T \tag{3.24}$$

Where $A = [u \quad 1]^T \, \Sigma^{-1} \, [u \quad 1]$, finally:

$$= [r_p \quad 1] \, A^{-1} \begin{bmatrix} r_p \\ 1 \end{bmatrix} \tag{3.25}$$

Using the previous result for A, note that:

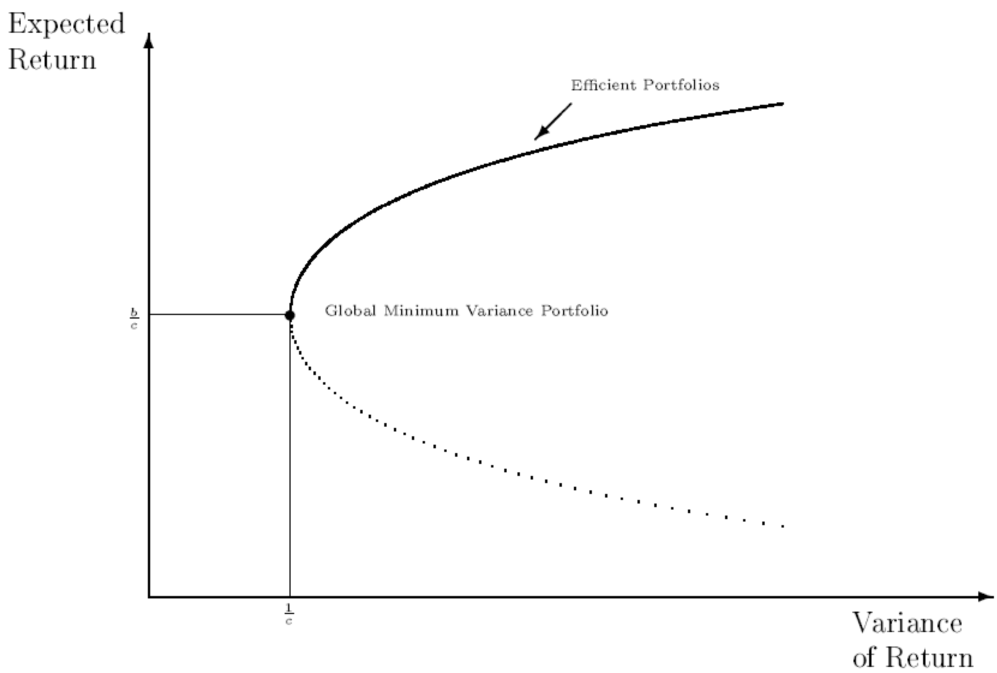
$$A^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix} \tag{3.26}$$

Which finally, allows to obtain:

$$\hat{\sigma_p}^2 = \frac{a - 2br_p + cr_p^2}{ac - b^2} \tag{3.27}$$

This last expression, represent a parabola in the plane of the portfolio expected returns, and variance $(r_p, \hat{\sigma_p}^2)$. This geometric space is call "variance portfolio frontier". The curve can be summarized in the following diagram:

Figure 3.1: The minimum variance portfolio frontier



Source: Lecture notes for the course; Investerings -og Finansieringsteori, (Lando and Poulsen, 2001)

The upper side of the curve, shows the so-call "efficient frontier", which summarize the portfolios whit the highest expected return for different level of risk (variance). The dotted curve, represent the "inefficient frontier". The figure 4.1, also include the global minimum variance portfolio, which represent the portfolio with the lowest variance between all the efficient portfolios. The mean of this portfolio can be found by minimizing 4.27, with respect to r_p, it easily lead to $r_g vm = \frac{b}{c}$, and by plugin this result in 4.27, it is obtained $\frac{1}{c}$. Thus, the portfolio weights give us: $\dot{w}_{gmv} = \frac{1}{c}\Sigma^{-1}1$. One result, that arise from the efficient portfolios representation, is a property call "two-fund separation", which state that any minimum variance portfolio, can be generated by a linear combination of other two different portfolios that belong to the efficient frontier. Let consider X_a and X_b, two minimum variance portfolios, with expected returns r_a and r_b, with $r_a \neq r_b$. Thus every portfolio that is a result of a linear combination of X_a and X_b, will also belong to the efficient frontier ($\alpha x_a + (1-\alpha)x_b$) for any $\alpha \epsilon [0;1]$ (omitted proof). Another interesting property that is a result of the previous derivation, is for every minimum variance portfolio (not for the global minimum variance), there exist a unique orthogonal portfolio, which also belong to the efficient frontier (omitted proof).

The previous case can be extended by included a "riskfree asset", which is an extra element in the returns vector, that include a deterministic value r_0. It will be denoted, the risky assets return relative to the risk-free asset, r_i^e as excess of return of risky assets over the risk-free asset. Using the same notation as before, it will be defined the average excess of return as u^e, and $/Sigma$ the variance. The vector w denote the weights $w_1, ...w_n$, corresponding to the risky assets. With this notation the average excess of return of the portfolio is:

$$r_p^e = w^T u^e \tag{3.28}$$

And the variance is;

$$\sigma_p^2 = w^T \Sigma w \tag{3.29}$$

The optimization problem can be set as:

$$min_w \ \frac{1}{2} w^T \Sigma W \tag{3.30}$$

$$subject\ to\ w^T u^e = r_p^e \tag{3.31}$$

Solving the problem following the same steps as for the case without risk-free asset. The portfolio weights can be solved as follow:

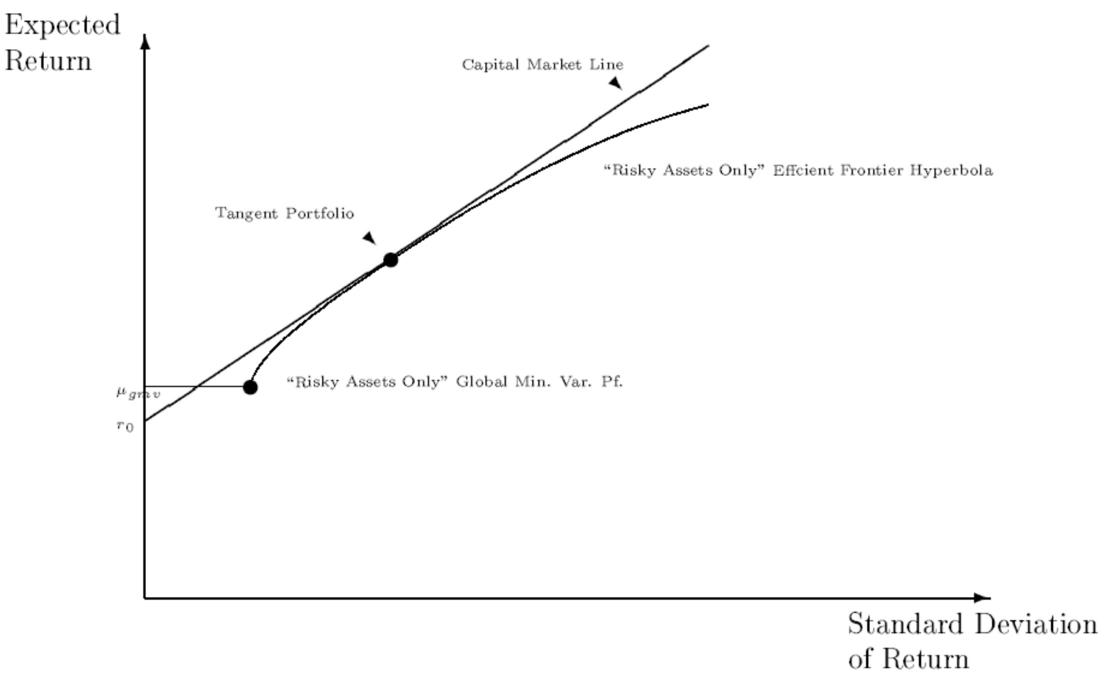
$$\hat{w} = \frac{r_p^e}{(u^e)^T \Sigma^{-1} u^e} \Sigma^{-1} u^e \tag{3.32}$$

And the variance of the minimum variance portfolios can be expressed as follow:

$$\hat{\sigma}_p^2 = \frac{(r_p^e)^2}{(u^e)^T \Sigma^{-1} u^e} \tag{3.33}$$

With, this equation it is stated the efficient frontier for the case of the portfolio optimization with a risk-free asset. For returns that are above the risk-free rate, the efficient frontier is a straight line with slope $\sqrt{(u^e)^T \Sigma^{-1} u^e}$ (solving 4.33 for volatility equal to zero). In the diagram shown below, the straight line represents the capital market line, the intercept of this line with the y-axis, represent the risk-free asset. The tangent portfolio is the minimum variance solution when all the assets allocation is done in risky instruments ($r_{tan}^T = 1$). The mean excess return for the tangent portfolio is $r_{tan}^T = \frac{u^T \Sigma - 1u}{1^T \Sigma - 1u}$ (proof omitted). From a mathematical perspective, $r_t an$ can be positive or negative, but it is a common situation that the risk-free asset return, is lower than the global minimum variance portfolio expected return. In which case, $r_{tan}^T > 0$. This portfolio is also, the asset allocation that offers the larger excess return per unit of risk (larger sharp ratio= $\frac{u_p - r_0}{\sigma_p}$).

Figure 3.2: The capital market line



Source: Lecture notes for the course; Investerings -og Finansieringsteori, (Lando and Poulsen, 2001)

3.2 Quadratic Utility Function

One of the most used formulation to described the utility function of a representative agent in financial economics, is the quadratic utility function. This functional form is also described in the seminal work of H. Markowitz (Markowitz, 1952). The theory describes the possibility of agents having different degrees of willingness to take risks. Based on this approach, one can describe the agents' utility function, adding a parameter (λ) that quantifies their willing to bear risk or their degree of risk aversion. A common utility function used in portfolio theory is the one that assumes quadratic preferences, which is described by the following expression:

$$U(W) = W - \frac{\lambda}{2}W^2, \ \lambda > 0 \tag{3.34}$$

Where W can be understood as the final agents wealth. To analyze the sensitivity of the utility function to changes in wealth, by taking, the function derivatives:

$$\begin{aligned} U'(W) &= 1 - 2, \\ U''(W) &= -2\lambda \end{aligned} \tag{3.35}$$

Analyzing the first derivative, it is observed that an additional constrain must be included, in order to insurance that the function is well defined (concave and with decreasing marginal returns). Thus:

$$\begin{aligned} U'(W) =& 1 - 2\lambda * W > 0 \\ \frac{1}{2\lambda} >& W \end{aligned} \tag{3.36}$$

To analyze the link between risk aversion and the functional form of the utility function, it will be used two measures described by Arrow (1971): absolute risk aversion (A(W)) and relative risk aversion (R(W)). In the context of portfolio theory both concepts represent, the change in the allocation from less risky assets to risky ones. However the relative risk aversion measure adjust the measure for the agents' wealth level. Thus, additionally to the computation of each one of them, it is relevant to study how they evolve when wealth changes. Thus:

$$\begin{aligned}
A(W) &= \frac{-U''(W)}{U'(W)} = \frac{2\lambda}{(1-2\lambda * W)} \\
A'(W) &= \frac{4\lambda^2}{(1-2\lambda * W)^2} > 0 \\
R(W) &= \frac{-U''(W) * W}{U'(W)} = \frac{2\lambda * W}{(1-2\lambda * W)} \\
R'(W) &= \frac{2\lambda}{(1-2\lambda * W)^2} > 0
\end{aligned} \tag{3.37}$$

From the above expressions it is observed that quadratic utility function shows, increasing absolute and relative risk aversion, meaning that, investors under this set of preferences will increase their asset allocation in risky assets as wealth increases. However, by analyzing the second derivatives it can be identified one of the main drawbacks of this functional form. Which is that the second derivative is decreasing in agent's wealth, this imply that richer people invest less in risky assets. This clearly is a counter-intuitive result (Arrow, 1971).

To link the quadratic utility functional form with the Markowitz portfolio theory, it is needed to include the portfolio return in the final agents wealth. To do so, it will be replace W for r_p. And by using the definition of variance as a function of expected values (Var(W)=E(W^2)-$E^2(W)$, it will be also assumed $E^2(W) = 0$). Using this results in the equation 3.34, and including algebraic notation, it is obtained the following expression (Engels, 2004):

$$U = r_p - \frac{\lambda * Var(r_p)}{2} = \mu^T w - \frac{1}{2}\lambda w^T \Sigma w \tag{3.38}$$

Where, θ = portfolio weights, Σ= matrix of variance and covariance, and $w^T \Sigma w$ = portfolio variance. Our objective is to maximize the investors utility function, assuming that all the wealth will be invested:

$$Max\{\mu^T w - \frac{1}{2}\lambda w^T \Sigma w\} \tag{3.39}$$

Subject to $1^T w = 1$.

The maximization of this problem can be solve using a Lagrangian as in equation 3.8. That is to say, it is needed to include the matrix defined in equation 3.17. Thus, it is found the optimal portfolio weights (intermediate steps omitted):

$$w_{\text{opt}} = \frac{1}{\lambda}\Sigma^{-}1(\mu + 1 + (\frac{-b}{c})) \tag{3.40}$$

Rearranging the above expression and using the previous results for the optimal weights of minimum variance and tangency portfolios (Engels, 2004), it is obtained (intermediate steps omitted):

$$w_{\text{opt}} = \frac{b}{\lambda} w_{\tan} + (1 - \frac{b}{\lambda}) w_{\text{minvar}} \tag{3.41}$$

It is observed, that the optimal portfolio weights are a linear combination of portfolio weights of the minimum variance portfolio and the tangency portfolio (intermediate steps omitted). Furthermore, computing the portfolio mean and variance it is obtained the following expressions (intermediate steps omitted):

$$\mu_{\text{opt}} = \mu^T w = \frac{d}{c\lambda} + \mu_{\text{minvar}} \tag{3.42}$$

and,

$$\sigma^2{}_{\text{opt}} = w^T \sigma w = \frac{d}{c\lambda^2} + \mu_{\text{minvar}} \tag{3.43}$$

Thus, when optimizing the portfolio under quadratic utility preferences, the portfolio mean and variance, are a function of the minimum variance portfolio, and the risk aversion parameter λ. Thus, agents with high levels of risk aversion, will prefer portfolios with lower expected return and volatility. In extreme cases, when λ is extremely high, the optimal portfolio in this case lead the same results as in the minimum variance case. In the opposite case, when λ is large enough, the portfolio weights will be similar to the tangency portfolio. This indicate us, that the minimal variance portfolio and the tangency, are special cases of the quadratic utility maximizing procedure (Engels, 2004).

3.3 Hierarchical Risk Parity

The hierarchical risk parity is a portfolio optimization methodology developed by Lopez de Prado (2016). This method aims to find a practical solution to the problem generated when dealing with matrices of variance and covariance that are not invertible, or generate ill-suited solutions after invertion is applied. As a requirement for the inversion, the matrix must be positive-definite, which imply that all eigenvalues need to be positive. The hierarchical risk parity (HRP), is based on an algorithm that aims to solve the optimal portfolio weights without the need invert the covariance matrix. As a result, one can expect that compared with the Markowitz optimization problem, the results, in this case, lead to more stable portfolios, with lower levels of concentration and a higher out of sample performance. The method is based on techniques of machine learning and graph theory to solve the optimal portfolio weights. The algorithm is based in the following three steps (Vyas, 2020):

3.3.1 Hierarchical Tree Clustering

In this step, the algorithm seek to create group of assets with similar correlation. Firstly, the correlation matrix, is transformed in a correlation-distance matrix, to do so each pair of assets is standarized in the following way:

$$D(i,j) = \sqrt{0.5 * (1 - \rho(i,j))} \tag{3.44}$$

Having this metric, a new distance is calculated, this time is used the Euclidean distance:

$$\overline{D}(i,j) = \sqrt{\sum_{k=1}^{N} (D(k,i) - D(k,j))^2} \tag{3.45}$$

The above equation explains that, assets which are close to each other (measured by $\overline{D}(i,j)$), based on the above distance measure, will have similar correlation with the rest of the portfolio. Thus, they can be cluster in the same group. The process is repeated in a iterative way. Lets describe the group of clusters as follow:

$$U[1] = argmin_{(i,j)} \overline{D}(i,j) \tag{3.46}$$

Where $U[1]$ is the first cluster. Lets consider the following matrix:

Figure 3.3: Example: Distance Matrix

	a	b	c	d	e
a	0	17	21	31	23
b	17	0	30	34	21
c	21	30	0	28	39
d	31	34	28	0	43
e	23	21	39	43	0

Source: The Hierarchical Risk Parity Algorithm: An Introduction, Hudson and Thames (Vyas, 2020)

In this case, the funds A and B have the minimum distance, so can be clustered. Afterwards, the columns of the funds A and B are remove, and a new matrix is generated, with the updated elements. To compute the new cluster, the following equation is applied:

$$\overline{D}(i, U[1]) = min(\overline{D}(i,a), \overline{D}(i,b)) \tag{3.47}$$

So for every fund, the minimum distance to the cluster a,b is computed:

$$\begin{aligned}\overline{D}(d, U[1]) &= min(\overline{D}(d, a), \overline{D}(d, b)) = min(31, 34) = 31\\ \overline{D}(c, U[1]) &= min(\overline{D}(c, a), \overline{D}(c, b)) = min(21, 30) = 21\\ \overline{D}(e, U[1]) &= min(\overline{D}(e, a), \overline{D}(e, b)) = min(23, 21) = 21\end{aligned} \tag{3.48}$$

Figure 3.4: Example: Distance Matrix (first iteration)

	(a,b)	c	d	e
(a,b)	0	21	31	21
c	21	0	28	39
d	31	28	0	43
e	21	39	43	0

Source: The Hierarchical Risk Parity Algorithm: An Introduction, Hudson and Thames (Vyas, 2020)

The above results, indicate us that, the minimum distance between the cluster a,b is with the funds C and E. It is updated the distance matrix with this result and it is obtained:

Figure 3.5: Example: Distance Matrix (second iteration)

	((a,b),c,e)	d
((a,b),c,e)	0	28
d	28	0

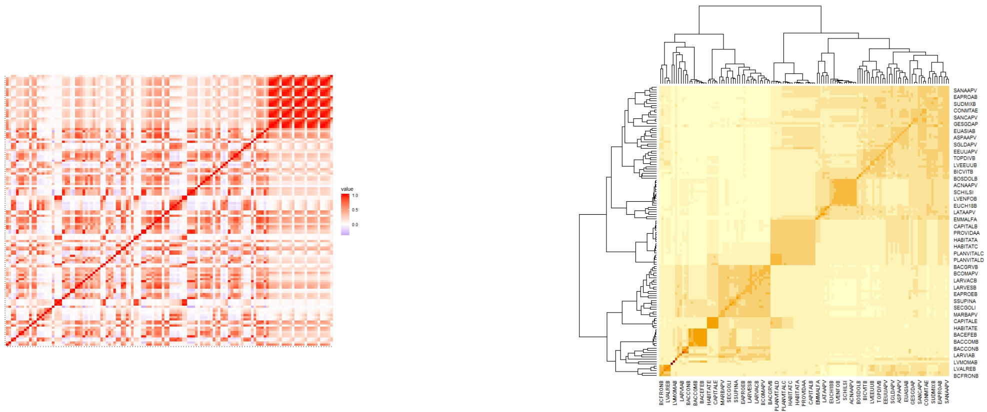
Source: The Hierarchical Risk Parity Algorithm: An Introduction, Hudson and Thames (Vyas, 2020)

The final result, of this part of the algorithm can be graphically described in a "dendogram", which is a representation of how the different elements of the matrix are sorted.

3.3.2 Matrix Seriation

In the original paper of Lopez de Prado (2016), it is mentioned as the second stage of the hierarchical risk parity process, the "quasi-diagonalization" method, which is part of the family algorithms of matrix seriation. This part of the process, seek to sort the elements of the matrix, in a way that similar funds are place together, and dissimilar are located far apart. The algorithm works recursively, placing funds in a way that the largest covariance are near the diagonal, and the smallest around this area. The name quasi-diagonal is given, as the elements that does not belong to the diagonal are close to zero, but normally different to zero.

Figure 3.6: Matrix Seriation Example



Source: Own elaboration

In the above picture, the darkest colours reflect higher correlations. In the left side, a group of funds is selected and displayed in a random way (fund names ommited). In the right side, the funds are sorted following the quasi-diagonalization procedure.

3.3.3 Recursive Bisection

In this part of the algorithm, the previous results are used to compute the portfolio weights. The process, first start by defining the portfolio weights:

$$W_i = 1, \forall i = 1, \ldots, N \tag{3.49}$$

The output of the hierarchical tree clustering, is one big cluster composed by sub-clusters. The big cluster, is divided in each one of their components. In this way each, component V, have a right and left neighbor clusters V_1 and V_2 (for simplicity it is assumed two clusters), for each sub-cluster is calculated its variance:

$$V_{adj} = w^T V w \tag{3.50}$$

where

$$w = \frac{diag[V]^{-1}}{trace(diag[V]^{-1})} \tag{3.51}$$

In this step, it is exploited the property of the previously defined quasi-diagonal matrix, since for a diagonal co-variance matrix the inverse-variance is the preferred allocation Vyas (2020). The above equation also describe one of the most remarkable properties of this method, which is to make compete the assets inside the cluster, and not against all the elements in the initial co-variance matrix. After this values are computed, a split factor is computed in the following way:

$$\alpha_1 = 1 \breve{} \frac{V_1}{V_1 + V_2}; \alpha_2 = 1 \breve{} \alpha_1 \tag{3.52}$$

The weights defined previously are re scaled, by the factor α_1, thus the new weights can be described as:

$$W_1 = \alpha_1 * W_1 W_2 \quad = \alpha_2 * W_2 \tag{3.53}$$

The process is repeated until all the weights are assigned to the funds that belong to the biggest cluster.

3.4 Performance and Risk Measures

In this section, are presented three performance measures that will be used to asses the results of the different portfolio. It is selected this measures with the idea of capture, three dimensions of the portfolio: one related to the risk-return relation (Sharpe ratio), other related to the performance relative to a benchmark (Treynor ratio), and one related to a measure of risk (Value-at-Risk).

3.4.1 Sharpe ratio

This measure was introduce by citesharpe. It consider the ratio between excess return (asset return minus the risk-free rate) and volatility İt is computed this measure based on the following expression:

$$Sharpe\ Ratio = \frac{r_i - r_f}{\sigma_i} \tag{3.54}$$

Where r_i denote the average return of the portfolio i, over a window of time of 250 observations (one year of business days). The r_f denote the average of the risk-free interest rate over the same window of time as before. For practical purposes, it will be assumed that this r_f is equal to zero, as the returns are computed in a daily basis, and the proxy of the free-risk interest rate approach to zero using this data frequency. σ_i, represent the standard deviation of the portfolio i, to compute it, it is used a window of 250 daily returns. The Sharpe ratio, measure the excess of return for unit of risk taken. Thus, high values of Sharpe ratios, will indicate more return for the risk taken. The main assumption of this measure is that, both the portfolio return and the standard deviation are good proxies of the realization of both variables.

3.4.2 Treynor Ratio

The Treynor ratio also known as the risk reward ratio, (Treynor and Black, 1973), measure the excess of return for unit of systematic risk taken. To do so, The excess of return is measure as the return of the portfolio over the risk-free rate. This result, is divided by the "beta" of the the portfolio with respect to a benchmark. The Treynor ratio can be summarize as follow:

$$Treynor\ Ratio = \frac{r_i - r_f}{\beta_i} \tag{3.55}$$

It will be keep the same assumptions and computation procedures as for the Sharpe ratio, regarding the variables r_i and r_f. The parameter beta will be calculated trough a linear regression (following the CAPM model):

$$r_i = \beta_i(r_m - r_f) + r_f + \epsilon_i \tag{3.56}$$

Where; r_m is the expected return of the market of reference (benchmark), r_f is the risk-free interest rate, which by simplicity it will be assume equal to zero, and r_i represent the portfolio return. Daily returns will be used of each one of the variables mentioned before, and an estimate window of 250 days.

3.4.3 Value at Risk

The Value-at-Risk (VaR) quantify how much the portfolio might lost over a given time horizon considering a given probability of occurrence. The method use in this thesis is the non parametric. The parametric VaR method was discarded, considering that fund returns do not follow a normal distribution and break most of the assumptions that the parametric VaR establish (see section 4, descriptive statistics). To compute the worst possible scenario given the portfolio returns, the historical returns are used. Afterwards it is identified the return that cut the empirical distribution in the the lowest 5th percentile. Thus, it is found the point that separate the The 95th percentile of the best returns from 5% of the worst returns. To do so, 250 historical observations are considered (Hendricks, 1996). Mathematically at a given level $\alpha(X)$, VaR is defined as follow:

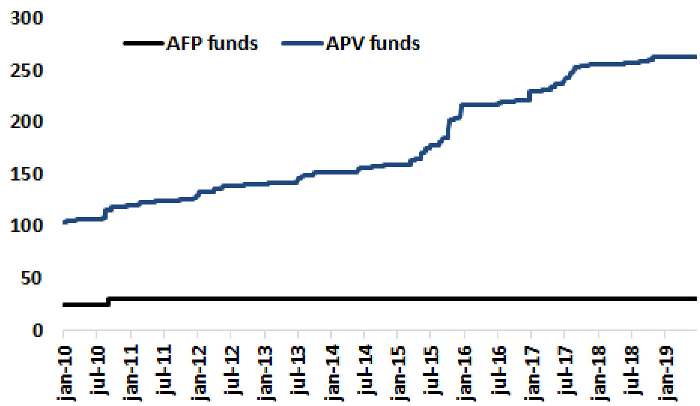
$$\mathrm{VaR}_\alpha(X) = \{y \mid \mathbb{P}(X \leq y) = \alpha\} \tag{3.57}$$

4 Data

4.1 Sample Description

The portfolio under consideration is composed of 25 AFP and 103 APV funds during the period, from 2-09-2010 to 18-06-2019. It has been included a stock index representative of the Chilean market, for the same period. This index is called IPSA, which is composed of the 40 stocks with the highest average annual trading volume in the Santiago Stock Exchange. The selection of the funds sample was done approximating the case of a saver that desire to select a portfolio of funds based on the entire pool of funds available. However, funds that were not available at the moment of the data selection (18-06-2019), plus the funds that were newly created, were discarded. The funds that were quoted in US dollars were converted to Chilean pesos. For the conversion, it was used the so-called "observed dollar", which corresponds to the average price of the market transactions, executed by financial institutions in the local currency spot market. The data was obtained from Bloomberg L.P. The data of the AFP funds was downloaded from the Chilean Pension regulator website (Superintendencia de Pensiones, 2019). The historical evolution of the number APV funds, and AFP funds can be described in the following graph: The above graph,

Figure 4.1: Number of AFP and APV funds (September 2019)



Own elaboration based on data published by Chilean Superintendency of Pensions Fund Administrators and Bloomberg L.P.

indicate us that the supply of APV funds, has experience a sustained growth, which has not been compensate by the supply of AFP funds, which has remain mostly unchanged. In September of 2019, the number of pension fund administrators was 6, including the following companies: Provida, Habitat, Cuprum, Planvital, Capital and Modelo. Each of these companies manages five different funds that account for a total of 30 funds. AFP Modelo was discarded, because of lack of historical data (created in September of 2019). The APV funds, for the same date were 263. These funds were mostly administrated by Banks, stockbroker houses, and financial intermediaries. Nevertheless, the sample selected considers 103 APV funds. These APV funds were distributed by the following financial intermediaries, banks: Bancoestado (5 funds), BCI (19 funds), BICE (2 funds), BTG Pactual (2 funds), Consorcio (4 funds), ITAU (9 funds) and Scotia Bank (14 funds). The following stockbroker houses also distributed APV funds: Larrain Vial (15 funds), Santander Asset Management (19 funds) and Security Zurich Asset Management (5 funds). The Chilean funds' regulators, define seven categories based on the mix of fixed income and variable income of each fund, the detail of this description can be found in the appendix, please see table A2.1. The detail of the APV funds, the companies that offer each fund and the type of fund can be seen in table A2.2 in the appendix. Related to the frequency of types of funds included in the sample, the largest proportion was identified for funds type 5 and type 3 with 61 and 25, components respectively.

4.2 Descriptive Statistics

The returns used in this research were calculated on a daily basis. The literature regarding the frequency of the return computation that must be used in the Markowitz type of optimizations is broad. Based on the problem that this thesis is scoping with, the approach developed by Hautsch et al. (2013), has been chosen. In this research, authors argue, that when trying to solve high-dimension portfolio optimization, which is characterized by a large number of assets (K), these must be relative to the historical observations (N). The uses of high-frequency data, such as daily returns can lead to improving the stability of mean covariance estimates. By computing less frequently funds' return, the number of observations "N", will approach the number of assets "K". When computing weekly returns, for the entire sample, the number of observations is 520, and for monthly returns, the sample is 120. The number of funds (128) remains unchanged during the period of analysis. Authors in Hautsch et al. (2013), expose that when "K" approaches to "N", optimization results lead to unstable results, and in the limit, the matrix of variance and covariance cannot be invertible. In relation to the selection of prices, nominal prices have been selected. As the portfolio allocation alternatives, these are analyzed from the perspective of a saver, in a scenario where, he or she is taking the investment choices in each period of time. This approach is assessed to be appropriated for the problem described. Additionally, Markowitz, describe the irrelevance of the use of nominal or real returns when using the Markowitz portfolio optimization algorithm.

In the detail of the descriptive statistics (see appendix table A3), the results indicates that the range for which the median daily return fluctuate is between -0.01% and 0.07%. The range for the standard deviation is between -0.0054% and 1.43%. Related to the second (skewness) and (kurtosis) third momentum. The data shows that these values fluctuate between -1.27 to 707 and -16 to 3.55, with most of the values around 71, and 1,5 respectively. As a reference, the normal distribution exhibit kurtosis of 3 and skewness of 0. The previous results indicate that most of the funds show, positive skewness (the size of the right side tail is larger than the left-handed tail) and positive kurtosis or leptokurtosis (most of the data around the mean). These results are aligned with previous results registered for individual assets and mutual funds (Pendaraki, 2012). Besides these results, the empirical literature has indicated that investors could include skewness in their preferences, thus preferences could not be quadratic. This result has been documented by (Scott and Horvath, 1980), which indicates that the expected utility could be positively related with expected return and skewness and negatively linked to variance and kurtosis.

The funds returns, can be analyze graphically, through a box plots:

Figure 4.2: AFP and APV Funds daily returns from 2010 to 2019

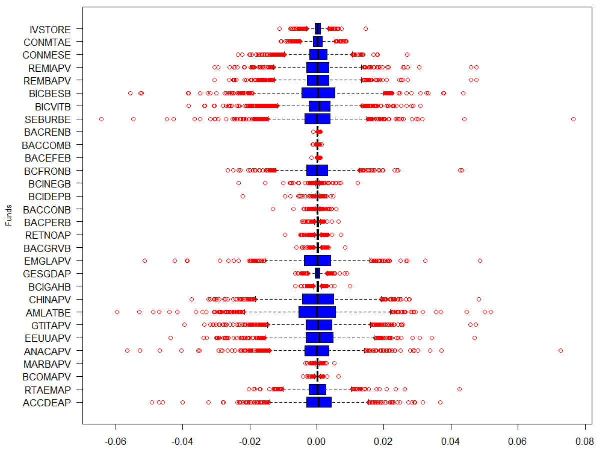


Figure 4.3: APV Funds

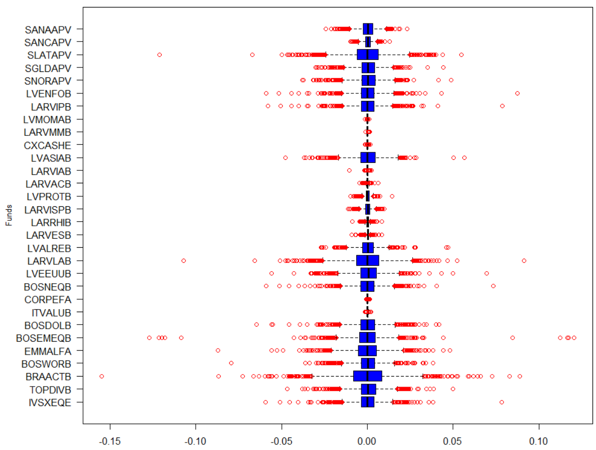
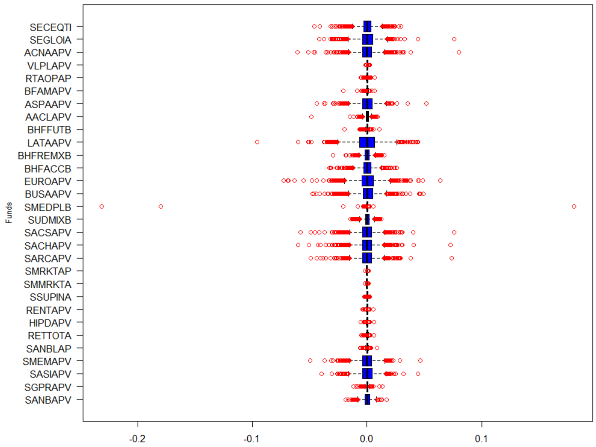


Figure 4.4: APV Funds



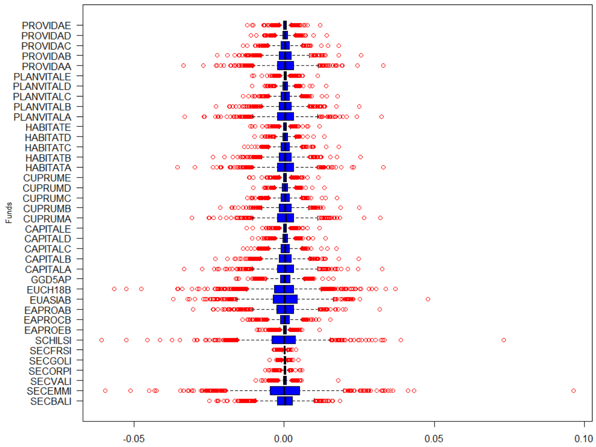


Figure 4.4: APV Funds **Figure 4.5:** APV and AFP Funds
Own elaboration based on data published by Chilean Superintendency of Pensions Fund Administrators and Bloomberg L.P.

The above graph, also shows how the AFP funds, exhibit approximately the same returns for the same kind of funds. One can also observe, some similarities in the sense that some APV funds, can be comparable with some APV funds. For instance, those APV funds mostly invest in money market or are classified between the first three categories by Chilean Financial Market Commission (see table A2.1). These funds exhibit similar returns as the equivalent AFP funds D or E. Another conclusion that arises from the boxplot graph, is the fact that risky AFP funds (A and B) exhibit a low level of data dispersion, compared with risky APV funds, which in extreme cases shows changes in intraday prices up to 20%.

The following graphs, describe the correlation matrix between funds, using two-sample size. The graph in the left side contains the full sample, whereas the graph in the right contains 80% of the sample size. We observe, high level of correlation between AFP funds (right side corner), which are aligned with the so call "herd effect" described in the literature Schlechter et al. (2019). Additionally, for both sample size, the correlation between AFP funds does not change. This suggests that the correlation between AFP funds and APV funds remain stable for two different windows of estimation.

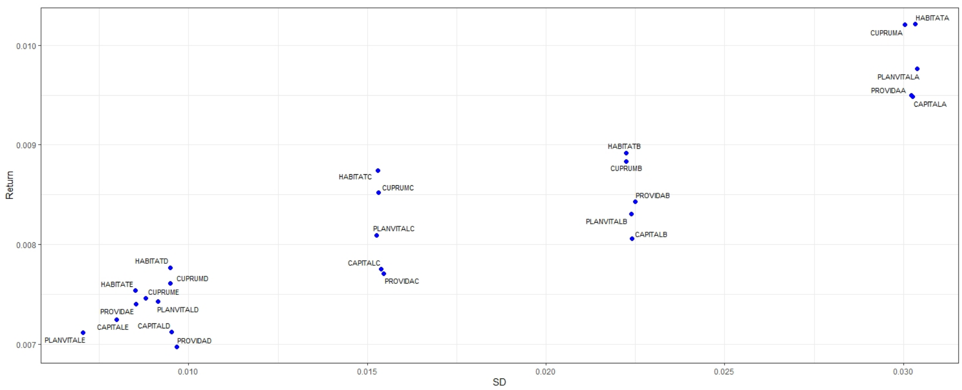
Figure 4.6: APV and AFP Funds correlation matrices

(a) 80 % of the sample **(b)** Full sample

Own elaboration based on data published by Chilean Superintendency of Pensions Fund Administrators and Bloomberg L.P.

Related to the risk-return relation, by using the entire sample to compute both measures, it is observed that the relationship between most risky funds and higher expected return is held for the type funds A, B and C. However for the less risky funds, the relation does not hold. In this regard, some AFP for fund E exhibit a higher expected return and volatility, than for fund D. This relation should be the opposite as the portfolio allocation of fund D, include a higher exposure to risky assets than fund E.

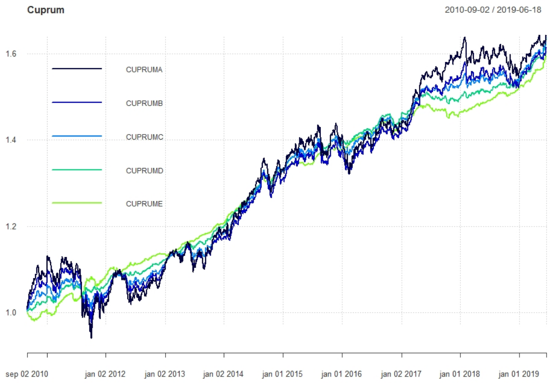
Figure 4.7: AFP Funds risk-return relation



Own elaboration based on data published by Chilean Superintendency of Pensions Fund Administrators and Bloomberg L.P.

By analyzing the evolution of the cumulative returns through time, we observe that the risk-return relation, between funds, confirms the results exposed in the literature review in the sense that the risk-return profile is not hold for all funds, during the period of analysis. Thus, the group of funds that are composed mostly by variable income assets yield lower cumulative returns, that funds that are invested mostly in fixed income products. This effect can be observed for the period between 2011 and 2013. During this period funds A, B and C, show a high level of volatility. But lower returns compared to the funds that are supposed to be less risky, namely funds D and E. Related to the relative performance of each pension fund manager, the administrator "Habitat", shows the highest expected return for each fund category. Followed by the administrator "Cuprum". This result has been also documented by Schlechter et al. (2019).

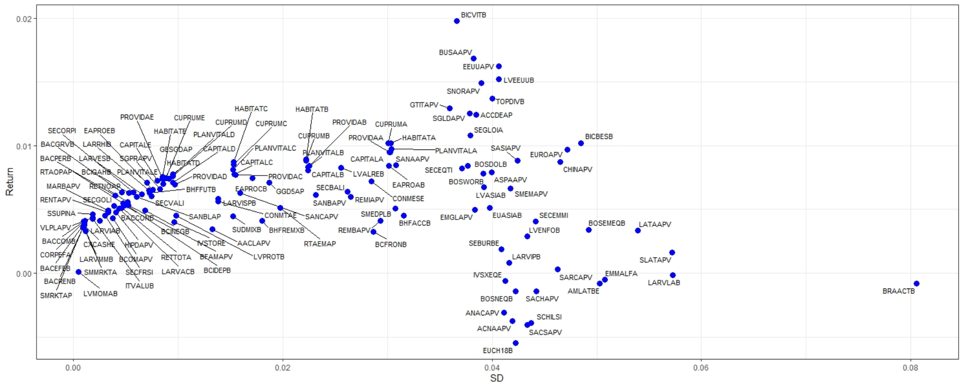
Figure 4.8: AFP Cuprum Funds Cumulative Returns



Own elaboration based on data published by Chilean Superintendency of Pensions Fund Administrators and Bloomberg L.P.

Finally, by analyzing the risk-return relation for the total number of funds, we observe two kinds of clusters. First, for the standard deviation between 0 and below 0.04, the funds exhibit a linear relationship between risk and return. However, for values of the standard deviation above 0.04, the relation turns unclear. By looking at some of the fund descriptions, one can observe that these are mostly focused on variable income. One possible explanation for this kind of result is that fund managers have a better know how when building up fixed-income funds. These results are compared, with the knowledge that managers have when doing asset allocation in stocks both, locally in Chile and in international markets. Other explanation can be related to the ability of the fund managers to compare themselves with their peers. For the case of low-risk fund, it seems that is easy to find benchmarks, as managers invest in similar assets. However, for variable income funds, this task is not that easy. Because the proportion of geographical allocation and currencies exposure differs from fund to fund (i.e EUROAPV and CHINAAPV, are both variable income funds, but with different geographical focus).

Figure 4.9: APV and AFP Funds risk-return relation



Own elaboration based on data published by Chilean Superintendency of Pensions Fund Administrators and Bloomberg L.P.

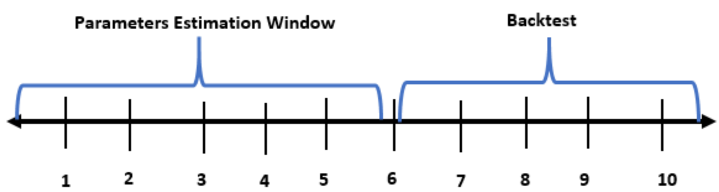
Overall, the previous description considers a picture of the funds' statistics. The reader must be aware that the relations described before, are dynamic and highly sensible to the sample used to estimate them. Nevertheless, the review of these figures provides a first overview of the problem faced by savers in the Chilean pension system. The high number of funds available in the system, the instability of the risk-return between funds, and the unexpected ex-post performance of funds (i.e. high risk deliver low returns) make the asset allocation problem a highly complex task.

5 Analysis

5.1 Empirical Implementation

In this section, it is explained how the theory described in section 3 has been implemented in practice. The programs that were used to compute the results were R Studio and Microsoft Excel. The portfolio weights were calculated considering three sample lengths: 60%, 70% and 80% of the dataset defined in section 4, which correspond to 1347, 1572 and 1796 observations (trading days) respectively. The first experiment defined consists of a static Markowitz optimization of the portfolio weights. That is to say, the minimum variance portfolio and the tangency portfolio are computed allowing and constraining short-selling, using first 60%, 70% and 80% of the data available. The analysis also included the so-called 1/n strategy, which is a portfolio designed with equal weights for each one of the assets available in the sample. Thus, every portfolio contribution will be equal to 1/128 (this strategy has been documented to be used by savers in defined contribution plans in EEUU, Benartzi and Thaler (2001)). The purpose of these exercises is to describe the case of an investor that can select their portfolio allocation once during the period of analysis. The use of different sample sizes is applied to check the sensitivity of the parameters to changes in the estimation of the inputs used in the algorithm (mean and variance). The performance and risk measures for this case have been computed using the full sample, as illustrated in the following picture:

Figure 5.1: Sample size static optimization

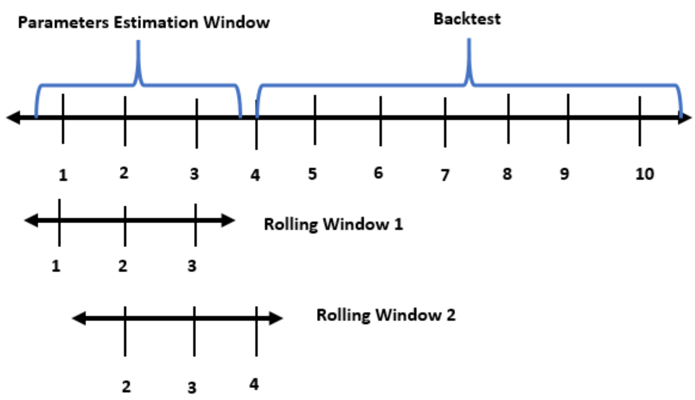


Source: Own elaboration

To analyse the effect of portfolio re-balancing on the performance and risk metrics, the static case is extended by including monthly portfolio re-balancing. Then again, it is computed the minimum variance and tangency portfolios allowing and constraining short-selling of funds. In this case, the equally weighted portfolio was excluded as it remains unchanged through time. The sampling procedure considers 60%, 70% and 80% of the data available. To assess the dynamic portfolio composition and performance, a sampling technique known as "rolling window" was implemented. This method consists on maintaining the sample length constant and move the data selected through time as new data emerges. This means that for each N-day of asset returns, an estimation window of length "K" is used, which can specifically take three values: 60%, 70% and 80% of the full sample (2245 observations). In this exercise monthly re-balancing is assumed to be execute every 25 days. Thus, starting at t = K+1, the optimization results are obtained from the previous K returns. With these results, the optimal portfolio weights are computed from K+1 to K+26. At that period of time the portfolio weights are re-estimated, using K+1 observations but dropping the first 25, and including the new data observed in the last 25 days previous to the re-balancing. For this day, the portfolio return is also calculated. The process continues, by moving "t" one day ahead until the final day is reached.

The length of the out-of-sample series is N-K returns for each strategy. For example, in a sample of ten observations, we use the first three to estimate the results for day number four. Then, as the information of number four is available, the consideration of day one is discarded and the data of day four is used in the new estimation. This process is summarized in the following figure:

Figure 5.2: Sample size rolling window optimization



Source: Own elaboration

Moreover, as described in chapter 3, the minimum variance and tangency portfolio are extreme cases that represent investors with extreme risk preferences. To describe more realistic investor preferences, the optimization of the utility function was applied as described in equation 3.39, with different risk aversion parameters that represent investors with a high level of risk appetite ($\lambda = 0$) to investors who bear less risk ($\lambda = 5000$). The sampling used in this case is the same as the one described in the above picture. Finally, the analysis is complemented by implementing a novel method which applied elements of machine learning to portfolio optimization. This approach, known as Hierarchical Risk Parity (Lopez de Prado, 2016), seeks to solve most of the potential pitfalls that emerge when solving optimal portfolio weights under the Markowitz framework. The steps of the algorithm were described in section 3.3 (methodology). The results of this computation are comparable with the minimum variance portfolio under constrained short selling. The sampling procedure used in this case is the same as the one described before –rolling window with different sample lengths.

5.2 Static Optimization

This section presents the results of solving the optimization problem as announced in the equation 3.8. The portfolio weights for the global minimum variance portfolio (GMVP) and for the tangency portfolio (short-selling constrained and unconstrained) are solved for the sample sizes of 60%, 70% and 80%. For each one of the sample sizes the following are solved: tangency portfolio with unconstrained short-selling, tangency portfolio with short selling constrained, GMVP without short-selling constrained and GMVP with short selling constrained. Additionally, the equally weighted portfolio (EW) was included in the analysis. In figures 5.4 and 5.5 (y-axis portfolio return and x-axis portfolio standard deviation) the results of the Markowitz efficient frontier are reported, for the two cases restricted and unconstrained short selling, respectively. To compute the tangency portfolio, we assume an interest rate equal to zero due to the requirement that for all the funds under optimization the risk-free interest rate must be equal to or lower than the rate of return of the assets used in the optimization (during the period of analysis). However, some funds under-perform the risk-free asset [2]. We observe in the graph below that the optimization allowing short-selling result in a higher expected return per unit of standard deviation. This compared with the case when short-selling of funds are not allowed. Additionally, the degree of concavity of the frontier is higher for the constrained case.

[2]. The average interest rate of the nominal bonds issued by the Central Bank of Chile (a proxy of free risk interest rate) for the maturities of 1,2,5 and 10 years, during the period under analysis was: 3.69%, 3.94%, 4.54% and 4.94% (see: https://bit.ly/34VzZsX. There are funds (please see table A3 with an average return over the period of 0.01%, which converted from daily to annual return, is 2.53%)

Figure 5.3: Markowitz efficient frontier (60% sample size)

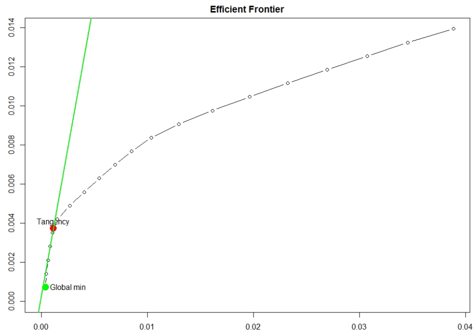


Figure 5.4: Constrained Optimization

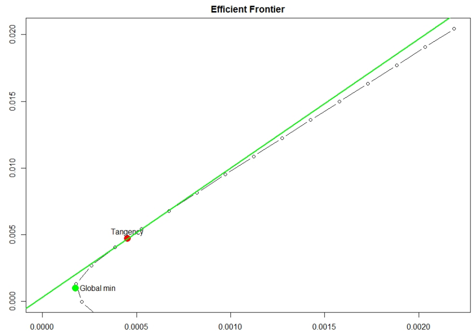


Figure 5.5: Unconstrained Optimization

Own elaboration

The statistical properties of the portfolios' returns are described in table A3. For the full sample, the largest and minimal daily return changes was observed for the equally weighted portfolio with -1.209% and 1.417%, respectively. Concerning the shape of the returns distributions, all the portfolios that resulted from the Markowitz optimization exhibit leptokurtosis (kurtosis higher than 3), which indicates that most of the returns are clustered around the mean. In the case of the equally weighted portfolio, it is observed a kurtosis value of 1.9, which indicates that under this empirical distribution, it is are more likely to observe outliers or values that are distant from the mean. As for the skewness of the data, from the Markowitz output, all values are between 1.2 and 2.6, which indicates that the mode is located to the right side of the mean. The exception is, once again, the equally weighted portfolio that shows a value of -0.359, indicating that the mode and the mean are close to each other. When computing the cumulative returns of each one of the 13 portfolios, one can observe that the portfolio returns are aligned with what is described in the portfolio theory. In other words, the portfolio that exhibits the highest cumulative return is precisely the one that, as a result of the optimization, has high exposure to the risky assets. This is the case of the tangency portfolio. Additionally, the portfolio that is meant to deliver the lowest possible risk, considering the set of funds, achieves its objective. This is the case of the minimum variance portfolio. It is observed that by imposing constraints on short-selling of funds, the cumulative return is negatively affected. For instance, when using 60% of the sample size to estimate the portfolio weights, adding short-selling constrains can reduce the final wealth by up to 10% (the difference between the tangency portfolio for the constrained case versus the unconstrained case). Although the selection of the data sample to estimate the optimal portfolio composition have an effect in the cumulative returns, the order of the portfolios performance remain unchanged for different sample sizes. That is to say, the most risky portfolio (tangency portfolio unconstrained) delivers the highest cumulative return, whereas the GMVP constrained delivers the lowest cumulative return. However, the constrained case delivered highest cumulative returns during the period of analysis, which is a non-intuitive result. Nevertheless, this is possibly based on the fact that the optimal portfolio is found in one period of time and the future performance is driven by random market changes. Ultimately, the evolution of cumulative returns for "the naive strategy" (1/n), is aligned with the research of Windcliff and Boyle (2004) and Benartzi and Thaler (2001), in the sense that, at the end of the period, the equally weighted portfolio outperforms the tangency portfolio.

Figure 5.6: Portfolio Cumulative Return (different sample sizes)

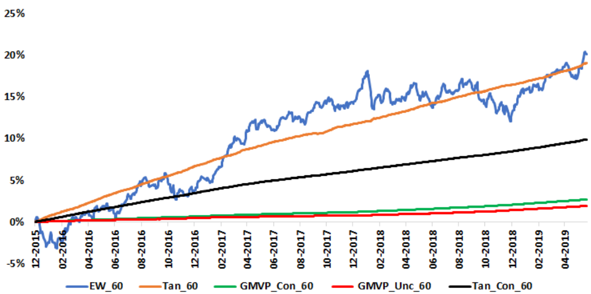


Figure 5.7: 60% Sample size

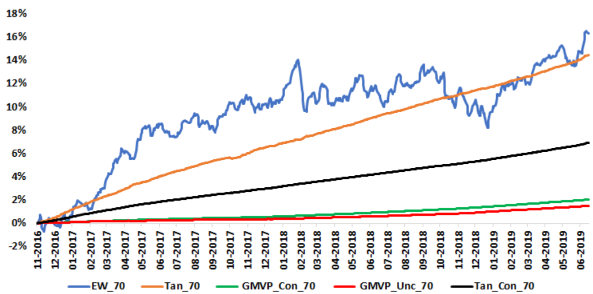


Figure 5.8: 70% Sample size

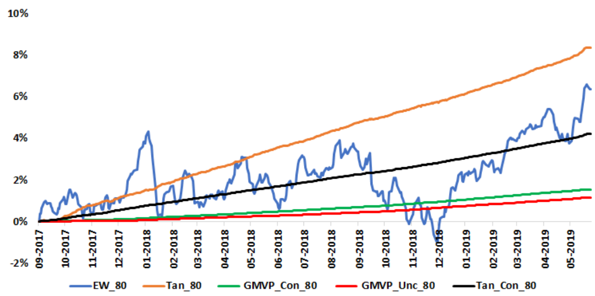


Figure 5.9: 80% Sample size

Source: Own elaboration

5.2.1 Performance Analysis: Sharpe Ratio

The results for the computation of the Sharpe ratio (SR) suggest that this metric is highly sensitive to market conditions, based on the changes that it exhibits when analysing year to year. Nevertheless, this level of sensitivity is not replicated when computing the portfolios using different sample sizes. This means that the hierarchy of results remains stable. When reviewing the result of specific portfolios, it is observed that the SR obtained for the equally weighted portfolio are not aligned with the Markowitz optimization portfolios. In this case, when looking at figure 5.9 (cumulative returns), the volatility exposure that the equally weighted portfolio capture does not compensate the extra returns that this strategy generates. However, the tangency portfolio (constrained case) exhibits the opposite behaviour, in which case, the gains that this strategy makes are compensated with low levels of volatility. This leads to higher Sharpe ratios when compared to other portfolios under analysis. Concerning the effect of short-selling constraints, it is detected that the order of the results is stable throughout time. This means that the highest Sharpe ratio is registered for the tangency portfolio with short-selling constraints; the second-highest values are observed for the GMVP portfolio without short-selling constraints; the third place is taken by the tangency portfolio without short-selling constraints. Finally, the GMVP portfolio with short-selling constraints delivers the lowest SR value. All in all, the hierarchy of risk-returns for the portfolios is not respected. As mentioned in the methodology section, it is expected that results are better for the unconstrained case than for the constrained ones, and for tangency portfolio versus the GMVP.

Table 5.1: Sharpe Ratio Static Optimization (Estimation Window: 60% Sample Size)

	Date	Equally Weighted	Tangency	GMVP	GMVP (Constrained)	Tangency (Constrained)
60 % Sample	2016 (Dic)	0.0065	0.0919	0.1022	0.0509	0.1096
	2017	0.0120	0.0588	0.0720	0.0211	0.0713
	2018	-0.0023	0.0617	0.0966	0.0482	0.0993
	2019 (Jan-Jun)	0.0061	0.0637	0.0962	0.0653	0.1020

Source: Own elaboration

Table 5.2: Sharpe Ratio Static Optimization (Estimation Window: 70% Sample Size)

	Date	Equally Weight	Tangency	GMVP	GMVP (Constrained)	Tangency (Constrained)
70 % Sample	2017 (Oct-Dic)	0.0119	0.0608	0.0715	0.0214	0.0696
	2018	-0.0022	0.0694	0.0960	0.0561	0.1002
	2019 (Jan-Jun)	0.0060	0.0685	0.0954	0.0688	0.1012

Source: Own elaboration

Table 5.3: Sharpe Ratio Static Optimization (Estimation Window: 80% Sample Size)

	Date	Equally Weight	Tangency	GMVP	GMVP (Constrained)	Tangency (Constrained)
80 % Sample	2018 (Sep-Dic)	-0.0020	0.0688	0.0954	0.0625	0.1001
	2019 (Jan-Jun)	0.0065	0.0714	0.0947	0.0743	0.1022

Source: Own elaboration

5.2.2 Performance Analysis: Treynor Ratio

As described in section 3.55, the Treynor ratio (TR) measures the excess of return generated by the portfolio (risk-free rate equal to zero is assumed) over the beta of the fund, calculated as fund return with respect to a benchmark, in this case, the so-called IPSA. It is observed that, high sensitivity of the results, with respect to the sample size, the portfolio types and years of analysis. Firstly, when using 60% of the sample for the case of the tangency portfolio (constrained) an extreme value in 2016 is detected. Additionally, most of the negative TR values are registered for the tangency portfolio (unconstrained), which indicates that the portfolio beta and the portfolio return have different sign. When computing the results using 70% of the sample, the largest TR values were observed for the Tangency (constrained) and GMVP (unconstrained) portfolios, which replicate the results obtained for Sharpe ratio, where these two portfolios performed the best. Finally, when computing the results using 80% of the sample size, one atypical value was detected for the tangency portfolio. During 2019, the return obtained for this portfolio goes beyond compensating the portfolio exposition to systemic risk.

Table 5.4: Treynor Ratio Static Optimization (Estimation Window: 60% Sample Size)

	Date	Equally Weighted	Tangency	GMVP	GMVP (Constrained)	Tangency (Constrained)
60 % Sample	2016 (Dic)	0.0017	-0.1139	0.1285	0.0510	-8.0529
	2017	0.0066	-0.0725	0.1470	0.0316	0.2278
	2018	-0.0006	-0.0704	0.2122	0.0341	0.3021
	2019 (Jan-Jun)	0.0026	0.2092	0.2581	0.0601	0.3249

Source: Own elaboration

Table 5.5: Treynor Ratio Static Optimization (Estimation Window: 70% Sample Size)

	Date	Equally Weight	Tangency	GMVP	GMVP (Constrained)	Tangency (Constrained)
70 % Sample	2017 (Oct-Dic)	0.0072	-0.1047	0.1316	0.0385	0.1512
	2018	-0.0006	-0.1506	0.2042	0.0468	0.2036
	2019 (Jan-Jun)	0.0026	0.1462	0.2457	0.0857	0.2110

Source: Own elaboration

Table 5.6: Treynor Ratio Static Optimization (Estimation Window: 80% Sample Size)

	Date	Equally Weight	Tangency	GMVP	GMVP (Constrained)	Tangency (Constrained)
80 % Sample	2018 (Sep-Dic)	-0.0005	0.0828	0.2454	0.0437	0.2504
	2019 (Jan-Jun)	0.0029	1.0152	0.2851	0.1293	0.2687

Source: Own elaboration

5.2.3 Performance Analysis: Value at Risk

The computation of the Value-at-Risk (VaR) was performed using the non-parametric method. Under this approach, we compute the 5th percentile of the daily returns using a sample window of 250 days. We then find the largest return of the group of the worst 5%. The results are aligned with the descriptive statistics, in the sense that the portfolios exhibiting kurtosis levels below 3 are more likely to show extreme values. That is the case of the equally weighted portfolio. In connection with the sensitivity of the estimates to the sample size, one can observe that the results are quite stable using different sample lengths. When analysing the variability of the results from year to year, in the Markowitz framework, the most extreme daily returns are observed for the tangency portfolio without short-selling constraints. However, these values are extremely low and close to zero. For the GMVP, the VaR values are above zero, though still very low. In both cases, these results suggest that in extreme conditions, and if the historical distribution behaves the same as the one used for the computation, the daily returns in the 5% of the cases can fall below the numbers reported in the tables. Overall, previous outcomes are repeated in the sense that the results exhibit the following order: 1) tangency portfolio with short-selling constrained, 2) GMVP without short-selling constrained, 3) GMVP with short-selling constrained, 4) tangency portfolio without short-selling, and 5) EW portfolio. These confirm the results in the sense that optimizing the portfolio one time does not guarantee that the hierarchical order of the results remains unchanged throughout time.

Table 5.7: VaR Static Optimization (Estimation Window: 60% Sample Size)

	Date	Equally Weighted	Tangency	GMVP	GMVP (Constrained)	Tangency (Constrained)
60 % Sample	2015 (Dic)	-0.4171%	0.0029%	0.0008%	-0.0039%	0.0037%
	2016	-0.3421%	0.0038%	0.0010%	-0.0017%	0.0070%
	2017	-0.2584%	-0.0066%	0.0004%	-0.0023%	0.0035%
	2018	-0.4372%	-0.0069%	0.0014%	-0.0025%	0.0032%
	2019 (Jan-Jun)	-0.3627%	-0.0058%	0.0019%	-0.0011%	0.0038%

Source: Own elaboration

Table 5.8: VaR Static Optimization (Estimation Window: 70% Sample Size)

	Date	Equally Weight	Tangency	GMVP	GMVP (Constrained)	Tangency (Constrained)
70 % Sample	2016 (Nov-Dic)	-0.3612%	0.0066%	0.0010%	-0.0021%	0.0072%
	2017	-0.2594%	-0.0041%	0.0004%	-0.0018%	0.0036%
	2018	-0.4607%	-0.0053%	0.0014%	-0.0014%	0.0034%
	2019 (Jan-Jun)	-0.3755%	-0.0056%	0.0018%	-0.0006%	0.0040%

Source: Own elaboration

Table 5.9: VaR Static Optimization (Estimation Window: 80% Sample Size)

	Date	Equally Weight	Tangency	GMVP	GMVP (Constrained)	Tangency (Constrained)
80 % Sample	2017 (Sep-Dic)	-0.2500%	0.0042%	0.0003%	-0.0022%	0.0036%
	2018	-0.4390%	-0.0069%	0.0013%	-0.0009%	0.0034%
	2019 (Jan-Jun)	-0.3609%	-0.0051%	0.0018%	-0.0002%	0.0041%

Source: Own elaboration

5.2.4 Allocation Analysis

The main characteristic of the portfolio allocation for the four portfolios under analysis is that all of them are highly concentrated i.e. in all of them, the majority of the portfolio is allocated to just four funds.

This result has been documented previously for portfolios optimized under the Markowitz framework (Lopez de Prado, 2016). Related to the effect of the sample size in the portfolio composition, it is recognized that this remains mostly unchanged. Associated with the structure of each portfolio, it is observed that the tangency portfolio under the unconstrained optimization is the one that gets the most short-sold amount. Obtaining 2.5 times the portfolio in the funds "SSUPINA" (type 2) and "LVMOMAB" (type 1), to buy the funds "BCIDEPB" (type 3) and "BACCOMB" (type 1). This choice remains the same regardless of the sample size used in the optimization. Related to the GMVP for the unconstrained case, the portfolio is mostly concentrated in two funds: the fund "BACCOMB" (type 1) which is sold to buy the fund "SSUPINA" (type 2). In both cases, the optimal choice is to buy a fund with higher risk exposure by funding the trade with a fund mostly composed by money-market assets. In the constrained case, the GMVP is mostly allocated in the fund "LVMOMAB" (type 1), and the tangency portfolio in this case also concentrated in one fund: "BACCOMB" (type 1). Finally, the funds that are part of the mandatory monthly contribution scheme (AFPs funds A,B,C,D,E) are represented through minor contributions in the tangency portfolio for the unconstrained case.

Figure 5.10: Portfolio Composition (different sample sizes)

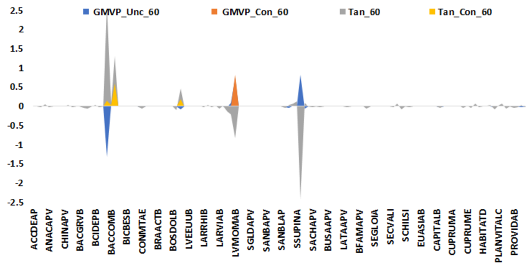


Figure 5.11: 60% Sample size

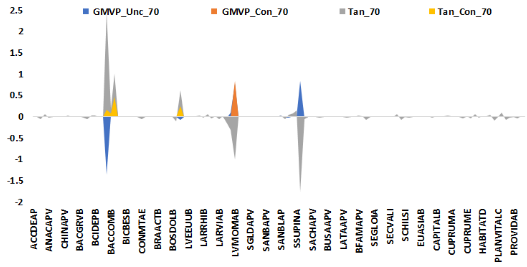


Figure 5.12: 70% Sample size

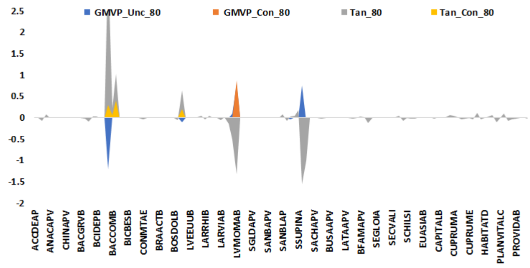


Figure 5.13: 80% Sample size

Source: Own elaboration

5.3 Rolling Window Optimization

In this section, it is extended the analysis of the static Markowitz portfolio optimization by solving the problem on a monthly basis. In other words, the optimization process is re-performed for weights of the GMVP and for the tangency portfolio (short-selling constrained and unconstrained), every 25 days. To compute the tangency portfolio, a risk-free interest rate equal to zero is assumed. The descriptive statistics of the portfolios are presented in table A3 in the appendix. When solving the optimization problem using 60% of the sample size, it is identified that the highest median return is shown by the tangency portfolio when short-selling of funds are allowed; this is 0.032% .For the case of the mean of the tangency portfolio for the constrained case, this is 0.016%. The mean return of both the constrained and unconstrained variants of the GMVP is identical at 0.01%. When the estimation window is extended, the result exhibits small changes whereas the order of the mean returns remains unchanged. In relation to the shape of the return's distribution, it is perceived that, compared with the static optimization, the kurtosis measure shows a different pattern. The series of return for all sample sizes show negative kurtosis (platykurtic), which indicates that the distribution tails are thinner compared to a normal distribution. In practical terms, this group of portfolios is less likely to observe extreme returns. Regarding skewness, most of the values are negative and around zero. This indicates that "normal" returns are more likely to be observed on the right side of the distribution (positive returns). However, extreme returns are more likely to be negative if the historical performance is repeated in the future. In terms of cumulative returns, the results are different as the static optimization case. Moreover, by using the rolling windows approach, one can detect that the theory is confirmed in the results. Thus, the tangency portfolio shows the highest cumulative return under the optimization that allows short selling. The second-highest cumulative return is exhibited by the tangency portfolio under the constrained optimization. Finally, the GMVP for the unconstrained case shows slightly better results than the GMVP constrained case, which is expected considering that constrained portfolios must perform equally or worse than unconstrained portfolios.

Figure 5.14: Portfolio Cumulative Return (different sample sizes/ rolling window sampling)

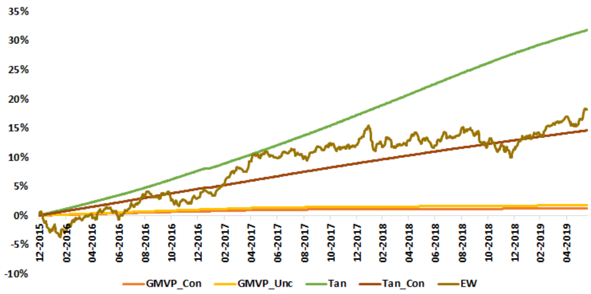


Figure 5.15: 60% Sample size

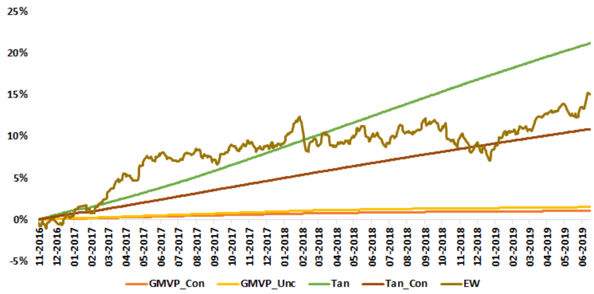


Figure 5.16: 70% Sample size

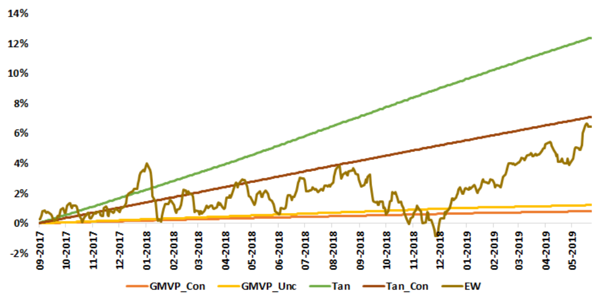


Figure 5.17: 80% Sample size

Source: Own elaboration

5.3.1 Performance Analysis: Sharpe Ratio

The Sharpe ratio analysis for the Markowitz optimization under rolling window sampling shows that the results are sensitive to the estimation window, which means that the results are sensible for different sample sizes and the SR also fluctuates when comparing portfolio types. The tangency portfolio delivers the largest return per unit of risk under unconstrained optimization. This pattern is repeated for all sample sizes and periods of analysis. The lowest Sharp ratios figures were calculated for the GMVP under the constrained optimization (for most of the cases).

Finally, the tangency portfolio, under the constrained optimization and the GMVP under the unconstrained case, shows more volatile results. For some periods, the GMVP (unconstrained) shows higher Sharpe ratios than the tangency portfolio (constrained). Compared to the case when it is estimated the portfolios using a static sample, the dynamic approach (rolling window) outperforms the static optimization in all the cases. This can be explained by the fact that the static optimization is exposed at every time to the market fluctuations as the portfolio is optimized once. In contrast, when using a rolling window, the optimal portfolio is solved for every period of time defined, in this case, with monthly frequency. As a result, one can expect that the portfolios return volatility should be reduced. Finally, the results indicate a dominance of the tangency portfolio over the GMVP. However, when considering the constrained and unconstrained optimization for each one of them, the results fluctuate from period to period. For instance, when using 70% of the sample size to estimate the portfolio weights, in 2018 the GMVP constrained outperformed GMVP unconstrained; however, in 2017 the result was the opposite. This showed that the SR can be susceptible to market changes that affect either the standard deviation or the return used to calculate it.

Table 5.10: Sharpe Ratio Rolling Window Optimization (Estimation Window: 60% Sample Size)

	Date	GMVP (Constrained)	GMVP	Tangency (Constrained)	Tangency
60 % Sample	2016 (Dic)	0.61	1.77	0.70	7.17
	2017	0.12	0.11	1.82	1.67
	2018	0.14	0.43	1.33	1.51
	2019 (Jan-Jun)	0.20	0.22	0.50	1.57

Source: Own elaboration

Table 5.11: Sharpe Ratio Rolling Window Optimization (Estimation Window: 70% Sample Size)

	Date	GMVP (Constrained)	GMVP	Tangency (Constrained)	Tangency
70 % Sample	2017 (Oct-Dic)	0.57	1.67	0.85	5.52
	2018	0.15	0.12	1.92	1.61
	2019 (Jan-Jun)	0.24	0.22	0.95	1.55

Source: Own elaboration

Table 5.12: Sharpe Ratio Rolling Window Optimization (Estimation Window: 80% Sample Size)

	Date	GMVP (Constrained)	GMVP	Tangency (Constrained)	Tangency
80 % Sample	2018 (Sep-Dic)	1.00	2.92	1.81	3.51
	2019 (Jan-Jun)	0.44	0.27	1.13	1.98

Source: Own elaboration

5.3.2 Performance Analysis: Treynor Ratio

The Treynor ratio was computed as in the equation 3.55, and the free risk interest rate equal to zero is assumed. The results suggest that there is not a unique portfolio that dominates over others for all time periods. Moreover, there is a high level of disparity between periods of time and allocation strategies. However, the highest return per unit of systemic risk was observed for the sample estimate using 60% of the data during the year 2018. The same portfolio exhibited the lowest ratio during 2018. These results indicate that this performance measure is highly sensitive to the parameters used in the estimation compared with the analysis of the Sharpe ratio, which remains mostly unchanged. We can conclude that the instability of the Treynor ratio can be attributed to the phenomena of beta values' sensitivity to the reference day used as been previously documented by (Sahadev et al., 2018), for example.

Table 5.13: Treynor Ratio Rolling Window Optimization (Estimation Window: 60% Sample Size)

	Date	GMVP	GMVP (Constrained)	Tangency	Tangency (Constrained)
60 % Sample	2016 (Dic)	0.80	1.72	-0.89	-12.73
	2017	0.56	0.44	-9.30	16.21
	2018	-180.21	15.85	-31.05	118.41
	2019 (Jan-Jun)	-1.47	-1.31	3.94	11.99

Source: Own elaboration

Table 5.14: Treynor Ratio Rolling Window Optimization (Estimation Window: 70% Sample Size)

	Date	GMVP	GMVP (Constrained)	Tangency	Tangency (Constrained)
70 % Sample	2017 (Oct-Dic)	4.406	11.445	-5.943	-64.599
	2018	-1.298	-1.199	-31.861	-517.866
	2019 (Jan-Jun)	-1.276	-1.112	7.430	11.150

Source: Own elaboration

Table 5.15: Treynor Ratio Rolling Window Optimization (Estimation Window: 80% Sample Size)

	Date	GMVP	GMVP (Constrained)	Tangency	Tangency (Constrained)
80 % Sample	2018 (Sep-Dic)	6.3	66.3	-11.6	-101.4
	2019 (Jan-Jun)	1.3	1.0	29.3	12.9

Source: Own elaboration

5.3.3 Performance Analysis: Value at Risk

The results were computed using the method described in section 3.4.3. They suggest that the rolling window optimization improves the ranking of portfolios based on this metric. As a consequence, the tangency portfolio under unconstrained optimization obtains the highest VaR results. The result is replicated on all periods of time under analysis and on all samples used in the estimation. Intuitively, this is a consequence of solving the optimization by prioritizing the highest Sharpe ratio, which leads to the riskiest result. On the opposite side, the result for the GMVP delivers the lowest VaR, which is the purpose of solving the optimization problem by prioritizing portfolio weights that produce an overall result with the lowest volatility. The GMPV in the constrained optimization, delivers similar results as in the unconstrained case. Finally, the tangency portfolio for the constrained case, exhibits results which are in the middle point of the GMVP and the unconstrained tangency portfolio. The order of the portfolio results is stable for the different years under analysis as well as for different estimation windows. By comparing the results of the rolling window optimization with the static case, we observe that the measure of VaR improves considerably. Firstly, the results are higher than zero for all the computations, which was valid for some periods of time in the static case, and for three out of five portfolios. Secondly, the order of the portfolios is stable under the rolling window optimization. This is a desirable result, considering that the GMVP should generate low but stable returns while, conversely, the tangent portfolio should generate more volatile results and higher returns. The results are consistent with a platykurtic and negatively skewed distribution of returns.

Table 5.16: VaR Ratio Rolling Window Optimization (Estimation Window: 60% Sample Size)

60 % Sample	Date	GMVP	GMVP (Constrained)	Tangency	Tangency (Constrained)
	2015 (Dic)	0.0030%	0.0041%	0.0249%	0.0167%
	2016	0.0023%	0.0036%	0.0253%	0.0168%
	2017	0.0003%	0.0005%	0.0318%	0.0150%
	2018	0.0002%	0.0005%	0.0296%	0.0129%
	2019 (Jan-Jun)	0.0003%	0.0006%	0.0235%	0.0121%

Source: Own elaboration

Table 5.17: VaR Ratio Rolling Window Optimization (Estimation Window: 70% Sample Size)

70 % Sample	Date	GMVP	GMVP (Constrained)	Tangency	Tangency (Constrained)
	2016 (Nov-Dic)	0.0026%	0.0035%	0.0251%	0.0164%
	2017	0.0019%	0.0031%	0.0260%	0.0160%
	2018	0.0006%	0.0007%	0.0276%	0.0139%
	2019 (Jan-Jun)	0.0006%	0.0007%	0.0249%	0.0131%

Source: Own elaboration

Table 5.18: VaR Ratio Rolling Window Optimization (Estimation Window: 80% Sample Size)

80 % Sample	Date	GMVP	GMVP (Constrained)	Tangency	Tangency (Constrained)
	2017 (Sep-Dic)	0.0021%	0.0029%	0.0248%	0.0157%
	2018	0.0017%	0.0029%	0.0254%	0.0149%
	2019 (Jan-Jun)	0.0013%	0.0017%	0.0239%	0.0142%

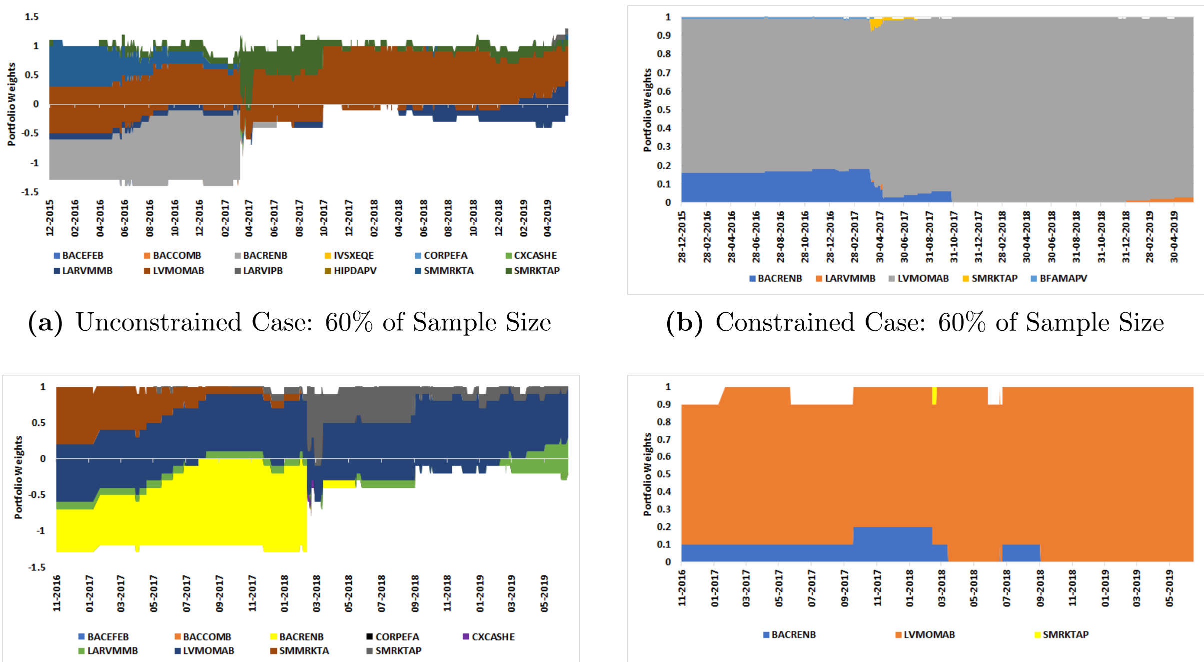
Source: Own elaboration

5.3.4 Allocation Analysis

In this section the results for the compositions of the GMVP and the tangency portfolio are presented for the constrained and unconstrained optimizations. The results indicate that the portfolio composition shows the same attribute that the portfolios obtained under static optimization. For most of the time, around 90% of the portfolio allocation is distributed between 5 funds or less. For the case of the constrained GMVP, the portfolio concentration is the most intensified. Two funds concentrated more than 90% of the portfolio during all the period of analysis, and for all the different sample sizes used in the optimization. These funds are "LVMOMAB" and "BACRENB", both of them categorized as type 1 for the Chilean Financial Market Commission (see table A2.1), which means that their asset allocation is based on short-term fixed income and money market assets. In relation to the GMVP under the unconstrained optimization, there are three funds that concentrate the majority of the portfolio. These funds are "BACEFEB" and

"LVMOMAB", which are funded by "BACRENB". All these funds are categorized as type 1. Finally, the portfolio weights for the unconstrained case, fluctuate dramatically when changing the window of estimation. In the constrained case, the weights are stable through time. For the tangency portfolio, we observe that the composition of the portfolio changes smoothly through time. In the constrained case, four funds compound the most substantial portfolio allocations for the different sample size. These funds are: "BACRENB", "BACEFEB", "BACCOMB" and "CORPEFA". All of them are categorized as type 1 by the Chilean Financial Market Commission. In the unconstrained case, we observe that the optimization is sensible to the sample size. The portfolio weights change dramatically through time. In the 60% sample, the fund "SMRKTAP" is used to buy funds: "SMMRKTA", "CORPEFA" and "BACEFEB". All of them are funds type 1. In the 70% sample, the fund "BACRENB" (type 1) is the source of funding, which is used to buy "LVMOMAB" and "SMMRKTA". Finally, a similar result is obtained when using 80% of the sample to estimate the parameters. In this case, "SMRKTAP" is used to buy funds: "SMMRKTA", "CORPEFA", "BACEFEB" and "BACRENB", all these funds are type 1.

Figure 5.18: Global Minimum Variance Portfolio Composition (different sample sizes)

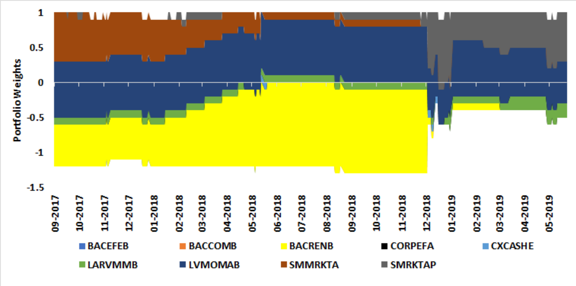


(a) Unconstrained Case: 60% of Sample Size

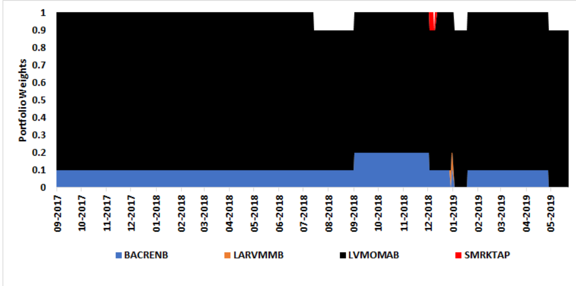
(b) Constrained Case: 60% of Sample Size

(c) Unconstrained Case: 70% of Sample Size

(d) Constrained Case: 70% of Sample Size



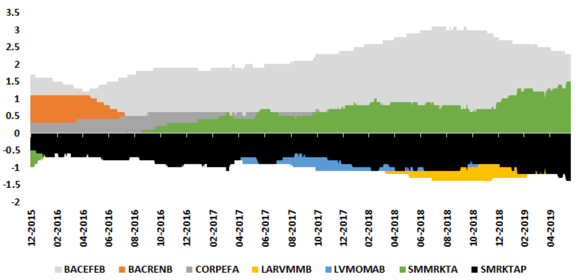
(e) Unconstrained Case: 80% of Sample Size



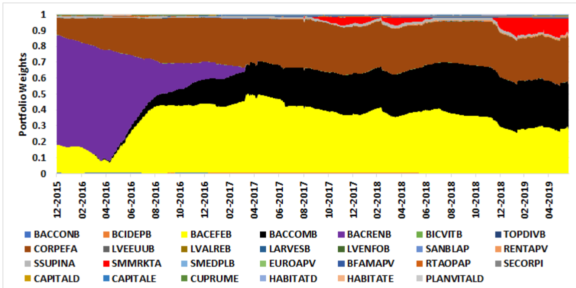
(f) Constrained Case: 80% of Sample Size

Source: Own elaboration

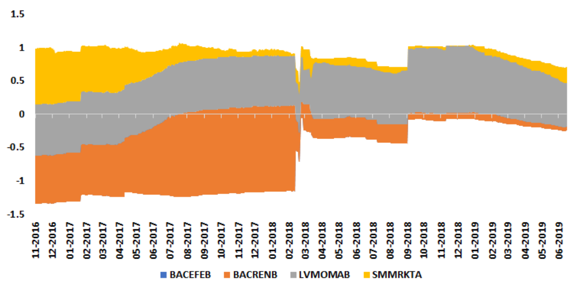
Figure 5.19: Tangency Portfolio Composition (different sample sizes)



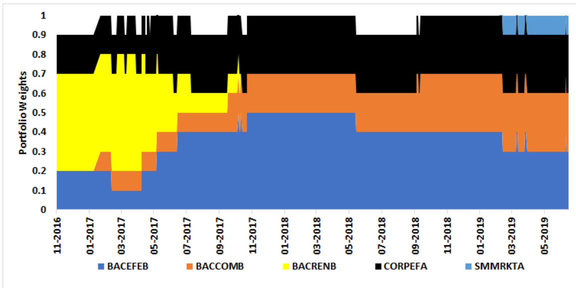
(a) Unconstrained Case: 60% of Sample Size



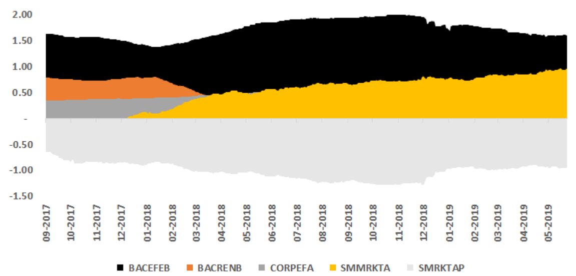
(b) Constrained Case: 60% of Sample Size



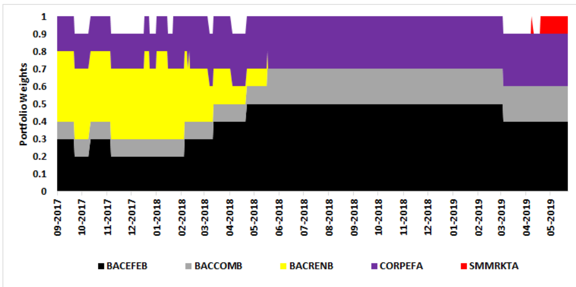
(c) Unconstrained Case: 70% of Sample Size



(d) Constrained Case: 70% of Sample Size



(e) Unconstrained Case: 80% of Sample Size



(f) Constrained Case: 80% of Sample Size

Source: Own elaboration

5.4 Quadratic Utility Function Optimization

This section presents the results of solving the optimization problem, assuming that savers preferences can be well-described by quadratic preferences. The portfolio weights are solved assuming no short selling of funds and risk aversion parameters that can be assumed to be proxies of extremely risk-averse agents ($\lambda \in [50 - 5000]$), medium level of risk aversion ($\lambda \in [1.5 - 5]$) and low level of risk aversion ($\lambda \in [0 - 0.8]$). In total, for each sample size (60%, 70% and 80%), nine cases of portfolio optimization were solved, using different risk aversion parameters. The descriptive statistics in this case (for further details see A3) reveal one of the attributes of the GMVP and the tangency portfolio described in Markowitz (1952), in the sense that both portfolios are specific cases of the optimization under quadratic preferences. This can be verified comparing the descriptive statistics of the rolling window optimization results for the GMVP and the tangency portfolio for the constrained optimization (no short selling) against the optimization under quadratic preferences. For each sample size, the mean and median returns of GVMP and tangency portfolio belong to the range of mean and median returns of the optimization under quadratic preferences, when using a risk aversion parameter from 0 to 5000. Additionally, for every sample size, the relation between risk aversion, volatility and return is decreasing. For instance, for the optimization under the 60% of the sample size, the median and the standard deviation for the portfolio with $\lambda = 0$ are 0.038% and 0.793%, respectively. Whereas for the portfolio with $\lambda = 5000$, these measures are 0.008% and 0.003%, respectively. This pattern is repeated for all sample sizes. In regard to the distributional properties of the portfolio returns, we observe that the kurtosis indicator is positive for all levels of risk aversion, and it is directly proportional to this measure; more specifically, all portfolios are leptokurtic. However, returns of portfolios optimized with larger risk aversion parameters are more leptokurtic than those in which the optimization considers low values for risk aversion parameters. This is the expected result, considering that the optimization under high values of risk aversion parameters, gives a higher penalty to volatility. Thus, one can expect that the return distributions fluctuate less, compared to the case with low levels of risk aversion parameters. The skewness test results for different portfolios returns show that most of them are left-skewed (negative skewness values), which implies that returns are more likely to be larger than the median. In practical terms, under this distribution shape, one can expect large movements in returns to be more frequent in the area of negative values (if the expected return is around zero). The visual inspection of the cumulative returns indicates that, overall, portfolios with large values of risk aversion parameters exhibit lower volatility. Regardless of the window size used to estimate the portfolio weights, the result is repeated, and it is aligned with the optimization equation that penalizes volatility the most when computing the optimal weights in the cases of high-risk aversion values. In spite of this result, when analysing the cumulative returns at different periods of time, it can be observed that in several time windows, the order of the portfolios based on the cumulative returns is not respected, showing a similar pattern previously described for AFP funds by (Schlechter et al., 2019). That is to say, portfolios that are supposed to be invested in low-risk assets, to exhibit low volatility, deliver higher cumulative returns compared with portfolios with low-risk aversion parameters with high-risk exposure. This effect is sensitive to the estimation window used to compute the optimal portfolio weights. For instance, when using 60% of the sample, during the first year of the estimation, the portfolios with high-risk aversion parameters obtain a higher cumulative return. In fact, the portfolios under low-risk aversion parameters reach a minimum level of -20% in October of 2016. The same negative trend is observed when computing the portfolio composition using 80% of the sample. During December 2018, the portfolios optimized under low-risk aversion coefficients show a lower cumulative return than portfolios computed with high-risk aversion coefficients. When the estimation includes 70% of the sample size, the risk lover portfolios exhibit higher cumulative returns than risk-averse allocation strategies. Finally, at the end of the period of analysis, the majority of the risk-bearing portfolios show higher performance, measured by cumulative returns, than risk-avoiding strategies.

Figure 5.20: Portfolio Cumulative Return (different sample sizes and risk aversion parameters)

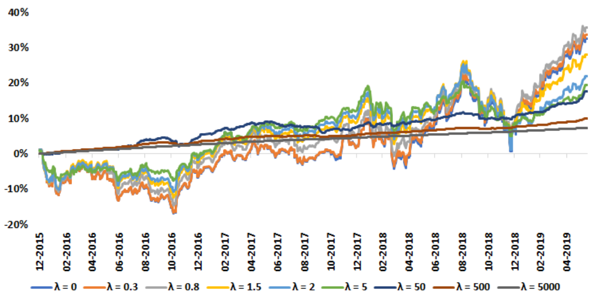


Figure 5.21: 60% Sample size

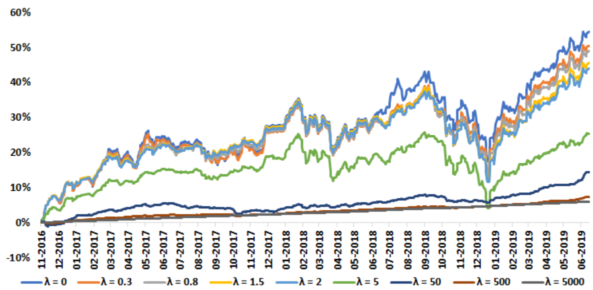


Figure 5.22: 70% Sample size

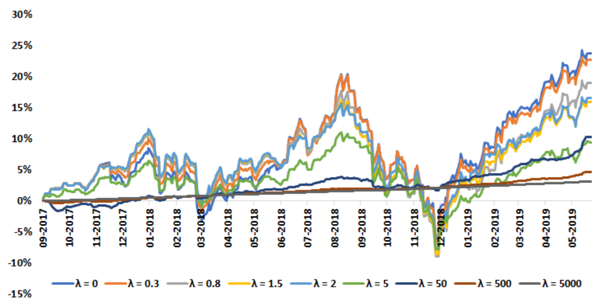


Figure 5.23: 80% Sample size

Source: Own elaboration

5.4.1 Performance Analysis: Sharpe Ratio

For each portfolio optimization under different risk aversion parameters, the Sharpe ratio is calculated, as stated in equation 3.54. This calculation is done for sample sizes of 60%, 70% and 80%. The results are reported by year. Some regularities were detected when analysing the results. For instance, for portfolios estimated under high-risk aversion parameters, SR is the highest. This is repeated for all sample sizes and years of analysis. This result is also comparable with previous findings from analysing the EW, tangency and GMVP portfolios. Moreover, when using low values for risk aversion parameter, the results approach to the EW portfolio; in the sense that the portfolio values are a highly affected by market parameters. Furthermore, for higher values of risk aversion the results are in similar to the GMVP and tangency portfolios. This in the sense that the results are stable to time and less volatile than the low risk aversion portfolios (or EW portfolio). It is also noted that the results are similar for risk aversion parameters below 5. This may suggest that there is a range of values that the risk aversion parameter can take, which leads to similar results. The breaking point in this trend is observed when increasing the risk aversion parameter from 50 to 500. It is shown that the Sharp ratio grows approximately four times on average, and when increasing the lambda value from 500 to 5000, the growth rate is around six times. This means that the relationship between risk aversion and Sharp ratio is positive and shows some level of convexity in the growth rate.

Table 5.19: Sharpe Ratio: Rolling Window Optimization for Different Risk Aversion Parameters (Estimation Window: 60% Sample Size)

	Date/Risk Aversion Parameter	λ=0	λ=0.3	λ=0.8	λ=1.5	λ=2	λ=5	λ=50	λ=500	λ=5000
60 % Sample	2016 (Dic)	0.001	0.001	0.002	0.002	0.003	0.002	0.012	0.031	0.217
	2017	0.003	0.003	0.004	0.006	0.007	0.010	0.006	0.019	0.131
	2018	0.000	0.000	0.000	-0.001	-0.002	-0.003	0.006	0.028	0.255
	2019 (Jan-Jun)	0.006	0.006	0.006	0.004	0.003	0.004	0.020	0.036	0.193

Source: Own elaboration

Table 5.20: Sharpe Ratio: Rolling Window Optimization for Different Risk Aversion Parameters (Estimation Window: 70% Sample Size)

	Date/Risk Aversion Parameter	λ=0	λ=0.3	λ=0.8	λ=1.5	λ=2	λ=5	λ=50	λ=500	λ=5000
70 % Sample	2017	0.0061	0.0059	0.0079	0.0099	0.0104	0.0113	0.0059	0.0215	0.1269
	2018	-0.0017	-0.0022	-0.0026	-0.0035	-0.0036	-0.0042	0.0052	0.0282	0.1554
	2019 (Jan-Jun)	0.0048	0.0051	0.0050	0.0044	0.0044	0.0031	0.0191	0.0358	0.1846

Source: Own elaboration

Table 5.21: Sharpe Ratio: Rolling Window Optimization for Different Risk Aversion Parameters (Estimation Window: 80% Sample Size)

80 % Sample	Date/Risk Aversion Parameter	λ=0	λ=0.3	λ=0.8	λ=1.5	λ=2	λ=5	λ=50	λ=500	λ=5000
	2018 (Sep-Dic)	-0.0018	-0.0026	-0.0040	-0.0041	-0.0039	-0.0034	0.0071	0.0329	0.2124
	2019 (Jan-Jun)	0.0048	0.0048	0.0041	0.0035	0.0037	0.0029	0.0192	0.0379	0.1640

Source: Own elaboration

5.4.2 Performance Analysis: Treynor Ratio

In general terms, the results are the equivalent as for the Sharpe ratio analysis, in the sense that, for values of lambda below 5, the ratio figures are similar and small. For lambda values above 5, the results reveal an increasing trend. Furthermore, the largest values are shown for the risk aversion parameter of 5000. These results are independent of the sample size used in the portfolio optimization and the year of analysis. This suggests that the highest return per unit of systematic risk exposure is achieved by the portfolios with high penalization to volatility, which is an expected outcome based on the fact that this type of portfolios does not exhibit an aligned level of volatility similar to the benchmark. Thus, they are not likely to show a significant degree of correlation. Additionally, for the portfolios with levels of risk aversion below 50, negative results are more frequently detected than in the case of Sharpe ratio. This indicates that the beta determines the sign of the result. Hence, the portfolio returns correlation with the benchmark returns in these cases is negative. Finally, results per year are mostly stable for each risk aversion case. This implies that the returns of the portfolio adjusted by systemic risk remains mostly the same for different periods of time. Nevertheless, there are some outliers in different periods. These differences are more prominently for risk aversion parameters of 1.5 and 5 when computing the portfolio weights with the 60% of the sample. For risk aversion parameters of 1.5 and 2, when calculating the portfolio weights with the 70% of the sample, the results indicate that the portfolio returns and exposure to systemic risk can be highly influenced by the window of estimate used in to compute the variables involved in the computation of the ratio.

Table 5.22: Treynor Ratio: Rolling Window Optimization for Different Risk Aversion Parameters (Estimation Window: 60% Sample Size)

60 % Sample	Date/Risk Aversion Parameter	λ=0	λ=0.3	λ=0.8	λ=1.5	λ=2	λ=5	λ=50	λ=500	λ=5000
	2016 (Dic)	0.0010	0.0011	0.0019	0.0022	0.0025	0.0020	0.0057	0.0187	0.0998
	2017	-0.0096	-0.0095	-0.0118	-0.0274	-0.0387	-0.1133	0.0262	0.0691	0.3594
	2018	-0.0004	-0.0011	-0.0010	-0.0469	-0.0103	-0.0043	0.0055	0.0300	0.1574
	2019 (Jan-Jun)	0.0354	0.0354	0.0354	0.0167	0.0073	0.0041	0.0157	0.0365	0.1512

Source: Own elaboration

Table 5.23: Treynor Ratio: Rolling Window Optimization for Different Risk Aversion Parameters (Estimation Window: 70% Sample Size)

70 % Sample	Date/Risk Aversion Parameter	λ=0	λ=0.3	λ=0.8	λ=1.5	λ=2	λ=5	λ=50	λ=500	λ=5000
	2017 (Oct-Dic)	-0.0910	-0.0839	-0.2750	-1.5670	-1.2610	-0.0470	0.0391	0.0890	0.9155
	2018	-0.0074	-0.0089	-0.0105	-0.0097	-0.0078	-0.0054	0.0040	0.0239	0.2076
	2019 (Jan-Jun)	0.0164	0.0186	0.0189	0.0123	0.0093	0.0033	0.0141	0.0338	0.2099

Source: Own elaboration

Table 5.24: Treynor Ratio: Rolling Window Optimization for Different Risk Aversion Parameters (Estimation Window: 80% Sample Size)

80 % Sample	Date/Risk Aversion Parameter	λ=0	λ=0.3	λ=0.8	λ=1.5	λ=2	λ=5	λ=50	λ=500	λ=5000
	2018 (Sep-Dic)	-0.0093	-0.0087	-0.0099	-0.0072	-0.0059	-0.0036	0.0049	0.0267	0.1298
	2019 (Jan-Jun)	0.0188	0.0151	0.0103	0.0050	0.0045	0.0027	0.0133	0.0336	0.1318

Source: Own elaboration

5.4.3 Performance Analysis: Value at Risk

The results of Value at Risk (VaR), in the case of portfolio optimization under quadratic preferences and different risk aversion parameters, show that worst scenarios are most likely to be observed in portfolios optimized under low-risk aversion parameters. This is a predictable result, considering that the function to be optimized in this case penalize volatility with a low parameter. Thus, the dispersion of returns is expected to be higher than in the case of the portfolio optimization that approaches high-risk aversion agents. It is appreciated (as for the Sharp and Treynor ratio results) that the group of portfolios which have been optimized under risk aversion parameters below 5, the results are similar. The main differences are evidenced for lambda parameters equal to or higher than 50. From the perspective of changes through time in VaR results, it is shown that for the group of portfolios with lambda below or equal than 5, the most extreme results are faced in 2016, 2018 and 2019. This suggests that the distribution of returns is wider for this period of time. On the opposite side, the results for the portfolios optimized with risk aversion parameters higher or equal than 50, the results are consistent through time, which is aligned with the fact that in the optimization algorithm, the volatility measure is heavily penalized. Overall, the results in the case of solving the problem using quadratic preferences, show more volatile returns when the risk aversion parameter is low (below 5), which can lead to more extreme scenarios. The latter occurs in comparison to risk aversion parameters higher than 5, which exhibit more stable distribution of returns.

Table 5.25: Value at Risk: Rolling Window Optimization for Different Risk Aversion Parameters (Estimation Window: 60% Sample Size)

60 % Sample	Date/Risk Aversion Parameter	λ=0	λ=0.3	λ=0.8	λ=1.5	λ=2	λ=5	λ=50	λ=500	λ=5000
	2015 (Dic)	-0.0697%	-0.0697%	-0.0360%	-0.0050%	-0.0053%	-0.0047%	-0.0144%	0.0055%	0.0087%
	2016	-1.1873%	-1.1873%	-1.1815%	-1.0451%	-0.9238%	-0.6070%	-0.1117%	-0.0186%	0.0062%
	2017	-0.7035%	-0.6978%	-0.6354%	-0.5142%	-0.4217%	-0.3831%	-0.1252%	-0.0292%	0.0024%
	2018	-1.6813%	-1.6483%	-1.5979%	-1.5855%	-1.4331%	-1.0506%	-0.1471%	-0.0228%	0.0038%
	2019 (Jan-Jun)	-1.4511%	-1.4511%	-1.4511%	-1.3897%	-1.1318%	-0.6288%	-0.0840%	-0.0169%	0.0046%

Source: Own elaboration

Table 5.26: Value at Risk: Rolling Window Optimization for Different Risk Aversion Parameters (Estimation Window: 70% Sample Size)

70 % Sample	Date/Risk Aversion Parameter	λ=0	λ=0.3	λ=0.8	λ=1.5	λ=2	λ=5	λ=50	λ=500	λ=5000
	2016 (Nov-Dic)	-0.6480%	-0.6480%	-0.4550%	-0.3920%	-0.3900%	-0.2490%	-0.3960%	-0.0870%	-0.0090%
	2017	-0.8410%	-0.7655%	-0.5330%	-0.3855%	-0.3810%	-0.3028%	-0.1300%	-0.0200%	0.0000%
	2018	-1.5820%	-1.5285%	-1.4880%	-1.3480%	-1.2550%	-1.1510%	-0.1500%	-0.0200%	0.0000%
	2019 (Jan-Jun)	-1.4300%	-1.3200%	-1.3200%	-1.2300%	-1.1560%	-0.8500%	-0.1200%	-0.0200%	0.0000%

Source: Own elaboration

Table 5.27: Value at Risk: Rolling Window Optimization for Different Risk Aversion Parameters (Estimation Window: 80% Sample Size)

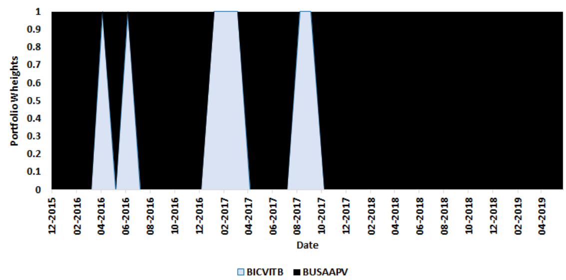
80 % Sample	Date/Risk Aversion Parameter	λ=0	λ=0.3	λ=0.8	λ=1.5	λ=2	λ=5	λ=50	λ=500	λ=5000
	2017 (Sep-Dic)	-0.006	-0.004	-0.003	-0.002	-0.002	-0.002	-0.002	0.000	0.000
	2018	-0.016	-0.016	-0.014	-0.012	-0.012	-0.009	-0.001	0.000	0.000
	2019 (Jan-Jun)	-0.014	-0.014	-0.013	-0.012	-0.011	-0.009	-0.001	0.000	0.000

Source: Own elaboration

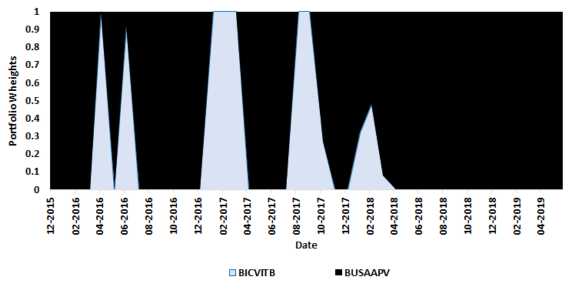
5.4.4 Allocation Analysis

In the following section, the results for different portfolio allocation are presented. The computation of the results is displayed by sample size and risk aversion parameters used in the optimization function. In general, the results show that the portfolios are highly unstable through time and heavily concentrated. As in for the GMVP and tangency portfolios analysed before, most of the portfolios are composed for less than five funds, which represent more than 90% of the allocation. For the portfolio optimized under risk aversion parameters of 0, 0.3, 0.8 and 2, the output funds with the largest weight through time are mostly unchanged; this result is stable for different sample sizes used in the optimization. The funds that obtain the largest allocation in most of the cases are: "BICVITB", "BUSAAPV" and "EEUUAPV", which are all classified as "Type 5" by the Chilean Market Commission. The three named funds have high exposure to variable income in EEUU. A result that is aligned with what should be a type of portfolio searched by an individual with a high willingness to take risk, which are more likely to choose risky assets. For portfolio composition optimized with lambda parameters of 5 or higher, the results suggest that the portfolio weights are highly sensitive to the window of estimation. By visual inspection, for the sample size of 60% of the data, it is observed that the main fund used during the period of the analysis is "BICVITB". However, as a result of the optimization, other funds appear, especially at the beginning of the period the mandatory pension funds: "HABITATC" and "HABITATD" are included. For the case of fund C, their composition is balanced between fixed income and variable income, whereas fund D is mostly invested in fixed income. When the optimization is solved with lambda equal to 10, the fund "BICVITB" still obtains most of the allocation during the period of analysis but the fund "HABITATE" obtains the second-largest allocation during most of the period. Finally, in the case of the portfolio solved with lambda equal to 50, it is observed that the major fund contributor to the portfolio through time is the fund "HABITATE", and the second-largest in the fund "SECORPI", which is classified as fund type 3 by the Chilean Market Commission.

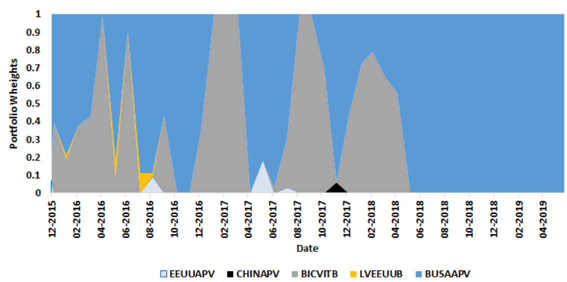
Figure 5.24: Portfolio Composition rolling window: Sample size 60% (different risk aversion parameters)



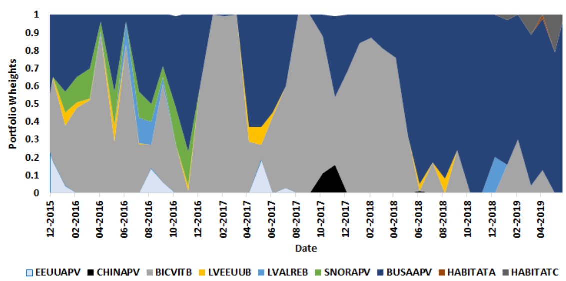
(a) $\lambda = 0$



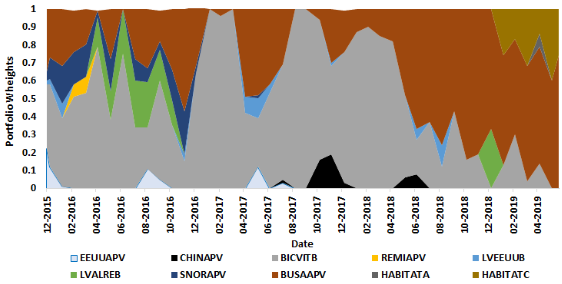
(b) $\lambda = 0.3$



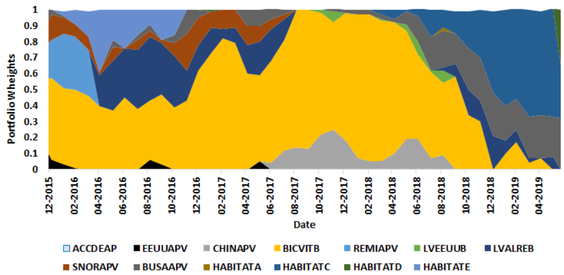
(c) $\lambda = 0.8$



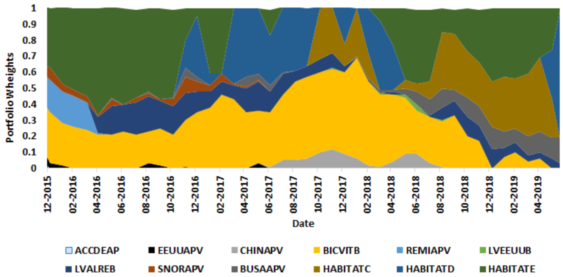
(d) $\lambda = 1.5$



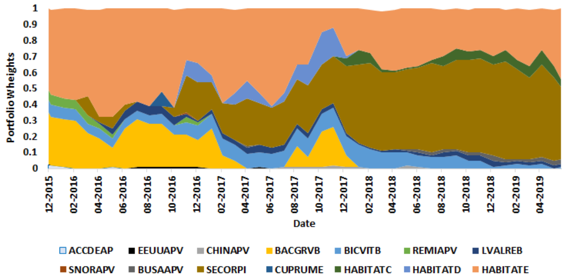
(e) $\lambda = 2$



(f) $\lambda = 5$



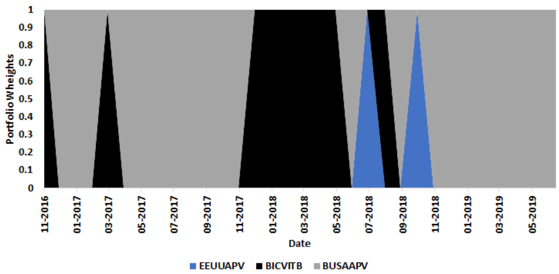
(g) $\lambda = 10$



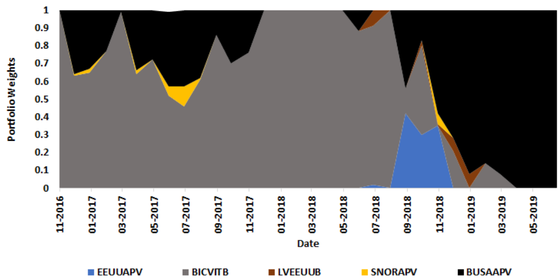
(h) $\lambda = 50$

Source: Own elaboration

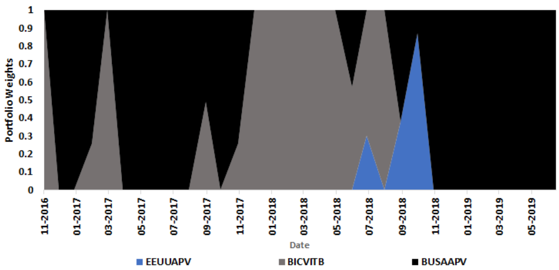
Figure 5.25: Portfolio Composition rolling window: Sample size 70% (different risk aversion parameters)



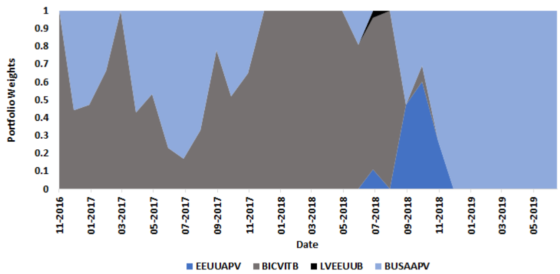
(a) $\lambda = 0$



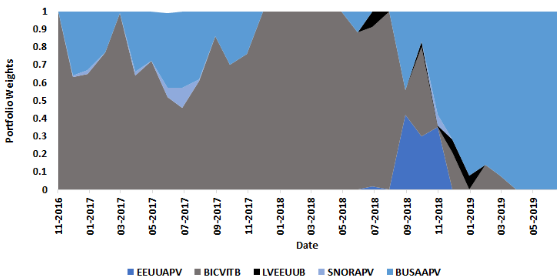
(b) $\lambda = 0.3$



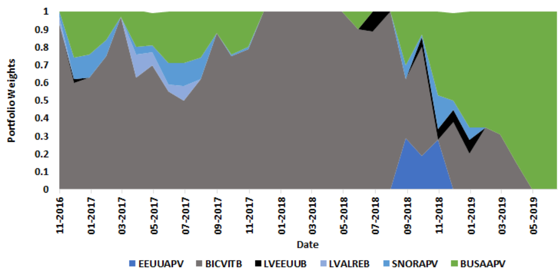
(c) $\lambda = 0.8$



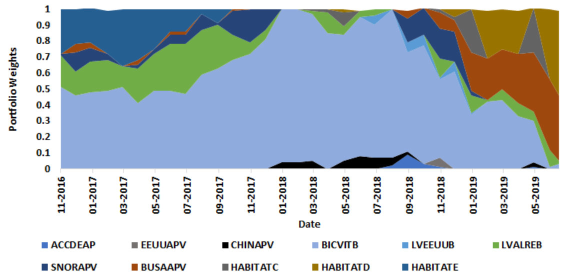
(d) $\lambda = 1.5$



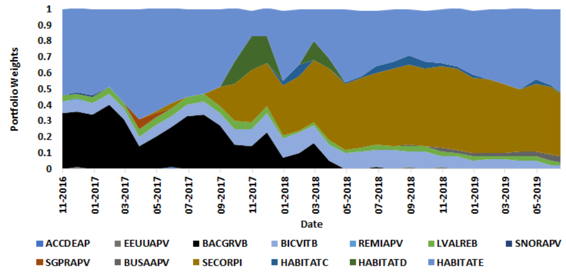
(e) $\lambda = 2$



(f) $\lambda = 5$



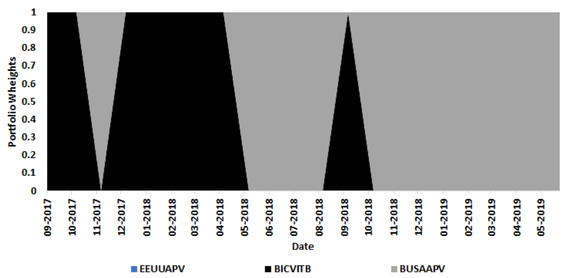
(g) $\lambda = 10$



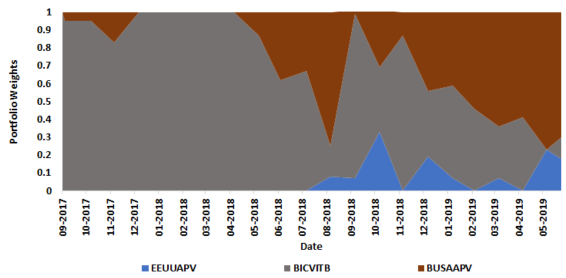
(h) $\lambda = 50$

Source: Own elaboration

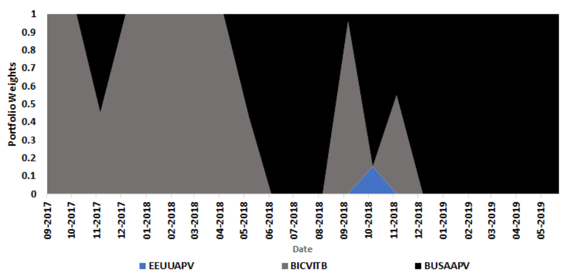
Figure 5.26: Portfolio Composition rolling window: Sample size 80% (different risk aversion parameters)



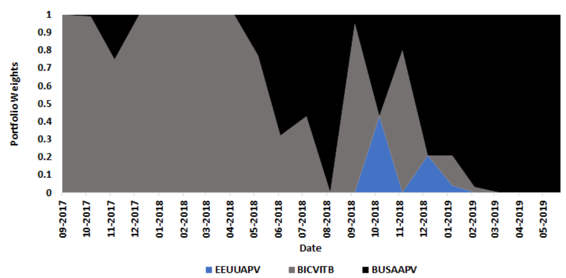
(a) $\lambda = 0$



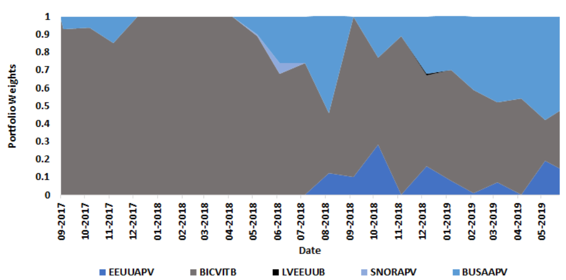
(b) $\lambda = 0.3$



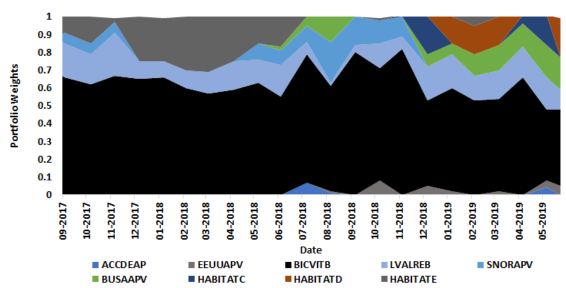
(c) $\lambda = 0.8$



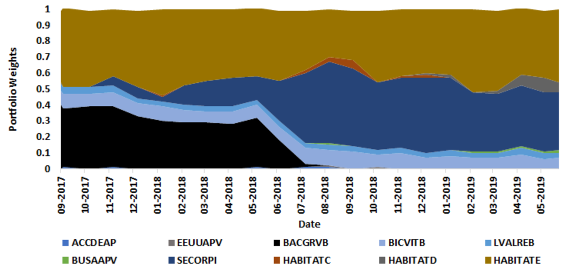
(d) $\lambda = 1.5$



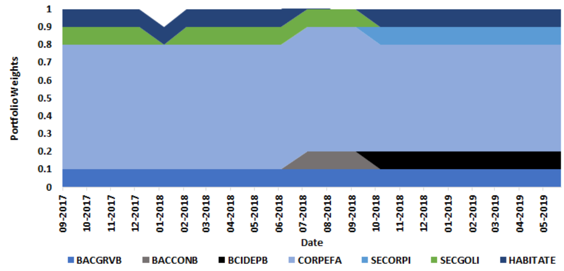
(e) $\lambda = 2$



(f) $\lambda = 5$



(g) $\lambda = 10$



(h) $\lambda = 50$

Source: Own elaboration

5.5 Hierarchical Risk Parity

In this section, the results of the computation of optimal portfolio weights under the Hierarchical Risk Parity (HRP) approach are presented. The algorithm described in section 3.3 is implemented for the optimization function described in equation 3.8. Thus, the results exposed in this section are comparable with the GMVP with short-selling constrains. Three sample sizes were used to calculate to compute the funds' allocations; these were 60%, 70% and 80% of the data available: 2245 daily returns. The descriptive statistics of portfolio's returns have shown similar characteristics in the three cases. The minimum value fluctuates between -0.01% and -0.015%, the median return variates between 0.004% and 0.003%. The main differences in the three portfolios are given by the maximum value reached by each one of them, HRP60 and HRP70 exhibit a maximum return of 0.076% and 0.072% respectively; whereas HRP80 shows a maximum value of 0.025%. The portfolios HRP60 and HRP70 exhibit positive kurtosis, with values larger than 3, which indicate that most of the data is concentrated around the median. The HRP80 portfolio shows a kurtosis of 2.1, which indicates that the distribution has in some extent, fat tails. The shape of portfolios return distribution suggests that most of the data is concentrated on the left side of the mean (positive skewness), which makes more likely to observe extreme values in the area of positive values –assuming that the true mean of the distribution is around zero. The cumulative return, in this case, indicates that when computing portfolio allocation under the samples size of 60%, 70% and 80%, one could have obtained 4.7%, 3.2% and 2.5% respectively; which compared to the GMVP cumulative return (shown in figure 5.14) represents a significant improvement in this metric. The best result of the GMVP was detected for the unconstrained portfolio when using a sample size for the estimation equal to the 60% of the 2245 daily returns. In this case, it obtained 1.8% of the cumulative return.

Figure 5.27: Portfolio Cumulative Return (different sample sizes)

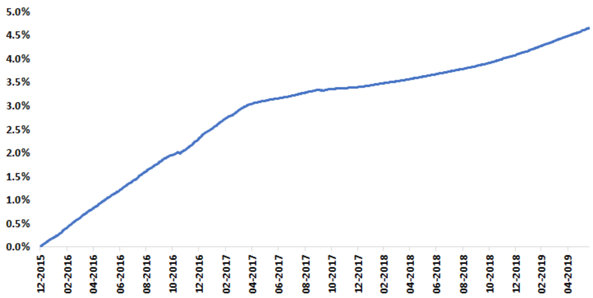


Figure 5.28: 60% Sample size

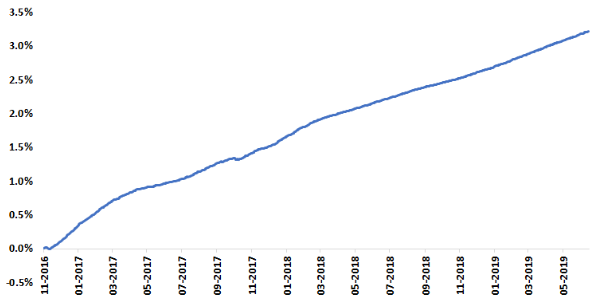


Figure 5.29: 70% Sample size

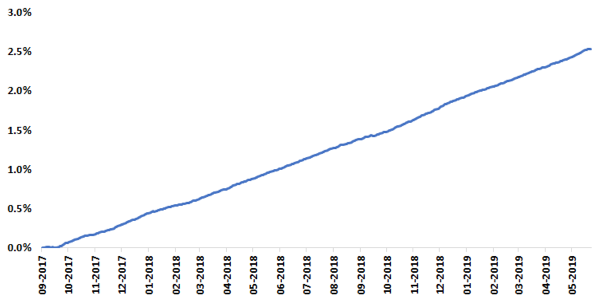


Figure 5.30: 80% Sample size

Source: Own elaboration

5.5.1 Performance Analysis: Sharpe Ratio

In this case, the Sharpe ratio analysis confirms the findings that we obtained before, in the sense that this measure is highly sensitive to the market conditions. Two extreme values are observed: the HRP70 portfolio in 2017 and the HRP 90 portfolio. In the former, the Sharpe ratio shows a minimal value of -4.75, whereas in the latter, the portfolio obtained 4.97. When comparing every specific period with the results of the GMVP estimated using the rolling window procedure, it is observed that the hierarchical risk parity method is superior for all cases but not for the portfolio estimated using 70% of the sample in 2017. The same conclusion is achieved when comparing the Sharpe ratios, of the portfolio optimization under quadratic preferences. For all the cases, except for HRP70 in 2017, the hierarchical risk parity portfolios beat shows higher Sharpe Ratios than the portfolio optimization under quadratic preferences.

Table 5.28: Sharpe Ratio Hierarchical Risk Parity Optimization (Estimation Window: 60%,70% and 80% Sample Size)

Date	HRPP 60	HRPP 70	HRPP 80
2016	1.65		
2017	0.66	-4.75	
2018	1.42	2.40	4.97
2019 (Jan-Jun)	1.42	2.16	3.31

Source: Own elaboration

5.5.2 Performance Analysis: Treynor Ratio

The Treynor Ratio for the HRP portfolios evidences a lower variability compared with previous results. Firstly, in all periods under analysis, the ratio is positive, reflecting that either the fund returns and the relation of the funds and the market were positive, or the returns and the beta were negative. Nevertheless, in both scenarios, the results indicate that the portfolio returns adjusted by systemic risk are positive. However, using the sign of the computation of the Sharpe ratios, we can conclude that for the portfolio HRP70 in 2017, the beta was negative. In all the other cases, both the beta of the portfolio and the return of the period under analysis were positive.

Table 5.29: Treynor Ratio Hierarchical Risk Parity Optimization (Estimation Window: 60%,70% and 80% Sample Size)

Date	HRPP 60	HRPP 70	HRPP 80
2016	0.347		
2017	0.128	0.137	
2018	1.464	0.476	0.297
2019 (Jan-Jun)	0.669	0.738	0.288

Source: Own elaboration

5.5.3 Performance Analysis: Value at Risk

The Value at Risk results suggest that under the HRP method, the expected worst returns are similar to those obtained for the GMV portfolios (constrained and unconstrained). Specifically, the results under HRP, belong the interval of results of GMVP with short selling constrained and without short-selling constrained. The figures indicate that there are fluctuations through periods but these are less pronounced when comparing portfolios in each period. The most extreme value is observed for the HRP 70, in 2017. In this case, the worst 5% of the daily returns can be lower than -0.0011%.

Table 5.30: Value at Risk Hierarchical Risk Parity Optimization (Estimation Window: 60%,70% and 80% Sample Size)

Date	HRPP 60	HRPP 70	HRPP 80
2015	0.0034%		
2016	0.0033%	0.0016%	
2017	0.0001%	-0.0011%	0.0012%
2018	0.0013%	0.0016%	0.0012%
2019 (Jan-Jun)	0.0016%	0.0018%	0.0018%

Source: Own elaboration

5.5.4 Allocation Analysis

From the results, two main differences arise compared to the optimization of the previous portfolio. Firstly, during most of the period under analysis, and for different sample sizes, the contribution of each asset remains relatively stable. Secondly, when more than one asset is selected, the portfolios became more diversified than in GMVP and in Tangency portfolios. For instance, in the sample that uses 80 % of the data, during most of the time, one fund account for around 50 % of the portfolio and eight other funds account for the other 50 %. Related to the specific asset contribution for each sample size, this remains mostly the same in all cases under analysis. Thus, the largest portfolio allocation is to "LVMOMAB", which is a fund type 1. The most frequent funds that are selected under the HRP algorithm (independent of the sample size used in the portfolio optimization) are: "BACCOMB","BACRENB","BACEFEB","CXCASHE" and "LARVMMB", all of them are categorized as type 1 funds by The Chilean Financial Market Commission. These kinds of funds are invested mostly in short-term debt with maturity less than or equal to 90 days. Finally, by comparing the results with the GMVP solved for the constrained case, and using the rolling window sampling method, it is noted that the main fund is the same as for HRP: "LVMOMAB". Additionally, none of the mandatory pension funds were selected as part of the portfolios under analysis.

Figure 5.31: Portfolio Composition Hierarchical Risk Parity Optimization (different sample sizes)

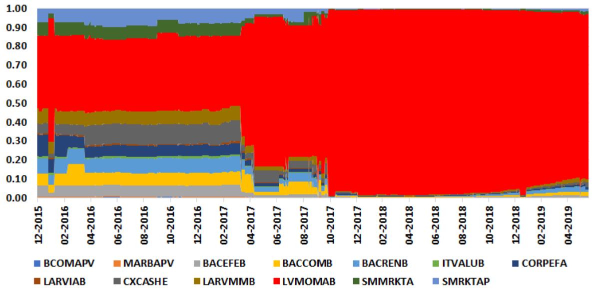


Figure 5.32: 60% Sample size

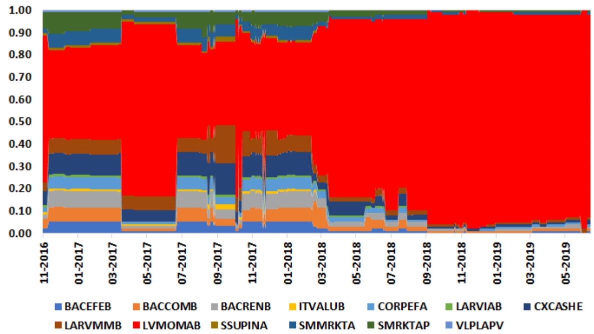


Figure 5.33: 70% Sample size

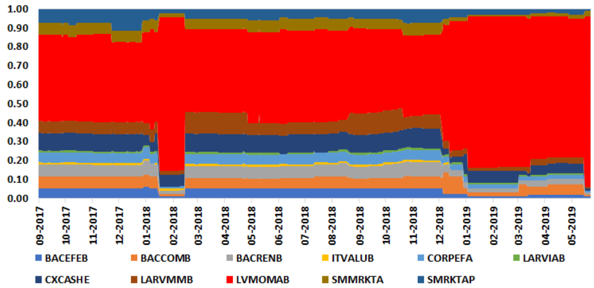


Figure 5.34: 80% Sample size

Source: Own elaboration

6 Discussion

This thesis was set forth with the purpose of describing the complexity of the portfolio allocation problem which savers in the Chilean Pension System face during the accumulation phase. Defined contribution plans are becoming a popular pension mechanism that alleviates the "pension problem" from governments, and the Chilean experience has illustrate the main challenge of this kind of pension system. However, a vital component of this saving mechanism is saver's decisions regarding their fund allocations. Nevertheless, as discussed in section 2, a country with a low level of financial literacy like Chile, the portfolio allocation is unlikely to fulfil any optimality criteria. Additionally, based on the characteristics of the system, and the variety and number of funds available to be chosen, even portfolio optimization methods could lead to wrong results. These could be unappropriated for the problem defined, or inadequate to be implemented in practice. This section will address most of the drawbacks of the portfolio optimization topics when applying them in the Chilean context. Additionally, methodological validity concerns will be outlined in section 6.2. Finally, practical applications of the data found and further extensions of this research will be described in section 6.3.

6.1 Portfolio Optimization Issues in the Chilean Pension System

The Modern Portfolio Theory (MPT) defines the portfolio optimization problem through a quadratic function with linear constraints. It specifically defines a trade-off between return and volatility. However, other dimensions of the returns distribution are omitted when the optimization problem is identified. As described in section 4, most of the assets used in the optimization procedures exhibit kurtosis and skewness with values that are different as those assumed in the normal distribution. Nevertheless, these attributes are not included in the function used to solve the portfolio allocation problem. But it may be reasonable to assume that investors likewise consider higher moments of the distribution when selecting funds (Harvey et al., 2010). This fact can lead to misinterpretation when analyzing the results, given the assumption that agents only include funds returns and variance when building portfolios.

In addition, when implementing MPT in practice, several problems arise. Firstly, the method assumes that returns and variances assets can be accurately estimated. A relevant component of the Markowitz optimization is the "plug-in" strategy, which uses the estimated variances and mean of returns in the mean-variance optimizer to calculate the optimal portfolio weights. When the data used in the estimation of these parameters follow a normal distribution, the accuracy of the estimated mean and variance can be achieved with relatively low sample size. However, when the data show fat-tailed distributions and show differences with a normal distribution, accurate results can be achieved just by using a large sample size. Furthermore, the literature suggests that in order to obtain reliable estimates for median a variance, when the data slightly deviate from a normal distribution, at least 1.000 years or 12.000 monthly returns are needed (DeMiguel and Nogales, 2009). It is also essential to link the optimization logic, stated in the mean-variance problem, with the lack of accuracy when computing the parameters used to solve the problem. The Markowitz method, select assets, that deliver the highest expected return, with low variance and negative correlation with other assets used in the optimization. These characteristics are the ones that make estimation inputs biased. For this reason, Michaud (1989) described the minimum variance portfolio optimizer as: "the estimation error maximizer algorithm". An additional drawback of the mean-variance efficient portfolios was evidenced when computing the portfolio composition through time presented in section 5. In almost, all the results under the Markowitz approach, the portfolio exhibited a high level of concentration, and high sensitivity to changes in the estimated parameters. The root of this problem can be attributed to differences in expected returns (Best and Grauer, 1991) or specific characteristics of the covariance matrix (Lopez de Prado, 2016). As described in the methodology section, the quadratic programming method requires the inversion of the covariance matrix, under the condition that all eigenvalues must be positive. However, despite the fact that the inversion procedure can be done in practice, the result could be ill-conditioned (prone to significant errors). The mathematical concept that described the sensitivity of a function for when changes to the inputs are done is "condition number" (Won et al., 2013). The condition number is the ratio between the maximum and minimum eigenvalue in a specific matrix. Thus, large condition numbers indicate high sensitivity to a function parameter changes. As Lopez de Prado (2016) shows, the condition number of a covariance or correlation matrix increases when: correlated investments are included and/or a large number of assets relative to the number of returns is used to compute the matrix. The results can be even more unstable when a large number of highly correlated assets is used to compute the matrix. When analyzing the characteristics of the funds selected to solve the portfolio optimization problem, one can link at least two of the conditions stated by Lopez de Prado (2016). Firstly, a large proportion of funds is used relative to the sample size. The number of funds used was 128, and the sample size was 2245. Secondly, when checking the correlations of the 128 funds, it is observed that the mandatory pension funds are highly correlated as a consequence of the incentive that they have to replicate other funds strategies, which lead to a so-called "herd behaviour" (Schlechter et al., 2019). Thus 25 funds out of 128, show a high level of correlation. Finally, the poor out-of-sample performance (outside the sample period used to estimate input parameters) of portfolio optimization under quadratic functions is a repetitive result documented in the literature; this result has also been observed in the portfolio performance analysis presented in section 5.

Overall, the problems associated with the Markowitz portfolio optimization can have detrimental consequences when designing a portfolio allocation strategy with retirement purposes, in the Chilean context. The instability of the results implies drastic changes in portfolio composition throughout time, which leads to buy or sell funds aggressively. This continuous portfolio re-balancing can lead to losing the tax benefit given to savers that choose APV funds. This tax deduction is conditional on not withdrawing funds before retirement age. This problem can be even worst, based on the fact that some of the APV funds have penalties for early liquidations. Additionally, the use of mandatory pension funds, include an additional constrain as savers get closer to the retirement age. These constrain limit the use of some risky funds based on the saver's age. Thus, there is a limit on the allocation that is feasible when accounting for the mandatory investments vehicles. Finally, the Markowitz type portfolios assume that the pool of assets used in the problem is perfectly divisible, which means that one could buy, for instance, half of an APV fund unit. This inconvenience can potentially be more prejudicial for savers with low saving amounts. That is the case of the fund "BACEFEB", which was included in the portfolio optimization results in several cases in section 5. This fund has a nominal value around 700.000 CLP (770 EUR), which make difficult for low-income savers to build a diversified portfolio that includes assets like these.

The alternative to the Markowitz optimization approach explored in this thesis was the Hierarchical Risk Parity algorithm, which partly mitigates some of the problems evidenced under the Markowitz portfolio optimization. Specifically, HRP compared to Markowitz optimization, delivers improvements in three aspects: 1) The results of the portfolio allocation exhibit a smooth pattern thoughout time, that is to say, the portfolio composition remains mostly unchanged. And observed changes, are in most of the cases, minimal portfolio changes. 2) The selection of assets gives a higher level of diversification, compared with GMVP. However, this result does not hold for all the period under analysis. 3) The out-of-sample performance is superior in most of the indicators, compared to other allocation strategies. These results, applied to the Chilean Pensions System, provide insights regarding the feasibility of generating a portfolio strategy that beat the GMVP, in a context where portfolio re-balancing is heavily penalized both through high transactions costs and through the opportunity cost of losing tax benefits. Furthermore, this kind of optimization may be desirable in a context were savers face a low level of financial literacy. In previous studies, a low level of financial literacy has been linked to limited involvement in investment choices and portfolio re-balancing, which eventually conducts to poor investment choices performance (Abreu and Mendes, 2010). For that reason, when choosing a possible allocation strategy it is appropriate to recognize the fact that in a low financial literate environment, the activity of portfolio re-balancing is unlikely to happen.

6.2 Validity Concerns

On the review of the results, one should be aware that these are susceptible to the methodology and assumptions used in the computation. For instance, the length of the in-sample and out-of-sample periods have been defined a priory, without theoretical justifications. However, different sample sizes have been considered to verify the sensitivity of the results to the sample used. The result of this sensitivity test reflects that, in most cases, the results are highly sensitive to the period of analysis. Additionally, the frequency of portfolio was chosen arbitrarily, assuming that savings in the mandatory saving scheme are added once a month to private accounts. Nevertheless, rebalancing in a context where transaction costs are relevant, and liquidation penalties exist are critical variables that can condition the results. However, there are also benefits of increasing the frequency of rebalancing, such as materializing capital gains or improving the risk control in the portfolio allocation (Bernstein, 1996). Furthermore, the extension of this research can address the question regarding the optimal frequency of rebalancing for savers under the pension funds scheme. Regarding the frequency for computing funds return, this was done on a daily basis following the approach developed by Hautsch et al. (2013). In their article, the authors described that for high-dimension portfolio optimization, the use of high-frequency data could improve the results. However, the literature is broad in this topic, and the results presented in section 5 can be affected by choosing a different frequency for computing returns.

Additionally, under the Markowitz framework, the assumption of agents quadratic utility can be susceptible to questioning. The main drawback of this functional form is the property of increasing absolute risk aversion, which intuitively implies that as wealth rises the risk-taking appetite decreases, which contradicts everyday experience. Moreover, this function exhibits "ultimate satiation", that is to say that beyond a specific point, return or money affects negatively the function output (Sarnat, 1974). Thus, the function must be constrained to a range of possible returns. In empirical testing, Wippern (1971) used the Sharpe-Lintner market model to show that returns that go beyond 1.3 standard deviations have been identified to provide negative marginal utility to investors.

In relation to the performance and risk measure indicators used in this thesis, a number of practical limitations can be identified. Specifically, the definition of sample size was chosen without previous justifications. For instance, to compute the Sharpe ratio, a sample size of 250 trading day was selected to estimate both, standard deviation and average daily return. However, the results can be highly sensitive to the chosen sample. Additionally, for the measures used to quantify portfolio performance and risk, there are also theoretical concerns. In the case of Sharpe ratio, the concerns have been focused on the measure of risk, used as the divisor. The measure of risk is normally computed by the standard deviation. This measure is supposed to be comparable between returns time series under analysis nevertheless, to make this measure meaning for all returns time series, standard deviation needs to have constant statistical characteristics (variance, skew and kurtosis). Moreover, the standard deviation is supposed to reflect data dispersion, but this is true when the data distribution is parameterizable, but when the distribution is non-stationary or non-parameterizable, this measure does not deliver any information (Harding, 2002). A similar conclusion emerges for Treynor ratio, as this measure computes the portfolios return for a unit of systemic risk, to quantify systemic risk the beta of the portfolios with respect to a benchmark was used. In relation to this, the literature has documented the so-call "reference-day risk", when computing beta. This risk arises when choosing specific estimate windows for this parameter, which can vary significantly based on the sample used to estimate it (Baker et al., 2016). Additionally, the selection of the benchmark to compute the sensitivity of portfolios to market changes was done arbitrarily, without theoretical support.

Finally, the use of Value-at-Risk as been questioned by academics and practitioners because this measure does not fulfil all the properties to make it a coherent risk measure. In this regard, a coherent risk measure is a function that satisfies four properties: 1) Monotonicity (2) Homogeneity 3) Translation invariance and 4) Sub-additivity. In the case of VaR the property of sub-additivity is not respected. As the VaR of a combined portfolio can be larger than the sum of the VaRs of its components (Artzner et al., 1999). Furthermore, VaR can provide a false sense of confidence, as it ignores the so-called "tail risk", which is the risk associate to extreme events. Additionally, it does not establish a measure of how much can be lost in the worst scenario. It defines a threshold where worst scenarios can be observed, but once that happens, this measure does not assess the size of the losses (Yamai and Yoshiba, 2005).

6.3 Practical Applications and Further Extensions

In a defined contribution pension system, saver have the entire responsibility of selecting funds and choosing investment strategies that fit them the best, the degree of information regarding investment vehicles characteristics, performance measures and investment costs. The ability of savers to assess that information is a needed condition for screening funds and select the best alternatives. In the Chilean context, these characteristics are not totally fulfilled (Landerretche and Martinez, 2013). As a result, the rise of financial advisors has satisfied the savers need for theoretically improve savings rate of return and participate actively in their pension fund management. In this context, the results exposed in section 5 can be used as a first attempt to understand the consequences and limitations of use optimality criteria to select pension funds. From a regulatory perspective, these results can be useful to assess the possible fund selection of savers based on different utility function assumptions as well as how fund management companies compete in specific segments of the risk-return spectrum. Furthermore, a relevant question that can be addressed for future research is whether; there is some level of segmentation of APV funds when serving segments of the market. That is to say when solving portfolios based on risk aversion measures, are there specific APV funds providers better performance associated with groups of funds with particular risk profiles? Additionally, the results show that the mandatory pension funds, in most of the simulations, are not included in the optimal portfolios. Especially the riskiest funds, A and B, were included in an optimal solution. However, the funds D and E were included in the portfolio of minimum variance and when solving portfolio weights under quadratic preferences with a high level of risk aversion. These indicate that the constrain of limited mandatory savings to five funds can be costly for some group of savers. Moreover, between the conclusions of the "Bravo" commission was mentioned the lack of competition between AFP funds and the entry barriers that newcomers AFP face (Barr and Diamond, 2016). On this basis, a possible solution to be explored is the inclusion of APV funds in the universe of funds that can be selected under the mandatory saving scheme. As the results suggest, some savers might be better off by using APV funds, instead of using forced saving in investment vehicles that do not match their risk-return preferences.

Finally, in a context of varied pensions funds options and savers characteristics, the use of financial advisors seems to be an alternative to savers' informational needs. However, there are specific challenges from the governance point of view and the incentives that fund companies could have when measuring their relative performance based on optimal portfolio optimization results; based on the fact that some funds systematically dominate the risk-return relation, when this relation is analyzed through time. As a consequence, the majority of market participants have no incentives to provide information regarding funds performance, as they underperform most of the time. When optimal portfolio weights are analyzed, few funds are included as part of portfolios.

In this regard, independent financial advisors can provide guidelines to savers about which funds fit them best. The active portfolio allocation under the Chilean pension scheme could be specially based on the characteristics of some population groups. As indicated in the literature review, the level of financial literacy and financial choices are strongly influenced by sociodemographic characteristics. Specifically, certain inequalities based on gender, pertinence to minorities, socioeconomic status, etc. Can lead specific population groups to be more likely to undertake wrong investment choices, which can mean to choose risky assets when they are risk-averse or low-risk assets when they have high-risk tolerance. Nevertheless, the advisory activity must be regulated. This thesis defines an option to provide financial advise based on quantitative measures and optimization criteria. However, as general conclusion from the exercises described in section 5, the regulation of for pension funds switching or financial advisory must consider the review of at least the following aspects: the methodology which is used to generate different allocation strategies, the backtesting assumptions implied when testing allocation strategies, assumptions used in the estimation of the parameters, between other variables.

Overall, the regulation must prioritize the standardization of methods used to define allocation rules. As it is observed in the results for quadratic optimization problems, these can be sensible to the window of estimation. In addition, the role of institutional investors in the Chilean capital market is crucial. As previous literature has documented, fund switching advises provided for financial counsellors, has generated massive funds flows, and aggressive asset buys and sells, which has affected asset prices and increases market volatility. Additionally, AFP managers prioritize liquid assets in their portfolios, as clients rebalancing is expected (Da et al., 2018). In this sense, portfolio allocations that are stable throughout time and based on savers risk profile can mitigate some of the problems of massive and coordinate funds switching. From a systemic perspective, regulators should also consider when approving new funds, the contribution of those to the overall system performance. For a period of four years, from July of 2015 to July 2019, around 100 new APV funds were included in the system. However, it is not clear if these new products are offering more of the same, less of the same or if they actually improve the overall performance of the system. The exercise performed in section 5, provides insights about the dynamic contribution, of funds under different optimization schemes. All in all, the algorithms under consideration choose a limited set of funds in most of the cases, which provides evidence that based on the risk-return relation, optimal values can be achieved with a limited set of funds.

Concerning the further extensions that can be explored in future research projects, these could be focus on relaxing the assumptions used in the different experiments described in point 5. For instance, the optimization problem could be solved by including transaction costs, using different utility functions, and adding another kind of assets available in the Chilean financial market. This can improve the accuracy of the result and could extend the conclusion regarding which assets should be chosen by which types of individuals. Additionally, the method of Hierarchical Risk Parity (HRP), could also be applied in the portfolio optimization under quadratic preferences, changing risk aversion parameters. This could extend the analysis by describing stable portfolios, for savers with different willingness to accept risks. Moreover, to understand the effect of pension funds limit regulations, in the optimization results the exercises described in part 5 can be applied in other pension fund system with different investment limits. Thus further research can compare the effect of restrictions in various defined contribution pension schemes.

7 Conclusion

This paper has described the complexity of the problem that savers in the Chilean pension system face when dealing with one of the most critical economic choices of their life: saving to retirement.

The case of Chile represents a unique context to analyze portfolio strategies that can be implemented under optimality criteria. Firstly, the main driver of the future pension payouts is the private contributions done during the savers working life. Secondly, the number of funds available make unfeasible for individuals to assess the risk-return performance of all the entire pool of assets, throughout time. Finally, the low levels of financial literacy that the country has shown in international test make it unrealistic to assume that savers are able to take optimal choices about saving strategies (Landerretche and Martinez, 2013).

At first, it was illustrated the characteristics of the Chilean pension system, and the different investment vehicles available as saving options for retirement. The system is structured by three pillars, these are solidarity, mandatory and volunteer. In this thesis, the portfolio allocation problem was solved by considering, funds that belong to the mandatory and volunteer pillars. Under the mandatory scheme, savers have the possibility of choosing between five founds whit different risk exposure. Whereas in the volunteer scheme, there are more than 250 funds that can be selected. Under the volunteer scheme, savers can obtain tax benefits in the function of their individual income tax and the amount saved per month. Nevertheless, these tax benefits can be only obtained without withdrawing resources before retirement.

The literature on the subject pension savings choices describes that Chilean workers have difficulties when assessing the information provided by the AFPs, in topics such as the amount saved, fees, types of funds, etc. Additionally, financial educations have been identified as the main driver to participate actively in investment pension activities. Such as pension fund selection, or fund company changes. Furthermore, specific segment of the population with higher levels of education, income and financial literacy have been identified have a high level of involvement in activities such as pension funds switching and the use of volunteer pension funds (Arenas et al. (2006), Mitchell et al. (2007), Landerretche and Martinez (2013) and Berstein and Ruiz (2005)).

All in all, the system characteristics, together to the users' profile, make the exercises of identifying optimal portfolio worth to be explored. The Markowitz portfolio theory was selected to analyze the portfolio composition under optimality criteria. Under this framework, some results are expected before the algorithm is implemented. As described in section 3 (methodology), the performance of the portfolio optimized, allowing funds short-selling must be superior to the case were short-selling is not allowed. And the tangency portfolio must exhibit higher returns than the GMVP, as by definition the first one delivers the highest Sharpe ratio, by selecting high risk-return funds whereas the second one is build-up to generate low-risk portfolios, by choosing low risk-return funds.

Under this framework, assumptions such as no transaction costs, assets divisibility and perfect capital markets, were considered. To compute model parameters in the optimal portfolio algorithm implementation, three sample sizes were defined; these were 60%, 70% and 80% of the data available (see section 4). At first, the optimization problem was stated, assuming a static approach. In that case, the portfolio was optimized once, and the out-of-sample performance was analyzed as if there was no changes on it. In this case was also included the naive strategy (1/n portfolio). As a result, the portfolios were highly affected by market changes, and in some cases performed in a counter-intuitive way. For instance, the constrained tangency portfolio exhibited better performance in the unconstrained case. However, for all risk-based metric, the EW portfolio obtained the worst performance, as a result of being the most exposed to volatility changes.

The static optimization was extended by including a dynamic approach. Using the "rolling window" sampling method (see section 5 analysis), optimal portfolios were computed as they were re-balanced every 25 days. In this case, notable improvements were observed compared to the static case. This, in a sense, that the hierarchy of the performance is respected for all sample-sized and portfolio types. Specifically, unconstrained portfolios performed better than constrained ones, and tangency portfolios exhibit higher returns than GMVP. This result remains the same for risk-return measures (cumulative returns, Sharpe ratio and VaR). Nevertheless, for the performance measure based on the benchmark, the results were highly volatile. This can be attributable to portfolios exposure to systemic risk.

As state in the background section, it is presumable that agents preferences for undertaking risk are different. For that reason, the portfolio analysis was extended by assuming individuals with quadratic preferences and different levels of risk aversion. The results show that the portfolios optimized under low-risk aversion measures delivers better performance measures (Sharpe ratio and cumulative returns), compare to those optimized with high levels of risk aversion.

Finally, as a result of the AFP funds limit set up, it is observed a high correlation between a large proportion of the assets used in the optimization. This has been attributed as a consequence of AFP funds limits set up, and minimum yield requirements. This phenomenon has been described in the literature as "funds herd effect". The "herd effect", evidenced for AFP funds affect the properties of the variance and covariance matrix used by Markowitz optimization methods. Specifically, when assets are highly correlated the matrix is described as "ill-conditioned "(prone to significant errors). This problem, generate unstable portfolios which require extreme dynamic rebalancing to remain optimal throughout time. However, this portfolios characteristic is a non-desirable under the Chilean pension savings scheme. This action implies losing tax benefits and incurring in transaction costs.

As a way to lead with these constraints, a method that assurance portfolio stability trough time was selected. The Hierarchical Risk Parity algorithm has been shown in the results, that deliver portfolios with several attributes that can be described as "tailor-made" for the Chilean context. These are portfolios stables though time, diversified and with relatively high out-of-sample performance.

This study has shown, that in the context of private pension plans, the portfolio optimization algorithms, are a powerful tool that can be used by savers to increase the risk-adjusted performance of their allocations. However, this research also highlights the difficulties of implement this kind of algorithms in practice. There are still open questions to be addressed in this topic of research. Especially in topics related to regulations applied to financial advisors that eventually use this kind of methodologies, to competition of funds providers for specific segments of the risk-return spectrum and the incentives that fund managers have to provide portfolio optimization results whenever these are not favourable to them. Finally, in most of the results less than 5 funds were selected as the optimal allocations. These results open the debate about whether or not more funds in the system improve savers allocations.

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Appendix

A Omitted tables

A1 Chilean Investment Regime

Table A1.1: Limits on structural investments (as percentage of the value of the fund)

	A fund		B fund		C fund		D fund		E fund	
Instrument	Min.	Max.	Min.	Max.	Min.	Max.	Min.	Max.	Min.	Max.
Issued by the General Treasury of the Republic, Central Bank of Chile, Ministry of Housing, Recognition bonds and other government securities.	30%	40%	30%	40%	35%	50%	40%	70%	50%	80%
Shared Limit: Foreign instruments and indirect investment abroad through investment and mutual funds	Minimum: 30% of VF (A+B+C+D+E); Maximum: 30% of VF (A+B+C+D+E)									
Limit per fund: Foreign instruments and indirect investment abroad through investment and mutual funds	45%	100%	40%	90%	30%	75%	20%	45%	15%	35%
Investment in foreign currency without exchange rate coverage	50% ofinvestment									
Restricted securities: low liquidity, less than BBB and less than 2 ratings.										
Variable income (maximum limit): [National securities + foreign securities] if they are capital + other public-offering instruments that are capital instruments under the supervision of the regulator.	80%		60%		40%		20%		5%	
	Fund A > Fund B > Fund C > Fund D > Fund E									
Variable income (minimum limit)	40%		25%		15%		5%		-	
Alternative assets: Real estate assets, private capital, private debt, infrastructure, etc.	5%	15%	5%	15%	5%	15%	5%	15%	5%	15%

Source: Investment Regime applicable to Chilean pension funds since November 2017, Superintendecia de Pensiones. And Schlechter et al. (2019).

A2 Type of Mutual Funds in Chile and Classification

Table A2.1: Types of Mutual Funds in Chile Based on the Portfolio Composition

Fund Type	Description
1	Short-term debt with maturity less than or equal to 90 days
2	Short-term debt with maturity less than or equal to 365 days
3	Medium and long-term debt with maturity larger than 356 days
4	Mixed (combination of the previous three categories plus variable income)
5	Mixed (combination of the first three categories but at least 90% of variable income)
6	Free investment (The investment limits are defined in the fund policies, and does not match any of the other cathegories)
7	Structured (seek a previously determined return fixed or variable after a specific period of time, through guaranteed investments).

Source: Mutual Funds Types Classification by Issuer, Financial Market Commission of Chile.

Table A2.2: Classification of Mutual Funds in Chile Based on the Portfolio Composition

Bloomberg ID	Company	Type	Bloomberg ID	Company	Type
ACCDEAP	BANCOESTADO	Type 5	CXCASHE	Larrain Vial	Type 1
RTAEMAP	BANCOESTADO	Type 3	LARVMMB	Larrain Vial	Type 1
BCOMAPV	BANCOESTADO	Type 3	LVMOMAB	Larrain Vial	Type 1
MARBAPV	BANCOESTADO	Type 3	LARVIPB	Larrain Vial	Type 5
ANACAPV	BANCOESTADO	Type 5	LVENFOB	Larrain Vial	Type 6
EEUUAPV	BCI	Type 5	SNORAPV	SANTANDER ASSET MANAGEMENT	Type 5
GTITAPV	BCI	Type 5	SGLDAPV	SANTANDER ASSET MANAGEMENT	Type 5
AMLATBE	BCI	Type 5	SLATAPV	SANTANDER ASSET MANAGEMENT	Type 5
CHINAPV	BCI	Type 5	SANCAPV	SANTANDER ASSET MANAGEMENT	Type 5
BCIGAHB	BCI	Type 3	SANAAPV	SANTANDER ASSET MANAGEMENT	Type 5
GESGDAP	BCI	Type 5	SANBAPV	SANTANDER ASSET MANAGEMENT	Type 5
EMGLAPV	BCI	Type 5	SGPRAPV	SANTANDER ASSET MANAGEMENT	Type 5
BACGRVB	BCI	Type 3	SASIAPV	SANTANDER ASSET MANAGEMENT	Type 5
RETNOAP	BCI	Type 3	SMEMAPV	SANTANDER ASSET MANAGEMENT	Type 5
BACPERB	BCI	Type 3	SANBLAP	SANTANDER ASSET MANAGEMENT	Type 3
BACCONB	BCI	Type 3	RETTOTA	SANTANDER ASSET MANAGEMENT	Type 3
BCIDEPB	BCI	Type 3	HIPDAPV	SANTANDER ASSET MANAGEMENT	Type 3
BCINEGB	BCI	Type 3	RENTAPV	SANTANDER ASSET MANAGEMENT	Type 3
BCFRONB	BCI	Type 2	SSUPINA	SANTANDER ASSET MANAGEMENT	Type 2
BACEFEB	BCI	Type 1	SMMRKTA	SANTANDER ASSET MANAGEMENT	Type 1
BACCOMB	BCI	Type 1	SMRKTAP	SANTANDER ASSET MANAGEMENT	Type 1
BACRENB	BCI	Type 1	SARCAPV	SANTANDER ASSET MANAGEMENT	Type 5
SEBURBE	BCI	Type 5	SACHAPV	SANTANDER ASSET MANAGEMENT	Type 5
GGD5AP	BCI	Type 5	SACSAPV	SANTANDER ASSET MANAGEMENT	Type 5
BICVITB	BICE	Type 5	SUDMIXB	SCOTIA	Type 5
BICBESB	BICE	Type 5	SMEDPLB	SCOTIA	Type 3
REMBAPV	BTG PACTUAL	Type 3	BUSAAPV	SCOTIA	Type 5
REMIAPV	BTG PACTUAL	Type 3	EUROAPV	SCOTIA	Type 5
CONMESE	CONSORCIO	Type 5	BHFACCB	SCOTIA	Type 5
CONMTAE	CONSORCIO	Type 5	BHFREMXB	SCOTIA	Type 5
IVSTORE	CONSORCIO	Type 5	LATAAPV	SCOTIA	Type 5
IVSXEQE	CONSORCIO	Type 5	BHFFUTB	SCOTIA	Type 3
TOPDIVB	ITAU	Type 5	AACLAPV	SCOTIA	Type 5
BRAACTB	ITAU	Type 5	ASPAAPV	SCOTIA	Type 5
BOSWORB	ITAU	Type 5	BFAMAPV	SCOTIA	Type 3
EMMALFA	ITAU	Type 5	RTAOPAP	SCOTIA	Type 3
BOSEMEQB	ITAU	Type 5	VLPLAPV	SCOTIA	Type 2
BOSDOLB	ITAU	Type 5	ACNAAPV	SCOTIA	Type 5
ITVALUB	ITAU	Type 2	SEGLOIA	SECURITY	Type 5
CORPEFA	ITAU	Type 1	SECEQTI	SECURITY	Type 5
BOSNEQB	ITAU	Type 5	SECBALI	SECURITY	Type 5
LVEEUUB	Larrain Vial	Type 5	SECEMMI	SECURITY	Type 5
LARVLAB	Larrain Vial	Type 5	SECVALI	SECURITY	Type 5
LVALREB	Larrain Vial	Type 5	SECORPI	SECURITY	Type 3
LARVESB	Larrain Vial	Type 5	SECGOLI	SECURITY	Type 3
LARRHIB	Larrain Vial	Type 3	SECFRSI	SECURITY	Type 3
LARVISPB	Larrain Vial	Type 6	SCHILSI	SECURITY	Type 5
LVPROTB	Larrain Vial	Type 6	EAPROEB	ZURICH ASSET MANAGEMENT	Type 5
LARVACB	Larrain Vial	Type 3	EAPROCB	ZURICH ASSET MANAGEMENT	Type 5
LARVIAB	Larrain Vial	Type 2	EAPROAB	ZURICH ASSET MANAGEMENT	Type 5
LVASIAB	Larrain Vial	Type 5	EUASIAB	ZURICH ASSET MANAGEMENT	Type 5
			EUCH18B	ZURICH ASSET MANAGEMENT	Type 5

Source: Mutual Funds Types Classification by Issuer, Financial Market Commission of Chile.

A3 Descriptive statistics

Table A3.1: Descriptive statistics of daily returns of APV and AFP funds.

Bloomberg ID	Minimum	1st Quartile	Media	Mean	3rd Quartile	Maximum	Std. Deviation	Kurtosis	Skewness
ACCDEAP	-4.93%	-0.31%	0.06%	0.04%	0.43%	3.69%	0.70%	4.796	-0.504
RTAEMAP	-2.03%	-0.24%	0.01%	0.02%	0.28%	4.25%	0.45%	5.428	0.393
BCOMAPV	-0.42%	0.00%	0.02%	0.02%	0.04%	0.65%	0.05%	21.464	0.173
MARBAPV	-0.34%	0.00%	0.02%	0.02%	0.04%	0.52%	0.04%	21.403	0.195
ANACAPV	-5.66%	-0.35%	0.00%	-0.01%	0.36%	7.27%	0.72%	10.121	-0.164
EEUUAPV	-4.36%	-0.33%	0.07%	0.05%	0.49%	4.70%	0.77%	2.681	-0.354
GTITAPV	-3.94%	-0.33%	0.04%	0.05%	0.44%	4.73%	0.74%	3.303	-0.212
AMLATBE	-5.96%	-0.53%	0.00%	0.00%	0.56%	5.19%	0.97%	3.104	-0.300
CHINAPV	-3.74%	-0.43%	0.02%	0.03%	0.51%	4.83%	0.81%	1.623	-0.164
BCIGAHB	-0.64%	-0.01%	0.03%	0.02%	0.06%	0.99%	0.07%	22.539	0.226
GESGDAP	-0.65%	-0.05%	0.02%	0.02%	0.09%	0.89%	0.13%	3.065	0.098
EMGLAPV	-5.12%	-0.38%	0.01%	0.01%	0.41%	4.86%	0.69%	4.527	-0.351
BACGRVB	-0.60%	0.00%	0.03%	0.02%	0.05%	0.84%	0.06%	21.839	-0.093
RETNOAP	-0.94%	0.00%	0.02%	0.02%	0.05%	0.71%	0.06%	38.549	-1.787
BACPERB	-0.46%	0.00%	0.02%	0.02%	0.05%	0.64%	0.05%	20.447	-0.360
BACCONB	-1.31%	0.00%	0.02%	0.02%	0.04%	0.58%	0.06%	160.624	-6.169
BCIDEPB	-2.20%	0.00%	0.02%	0.02%	0.04%	0.48%	0.07%	477.140	-16.633
BCINEGB	-2.34%	0.00%	0.02%	0.02%	0.03%	1.22%	0.09%	260.329	-10.026
BCFRONB	-2.66%	-0.30%	0.00%	0.02%	0.33%	4.33%	0.55%	4.434	0.299
BACEFEB	-0.15%	0.01%	0.01%	0.01%	0.02%	0.11%	0.01%	28.020	0.623
BACCOMB	-0.13%	0.01%	0.01%	0.01%	0.02%	0.14%	0.01%	22.148	0.777
BACRENB	-0.10%	0.01%	0.01%	0.01%	0.01%	0.11%	0.01%	14.089	1.574
SEBURBE	-6.42%	-0.35%	0.00%	0.00%	0.39%	7.64%	0.75%	11.120	-0.234
BICVITB	-3.83%	-0.23%	0.08%	0.05%	0.40%	3.09%	0.66%	3.879	-0.622
BICBESB	-5.57%	-0.45%	0.04%	0.03%	0.53%	4.36%	0.86%	3.366	-0.467
REMBAPV	-3.06%	-0.29%	0.02%	0.03%	0.36%	4.75%	0.61%	4.613	0.157
REMIAPV	-3.06%	-0.30%	0.01%	0.02%	0.36%	4.76%	0.61%	4.963	0.212
CONMESE	-2.34%	-0.21%	0.04%	0.02%	0.30%	2.69%	0.46%	2.257	-0.456
CONMTAE	-1.07%	-0.10%	0.02%	0.02%	0.16%	0.87%	0.23%	1.538	-0.337
IVSTORE	-1.11%	-0.06%	0.02%	0.02%	0.10%	1.46%	0.16%	6.746	-0.267
IVSXEQE	-5.94%	-0.36%	-0.01%	-0.01%	0.38%	7.85%	0.74%	11.242	-0.060
TOPDIVB	-4.68%	-0.35%	0.05%	0.05%	0.50%	5.00%	0.81%	3.056	-0.275
BRAACTB	-15.48%	-0.79%	0.01%	0.00%	0.84%	8.89%	1.58%	6.565	-0.445
BOSWORB	-7.93%	-0.35%	0.03%	0.02%	0.43%	3.82%	0.75%	7.474	-0.676
EMMALFA	-8.72%	-0.52%	-0.01%	-0.01%	0.54%	4.82%	0.97%	5.047	-0.518
BOSEMEQB	-12.71%	-0.44%	0.00%	0.01%	0.48%	12.05%	1.17%	45.975	-0.064
BOSDOLB	-6.45%	-0.38%	0.00%	0.01%	0.44%	4.20%	0.80%	6.314	-0.534
ITVALUB	-0.14%	0.00%	0.01%	0.02%	0.03%	0.21%	0.02%	5.650	0.379
CORPEFA	-0.06%	0.01%	0.01%	0.02%	0.02%	0.16%	0.01%	21.953	3.098
BOSNEQB	-5.89%	-0.39%	-0.01%	-0.01%	0.39%	7.33%	0.75%	8.691	-0.070
LVEEUUB	-5.58%	-0.38%	0.07%	0.05%	0.52%	6.96%	0.83%	5.096	-0.135
LARVLAB	-10.73%	-0.64%	0.02%	0.00%	0.68%	9.15%	1.18%	5.856	-0.354
LVALREB	-2.71%	-0.29%	0.03%	0.03%	0.34%	4.71%	0.59%	5.189	0.253
LARVESB	-0.92%	-0.01%	0.03%	0.02%	0.06%	0.84%	0.08%	18.877	-0.510
LARRHIB	-0.87%	-0.02%	0.03%	0.02%	0.06%	0.86%	0.10%	15.513	-0.901
LARVISPB	-1.13%	-0.10%	0.02%	0.02%	0.16%	0.99%	0.23%	1.657	-0.309
LVPROTB	-0.97%	-0.06%	0.02%	0.02%	0.11%	1.46%	0.16%	5.977	-0.205
LARVACB	-0.46%	0.00%	0.02%	0.02%	0.04%	0.63%	0.06%	15.837	-0.230
LARVIAB	-1.04%	0.00%	0.01%	0.01%	0.02%	0.26%	0.03%	579.633	-16.815
LVASIAB	-4.77%	-0.39%	0.02%	0.01%	0.48%	5.66%	0.78%	3.607	-0.196
CXCASHE	-0.12%	0.01%	0.01%	0.01%	0.02%	0.17%	0.01%	33.314	1.451
LARVMMB	-0.11%	0.01%	0.01%	0.01%	0.01%	0.13%	0.01%	27.048	1.647
LVMOMAB	-0.13%	0.00%	0.00%	0.00%	0.00%	0.12%	0.01%	383.689	5.593
LARVIPB	-5.80%	-0.36%	-0.01%	-0.01%	0.38%	7.86%	0.73%	11.277	-0.009
LVENFOB	-5.90%	-0.36%	-0.01%	0.00%	0.41%	8.74%	0.74%	14.060	0.163
SNORAPV	-3.80%	-0.32%	0.05%	0.05%	0.45%	4.89%	0.73%	3.107	-0.297
SGLDAPV	-3.00%	-0.30%	0.04%	0.04%	0.42%	4.42%	0.64%	2.679	-0.182
SLATAPV	-12.13%	-0.60%	0.00%	0.00%	0.62%	5.47%	1.19%	6.346	-0.667
SANCAPV	-0.99%	-0.11%	0.02%	0.02%	0.17%	1.29%	0.25%	1.251	-0.193
SANAAPV	-2.41%	-0.24%	0.04%	0.02%	0.31%	2.33%	0.48%	1.798	-0.333
SANBAPV	-1.87%	-0.18%	0.03%	0.02%	0.24%	1.70%	0.35%	1.608	-0.319
SGPRAPV	-1.17%	-0.03%	0.02%	0.02%	0.08%	1.33%	0.12%	17.579	-0.183

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Table A3.1 – *Continued from previous page*

Bloomberg ID	Minimum	1st Quartile	Media	Mean	3rd Quartile	Maximum	Std. Deviation	Kurtosis	Skewness
SASIAPV	-3.97%	-0.36%	0.02%	0.02%	0.44%	4.44%	0.68%	2.217	-0.173
SMEMAPV	-4.95%	-0.36%	0.00%	0.02%	0.41%	4.66%	0.67%	3.606	-0.304
SANBLAP	-0.61%	-0.01%	0.02%	0.02%	0.06%	0.87%	0.08%	16.423	-0.588
RETTOTA	-0.48%	0.00%	0.02%	0.02%	0.05%	0.63%	0.06%	13.496	-0.261
HIPDAPV	-0.52%	0.00%	0.02%	0.02%	0.04%	0.60%	0.05%	16.511	-0.340
RENTAPV	-0.38%	0.00%	0.02%	0.02%	0.04%	0.58%	0.04%	24.365	0.228
SSUPINA	-0.20%	0.00%	0.01%	0.02%	0.03%	0.26%	0.03%	11.782	0.404
SMMRKTA	-0.15%	0.01%	0.01%	0.01%	0.01%	0.13%	0.01%	36.534	0.609
SMRKTAP	-0.16%	0.01%	0.01%	0.01%	0.01%	0.13%	0.01%	41.656	0.293
SARCAPV	-4.90%	-0.38%	0.00%	0.00%	0.38%	7.39%	0.72%	8.612	0.010
SACHAPV	-6.01%	-0.38%	-0.01%	-0.01%	0.39%	7.31%	0.74%	9.361	-0.172
SACSAPV	-5.82%	-0.39%	0.00%	-0.01%	0.40%	7.60%	0.77%	8.619	-0.047
SUDMIXB	-1.42%	-0.14%	0.02%	0.01%	0.19%	1.25%	0.29%	1.975	-0.389
SMEDPLB	-23.12%	-0.01%	0.02%	0.01%	0.04%	18.05%	0.73%	778.538	-14.135
BUSAAPV	-4.73%	-0.35%	0.06%	0.06%	0.49%	4.92%	0.84%	4.260	-0.220
EUROAPV	-7.27%	-0.45%	0.03%	0.03%	0.57%	6.40%	0.99%	6.638	-0.565
BHFACCB	-3.27%	-0.28%	0.04%	0.01%	0.34%	2.64%	0.58%	2.586	-0.465
BHFREMXB	-2.98%	-0.16%	0.02%	0.01%	0.20%	1.51%	0.33%	5.068	-0.708
LATAAPV	-9.57%	-0.62%	0.02%	0.01%	0.68%	4.47%	1.18%	3.207	-0.386
BHFFUTB	-1.94%	-0.01%	0.02%	0.02%	0.06%	1.07%	0.10%	89.880	-3.918
AACLAPV	-4.86%	-0.08%	0.01%	0.01%	0.12%	0.93%	0.21%	122.644	-5.713
ASPAAPV	-4.36%	-0.39%	0.02%	0.02%	0.46%	5.18%	0.72%	3.049	-0.225
BFAMAPV	-2.05%	-0.01%	0.02%	0.02%	0.04%	0.65%	0.08%	267.849	-10.240
RTAOPAP	-0.54%	0.00%	0.02%	0.02%	0.04%	0.68%	0.06%	18.286	0.089
VLPLAPV	-0.12%	0.00%	0.01%	0.02%	0.03%	0.27%	0.03%	7.012	0.819
ACNAAPV	-6.05%	-0.40%	0.00%	-0.01%	0.39%	8.02%	0.76%	10.641	-0.045
SEGLOIA	-4.14%	-0.37%	0.04%	0.04%	0.50%	7.60%	0.76%	6.323	0.158
SECEQTI	-4.59%	-0.29%	0.03%	0.02%	0.37%	3.00%	0.64%	4.125	-0.547
SECBALI	-2.50%	-0.22%	0.03%	0.02%	0.27%	1.85%	0.43%	2.685	-0.502
SECEMMI	-5.97%	-0.47%	0.00%	0.01%	0.51%	9.64%	0.94%	7.558	0.246
SECVALI	-0.94%	-0.04%	0.02%	0.02%	0.08%	1.80%	0.13%	19.476	0.559
SECORPI	-0.62%	-0.01%	0.03%	0.02%	0.06%	0.60%	0.07%	10.135	-0.436
SECGOLI	-0.49%	0.00%	0.02%	0.02%	0.05%	0.47%	0.05%	10.642	-0.508
SECFRSI	-0.34%	0.00%	0.02%	0.02%	0.04%	0.40%	0.05%	9.090	-0.655
SCHILSI	-6.08%	-0.40%	-0.01%	-0.01%	0.38%	7.32%	0.77%	8.623	-0.105
EAPROEB	-0.91%	-0.02%	0.02%	0.02%	0.07%	1.19%	0.11%	14.440	-0.434
EAPROCB	-1.39%	-0.12%	0.02%	0.02%	0.18%	1.53%	0.27%	2.658	-0.384
EAPROAB	-3.03%	-0.24%	0.03%	0.02%	0.31%	3.19%	0.49%	3.243	-0.378
EUASIAB	-3.70%	-0.37%	0.02%	0.02%	0.44%	4.80%	0.71%	2.537	-0.235
EUCH18B	-5.67%	-0.32%	0.00%	-0.02%	0.31%	3.71%	0.70%	6.792	-0.624
GGD5AP	-1.61%	-0.13%	0.02%	0.02%	0.19%	1.65%	0.29%	2.692	-0.236
CAPITALA	-3.34%	-0.23%	0.04%	0.03%	0.32%	3.27%	0.50%	3.723	-0.377
CAPITALB	-2.29%	-0.17%	0.04%	0.03%	0.24%	2.49%	0.36%	3.439	-0.317
CAPITALC	-1.37%	-0.11%	0.03%	0.03%	0.18%	1.74%	0.25%	3.171	-0.178
CAPITALD	-1.06%	-0.06%	0.03%	0.03%	0.12%	1.39%	0.16%	6.035	-0.004
CAPITALE	-1.25%	-0.02%	0.03%	0.03%	0.08%	1.18%	0.11%	20.343	-0.714
CUPRUMA	-3.07%	-0.23%	0.05%	0.03%	0.32%	3.20%	0.49%	3.449	-0.329
CUPRUMB	-2.13%	-0.17%	0.04%	0.03%	0.24%	2.49%	0.36%	3.242	-0.267
CUPRUMC	-1.24%	-0.10%	0.03%	0.03%	0.18%	1.77%	0.24%	2.932	-0.096
CUPRUMD	-1.04%	-0.06%	0.03%	0.03%	0.12%	1.33%	0.15%	4.899	0.040
CUPRUME	-1.16%	-0.02%	0.03%	0.03%	0.08%	1.14%	0.12%	18.460	-0.959
HABITATA	-3.56%	-0.23%	0.04%	0.03%	0.32%	3.31%	0.50%	4.154	-0.397
HABITATB	-2.38%	-0.16%	0.04%	0.03%	0.24%	2.55%	0.36%	3.697	-0.306
HABITATC	-1.33%	-0.10%	0.03%	0.03%	0.18%	1.81%	0.24%	3.162	-0.113
HABITATD	-0.98%	-0.05%	0.03%	0.03%	0.12%	1.34%	0.15%	5.137	0.084
HABITATE	-1.12%	-0.02%	0.03%	0.03%	0.08%	1.17%	0.12%	18.366	-0.727
PLANVITALA	-3.32%	-0.23%	0.04%	0.03%	0.32%	3.24%	0.50%	3.937	-0.365
PLANVITALB	-2.27%	-0.16%	0.03%	0.03%	0.24%	2.50%	0.36%	3.708	-0.318
PLANVITALC	-1.37%	-0.10%	0.03%	0.03%	0.17%	1.78%	0.24%	3.603	-0.139
PLANVITALD	-1.07%	-0.06%	0.03%	0.03%	0.11%	1.40%	0.15%	6.600	0.123
PLANVITALE	-1.06%	-0.02%	0.03%	0.03%	0.07%	1.12%	0.10%	18.926	-0.517
PROVIDAA	-3.35%	-0.23%	0.04%	0.03%	0.32%	3.30%	0.49%	3.650	-0.351
PROVIDAB	-2.23%	-0.16%	0.04%	0.03%	0.24%	2.56%	0.36%	3.285	-0.262
PROVIDAC	-1.36%	-0.11%	0.03%	0.03%	0.18%	1.82%	0.25%	3.189	-0.068

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Table A3.1 – *Continued from previous page*

Bloomberg ID	Minimum	1st Quartile	Media	Mean	3rd Quartile	Maximum	Std. Deviation	Kurtosis	Skewness
PROVIDAD	-1.11%	-0.06%	0.03%	0.03%	0.12%	1.43%	0.16%	6.218	0.010
PROVIDAE	-1.23%	-0.02%	0.03%	0.03%	0.08%	1.19%	0.12%	17.542	-0.756

Source: Own elaboration based on Bloomberg L.P data and AFP funds data from Superintendence of Pensions of Chile.

Table A3.2: Descriptive statistics of Portfolios

Portfolio	Minimum	1st Quar.	Media	Mean	3rd Quar.	Maximum	Std. Dev.	Kurtosis	Skewness
EW (static Opt.)	-1.209%	-0.106%	0.029%	0.020%	0.166%	1.417%	0.240%	1.999	-0.359
Tan60 (Static Unconst. Opt.)	-0.063%	0.014%	0.023%	0.027%	0.035%	0.176%	0.021%	4.092	1.240
GMVP60 (Static const. Opt.)	-0.122%	0.002%	0.003%	0.004%	0.004%	0.106%	0.005%	264.204	1.300
GMVP60 (Static unconst. Opt.)	-0.083%	0.001%	0.002%	0.003%	0.004%	0.088%	0.005%	105.543	2.168
Tan60 (Static const. Opt.)	-0.097%	0.008%	0.011%	0.014%	0.016%	0.112%	0.011%	12.161	1.644
Tan70 (Static unconst. Opt.)	-0.067%	0.016%	0.025%	0.029%	0.036%	0.237%	0.022%	7.280	1.599
GMVP70 (Static const. Opt.)	-0.123%	0.002%	0.003%	0.004%	0.004%	0.107%	0.005%	280.462	1.625
GMVP70 (Static unconst. Opt.)	-0.084%	0.000%	0.002%	0.002%	0.004%	0.090%	0.005%	120.742	2.448
Tan70 (Static const. Opt.)	-0.097%	0.008%	0.011%	0.015%	0.016%	0.119%	0.011%	12.997	1.733
Tan 80 (Static Unconst. Opt.)	-0.082%	0.017%	0.027%	0.032%	0.040%	0.228%	0.025%	5.289	1.525
GMVP80 (Static const. Opt.)	-0.123%	0.002%	0.002%	0.003%	0.004%	0.110%	0.005%	299.508	2.139
GMVP80 (Static unconst. Opt.)	-0.087%	0.000%	0.002%	0.002%	0.003%	0.091%	0.005%	140.016	2.661
Tab 80 (Static const. Opt.)	-0.105%	0.008%	0.012%	0.015%	0.016%	0.113%	0.011%	13.429	1.544
GMVP60 (Roll. const. Opt.)	0.000%	0.000%	0.001%	0.001%	0.002%	0.003%	0.001%	-1.459	0.326
GMVP60 (Roll. unconst. Opt.)	0.000%	0.001%	0.001%	0.002%	0.004%	0.004%	0.001%	-1.681	0.371
Tan60 (Roll. unconst. Opt.)	0.023%	0.027%	0.032%	0.031%	0.034%	0.036%	0.004%	-1.119	-0.555
Tan60 (Roll. const. Opt.)	0.012%	0.014%	0.016%	0.015%	0.017%	0.017%	0.002%	-1.370	-0.389
GMVP70 (Roll. const. Opt.)	0.001%	0.001%	0.002%	0.002%	0.002%	0.003%	0.001%	-1.531	0.055
GMVP70 (Roll. unconst. Opt.)	0.001%	0.001%	0.002%	0.002%	0.003%	0.004%	0.001%	-1.799	-0.102
Tan70 (Roll. const. Opt.)	0.025%	0.026%	0.029%	0.028%	0.031%	0.031%	0.002%	-1.518	-0.229
Tan 70 (Roll. unconst. Opt.)	0.013%	0.014%	0.016%	0.015%	0.016%	0.017%	0.001%	-1.252	-0.461
GMVP80 (Roll. const. Opt.)	0.001%	0.002%	0.002%	0.002%	0.002%	0.002%	0.000%	-1.001	-0.444
GMVP80 (Roll. unconst. Opt.)	0.002%	0.003%	0.003%	0.003%	0.003%	0.003%	0.001%	-0.300	-1.217
Tan80 (Roll. unconst. Opt.)	0.024%	0.025%	0.026%	0.026%	0.027%	0.028%	0.001%	-1.054	0.210
Tan80 (Roll. const. Opt.)	0.014%	0.015%	0.016%	0.015%	0.016%	0.016%	0.001%	-1.002	-0.720
$\lambda = 0$ 60% Sample size	-3.427%	-0.346%	0.038%	0.024%	0.427%	4.583%	0.793%	3.202	-0.269
$\lambda = 0.3\%$ 60% Sample size	-3.387%	-0.331%	0.033%	0.025%	0.427%	4.583%	0.774%	3.248	-0.264
$\lambda = 0.8\%$ 60% Sample size	-3.387%	-0.283%	0.052%	0.028%	0.407%	4.583%	0.720%	4.046	-0.259
$\lambda = 1.5\%$ 60% Sample size	-3.079%	-0.269%	0.055%	0.023%	0.359%	4.583%	0.662%	5.236	-0.275
$\lambda =$ 60% Sample size	-2.839%	-0.232%	0.068%	0.018%	0.332%	4.136%	0.609%	5.374	-0.392
$\lambda = 5$ 60% Sample size	-2.563%	-0.168%	0.033%	0.016%	0.240%	1.824%	0.438%	4.509	-0.904
$\lambda = 50$ 60% Sample size	-0.587%	-0.023%	0.019%	0.018%	0.068%	0.607%	0.094%	6.227	-0.398
$\lambda = 500$ 60% Sample size	-0.163%	0.002%	0.012%	0.011%	0.022%	0.142%	0.022%	10.002	-1.222
$\lambda = 5000$ 60% Sample size	-0.011%	0.006%	0.008%	0.008%	0.010%	0.025%	0.003%	5.294	-0.153
$\lambda = 0$ 70% Sample size	-2.990%	-0.278%	0.070%	0.055%	0.420%	4.580%	0.735%	4.481	-0.156
$\lambda = 0.3$ 70% Sample size	-3.050%	-0.260%	0.060%	0.052%	0.410%	4.580%	0.714%	5.084	-0.186
$\lambda = 0.8$ 70% Sample size	-3.040%	-0.200%	0.055%	0.052%	0.380%	4.580%	0.667%	6.716	-0.253
$\lambda = 1.5$ 70% Sample size	-2.640%	-0.180%	0.060%	0.050%	0.348%	4.130%	0.624%	6.267	-0.389
$\lambda = 2$ 70% Sample size	-2.640%	-0.168%	0.050%	0.049%	0.330%	3.750%	0.598%	5.801	-0.515
$\lambda = 5$ 70% Sample size	-2.580%	-0.130%	0.045%	0.031%	0.260%	2.870%	0.486%	8.106	-1.075
$\lambda = 50$ 70% Sample size	-0.520%	-0.020%	0.020%	0.020%	0.070%	0.640%	0.101%	5.819	-0.315
$\lambda = 500$ 70% Sample size	-0.130%	0.000%	0.010%	0.011%	0.020%	0.150%	0.022%	7.687	-0.581
$\lambda = 5000$ 70% Sample size	-0.010%	0.010%	0.010%	0.009%	0.010%	0.030%	0.004%	4.745	-1.546
$\lambda = 0$ 80% Sample size	-2.790%	-0.297%	0.077%	0.042%	0.423%	4.583%	0.762%	4.765	-0.210
$\lambda = 0.3$ 80% Sample size	-2.644%	-0.288%	0.073%	0.040%	0.401%	3.310%	0.726%	2.522	-0.471
$\lambda = 0.8$ 80% Sample size	-2.644%	-0.268%	0.071%	0.032%	0.374%	2.723%	0.695%	2.322	-0.615
$\lambda = 1.5$ 80% Sample size	-2.644%	-0.203%	0.062%	0.028%	0.366%	2.557%	0.657%	2.775	-0.761
$\lambda = 2$ 80% Sample size	-2.644%	-0.185%	0.060%	0.030%	0.370%	2.510%	0.645%	2.988	-0.809
$\lambda = 5$ 80% Sample size	-2.444%	-0.134%	0.042%	0.019%	0.269%	2.393%	0.495%	5.925	-0.987
$\lambda = 50$ 80% Sample size	-0.442%	-0.021%	0.023%	0.021%	0.067%	0.678%	0.097%	7.418	0.286
$\lambda = 500$ 80% Sample size	-0.087%	0.001%	0.009%	0.010%	0.019%	0.153%	0.020%	10.765	0.354
$\lambda = 5000$ 80% Sample size	-0.004%	0.006%	0.007%	0.007%	0.008%	0.029%	0.003%	15.653	1.182
HRP60	-0.010%	0.002%	0.003%	0.005%	0.007%	0.076%	0.005%	40.128	3.898
HRP70	-0.015%	0.002%	0.003%	0.005%	0.007%	0.072%	0.005%	65.840	4.921
HRP80	-0.011%	0.003%	0.004%	0.006%	0.007%	0.025%	0.004%	2.177	0.918

Source: Own elaboration based on Bloomberg L.P data and AFP funds data from Superintendence of Pensions of Chile.