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Equilibrium arrivals to a last-come first-served preemptive-resume queue*

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Abstract

We consider a queueing system where a single server opens and serves users according to the last-come first-served discipline with preemptive-resume (LCFS-PR). A finite number of strategic users must choose individually when to arrive at the server. We allow for general classes of user preferences and service time distributions and show existence and uniqueness of a symmetric Nash equilibrium. Furthermore, we show that no asymmetric equilibrium exists, if the population consists of only two users, or if arrival strategies satisfy a mild regularity condition. Based on the constructive existence proof for the symmetric equilibrium, we provide a numerical example in which we compute the symmetric equilibrium strategy and compare the resulting social efficiency to that obtained if users are instead served on a first-come first-served (FCFS) basis.

Keywords Queueing · Strategic arrivals · Nash equilibrium · LCFS-PR · FCFS

JEL Classification C72 · D62 · R41

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1 Introduction

In a variety of situations in which multiple users demand a service that is made accessible at a certain time, the peak demand for service often exceeds the capacity to provide it. Examples of such situations include users purchasing popular concert tickets, accessing a website at the release of a new product or the start of a sale (e.g., Black Friday), or conducting online financial transactions when a bank or stock market opens. To cope with excess demand, the provision of service to users is often managed with a queueing system. The way a queue is managed affects the behaviour of users and, consequently, the waiting time that users face, and inefficient queueing leads to both frustration for the unlucky user and costs to society. Therefore, the study of how strategic users behave when faced with specific queueing systems and of the implied social welfare loss is important for the design and evaluation of queueing systems.

This paper considers a queueing system with a single server that opens at a given point in time and manages the demand for some service. A finite number of users choose independently when to arrive at the system. Users prefer to complete service early rather than later, and they dislike waiting in the queue. The service time requirements of users are identically and independently distributed. The order in which waiting users are served is determined by the Last-Come-First-Served service discipline with preemptive resume (LCFS-PR).

The LCFS-PR discipline admits any newly arrived user into service immediately, possibly preempting the service progress of another user. The preempted user on the other hand joins the queue where later arrivals are prioritized over earlier arrivals. When a preempted user re-enters service, her service is resumed from the point of interruption. Whereas the most frequently used (and studied) discipline is the First-Come First-Served (FCFS) discipline, papers studying equilibrium and efficiency properties of alternative disciplines such as the Last-Come First-Served (LCFS) and LCFS with preemptive resume (LCFS-PR) show that these may in some settings result in superior outcomes. In Hassin (1985) and Platz and Østerdal (2017) different environments are studied in which LCFS(-PR) disciplines are shown to be socially optimal for general classes of user preferences and service time distributions. However, in a situation with a finite number of strategic users and LCFS-PR, the question of existence (and uniqueness) of equilibria and of whether LCFS-PR generally outperforms FCFS have remained open. In this paper, we answer the first question affirmatively (with some qualifications) and the last question negatively.

The strategic choices of arrivals to queues have been studied for almost half a century (see Hassin, 2016, for an extensive survey). The problem was first approached by considering a fluid model for congestion dynamics that studied the equilibrium arrival behavior of a continuum of users (Vickrey, 1969). In this model, each user must choose jis/her arrival time to a continuously open bottleneck, and each user has a preferred time for passing the bottleneck and will incur a cost from being early or late. Similar fluid models have been studied further and extended in various directions, e.g. to treat heterogeneous users (Arnott et al., 1989), elastic user demand (Arnott et al., 1993), and hypercongestion (Verhoef, 2003).

The study of strategic arrivals in queueing systems where the server has a limited service period (i.e. the server admits an opening and/or closing time) was first formulated by Glazer and Hassin (1983). They consider a Poisson-distributed number of identical users with exponential service requirements that arrive at a server with a known opening and closing time and wish to minimize their own waiting time (Glazer and Hassin, 1983). This work showed that in a symmetric equilibrium under FCFS, the users arrive according to a continuous distribution function that extends over a finite interval before and after the opening time. Several variations of this model have since been considered, e.g., to treat bulk service (Glazer and Hassin, 1987), no arrivals prior to opening (Hassin and Kleiner, 2011) and discrete arrival times and deterministic service times (Rapoport et al., 2004; Seale et al., 2005; Stein et al., 2007). Whereas the aforementioned studies assume that users only want to minimize their wait in the queue, another body of literature studies environments where users also care about being served at an early time. This type of preference has been modelled as a tardiness cost that increases the later the user is admitted into service. The equilibrium behavior induced by such user preferences has been studied for several variants of assumptions. Specifically, the symmetric equilibrium has been studied for a Poisson-distributed number of identical users with exponential service time requirements and multilinear costs of waiting and tardiness in time, and it has been studied in settings both with and without early arrivals, as have the fluid analogues of these models (Jain et al., 2011; Haviv, 2013). A complete analysis of the existence and uniqueness of the equilibrium for a general population size with multilinear costs and exponential service times showed that there always exists an equilibrium, and that it is in fact symmetric (Juneja and Shimkin, 2013). Lastly, the existence and uniqueness of a symmetric equilibrium was established for more general classes of utility functions and service time distributions (Breinbjerg, 2017).

The above-mentioned studies all consider queueing environments that employ the FCFS service discipline. Though the FCFS discipline is intuitively fair and reasonable to most people, it is as mentioned not necessarily the most socially efficient way of settling a queue (Hassin, 1985). In both theoretical analysis (e.g., Glazer and Hassin (1983), Breinbjerg (2017) and others mentioned above) and in empirical experiments (e.g., Rapoport et al. (2004), Seale et al. (2005)), it has been found that under FCFS, users tend to show up (too) early, which may lead to excess waiting time. Therefore, employing a service discipline that induces users to spread out their arrivals in order to avoid arriving at the same time or just before other users, may improve on overall efficiency. In particular, in queueing environments where the server opens at a given point in time, and a continuum of users choose their arrival time in a setting where they incur costs from queueing and being served late, the FCFS discipline provides the lowest level of social efficiency among all work-conserving disciplines, whereas the LCFS discipline provides the highest (Platz and Østerdal, 2017).[1] Furthermore, empirical support for the greater social efficiency of LCFS compared to FCFS has been established in an experimental setting for a queueing environment with a very small (three-user) population size, where each user chooses arrival time from a finite set of time slots (Breinbjerg et al., 2016).

In this paper, we consider a queueing environment where a finite number of users with identical preferences, composed of waiting costs and tardiness costs, choose when to arrive at a single-server facility that opens at a commonly known point in time and serves users on a LCFS-PR basis. We restrict attention to facilities with no closing time and do not allow users to leave the queue once they have arrived. Our main findings are the following: First, we provide a few results on the properties of equilibria in general. Second, we develop a constructive procedure that establishes the existence of a symmetric mixed Nash equilibrium, and we show that this is the unique symmetric equilibrium.[2] Furthermore,

[1]For the fluid model, it has been show for varying degrees of random sorting, ranging from FCFS to a completely random service order, that the choice of service discipline does not play a role for the properties of social efficiency (de Palma and Fosgerau, 2013).

[2]When interpreting a symmetric mixed strategy equilibrium, we do not necessarily expect users in real life queueing settings to fully randomize accordingly. Instead, the mixed strategy may be interpreted as a user's expectation about the arrival decisions of others. Empirical support for this interpretation is found in Rapoport et al. (2004) and Stein et al. (2007), who find that overall behaviour in a considered queueing game is represented well by the mixed equilibrium strategy, while it does not reflect individual behaviour.

we show that no asymmetric equilibrium exists, if the population is of size two, or if all possible arrival strategies have a finite number of inflection points. Using a numerical method based on the constructive procedure from the existence proof, we provide an example of a symmetric equilibrium as an illustration. We calculate the social efficiency of the resulting symmetric equilibrium and compare it to the social efficiency when users are served on a first-come first-served basis. The example shows that social efficiency under LCFS-PR may be lower than under FCFS, meaning that the efficiency loss due to preemption may outweigh the efficiency gain from less initial congestion under LCFS-PR. The paper is organized as follows: Section 2 formalizes the queueing environment and model assumptions. Section 3 defines the relevant notion of an equilibrium, presents the equilibrium properties of the queueing model, and in section 3.3, provides the proof of existence and uniqueness of a symmetric equilibrium. In Section 3.4, we provide insights on the (non)-existence of asymmetric equilibria. Section 4 presents a numerical method to compute the symmetric equilibrium and in an example compares the resulting social efficiency with that obtained in a corresponding queueing system that employs the FCFS service discipline. We conclude the paper in Section 5 with a brief summary and future research directions. Proofs that require technical notation for the stochastic queueing dynamics are relegated to the Appendix.

2 Model[3]

A finite user population N of size $n \geq 2$ must obtain service by a single-server facility. The facility opens for service at time 0 and does not close until all users have been served. The facility serves one user at a time according to a work-conserving LCFS-PR regime. We assume that users cannot queue up at the facility before opening time, and moreover, we assume that a user cannot leave the queue once arrived. Note that even when early arrivals are allowed, it can never be optimal for a user to arrive before opening time under a LCFS service discipline, since later arriving users will be prioritized once service starts. Therefore, the equilibrium strategy derived in the present paper will be identical to the equilibrium in a LCFS-PR queueing game in which early arrivals are allowed.

[3]With the exception of the considered service discipline, the queueing game of this paper closely resembles that presented in Breinbjerg (2017).

Strategy of arrival. Suppose that each user $i \in N$ independently arrives according to a cumulative distribution function F_i that assigns to each point in time t the probability that i has arrived by time t. Since users cannot show up before opening time, $F_i(t) = 0$ for $t < 0$. We refer to F_i as a *strategy.* Let $\mathcal{S}(F_i)$ denote the support of strategy F_i. Thus, $\mathcal{S}(F_i)$ is the smallest closed set such that $\int_{\mathcal{S}(F_i)} \mathrm{d}F_i(t) = 1$. The collection of strategies of all users is given by the *arrival profile* $\mathcal{F} = \{F_i\}_{i \in N}$. The notation $\mathcal{F}^{-i}$ will be used to denote the collection of strategies for all users except user i.

Time of departure. Given an arrival profile $\mathcal{F}$, we consider the probabilities associated with the time at which a given user has completed her service and departs the system. Assume that the amount of time required for the facility to complete the service of each user is independently and identically distributed according to a cumulative distribution function S, where S is absolutely continuous and has finite, positive moments.

Let D_i denote the ex-ante *cumulative departure time distribution* for user i, such that $D_i(d \mid t, \mathcal{F}^{-i})$ is the probability that user i has departed the system by time $d \in \mathbb{R}$, given that she arrived at time t, and the $n-1$ other users arrive according to $\mathcal{F}^{-i}$. Note that $\lim_{d \to \infty} D_i(d \mid t, \mathcal{F}^{-i}) = 1$ for all t since the user population is finite, the service time distribution S has finite moments, and LCFS-PR is work-conserving. Note also that $D_i(d \mid t, \mathcal{F}^{-i}) = 0$ for all $d \leq t$.

Utility function. We assume that all users have identical preferences and that each user prefers early service to later service and dislikes spending time in the queue. To capture such preferences, let $V(t, d)$ be a real-valued function representing the utility of a user who arrives at time t and departs from the system at time d after waiting in the queue (and receiving service) for a total of $d - t$ time units. We assume that V is continuous for all $d \geq t$ and strictly decreasing in both the departure time d and the waiting time $d - t$. Moreover, V is bounded from above, and $\lim_{t \to \infty} V(t, t) = -\infty$.

We assume that every user aims to maximize her expected utility with respect to the timing of arrival. For a given collection of strategies $\mathcal{F}^{-i}$, let U_i denote the expected utility of user i who arrives at time t or according to F_i, when the $n-1$ other users arrive according to $\mathcal{F}^{-i}$, so that

$$U_i(t, \mathcal{F}^{-i}) = \int_t^{\infty} V(t, d) \ \mathrm{d}D_i(d \mid t, \mathcal{F}^{-i}) \tag{1}$$

is the expected utility of arriving at time t, when the other users arrive according to

$\mathcal{F}^{-i}$. Here $\int$ is the Lebesgue integral over the cumulative departure time distribution D. Furthermore, the expected utility of arriving according to strategy F_i is given by

$$U_i(F_i, \mathcal{F}^{-i}) = \int_0^{\infty} U_i(t, \mathcal{F}^{-i}) dF_i. \quad (2)$$

A LCFS-PR queueing game is thus represented by a tuple $\mathcal{G} = \langle\, n, V, S \,\rangle$.

3 Equilibrium analysis

In this section, we start by defining the notion of an *equilibrium* and establish some general properties of equilibrium arrival profiles in Section 3.1. In Section 3.2, we present our main results in Theorem 1. The theorem establishes the existence and uniqueness of a symmetric Nash equilibrium as well as two general properties of such an equilibrium. The proof of Theorem 1 is presented in Section 3.3. Subsequently in Section 3.4, we show that no asymmetric equilibrium exists if we impose certain regularity conditions on the arrival strategies of individuals or if $n = 2$.

3.1 General properties of equilibrium arrival profiles

To study the strategic arrivals of users in a queueing game $\mathcal{G}$, we adopt the standard Nash equilibrium concept and say that the arrival profile $\mathcal{F} = \{F_1, \ldots, F_n\}$ constitutes an equilibrium, if it holds that no individual user can obtain strictly higher expected utility by changing her arrival strategy unilaterally. Since the expected utility for player i of arriving at time t must be the same for every t in the support of F_i, it follows from the definition of U_i that we may alternatively characterize an equilbrium as follows:

Definition 1 *The arrival profile $\mathcal{F} = \{F_1, \ldots, F_n\}$ constitutes an equilibrium, if it holds for all $i \in N$ that $U_i(t, \mathcal{F}^{-i}) \geq U_i(s, \mathcal{F}^{-i})$ for every $t \in \mathcal{S}(F_i)$ and every $s \in \mathbb{R}$.*

Next, we present two general properties that apply to any equilibrium. We start by establishing a result that links the cumulative departure time distribution D and expected utility U.

Lemma 1 *Consider a queueing game $\mathcal{G}$, and let $\mathcal{F}$ and $\tilde{\mathcal{F}}$ be two distinct arrival profiles. If it holds for user i that $D_i(d \mid t, \mathcal{F}^{-i}) \geq D_i(d \mid t, \tilde{\mathcal{F}}^{-i})$ for all $d \in \mathbb{R}$, then $U_i(t, \mathcal{F}^{-i}) \geq U_i(t, \tilde{\mathcal{F}}^{-i})$ for any $t \in \mathbb{R}$. Furthermore, if strict inequality holds for some d, then $U_i(t, \mathcal{F}^{-i}) > U_i(t, \tilde{\mathcal{F}}^{-i})$.*

This lemma follows immediately once we note that the utility function V is monotonically decreasing in the departure time.

The next result addresses the continuity of the equilibrium strategies.

Lemma 2 *Consider a queueing game $\mathcal{G}$, and let $\mathcal{F}$ be an equilibrium arrival profile for $\mathcal{G}$. Then for every $F_i \in \mathcal{F}$, we have that $F_i(t) = \lim_{s\uparrow t} F_i(s)$ for all $t \in \mathbb{R}$.*

Proof. First note that for any $t \geq 0$, a user arriving immediately after time t will start service instantaneously, thereby preempting the service progress of a user already residing in the queue. Therefore, $D_i(d \mid t, \mathcal{F}^{-i}) \leq \lim_{s\downarrow t} D_i(d \mid s, \mathcal{F}^{-i})$ for all $t \geq 0, d > t$. Next, to prove the statement by contradiction, suppose for some $t \geq 0$ and some $F_j \in \mathcal{F}$, $j \neq i$, that F_j has a point of upwards discontinuity such that $F_j(t) > \lim_{s\uparrow t} F_j(s)$. Then, for user i, there will exist a $d > t$, such that the probability of departing the system before d will be strictly smaller, when i arrives at the jump, where there is a positive probability that j arrives at the same time, than when i arrives immediately after the jump where the risk of getting preemted by j is smaller. That is, the inequality $D_i(d \mid t, \mathcal{F}^{-i}) \leq \lim_{s\downarrow t} D_i(d \mid s, \mathcal{F}^{-i})$ will be strict for some d, implying that $U_i(t, \mathcal{F}^{-i}) < \lim_{s\downarrow t} U_i(s, \mathcal{F}^{-i})$ by Lemma 1. Therefore, if F_j has a point of upwards discontinuity at t, then no other players will arrive at or immediately before t in equilibrium. However, when there exists an $\epsilon > 0$ such that none of the other players arrive in the interval $[t-\epsilon, t]$, then $U_j(t-\epsilon, \mathcal{F}^{-j}) > U_j(t, \mathcal{F}^{-j})$, implying that j can increase her expected utility by arriving at $t-\epsilon$ instead of t. This, however, contradicts that $\mathcal{F}$ is an equilibrium profile and proves that no equilibrium strategy can have a point of upwards discontinuity. Therefore, $F_i(t) = \lim_{s\uparrow t} F_i(s)$ for all $t \in \mathbb{R}$ and all $F_i \in \mathcal{F}$. □

Since F_i is right-continuous by definition, it follows from Lemma 2 that any equilibrium strategy F_i is continuous at all $t \in \mathbb{R}$.[4]

The next result establishes that in equilibrium, the users arrive at the facility within some bounded interval of time.

Lemma 3 *Consider a queueing game $\mathcal{G}$, and let $\mathcal{F}$ be an equilibrium arrival profile for $\mathcal{G}$. Then $\mathcal{S}(F_i)$ is a compact set for all $F_i \in \mathcal{F}$.*

[4]By a similar argument, it can be showed that in the setting where early arrivals are allowed, any equilibrium strategy F_i must have $F_i(s) = 0$ for all $s \leq 0$, since a user can arrive immediately after opening time and start service instantaneously.

Proof. Since $t \geq 0$, the support $\mathcal{S}(F_i)$ of F_i is bounded from below at 0. Moreover, $\mathcal{S}(F_i)$ is also bounded from above. To see this, assume on the contrary that $\inf\{t|F_i(t) = 1\} = \infty$ for some $F_i \in \mathcal{F}$. Now, since $\lim_{t\to\infty} D_i(d|t, \mathcal{F}^{-i}) = 0$ for all $d < \infty$ and $\lim_{t\to\infty} V(t,t) = -\infty$, it follows that if $\mathcal{F}$ represents an equilibrium, then $U_i(t, \mathcal{F}^{-i}) = -\infty$ for all $t \in \mathcal{S}(F_i)$. This, however, leads to a contradiction. Since the user population is finite, the service time distribution S has finite moments, and the LCFS-PR discipline is work-conserving, there must exist a $d < \infty$ such that $D_i(d|0, \mathcal{F}^{-i}) = 1$. This in turn implies that $U_i(0, \mathcal{F}^{-i}) > -\infty$, a contradiction. The support $\mathcal{S}(F_i)$ is therefore bounded. Since $\mathcal{S}(F_i)$ is closed by definition, it follows immediately from the Heine–Borel theorem that $\mathcal{S}(F_i)$ is compact. □

3.2 Symmetric equilibrium

We now consider a situation in which $\mathcal{F}$ is a collection of strategies such that $F_i = F_j = F$ for all $F_i, F_j \in \mathcal{F}$. The main results are summarized in the theorem below in which existence and uniqueness of an equilibrium are established, and some general properties of the symmetric equilibrium strategy in a queueing game $\mathcal{G}$ are presented.

Theorem 1 *For any queueing game $\mathcal{G}$, there exists one and only one strategy F that constitutes a symmetric equilibrium. Moreover, the following properties hold for F:*

(i) $F(t)$ is continuous at all $t \in \mathbb{R}$ and has $F(0) = 0$.

(ii) The support $\mathcal{S}(F)$ of F is a closed interval $[0, b]$.

Intuitively speaking, Theorem 1 says that in equilibrium, the users will arrive according to a continuous and strictly increasing distribution function that extends over a finite interval of time starting at the opening time.

3.3 Proof of Theorem 1

This section is devoted to the proof of Theorem 1 which proceeds through several lemmas. We start by noting that Part (i) of Theorem 1 follows from Lemma 2. The next result addresses the monotonicity of an equilibrium strategy.

Lemma 4 *Consider a queueing game $\mathcal{G}$, and let F be an equilibrium strategy for $\mathcal{G}$. Then $\mathcal{S}(F)$ is a connected set.*

Proof. Since F is an equilibrium strategy, it follows from Lemma 3 that F has a bounded support $\mathcal{S}(F)$ with supremum $0 < b < \infty$, and it follows from Lemma 2 that F is everywhere continuous and has $F(s) = 0$.

Now, suppose that $\mathcal{S}(F)$ is not a connected set, implying that $\mathcal{S}(F)$ can be covered by the union of two disjoint nonempty open subsets. This implies that there exists an interval $0 \leq t_1 < t_2 \leq b$ such that $F(t_1) = F(t_2)$. However this leads to a contradiction of the equilibrium definition since $U(t_1, F) > U(t_2, F)$. To see this, note that any user who arrives at time t_1 will start service instantaneously according to the LCFS-PR service discipline. Since no other users arrive in the time interval $[t_1, t_2]$, and V is strictly decreasing in departure time, it follows that $U(t_1, F) > U(t_2, F)$. Hence, a strategy F with a support $\mathcal{S}(F)$ that is not a connected set cannot be an equilibrium strategy. $\square$

Part (*ii*) of Theorem 1 now follows immediately from Lemmas 3 and 4. We next address the existence of an equilibrium strategy for a queueing game $\mathcal{G}$.

Lemma 5 *For any queueing game $\mathcal{G}$, there exists a strategy F that constitutes a symmetric equilibrium.*

Proof. We constructively prove this claim by defining a family of cumulative distribution functions $\{X_b\}_{0<b<\infty}$, where $X_b(s) = 1$ for all $s \geq b$. We then show that there exists a member of the family $\{X_b\}$ such that the arrival profile $\mathcal{F} = \{X_b, \ldots, X_b\}$ is a symmetric equilibrium. We will abuse notation and let $U_i(t, X_b)$ denote the expected utility of arriving at time t, when everyone else arrives according to X_b, i.e., when $\mathcal{F}^{-i} = \{X_b, \ldots, X_b\}$. Specifically, we will show that there is a b such that X_b satisfy the following criteria: $U_i(t, X_b) \geq U_i(s, X_b)$ for all $s, t \in [0, b]$, and $U_i(b, X_b) \geq U_i(q, X_b)$ for all $q \geq b$.[5]

From the point of view of user i, we are going to think of b as the earliest point in time, where the $n-1$ other users have already arrived at the facility with certainty. Therefore, if the other $n-1$ users arrive according to the strategy X_b, then the remaining user i can arrive at time b and start service instantaneously without being preempted, thus obtaining an expected utility of $U_i(b, X_b)$. In fact, for any cdf X that has no jump at time b, and for which $X(b) = 1$, we have that $U_i(b, X) = V(b, b)$, which will prove to be a useful observation.

[5]Note that any X_b satisfying this criteria will also satisfy the criteria of the equilibrium definition (Definition 1).

We construct the cumulative distribution function X_b as the limit of a convergent and recursive sequence of cumulative distribution functions, $\{X_{b,h} \mid 0 < b < \infty\}_{h\in\mathbb{N}}$, indexed by the non-negative integer h, to be defined in what follows.

For a given $0 < b < \infty$ and $h \in \mathbb{N}$, let $X_{b,h} : [0, \infty) \to [0, 1]$ be a function where $X_{b,h}(s) = 1$ for all $s \geq b$. In order to define the recursive sequence $\{X_{b,h} \mid 0 < b < \infty\}_{h\in\mathbb{N}}$, we start by introducing some notation.

As before, $U_i(t, X)$ denotes the expected utility for user i of arriving at time t, when everyone else arrives according to X. Now, for $x \in [0, X(t)]$, we let $\tilde{U}_i(t, X, x)$ denote user i's expected utility of arriving at time t, when the arrival strategy of each of the other $n-1$ users follows X except that: their probability of having arrived before t is $X(t) - x$, their probability of arriving exactly at t is x, and when arriving at t they will be prioritized before i.[6] Since we consider only symmetric equilibria, we will for ease of exposition simply denote U_i by U in the remainder of this proof.

Next, we define the starting term $X_{b,0}$ such that we ensure that $U(t, X_{b,0}) \geq U(b, X_b)$ for all $t < b$. To achieve this, we will consider the expected utility $\tilde{U}(t, 1, x)$ of arriving at time t, when the $n-1$ other users have arrived before user i with probability $1 - x$, and they arrive at t and be prioritized before i with probability x. Note that $\tilde{U}(t, 1, x)$ is strictly decreasing in x. In particular, the higher the probability of other users arriving at t and being prioritized before i, the longer user i is expected to wait in line before service completion.

By varying the probability of the $n-1$ other users arriving at time t and being prioritized before i, we can determine the maximal jump, x, such that the expected utility, $\tilde{U}(t, 1, x)$, is as least as great as the expected utility, $U(b, X_b)$, of arriving at time b and being serviced immediately. We denote this maximal x by $x^t_{b,0}$ and defined it as[7]:

$$x^t_{b,0} = \max\left\{x \in [0,1] \mid \tilde{U}(t, 1, x) \geq U(b, X_b)\right\}. \tag{3}$$

Next, by defining the maximal size of this jump for each t, we can construct the function $X_{b,0}$ such that at every point in time $0 < t < b$, the cdf is given by $1 - x^t_{b,0}$. We

[6]Note that this corresponds to a situation in which user i arrives at t, and an expected share x of the $n-1$ other users arrive immediately after t, and all user are serviced on a LCFS-PR basis.

[7]Since the expected utility is strictly decreasing in d, the utility of arriving at $0 \leq t < b$ and being serviced immediately with certainty, is always greater than $U(b, X_b)$. Thus, since $x = 0$ corresponds to the situation where a given user arriving at t is serviced immediately upon arrival, we must have $x^t_{b,0} \geq 0$ for all $0 \leq t \leq b$.

define the starting cdf $X_{b,0}$ of the sequence of recursive functions $X_{b,0}, X_{b,1}, X_{b,2}, \ldots$ as follows:

$$X_{b,0}(t) = \begin{cases} 1 - x_{b,0}^{t} & \text{for } t \in [0, b) \\ 1 & \text{for } t \geq b. \end{cases} \tag{4}$$

Figure 1 graphically illustrates an example of $X_{b,0}$.

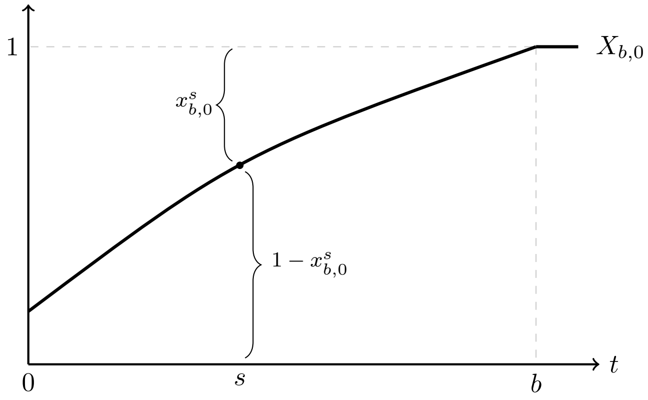

Fig. 1: *Example of $X_{b,0}$:* The constant b is an arbitrary fixed point in time for which $X_{b,0}(t) = 1$ for all $t \geq b$. $X_{b,0}$ is continuous and strictly increasing over the time interval $[0, b]$.

Now, since we know that $\tilde{U}(t, 1, x) \geq U(b, X_b)$, it must also hold that $U(t, X_{b,0}) \geq U(b, X_b)$. To see this, note that for a given user i, it must be that in the former situation, the probability of another user arriving immediately after i is $x_{b,0}^{t}$, whereas in the latter case when the others arrive according to $X_{b,0}$, the probability of another user arriving immediately after i is lower, and the arrival strategy will be spread over a larger interval of time. Therefore, the probability of departing the system at any given time is as least as great under $X_{b,0}$, and it follows from Lemma 1 that $U(t, X_{b,0}) \geq \tilde{U}(t, 1, x)$. By construction, it therefore follows that $U(t, X_{b,0}) \geq U(b, X_b)$.

Note that if we let $A_0(b)$ denote the latest point in time $t > 0$ such that $X_{b,0}(t) = 0$, if such a t exists, and otherwise let $A_0(b) = 0$, then since the utility function V is strictly decreasing in both departure and waiting time, it follows by construction that $X_{b,0}$ is strictly increasing over the time interval $[A_0(b), b]$ and non-decreasing over the interval $[0, b]$.

Next, we move on to characterize the recursive statement of $X_{b,h}$ for each $h > 0$ in a similar fashion. First, suppose that $X_{b,h-1}$ has been defined for $h > 0$. Consider a point in time t, $0 \leq t < b$, and let $\tilde{U}(t, X_{b,h-1}, x)$ be the expected utility of a user that arrives at t, when the probability of each of the $n-1$ other users having already arrived is $X_{b,h-1}(t) - x$, and the probability of each of the other users arriving exactly at time t and being prioritized for service over i is x.

We denote by $x^t_{b,h}$ (where $x^t_{b,h} \leq X_{b,h-1}(t)$) the maximal probability of each of the $n-1$ other users arriving at time t (the maximal jump at t), when the expected utility $\tilde{U}(t, X_{b,h-1}, x)$ for user i of arriving at t must be at least as high as the expected utility from arriving at time b and being serviced immediately. The maximal jump is thus defined as:

$$x^t_{b,h} = \max\left\{x \in [0, X_{b,h-1}(t)] \mid \tilde{U}(t, X_{b,h-1}, x) \geq U(b, X_b)\right\} \tag{5}$$

We define $X_{b,h}(t)$ as follows:

$$X_{b,h}(t) = \begin{cases} X_{b,h-1}(t) - x^t_{b,h} & \text{for } t \in [0, b) \\ 1 & \text{for } t \geq b. \end{cases} \tag{6}$$

As in the previous section, this construction ensures that $U(t, X_{b,h}) \geq \tilde{U}(t, X_{b,h-1}, x) \geq U(b, X_b)$. Note also that if we let $A_h(b)$ denote the latest point in time $t > 0$ where $X_{b,h}(t) = 0$, if such a t exists and otherwise, let $A_h(b) = 0$, then $X_{b,h}$ is, by construction, strictly increasing over the time interval $[A_h(b), b]$.

The recursive process yields the sequence $X_{b,0}, X_{b,1}, X_{b,2}, \ldots$ which is bounded and monotonically decreasing with $X_{b,0}(t) \geq X_{b,1}(t) \geq \ldots$ over $h \in \mathbb{N}$ and for all $t \in \mathbb{R}$. It thus follows by the monotone convergence theorem that the sequence is convergent. Let $X_b(t) = \lim_{h\to\infty} X_{b,h}(t)$ denote the limit of the sequence at each t, and let $A(b) = \lim_{h\to\infty} A_h(b)$. Figure 2 graphically illustrates an example of a recursive sequence $X_{b,0}, X_{b,1}, \ldots$ that converges towards the limit X_b.

So far b has been fixed. We now define a family of functions $\{X_b\}_{0<b<\infty}$ such that for each b, X_b is the limit of the convergent and recursive sequence $\{X_{b,h} \mid 0 < b < \infty\}_{h\in\mathbb{N}}$. For each member of $\{X_b\}$, we examine whether it represents an equilibrium strategy. First, we note that X_b is by construction a cumulative distribution function for any $0 < b < \infty$. Second, note that since X_b must satisfy the criteria that $U_i(t, X_b) \geq U_i(s, X_b)$ for all $s, t \in [0, b]$, we must have $U(0, X_b) = U(A(b), X_b)$. However, this only holds for values of b such that $X_b(0) = 0$ and $A(b) = 0$.[8] We make the following observations:

[8]The former follows from Lemma 2. To see the latter, note that if $A(b) > 0$, then $X_b(t) = 0$ for $t \in [0, A(b)]$. Then, since no other users arrive in the interval from 0 to $A(b)$, a given user can arrive at $t = 0$ and be serviced immediately without risk of being preempted before time $A(b)$. Therefore, $U(0, X_b) > U(A(b), X_b)$.

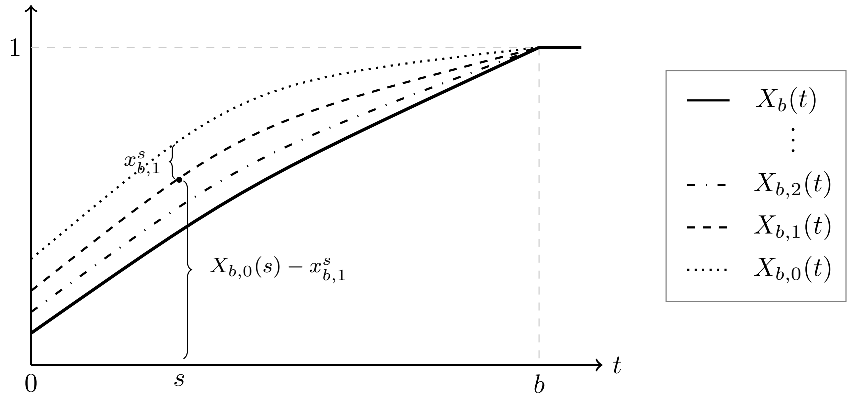

Fig. 2: *Example of a recursive sequence* $X_{b,0}, X_{b,1}, \ldots$*:* As the number of iterations h increases, the hth recursively stated term $X_{b,h}$ converges towards the limit X_b. Note that X_b is continuous and strictly increasing over $[A(b), b]$, where $A(b) = 0$ in this particular case.

(i) For b sufficiently close to 0, $X_b(0) > 0$, implying $A(b) = 0$.

(ii) For b sufficiently close to ∞, $A(b) > 0$, and $X_b(t) = 0$ for $t \in [0, A(b)]$.

(iii) $X_b(0)$ and $A(b)$ are continuous at all b and are monotonically decreasing and monotonically increasing, respectively.

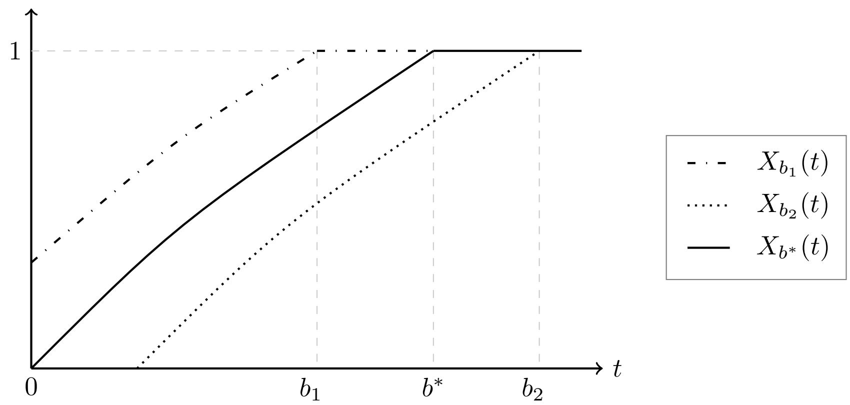

Fig. 3: *Example of* $\{X_b\}_{0<b<\infty}$: All members of $\{X_b\}$ for which $b \neq b^*$ yields $X_b(0) > 0$ or $A(b) > 0$. Only the member of $\{X_b\}$ for which $b = b^*$ represents an equilibrium strategy as $X_b^*(0) = 0$ and $A(b^*) = 0$.

Combining (i), (ii) and (iii), there must exist $b = b^*$ such that $X_{b^*}(0) = 0$ with $A(b^*) = 0$. It therefore immediately follows that X_{b^*} represents an equilibrium strategy for a symmetric equilibrium. Figure 3 graphically illustrates an example of such X_{b^*}

□

We next address the uniqueness of an equilibrium strategy.

Lemma 6 *For any queueing game $\mathcal{G}$, there exists at most one strategy F that constitutes a symmetric equilibrium.*

Proof. We prove this by contradiction. Let F and $\tilde{F}$ be two distinct symmetric equilibrium strategies such that $F \neq \tilde{F}$. Let $b = \min\{t \mid F(t) = 1\}$ and $\tilde{b} = \min\{t \mid \tilde{F}(t) = 1\}$. It then follows from Theorem 1 that both are strictly increasing with supports $[0, b]$ and $[0, \tilde{b}]$, respectively. We distinguish between three cases:

$b < \tilde{b}$: It immediately follows that $U(t, F) > U(t, \tilde{F})$ for all $t \in [0, b]$. Let $s = \max\{t \mid F(t) = \tilde{F}(t), 0 \leq t < b\}$ be the latest point in time at which the two strategies intersect. Note that s exists and is uniquely determined since F and $\tilde{F}$ are continuous, $F(0) = \tilde{F}(0)$, and $b < \tilde{b}$. It then follows that the expected share of users arriving from time s up until time b is strictly larger under F than under $\tilde{F}$. Therefore, $D(d \mid s, F) \leq D(d \mid s, \tilde{F})$ for all d with strict inequality at some d, implying that $U(s, F) < U(s, \tilde{F})$ by Lemma 1. This contradicts the assumption that F provides higher expected utility than $\tilde{F}$ and proves that F and $\tilde{F}$ cannot both be equilibrium strategies.

$b > \tilde{b}$: The case is symmetric to that of $b < \tilde{b}$ and thus omitted.

$b = \tilde{b}$: In this case, it immediately follows that $U(t, F) = U(t, \tilde{F}) = U(b, F)$ for all $t \in [0, b]$. Let F be the symmetric equilibrium strategy constructed from the procedure in Lemma 5 and recall that $F(0) = \tilde{F}(0) = 0$. Then by construction, F first-order stochastically dominates $\tilde{F}$ in the sense that $F(t) \geq \tilde{F}(t)$ for all $t \in [0, b]$ with strict inequality at some t. To see this recall that $F(t) = \lim_{h\to\infty} X_{b,h}(t)$ for the recursive sequence $X_{b,0}, X_{b,1}, \ldots X_{b,h}$ with $X_{b,0}(t) \geq X_{b,1}(t) \geq, X_{b,2}(t), \ldots$. From the definition of $X_{b,0}$, it immediately follows that $\tilde{F}(t) \leq X_{b,0}(t)$ for every $t \in [0, b]$, since otherwise, $U(t, \tilde{F}) > U(b, \tilde{F})$. Next, to show that $\tilde{F}(t) \leq X_{b,1}(t)$, assume on the contrary that $X_{b,0}(s) > \tilde{F}(s) > X_{b,1}(s)$ for some $s \in]0, b[$. Let $\tilde{U}(s, X_{b,0}, X_{b,0} - X_{b,1})$ denote the expected utility of arriving at s, when the probability of another user having already arrived is $X_{b,1}(s)$, and the probability for each of the other $n - 1$ users of arriving immediately after s is $X_{b,0}(s) - X_{b,1}(s)$. Then it follows from the procedure that $\tilde{U}(s, X_{b,0}, X_{b,0} - X_{b,1})) \geq U(b, F) = U(b, \tilde{F})$. However, since $\tilde{F}(s) > X_{b,1}(s)$, and $X_{b,0}(t) > \tilde{F}(t)$ for all $t \in [0, b]$, it follows that under $\tilde{F}$, fewer users will arrive after s, and they willl arrive at a slower rate than under F.

Therefore, $U(s, \tilde{F}) > \tilde{U}(s, X_{b,0}, X_{b,0} - X_{b,1}) \geq U(b, \tilde{F})$, which contradicts that $\tilde{F}$ is an equilibrium strategy. Thus, $\tilde{F}(t) \leq X_{b,1}(t)$ for all $t \in [0, b]$. Recursively applying this argument for each element in the sequence, we arrive at the desired result. It now follows that $D(d \mid 0, F) \leq D(d \mid 0, \tilde{F})$ for all d with strict inequality at some d, implying $U(0, F) < U(0, \tilde{F})$. This contradicts that $U(t, F) = U(t, \tilde{F})$ for all $t \in [0, b]$.

To conclude, there cannot exist two distinct strategies F and $\tilde{F}$ with $F \neq \tilde{F}$ that both constitute a symmetric equilibrium. □

Lemmas 5 and 6 complete the proof of Theorem 1.

3.4 Asymmetric equilibria

Let $\mathcal{F}$ be an equilibrium arrival profile. Then we say that $\mathcal{F}$ is an *asymmetric* equilibrium, if there exists $F_i, F_j \in \mathcal{F}$ such that $F_i \neq F_j$. In this section, we show that no asymmetric equilibrium exists, if $n = 2$, or if all arrival strategies have a finite number of inflection points.

Theorem 2 *Let $\mathcal{G} = \langle n, V, S \rangle$ be a queueing game. If $n = 2$, then no asymmetric equilibrium exist.*

Proof. To prove the theorem by contradiction, assume that $\mathcal{F}$ is an asymmetric equilibrium for some queueing game $\mathcal{G}$ with $N = \{1, 2\}$. It follows from the equilibrium definition that $U_i(s, \mathcal{F}^{-i}) = U_i(t, \mathcal{F}^{-i})$ for all $s, t \in \mathcal{S}(F_i)$ and $i \in \{1, 2\}$. Next, we show that $\mathcal{S}(F_1) = \mathcal{S}(F_2)$. To arrive at a contradiction, let $b = \max\{b_1, b_2\}$ and assume that there exists an $s \in]0, b[$ and an $\epsilon > 0$ such that a) $[s, s + \epsilon] \subset \mathcal{S}(F_1)$, $[s, s + \epsilon] \subset [0, b] \setminus \mathcal{S}(F_2)$, or b) $[s, s + \epsilon] \subset [0, b] \setminus \mathcal{S}(F_1)$, $[s, s + \epsilon] \subset \mathcal{S}(F_2)$.

Consider case a). Since there is no risk of user 2 arriving in the time interval $[s, s + \epsilon]$ it follows that $D_1(d \mid s, F_2) \geq D_1(d \mid s + \epsilon, F_2)$ for all d with strict inequality at some d, which then (due to Lemma 1) implies that $U_1(s, F_2) > U_1(s + \epsilon, F_2)$. This contradicts that F_1 is an equilibrium strategy. A symmetric argument applies to case b). Therefore, we must have $b_1 = b_2 = b$ and $\mathcal{S}(F_1) = \mathcal{S}(F_2) = [0, b]$, which in turn implies $U_1(t, F_2) = U_2(t, F_1) = U_1(b, F_2)$, for all $t \in [0, b]$.

It remains to be shown that $F_1 = F_2$. Since $U_1(t, F_2) = U_1(b, F_2)$, for all $t \in [0, b]$, the arrival strategy F_2 is also a best response to F_2, implying that $\mathcal{F} = \{F_2, F_2\}$ must be a symmetric equilibrium strategy. However, applying the symmetric argument for F_1

implies that $\mathcal{F}' = \{F_1, F_1\}$ is also a symmetric equilibrium, thereby contradicting Theorem 1. $\square$

Next, if we restrict the possible set of strategies to include only functions with a finite number of inflections points, then we ensure that no asymmetric equilibria exist.

Theorem 3 *Let $\mathcal{G} = \langle n, V, S \rangle$ be a queueing game, and restrict the set of admissible strategies to those with a finite number of inflection points. Then no asymmetric equilibrium exists.*

Proof. To prove the theorem by contradiction, assume that $\mathcal{F}$ is an asymmetric equilibrium for some queueing game $\mathcal{G}$, and assume that every $F_i \in \mathcal{F}$ has a finite number of inflection points. As before, it follows from the equilibrium definition that $U_i(s, \mathcal{F}^{-i}) = U_i(t, \mathcal{F}^{-i})$ for all $s, t \in \mathcal{S}(F_i)$ and all $i \in N$. Furthermore, the expected utility in equilibrium must be the same for all users. To see this, let $b_i = \min\{t | F_i(t) = 1\}$ for all $i \in N$ and assume on the contrary that there exists a pair of users $i, j \in N$, such that $U_j(b_j, \mathcal{F}^{-j}) > U_i(b_i, \mathcal{F}^{-i})$. Then there exists an $\epsilon > 0$ sufficiently small such that $U_i(b_j + \epsilon, \mathcal{F}^{-i}) > U_i(b_i, \mathcal{F}^{-i})$, contradicting equilibrium.

From Lemma 3, we know that there exists an earliest point in time $b = \max_i b_i$, such that all n users have arrived with certainty. Let $N_b \subseteq N$ be the set of users i for whom $b_i = b$. Furthermore, let $s < b$ be the earliest point in time such that all users in $N \setminus N_b$ have arrived with certainty. Next, consider two users i, j such that $i \in N_b, j \in N \setminus N_b$. Then $b_i = b$ and $b_j = s$. Now there exists $0 < \epsilon < b - s$ such that $U_i(s + \epsilon, \mathcal{F}^{-i}) > U_j(s, \mathcal{F}^{-j}) = U_i(b, \mathcal{F}^{-i})$, where the inequality follows since i by arriving immediately after s preempts any user currently in service and furthermore faces a lower risk of being preempted herself, since one user less can potentially arrive in the time interval from s to b, thereby contradicting equilibrium. Thus, for $\mathcal{F}$ to be an equilibrium arrival profile, the upper bound of $\mathcal{S}(F_i)$ must be the same for all $i \in N$. That is, there exists a b such that $b_i = b$ for all $i \in N$.

It remains to be shown that $F_i = F_j$ for all pair of users $i, j \in N$. To prove this by contradiction, consider $i, j \in N$ with $F_i \neq F_j$, and let $\underline{t}$ be the latest point in time such that the two strategies cross and such that there exists an $s > \underline{t}$ where the strategies differ. Furthermore, let $\bar{t} > \underline{t}$ denote the earliest point in time where the two strategies cross after $\underline{t}$. Since $F_i(0) = F_j(0) = 0$, $F_i(b) = F_j(b) = 1$, $F_i \neq F_j$, and each arrival strategy has only finitely many inflection points, we know that $\underline{t}$ and $\bar{t}$ exist. Then, one of the following cases must hold:

(i) $F_i(t) \geq F_j(t)$ for all $t \in [\underline{t}, b]$, with strict inequality for $t \in]\underline{t}, \bar{t}[$

(ii) $F_i(t) \leq F_j(t)$ for all $t \in [\underline{t}, b]$, with strict inequality for $t \in]\underline{t}, \bar{t}[$

For case (i), we distinguish between two subcases depending on whether there exists an s, $\underline{t} < s < \bar{t}$ such that $s \in \mathcal{S}(F_i) \cap \mathcal{S}(F_j)$:

(a) There exists s, $\underline{t} < s < \bar{t}$ such that $s \in \mathcal{S}(F_i) \cap \mathcal{S}(F_j)$. Since $F_i(b) - F_i(t) \leq F_j(b) - F_j(t)$ for all $t \in [s, b]$ with strict inequality at some t, it follows that $D_i(d \mid s, \mathcal{F}^{-i}) \leq D_j(d \mid s, \mathcal{F}^{-j})$ for all $d > s$, with strict inequality at some d. Hence $U_i(s, \mathcal{F}^{-i}) < U_j(s, \mathcal{F}^{-j})$, and since $s \in \mathcal{S}(F_i) \cap \mathcal{S}(F_j)$, this contradicts that $U_i(t, \mathcal{F}^{-i}) = U_j(t, \mathcal{F}^{-j})$ for all $t \in \mathcal{S}(F_i) \cap \mathcal{S}(F_j)$ and all i, j.

(b) $s \notin \mathcal{S}(F_i)$ or $s \notin \mathcal{S}(F_j)$ for all s, $\underline{t} < s < \bar{t}$. Then there must exist an $\epsilon > 0$ such that $t \notin \mathcal{S}(F_i)$ for all $t \in]\bar{t} - \epsilon, \bar{t}[$ whereas $\bar{t} - \epsilon \in \mathcal{S}(F_i)$. Furthermore, there exists $\epsilon' < \epsilon$ such that $[\bar{t} - \epsilon', \bar{t}[\subset \mathcal{S}(F_j)$. Together, this implies that there exists $\delta > 0$ (sufficiently small) such that $U_j(\bar{t} - \epsilon + \delta, \mathcal{F}^{-j}) > U_i(\bar{t} - \epsilon, \mathcal{F}^{-i})$. To see this note first that by arriving at $\bar{t} - \epsilon + \delta$, user j will preempt any service progress of user i. Second, since $\mathcal{F}^{-i} = (\mathcal{F}^{-\{i,j\}}, F_j)$ and $\mathcal{F}^{-j} = (\mathcal{F}^{-\{i,j\}}, F_i)$, users i and j will face the same arrival patterns of users in $N \setminus \{i, j\}$, but whereas j does not risk user i arriving in the interval $]\bar{t} - \epsilon, \bar{t}[$, user i faces a positive risk of j arriving before $\bar{t}$. Therefore, $U_j(\bar{t} - \epsilon + \delta, \mathcal{F}^{-j}) > U_i(\bar{t} - \epsilon, \mathcal{F}^{-i}) = U_j(\bar{t} - \epsilon', \mathcal{F}^{-j})$, where the equality holds since $\bar{t} - \epsilon \in \mathcal{S}(F_i)$ and $\bar{t} - \epsilon' \in \mathcal{S}(F_j)$. This, however, contradicts that F_j is an equilibrium strategy.

Symmetric arguments holds for case (ii), thereby proving that no asymmetric equilibrium exists. □

4 Computational results

This section develops a method to numerically compute a symmetric equilibrium strategy for a queueing game $\mathcal{G}$, so as to provide an example (Section 4.1). We subsequently compute the social efficiency of the computed equilibrium example and compare it to the social efficiency obtained in a companion paper by Breinbjerg (2017), where users are served on a FCFS basis and cannot arrive before opening time (Section 4.2). To obtain tractable numerical solutions for the LCFS-PR game, we restrict attention to games with two users and exponential service times.

We start by deriving a mathematical expression for D in the LCFS-PR queueing game with two users.

Lemma 7 *Consider a queueing game $\mathcal{G}$ in which $n = 2$, and S is independently, identically and exponentially distributed. Let F be a strategy with $b = \min\{t \mid F(t) = 1\} < \infty$, and let I_a denote the size of a jump discontinuity of F at point a. Then the cumulative departure time distribution D can be expressed as*

$$\begin{aligned} D(d \mid t, F) = &\sum_{a \leq t} I_a G(d-t;\mu) + \int_0^t f_+(a) G(d-t;\mu) \mathrm{d}a \\ &+ \sum_{t<a<b} I_a \left(G(a-t;\mu) G(d-t;\mu) + (1 - G(a-t;\mu)) H(d-a;2,\mu) \right) \\ &+ \int_t^b f_+(a) \left(G(a-t;\mu) G(d-t;\mu) + (1 - G(a-t;\mu)) H(d-a;2,\mu) \right) \mathrm{d}a \end{aligned}$$

for each $t \geq 0$, where G is the exponential cumulative distribution function, and H is the cumulative distribution function of the Erlang distribution, the latter two defined, respectively, as:

$$G(x;\mu) = \begin{cases} 1 - e^{-\mu x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$H(x;k,\mu) = \begin{cases} 1 - \sum_{m=0}^{k-1} \frac{1}{m!} e^{-\mu x} (\mu x)^m & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

for any $x \in \mathbb{R}$.

The proof is postponed to Appendix A.1 as it requires additional notation to describe the stochastic queueing processes. Specifically, we prove Lemma 7 by defining some basic queueing relations for our system and deriving the cumulative departure time process using sample-path techniques. Such a sample-path approach is commonly used in the queueing literature to describe the transient states of a queueing system. As an example, Juneja and Shimkin (2013) use a similar sample-path approach to derive relevant queueing relations for a corresponding queueing game where user are served on a FCFS basis.

4.1 Numerical procedure and an example

We now present the numerical method used to compute an example equilibrium strategy. The method is a discretized variant of the constructive proof of Lemma 5. Figure 4 depicts a flowchart of the general numerical procedure. For a given set of inputs, the method performs a search for the value b that induces a function X_b which constitutes an equilibrium strategy. Note that the number of required iterations for the search of b to converge is a function of the tolerance parameter ϵ. For any equilibrium strategy with $b \neq 1$, the search method (which combines a linear and binary search) requires multiple, and possibly many, iterations of b before convergence.

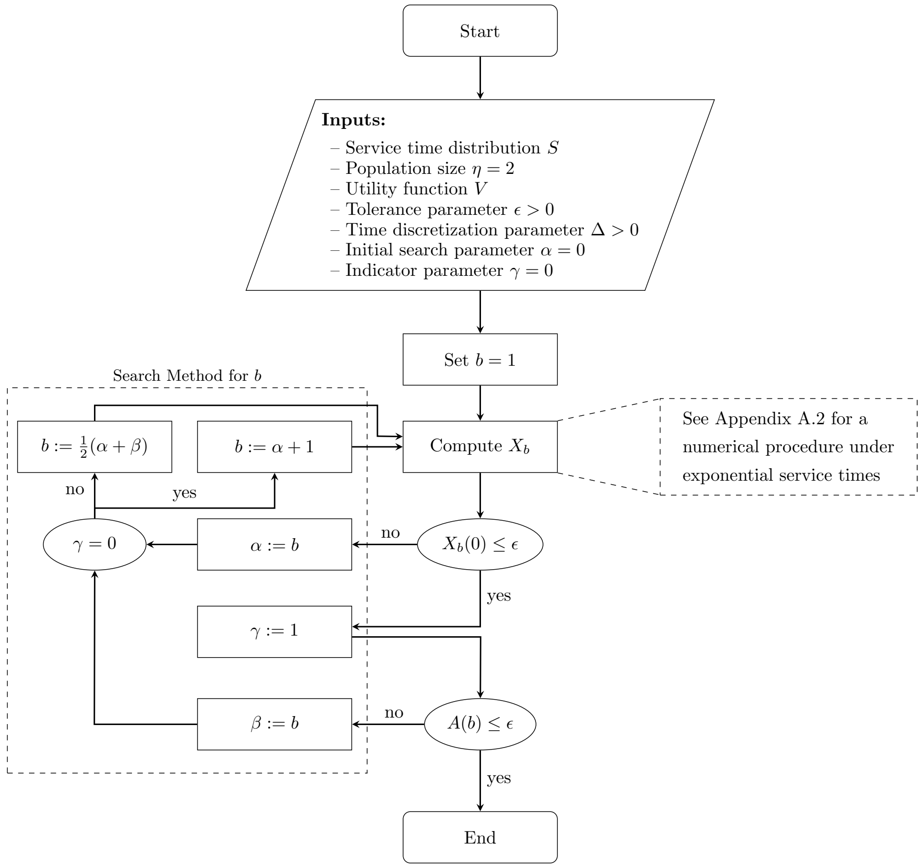

Fig. 4: *Flowchart of the numerical procedure:* Each geometric shape represents an action within the method. That is, the rounded squares are the start and ending, the trapezium is the exogenous inputs, the squares are steps in the process, and circles are binary decisions (yes/no) based on a question. The arrows indicate the flow from one action to another. Note that := is the assignment operator that changes the value of an existing variable.

Next, we apply the numerical procedure to compute an equilibrium strategy for a population of size two under exponential service times and for a specific utility function.

Figure 5 depicts the equilibrium strategy for the utility function $V(t,d) = -d^{0.5}(d-t)^{0.8}$. The figure illustrates a recursive sequence $\{X_{b,h}\}_{h\in\mathbb{N}}$ for $b = 3.8$ that represents an equilibrium strategy. Intuitively speaking, the symmetric equilibrium prescribes a strategy such that each user arrives according to a continuous and strictly increasing distribution function that extends over the interval from the opening time and up until time 3.8.

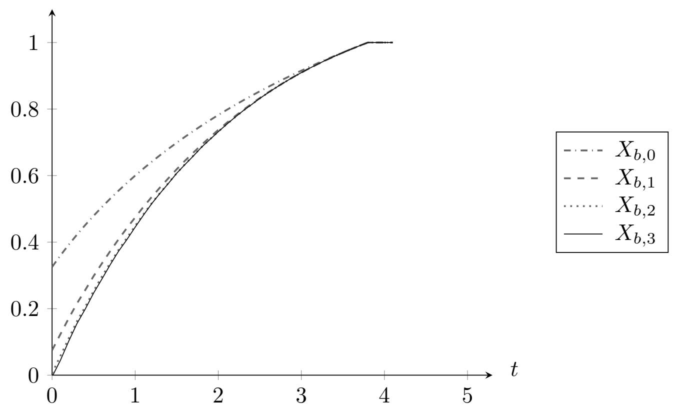

Fig. 5: *Numerically computed equilibrium strategy:* An approximated symmetric equilibrium in the queueing game $\mathcal{G}$, where: $n = 2$; $V(t,d) = -d^{0.5}(d-t)^{0.8}$; S is identical, independently and exponentially distributed with rate $\mu = 1$; $\Delta = 0.1$; and $\epsilon = 0.01$. The function $X_{b,3}$ represents an equilibrium strategy in the sense that it approximates the convergent limit of the recursive sequence $\{X_{b,h}\}_{h\in\mathbb{N}}$ with respect to the tolerance parameter ϵ.

4.2 Social inefficiency

We measure the social inefficiency of the equilibrium under LCFS-PR by comparing the aggregate expected utility in the Nash equilibrium to that of a socially optimal solution in which there is no waiting time or idle time. For the considered queueing game with only two users, the socially optimal solution is one in which one user starts service at time 0, and the other starts service immediately after the departure of the first user with no idleness at the server. Formally, let $\mathbf{W}$ denote the (random) sum of the two users' utilities in the socially optimal solution. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be the (random) independent, identically and exponentially distributed service time requirements of the users. The expected value of $\mathbf{W}$ conditional on $\mathbf{S}_1$ and $\mathbf{S}_2$ is then given by

$$\mathrm{E}[\mathbf{W} \mid \mathbf{S}_1, \mathbf{S}_2] = V(0, \mathbf{S}_1) + V(\mathbf{S}_1, \mathbf{S}_1 + \mathbf{S}_2),$$

where E is the expectation operator. Note that the sum of two independent and identically exponentially distributed variables follows an Erlang distribution with shape 2. Let $g(x;\mu)$

and $h(x;\mu)$ denote the density function at x for the exponential and Erlang distribution with rate μ and shape 2, respectively. Then the total expected utility for the socially optimal solution is given by

$$\mathrm{E}[\mathbf{W}] = \int_0^\infty \int_0^\infty (V(0,s) + V(s,z))\, g(s;\mu)h(z;\mu)\mathrm{d}s\mathrm{d}z. \tag{7}$$

Let U^* denote the expected utility for any of the users as induced by the equilibrium strategy. Then, we may consider the ratio between the total expected utility in equilibrium and in the socially optimal solution as a measure of the social inefficiency of the equilibrium solution:

$$\frac{2U^*}{\mathrm{E}[\mathbf{W}]}. \tag{8}$$

Table 1 reports the approximated value of this ratio in the specific queueing game considered in Figure 5, for which $n = 2$, the utility function is given by $V(t,d) = -d^{0.5}(d-t)^{0.8}$, and the service time requirement S is exponentially distributed with rate 1. The table also reports the approximated ratio obtained in a companion paper by Breinbjerg (2017), where users facing a closely related queueing game, cannot arrive before opening time, and are served on a FCFS basis. They find that the symmetric equilibrium prescribes a strategy such that both users arrive at opening time, i.e., $t = 0$, with certainty. When comparing the two approximated values, we find that the FCFS queueing discipline yields a lower efficiency cost compared to that of the LCFS-PR queueing discipline. This means that the arrival time incentives provided under the FCFS service discipline, when users cannot arrive before opening time, lead to higher social efficiency than the incentives induced by the LCFS-PR discipline. This result for the two-person case is in contrast to Platz and Østerdal (2017), who establish the equilibrium utility under FCFS and LCFS as a low and high social efficiency bound, respectively.

Table 1: Equilibrium utility and social inefficiency of LCFS-PR and FCFS

	U^*	$\mathrm{E}[\mathbf{W}]$	$\frac{2U^*}{\mathrm{E}[\mathbf{W}]}$
LCFS-PR	-2190	-2.129	2.057
FCFS	-1925	-2.129	1.808

Note: Approximated values for a queueing game with $n = 2$, $V(t,d) = -d^{0.5}(d-t)^{0.8}$, S exponentially distributed with rate $\mu = 1$, and $\Delta = 0.1$, $\epsilon = 0.01$ under the LCFS-PR and FCFS service discipline, respectively.

Since preemption in the current model increases mean sojourn time for a given arrival distribution, we might expect preemption to be a cause of the greater social inefficiency under LCFS-PR. The effect of preemption on efficiency will, however, also be influenced by the change in equilibrium behaviour that results from employing a different service discipline. Whereas a general analysis of the effects of preemption on social efficiency is beyond the scope of this paper, we note the following:

For any two-person queueing game, the equilibrium under the LCFS discipline without preemption is the same as under the FCFS discipline. Since preemption is not allowed for either discipline, they govern the two-person queue in the same way. Under either discipline, a user will face the following set of possible outcomes: she arrives first, starts service immediately and finishes service without interruptions; she arrives while the other user is in service and must wait her turn; she arrives after the other user has left the system and starts service right away.[9]

Therefore, when considering games with two players, equilibrium under LCFS (no preemption) can be represented by the equilibrium under FCFS, and it follows immediately from table 1 that social inefficiency is lower for LCFS without preemption than for LCFS-PR in our example. Furthermore, the fact that the equilibrium in the two-person case is always the same under service disciplines LCFS and FCFS is in line with the result from Platz and Østerdal (2017).

5 Conclusion

We have examined the strategic choices of a population of users that independently choose when to arrive at a queueing system that employs the LCFS-PR service discipline, when each user prefers earlier service and dislikes spending time in the queue. Our main contribution consists of establishing the existence and uniqueness of a symmetric mixed Nash equilibrium for general classes of preferences and service time distributions. Whereas the constructive procedure provided in the existence proof advices an approach to solve the problem that could in principle be applied for any n, the computations quickly become cumbersome. We provide a numerical method to compute the equilibrium in a two-user

[9]In fact, this will be the case for any work-conserving discipline without preemption, and we may think of FCFS as representing a larger family of disciplines (without preemption) that all induce the same equilibrium in a two player setting. Likewise, we can think of LCFS-PR as representing a larger family of disciplines (with preemption) that all govern the two-person queue in the same way.

setting and present an example.

The numerical example shows that the LCFS-PR service discipline may provide incentives for arrival profiles that lead to lower social efficiency compared to the incentives provided by the FCFS discipline. A likely explanation for this is the additional inefficiency caused by the property of preemption. As any newly arrived user may preempt the service progress of another user, the users must in equilibrium arrive according to a distribution function that extends over an 'excessively' large interval of time, in order to mitigate the expected disutility of being preempted after arrival. A further and more comprehensive study of the impact of preemption on social efficiency proposes an interesting avenue for future research. In particular, it remains an open question to determine the optimal service discipline in the current setting. Furthermore, the differences in social efficiency induced by the FCFS and LCFS-PR service disciplines could be studied in more detail. This could be done for various combinations of user population sizes, service times distributions (e.g. non-exponential distributions with decreasing hazard rate, heavy tailed, etc.) and utility functions. By establishing existence and uniqueness of a symmetric equilibrium under general preferences and service time requirements, we have provided an important starting point for the search for comprehensive methods to numerically solve the problem for varying populations sizes and specific user preferences.

Whereas we in the current paper restrict attention to queueing environments in which users have preferences over early service completion and waiting time, the basic approach of the constructive proof of existence carries over to variants of this assumption. For example, one might consider a generalization in the form of target time preferences, where users have a target time at which they wish to get to their destination, and are accordingly penalized for being too early or too late. The tardiness cost considered in the present paper is a special case of such preferences, where the target time equals the opening time of the facility. Alternatively, users may not (only) care about early service completion, but rather about the number of users who arrived ahead of them. This is the case for concerts or flights with unmarked seats, where there is no actual penalty for being served late unless other users have arrived and taken hold of the better seats. This disutility is modelled as an order penalty and was introduced by Ravner (2014). In both these cases, the basic approach of the constructive proof can be applied. Note, however, that in the order penalty setting, the inclusion of a tardiness cost alongside the order penalty is necessary to ensure a compact support of the arrival distribution.[10]

Finally, the LCFS-PR queueing game may be further extended in other important directions. One is the consideration of heterogeneous (multiclass) users. This has for example been considered in a FCFS queueing environment by Guo and Hassin (2012). A second direction is the consideration of queueing games with multiple servers. This has recently been considered by Haviv and Ravner (2015) who examine a multi-server system with no queue buffer, where users are interested in maximizing the probability of obtaining service.

A Appendix

A.1 Proof of Lemma 7

We start the proof by the following observation: A user's waiting time when queueing under the LCFS-PR service discipline is independent of the queue length she faces upon arrival, since the discipline allows the user to suspend the whole queue until after she completes her service. Thus, without loss of generality, we may say that the user arrives at an idle server. A user's waiting time in a LCFS-PR queue is then identical to the period of time between when the user arrives to an empty system and when she departs, leaving behind an empty queue. A user therefore only cares about the expected share of $n-1$ users that may arrive after her arrival and their respective service time requirements.

To capture such a situation, we start by introducing some notation. Let $\mathbf{A}$ denote the (random) arrival time of one of the two users, and let $\{\mathbf{S}_j\}_{j\in\{1,2\}}$ be a sequence of (random) service time requirements, such that $\mathbf{S}_j$ is the service time of the jth user to start service. For any user $i \in \{1,2\}$, let $\mathbf{R}_i$ denote the (random) residual service time of user i, if i is preempted prior to service completion. Moreover, let $\mathbf{D}_i(t)$ denote the (random) departure time of user i, when she arrives at time t, and the other user arrives at $\mathbf{A}$. The departure time of user i satisfies for each $t \geq 0$ the following sample path relation:

$$\mathbf{D}_i(t) = \begin{cases} \mathbf{S}_2 + t & \text{if } \mathbf{A} \leq t \\ \mathbf{1}_{\{\mathbf{S}_1+t\leq \mathbf{A}\}}(\mathbf{S}_1+t) + \mathbf{1}_{\{\mathbf{S}_1+t>\mathbf{A}\}}(\mathbf{A}+\mathbf{S}_2+\mathbf{R}_i) & \text{if } \mathbf{A} > t \end{cases}$$

[10] In the setting with an order penalty and a waiting cost but no tardiness cost, the expected utility of arriving at any point in time, where all other users have arrived with certainty, is equal to the expected utility of arriving to an empty server with no risk of being preempted while incurring the maximum order penalty.

Intuitively, the sample path above describes the possible outcomes, depending on whether user i is preempted or not prior to service completion. That is, in the event of $[\mathbf{A} \leq t]$, user i possibly preempts the user already residing in the queue and completes her service after $\mathbf{S}_2$ time units. In the event of $[\mathbf{A} > t]$, user i is the first to arrive at the system and is *possibly* preempted prior to service completion. That is, in the event that $[\mathbf{S}_1 + t \leq \mathbf{A}]$, user i departs the system at time $\mathbf{S}_1 + t$ before the other user arrives at $\mathbf{A}$. Otherwise, if $[\mathbf{S}_1 + t > \mathbf{A}]$, then user i is preempted and does not depart the system until the other user has completed service, and i has completed her residual service requirement, i.e. she departs at time $\mathbf{A} + \mathbf{S}_2 + \mathbf{R}_i$.

Fix a strategy F, and let $\mathbf{A} \sim F$. Moreover, let $\mathbf{S}_j \sim G$ for any j where G is the exponential cumulative distribution function, such that for any $x \in \mathbb{R}$

$$G(x;\mu) = \begin{cases} 1 - e^{-\mu x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

and $g(x;\mu)$ denotes the density of G at x. We characterize the probability of the event $[\mathbf{D}_i(t) \leq d]$ conditional on $\mathbf{A}$, $\mathbf{S}_1$ and $\mathbf{S}_2$, which equals zero for any $d < t$, and is otherwise given by

$$\begin{aligned} \Pr\{\mathbf{D}_i(t) \leq d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\} &= \mathrm{E}\left[\mathbf{1}_{\{\mathbf{D}_i(t)\leq d\}} \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\right] \\ &= \mathbf{1}_{\{\mathbf{A}\leq t\}} G(d-t;\mu) \\ &\quad + \mathbf{1}_{\{\mathbf{A}>t\}}\left[\mathbf{1}_{\{\mathbf{S}_1+t\leq \mathbf{A}\}} G(d-t;\mu) + \mathbf{1}_{\{\mathbf{S}_1+t>\mathbf{A}\}} H(d-\mathbf{A};2,\mu)\right] \end{aligned}$$

for any $0 \leq t \leq d$, where H denotes the cumulative distribution function of the Erlang distribution defined by

$$H(x;k,\mu) = \begin{cases} 1 - \sum_{m=0}^{k-1} \frac{1}{m!} e^{-\mu x} (\mu x)^m & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Note that the memoryless property of the exponential distribution implies that the distribution of the residual service times does not depend on how long a user has been in service prior to preemption, since the remaining time is still probabilistically the same as at her arrival time. That means that in case she is preempted, user i's departure time is the sum of two independent, identically and exponentially distributed variables (or equivalently, Erlang distributed with shape 2) with location at time $\mathbf{A}$. Consequently, the conditional probability $\Pr\{\mathbf{D}_i(t) \leq d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\}$ is independent of $\mathbf{S}_2$.

We next characterize D by marginalizing out the variables $\mathbf{A}$ and $\mathbf{S}_1$ such that

$$\begin{aligned}D(d \mid t, F) &= \Pr\{\mathbf{D}_i(t) \le d\} \\ &= \int_{\mathcal{S}(F)} \int_{\mathcal{S}(G)} \Pr\{\mathbf{D}_i(t) \le d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\} \mathrm{d}G(s-t;\mu)\mathrm{d}F(a) \\ &= \int_0^b \int_0^\infty \Pr\{\mathbf{D}_i(t) \le d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\} g(s-t;\mu)\mathrm{d}s\mathrm{d}F(a)\end{aligned}$$

for each $t \ge 0$ where all integrals are Lebesgue integrals, and the supremum of the support $\mathcal{S}(F)$ is given by b. Since F might have points of discontinuity, let I_a denote the jump size of F at the point in time a, so

$$I_a = \begin{cases} F(a) - \lim_{s\uparrow a} F(s) & \text{if } F(a) - \lim_{s\uparrow a} F(s) > 0 \\ 0 & \text{otherwise} \end{cases}$$

for any $a \in \mathbb{R}$. Then we may express D as follows

$$\begin{aligned}D(d \mid t, F) = &\sum_{a \le b} I_a \int_0^\infty \Pr\{\mathbf{D}_i(t) \le d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\} g(s-t;\mu)\mathrm{d}s \\ &+ \int_0^b \int_0^\infty \Pr\{\mathbf{D}_i(t) \le d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\} g(s-t;\mu) f_+(a)\mathrm{d}s\mathrm{d}a.\end{aligned}$$

We next insert the expression for $\Pr\{\mathbf{D}_i(t) \le d \mid \mathbf{S}_1, \mathbf{S}_2, \mathbf{A}\}$ and divide the expression in the two intervals of $(-\infty, t]$ and $(t, b]$, respectively:t

$$\begin{aligned}D(d \mid t, F) = &\sum_{a \le t} I_a G(d-t;\mu) + \int_0^t f_+(a) G(d-t;\mu)\mathrm{d}a \\ &+ \sum_{t<a\le b} I_a \left(\int_0^a g(s-t;\mu)G(d-t;\mu)\mathrm{d}s + \int_a^\infty g(s-t;\mu)H(d-a;2,\mu)\mathrm{d}s \right) \\ &+ \int_t^b f_+(a) \left(\int_0^a g(s-t;\mu)G(d-t;\mu)\mathrm{d}s + \int_a^\infty g(s-t;\mu)H(d-a;2,\mu)\mathrm{d}s \right) \mathrm{d}a.\end{aligned}$$

The claim of Lemma 7 now follows immediately once we note that $\int_0^a g(s-t;\mu)ds = G(a-t;\mu)$ and $\int_a^\infty g(s-t;\mu)ds = 1 - G(a-t;\mu)$. □

A.2 Numerical procedure

We here present a numerical method to compute X_b for a given b. Note that V, n, μ, ϵ, Δ and b are exogenous inputs to the procedure:

1. Let $\mathcal{T}_\Delta^b = \{t \in \{0, 1, 2, \dots\} : t\Delta < b\}$ be a discretization of the interval $[0, b)$ wrt. Δ.

2. Let $X_{b,h}(s) = 1$ for all $s \ge b$ and all $h \in \mathbb{N}$

3. Compute $U(b, X_{b,h}) = \int_b^\infty V(b,d)\mathrm{d}D(d \mid b, X_{b,h})$ according to the expression of D in Lemma 7 (note that $U(b, X_{b,h})$ is the same for all $h \in \mathbb{N}$).[11]

4. Let $h = 0$ and sequentially compute $X_{b,0}(t)$ for each $t \in \mathcal{T}_\Delta^b$ according to equation (4).

5. Assign $h := h + 1$ and sequentially compute $X_{b,h}(t)$ for each $t \in \mathcal{T}_\Delta^b$ according to equation (6).

6. If $X_{b,h}(t) - X_{b,h-1}(t) \leq \epsilon$ for all $t \in \mathcal{T}_\Delta^b$, then let $X_b = X_{b,h}$ and stop the procedure.

7. Else, go back to step (5) and begin the next iteration of h.

Remark 1 Although the considered queueing game allows users to arrive on the real line, we restrict ourselves only to compute the image of $X_{b,h}$ at non-negative points in time. This is done to save computing time, as we know the equilibrium strategy constitutes a distribution function with support on the non-negative real line.

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[11] We can further reduce the expression of D in the context of the constructive procedure in Lemma 5. That is, for a given user the procedure considers the other user's strategy to be such that only one jump occurs immediately after the point in time that she arrives. Thus, the D in this context can be expressed as

$$D(d \mid t, F) = F(t)G(d-t;\mu) + \lim_{s \downarrow t}(F(s) - F(t))H(d-t;2,\mu) + \int_t^b f_+(a)\left[G(a-t;\mu)G(d-t;\mu) + (1 - G(a-t;\mu))H(d-a;2,\mu)\right]\mathrm{d}a.$$

Since H and G are everywhere continuous and have $\lim_{a \downarrow t} G(a-t;\mu) = 0$ and $\lim_{a \downarrow t} H(a-t;2,\mu) = 0$, respectively, we also note that D is continuous and everywhere differentiable with respect to d. Let $D_i'(d \mid t, F) = \frac{\mathrm{d}}{\mathrm{d}d} D(d \mid t, F)$ be the derivative of D with respect to d, then $U(t, F) = \int_t^\infty V(t,d)D_i'(d \mid t, F)\mathrm{d}d$.

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