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The SABR model in a negative interest rate framework

Theory and practice

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Abstract

This thesis focused on the stochastic volatility mode, the SABR model. This model is well known and has been used in financial institutions since its inception. However, the model is unable to work if interest rates are negative. Some refinements have been proposed to change the workings of the model to incorporate negative rates. The thesis looked at these refinements and tested whether the Normal SABR model and the Shifted SABR model were able to produce implied volatilities similar to the ones observed in the market for both caplets and swaptions.

The results showed us that, in a negative interest rate environment, these modifications to the SABR model can produce implied volatilities that are very close to the market volatilities for both caplets and swaptions. Furthermore, the thesis looked at Obłój's refinement, which states that the SABR model is unable to produce accurate volatilities for options with low strikes and long maturities. It was found that this method was unable to produce better fitting volatilities for those options than the Shifted SABR model.

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1 Introduction

1.1 Background and motivation

Due to economic conditions prevailing after the global financial crisis, central banks were forced to lower interest rates. However, that strategy didn't work and as a result,they were forced to push them into negative values. This has had an enormous effect on the financial products that rely on the current rate of interest, the interest rate derivatives. As the underlying assumptions of the widely used models assumed that interest rates could never become negative, reform had to be sparked (Russo and Fabozzi, 2017).

Several changes to the existing models were made in order so they could work in this new interest rate regime. Especially for the SABR model, these changes are not as extensive as one would expect. They are the Normal SABR model, which is a SABR model with a β parameter fixed to zero whereas the Shifted SABR model is a natural extension of the original SABR model with a displacement parameter to make the negative interest rates positive (Crispoldi et al., 2016).

For these models to replace the widely used SABR model, they must be tested rigorously.

1.2 Research question

Does the fact that interest rates are negative have any effect on the performance of SABR models?

However, in light of the COVID-19 pandemic of 2020, the author was unable to access the data needed to perform the empirical analysis. However, as data was only collected data from one single date, we are unable to compare the performance of the model when interest rates are negative to when they were positive rates. The question will be refined to the following:

Do changes to the SABR model allow it to accurately produce implied volatilities when interest rates are negative?

With the changes being the addition of the displacement parameter and the fixing of the β parameter as discussed prior. As a result, we can perform an analysis to see the performance of the negative rates capable SABR models and to test whether they manage to model implied volatilities accurately.

1.3 Structure of the thesis

To answer the question which was posed, we need to take a step back and start by laying down the theoretical foundations that the thesis is built upon. This includes the definition of interest rates, derivatives and their uses. Furthermore, we will discuss the different techniques required for pricing interest rate derivatives.

Within that section, we discuss different interest rate models, from Black's 76 model to the Libor Market model and look at the advantages and disadvantages each model has. To solve the main problem of the models within the Black's framework, we introduce local volatility models. However, those models have inherent flaws with regards to risk management which makes them impractical for usage. These problems were solved by the SABR model.

The SABR model will be discussed thoroughly and we will look at it extensively to gain an understanding of the function of the model. We look at the reasons why the model has been widely used by financial institutions and how it manages to solve the problems of the local volatility models. We will then look at the changes that have been proposed to make the SABR model function within a negative interest rate framework and see if they are viable for usage within the financial sector. Furthermore, we look at the research surrounding the model and the ways that have been attempted to change the model to allow it to incorporate negative interest rates.

The third chapter of the thesis explains the empirical analysis that will be performed. We present the data that has been collected and describe it in detail. Furthermore, we state the methodology that the analysis is built upon and what actions will be performed to answer the research question.

The fourth chapter describes the results of the analysis. We show how well the modified variants of the SABR model handle implied volatilities. To conclude, we give discussions to our findings.

2 Theory and literature review

2.1 Interest rates and interest rate derivatives

The concept of interest rates is well known in the fields of macro- and microeconomics. It is the rate of which the deposit of a bank account grows in a given period or the cost of borrowing money. However, as simple as those definition may be, there are many ways to calculate interest and it is vital to know the differences between methodologies and the advantages and disadvantages of each methodology (Brigo and Mercurio, 2007).

2.1.1 Mathematical framework

This sub-chapter contains some preliminary and relevant concept that will be relevant for the understanding, arguments, and developments for the following chapters.

- **Bank account** ($B(t)$). To understand the time value of money and interest rates, we must think of a bank account and how the deposit in it grows over time. The value at time t of a bank account, $B(t)$, represents a zero-risk investment, which is continuously compounded at the r_t rate at time $t \geq 0$. If we assume that $B(0) = 1$ and the value of the bank account changes according to the following differential equation:

$$dB(t) = r_t B(t) dt \tag{2.1}$$

  Where r_t is a positive function of time. Solving the differential equation by integration yields us:

$$B(t) = B(0) \exp \left(\int_t^T r_u du \right) \tag{2.2}$$

  Which means that the bank account accrues interest continuously at the rate r_t, which is the instantaneous spot rate also known as the short rate (Brigo and Mercurio, 2007).

- **Discount factor** ($D(t, T)$). The discount factor denotes the value, at time t, of receiving one unit of currency at time T. This can be used to find the present value of a payment arriving at time T as seen from time t (Brigo and Mercurio, 2007). We can view this discount factor as a stochastic process if we have a continuously compounding interest rate, it would be written as

$$D(t,T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r_u du\right) \tag{2.3}$$

Which means that the discount factor decreases continuously at the rate r_t (Brigo and Mercurio, 2007).

- **Zero-coupon bond price ($P(t,T)$)** The risk-free zero-coupon bond (ZCB) is a financial contract that pays out its face value upon maturity with certainty, meaning that there is no risk whether the payment will be made. As it is possible to trade such an asset, we can assign a market value to it. We define $P(t,T)$ to be the price observed at time t for a zero-coupon bond maturing at time T. Furthermore, as there is no credit risk for zero-coupon bonds, the price of $P(T,T)$ must be 1 for all T. (Brigo and Mercurio, 2007).

  Likewise, we can deduct that a payment of \$1 at time T must have a market value of $P(t,T)$ as seen from time t and that $P(t,T) \leq 1$ if we assume that time-value-of-money is positive. Knowing this, we can use a replication argument as described by Hull (2018), to calculate the $P(t,T)$ if we know the $P(0,t)$ and $P(0,T)$. Because if we buy 1 unit of $P(0,T)$ and finance that by shorting $P(0,T)/P(0,t)$ units of the ZCB maturing at time t. As a result, at time t we are forced to pay $P(0,T)/P(0,t)$ while we are guaranteed to receive \$1 at time T, effectively creating a ZCB with the payment schedule of (t,T) and as a result, the price of that bond must be:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \tag{2.4}$$

  From the price $P(t,T)$ we can find the zero-coupon rate between the fixing (t,T). We can denote the rate in different ways, with the two most popular being, discrete compounding and continuous compounding. Letting $r_{cont}(t,T)$ and $r_{disc}(t,T)$ denote the zero-coupon rates using continuous and simply compounding respectively, we can calculate them as:

$$P(t,T) = \exp(-r_{cont}(t,T) * (T-t)) \tag{2.5}$$

$$P(t,T) = \frac{1}{(1 + r_{disc}(t,T) * (T-t))} \tag{2.6}$$

If we can observe a set of zero-coupon bond prices, we can calculate the zero-coupon rates as there is a one-to-one mapping between the prices and the rates. For a given observation time t, we call any mapping from T to $r(t,T)$ a zero-coupon *yield curve* or the *term structure of rates* (Brigo and Mercurio, 2007).

- **Risk-neutral measure** $\mathbb{Q}$: When pricing derivatives, we must assume that there are no frictions in financial markets. If markets contain frictions, players in the market can obtain risk-free profits, such a strategy would be referred to as an arbitrage (Brigo and Mercurio, 2007).

  The risk-neutral measure $\mathbb{Q}$ has the bank account as the numeraire. Under the risk-neutral measure and in the absence of frictions the contingent claim $V(t)$ is valued as:

$$V(t) = B(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{V(T)}{B(T)}|\mathcal{F}_t\right] \tag{2.7}$$

  where $\mathcal{F}$ is a filtration from time t to time T. $\mathcal{F}_t$ can be thought of as the information available at time t (Andersen and Piterbarg, 2010).

- **T-forward measure** $\mathbb{F}^T$: Using a Zero-coupon bond $P(t,T)$ is used as the numeraire. Using this measure and with the absence of arbitrage, we can value a contingent claim $P(t,T)$ as

$$V(t) = P(t,T)\mathbb{E}^{\mathbb{F}^T}\left[\frac{V(T)}{P(T,T)}|\mathcal{F}_t\right] \tag{2.8}$$

  and as we know $P(T,T)$ is equal to 1, we can shorten that formula to

$$V(t) = P(t,T)\mathbb{E}^{\mathbb{F}^T}[V(T)|\mathcal{F}_t] \tag{2.9}$$

(Brigo and Mercurio, 2007)

- **Martingales** Using these two measures, we obtain the fundamental theorem of derivatives pricing, that the market is arbitrage-free when there exists a martingale measure. Which means that in the absence of arbitrage we can price derivatives as the expected values of their payouts.

  A stochastic variable that is a martingale can be written as

$$df_t = c(\cdot)dW_t \tag{2.10}$$

  where the W_t is a standard Brownian motion and f_t is a martingale process under the filtration $\mathcal{F}_t$. The uncertain growth of f_t is equal to the Brownian motion development multiplied to a process $c(\cdot)$, which any interest rate model needs to define so that the process becomes a martingale. This has been named the Martingale representation theorem (Andersen and Piterbarg, 2010).

2.1.2 Interest rates concepts

As interest is accrued over a period of time, the day count convention measures this interval. The time to maturity is the time between two dates measured in years. As there are many market conventions on how to measure the length between two discrete time points, as a result, it is important to know the different conventions and know when to apply them (Brigo and Mercurio, 2007).

In this thesis, we define δ_i as the length of time between $[T_S, T_E]$ over the length of a year. Where T_S and T_E denote the start date and the end date, respectively. There are many ways to calculate this coverage. Brigo and Mercurio (2007) give three examples of how to calculate the day count:

- **Act/360:** Where we define the year as being 360 days long and is calculated as $\delta_i = \frac{T_E - T_S}{360}$
- **Act/365:** Where we define the year as being 365 days long and is calculated as $\delta_i = \frac{T_E - T_S}{365}$
- **30/360** With this convention, we assume that every month is 30 days long and years have

360 days. It is calculated using the formula:

$$\delta_i = \frac{1}{360}[(\text{Year}(T_E) - \text{Year}(T_S)) * 360 + (\text{Month}(T_E) -$$
$$\text{Month}(T_S)) * 30 + \min(30, Day(T_E)) - \min(30, Day(T_S))]$$

The rate that most interest rate derivative contracts are written on is called the LIBOR, which stands for the *London Interbank Offered Rate.* Its name differs by location, the L is replaced by the the first letter of the name of the countries' capital with some exceptions. The most notable of which is the EURIBOR, the interbank offered rates for the Euro area (Hull, 2018).

The LIBOR fixings are decided by a few select banks in each country and are supposed to reflect the rates those prime banks can borrow money uncollateralized in each currency from each other for a particular period and currency. The LIBOR rates are fixed daily for multiple maturities, starting from overnight rates and the highest being a whole year (Hull, 2018).

LIBOR fixings are simply-compounded rates, where we denote an interest payment at maturity as $\delta * N * L(0,T)$, where the N is the notional, δ equals the tenor, and $L(0,T)$ equals the LIBOR fixing (Brigo and Mercurio, 2007). We can calculate the LIBOR rate if we have the price of a ZCB by rearranging equation 2.5 and obtaining:

$$L(0,T) = \frac{1}{P(0,T) * \delta} - 1 \tag{2.11}$$

The LIBOR fixings are sometimes thought of as a proxy for the *risk-free rate* but after the global financial crisis, the LIBOR fixings have been criticized for being a fictitious number, as suspicions have arisen that the method which calculates the LIBOR fixings has been abused by financial institutions. Why would financial institutions manipulate their LIBOR fixings? Two possible reasons have been suggested. The first being to lower their borrowing costs while the other reason being to profit from transactions that depend on LIBOR fixings (Hull, 2018).

Forward rates are interest rates that are locked in today, at time t, that starts accruing at time T and matures at time $T + \delta$, given that $t \leq T \leq T + \delta$. We denote the forward LIBOR rate as $F(t, T, T + \delta)$ (Brigo and Mercurio, 2007). We already know how to find the forward price of a zero-coupon bond (Equation 2.6). Therefore we have to find the rate that corresponds to that price. This is done as:

$$F(t, T, T+\delta) = \frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right) \tag{2.12}$$

2.1.3 Interest rate derivatives

The instruments that will be covered in this section are those that are those *vanilla* interest rate derivatives that the analysis in the later chapters will focus on. Derivative contracts are financial instruments whose value depends on the value of other instruments (Hull, 2018).

2.1.3.1 Forward rate agreements

A forward rate agreement (FRA) is an over-the-counter (OTC) interest rate derivative. It can be described as two parties agree to pay and receive the difference between a pre-determined fixed-rate and an LIBOR fixing of a specific period on a given notional, denoted as N. In short, entering into such a contract gives you a way to lock in a future funding rate. The FRA is an OTC derivative, which means that the derivative is traded between two parties without an intermediary (Hull, 2018).

The payoff of such a contract is:

$$\text{Payoff}_T^{FRA} = \frac{N\delta(L(T,T+\delta) - K)}{1 + \delta L(T, T+\delta)} \tag{2.13}$$

where the K is the pre-determined fixed-rate. Using that payoff, we can discount it to time t by multiplying it with $P(t,T)$ (Hull, 2018)

To trade such a contract, the initial present value of the contract (*for both parties*) must be equal to zero. As a result, the pre-determined fixed-rate must be the forward rate as seen from time 0 (Hull, 2018).

2.1.3.2 Interest rate swaps

An interest rate swap (IRS) is an OTC derivative where interest rates are exchanged between two parties. One of the parties pays a floating interest rate to the other party while receiving a fixed interest rate from the other. The swaps are quoted with the respect to the fixed-rate, i.e. whether you pay it (payer swap) or you receive it (receiver swap). The interest rate swap offers the possibility to exchange a sequence of floating forward $F(t, T_i, T_{i+1}), i = 0, 1, \ldots n-1$ rate against a fixed-rate K over a time interval $[T_S, T_E]$ (Hull, 2018).

The fixed leg will have the same starting dates as the floating leg but due to the different usage of day count conventions and frequency of the payments, i.e. in Euro IRS, the convention is to trade the floating leg with an Act/360 day count and is paid every 6 months. Whereas the fixed leg has a 30/360 day count and a payment once a year. As a result, the dates of payments (usually) are not the same. Therefore, we must differ the δ_i for each leg. We create two sets of δ, one for the floating and the other for the fixed leg (Hull, 2018).

To find the present values of the fixed and floating legs we write it as:

$$PV_t^{Fixed} = \sum_{i=S+1}^{E} \delta_i^{Fixed} K N_i P(t, T_i) \tag{2.14a}$$

$$PV_t^{Float} = \sum_{i=S+1}^{E} \delta_i^{Floating} F(t, T_{i-1}, T_i) N_i P(t, T_i) \tag{2.14b}$$

For the swap to be traded, the net present value (NPV) upon inception must be equal to zero. This is done to eliminate any credit risk at inception for either party and as a result, PV_{Fixed} and $PV_{Floating}$ must be equal (however, financial actors can trade non-zero NPV swaps if they want, at a price) (Brigo and Mercurio, 2007).

For those terms to match, one must choose the K that makes the contract 0 NPV. This is done by finding the *par swap rate* $S(t, T_S, T_E)$ which can be obtained using the formula:

$$S(t, T_S, T_E) = \frac{P(t, T_S) - P(t, T_E)}{\sum_{S=i+1}^{E} \delta_i^{Fixed} N_i P(t, T_i)} \tag{2.15}$$

With a further deviation of the par swap rate, we can deduct that it is a weighted average of forward rates (Brigo and Mercurio, 2007). Therefore, we can formulate the present value of the payer and the receiver swaps as:

$$\text{PV}_{Payer} = A(t, T_S, T_E)(S(t, T_S, T_E) - K) \tag{2.16a}$$

$$\text{PV}_{Receiver} = A(t, T_S, T_E)(K - S(t, T_S, T_E)) \tag{2.16b}$$

$$A(t, T_S, T_E) = \sum \delta_i^{Fixed} P(t, T_i) \tag{2.16c}$$

where $A(t, T_S, T_E)$ denotes the swap's annuity factor (Brigo and Mercurio, 2007).

2.1.3.3 Caps and floors

An important class of derivatives are options, they give the owner the right but not the obligation to exercise the security. Options are categorized in two ways, call options, a contract that can be exercised only if the value underlying has reached a pre-determined point at the time of expiry. This pre-determined point is called the strike price of the option and is denoted by K. Likewise, with the other type of options, put options. Which only produce a payoff if the underlying is under the strike. In the interest rate world, these are called caps and floors, they are portfolios composed of caplets and floorlets, respectively, which are options on interest rates (Hull, 2018).

If a firm funds itself on a floating rate is concerned with the possibility of higher rates, which would increase their funding costs. They could enter into a payer IRS, which would eliminate the interest rate exposure by eliminating the benefits of a decrease in interest rates. Therefore, there is a clear benefit in owning an interest rate cap, which negates the negative effects of an increase in rates while still giving the upside of a decrease in rates. Floors, on the other hand, can be thought of as insurance for a buyer of floating rate debt for the opposite reason (Hull, 2018).

To price these caps/floor, we must find the individual payoff of every single caplet/floorlet, which can be calculated as

$$\text{Payoff}_{caplet} = \delta(F(T, T, T + \delta) - K)^+ \tag{2.17a}$$

$$\text{Payoff}_{floorlet} = \delta(K - F(T, T, T + \delta))^+ \tag{2.17b}$$

Where the caplet is fixed-in-advanced and paid in arrears. Meaning that the caplet is exercised at maturity and settled 6 months after. Caps/floors are quoted in such a way that describes their start and their maturity, i.e. $5x10$, means that the cap begins in 5 years and ends in 10 years from the start of the contract (Andersen and Piterbarg, 2010).

The present value of a single cap/floor as described by Brigo and Mercurio (2007), can be thought of as the sum of the present values of the caplets/floorlets:

$$\mathrm{PV}_{cap} = \sum_{S=i+1}^{E} P(t,T_i)N\delta_i(F(t,T,T+\delta_i)-K)^+ \tag{2.18a}$$

$$PV_{floor} = \sum_{S=i+1}^{E} P(t,T_i)N\delta_i(K-F(T,T,T+\delta))^+ \tag{2.18b}$$

Caps for the Euro are quoted against 3M forward rates up to 2 years maturity and after that, they are quoted against the 6M forward rate. Meaning that the 2-year cap includes seven caplets while the 3-year cap contains five caplets (Kienitz, 2013).

2.1.3.4 Swap options (Swaptions)

A European swap option (or swaption) gives the holder the right but not the obligation to enter into an IRS upon the maturity of the option at a pre-specified swap rate. A payer swaption gives the holder the possibility to enter into a payer IRS and the receiver swaption gives the possibility of entering into a receiver swap. The underlying length of the IRS is called the tenor of the swaption. We denote the exercise date of the option as T_S and the maturity of the swap by T_E (Brigo and Mercurio, 2007).

Swaptions are settled by either a *physical* or *cash* settlement. The physical settlement means that at if the option is exercised the trade between the buyer and the seller is an actual interest rate swap. However, regarding cash-settled swaptions, the owner of the option is paid a cash settlement which equals the present value of the IRS. However, a more in-depth discussion of cash-settled swaptions will not be covered in this thesis (Andersen and Piterbarg, 2010).

In order to price swaptions, the swap measure must be discussed. If we look at a linear collection of zero-coupon bonds, we get the annuity factor as seen in Equation 2.16c. If we use this annuity factor as the numeraire asset we obtain the *swap measure* $\mathbb{Q}^A$ and in absence of arbitrage

$$V(t) = A(t, T_S, T_E)\mathbb{E}^{\mathbb{Q}^A}\left[\frac{V(T)}{A(t, T_S, T_E)}\right] \tag{2.19}$$

As we find that the forward swap rate $S(t, T_S, T_E)$ is a martingale under $\mathbb{Q}^A$ (Andersen and Piterbarg, 2010).

To find the present value of the physically settled swaption, we need to look at the value of the underlying IRS at exercise date of the option. If we look back to equations 2.16a and 2.16b, we can see the formula for present values of interest rate swaps. To find the present value of the option we add a term to those formulae, represented by the $^+$, which means that the number only has a value if it is positive. We take the expectation with the appropriate swap measure (Brigo and Mercurio, 2007).

$$\text{Payer swaption } PV_t = A(t, T_S, T_E)E_t^{\mathbb{Q}^A}(S(t, T_S, T_E) - K)^+ \tag{2.20a}$$

$$\text{Receiver swaption } PV_t = A(t, T_S, T_E)E_t^{\mathbb{Q}^A}(K - S(t, T_S, T_E))^+ \tag{2.20b}$$

where $A(t, T_S, T_E) = \sum \delta_i^{Fixed} P(t, T_i)$ denotes the swap's annuity factor as seen from t. We can see from equations 2.20, that a payer swaption is comparable to a call option and a receiver swaption is a put option. Swaptions are denoted by their option expiry and tenor in such a way that for example, a 10-year option on a 10-year swap would be noted as $10Y10Y$ swaption (Andersen and Piterbarg, 2010).

The swaption is considered to be at-the-money (ATM) when the strike price of the option is equal to the forward swap rate. When regarding payer swaptions, when the strike is above the forward swap rate, the option is considered in-the-money (ITM) and it is under the forward swap rate, it is considered out-of-the-money (OTM). As we already know that an interest rate swap can be viewed as a portfolio of LIBOR forwards, we can deduct that a swaption is an option on that portfolio. (Brigo and Mercurio, 2007).

2.2 Interest rate models

This chapter introduces conventional and unconventional interest rate models that are used in the pricing and hedging of financial instruments. These models will attempt in one way or another model the short rate or the forward rate and provide close-ended solutions to cap/floor and swaption pricing.

2.2.1 Black's Model (1976)

Fischer Black (1976) introduced a model that attempts to model forward prices of commodities and states that the nature of the forward LIBOR rate or forward swap rate F_t can be described by the stochastic differential equation:

$$dF_t = \sigma_B F_t dW_t \tag{2.21}$$

Where σ_B is the *constant* lognormal volatility and W_t is a Brownian motion under the T-forward measure $\mathbb{F}^T$. This model is called the Black's 76 model or Black's Model. It differs from the Black Scholes Merton model by using the forward rate instead of the spot rate and as a result, makes it quite useful to price interest rate derivatives (Russo and Fabozzi, 2017).

If we integrate equation 2.21 we obtain:

$$F_t = F_0 * \exp\left(\sigma W_t - \frac{\sigma^2 t}{2}\right) \tag{2.22}$$

Which implies that the forward rate is lognormally distributed and as such must be a positive number (Hull, 2018). Using a T-maturity ZCB $P(t,T)$ as the numeraire asset for the T-forward measure $\mathbb{F}^T$, which contains that in the units of that numeraire, $F(t,T_S,T_E)$ is a tradable asset, then the Black formula states *today's* price, at time t for a caplet/floorlet on a forward rate $F(t,T_S,T_E)$ with the strike K and the notional N is given by:

$$V_{caplet}^{Black}(T,K,F_0,\sigma) = N\delta(T_S,T_E)P(t,T_E)[F_0\Phi(d_1) - K\Phi(d_2)] \tag{2.23a}$$

$$V_{floorlet}^{Black}(T, K, F_0, \sigma) = N\delta(T_S, T_E)P(t, T_E)[K\Phi(-d_2) - F_0\Phi(-d_1)] \tag{2.23b}$$

Where Φ is the normal cumulative distribution and d is represented with:

$$\begin{aligned} d_1 &= \frac{\log(F_0/K) + (\sigma_B^2/2)}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned} \tag{2.24}$$

(Brigo and Mercurio, 2007).

To price these cap- or floorlets, we start by calculating the relevant LIBOR forward rate and the discount factor. We continue by inserting those values into equations 2.23a or 2.23b, with the strike K, time to expiry T_E and the volatility σ_B. All but the volatility is easily observable in the market and since option prices are increasing in σ, means that there must be a single unique $\bar{\sigma}_B$ that matches any observable market price, it is called the implied volatility (Brigo and Mercurio, 2007).

As we know that caps/floors are a portfolio of caplets/floorlets thus the sum of the present values of the caplets must equal the present value of the cap.

$$V_{cap\ \ t}^{Black} = N \sum_{i=S+1}^{E} \delta_i P(t, T_i)[F(t, T_{i-1}, T_i)\Phi(d_1) - K\Phi(d_2)] \tag{2.25a}$$

$$V_{floor\ \ t}^{Black} = N \sum_{i=S+1}^{E} P(t, T_i)\delta_i[K\Phi(-d_2) - F(t, T_{i-1}, T_i)\Phi(-d_1)] \tag{2.25b}$$

where the δ_i is the day-count convention for the period starting at T_{i-1} and ending in T_i (Brigo and Mercurio, 2007).

Regarding European swaptions, the Black model has a way to price them.

If we consider a swap that starts in T_S and matures at T_E, which has a forward swap rate $S(t, T_S, T_E)$ observed at time t. The formula for this swap rate can be seen in equation 2.15,If we replace the F_t in Equation 2.21 for the forward swap rate as: $dS(t, T_S, T_E) = \sigma S_(t, T_S, T_E)dW_t$, under the swap measure $\mathbb{Q}^A$ (Crispoldi et al., 2016). We can price swaptions as:

$$V^{\text{Black}}_{\text{payer swaption}_t} = N \cdot A(t, T_S, T_E)[S(t, T_S, T_E)\Phi(d_1) - K\Phi(d_2)] \tag{2.26a}$$

$$V^{\text{Black}}_{\text{receiver swaption}_t} = N \cdot A(t, T_S, T_E)[-S(t, T_S, T_E)\Phi(-d_1) + K\Phi(-d_2)] \tag{2.26b}$$

Where d_1 and d_2 are defined in equation 2.24 and $A(t, T_S, T_E)$ is the swap's annuity factor and is defined as $\sum_{i=S+1}^{E} \delta_i P(t, T_i)$. Similarly, with caps/floors, swaptions have an implied volatility that can be deducted from market prices to price them (Brigo and Mercurio, 2007).

However, the Black model has two inherent flaws. Firstly, the basic premise of the σ_B is assumed to be constant and independent of K and the underlying. Whereas it is a stylized fact in the market for years that implied volatilities exhibit a dependence on their strikes, this is known as the **volatility smile**. Which can be seen from market prices as there is a different implied volatility to each strike (Brigo and Mercurio, 2007). Secondly, it does not allow interest rates to become negative as they are assumed to be lognormally distributed (Russo and Fabozzi, 2017).

2.2.2 Shifted Black model

To solve the problem that Black's model has with negative interest rates; the shifted Black model was created. It is a variation of the Black model which can price interest rate derivatives when forward rates are negative. It does this by adding a shift parameter to the forward rate process and thus making the value positive and as a result, we can continue to use the Black model (Russo and Fabozzi, 2017). The stochastic process for the forward rate thus becomes:

$$dF_t = \sigma_B(F_t + s)dW_t \tag{2.27}$$

where σ_B is the constant lognormal volatility, which must be a positive number. Furthermore, s is a constant displacement parameter, which should be chosen in a way that $F_t + s$ and $K + s$ will both be positive. However, the shift parameters should not be chosen at a too high of a level as it may disrupt the Black formula (Kienitz, 2014).

To price the interest rate options, we can adjust the Black model to account for the shift, this is done by adding a shift operator into the relevant pricing equations, those being Equations 2.23, 2.25, 2.26, 2.24. Using these formulae, exchange the $F(t, T_S, T_E)$ and K with $F(t, T_S, T_E) + s$

and $K + s$, respectively (Russo and Fabozzi, 2017).

Even though this model *solves* the problem with interest rates being unable to become negative in the Black model. However, it still carries the constant volatility problem of the Black model, as a result, it is still an incomplete pricing model. Furthermore, the *shift* parameter is subject to change if the forward rate moves, which would require a new calibration for the volatilities, which could be troublesome (Kienitz, 2014).

2.2.3 Bachelier's (Normal) model

The normal model was introduced in 1900 by Bachelier (1900), this model attempts to model the instantaneous forward rate under a *T-forward measure*, which follows the process:

$$dF_t = \sigma_N dW_t \tag{2.28}$$

where σ_N denotes the *constant* volatility of the instantaneous forward rate under normal specification, which is different from Black's volatility. The difference is that Black's volatility measures the annual volatility of the underlying in percentages and the Normal volatility measures the basis point changes in the underlying (Crispoldi et al., 2016).

If we solve equation 2.28, we obtain:

$$F_t = F_0 + \sigma_N W_t \tag{2.29}$$

Which means that the forward rate follows a normal distribution, which means that this model can model negative rates, however, it has the drawback of the possibility of becoming arbitrarily negative. As this model can produce option prices to options with negative strikes, it has become standard for brokers to quote interest rate derivatives with the normal volatility (Crispoldi et al., 2016).

Using the Normal model, we can price caplets and floorlets by applying the fundamental theorem of derivative pricing under a *T-forward measure*. We use:

$$V^{Normal}_{caplet\quad t} = N\delta P(t, T_E) \cdot \sigma_N \sqrt{T_S}(d \cdot \Phi(d) + \phi(-d)) \tag{2.30a}$$

$$V^{Normal}_{floorlet\ t} = N\delta P(t, T_E) \cdot [\sigma_N \sqrt{T_S}(-d \cdot \Phi(-d) + \phi(-d))] \tag{2.30b}$$

Where we define d as:

$$d = \frac{F(t, T_S, T_E) - K}{\sigma_N \sqrt{T_S}} \tag{2.31}$$

where Φ and ϕ are the normal cumulative distribution and the normal probability density, respectively. With this information, we can aggregate the caplets and floorlets in each cap or floorlet to price them (Crispoldi et al., 2016).

$$V^{Normal}_{cap\quad t} = N \sum_{i=S+1}^{E} \delta_i \cdot [\sigma_N \sqrt{T_{i-1}}(-d_i \cdot \Phi(-d_i) + \phi(-d_i))] \tag{2.32a}$$

$$V^{Normal}_{floor\quad t} = N \sum_{i=S+1}^{E} \delta_i \cdot [\sigma_N \sqrt{T_{i-1}}(-d_i \cdot \Phi(-d_i) + \phi(-d_i))] \tag{2.32b}$$

Similarly we can rewrite the normal process with the forward swap rate and end up with $dS(t, T_S, T_E) = \sigma dW_t$, under the swap measure $\mathbb{Q}^A$ (Crispoldi et al., 2016). Therefore, we can derive the Normal model formula for the present value of them as:

$$V^{\text{Normal}}_{\text{payer swaption}_t} = A(t, T_S, T_E) \cdot [\sigma_N \sqrt{T_S}(d \cdot \Phi(d) + \phi(-d))] \tag{2.33a}$$

$$V^{\text{Normal}}_{\text{receiver swaption}_t} = A(t, T_S, T_E) \cdot [\sigma_N \sqrt{T_S}(-d \cdot \Phi(-d) + \phi(-d))], \text{ where} \tag{2.33b}$$

$$d = \frac{S(0, T_S, T_E) - K}{\sigma_N \sqrt{T_S}} \tag{2.34}$$

(Crispoldi et al., 2016)

2.2.4 Risk management within the constant volatility models

To understand how to manage risk, we must look at the indicators that come with the models that we have discussed. These indicators have been called the *Greeks*, as they are each represented by a Greek letter. The most important of which regarding this thesis are Delta and Vega (Hull, 2018).

The delta of an option is the rate of change in the price of the option with regards to the price of the underlying, in mathematical terms, the Delta is the derivative of the option price with regards to the underlying, usually denoted as:

$$\Delta = \frac{\delta V}{\delta F_t} \tag{2.35}$$

which means that we expect the forward rate for caps or swap rate for the swaptions to increase, the value of the payer option increases and decreases for a receiver option. In the Black's model, we would calculate the Delta as $\Delta = \Phi(d_1)$ where the d_1 is Equation 2.24 (Hull, 2018).

However, in practice, for products that rely on the value of interest rates, the market standard is to make an upward parallel shift in the zero curve and calculate the difference in the option price. Such a calculation is called a DV01 (Hull, 2018).

We use this number (DV01 or Delta) as the negative number of units of the underlying that we have to trade to become delta hedged. As a result, our portfolio is protected against changes in the underlying asset, this position has to be re-calibrated now and then to stay hedged. To maintain a perfect hedge, one would have to trade continuously, which would be quite expensive to maintain (Hull, 2018).

Vega, while not being an actual Greek letter, is an important Greek with regards to options trading. It is the rate of change in the value of the option with regards to the implied volatility. Which is quite counter-intuitive within the Black framework, as implied volatility is assumed to be constant. Vega increases with time to maturity and is the highest at the ATM point, the Vega is calculated as:

$$\wedge = \frac{\delta V}{\delta \sigma} = F_0 \sqrt{T} \phi(d_1) \tag{2.36}$$

(Hull, 2018).

2.2.5 Short-rate models (One-factor)

Until now, the focus of the chapter has been regarding vanilla models that can be expressed as an expectation of a single random variable, namely, the forward swap rate or forward rate. In this section, we will attempt to model the short rate r_t with two basic one-factor short-rate models that can model negative interest rates. Those models being, the Vasicek- and Hull-White model.

2.2.5.1 Vasicek Model

In his paper, Vasicek (1977) notes that empirically, interest rates exhibit signs of *mean reversion*. Which means that if rates become unusually high by historical standard, it will likely fall in the future (and vice versa if it is low). To implement these findings, he created an interest rate model that is driven by a single variable, the short rate. Which follows a one-factor Ornstein-Uhlenbeck process in the risk-neutral measure $\mathbb{Q}$

$$dr_t = k[\theta - r_t]dt + \sigma dW_t, r(0) = r_0 \tag{2.37}$$

where r_0, k, θ, and σ are positive constants and W_t is a Brownian motion. An Ornstein-Uhlenbeck process is *mean-reverting*, in a way that the short rate r_t tends to return to the *long-term value* θ at the rate of k (Vasicek, 1977).

If we integrate Equation 2.37 for each $s \leq t$

$$r(t) = r(s)e^{-k(t-s)} + \theta(1 - e^{-k(t-s))}) + \sigma \int_s^t e^{-k(t-u)} dW(u) \tag{2.38}$$

Which means that the rate $r(t)$ is normally distributed which means that rates can become negative. This was considered as a drawback of the model before the introduction of negative rates (Brigo and Mercurio, 2007). Under the Vasicek model, we can find the prices for caps and floors using the notation from Brigo and Mercurio (2007):

$$V^{\text{Vasicek}}_{cap\ \ t} = N \sum_{i=S+1}^{E} (P(t, T_{i-1})\Phi(\sigma_i - h_i) - (1 - \delta_i K)P(t, T_i)\Phi(-h_i)) \tag{2.39a}$$

$$V_{floor}^{\text{Vasicek}}{}_t = N \sum_{i=S+1}^{E} \left(-P(t,T_{i-1})\Phi(\sigma_i - h_i) + (1-\delta_i K)P(t,T_i)\Phi(-h_i)\right) \tag{2.39b}$$

where Φ denotes the normal distribution and

$$P(t,T) = A(t,T)\exp\left(-r_t B(t,T)\right), \tag{2.40a}$$

$$A(t,T) = \exp\left[(\theta - \frac{\sigma^2}{2k^2})(B(t,T) - T + t) - \frac{\sigma^2}{4k}B^2(t,T)\right] \tag{2.40b}$$

$$B(t,T) = \frac{1}{k} \cdot (1 - \exp(-k(T-t)), \tag{2.40c}$$

$$\sigma_i = \sigma\sqrt{\frac{1-\exp(-2k(T_{i-1}-t))}{2k}}B(T_{i-1},T_i) \tag{2.40d}$$

$$h_i = \frac{1}{\sigma_i}\log\left(\frac{(1+\delta_i K)P(t,T_i)}{P(t,T_{i-1})}\right) + \frac{\sigma_i}{2} \tag{2.40e}$$

However, Andersen and Piterbarg (2010) state that the Vasicek model is rarely calibrated well towards the observed prices of caps and swaptions in the market, which makes it a poor pricing model. Furthermore, the Vasicek model is endogenous term structure model, which means that the current term structure of interest rate is an output of the model, not an input. As a result, it is unable to fit the initial term structure and a predefined future behaviour of the volatility of the short rate at the same time (Brigo and Mercurio, 2007).

2.2.5.2 Hull-White one-factor model

In their paper, Hull and White (1990) present their extension of the Vasicek model, it is referred to as the Hull-White one-factor model. This model was created to fit the Vasicek model to the initial term structure of rates, which is the θ term. It can be described as the Vasicek model with a time-dependent reversion level. This model assumes that instantaneous short-rate has the process:

$$dr_t = k(\theta_t - ar_t)dt + \sigma dW_t \tag{2.41}$$

where a and σ are constants, the a defines the mean reversion rate of the model. Hull (2018) shows that we can calculate θ_t, using the instantaneous forward rates of the markets $F^m(0,t)$

and the market discount factors $P^m(0,t)$:

$$\theta_t = F^m(0,t) + aF^m(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{2.42}$$

Which gives us the impression that on average the process r at time t follows the slope of the initial instantaneous forward rate curve. If the process deviates from that curve, it reverts to the long term value at the rate of a. To price bond options using the Hull-White model, we must define:

$$P(t,T) = A^m(t,T)e^{-B(t,T)r(t)} \tag{2.43}$$

$$A^m(t,T) = \frac{P^m(0,T)}{P^m(0,t)} + \exp\left(B(t,T)F^m(0,t) - \frac{\sigma^2}{4a}(1 - e^{-2at})B(t,T)^2\right) \tag{2.44}$$

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{2.45}$$

Using these formulae for bond pricing, we can define the Hull-White cap and floor prices:

$$V_{cap\ \ t}^{HW} = N \sum_{i=S+1}^{E} \left(P(t,T_{i-1})\Phi(\sigma_i - h_i) - (1 - \delta_i K)P(t,T_i)\Phi(-h_i)\right) \tag{2.46a}$$

$$V_{floor_t}^{HW} = N \sum_{i=S+1}^{E} \left(-P(t,T_{i-1})\Phi(\sigma_i - h_i) + (1 - \delta_i K)P(t,T_i)\Phi(-h_i)\right) \tag{2.46b}$$

Where we can use the same h_i and σ_i as in the Vasicek model using Equations 2.40e and 2.40d respectively (Hull, 2018).

The Hull-White model has been described as the most important interest-rate models, it has its drawbacks. The volatility of the forward rate is a declining function with regards to time, which goes against empirical evidence (Hull, 2018).

2.2.5.3 Swaption pricing with short-rate models

Pricing swaptions is quite more cumbersome for the one-factor models, this is because ZCB prices and r are inversely correlated. As a result, the short-rate models can price the coupon-bearing bond as the sum of European options on ZCB using these steps using Jamshidian's (1989) decomposition:

1. Find r^* as the value of r where the price of the coupon-bearing bond is the same as the strike price of the option on the coupon-bearing bond at the option maturity T_S.
2. Classify the coupon bearing bond as a collection of the zero-coupon bonds and find the price of the options on those ZCB. The strike price of the options will be the same as the value of the ZCB at time T_S when $r = r^*$
3. Create a function that takes the price of the coupon-bearing bond and subtracts it with sum of the ZCB options. Set the output of the function to zero by solving for r (Hull, 2018).

As we already know, a collection of ZCBs options is a coupon-bearing bond option, which is the same as a swaption, and it can be seen that this process could get quite cumbersome if calculating for multiple swaptions. This method works for all the short rate models that have been covered in this section (Hull, 2018).

2.2.6 Libor Market Model

According to Brigo and Mercurio (2007), short-rate models were used to price and hedge interest rate derivatives until the introduction of the market models. The Libor market model, or the Log-normal Forward LIBOR model or the Brace-Gatarek-Musiela 1997 Model (BGM model), named after the authors. It will be referred to as the LMM and will be briefly featured here.

The advantages of the LMM is that each forward rate is modelled by a log-normal process under the T-forward measure for the maturity T_k. The process of a single forward rate can be described as

$$dF_k(t) = \sigma_k(t)F_k(t)dZ_k(t), t \leq T_{K-1} \tag{2.47}$$

where the set $E = T_0, T_1, ...T_E$ as the expiry dates and $F_k(t) = F(t, T_{k-1}, T_k)$ as the forward rate for time t between T_{k-1} and T_k and the $Z_k(t)$ is an M-dimensional column vector of Brownian motion with the instantaneous covariance $\rho = (\rho_{i,j})_{i,j=1,\ldots,M}$

$$dZ^k(t)dZ^k(t)' = \rho dt \tag{2.48}$$

and $\sigma_k(t)$ is the horizontal M-vector for the volatility coefficient for the forward rate $F_k(t)$ (Brigo and Mercurio, 2007)

The selection of the appropriate instantaneous covariance structure is very important for the LMM as it can result in errors with calibration, the instantaneous correlation, terminal correlation, and the smoothing the caplet volatilities' evolution over time. If we assume that the volatility $\sigma_k(t)$ is a piecewise constant and we define $\sigma_k(t) = \sigma_{k,t}$, meaning that we can specify the volatility among all the forward rates $F_k(t)$ (Brigo and Mercurio, 2007).

To use the LMM to price interest rate derivatives, the first step is to calibrate it using caplet volatilities by stripping them from market cap quotes. Brigo and Mercurio (2007) show that the price of a T_{i-1} caplet in the LMM framework coincides with the Black caplet formula of Equation 2.25a and as a result, we can price caplets with the LMM as:

$$V_{Caplet}^{LMM}(0, T_{i-1}, T_i) = P(0, T_i)\delta_i[F_0\Phi(d_1) - K\Phi(d_2)] \tag{2.49a}$$

$$d_1 = \frac{\ln(F/K) + Tv^2/2}{v\sqrt{T}} \tag{2.49b}$$

$$d_2 = \frac{\ln(F/K) - Tv^2/2}{v\sqrt{T}} \tag{2.49c}$$

we define $v_{T_{i-1-caplet}}$ as the termed T_{i-1} caplet volatility, which is the average percentage variance of the forward rate $F_i(t)$ and the volatility can be described as:

$$v_{T_{i-1}}^2 = \frac{1}{T_{i-1}} \int_0^{T_{i-1}} \sigma_i(t)^2 dt = \frac{1}{T_{i-1}} \sum_{j=1}^{i} (T_{j-1} - T_{j-2})\sigma_{i,j}^2 \tag{2.50}$$

Brigo and Mercurio (2007) continue this discussion by listing the different assumptions that the piecewise constant instantaneous volatilities can take within the model. For example, it could be that the volatility σ_{i,T_E} could be dependent on time to maturity, only the maturity of the relevant forward rate, etc. This means that the LMM can view these volatilities in many ways.

Furthermore, using the inherent relationship between the forward rates and swap rates, we can calculate the swaption volatility using the LMM framework. Using that swaption volatility, we can price payer swaptions as:

$$(v_{T_S,T_E}^{Black})^2 = \sum_{i,j=T_S+1}^{T_E} \frac{w_i(t)w_j(t)F_i(t)F_j(t)\rho_{i,j}}{S(t,T_S,T_E)} \int_0^{T_E} \sigma_i(t)\sigma_j(t)dt \tag{2.51}$$

where

$$w_i = \frac{P(t,i)\delta_i}{\sum_{k=1}^{n} P(T_k,T_{k+1})\delta_{T_k,T_{k+1}}} \tag{2.52}$$

$$S(t,T_S,T_E) = \frac{P(t,T_S) - P(t,T_E)}{\sum_{i=T_S+1}^{T_E} \delta_i P(t,T_i)} = \sum_{i=T_S+1}^{T_E} w_i(t)F_i(t) \tag{2.53}$$

(Brigo and Mercurio, 2007)

As we can tell from one of the names that the LMM has, it assumes that the forward rate is lognormally distributed and as such cannot become negative. Furthermore, according to Hagan and Lesniewski (2008), the volatility smile structures that the LMM produces are rather rigid and do not match the smiles observable in the markets for vanilla caps/floors and swaptions. Which makes it a poor volatility estimator for caplets and swaptions.

This was a brief overview of the LMM, for further literature for it, the author points the reader to Brigo and Ricardo (2007).

2.2.7 Comparison of the models

As we can see, all of the models that are built around Black's framework, follow a common thread, they all require the *constant* volatility parameter and require that it is independent of the forward rate and the strike of the option.

The short rate models that we covered in this section, were the Vasicek model and the Hull-White model. The Vasicek is an equilibrium model, which has the disadvantages that the initial term structure is the output of the model rather than the input. The Hull-White model is a model that is analytically tractable and includes mean-reversion. Both models can price caps, floors and swaptions, however, the calculation for swaptions can be quite cumbersome. Many other short-rate models exist, such as the Cox-Ingersoll-Ross (CIR) model, Ho-Lee model (Brigo and

Mercurio, 2007).

The LIBOR market model stands out as it is based around the fact that it models forward rates, where each forward rate has its volatility, the caplet volatility. In this model, all forward rates are correlated with each other. However, the LMM is unable to produce a well-rounded volatility smile to fit the market, therefore we must look toward models that are designed to function with stochastic volatility.

Nonetheless, these models that were featured in this section, have been used within the markets and academia. The LIBOR market model is without a doubt the most useful of the models and the framework that has been built around it is extensive (Brigo and Mercurio, 2007).

2.3 Stochastic volatility models

As we have seen from the previous chapter, there are numerous models used to price interest rate derivatives. None of them are without their faults, regarding the Black model and its variants. They all require the volatility parameter to be constant as has been stated before, options with different strikes require different volatility parameters to match the prices observed in the market.

2.3.1 Volatility smiles

Volatility smiles are the plots of the observed implied volatility of an option in the market and its strike price, as when you draw a line between the volatilities, it often shows a smile (or a skew, depending on markets). Using the data presented in the next chapter (ICAP, 2020b), we can visualize the smile of a 3Y5Y swaption in Figure 2.1.

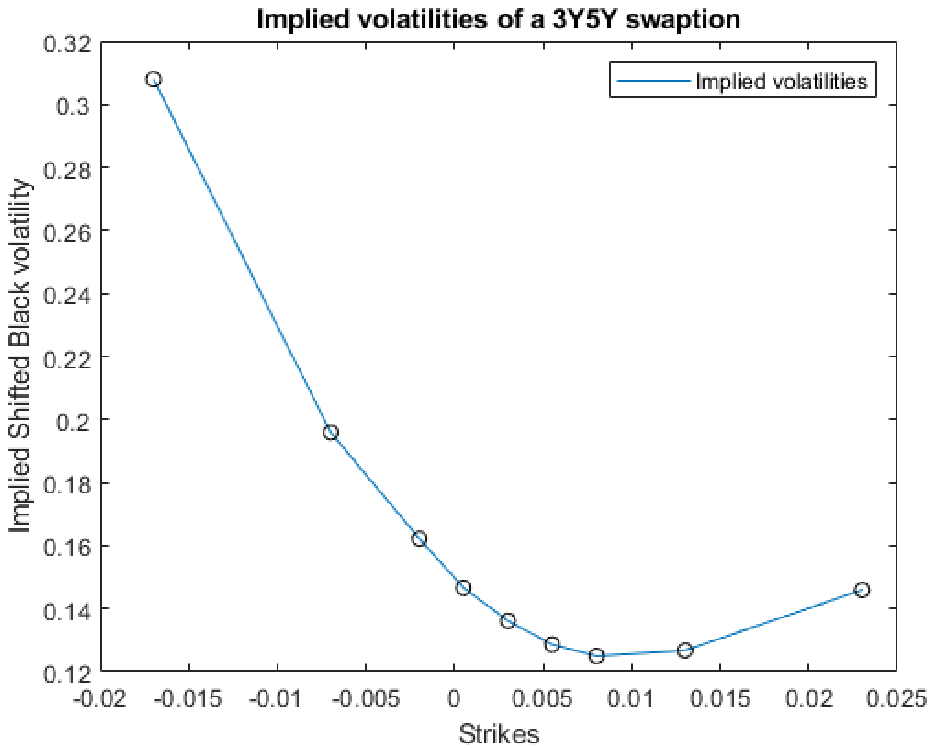

Figure 2.1: 3Y5Y Swaption smile

As we can see from the figure, the volatility is not constant and has a clear dependence on strikes as the volatility changes with regards to strikes. The line between the points in the graph was interpolated between the 9 observations. It is clear that the market differentiates between in-the-money options and out-of-the-money options, as we can see from the figure, in-the-money options are more valuable than the out-of-the-money ones.

2.3.2 Local volatility model

Derman and Kani (1994) and Dupire (1994) created a self-consistent models to obtain volatilities for any strike, which would deal with the smiles and skews found in the interest rates markets. These models uses market prices of options to find a local specification of the underlying process, this is done in a way that the model output volatilities will match the market implied volatilities. We can describe the forward rate process under the local volatility model under a T-forward measure as

$$dF_t = \sigma(F_t, t)dW_t \tag{2.54}$$

where we define $C(F_t, t)$ as a deterministic volatility coefficient which is dependent on the forward rate F and time t. This coefficient will be determined numerically from the smile. For a full account of the function of the model and its derivations, readers can look to Hagan *et al.* (2002).

These models have widely been used as they can accurately price options and work with the smile elements of the volatilities. However, as Hagan *et al.* (2002) explain, the model is unable to provide the user with a good indicator to hedge his position, as these models predict a dynamic evolution for the smile that is contrary to what the inputs of the model suggest.

As has been stated in this chapter, volatility is dynamic and moves with changes in the forward rate; the local volatility models predicted the volatility smile to move in the opposite direction than the direction that the markets expected. This caused the *Delta* and *Vega* risks metrics under the local volatility models to wrong. As a result, we can see that the local volatility models are suited for pricing but not for risk management (Hagan et al., 2002).

2.3.3 SABR model

The natural continuation from the local volatility models would be a model with the same pricing capabilities but with the ability to accurately be used for risk management. The **SABR model** fills that role. The name of the model stands for the parameters within the model, **the Stochastic Alpha Beta Rho**.

Hagan, Kumar, Lesniewski, and Woodward (2002) propose a stochastic volatility model that models the forward price of a single asset (LIBOR, swap rate etc.) under the assets canonical measure, with a stochastic volatility of the said asset, denoted as σ. As a result, it is a two-factor model where both F_t and σ_t are stochastic variables that develop over time by the following system of stochastic differential equations:

$$dF_t = \hat{\alpha}_t F_t^{\beta} \cdot dW_t, \quad F_0 = f \tag{2.55a}$$

$$d\hat{\alpha}_t = \nu\alpha \cdot dZ_t, \quad \hat{\alpha_0} = \alpha \tag{2.55b}$$

under the T-forward measure, they are correlated by

$$dW_t \cdot dZ_t = \rho dt \tag{2.55c}$$

where $0 \leq \beta, \alpha > 0$ and $-1 \leq \rho \leq 1$. W_t and Z_t are two correlated Wiener processes that have the correlation coefficient ρ under the T-forward measure $\mathbb{F}^T$ and ν is the *volatility-of-volatility* of the forward rate and the β is the *power* parameter of the model. All of the parameters in the model are constants and they are all specific to a distinct forward rate (Hagan et al., 2002).

2.3.3.1 The SABR parameters

In this section, we will look at the parameters of the SABR model to gain a better understanding of the model; we can examine the different effects of changing each parameter has to the smile.

- β: This is the *power* parameter. It controls the **steepness** of the skew. Due to the similar effect on the skew as ρ, this parameter is often fixed before the calibration of the other parameters. The β parameters should be fixed to fit within $0 \leq \beta \leq 1$ (West, 2005).

  Hagan et al (2002) recommended in their paper that the β parameter be observed from the historical values of the *backbone*. Such a calculation can be done by regressing the historical $\log \sigma_{ATM}$ with the $\log f$ linearly. However, such is usually not done in practice (West, 2005).

  As West (2005) shows in his paper, estimating the β parameter is very time-sensitive and deteriorates towards zero as the options draw to expiry. He notes that by fixing the β parameter to a single value, rather than having it dependent on time, there was little need to re-calibrate the model as often as it was required had the β parameter been fluid.

  Usually, it is fixed to one of three values, first of which is 0, where we obtain a stochastic Normal model, in that model the forward process is stochastic normally distributed with a mean of zero and a standard deviation that is lognormally distributed. As the process is

$$dF_t = \sigma W_t \tag{2.56}$$

  Using this β therefore, allows the forward process to have negative values, which is very relevant for the current economic conditions. By fixing the β parameter to zero, allows us to model the *backbone* in a way that the whole volatility smile moves with the change in the forward rate (Hagan et al., 2002).

Fixing the β to 0.5 gives us a model that has shares similarities to the Cox-Ingersoll-Ross model (CIR), which is a short-rate model. We can see that the process becomes

$$dF_t = \sigma \sqrt{F_t} dW_t \tag{2.57}$$

and as a result, we can deduct that the current level of the price process is the square root of the F_t. This results in the model not being able to model negative interest rates (Hagan et al., 2002).

If the β is fixed to 1, we obtain a model that is similar to the Black's model as the forward process is lognormally distributed. This gives the backbone effect of the volatility smile moving horizontally with the change in the forward rate, with the ATM volatility being constant (Hagan et al., 2002).

To see the effect that the β parameter has on the SABR volatility smile, Figure 2.2 shows the difference when changing the parameter. It can be seen that as the β parameter increases the output of the SABR model has a flatter curve and the lower β results in a steeper curve. Note that the other parameters for the different β parameters from 0.5 are not calibrated with the different β, which explains the inaccuracies in modelling the market volatilities.

- α: This is the starting point of the volatility parameter; it influences the **level** of the skew. A small change in this parameter will shift the smile in the direction of the change; a positive change will increase the values of the smile while a negative change decreases the values of the smile (Hagan et al., 2002). This can be seen in Figure 2.3.

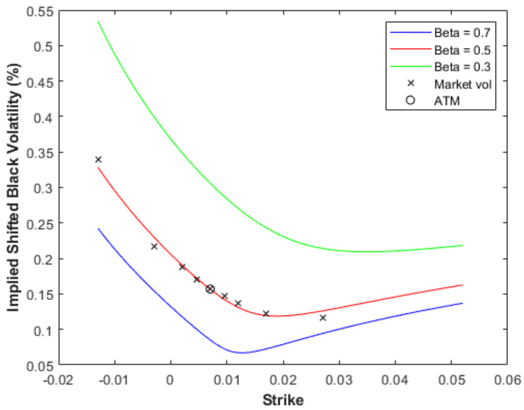

Figure 2.2: Changing β for a 30Y2Y swaption

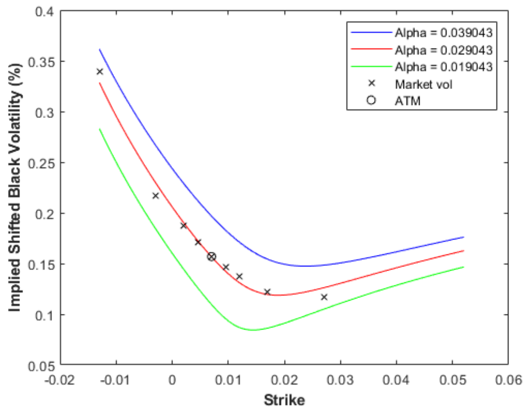

Figure 2.3: Changing α for a 30Y2Y swaption

As can be seen in this figure, the alpha parameter influences the level of the output smile.

- ν: The volatility-of-volatility parameters affects the **curvature** of the smile. This parameter should be calibrated with the bounds that it is equal to or greater than zero (Hagan et al., 2002). Attempting to visualize the change of the vol-of-vol results in Figure 2.4. We can see that the difference in the different smiles is quite large, as ν increases the smile displays a lot of curvature and when ν is reduced we can see that the curvature decreases as well.

- ρ: As noted before, the correlation parameter also influences the **steepness** of the skew (similarly to β) (Hagan et al., 2002). To visualize this parameter, we can see the changes in the output of the SABR model in Figure 2.5. Looking over the figure, we can see that the steepness of the smile does in fact change with regards to ρ. However, we can see that the effect is again very similar to the β changes in Figure 2.2 and that is the reason why these parameters are calibrated separately.

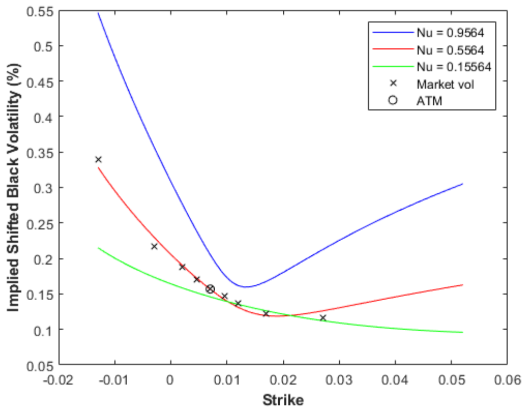

Figure 2.4: Changing ν for a 30Y2Y swaption

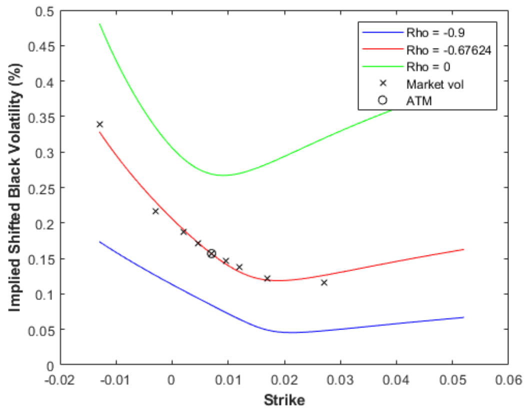

Figure 2.5: Changing ρ for a 30Y2Y swaption

2.3.3.2 Usage of SABR

The function of the SABR is different from *most* other stochastic volatility models, (such as Heston etc), it differs in the way that the SABR model does not produce option prices. Rather, it produces an estimate of the implied volatility curve and is used in the Black- or Normal model to obtain a price for a specific interest rate derivative. This approximation is called the Hagan formula (Hagan et al., 2002).

$$\sigma_{Black}(K, f, T_S) = \frac{\alpha}{(f \cdot K)^{(1-\beta)/2} \cdot \left\{1 + \frac{(1-\beta)^2}{24}\log^2(f/K) + \frac{(1-\beta)^4}{1920}\log^4(f/K) + \dots\right\}} \cdot \left(\frac{z}{x(z)}\right) \cdot \left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{v^2}{(f \cdot K)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\alpha\nu}{(fK)^{1-\beta}/2} + \frac{2-3\rho^2}{24}\nu^2\right] T_S + \dots\right\} \tag{2.58}$$

$$z = \frac{\nu}{\alpha}(f \cdot K)^{(1-\beta)/2} \ln\left(\frac{f}{K}\right) \tag{2.59}$$

$$x(z) = \ln\left[\frac{\sqrt{1 - 2\rho z + z^2]} + z - \rho}{1-\rho}\right] \tag{2.60}$$

However, when $f = K$ (ATM options), Equation 2.58 can be simplified to:

$$\sigma_{Black}^{ATM} = \sigma_{Black}(f, f) = \frac{\alpha}{f^{(1-\beta)}} \cdot \left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4}\frac{\rho\beta\alpha\nu}{f^{1-\beta}} + \frac{2-3\rho^2}{24}\nu\right] T_S\right\} \tag{2.61}$$

The Hagan formula is, therefore, a closed-form approximation of Black's implied volatility as a function of the forward rate, the strike, the maturity T_E and the SABR parameters. Which we plug into the Black pricing formula for caps/floors and swaptions (equations 2.25 & 2.26) to obtain the prices of the instruments (Hagan et al., 2002).

Similarly, there exists a formula to calibrate an implied normal volatility smile as seen in (Hagan et al., 2002).

$$\sigma_N(K,f) = \alpha(fK)^{\frac{\beta}{2}} \frac{1 + \frac{1}{24}\log^2\frac{f}{K} + \frac{1}{1920}\log^4\frac{f}{K} + \dots}{1 + \frac{(1-\beta)^2}{24}\log^2\frac{f}{K} + \frac{(1-\beta)^4}{1920}\log^4\frac{f}{K}} * \\ (\frac{z}{x(z)}) * \left\{1 + \left[\frac{-\beta(2-\beta)}{24}\frac{\alpha^2}{(fk)^{1-\beta}} + \frac{1}{4}\frac{\beta\rho\alpha\nu}{(fK)^{\frac{(1-\beta)}{2}}} + \frac{2-3\rho^2}{24}\nu^2\right]\right\} * T_S + \dots \tag{2.62}$$

2.3.3.3 Calibrating a SABR model

As stated before, for each option maturity and underlying, the SABR model must be calibrated. We calibrate the SABR model towards the volatilities observed in the market. There are two methods to calibrate the model and for both, it is required for the parameter β to be fixed beforehand.

- The first method - Estimating α, ρ, and ν. Once $\hat{\beta}$ has been chosen, we need to estimate α, ρ, and ν. This will be performed by minimizing the errors between the model and market volatility $\bar{\sigma_t}$ using a sum of squared errors (SSE), which produces

$$\Omega = (\hat{\alpha}, \hat{\rho}, \hat{\nu}) = \arg\min_{\Omega} \sum_{i=1}^{n} (\bar{\sigma_i} - \sigma_{\text{Black}}(K_i, f, \Omega))^2 \quad \text{s.t.} \tag{2.63}$$

$$-1 \leq \rho \leq 1$$
$$0 \leq \alpha$$
$$0 \leq \nu$$

Then we take the $\hat{\alpha}, \hat{\rho}, \hat{v}$ and plug them into equation 2.58 to obtain σ_{Black} and plug it into Black's formula to obtain the price of the option (Crispoldi et al., 2016).

- The second method - Estimating ρ and ν. Another way to calibrate the SABR model, which was proposed by West (2005) is to calculate the value of α before parameterizing the model and as a result, we only estimate ρ and ν. This is done if market data for ATM implied volatility is available. We can obtain $\hat{\alpha}$ from equation 2.61 (or its Bachelier equivalent) by inverting that formula and taking note that σ is the root of the cubic equation

$$\left(\frac{(1-\beta)^2}{24f^{2-2\beta}}T_S\right)\alpha^3 + \left(\frac{\beta\rho v}{4f^{1-\beta}}T_S\right)\alpha^2 + \left(1+\frac{2-3\rho^2}{24}v^2T_S\right)\alpha - \sigma^{ATM}f^{1-\beta} = 0 \quad (2.64)$$

As a result, we can perform the same minimization but with $\alpha(\rho, v)$ as a function of ρ and ν. Therefore, equation 2.63 becomes:

$$(\hat{\sigma}, \hat{\rho}, \hat{\nu}) = \underset{(\sigma,\rho,\nu)}{\arg\min} \sum_{i=1}^{n} (\bar{\sigma}_i - \sigma_{\text{Black}}(K_i, f, (\alpha(\rho, v), \rho, \nu)))^2 \quad (2.65)$$

This approximation will take more time to compute as every iteration of the algorithm must produce ρ and ν but must also use a root-finding algorithm to obtain σ from equation 2.64 that uses input from f, K, σ^{ATM}, T_S and the other SABR parameters β, ρ, v. In the end, we obtain the relevant parameters and plug them into equation 2.58, similarly as in the previous method and obtain the option price (West, 2005).

2.3.4 Shifted SABR model

Since the SABR model is unable to model negative forward rates and that when the forward rate approaches zero the model becomes increasingly inaccurate. This is similar to the problem with Black's model, as a result, we can implement the shift solution to the SABR model that we have previously discussed. The result being the Shifted SABR model (Antonov et al., 2019).

The Shifted SABR model has the following processes:

$$dF_t = (F_t + s)^{\beta}\sigma_t dW_t \quad (2.66a)$$

$$d\sigma_t = \alpha \sigma_t dZ_t \tag{2.66b}$$

$$dW_t dZ_t = \rho dt \tag{2.66c}$$

Where the s is a displacement parameter (Antonov et al., 2019).

This has the same problem as the shifted Black model, where if the choice of s has been decided and interest rates go further below the shift parameter, then the displacement parameter must be chosen again, which results in the model parameters having to be calibrated again. As a result, this could influence the parameters in such a way that it could change the values of the options or the Greeks, which could prove costly for trading divisions (Antonov et al., 2019).

2.3.5 Obłój's refinement of the SABR model

Obłój (2008) claims that the Hagan formula is known to produce incorrect prices when strikes are low and maturities are long. To fix this problem, he proposed a correction which is now called Obłój's refinement.

Obłój's refinement can be summed up in these formulae:

$$\sigma_B(x, T_S) = I_B^0(x)(1 + I_H^1(x)T_S) \tag{2.67a}$$

$$I_H^1 = \frac{(\beta-1)^2}{24} \frac{\alpha^2}{(fK)^{1-\beta} + \frac{1}{4}\frac{\rho\nu\alpha\beta}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}} \nu^2 \tag{2.67b}$$

where $x = \log(f/K)$ and if $x = 0$ then

$$I_B^0(0) = \alpha K^{\beta-1} \tag{2.68a}$$

The Obłójs method has a difference depending on the value of the β parameter, if it is equal to 1 then

$$I_B^0(x) = \frac{\nu x}{\log\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)} \tag{2.68b}$$

$$\text{where } z = \frac{\nu x}{\alpha} \tag{2.68c}$$

and the other case, where β is less than one

$$I_B^0(x) = \frac{\nu x}{\log\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)} \tag{2.68d}$$

$$\text{where } z = \frac{\nu(f^{1-\beta} - K^{1-\beta})}{\alpha(1-\beta)} \tag{2.68e}$$

Using these formulae, we have gained a new way to calculate the implied Black volatility and whether Obłój's refinements do indeed improve the volatility fit will be tested later on (Obł ój, 2008).

2.3.6 Risk management under the SABR model

As we noted earlier, the local volatility models proved to be ineffective for hedging, the SABR model was created to fix that problem and therefore it might be relevant to go over the hedging capabilities of the SABR model.

The SABR Greeks uses the original Greek names but they have been changed. In the original SABR paper, Hagan *et al* (2002) described the Delta as the change in the current value of the option when the forward rate is increased by δf while keeping the other parameters fixed, notably the α_t. Furthermore, as we expect implied volatility to move with the underlying the SABR delta is equal to:

$$\Delta = \frac{\delta V}{\delta f} + \frac{\delta V}{\delta \sigma}\frac{\delta \sigma}{\delta f} \tag{2.69}$$

The Vega, the change in the value of the option with regards to the underlying volatility is also calculated by the change in the α parameter as $\alpha_T = \alpha_t + \delta\alpha$ while keeping $f_T = f_t$. Then we can calculate the Vega as:

$$\wedge = \frac{\delta V}{\delta \sigma}\frac{\delta \sigma}{\delta \alpha} \tag{2.70}$$

These have been called the classic SABR Greeks, Barlett (2006) derived new risk formulas in his paper and they have been called the Barlett Greeks. As we know from Equation 2.55, α and f are correlated, which implies that a change in α might occur as f changes. As a result, we should expect a change of δ_f for α when f changes by Δ.

To obtain $\delta_f \alpha$, Barlett (2006) rewrote the SABR dynamics in terms of independent Brownian motions, W_t and Z_t.

$$dF_t = \hat{\alpha}_t F_t^{\beta} dW_t \tag{2.71}$$

$$d\hat{\alpha}_t = \nu \alpha_t (\rho dW_t + \sqrt{1-\rho^2} dZ_t) \tag{2.72}$$

Which can be put together to form

$$d\hat{\alpha}_t = \frac{\rho\nu}{F^{\beta}} dF_t + \nu \alpha_t \sqrt{1-\rho^2} dZ_t \tag{2.73}$$

Which states that the time evolution of α can be decomposed into two independent components: the change of F_t and the other being the idiosyncratic changes in α_t (Bartlett, 2006). Then we can think of average changes in α as

$$\delta_f \alpha = \frac{\rho\nu}{f^{\beta}} \Delta f \tag{2.74}$$

Which makes the change in the option value

$$\Delta V = \frac{\delta V}{\delta f} + \frac{\delta V}{\delta \sigma} \left(\frac{\delta \sigma}{\delta f} + \frac{\delta \sigma}{\delta \alpha} \frac{\rho \nu}{f^{\beta}} \right) \delta f \tag{2.75}$$

Which makes the delta risk calculated as:

$$\Delta = \frac{\delta V}{\delta f} + \frac{\delta V}{\delta \sigma} \left(\frac{\delta \sigma}{\delta f} + \frac{\delta \sigma}{\delta \alpha} \frac{\rho \nu}{f^{\beta}} \right) \tag{2.76}$$

As we can see, this is almost the same as the original SABR delta but with the addition of the term (Bartlett, 2006)

$$\frac{\delta V}{\delta \sigma}\frac{\delta \sigma}{\delta \alpha}\frac{\rho \nu}{f^{\beta}} \tag{2.77}$$

Moving towards the Vega risk, Barlett (2006) derived it in a similar way to the delta. We should expect there to be some movement in the underlying if the volatility changes. Therefore, we would expect f to move by $\delta_{\alpha} f$ on average when α changes by $\Delta\alpha$.

$$\delta_{\alpha} f = \frac{\rho f^{\beta}}{\nu} d\alpha_t \tag{2.78}$$

and then we can find the vega risk as:

$$\wedge = \frac{\delta V}{\delta \sigma}\left(\frac{\delta \sigma}{\delta \alpha} + \frac{\delta \sigma}{\delta f}\frac{\rho f^{\beta}}{\nu}\right) \tag{2.79}$$

(Bartlett, 2006)

We have derived the two Greeks that change the most with the SABR model and the two that Barlett refined to fully use the properties of the SABR model (Bartlett, 2006).

2.3.7 Further research in the SABR model

Even though interest in stochastic volatility models is considerable, very little empirical research has been done on the SABR model. Wu (2012) performed a comprehensive empirical study regarding the pricing and hedging capabilities of the SABR model using interest rate caps. He found out that the SABR model manages to fit the implied volatility smile quite well and that it represents the dynamics of the smile over time very well. Furthermore, he found that daily re-calibration of the SABR parameters was essential to obtain the best possible pricing performance. He decided to implement Obłój's refinement in his analysis.

Furthermore, Wu tested the usage of a SABR model where the ν or ρ parameters are constant and calibrated after the fact and tested to see whether those parameters were necessary to be re-calibrated every day. However, this study was performed in a positive interest rate regime and does not explore the shifted or the Normal variants of the SABR model (Wu, 2012).

Antonov (2019) created a new SABR model that is specifically designed to model negative interest rates, it is called the Free-boundary SABR model. It tries to avoid the need to choose a *shift* parameter by forcing the forward rate to be positive by making the forward rate process dependent on the absolute value of the forward rate.

2.4 Summary of chapter

In this chapter, we have gone over the basics of interest rate derivatives, their properties, usages, and many ways to value them using different models. We discussed the advantages and disadvantages of each model. The models with Black's framework assumed that the volatility parameter would be constant for all strikes, something that empirical evidence points against.

To solve that issue, we introduced models that assume that volatility is a stochastic process, this is the SABR model. We discussed its properties and the underlying faults of the model, the biggest being that interest rates cannot become negative. Therefore, the model is unusable when interest rates are negative, adjustments to the model have been made to allow it to include negative rates.

The Shifted SABR model introduces a displacement parameter which pushes the forward rate into positive space and the Normal SABR model, which forces the β parameter to be zero to allow the forward rate to become negative. Furthermore, we discussed Obłój's refinement of the SABR model which improves performance with options that have low strikes and long maturities.

We discussed the ways that the SABR model can be used in risk management by showing the Greeks that were presented in the original SABR paper by Hagan and the ones refined by Barlett. Lastly, we talked about research that has been performed with or on the SABR model, not much has been empirically researched on the performance of the model but numerous studies have attempted to alter the model to fit specific needs.

3 Data & methodology

3.1 Data

This sub-chapter is devoted to the description of the data used throughout the thesis.

3.1.1 Spot rates

The spot rates that will be used for the discounting and forwarding of the thesis were collected the Euro area yield curves for January the 16^{th} of 2020, these figures were obtained from the European Central Bank on the 10^{th} of March 2020 (European central bank, 2020). We can see these spot rates in Table A1.1 in the Appendix.

3.1.2 Implied volatilities

The author has acquired data for EUR caps in normal volatilities, which can be seen in Table A1.2 in the Appendix. This normal cap data is sourced from ICAP, retrieved on the 10^{th} of March 2020 for the trading day of 16^{th} of January 2020. It will be used as the basis for the analysis (ICAP, 2020a).

Since the *Bloomberg* terminal does not offer to *shift* volatilities (to the author's knowledge), we can translate between the normal volatilities and the shifted black volatilities. This is performed by calculating the cap price using the Bachelier pricing formula, Equation 2.32a and then calculating another value with the Black pricing formula (equation 2.25) and adding a shift factor of 2% with an unknown volatility. Finally, we take the difference from these two values and set it to zero by changing the implied shifted black volatility. Performing this for all the normal cap volatilities will give us a shifted black volatility surface.

Furthermore, the author has acquired normal volatilities for swaptions, they can be seen in Table A2.1, which was sourced from ICAP as well (ICAP, 2020b). This data was plentiful and had volatilities away from the money as well. However, in the next chapter, we will be focusing on a single section of that table, where the tenor of the swaption is 5 years. Which can be seen in Table A1.4, also in the Appendix.

This problem with the unavailability of the shifted black volatilities remains with swaptions, as a result, a similar conversion has been performed to obtain shifted black volatilities for swaptions as was done with the caps. As we know that implied volatilities have a one-to-one relationship with prices, we can convert the normal volatilities into EUR prices and doing the same for an unknown Shifted Black volatility using the Shifted Black model and setting the equation equal to zero by changing the Shifted volatility. For this instrument, we use a shift parameter of 3%, as we have strikes that go beyond the -2% mark. The result of the conversion for the 5-year tenor swaption can be seen in Table A1.5 in the appendix.

3.2 Empirical model

In this section of the thesis, we will discuss the empirical model that will be tested throughout the thesis and explain exactly the thought process and the way that we will attempt to answer the research question.

3.2.1 Stripping cap volatility

As we know from the previous chapter, a cap has a *flat* volatility that is quoted for the entire cap but each caplet within that cap has a specific *spot* volatility. The process of calculating the actual implied spot volatility of caplets from the market quotes is called *caplet stripping.* The output from this process is a sequence of caplet volatilities for all maturities possible for caps, i.e. from a year to 30 years. In their paper on volatility cube construction, Hagan and Konikov (2004) discuss the different methods of *caplet stripping*, one of which is called bootstrapping, it assumes that caplet volatilities are a function of time to maturity. As a result, there must be a volatility term structure, which is without strike dependence. The technique works as follows:

- 1: For a given strike, sort the cap prices in ascending order of maturity.
- 2: Find the difference between the caps

$$Cap^{Mkt}(t, T_i, N, K) - Cap^{Mkt}(t, T_{i-1}, N, K), i = 1, \dots, N \tag{3.1}$$

where $CAP_0^{Mkt} = 0$ and N is the number of observed cap prices who have the strike K

- 3: Partition the caplets in a way that they are sorted by the relevant price difference (where the first set of caplets correspond to the caplets in the shortest cap)
- 4: Start with the first caplet to have the same volatility as the first cap. For each partition, assign a common volatility and solve by using a one-dimensional root finder for the price difference.

$$Cap(t, T_i, N, K) - Cap(t, T_{i-1}, N, K) = \sum_{j=i+1}^{n} Caplet(t, T_j, N, K, \sigma(K, i)) \tag{3.2}$$

This way works as it decouples the problem and allows the root-finding algorithm to work at different expiry to be done in parallel. This method can only fail if a specific price difference is less than the intrinsic value of the set of caplets (Hagan and Konikov, 2004).

However, this method does not apply to ATM caps, as its strike differs on maturity, therefore, to solve this problem, we can use another method that works for ATM caps, as ATM caps have a strike that is equal to the forward swap rate $S(t, T_S, T_E)$. According to Schoenmakers (2005), a given ATM caplet volatility is calculated using this method:

As ATM cap volatility is given as a flat volatility, we can view an ATM cap as the following

$$Cap_{t,k}^{ATM} = \sum_{j=1}^{k-1} Caplet(F_j, S_{1,k}, \sigma_{1,k}^{ATM}, T_j) \tag{3.3}$$

for $k = 2, \ldots, n$. As we know that $\sigma_{1,k}^{ATM}$ is a flat volatility. Using this formula, we want to find a sequence of caplet volatilities σ_j such that

$$Cap_{t,k}^{ATM} = \sum_{j=1}^{k-1} Caplet(F_j, S_{1,k}, \sigma_j, T_j) \tag{3.4}$$

Therefore, we should be able to solve from

$$Cap_{1,2}^{ATM} = Caplet(F_1, S_{1,2}, \sigma_{1,2}^{ATM}, T_1) = Caplet_1(F_1, F_1, \sigma_1, T_1) \tag{3.5}$$

As $S_{1,2} = F_1$, we have $\sigma_1 = \sigma_{1,2}^{ATM}$. As a result, for some k with $k < n$, the volatilities of σ_j,

$j = 1, \ldots, k-1$ are already obtained (Schoenmakers, 2005). Therefore, we can calculate σ_k from the equation

$$Cap_k(F_k, F_{1,k+1}, \sigma_k, T_k) = Cap_{1,k+1}^{ATM} - \sum_{j=1}^{k-1} Caplet_j(F_j, F_{S,k+1}, \sigma_j, T_j) \tag{3.6}$$

3.2.2 Calibrating the SABR model

After we have obtained the separate spot volatilities and managed to create a caplet volatility surface, we can calibrate the models described in the previous chapter to obtain a wider choice of volatilities. Using the methodology discussed in the previous chapter regarding the two different methods of calibrating the SABR model, we will perform both methods and see which method performs better. This will be checked visually using graphs and by calculating the errors that each model produces. The error term will be defined as the absolute sum of the difference between the market volatilities and the model volatilities, mathematically as for the first method as $\sum_i^n(|\sigma_{Market} - \sigma(\alpha, \rho, \nu, K_i, f, \beta|)$ and for the second method is $\sum_i^n(|\sigma_{Market} - \sigma(\alpha(\rho, \nu, \sigma_{ATM}), \rho, \nu, K_i, f, \beta|)$.

Furthermore, the constraints that the model has to obey are $\alpha \leq 0$, $v \leq$0, and $-1 \leq \rho \leq 1$. This is the case for the Shifted SABR model with the addition of the shift parameter, which in this case is fixed at 2% for the calibration of the SABR model for the caplets, where it will be 3% for the swaptions. As for the Normal SABR model, there exists no shift parameter and the β parameter must be fixed to zero as to allow forward rates to become negative as was discussed in the previous chapter.

Furthermore, we will be testing to see if the Obłój's method proves to be a better fit to the market volatilities and see if it improves the SABR output for options with low strikes and high maturities. His method will be used with a shift parameter for the analysis to function. Therefore, we must change the f in Equation 2.67 to a $\hat{f}$, which will denote $f + s$, which is the underlying forward rate plus a displacement parameter.

3.3 Limitations and delimitations of the analysis

The biggest problem with this thesis would be the lack of data. Due to the lockdown caused by the COVID-19 pandemic, the author was unable to gather the necessary data to do a comparison of the SABR model output during different interest rate regimes. As we only have implied volatilities for interest rate derivatives when the rates are negative. Preferably, the author would have liked to have multiple data points to obtain a better picture of the effect of the interest rates becoming negative.

As we only have a sample of one day for data, it might be hard to jump to conclusions for the model. The topic must be explored better to gain a greater understanding of the workings of the SABR model in a negative interest rate regime.

The description of the caplet stripping process did not mention that it was made for USD caps. There is a difference between the EUR cap and the USD, as was discussed in the previous chapter. However, using the methodology described in this chapter, we will assume that these caps are quoted against the 6M forward rate.

With regards to the delimitations, the basis of the analysis is to compare the implied volatility outputs of the different SABR models, we assume that there is no arbitrage and that there is a one-to-one relationship between implied volatilities and prices. Without that assumption, the transformed normal volatility to shifted black volatility might not match.

3.4 Summary of chapter

In this chapter, we discussed the different data that will be used in the analysis and how the data was prepared for the analysis. We have collected normal cap volatilities for euro caps, we use a solver to obtain Shifted Black volatilities with a shift factor of 2%. The cap volatilities have 16 maturities and 9 strikes for each. Furthermore, we have obtained normal swaption volatilities, performing a similar process as the caps we obtain Shifted Black swaption volatilities, using a shift parameter of 3%. These swaption volatilities are quoted in relevant to their position toward the ATM rate, the strikes are $\pm 200, \pm 100, \pm 50$, and ± 25 bps from the ATM point, with the ATM point being included.

Furthermore, we describe the methods that we will be performing in the upcoming analysis, such as the bootstrapping of the caps. Where we transform cap volatilities into caplet volatilities, for ATM and non-ATM caps. We also discussed the way that the SABR calibration will be performed and the limitations and delimitations of the analysis.

4 Analysis

This chapter will describe the main results of the empirical research using the methods and data described in the previous chapter.

4.1 Normal SABR model for caplets

To use the Normal SABR model for caplets, we need to differentiate between cap and caplet volatilities.

4.1.1 The volatility term structure of normal volatilities

In the previous chapter the process of bootstrapping caplet volatilities from cap volatilities was explained and the reasons why such an operation is integral to this thesis. The transformation of the cap volatilities found in Table A1.2 to caplet volatilities has been performed and the results can be seen in Figure 4.1 for the normal caplet volatilities with the ATM normal volatilities. Due to the nature of the bootstrap approach, we obtain one volatility term for each strike.

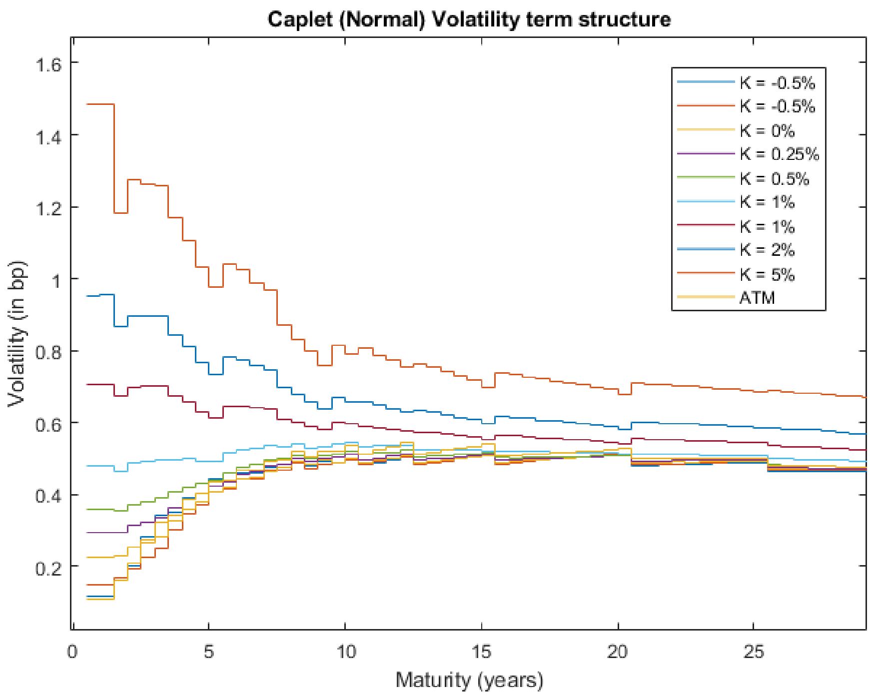

Figure 4.1: Normal EUR caplet volatilities

Using the caplet volatility stripper algorithm, we obtained 60 different normal caplet volatilities for each strike. Which we will use to calibrate the Normal SABR model. As we can see in the figure, the normal volatility dynamic evolution is monotonous with time to maturity, as the different volatility terms never cross each other. To test the accuracy of the models, we must at first calibrate the SABR models to extract the output. Then we can compare that output to the caplet volatilities and see how close and accurate the SABR models perform.

4.1.2 Calibrating the Normal SABR model for caplets

Using the caplet volatilities obtained in the previous section, we can calibrate the Normal SABR model to create a volatility surface for the caplets. Using the methodology described in the previous chapter, we have obtained sets of SABR parameters using the second method of calibration of the model. These parameters for the Normal SABR model can be seen in Table 4.1 using method 1 and Table 4.2 for method 2. Note that select maturities were selected to be placed in the table.

Table 4.1: Normal SABR parameters using method 1 ($\beta = 0$)

T (Years)	α	ρ	ν
1	0.0020	-0.9174	1.8320
2	0.0022	-0.6666	1.0895
3	0.0028	-0.5703	0.9193
4	0.0033	-0.2373	0.5981
5	0.0038	-0.0975	0.4516
7	0.0044	0.1357	0.3699
8	0.0048	0.2092	0.2597
9	0.0048	0.2106	0.2191
10	0.0049	0.2525	0.2259
12	0.0049	0.1837	0.2138
15	0.0049	0.1086	0.1914
20	0.0049	0.0875	0.1807
25	0.0048	0.2076	0.1825
30	0.0045	0.2742	0.1712

Table 4.2: Normal SABR parameters using method 2 ($\beta = 0$)

T (Years)	α	ρ	ν
1	0.0010	-0.1693	1.1266
2	0.0019	-0.3683	0.8751
3	0.0029	-0.5715	0.9094
4	0.0035	-0.3587	0.6061
5	0.0040	-0.1632	0.4536
7	0.0046	0.0402	0.3761
8	0.0050	0.1293	0.2534
9	0.0051	0.1276	0.2111
10	0.0052	0.1995	0.2075
12	0.0053	0.1425	0.1914
15	0.0052	0.0179	0.1848
20	0.0050	-0.0622	0.1983
25	0.0046	0.0622	0.2076
30	0.0043	0.1259	0.2074

Looking over the values of the parameters, we can see that α parameter changes very little over time and is very small in general for both methods. The parameter ρ is very volatile and changes rapidly over time and the parameter ν seems to decrease with regards to time and starts very high with the 1-year maturity for both methods.

Using these parameters, we can construct a volatility smile for each set of parameters and as a result, we can construct a caplet volatility surfaces, one for each method. They can be seen in Figure 4.2.

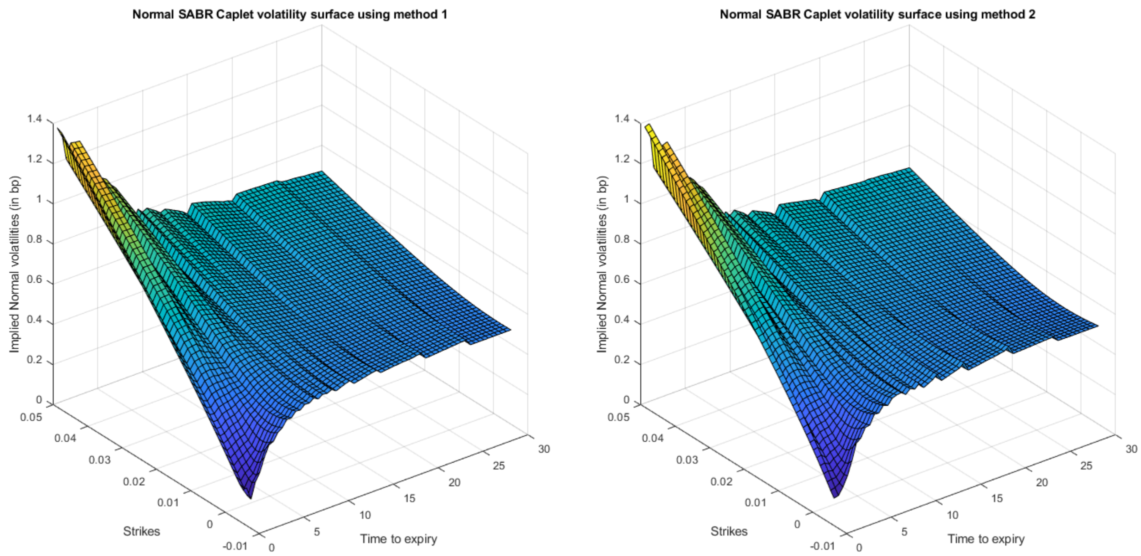

Figure 4.2: Caplet volatility surfaces obtained from the Normal SABR model, depending on method

As we can see from these figures, the Normal SABR model can create volatility surfaces for caplets. However, it does not tell us much about the difference between the two methods or the accuracy of the Normal SABR model. To clarify the difference, we must compare the values of the caplet volatilities as seen in Figure 4.1 with the appropriate values of the Normal SABR volatilities.

In Figure 4.3, we can see four different volatility smiles, each with a different maturity. This is to visualize how close each method can produce market volatilities at different strikes.

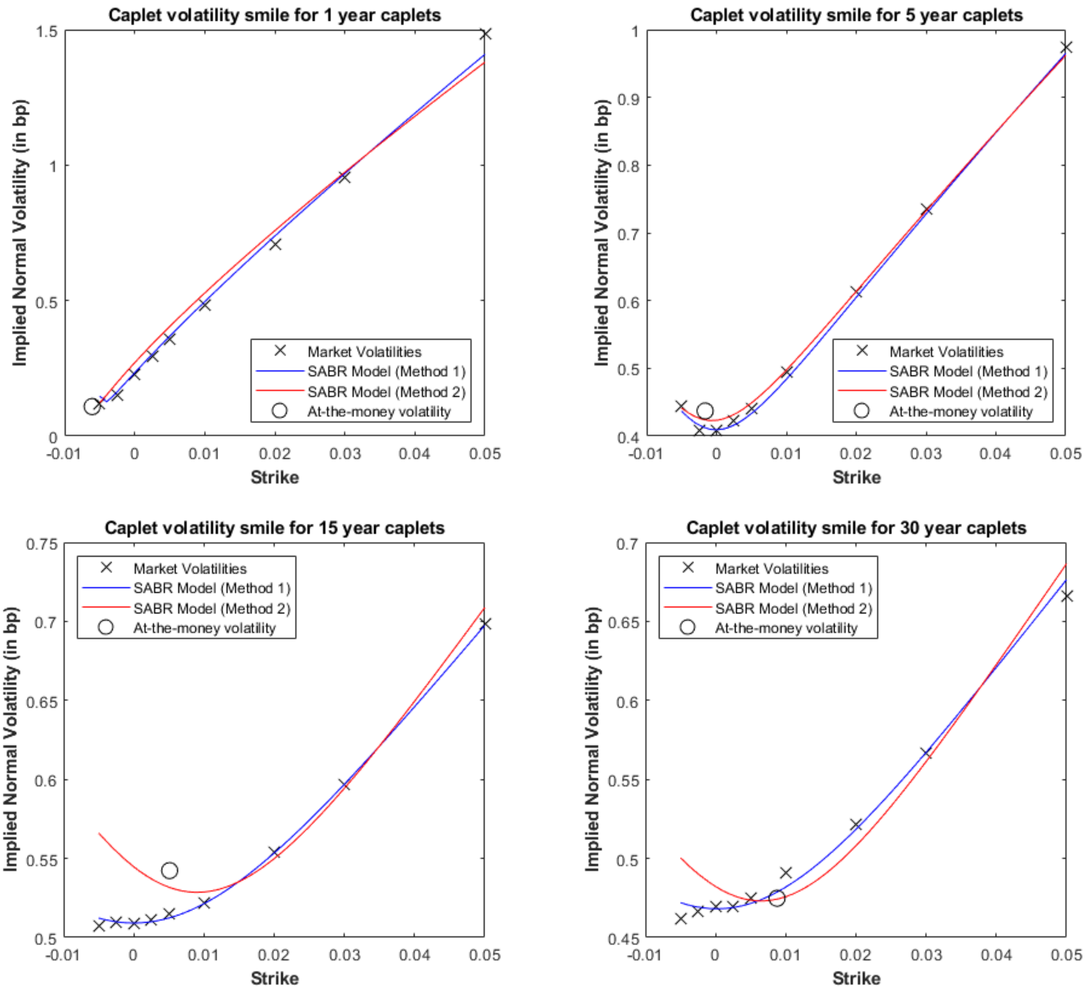

Figure 4.3: Four caplet volatility smiles generated with the Normal SABR model, comparing methods to the market volatilities

As we can see from the figure, the first method is very close to matching all of the market volatilities for all maturities and across strikes. The second method seems to have modelled the volatilities finely, however, there are some errors with the low strikes. To fully acknowledge each method's good and bad qualities, we should look at Figure 4.4 where the error term, as described in the previous chapter has been calculated and plotted against time to expiry.

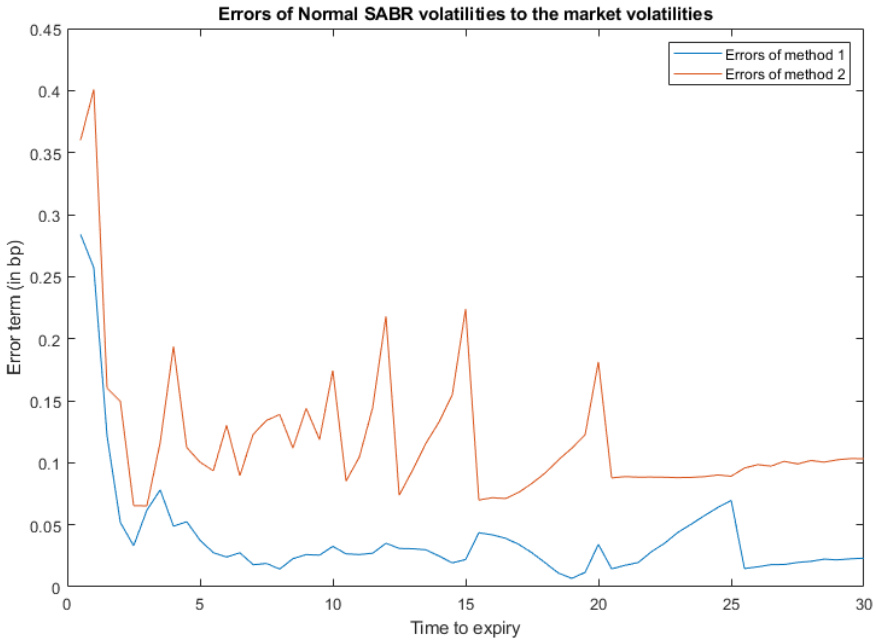

Figure 4.4: Errors of Normal SABR volatilities to the market volatilities (in bp)

These results show us that the first method is a little inaccurate with short maturities but almost no errors as the time to expiry increases. Method two, on the other hand, is not as accurate as method one for the normal caplets but we can observe the same with the first method as the errors deteriorate with time. However, we can conclude that the first method of the Normal SABR model is superior to the second method, as the errors are consistently lower for all strikes and expiry. As a result, the first method should be used to model normal caplet volatilities.

4.2 Shifted SABR model for caplets

4.2.1 The volatility term structure for shifted black volatilities

We start by taking the implied black cap volatilities from Table A1.3 in the previous chapter and transforming those cap volatilities into caplet volatilities. This is done by performing the caplet stripping methods described in the previous chapter. We obtain caplet volatilities for 60 different maturities and across 9 strikes and the ATM volatilities. These caplet volatilities can be seen in Figure 4.5.

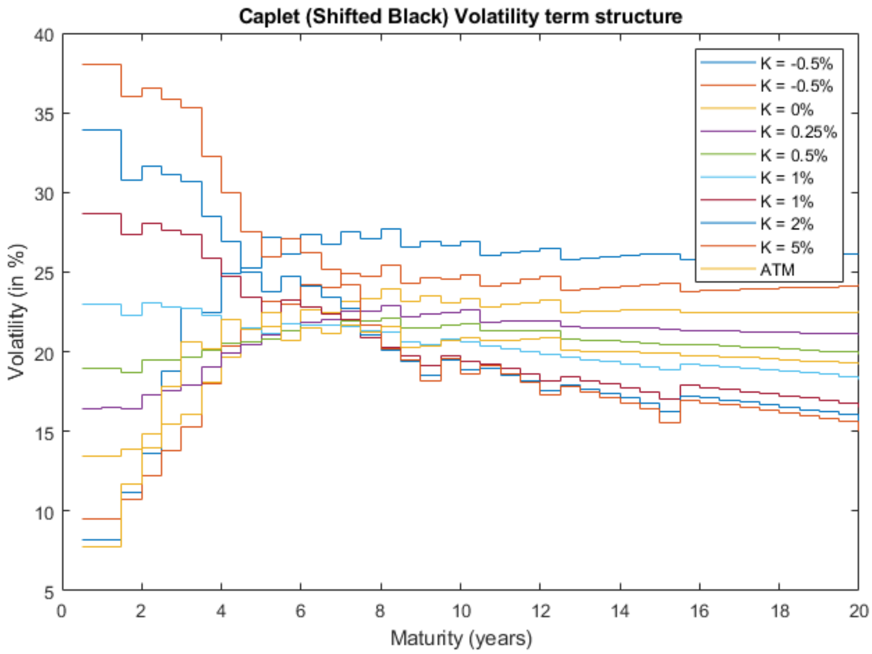

Figure 4.5: Shifted Black EUR caplet volatilities

As we can see in the figure, the shifted Black dynamic evolution is characterized by the switch of the low strikes with the high strikes when the time to maturity increases. As the high strike starts with the highest value and drops as time increases, while the low strikes do the opposite.

4.2.2 Calibrating the Shifted SABR model for caplets

Using the caplet shifted black volatilities, we can run the SABR calibration for the Shifted SABR model for all maturities. Table 4.3 shows the parameters of the calibration using the first method and Table 4.4 shows the parameters using the second method.

Table 4.3: Shifted SABR parameters using method 1 ($\beta = 0.5$)

T (Years)	α	ρ	ν
1	0.01425	-0.82112	1.44323
2	0.01624	-0.55601	0.96182
3	0.02372	-0.61713	0.91823
4	0.02674	-0.36911	0.57162
5	0.02986	-0.33569	0.44742
7	0.03159	-0.18950	0.35551
8	0.03352	-0.25208	0.25463
9	0.03319	-0.28708	0.21565
10	0.03329	-0.24310	0.21547
12	0.03281	-0.32855	0.21057
15	0.03211	-0.43523	0.20872
20	0.03152	-0.48344	0.22475
25	0.03089	-0.35555	0.20813
30	0.02831	-0.27271	0.18653

Table 4.4: Shifted SABR parameters using method 1 ($\beta = 0.5$)

T (Years)	α	ρ	ν
1	0.00866	-0.55237	1.18976
2	0.01571	-0.52532	0.94433
3	0.02499	-0.65161	0.94706
4	0.02670	-0.36688	0.57097
5	0.02884	-0.30020	0.44631
7	0.02888	-0.21295	0.41082
8	0.03035	-0.28714	0.33407
9	0.02959	-0.33730	0.32067
10	0.03057	-0.29391	0.29724
12	0.03177	-0.34011	0.24296
15	0.03091	-0.43658	0.24431
20	0.02993	-0.48440	0.26707
25	0.03232	-0.35308	0.16879
30	0.02130	-0.34913	0.33217

According to these tables, we can see that the parameters are quite similar between methods, the α and ρ parameters start with low values and begin to increase until reaching a peak and then starting to decrease in value. The ν parameter starts high for both methods and decreases with regards to time.

We can use these parameters to generate volatility surfaces for caplets. This can be seen in Figure 4.6.

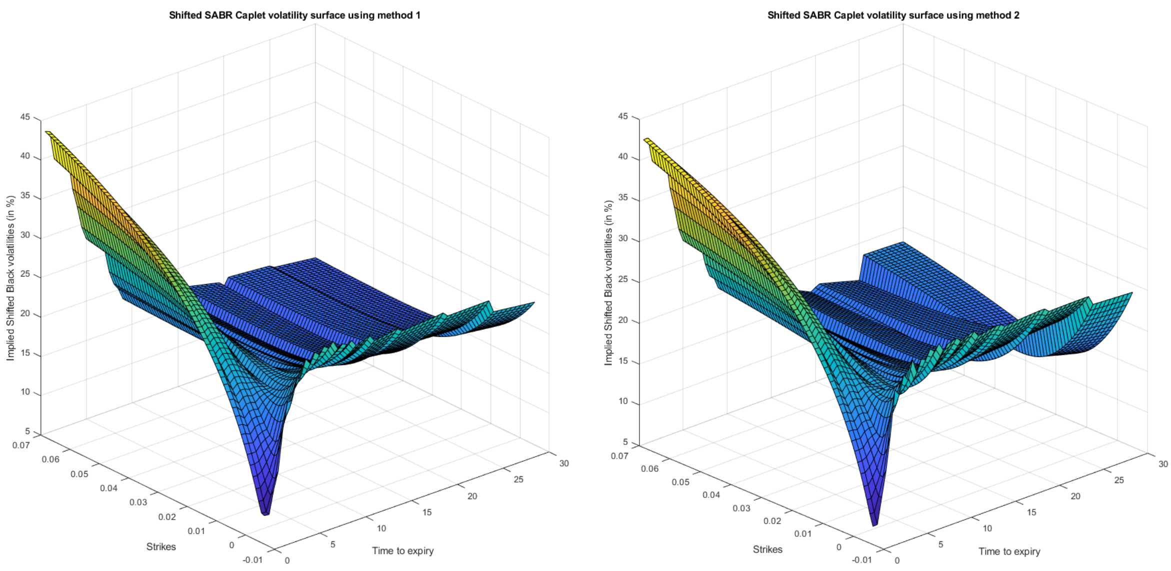

Figure 4.6: Caplet volatility surface obtained from the Shifted SABR model, the first method on the left and the second on the right

We can see that there is some difference between the two methods, the higher maturities which can be seen as being slightly higher on the left side of the graph. However, to investigate this further, it is better to view the volatility smiles with different maturities to visualize which method is better at producing the volatility smile depending on the strike. This can be seen in Figure 4.7.

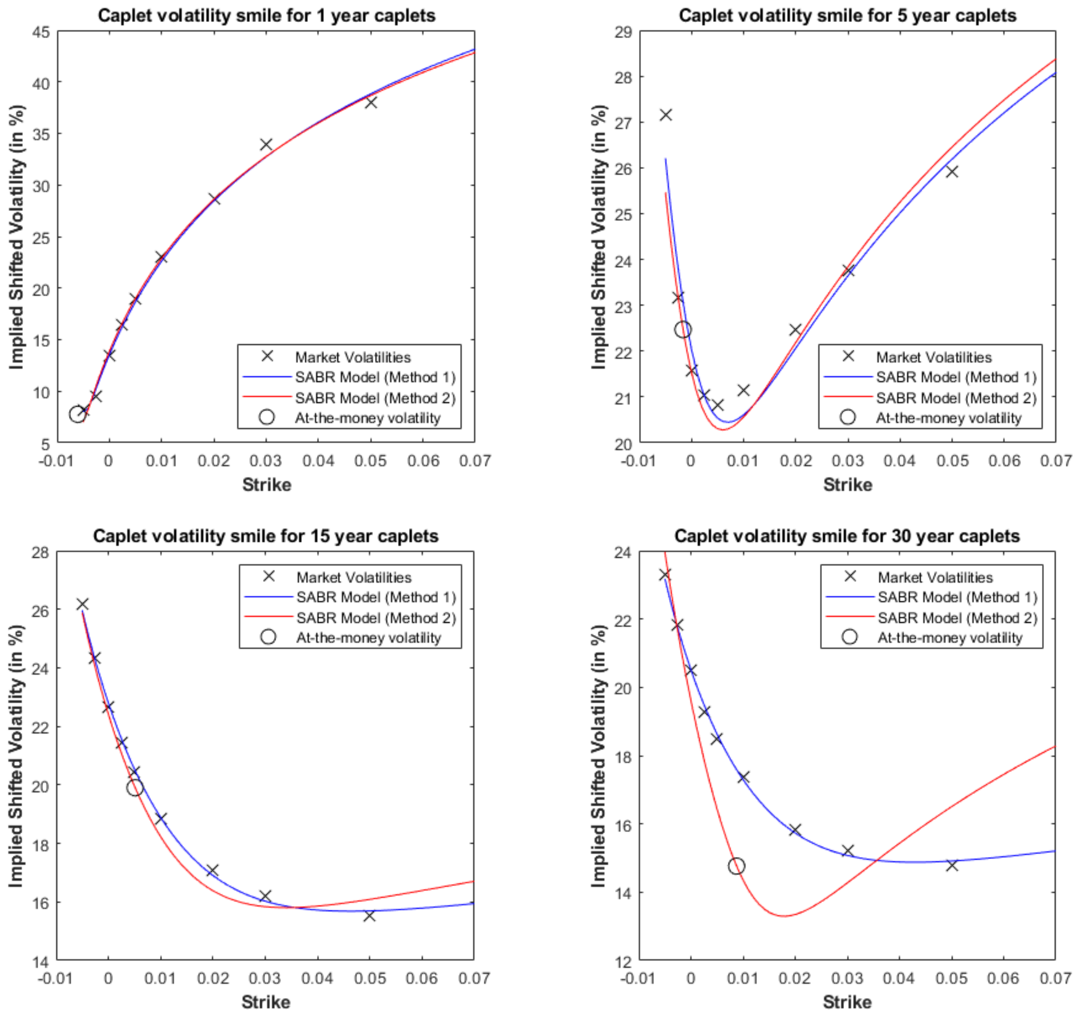

Figure 4.7: Four caplet volatility smiles generated with the Shifted SABR model, comparing methods to the market volatilities

Looking at the different volatility smiles, it is quite clear that method 1 is the superior method to model the volatility smile as it deviates less from real market volatilities. However, it is not perfect as we can see with the 5-year caplet, the method one doesn't match all of the caplet volatilities, while method 2 does. Method 2 is, clearly influenced by the ATM volatility, which again is not in line with the market volatilities in the longer maturity caplets. Due to the great differences between them, one should cast doubt on the method that was used to calculate it.

To gain a better understanding of the accuracy of both methods, we can look at the error term between the two methods and the original caplet volatility. This can be seen in Figure 4.8.

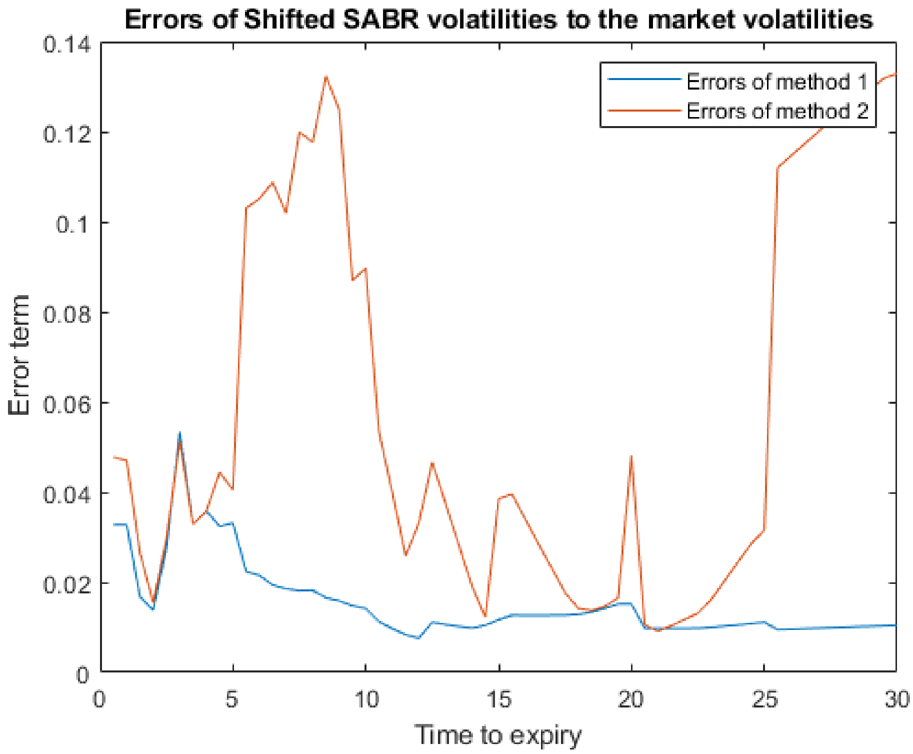

Figure 4.8: Errors of Shifted SABR volatilities to the market volatilities

We can see, method one performs much better than method two, this is in line the results with the analysis of the normal SABR model. Again, method one has some errors when the time to expiry is low, but in general, it manages to model the volatility smiles with far lower error rates than the second method. At some points, we can see that method two performs similarly to the first method, only at some points as the errors are bigger at most. Therefore, we can deduct that the first method is the superior method for modelling shifted Black caplet volatilities with the Shifted SABR model as it is much more constant and produces fewer errors.

4.2.2.1 Obłój's method for the Shifted SABR model

As discussed in the methodology chapter, Obłój found that the SABR model was not modelling volatilities for low strike, long maturity options well enough and decided to attempt to improve the model. Using the Shifted Black caplet volatilities in 4.5 to test whether Obłój's refinement can improve the long maturity, low strikes compared to the Shifted SABR model. We begin by estimating the SABR parameters.

Table 4.5: Shifted SABR parameters with Obłój method using method 1 ($\beta = 0.5$)

T (Years)	α	ρ	ν
1	0.01289	-0.76197	1.31093
2	0.01582	-0.52076	0.90873
3	0.02349	-0.60494	0.88490
4	0.02662	-0.36137	0.55588
5	0.02980	-0.33318	0.43833
7	0.03155	-0.18871	0.34987
8	0.03350	-0.25252	0.25131
9	0.03318	-0.28800	0.21315
10	0.03329	-0.24384	0.21300
12	0.03281	-0.32992	0.20885
15	0.03213	-0.43676	0.20731
20	0.03155	-0.48503	0.22334
25	0.03091	-0.35678	0.20681
30	0.02832	-0.27360	0.18534

Table 4.6: Shifted SABR parameters with Obłój method using method 2 ($\beta = 0.5$)

T (Years)	α	ρ	ν
1	0.00864	-0.51013	1.13867
2	0.01570	-0.50059	0.91188
3	0.02495	-0.64238	0.92228
4	0.02674	-0.35868	0.55912
5	0.02888	-0.29594	0.43890
7	0.02893	-0.21185	0.40620
8	0.03038	-0.28709	0.33052
9	0.02962	-0.33810	0.31785
10	0.03060	-0.29487	0.29483
12	0.03179	-0.34139	0.24137
15	0.03093	-0.43791	0.24289
20	0.02996	-0.48575	0.26564
25	0.03233	-0.35393	0.16813
30	0.02133	-0.35042	0.33096

As we can see from these tables, the parameters for the Obłój version of the SABR model are almost the same as the shifted Black parameters from Tables 4.3 and 4.4. It should be quite obvious that the output of the SABR model should be similar to the Shifted Black. As a result, we can skip seeing the surface and look at different smiles, especially the ones with a higher time to expiry as Obłój's refinement attempts to improve performance for them.

To visualize this, we have plotted the output from Obłój's method with the market volatilities to see if it manages to come close to them. Furthermore, we have added the Shifted SABR volatilities, to compare the methods. This can be seen in Figure 4.9.

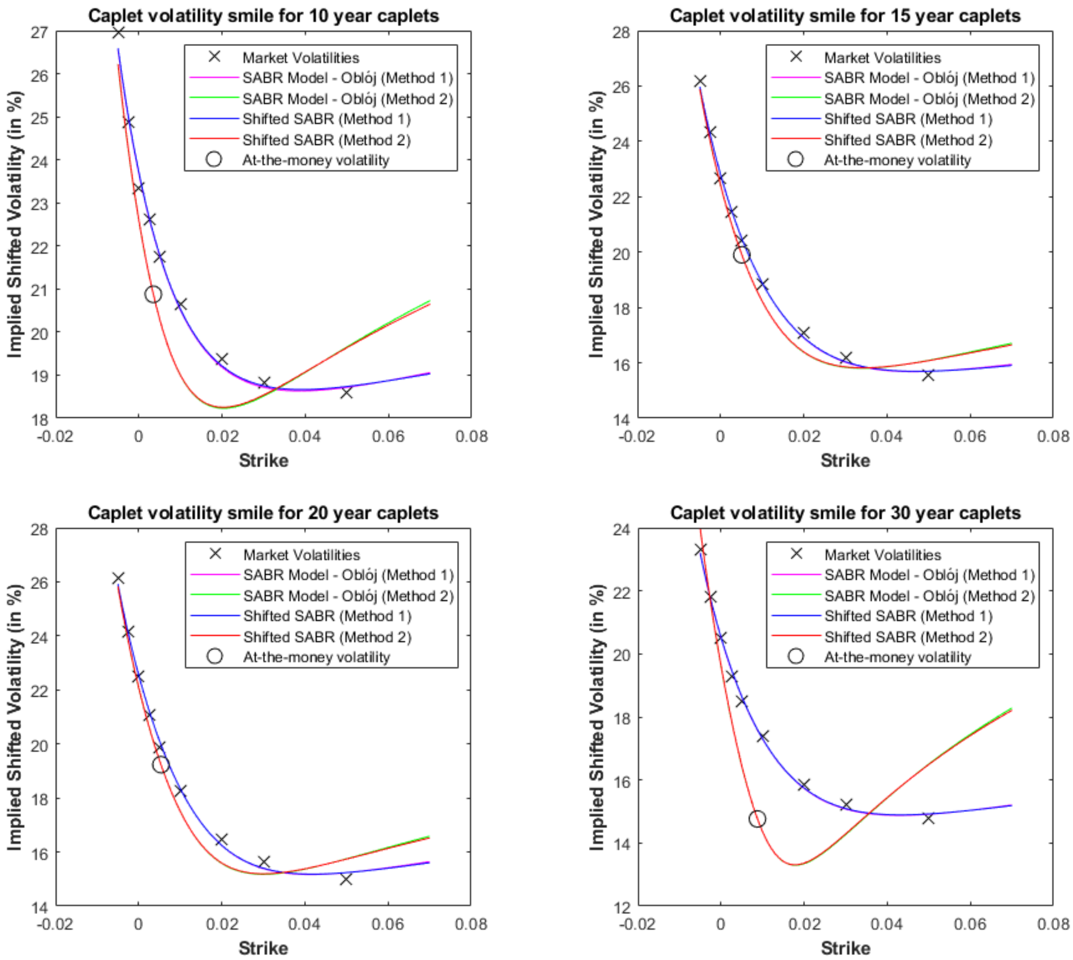

Figure 4.9: Four caplet volatility smiles generated with the Shifted SABR model and Obłój's refinement, comparing methods to the market volatilities

As we can see from the figure, the Obłój volatilities are almost the same as the Shifted SABR volatilities and as a result, they clash with each other and we cannot see much difference. This should have been obvious since both sets of parameters were quite similar. As we cannot see much difference between the two models, we calculated the error term, which can be seen in Table 4.10.

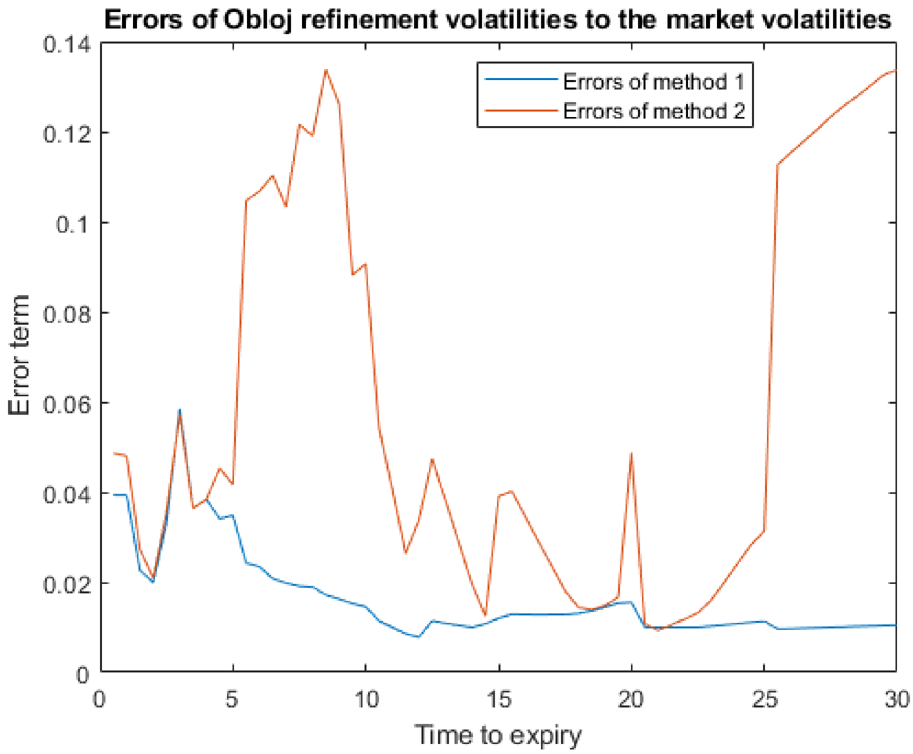

Figure 4.10: Errors of Obłój's refinement of SABR volatilities to the market volatilities (in bp)

Due to the volatilities being extraordinary similar, we are unable to determine from the figure, whether Obłój's refinement has more or less errors than the Shifted SABR model. Therefore we have taken the mean errors of both methods for both models. Shifted Black using the first method has an mean error of 0.015455 while Obłój using the first method has 0.016513, the second method is 0.057583 and 0.058575 for Shifted SABR and Obłój respectively.

As we can see using Obłój's method yields worse results than the first method for the Shifted SABR model. However, the difference is really small and it does an excellent job of modelling the caplet volatility regardless of the small discrepancies. Therefore, we can conclude that Obłój's method is a viable strategy to model implied volatilities, however, with a small cost of accuracy.

4.2.3 Conclusion for caplet calibration

To conclude the caplet volatility SABR models. It is quite clear that all models and methods discussed in this sub-chapter can model the volatility smile. However, the Shifted SABR model using method one is the best way to create a volatility surface is you have only Shifted Black volatilities if interest rates are negative, if they were positive, the original SABR model would replace the Shifted one. If you have normal implied volatilities, then the Normal SABR model, using method 1 would be the best choice. The Obłój method did not prove to improve the Shifted SABR model's performance. Nonetheless, it remains a viable model to use as the difference between it and the Shifted SABR model was quite small. We can see the calculated error terms for all models and methods in Table A3.1 in the Appendix.

4.3 SABR model for swaptions

Performing a similar analysis for swaption will be easier to perform. As we have swaption volatilities that can be calibrate the SABR models directly with no need for any bootstrapping. Furthermore, the second method should be more accurate as the collected volatilities are based around the ATM volatility. We will only show the process for 5-year tenor swaptions while concluding with the entire data-set at the end of this sub-chapter.

4.3.1 Normal SABR model for swaptions

Using the Normal swaption volatilities that we discussed in the previous chapter we can input them into the calibration of the Normal SABR model. What we get is a set of parameters, one for each expiry of the swaptions. These parameters are the SABR parameters and when put in the Normal Hagan Formula (Equation 2.62) for a set of strikes will give us a set of Normal volatilities for corresponding to each of those strikes.

As with the caplets, the β parameter is fixed for the Normal SABR model. Performing the calibration for the Normal SABR model yields us the parameters that can be found in Table 4.7 using the first method and Table 4.8 using the second method.

Table 4.7: Normal swaption parameters using method 1 ($\beta = 0$)

T (Years)	α	ρ	ν
0.25	0.00315	0.35422	0.99971
0.5	0.00320	0.34343	0.81808
1	0.00337	0.33133	0.58053
2	0.00374	0.28855	0.41080
3	0.00407	0.27744	0.35882
4	0.00432	0.26216	0.31958
5	0.00449	0.24794	0.28951
7	0.00486	0.27020	0.22771
10	0.00500	0.22263	0.21930
15	0.00497	0.22730	0.16462
20	0.00479	0.23796	0.16937
30	0.00441	0.24412	0.17560

Table 4.8: Normal swaption parameters using method 2 ($\beta = 0$)

T (Years)	α	ρ	ν
0.25	0.00234	0.78618	1.28813
0.5	0.00262	0.78328	0.99627
1	0.00293	0.75056	0.61189
2	0.00338	0.45539	0.25628
3	0.00379	0.41091	0.22722
4	0.00410	0.36839	0.21028
5	0.00434	0.33229	0.19723
7	0.00457	0.28495	0.20307
10	0.00499	0.31147	0.14244
15	0.00486	0.22693	0.15817
20	0.00469	0.22717	0.17166
30	0.00446	0.24334	0.16897

We can see that there is some difference between the two methods for the Normal SABR model. The α parameters are quite similar for both models, as they both have a concave path. The ρ parameter is decreasing with time to expiry with both models but starts much higher with the second method, the same can be said about the ν parameter. As we can see that the models produce different parameters, the biggest question regarding their fit, to answer that, we can look at the errors of the model which can be seen in Figure 4.11.

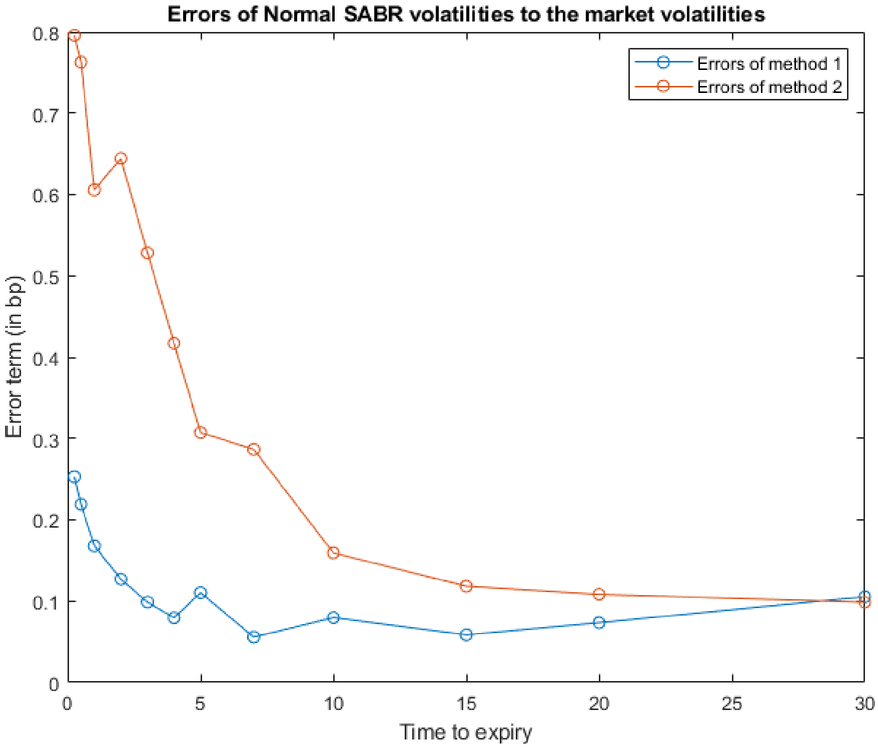

Figure 4.11: Errors of Normal SABR volatilities to the market volatilities (in bp)

As we can see from this figure, the first method seems to outperform the second one, as it has constantly lower errors. This is consistent with the results for the Normal SABR model for the caplets. As we only have one of each value for an expiry time, we can further investigate the differences for each strike. This can be seen in Figure 4.12, where we compare the outputs of the methods to the market volatilities for different expires.

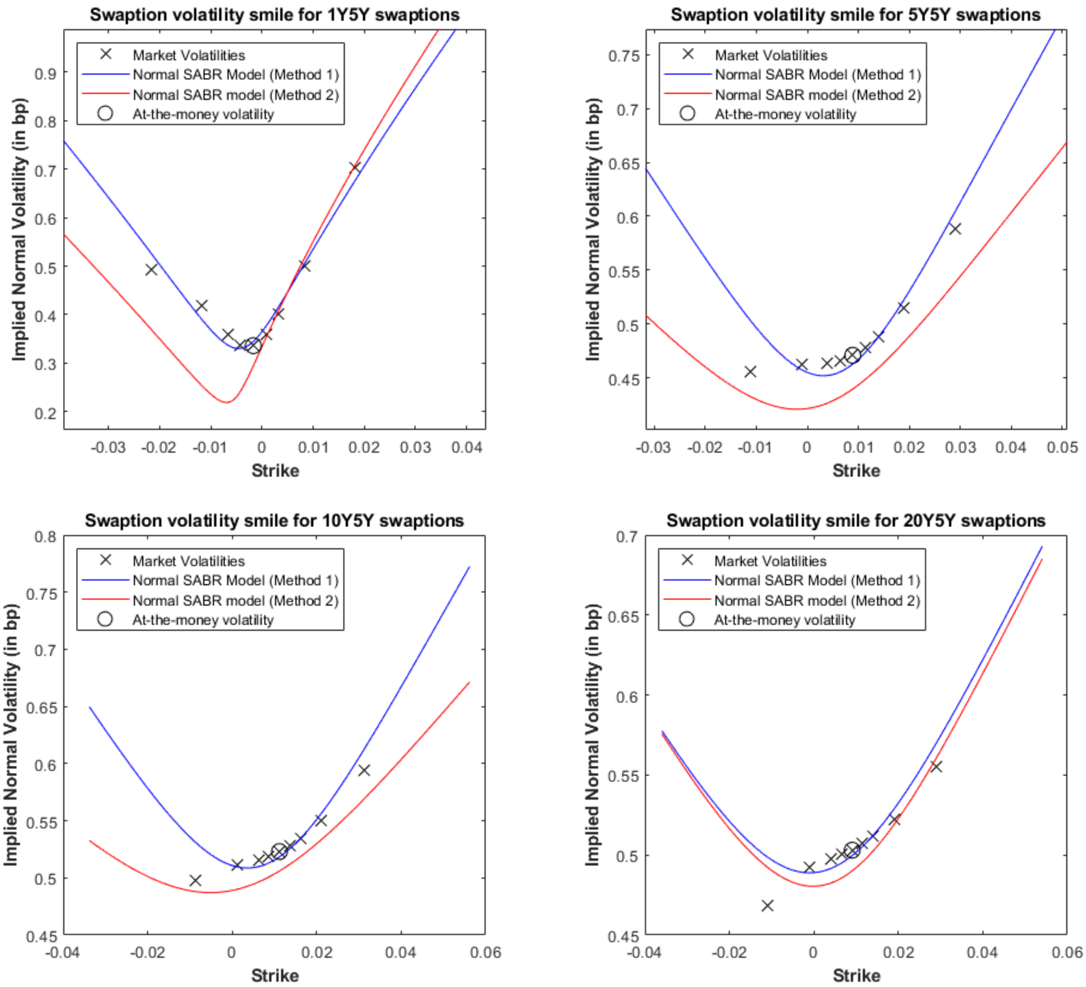

Figure 4.12: Four swaption volatility smiles with different expiry and 5-year tenor using the Normal SABR model and comparing methods

As we can see from the figure the first method can produce volatilities that match the market volatilities with relatively little errors. The second method, on the other hand, hasn't performed well, in the $2Y5Y$ and the $10Y5Y$ swaption volatility smiles it doesn't seem to match any of the market volatilities. It is clear that the evidence points towards method 2 of the Normal SABR model not work properly, especially when compared to method one. As a result, we should use method one to model normal swaption volatilities as they provide a good fit.

4.3.2 Shifted SABR model for swaptions

Continuing in the same path, we obtain Shifted SABR parameters by calibrating the Shifted SABR model for the swaption volatilities. Using method one, we obtain the Shifted SABR parameters presented in Table 4.9 and 4.10 shows the parameters using method two.

Table 4.9: Shifted swaption parameters using method 1 ($\beta = 0.5$)

T (Years)	α	ρ	ν
0.25	0.01981	0.16334	0.73118
0.5	0.01781	-0.06284	0.80858
1	0.01882	-0.12567	0.61587
2	0.02127	-0.26161	0.51340
3	0.02351	-0.33658	0.51041
4	0.02572	-0.41170	0.50872
5	0.02781	-0.48031	0.48735
7	0.02960	-0.53788	0.54136
10	0.02986	-0.57778	0.50291
15	0.02871	-0.60714	0.45111
20	0.02786	-0.60575	0.43301
30	0.02665	-0.60846	0.40708

Table 4.10: Shifted swaption parameters using method 2 ($\beta = 0.5$)

T (Years)	α	ρ	ν
0.25	0.01069	0.76033	1.34295
0.5	0.01401	-0.74927	0.81069
1	0.01625	-0.73010	0.57456
2	0.01992	-0.74890	0.46019
3	0.02326	-0.78446	0.45062
4	0.02653	-0.81780	0.44726
5	0.02922	-0.84452	0.42344
7	0.03390	-0.87391	0.52948
10	0.03749	-0.90162	0.59280
15	0.02327	-0.62376	0.50003
20	0.02039	-0.46616	0.49348
30	0.02123	-0.46597	0.42710

As we can see from the table, the α parameters are similar to the Normal SABR for the swaptions, as both methods rise at first and then decrease with regards to the time to expiry. The ρ parameters are quite different between models, both start positive and quickly become negative, the first method decreases further while the second method starts to increase after the 7-year to expiry mark. The ν parameter decreases with time for both models. To gain a sense of the fit of these models, the errors of each method can be seen in Figure 4.13.

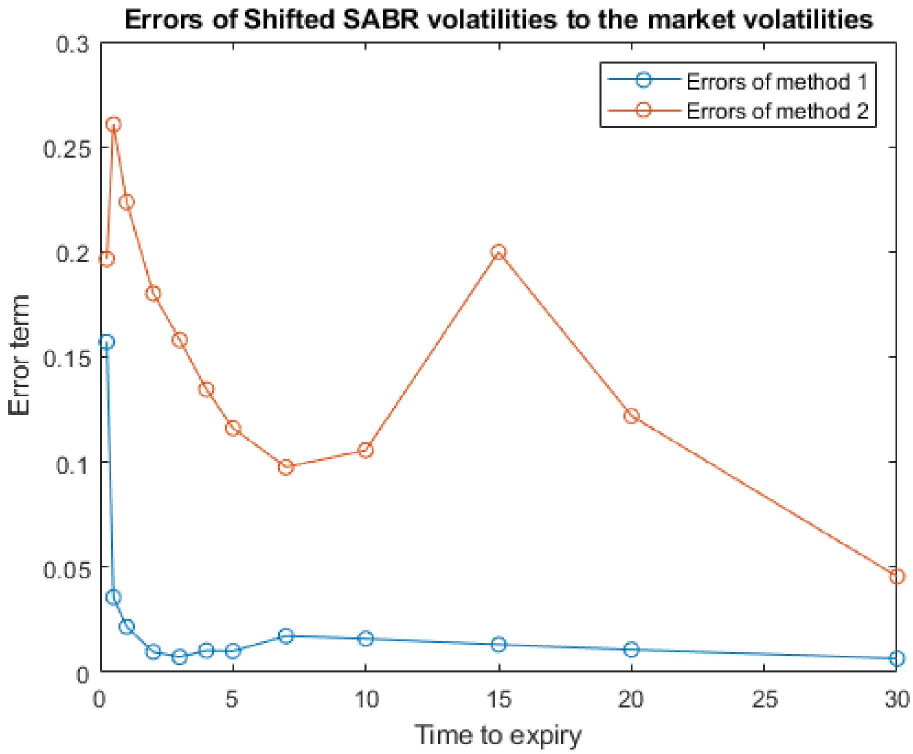

Figure 4.13: Errors of Shifted SABR volatilities to the market volatilities

This figure shows us that the first method fits the market volatilities much better than the second method. The first method seems to have its weaknesses when the time to expiry is low and the second method works best at the highest time to expiry. To visualize the difference and determine the performance of the models with regards to strikes we can look at four different swaption volatility surfaces extracted from the Shifted SABR model on Figure 4.14

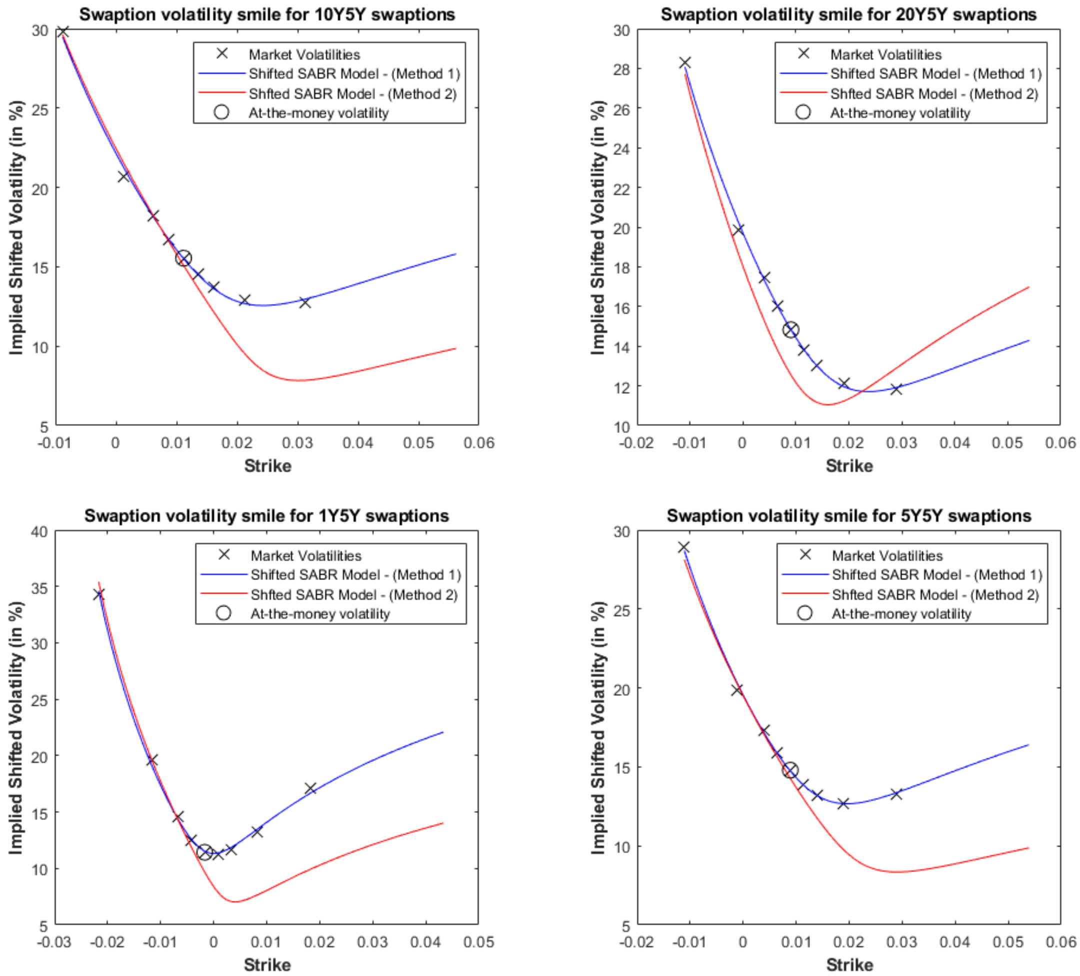

Figure 4.14: Four swaption volatility smiles with different expiry and 5-year tenor using the Shifted SABR model and comparing methods

We can see that the first method seems to fit the market volatilities quite well for every strike and expiry, which is in line with the previous findings. The second method manages to fit the lower strikes well but is unable to continue that after the ATM volatility has been reached, this applies to the $1Y5Y$, $5Y5Y$, and the $10Y5Y$ but not the $20Y5Y$ swaption. Where the second method is unable to produce volatilities close to the market volatilities at the low strikes.

4.3.2.1 Obłój's method for the Shifted SABR model for swaptions

Finally, we attempt to model the swaption volatilities using Obłój's method, this is performed in a similar way that other analyses for the swaption volatilities. We begin by obtaining the SABR parameters, the parameters for Obłój's refinement can be seen using the first method in Table 4.11 and for the second method in Table 4.12.

Table 4.11: Shifted Obłój swaption parameters using method 1 ($\beta = 0.5$)

T (Years)	α	ρ	ν
0.25	0.01992	0.16874	0.71849
0.5	0.01791	-0.05685	0.79699
1	0.01888	-0.12144	0.60837
2	0.02130	-0.25937	0.50870
3	0.02353	-0.33527	0.50692
4	0.02574	-0.41102	0.50611
5	0.02783	-0.48003	0.48534
7	0.02961	-0.53775	0.53970
10	0.02988	-0.57774	0.50125
15	0.02873	-0.60718	0.44943
20	0.02788	-0.60580	0.43138
30	0.02668	-0.60856	0.40533

Table 4.12: Shifted Obłój swaption parameters using method 2 ($\beta = 0.5$)

T (Years)	α	ρ	ν
0.25	0.01096	0.76182	1.314321
0.5	0.01411	-0.74866	0.793436
1	0.01626	-0.72773	0.565577
2	0.01991	-0.74699	0.455076
3	0.02324	-0.78340	0.446846
4	0.02651	-0.81724	0.444551
5	0.02920	-0.84418	0.421461
7	0.03387	-0.87381	0.527471
10	0.03727	-0.90110	0.587042
15	0.02335	-0.62489	0.497233
20	0.02043	-0.46599	0.491618
30	0.02126	-0.46535	0.425492

As we can see, these parameters are very similar to the Shifted SABR parameters found in Tables 4.9 and 4.12 and as a result, we would expect the SABR model to give similar outputs. To investigate the accuracy of Obłój's method, Figure 4.15 shows the differences between both of its methods and the market volatilities.

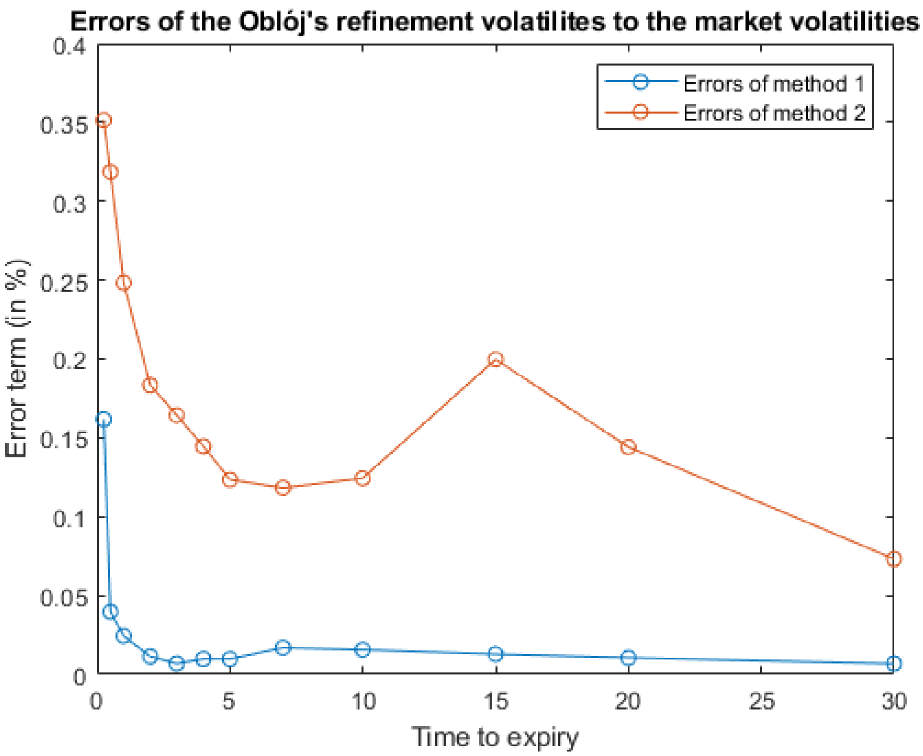

Figure 4.15: Errors of Shifted Obłój SABR volatilities to the market volatilities

As we can see, the errors that the Obłój's method produces is extremely similar to the Shifted SABR errors that can be found in Figure 4.13, which is unsurprising considering that they have almost identical parameters. However, these prove to be slightly worse fitted than the Shifted SABR model. To compare them closely, Figure 4.16 shows the plotted volatility smile using the Obłój's refinement and the Shifted SABR model.

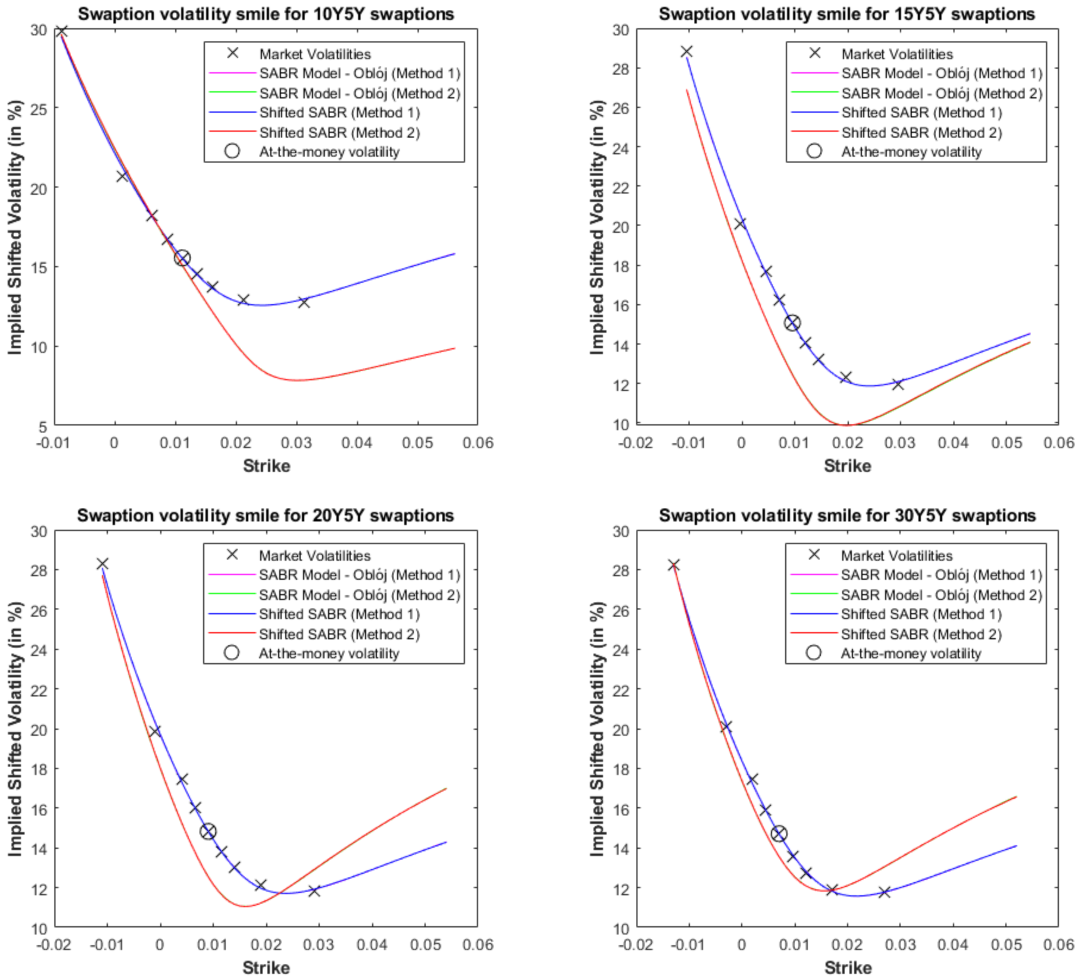

Figure 4.16: Four swaption volatility smiles with different expiry and 5-year tenor using Obłój's refinement and comparing methods

As we can see, the figure has been plotted to show the volatility smiles with the highest maturities. However, as the values of both smiles are very similar the figure is unable to show both smiles as they overlap each other To investigate whether Obłój's refinement manages to improve the underlying problem that the low strike and high maturity SABR outputs which he described to be inefficient.

As a result, the same compliments and criticism can be applied to Obłój's method as that the Shifted SABR had. We calculate the mean errors of these models depending on method, Shifted SABR for method 1 had 0.026338 and for method 2 had 0.1825, Obłój on the other hand had for the first method: 0.02717 and for the second 0.1828. As a result, we can conclude that the Shifted SABR model performs better than the Obłój's method.

4.3.3 Conclusion for the SABR model for swaptions

To conclude the analysis for the swaption volatility, I would like to present the averages of the error term produced by each SABR model with regards to the tenor of the swaption. As the results from every calibration that was performed on the swaptions gave similar results to the 5-year tenor swaption results presented above. For every model, using the second method produced more errors as the output is quite far from the market volatilities. The mean of the error terms for each tenor can be seen in Figure 4.17

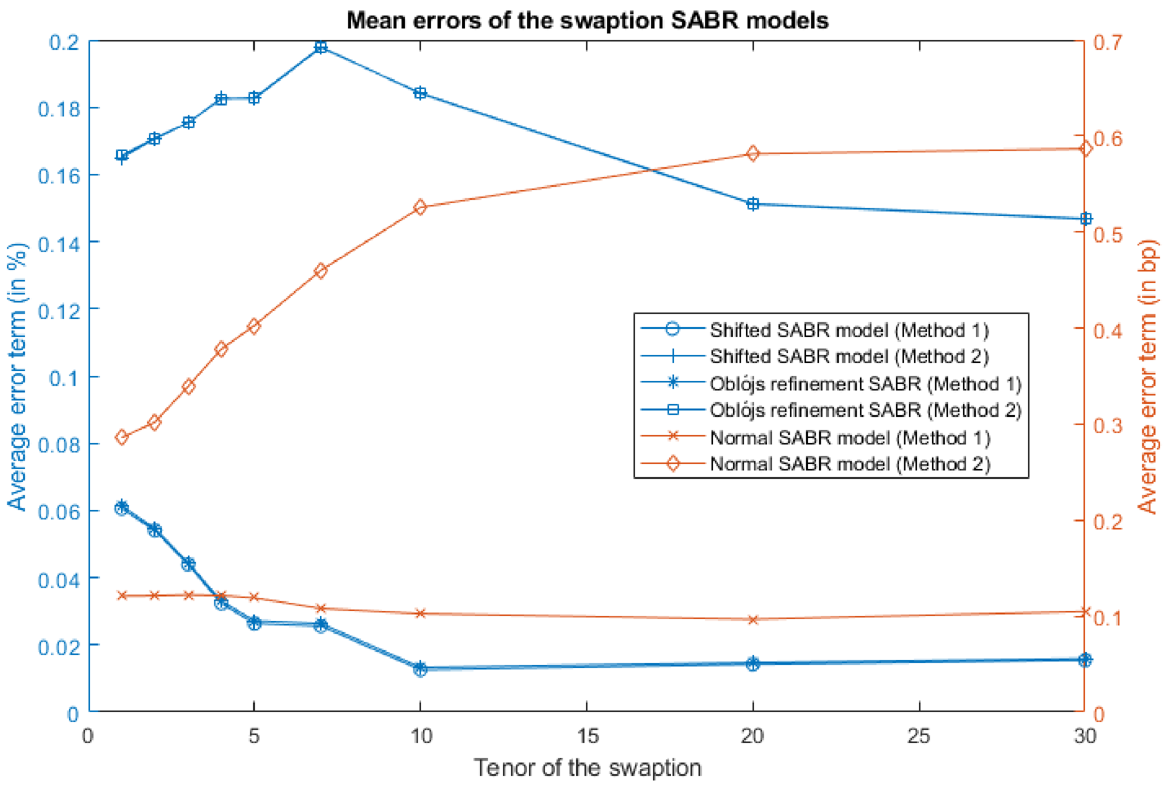

Figure 4.17: Mean errors for each tenor of the swaption SABR models

This figure shows us the mean of the error term depending on the method and the model.

Although hard to see, we can differentiate the volatilities quoted in percentage terms as the blue ones, whose y-axis is to the left, the normal quoted volatilities have a red colour, and the y-axis is on the right for them. According to the figure, we can see that the results that we saw with the 5-year tenor swaption are in line with the other tenors. We can conclude that the first method of the Shifted SABR model is the best model of those discussed to model shifted black swaption volatilities and the Normal SABR using the first method is the best for normal quoted volatilities.

Furthermore, we can see all of the errors that the models and both of their methods produced in Table A3.2 in the Appendix.

4.4 Summary of chapter

In this chapter the main part of the analysis was performed, using the data that was described in the previous chapter, we began by using caps. As caps have a flat volatility, we needed to change that into spot volatility, using the methodology described in the previous chapter we obtain spot volatility, also known as caplet volatilities. Using these volatilities we calibrated the Normal and the Shifted SABR models to test how close the output of those models came to matching the caplet volatilities. We found that using the first method where we estimate the α, ρ, and ν parameters in the calibration worked very well and gave excellent results.

The same could not be said about the second method, where we calibrate the ρ and the ν parameters with the ATM volatility and calculate α from them. The results conclude that the second method is unable to recreate the exact volatilities found in the market. The reason for this is not exactly known, but it might have to do with the fact that the ATM volatilities don't seem to be in line with the caplet volatilities, thus shifting the SABR model output.

Performing the same analysis for swaptions yielded similar results to the caplet analysis. The first method managed to model the market volatilities with very good accuracy. The second method had the same issue as with the caplets. However, in this case, the issue with the ATM volatilities being not in line with the market volatilities is not present as we are using ATM volatilities and strikes differing on the distance from the ATM strike. Which casts doubt on the reasoning for the errors of the second method.

Furthermore, Obłój's refinement was tested for both caplets and swaptions to see if it had any effect on the pricing accuracy of the Shifted SABR model for options with long maturities and low strikes. This was done for both caplets and swaptions. The results show that this refinement does not seem to improve the accuracy of the Shifted SABR model for the long maturity and low strike options.

5 Discussion

The analysis performed in this thesis regarding the performance of the SABR model variants when interest rates are negative has led us to the conclusion that both the alternatives for the SABR model designed for negative rates can model implied volatilities very well. This is only the case when method one of the SABR model is used, the thesis showed that method two is unreliable and produces volatilities far from the market.

5.1 Comparison with prior research

As there is no prior research about the functionality of the SABR model within a negative interest rate regime, it is hard to find direct comparisons to this thesis.

The only that comparable research would be the one made by Wu in 2012 as it was an empirical study on the functionality of the SABR model. Wu concluded that the SABR model was an excellent model to estimate implied volatilities. This thesis came to the same conclusion as Wu regarding the SABR model, despite us not testing the SABR model itself, rather the negative interest rate variants of the SABR model. Nonetheless, we come to the same conclusion regarding the functionality of the model.

Obłój tweaked the Hagan formula to improve performance for low strikes and long maturity options. In this thesis, the evidence points towards the opposite being the case, the Obłój's method produced slightly more errors than the first method for the Shifted SABR model, however, it improved performance using the second method on some data points but not consistently enough be recommended over the Shifted SABR model. However, the method may be plausible when the rates that we are comparing to are not as low as they are now, meaning that the model is better suited to produce volatilities at strikes far from the ATM strike.

5.2 General discussions

As these results seem fine, they are of course not exactly what the thesis proposed, as it was supposed to be an empirical analysis regarding the functionality of the SABR model within different interest rate regimes and see whether the model would change depending on the state of the interest rates, whether they are low, high or negative. As described in the third chapter, this proved to be impossible due to the COVID-19 pandemic of 2020, where the author's data-collecting ability was hindered after a lockdown was enforced to limit the spread of the pandemic.

Due to the workings of the SABR model, to calibrate the model and obtain the parameters, one needs to have market implied volatilities. Then the model finds parameters that fit those market volatilities and using those parameters to find more volatilities outside of those that we already have from the market. Therefore, it could be said that the analysis in this paper was, in fact, a test for calibration, not the actual pricing purposes of the models. However, this could be countered by stating that the results show us that the calibration failed for the second method but not the first while using the same calibration algorithm.

Of course, the result that we concluded to in the thesis, technically only shows that the SABR model works in a negative interest rate environment on the date of January the 16th. To conclude about the model in general, it is required to perform an analysis that tests the difference between multiple dates and different interest rate regimes.

5.3 Possible further studies for the future

If possible, it would be interesting to complete the analysis that was originally planned to be performed in this thesis, which was to perform the analysis that was performed on a larger scale. Comparing the results throughout different interest rate regimes, positive and negative and see how they could compare.

Furthermore, there have been some new SABR models developed over the years that were not featured in this thesis and it would be interesting to see whether they can improve the fit of the volatilities to the market volatilities. Doing a study where comparing the pricing capabilities of the SABR model to the Libor market model could be very interesting. Also, incorporating the LMM SABR model for comparison to see how accurately it manages to produce implied volatilities and pricing the options, would be very interesting.

6 Conclusion

In this thesis, we have introduced the mathematical framework and the market practices to use and price vanilla interest rate derivatives, such as the interest rate swaps, caps, floors, and swaptions. We explained the models commonly used and guided the reader to realize the inherent flaws within the Black's 76 model. To solve these issues, we introduced the local volatility models, while an excellent pricing model it had inherent flaws which made it unusable for risk management. As a result, the stochastic volatility model, the SABR model was developed, a model that can price options with the usage of Black's 76 model or the Normal Bachelier model, while being a valid tool for risk management.

The SABR model is not perfect as it is unable to model negative interest rates. The thesis tested whether it could produce implied volatilities effectively when certain modifications have been applied to it so that it can function with negative rates. Furthermore, the thesis tested whether Obłój's refinement improved the performance of the SABR model for options with long maturities and low strikes.

As the research question put forward at the beginning of the thesis asked whether the changes to the SABR model allow it to accurately produce implied volatilities when interest rates are negative? To that question, the evidence put forth in this thesis seems to suggest that, yes, the Normal and Shifted SABR models can model implied volatilities when interest rates are negative.

However, as this thesis only tested a single day of trading, it should be noted that we cannot conclude that either model is capable of *consistently* modelling implied volatilities in a negative interest rate regime. Nonetheless, the evidence presented in this thesis showed that these methods were effective at producing implied volatilities that matched the ones observed in the market for this single day of trading. As a result, we can conclude that the Shifted and Normal SABR models can produce implied volatilities accurately when interest rates are negative.

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Appendix

A1 Collected data for the analysis

Table A1.1: Spot rates on 16$^{\text{th}}$ of January 2020

Maturity	Spot Rates
6Mo	-0.600837
1Yr	-0.610152
2Yr	-0.607273
3Yr	-0.581843
4Yr	-0.54053
5Yr	-0.488901
6Yr	-0.431343
7Yr	-0.371166
8Yr	-0.310758
9Yr	-0.251771
10Yr	-0.195281
12Yr	-0.092062
15Yr	0.036291
20Yr	0.189322
25Yr	0.289894
30Yr	0.35883

Table A1.2: Implied Normal volatilities for EUR caps

Expiry	ATM	ATM Strike	-0.50%	-0.25%	0.00%	0.25%	0.50%	1.00%	2.00%	3.00%	5.00%
1Yr	10.8	-0.61%	11.8	14.9	22.5	29.4	35.9	48.1	70.7	95.4	148.3
18Mo	13.14	-0.61%	14	16.4	23.1	29.4	35.4	46.7	67.7	87.5	124.8
2Yr	15.64	-0.6%	16.3	18.3	24.6	30.8	36.8	48.3	69.5	89.4	127
3Yr	22.08	-0.53%	23.3	22	26.9	32.6	38.3	49.3	70.1	89.6	126.2
4Yr	27.44	-0.4%	28	26.7	30.8	35.5	40.2	49.6	67.7	84.9	117.2
5Yr	31.99	-0.3%	31.9	30.7	34.1	38	41.8	49.5	64.7	79.4	107.6
6Yr	35.66	-0.13%	34.8	33.8	36.8	40.2	43.7	50.6	64.7	78.6	105.2
7Yr	38.66	-0.01%	37.1	36.3	38.9	42.1	45.2	51.5	64.4	77.2	101.9
8Yr	41.25	0.11%	39	38.4	40.8	43.7	46.5	52.1	63.2	74.3	96
9Yr	43.22	0.21%	40.4	39.9	42.2	44.8	47.3	52.3	61.9	71.4	90.5
10Yr	44.93	0.32%	41.6	41.2	43.3	45.8	48.1	52.7	61.4	70	87.4
12Yr	47.22	0.43%	43.3	43.1	44.9	47	49.1	53	60.3	67.7	83.1
15Yr	49.11	0.55%	45	44.9	46.3	48	49.7	52.8	58.8	65	78.2
20Yr	50.27	0.65%	46.7	46.6	47.6	48.8	50	52.4	57.3	62.6	74.2
25Yr	50.32	0.61%	47.2	47.2	48	49	50	52	56.5	61.4	72.3
30Yr	49.82	0.78%	47	47.1	47.8	48.6	49.5	51.4	55.5	60.2	70.6

Table A1.3: Implied Shifted Black volatilities for EUR caps

Expiry	ATM	ATM Strike	-0.50%	-0.25%	0.00%	0.25%	0.50%	1.00%	2.00%	3.00%	5.00%
1Yr	7.76	-0.61%	8.0	10.0	12.0	15.0	18.0	22.0	29.0	35.0	40.0
18Mo	9.42	-0.61%	9.7	10.5	13.8	16.5	18.7	22.3	27.4	33.0	38.0
2Yr	10.90	-0.6%	11.3	11.7	14.7	17.2	19.4	23.1	28.2	31.8	36.7
3Yr	14.73	-0.53%	15.9	13.8	15.7	17.9	19.8	23.1	27.9	31.3	36.0
4Yr	17.22	-0.4%	18.7	16.3	17.5	18.9	20.2	22.6	26.2	28.8	32.5
5Yr	18.87	-0.3%	20.9	18.5	19.1	19.9	20.6	22.1	24.5	26.4	29.3
6Yr	19.67	-0.13%	22.3	19.9	20.1	20.5	21.0	22.0	23.9	25.4	27.8
7Yr	20.23	-0.01%	23.3	21.0	20.8	21.1	21.3	21.9	23.2	24.5	26.4
8Yr	20.58	0.11%	24.1	21.8	21.5	21.5	21.5	21.7	22.3	23.1	24.4
9Yr	20.63	0.21%	24.5	22.3	21.8	21.7	21.6	21.5	21.5	21.8	22.5
10Yr	20.72	0.32%	24.8	22.6	22.1	21.8	21.6	21.3	21.0	21.0	21.4
12Yr	20.80	0.43%	25.2	23.1	22.3	21.9	21.5	20.9	20.1	19.8	19.9
15Yr	20.59	0.55%	25.4	23.3	22.4	21.7	21.2	20.3	19.1	18.6	18.3
20Yr	20.21	0.65%	25.6	23.5	22.4	21.5	20.8	19.7	18.2	17.5	17.0
25Yr	20.18	0.61%	25.6	23.6	22.4	21.4	20.7	19.4	17.9	17.1	16.6
30Yr	19.08	0.78%	25.1	23.3	22.0	21.0	20.2	19.0	17.4	16.7	16.1

Table A1.4: Implied Normal volatilities for EUR swaptions

Exp x Tenor	-200bps	-100bps	-50bps	-25bps	ATM	25bps	50bps	100bps	200bps	ATM STR
3Mo X 5Yr	63.96	50.51	38.3	32.54	30.5	35.69	43.68	60.35	91.93	-0.0031
6Mo X 5Yr	57.67	46.62	36.88	32.67	31.5	35.43	41.77	55.7	82.74	-0.0027
1Yr X 5Yr	49.19	41.95	35.87	33.74	33.6	35.92	39.99	49.89	70.37	-0.0017
2Yr X 5Yr	45.79	41.94	38.64	37.71	37.8	39.05	41.37	47.81	62.85	0.0005
3Yr X 5Yr	46.36	43.76	41.74	41.33	41.7	42.67	44.46	49.57	62.27	0.0030
4Yr X 5Yr	46.39	45.23	44.33	44.3	44.8	45.58	46.94	50.79	60.93	0.0060
5Yr X 5Yr	45.54	46.25	46.34	46.61	47.1	47.83	48.83	51.51	58.81	0.0089
7Yr X 5Yr	48.18	48.98	49.27	49.58	50.1	50.7	51.54	53.76	59.88	0.0123
10Yr X 5Yr	49.79	51.1	51.59	51.91	52.3	52.83	53.44	55.01	59.4	0.0112
15Yr X 5Yr	48.69	50.47	51	51.29	51.6	51.97	52.4	53.47	56.52	0.0096
20Yr X 5Yr	46.86	49.23	49.79	50.06	50.3	50.72	51.15	52.24	55.48	0.0091
30Yr X 5Yr	43.33	46.98	47.27	47.38	47.7	47.77	48.13	49.27	53.35	0.0070

Table A1.5: Implied Shifted Black volatilities for EUR swaptions

Term x Tenor	-200bps	-100bps	-50bps	-25bps	ATM	25bps	50bps	100bps	200bps
3Mo X 5Yr	35.0	24.0	15.8	12.2	10.5	11.3	13.4	17.6	22.6
6Mo X 5Yr	40.6	22.0	15.1	12.2	10.8	11.2	12.3	14.9	20.3
1Yr X 5Yr	34.3	19.6	14.6	12.5	11.4	11.3	11.7	13.3	17.1
2Yr X 5Yr	31.2	19.2	15.3	13.7	12.6	12.0	11.9	12.5	15.0
3Yr X 5Yr	30.8	19.6	16.2	14.7	13.6	12.9	12.5	12.7	14.6
4Yr X 5Yr	30.1	19.8	16.9	15.4	14.3	13.5	12.9	12.7	14.0
5Yr X 5Yr	28.9	19.9	17.3	15.9	14.8	13.9	13.2	12.7	13.3
7Yr X 5Yr	29.6	20.4	17.8	16.4	15.3	14.3	13.6	12.9	13.2
10Yr X 5Yr	29.8	20.7	18.2	16.7	15.5	14.5	13.7	12.9	12.8
15Yr X 5Yr	28.8	20.1	17.7	16.3	15.1	14.0	13.2	12.3	11.9
20Yr X 5Yr	28.3	19.9	17.5	16.0	14.8	13.8	13.0	12.1	11.8
30Yr X 5Yr	28.2	20.1	17.4	15.9	14.7	13.6	12.8	11.9	11.8

A2 Full swaption dataset

Table A2.1: Obtained implied Normal swaption volatilities

Term x Tenor	-200bps	-100bps	-50bps	-25bps	ATM	25bps	50bps	100bps	200bps	ATM STR
3Mo X 1Yr	41.64	32.09	21.17	14.94	10.8	18.34	26.18	40.53	66.53	-0.0063
6Mo X 1Yr	38.33	29.99	20.45	15.14	12.3	18.02	24.78	37.41	60.41	-0.0063
1Yr X 1Yr	35.26	28.55	21.09	17.38	15.9	19.72	24.88	35.23	54.54	-0.0058
2Yr X 1Yr	32.73	29.16	25.55	24.44	24.5	26.36	29.15	35.9	49.99	-0.0039
3Yr X 1Yr	38	35.8	33.23	32.54	32.7	33.84	35.88	41.46	54.44	-0.0019
4Yr X 1Yr	40.75	40.39	38.8	38.45	38.7	39.45	40.9	45.19	56.23	0.0003
5Yr X 1Yr	41.63	42.95	42.47	42.48	42.6	43.43	44.44	47.41	55.7	0.0027
7Yr X 1Yr	45.78	47.16	47.44	47.75	48.3	48.93	49.84	52.28	58.97	0.0086
10Yr X 1Yr	46.95	49.69	50.77	51.35	52.1	52.7	53.48	55.26	59.66	0.0129
15Yr X 1Yr	47.21	50.01	50.99	51.46	51.9	52.43	52.95	54.09	56.87	0.0102
20Yr X 1Yr	45.55	48.79	49.75	50.2	50.8	51.14	51.65	52.81	55.79	0.0095
30Yr X 1Yr	41.91	46.48	47.25	47.57	47.7	48.27	48.71	49.87	53.4	0.0070
3Mo X 2Yr	47.39	36.07	23.98	17.26	14	23.09	32.24	49.01	79.38	-0.0058
6Mo X 2Yr	42.2	33.13	23.44	18.33	16.2	21.66	28.52	41.69	65.88	-0.0056
1Yr X 2Yr	35.77	29.64	23.34	20.72	20.4	23.6	28.14	37.8	56.3	-0.0048
2Yr X 2Yr	37.25	33.32	29.52	28.34	28.4	30.06	32.84	39.85	54.86	-0.0029
3Yr X 2Yr	40.92	38.51	35.89	35.21	35.3	36.53	38.58	44.28	57.71	-0.0008
4Yr X 2Yr	41.86	41.36	39.88	39.61	39.8	40.75	42.24	46.55	57.62	0.0015
5Yr X 2Yr	42.61	43.4	43.02	43.1	43.4	44.16	45.2	48.19	56.4	0.0041
7Yr X 2Yr	46.05	47.12	47.4	47.72	48.2	48.94	49.87	52.32	59	0.0108
10Yr X 2Yr	47.13	49.62	50.63	51.19	51.9	52.5	53.27	55.04	59.48	0.0124
15Yr X 2Yr	46.92	49.72	50.7	51.16	51.6	52.13	52.64	53.76	56.47	0.0099
20Yr X 2Yr	45.25	48.53	49.5	49.94	50.4	50.85	51.35	52.46	55.28	0.0094
30Yr X 2Yr	41.98	46.48	47.21	47.51	47.7	48.17	48.6	49.73	53.26	0.0070
3Mo X 3Yr	52.89	40.86	28.58	22.11	19.7	27.03	35.89	52.73	83.54	-0.0050
6Mo X 3Yr	47.47	37.76	27.95	23.13	21.5	26.21	32.93	46.46	71.74	-0.0047
1Yr X 3Yr	40.86	34.39	28.14	25.72	25.4	28.25	32.61	42.43	61.8	-0.0039
2Yr X 3Yr	40.78	36.97	33.38	32.32	32.6	33.87	36.47	43.29	58.44	-0.0018
3Yr X 3Yr	43.08	40.85	38.42	37.82	37.8	39.11	41.05	46.56	59.84	0.0004
4Yr X 3Yr	43.92	43	41.69	41.49	41.7	42.67	44.1	48.27	59.09	0.0028
5Yr X 3Yr	43.66	44.34	44.13	44.29	44.6	45.42	46.44	49.32	57.21	0.0056
7Yr X 3Yr	46.79	47.74	48.03	48.36	48.8	49.55	50.45	52.82	59.31	0.0122
10Yr X 3Yr	48.12	50.19	51.03	51.51	52.2	52.7	53.42	55.12	59.53	0.0120
15Yr X 3Yr	47.57	50.03	50.86	51.26	51.9	52.13	52.62	53.72	56.55	0.0097
20Yr X 3Yr	45.91	48.9	49.72	50.11	50.7	50.94	51.41	52.51	55.47	0.0093
30Yr X 3Yr	42.56	46.76	47.37	47.6	47.8	48.17	48.58	49.71	53.43	0.0070
3Mo X 4Yr	59.41	46.65	34.31	28.2	26.3	32.11	40.56	57.46	88.94	-0.0041
6Mo X 4Yr	53.24	42.87	33.03	28.54	27.4	31.37	37.89	51.71	78.08	-0.0037
1Yr X 4Yr	45.49	38.66	32.49	30.23	30.2	32.53	36.73	46.61	66.66	-0.0028
2Yr X 4Yr	43.37	39.63	36.24	35.28	35.6	36.74	39.18	45.8	60.88	-0.0007
3Yr X 4Yr	44.7	42.46	40.21	39.69	39.9	40.98	42.85	48.16	61.21	0.0016
4Yr X 4Yr	45.49	44.33	43.19	43.06	43.6	44.27	45.66	49.67	60.21	0.0043
5Yr X 4Yr	44.6	45.26	45.21	45.43	45.8	46.61	47.62	50.4	57.98	0.0074
7Yr X 4Yr	47.53	48.39	48.68	49.01	49.6	50.16	51.03	53.33	59.63	0.0124
10Yr X 4Yr	48.91	50.59	51.26	51.67	52.2	52.72	53.39	55.03	59.42	0.0115
15Yr X 4Yr	48.12	50.23	50.91	51.26	51.7	52.03	52.49	53.58	56.52	0.0096
20Yr X 4Yr	46.31	48.98	49.67	50	50.4	50.75	51.2	52.3	55.39	0.0092
30Yr X 4Yr	42.83	46.76	47.19	47.37	47.7	47.85	48.23	49.37	53.26	0.0070

3Mo X 5Yr	63.96	50.51	38.3	32.54	30.5	35.69	43.68	60.35	91.93	-0.0031
6Mo X 5Yr	57.67	46.62	36.88	32.67	31.5	35.43	41.77	55.7	82.74	-0.0027
1Yr X 5Yr	49.19	41.95	35.87	33.74	33.6	35.92	39.99	49.89	70.37	-0.0017
2Yr X 5Yr	45.79	41.94	38.64	37.71	37.8	39.05	41.37	47.81	62.85	0.0005
3Yr X 5Yr	46.36	43.76	41.74	41.33	41.7	42.67	44.46	49.57	62.27	0.0030
4Yr X 5Yr	46.39	45.23	44.33	44.3	44.8	45.58	46.94	50.79	60.93	0.0060
5Yr X 5Yr	45.54	46.25	46.34	46.61	47.1	47.83	48.83	51.51	58.81	0.0089
7Yr X 5Yr	48.18	48.98	49.27	49.58	50.1	50.7	51.54	53.76	59.88	0.0123
10Yr X 5Yr	49.79	51.1	51.59	51.91	52.3	52.83	53.44	55.01	59.4	0.0112
15Yr X 5Yr	48.69	50.47	51	51.29	51.6	51.97	52.4	53.47	56.52	0.0096
20Yr X 5Yr	46.86	49.23	49.79	50.06	50.3	50.72	51.15	52.24	55.48	0.0091
30Yr X 5Yr	43.33	46.98	47.27	47.38	47.7	47.77	48.13	49.27	53.35	0.0070
3Mo X 7Yr	75.63	58.22	44.48	38.02	35	40.39	48.83	66.9	101.42	-0.0009
6Mo X 7Yr	67.15	53.28	42.62	38.02	36	39.97	46.34	60.95	89.79	-0.0004
1Yr X 7Yr	55.87	47.28	40.88	38.56	37.7	39.84	43.53	53.2	73.99	0.0008
2Yr X 7Yr	49.53	45.21	42.2	41.34	41.3	42.29	44.21	49.91	64	0.0034
3Yr X 7Yr	49.72	46.53	44.54	44.07	44.1	45.01	46.5	51.02	62.9	0.0061
4Yr X 7Yr	49.12	47.42	46.39	46.24	46.3	47.15	48.28	51.67	61.11	0.0082
5Yr X 7Yr	47.92	48.01	47.83	47.95	48.1	48.84	49.65	51.99	58.76	0.0099
7Yr X 7Yr	49.99	50.19	50.2	50.36	50.6	51.2	51.9	53.89	59.72	0.0118
10Yr X 10Yr	53.13	52.97	52.71	52.7	52.5	53.06	53.49	54.87	59.53	0.0104
15Yr X 7Yr	49.82	51.04	51.29	51.46	51.4	51.99	52.4	53.53	57.09	0.0095
20Yr X 7Yr	48.1	49.84	50.01	50.11	50.1	50.52	50.87	51.95	55.64	0.0087
30Yr X 7Yr	44.38	47.4	47.17	47.04	46.8	47.06	47.32	48.49	53.27	0.0070
3Mo X 10Yr	90.76	66.72	50.76	43.15	39.5	44.65	53.86	73.81	112.01	0.0031
6Mo X 10Yr	78.63	60.15	48.12	42.85	40.5	43.83	50.37	65.85	96.73	0.0036
1Yr X 10Yr	63.72	52.68	45.76	43.1	41.9	43.23	46.43	55.68	76.5	0.0045
2Yr X 10Yr	53.14	48.29	45.42	44.52	44.2	44.85	46.25	50.91	63.49	0.0063
3Yr X 10Yr	52.73	49.05	46.99	46.39	46.2	46.74	47.81	51.48	62.05	0.0078
4Yr X 10Yr	51.95	49.66	48.36	48.02	47.9	48.37	49.16	51.88	60.26	0.0090
5Yr X 10Yr	50.8	50.08	49.49	49.37	49.3	49.72	50.26	52.09	58.1	0.0100
7Yr X 10Yr	52.34	51.77	51.33	51.27	51.2	51.64	52.13	53.77	59.22	0.0113
10Yr X 10Yr	53.13	52.97	52.71	52.7	52.5	53.06	53.49	54.87	59.53	0.0104
15Yr X 10Yr	51.63	52.02	51.85	51.84	51.5	52.15	52.52	53.75	58.04	0.0093
20Yr X 10Yr	50.04	50.86	50.46	50.3	49.8	50.3	50.54	51.61	55.99	0.0081
30Yr X 10Yr	45.7	47.68	46.71	46.23	45.2	45.7	45.82	47.03	52.82	0.0070
3Mo X 20Yr	96.54	71.94	57	49.79	44.7	45.83	51.61	67.26	99.5	0.0065
6Mo X 20Yr	90.27	68.8	55.91	49.88	45.6	46.34	50.85	64.25	93.02	0.0067
1Yr X 20Yr	72.58	59.65	52.2	49.03	46.6	46.77	48.56	55.89	74.82	0.0072
2Yr X 20Yr	59.38	53.51	50.16	48.81	47.6	47.61	48.05	50.91	61.12	0.0079
3Yr X 20Yr	57.56	53	50.38	49.33	48.3	48.4	48.71	50.93	59.49	0.0086
4Yr X 20Yr	55.48	52.31	50.39	49.64	48.8	48.95	49.16	50.78	57.58	0.0092
5Yr X 20Yr	53.18	51.49	50.27	49.79	48.9	49.33	49.45	50.52	55.46	0.0097
7Yr X 20Yr	52.84	51.6	50.64	50.27	49.3	49.99	50.16	51.23	55.92	0.0101
10Yr X 20Yr	51.92	51.47	50.86	50.64	49.7	50.6	50.84	51.91	56.29	0.0093
15Yr X 20Yr	49.26	49.67	49.23	49.06	47.8	49.03	49.24	50.2	54.24	0.0082
20Yr X 20Yr	47.95	48.42	47.55	47.16	45.7	46.77	46.89	47.92	52.78	0.0076
30Yr X 20Yr	43.39	45.16	43.72	42.99	41.3	42.07	42.09	43.37	49.74	0.0070

3Mo X 30Yr	98.21	73.29	58.16	50.77	45.3	45.84	51.23	66.52	98.35	0.0070
6Mo X 30Yr	91.85	70.16	57.11	50.9	46.2	46.38	50.45	63.39	91.67	0.0071
1Yr X 30Yr	73.75	60.74	53.18	49.87	47.1	47.09	48.57	55.49	74	0.0074
2Yr X 30Yr	59.63	54.21	50.96	49.59	48.1	48.14	48.32	50.6	59.83	0.0078
3Yr X 30Yr	57.37	53.25	50.7	49.63	48.2	48.49	48.61	50.38	58.12	0.0082
4Yr X 30Yr	54.94	52.18	50.32	49.54	48.2	48.69	48.77	50.06	56.22	0.0086
5Yr X 30Yr	52.32	50.98	49.8	49.29	48	48.73	48.76	49.61	54.11	0.0089
7Yr X 30Yr	51.25	50.29	49.31	48.9	47.4	48.52	48.63	49.57	54.05	0.0091
10Yr X 30Yr	49.99	49.67	48.97	48.71	47.1	48.58	48.79	49.86	54.38	0.0086
15Yr X 30Yr	46.89	47.45	46.93	46.72	44.9	46.67	46.9	47.99	52.46	0.0078
20Yr X 30Yr	45.25	45.88	44.95	44.54	42.4	44.21	44.4	45.65	51.06	0.0074
30Yr X 30Yr	40.74	42.19	40.83	40.2	37.9	39.56	39.76	41.36	48.01	0.0070

A3 Error terms

Table A3.1: Error term for caplets

0.5	0.28409	0.35994	0.03274	0.04774	0.03938	0.04863
1	0.25724	0.40084	0.03276	0.04704	0.03930	0.04800
1.5	0.12154	0.16026	0.01684	0.02657	0.02270	0.02724
2	0.05187	0.14947	0.01384	0.01543	0.01994	0.02098
2.5	0.03306	0.06537	0.02753	0.02962	0.03313	0.03551
3	0.06181	0.06529	0.05328	0.05133	0.05845	0.05725
3.5	0.07812	0.11631	0.03280	0.03281	0.03647	0.03642
4	0.04892	0.19369	0.03581	0.03583	0.03841	0.03838
4.5	0.05249	0.11220	0.03239	0.04438	0.03409	0.04531
5	0.03754	0.10053	0.03317	0.04052	0.03485	0.04165
5.5	0.02752	0.09351	0.02237	0.10301	0.02425	0.10473
6	0.02402	0.13010	0.02154	0.10511	0.02347	0.10674
6.5	0.02749	0.08964	0.01936	0.10877	0.02082	0.11030
7	0.01777	0.12284	0.01856	0.10192	0.01982	0.10329
7.5	0.01886	0.13407	0.01813	0.11990	0.01912	0.12157
8	0.01426	0.13886	0.01820	0.11760	0.01893	0.11900
8.5	0.02275	0.11175	0.01656	0.13233	0.01716	0.13377
9	0.02597	0.14378	0.01590	0.12483	0.01629	0.12605

Caplet Maturity	Normal 1	Normal 2	Shifted 1	Shifted 2	Obłój 1	Obłój 2
9.5	0.02553	0.11868	0.01479	0.08699	0.01529	0.08816
10	0.03265	0.17432	0.01421	0.08967	0.01459	0.09070
10.5	0.02657	0.08525	0.01126	0.05345	0.01142	0.05433
11	0.02603	0.10462	0.00972	0.03984	0.00997	0.04057
11.5	0.02708	0.14430	0.00833	0.02583	0.00853	0.02643
12	0.03516	0.21790	0.00759	0.03321	0.00781	0.03382
12.5	0.03093	0.07380	0.01112	0.04669	0.01143	0.04740
13	0.03070	0.09386	0.01063	0.03755	0.01089	0.03818
13.5	0.02987	0.11551	0.01021	0.02837	0.01043	0.02891
14	0.02485	0.13287	0.00982	0.01915	0.01000	0.01961
14.5	0.01930	0.15480	0.01047	0.01229	0.01076	0.01255
15	0.02197	0.22385	0.01169	0.03854	0.01203	0.03909
15.5	0.04357	0.06991	0.01269	0.03958	0.01292	0.04019
16	0.04191	0.07182	0.01266	0.03406	0.01286	0.03462
16.5	0.03916	0.07114	0.01263	0.02852	0.01281	0.02903
17	0.03423	0.07612	0.01269	0.02317	0.01284	0.02364
17.5	0.02744	0.08341	0.01272	0.01770	0.01284	0.01812
18	0.01924	0.09179	0.01290	0.01421	0.01312	0.01444
18.5	0.01102	0.10242	0.01343	0.01374	0.01368	0.01393
19	0.00681	0.11169	0.01427	0.01467	0.01449	0.01482
19.5	0.01165	0.12253	0.01519	0.01657	0.01539	0.01669
20	0.03417	0.18127	0.01521	0.04815	0.01557	0.04878
20.5	0.01445	0.08779	0.00976	0.01056	0.01003	0.01081
21	0.01733	0.08880	0.00973	0.00910	0.00999	0.00927
21.5	0.01957	0.08843	0.00977	0.01039	0.01001	0.01055
22	0.02833	0.08847	0.00976	0.01173	0.01000	0.01187
22.5	0.03524	0.08832	0.00978	0.01317	0.01002	0.01320
23	0.04396	0.08803	0.01000	0.01614	0.01026	0.01605
23.5	0.05056	0.08821	0.01029	0.02025	0.01054	0.02003
24	0.05746	0.08882	0.01054	0.02443	0.01079	0.02422

Caplet Maturity	Normal 1	Normal 2	Shifted 1	Shifted 2	Obłój 1	Obłój 2
24.5	0.06400	0.09013	0.01083	0.02847	0.01106	0.02827
25	0.06968	0.08912	0.01112	0.03162	0.01135	0.03143
25.5	0.01472	0.09572	0.00950	0.11198	0.00962	0.11269
26	0.01612	0.09845	0.00959	0.11465	0.00970	0.11537
26.5	0.01789	0.09728	0.00970	0.11721	0.00981	0.11795
27	0.01799	0.10116	0.00980	0.11975	0.00990	0.12049
27.5	0.01966	0.09906	0.00992	0.12255	0.01000	0.12330
28	0.02055	0.10178	0.01001	0.12498	0.01008	0.12575
28.5	0.02231	0.10049	0.01016	0.12711	0.01022	0.12790
29	0.02174	0.10240	0.01024	0.12948	0.01029	0.13026
29.5	0.02257	0.10337	0.01034	0.13182	0.01038	0.13261
30	0.02307	0.10320	0.01046	0.13287	0.01049	0.13366

Table A3.2: Error terms for swaptions

	Swaption 1-year tenor error terms					
T	Black 1	Black 2	Obłój 1	Obłój 2	Norm 1	Norm 2
0.25	0.16095371	0.147158	0.171065	0.157155	0.215627	0.29771908
0.5	0.07033898	0.14508	0.078643	0.14755	0.19708	0.29889097
1	0.02634943	0.183407	0.0306	0.183556	0.17352	0.37081628
2	0.01262547	0.164471	0.010085	0.165594	0.124684	0.427755
3	0.02478959	0.162995	0.02237	0.162192	0.118646	0.48705882
4	0.03392365	0.151836	0.032088	0.150568	0.10827	0.41893251
5	0.0373289	0.135874	0.035979	0.134999	0.115197	0.32556158
7	0.06956051	0.149276	0.068691	0.148601	0.060782	0.27837273
10	0.07546808	0.15103	0.074835	0.149849	0.084985	0.225934
15	0.07492469	0.268468	0.074155	0.268298	0.065114	0.10686011
20	0.0738723	0.198125	0.073083	0.198025	0.079205	0.09285819
30	0.06658267	0.121707	0.065617	0.12111	0.107434	0.09987956
mean	0.06055983	0.164952	0.061434	0.165625	0.120879	0.28588657
	Swaption 2-year tenor error terms					
T	Black 1	Black 2	Obłój 1	Obłój 2	Norm 1	Norm 2
0.25	0.15861433	0.169792	0.167118	0.173328	0.233758	0.301627
0.5	0.06290092	0.198235	0.069902	0.198651	0.203444	0.3738089
1	0.01864714	0.217534	0.021467	0.220688	0.159249	0.34863048
2	0.01643894	0.172952	0.013548	0.1732	0.132318	0.47769003
3	0.02824897	0.168585	0.026085	0.16776	0.120803	0.53872729
4	0.03180654	0.154174	0.030275	0.153126	0.102646	0.43531075
5	0.03771908	0.137094	0.03657	0.136356	0.10935	0.32510077
7	0.0642784	0.146228	0.063575	0.145595	0.056597	0.28917573
10	0.06234359	0.145479	0.061725	0.145308	0.083359	0.22816255
15	0.05889105	0.244572	0.058159	0.244249	0.065647	0.10662033
20	0.05723146	0.183785	0.056481	0.183659	0.079998	0.09888299
30	0.05153024	0.108942	0.050601	0.108381	0.108054	0.09883762
mean	0.05405422	0.170614	0.054625	0.170858	0.121269	0.3018812
	Swaption 3-year tenor error terms					
T	Black 1	Black 2	Obłój 1	Obłój 2	Norm 1	Norm 2
0.25	0.16234622	0.229307	0.169612	0.231901	0.245609	0.4387124
0.5	0.06876039	0.267008	0.074429	0.2662	0.213074	0.49020087
1	0.01973646	0.234669	0.022081	0.237205	0.167274	0.46412056
2	0.01185224	0.176108	0.01004	0.175552	0.130983	0.54538606
3	0.01809952	0.167337	0.016347	0.166929	0.119215	0.55433222
4	0.02580242	0.151164	0.024543	0.150288	0.092496	0.4424981
5	0.02684803	0.130808	0.025971	0.130198	0.108038	0.32349085
7	0.04692809	0.134123	0.046339	0.133658	0.055456	0.2918994
10	0.04388917	0.133867	0.043299	0.133588	0.08237	0.23089131
15	0.03986759	0.218068	0.039185	0.217475	0.063931	0.0983373
20	0.03771097	0.167957	0.037009	0.167793	0.07919	0.09174863
30	0.02551728	0.093491	0.024727	0.093492	0.108054	0.09883762
mean	0.04394653	0.175326	0.044465	0.175356	0.122141	0.33920461

	Swaption 4-year tenor error terms					
T	Black 1	Black 2	Obłój 1	Obłój 2	Norm 1	Norm 2
0.25	0.16513538	0.30045	0.170696	0.299606	0.254677	0.64245722
0.5	0.0346547	0.338585	0.039445	0.337343	0.21874	0.64979383
1	0.0189342	0.240285	0.021942	0.242426	0.168293	0.55909855
2	7.57E-03	0.178138	0.008548	0.178179	0.128058	0.57589519
3	0.01034042	0.165154	0.00958	0.164793	0.111567	0.53955898
4	0.01809607	0.146405	0.017076	0.145656	0.08733	0.42246733
5	0.01674401	0.125482	0.016058	0.125362	0.108035	0.31739466
7	0.03124319	0.126025	0.030705	0.125583	0.056467	0.28208269
10	0.02773917	0.125779	0.027175	0.125377	0.08092	0.23968413
15	0.0238766	0.209631	0.023245	0.209325	0.06083	0.10831305
20	0.02092443	0.154865	0.020281	0.154679	0.076012	0.10069054
30	0.01278034	0.080884	0.012177	0.080899	0.10605	0.09881199
mean	0.03233663	0.18264	0.033077	0.182436	0.121415	0.37802068
	Swaption 5-year tenor error terms					
T	Black 1	Black 2	Obłój 1	Obłój 2	Norm 1	Norm 2
0.25	0.1572	0.3525	0.1617	0.3514	0.2528	0.79562407
0.5	0.0357	0.3160	0.0397	0.3186	0.2189	0.76272031
1	0.0217	0.2465	0.0244	0.2483	0.1680	0.60570099
2	0.0097	0.1825	0.0113	0.1833	0.1270	0.64429152
3	0.0073	0.1647	0.0069	0.1644	0.0987	0.5281925
4	0.0104	0.1448	0.0098	0.1446	0.0795	0.41710499
5	0.0101	0.1236	0.0097	0.1235	0.1103	0.3073069
7	0.0173	0.1187	0.0169	0.1183	0.0558	0.28653931
10	0.0160	0.1242	0.0156	0.1244	0.0797	0.15928481
15	0.0132	0.1999	0.0128	0.1997	0.0586	0.11833965
20	0.0108	0.1443	0.0105	0.1441	0.0735	0.10831894
30	0.0066	0.0729	0.0068	0.0732	0.1053	0.098592
mean	0.02633866	0.182553	0.027179	0.182811	0.119001	0.402668
	Swaption 7-year tenor error terms					
T	Black 1	Black 2	Obłój 1	Obłój 2	Norm 1	Norm 2
0.25	0.16343609	0.472296	0.166714	0.474361	0.247339	1.03093797
0.5	0.03250529	0.339561	0.035488	0.338787	0.217185	0.97263635
1	0.02254555	0.254033	0.024454	0.25397	0.163437	0.76213538
2	0.01128556	0.179152	0.012344	0.179384	0.102446	0.62995486
3	5.06E-03	0.162052	5.22E-03	0.161925	0.087287	0.53662287
4	6.79E-03	0.146077	6.52E-03	0.145997	0.074279	0.43816673
5	6.86E-03	0.134323	6.79E-03	0.134598	0.049802	0.33039403
7	0.01013412	0.132497	0.009926	0.132874	0.054969	0.21975501
10	0.01287073	0.150995	0.012611	0.151868	0.072194	0.22789486
15	8.65E-03	0.196669	8.47E-03	0.196038	0.05296	0.15091312
20	0.00826413	0.131434	0.00867	0.131225	0.0698	0.12025819
30	0.01882884	0.073831	0.019411	0.074074	0.102585	0.10143309
mean	0.02560219	0.197743	0.026384	0.197925	0.107857	0.46009187

	Swaption 10-year tenor error terms					
T	Black 1	Black 2	Obłój 1	Obłój 2	Norm 1	Norm 2
0.25	0.02222823	0.430026	0.024591	0.429595	0.212045	1.29492084
0.5	0.01858171	0.368607	0.020421	0.368044	1.83E-01	1.21034373
1	0.01480812	0.251593	0.015981	0.250725	1.40E-01	1.08278208
2	0.01141445	0.175063	0.012064	0.175372	0.090233	0.60162979
3	0.00771602	0.161104	0.008238	0.161204	0.073973	0.51644665
4	0.00653565	0.13772	0.00696	0.137878	0.080095	0.42979521
5	0.00720352	0.109581	0.007558	0.109567	0.050186	0.24576028
7	4.95E-03	0.117142	5.24E-03	0.117115	0.056221	0.23723666
10	0.00612579	0.130662	0.006422	0.13055	0.072194	0.22789486
15	0.00908467	0.161026	0.009433	0.16061	0.099609	0.19881536
20	0.01434713	0.104668	0.01472	0.1044	0.067802	0.15352174
30	0.02782518	0.063151	0.028255	0.06332	0.105548	0.10951312
mean	0.01256867	0.184195	0.013323	0.184032	0.102581	0.52572169
	Swaption 20-year tenor error terms					
T	Black 1	Black 2	Obłój 1	Obłój 2	Norm 1	Norm 2
0.25	0.01459479	0.327926	0.016054	0.327554	0.158582	1.86343952
0.5	0.01360155	0.302357	0.01489	0.301744	0.149131	1.59484479
1	0.01255713	0.21194	0.013409	0.211298	0.117283	0.93764467
2	0.01106055	0.147029	0.011563	0.147141	0.076659	0.47371925
3	0.00974046	0.133334	0.010169	0.133419	0.067176	0.40475751
4	0.00898824	0.109988	0.009348	0.110213	0.088071	0.33205207
5	0.00942411	0.098393	0.009721	0.09835	0.055343	0.27072646
7	0.00932525	0.105939	0.009597	0.105869	0.064499	0.27362035
10	0.01114528	0.109914	0.011399	0.109802	0.078985	0.2634685
15	0.01521849	0.123384	0.015463	0.12305	0.108386	0.23601908
20	0.02044472	0.090995	0.020724	0.090896	0.079213	0.20202555
30	0.03437564	0.054272	0.034667	0.05439	0.117907	0.12383179
mean	0.01420635	0.151289	0.014751	0.151144	0.09677	0.5813458
	Swaption 30-year tenor error terms					
T	Black 1	Black 2	Obłój 1	Obłój 2	Norm 1	Norm 2
0.25	0.0153319	0.313207	0.016703	0.312838	0.156325	1.83888186
0.5	0.01476069	0.286339	0.01598	0.285906	0.148255	1.56713471
1	0.01350475	0.201527	0.014319	0.20089	0.118016	0.92511131
2	0.011868	0.137854	0.01234	0.137909	0.076696	0.4466106
3	0.01128022	0.12443	0.011686	0.124563	0.07434	0.39358111
4	0.01090626	0.107632	0.011248	0.107577	0.093771	0.33529652
5	0.01085785	0.09717	0.011139	0.097111	0.06378	0.27593555
7	0.01133761	0.10744	0.011598	0.107355	0.074091	0.29474021
10	0.01306909	0.116125	0.013312	0.116046	0.088232	0.30580162
15	0.0170111	0.123222	0.017233	0.12288	0.11472	0.2710464
20	0.0226354	0.091358	0.022882	0.091296	0.131872	0.24144656
30	0.03240819	0.055872	0.032657	0.055995	0.11784	0.14551912
mean	0.01541426	0.146848	0.015925	0.146697	0.104828	0.5867588