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Likelihood-based inference for a frailty-copula model based on competing risks failure time data

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Abstract

A competing risks phenomenon arises in industrial life tests, where multiple types of failure determine the working duration of a unit. To model dependence among marginal failure times, copula models and frailty models have been developed for competing risks failure time data. In this paper, we propose a frailty-copula model, which is a hybrid model including both a frailty term (for heterogeneity among units) and a copula function (for dependence between failure times). We focus on models that are useful to investigate the reliability of marginal failure times that are Weibull distributed. Furthermore, we develop likelihood-based inference methods based on competing risks data, including accelerated failure time models. We also develop a model-diagnostic procedure to assess the adequacy of the proposed model to a given dataset. Simulations are conducted to demonstrate the operational performance of the proposed methods, and a real dataset is analyzed for illustration. We make an R package “gammaGumbel” such that users can apply the suggested statistical methods to their data.

KEYWORDS: Competing Risk; Copula; Frailty; Reliability; Weibull distribution

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1. INTRODUCTION

Competing risks often arise in experimental/field life tests, where multiple types of failure determine the working duration of a unit. For instance, a maintenance center of mechanical units may classify the reasons of failure into two types: the confirmed failure and the apparent failure ^[1,2]. The interest lies in the estimation of the distribution of the confirmed failure time as the apparent failure might be due to the erroneous report by operators. The secondary interest is the estimation of the dependence between the confirmed failure and apparent failure, as the latter may be a signal of the former. In this example, the observed failure time is actually times to removal of units that were assumed to be failed. The failure times have *competing risks* relationship as the early occurrence of one event type prevents the observations of other event types. In competing risks data we therefore only observe the time of failure and the type of failure (i.e., the type of the first-occurring failure).

Most competing risks analyses of components' lifetime are based on parametric latent lifetime models ^[3], where a variety of parametric models have been employed. These include bivariate Weibull models ^[4,5], bivariate normal models ^[4,6,7], independent Lindley models ^[8], bivariate Pareto models ^[9,10], bivariate piecewise exponential models ^[11], and Marshall-Olkin Weibull models ^[12].

This paper considers copula-based multivariate failure time models, which permit for more flexible dependencies between the different types of failures than the standard models. The copula-graphic estimator has been proposed to analyze competing risks data under semiparametric latent lifetime models ^[13-17]. Semiparametric maximum likelihood estimators have been introduced for Cox-type regression analysis ^[18-20]. Parametric maximum likelihood estimators have been proposed for analyzing competing risks data for engineering applications ^[21-25] and medical applications ^[19,26]. The main strategy of copula models is to make inference on marginal (latent) failure time distributions by imposing a copula structure among different failure types. The competing risks model is identified under an assumed copula ^[13].

Another strategy for modelling competing risks is based on shared frailties of Liu ^[27] or mixed

models of Lo et al. ^[28]. The frailty model considers an unobserved frailty factor, which characterizes frail units via a large value (easy to fail) or robust units via a small value (difficult to fail). Liu proposed a gamma frailty model for reliability theory, where the conditional failure times follow a log-location-scale model. The mixed model of Lo et al. is more general than the frailty model, but the essential ideas of the two models are equivalent.

It should be emphasized that the copula and frailty models base on different approaches. The frailty (or mixed) model considers a conditional model (given a frailty term), which is different from the copula model that considers marginal (unconditional) models. For instance, a frailty model with the conditional piecewise exponential model of Lo et al. ^[28] does not have piecewise constant hazards in their marginal. Hence, such a model is not equivalent to a copula model with marginal piecewise exponential models ^[29], even though the two models can have the same copula. Consequently, one cannot directly compare the frailty model and the copula model since their parametrizations and interpretations are different (Section 3.3.4 of Duchateau and Janssen ^[30]).

A key in distinguishing between the frailty model and the copula model is the parameter interpretation. In the frailty model, the variance parameter has the interpretation of the degree of unobserved “heterogeneity” among units ^[30]. Hence, the variance parameter in the frailty distribution is not purely representative of “dependence” among units, as it also influences the marginal distributions. In general, the copula model is suitable for purpose of studying dependence among units while the frailty model is suitable for purpose of studying heterogeneity among units.

The objective of this paper is to introduce a new lifetime model for reliability analysis, which includes both a frailty term (for heterogeneity) and a copula function (for dependence). In the literature, such a hybrid model is rarely considered, since frailty and copula models were separately studied. In contrast, Emura et al. ^[19] considered the joint frailty-copula model for “clustered” survival data. Their model adopted frailty factors to characterize heterogeneity among clusters, and then applied a copula to characterize residual dependence of failure times within a cluster. Our proposed lifetime model is derived by integrating out the clustering effects in the model of Emura et

al. ^[19]. Toward our objective of analyzing competing risks, we deal with the identifiability problem, which was not present in the setting of Emura et al. ^[19].

This paper is organized as follows. Section 2 presents our proposed methods. Section 3 introduces our developed R package for implementing the proposed methods. Section 4 provides simulation studies to check the performance of the proposed methods. Section 5 includes an application of the proposed method to real data. Section 6 concludes with discussions.

2. PROPOSED METHODS

This section describes our proposed models for competing risks data analysis.

2.1 Competing risks data

Let T_{ij} be continuous failure time of a unit $i \in \{1, \dots, n\}$ with failure type $j = 1, 2$. Let C_i be a fixed length of testing duration (Type I censoring) or random censoring time. Under competing risks, one can observe the first occurring event time $T_i = \min(T_{i1}, T_{i2}, C_i)$ and one of the three event types (Event 1, Event 2, and Censoring). We define two event indicators $\delta_i = \mathbf{I}(T_i = T_{i1})$ and $\delta_i^* = \mathbf{I}(T_i = T_{i2})$, where $\mathbf{I}(\cdot)$ is the indicator function (taking 0 or 1). What we observe are the triplets $\{(T_i, \delta_i, \delta_i^*) : i = 1, \dots, n\}$.

We assume that the units under an experiment are heterogeneous in terms of their reliability. In particular, the unobserved heterogeneity induces a unit to be more fragile (more likely to fail) or more robust (less likely to fail). Fragile items may have short failure times for Events 1 and 2, while robust items may have long failure times. Hence, this heterogeneity yields positive correlation between T_{i1} and T_{i2} ^[30]. An example for heterogeneity is unobserved quality of the assembled components or the assembly process. Ignorance of frailty renders results of statistical analysis inconsistent.

2.2 Proposed model

To permit for heterogeneity in our statistical model, it is incorporated by means of an unobserved frailty term Z_i , which is a positive random variable. We specifically consider a gamma frailty model with probability density function

$$f_z(z) = \frac{1}{\Gamma(1/\eta)\eta^{1/\eta}} z^{1/\eta-1} \exp\left(-\frac{z}{\eta}\right), \quad \eta > 0, \quad z > 0,$$

with mean=1 and variance= η . Z_i affects the two failure types. A unit is said to be robust if it has a small value of Z_i , while it is fragile for larger values of Z_i . The variance η represents the degree of heterogeneity.

Before deriving our model, we first introduce a shared frailty model ^[30]. Define the conditional survival function of T_{ij} given Z_i by a Weibull distribution

$$\begin{aligned} S_{T_{ij}|Z_i}(t | Z_i = z) &= P(T_{ij} > t | Z_i = z) = P(T_{ij} > t | Z_i = 1)^z \\ &= \left(\exp \left[- \left\{ \frac{t}{\exp(\mu_j)} \right\}^{\frac{1}{\sigma_j}} \right] \right)^z = \exp \left[-z \left\{ \frac{t}{\exp(\mu_j)} \right\}^{\frac{1}{\sigma_j}} \right], \end{aligned}$$

where μ_j is a location parameter and σ_j is a scale parameter. The model implies that the unobserved heterogeneity value $Z_i = z$ acts on the scale without affecting the shape; see some recent discussions in Vallejos and Steel ^[31] about the heterogeneity modeling for the Weibull lifetime.

Liu ^[27] proposed to analyze competing risks data under the assumption that the pairs (T_{i1}, T_{i2}) are conditionally independent, i.e.

$$P(T_{i1} > t_1, T_{i2} > t_2 | z) = \exp \left[-z \left\{ \frac{t_1}{\exp(\mu_1)} \right\}^{\frac{1}{\sigma_1}} \right] \exp \left[-z \left\{ \frac{t_2}{\exp(\mu_2)} \right\}^{\frac{1}{\sigma_2}} \right]. \quad (1)$$

In the following, we extend this model for the analysis of competing risks data by relaxing the conditional independence assumption along the lines of Emura et al. ^[19]. In particular, we permit

competing failure times to be correlated conditional to Z_i . This is an important generalization as there should be no reason for conditional independence always to hold. An example is that confirmed and apparent failure times may well be positively correlated conditional on the item's quality. In this case, the latter could be used as a signal of the former. We suggest using a copula function to model such a dependency. In particular, we consider the Gumbel copula ^[32,33]

$$C_\theta(u, v) = \exp\left[-\left\{(-\log u)^{\theta+1} + (-\log v)^{\theta+1}\right\}^{\frac{1}{\theta+1}}\right],$$

where $\theta \geq 0$ is a dependence parameter.

By having specified the risk specific conditional survival and the dependence structure between risks, we propose a bivariate survival model that generalizes the model of Liu via an additional parameter θ :

$$\begin{aligned} S_{T_{i1}, T_{i2}|Z_i}(t_1, t_2 | z) &= P(T_{i1} > t_1, T_{i2} > t_2 | z) \\ &= \exp\left\{-\left(\left[-\log\{S_{T_{i1}|Z_i}(t_1 | z)\}\right]^{\theta+1} + \left[-\log\{S_{T_{i2}|Z_i}(t_2 | z)\}\right]^{\theta+1}\right)^{\frac{1}{\theta+1}}\right\} \\ &= \exp\left(-z \left[\left\{\frac{t_1}{\exp(\mu_1)}\right\}^{\frac{\theta+1}{\sigma_1}} + \left\{\frac{t_2}{\exp(\mu_2)}\right\}^{\frac{\theta+1}{\sigma_2}}\right]^{\frac{1}{\theta+1}}\right). \end{aligned} \quad (2)$$

Accordingly, the above model reduces to model (1) for $\theta = 0$. It reduces to the Gumbel copula model for bivariate competing risks for $z = 1$ ^[21].

Model (2) is not directly applicable in statistical analysis as it includes the unobserved factor $Z_i = z$. By integrating out z , we arrive at the bivariate survival function

$$\begin{aligned} S_{T_{i1}, T_{i2}}(t_1, t_2) &= P(T_{i1} > t_1, T_{i2} > t_2) = \int_0^\infty S_{T_{i1}, T_{i2}|Z_i}(t_1, t_2 | z) f_Z(z) dz \\ &= \left(1 + \eta \left[\left\{\frac{t_1}{\exp(\mu_1)}\right\}^{\frac{\theta+1}{\sigma_1}} + \left\{\frac{t_2}{\exp(\mu_2)}\right\}^{\frac{\theta+1}{\sigma_2}}\right]^{\frac{1}{\theta+1}}\right)^{-\frac{1}{\eta}}. \end{aligned} \quad (3)$$

Here, the assumptions of the Gumbel copula and gamma frailty are essential to obtain model (3)

since other copulas or other frailty distributions may not reach a closed form. This model can be shown to be equivalent to the model of Lu and Bhattacharyya ^[34] after some re-parameterizations, although, their derivation is remarkably different from ours. The marginal survival and the p -th quantile function respectively follow those of the Burr XII model ^[35],

$$S_{T_{ij}}(t) = \left[1 + \eta \left\{ \frac{t}{\exp(\mu_j)} \right\}^{\frac{1}{\sigma_j}} \right]^{-\frac{1}{\eta}}, \quad t_{p,j} = \exp(\mu_j) \left\{ \frac{(1-p)^{-\eta} - 1}{\eta} \right\}^{\sigma_j},$$

for $j = 1$ and 2 . Also, the mean and variance are

$$E(T_{ij}) = \frac{\exp(\mu_j) \Gamma(1/\eta - \sigma_j) \Gamma(\sigma_j + 1)}{\eta^{\sigma_j + 1} \Gamma(1/\eta + 1)},$$

$$Var(T_{ij}) = \frac{\exp(2\mu_j)}{\eta^{2\sigma_j + 2}} \left[\eta \frac{\Gamma(1/\eta - 2\sigma_j) \Gamma(2\sigma_j + 1)}{\Gamma(1/\eta + 1)} - \left\{ \frac{\Gamma(1/\eta - \sigma_j) \Gamma(\sigma_j + 1)}{\Gamma(1/\eta + 1)} \right\}^2 \right].$$

The Burr model is often used in reliability theory ^[36-38].

We adopt Kendall's tau of the proposed model to measure dependency between T_{i1} and T_{i2} . The reason is that Kendall's tau has a simple form and is not affected by the marginal distributions. Using the properties of Archimedean copulas ^[33], Kendall's tau for a pair (T_{i1}, T_{i2}) is

$$\tau_{\theta, \eta} = 1 - \frac{2}{(\theta + 1)(\eta + 2)}, \quad \theta \geq 0, \quad \eta > 0.$$

Under the model of Liu, we have $\tau_{0, \eta} = \eta / (\eta + 2)$. The conditional Kendall's tau is

$$\tau_{\theta}^Z = \tau_{\theta, \eta=0} = \frac{\theta}{\theta + 1} < \tau_{\theta, \eta}, \quad \theta \geq 0, \quad \eta > 0.$$

This is relevant for the conditional model $(T_{i1}, T_{i2} | Z_i)$ and is equal to Kendall's tau for the Gumbel copula.

The proposed model (2) can be extended to include covariates. In particular, we consider an accelerated life-test experiment, where the failure times of units are measured under several different stress levels. We consider an experimental region $[s_0, s_h]$, where s_0 is the normal use

condition and s_h is the highest level of the accelerating variable. Following Liu ^[27], we use the standardized stress level x defined as follows

$$x = (s - s_0) / (s_h - s_0), \quad s \in [s_0, s_h].$$

We redefine model (3) by setting $\mu_j \equiv \beta_{j0} + \beta_{j1}x$. This model gives the Weibull accelerated failure time (AFT) model, where the location parameter depends on a stress level x . The resulting bivariate survival function is

$$S_{T_{i1}, T_{i2}}(t_1, t_2 | x) = \left(1 + \eta \left[\left\{ \frac{t_1}{\exp(\beta_{10} + \beta_{21}x)} \right\}^{\frac{\theta+1}{\sigma_1}} + \left\{ \frac{t_2}{\exp(\beta_{20} + \beta_{21}x)} \right\}^{\frac{\theta+1}{\sigma_2}} \right]^{\frac{1}{\theta+1}} \right)^{-\frac{1}{\eta}}. \quad (4)$$

By setting $\theta = 0$, model (4) reduces again to the AFT model of Liu.

In reliability analysis of manufactured items, quantiles, such as those for $p = 0.01$ and 0.1 , are chosen to assess the quality of the items, which is related to lower tolerance limits ^[39]. Engineers often investigate the lower quantile under the normal condition ($x = 0$):

$$t_{p, j|x=0} = \exp(\beta_{j0}) \left\{ \frac{(1-p)^{-\eta} - 1}{\eta} \right\}^{\sigma_j}, \quad j = 1, 2.$$

2.3 Identifiability

Recall that competing risks data provide the observation of either T_{i1} or T_{i2} (but not for both) for each i . Since T_{i1} and T_{i2} are never observed simultaneously, the identification of their correlation is challenging ^[4,6,7,40]. This leads to some problem for the joint identifiability of θ and η . The simplest solution to the problem is to assume the value $\theta=0$ and estimate only η . We propose a more general identifying assumption by allowing η to be unknown and $\theta = \omega(\eta)$ to be restricted with a link function $\omega(\cdot)$. The following proposition justifies the identifiability:

Proposition 1: Under the link function $\theta = \omega(\eta)$, the parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2, \eta, \theta)$ in model (3) are identifiable from the observed $T_i = \min(T_{i1}, T_{i2})$ and $\delta_i = \mathbf{I}(T_i = T_{i1})$.

The proof of Proposition 1 is given in Appendix A.

Under the link $\theta = \omega(\eta)$, Kendall's tau is the function of η

$$\tau_\eta = 1 - \frac{2}{\{\omega(\eta) + 1\}(\eta + 2)}, \quad \eta \in \left[\eta \mid \{\eta > 0\} \cap \{\omega(\eta) \geq 0\} \right].$$

The link function includes the following examples:

- $\theta = \theta_0$ (assumed value): Kendall's tau takes values in $(\theta_0 / (\theta_0 + 1), 1)$, and is given by

$$\tau_\eta = 1 - \frac{2}{(\theta_0 + 1)(\eta + 2)}, \quad \eta > 0.$$

This includes the model of Liu ^[27] by setting $\theta_0 = 0$.

- $\theta = \eta$ (identity link): Kendall's tau takes values in $(0, 1)$, and is given by

$$\tau_\eta = 1 - \frac{2}{(\eta + 1)(\eta + 2)}, \quad \eta > 0.$$

- If $\theta = \eta / 2$ (linear link 1): Kendall's tau takes values in $(0, 1)$, and is given by.

$$\tau_\eta = 1 - \frac{4}{(\eta + 2)^2}, \quad \eta > 0.$$

- $\theta = \eta + 1$ (linear link 2): Kendall's tau takes values in $(0.5, 1)$, and is given by

$$\tau_\eta = 1 - \frac{2}{(\eta + 2)^2}, \quad \eta > 0.$$

- If $\theta = 1/\eta - 1$ (*inverse link*), Kendall's tau takes values in $(1/3, 1)$, and given by

$$\tau_\eta = \frac{2-\eta}{2+\eta}, \quad 0 < \eta \leq 1.$$

This is the same as Kendall's tau for the one parameter copula of the “BB1 family” after some re-parametrizations (see Appendix A). The corresponding copula function does not have a specific name, but it appears in Equation (4.2.14) in Table 4.1 of Nelsen ^[33].

Figure 1 shows how the link function helps restricting the parameter space for θ and η by making the two-dimensional space into a one-dimensional space. While this strong restriction seems unavoidable for the sake of identifiability, the link function $\theta = \omega(\eta)$ can still allow flexible shapes, including increasing, constant, or decreasing functions in η .

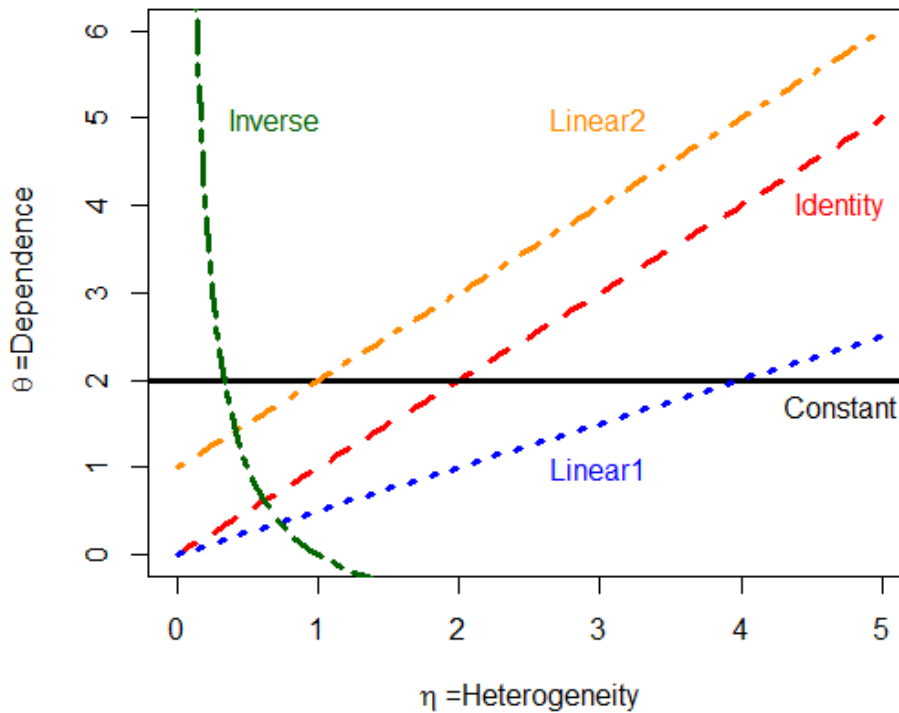


FIGURE 1: The link function $\theta = \omega(\eta)$ that restricts the parameter space of θ and η .

2.4 Likelihood-based inference

To estimate the parameters of model (3), we propose a likelihood-based method. The sub-density function $f(t, j)$ for the j -th failure type is

$$\begin{aligned} f(t, j) &= - \left. \frac{\partial S_{T_{i_1}, T_{i_2}}(t_1, t_2)}{\partial t_j} \right|_{t_1=t_2=t} \\ &= \left(1 + \eta \left[\left\{ \frac{t}{\exp(\mu_1)} \right\}^{\frac{\theta+1}{\sigma_1}} + \left\{ \frac{t}{\exp(\mu_2)} \right\}^{\frac{\theta+1}{\sigma_2}} \right]^{\frac{1}{\theta+1}} \right)^{-\frac{1+\eta}{\eta}} \\ &\quad \times \left[\left\{ \frac{t}{\exp(\mu_1)} \right\}^{\frac{\theta+1}{\sigma_1}} + \left\{ \frac{t}{\exp(\mu_2)} \right\}^{\frac{\theta+1}{\sigma_2}} \right]^{\frac{1}{\theta+1}-1} \frac{t^{\frac{\theta+1}{\sigma_j}-1}}{\sigma_j \exp \left\{ \mu_j \left(\frac{\theta+1}{\sigma_j} \right) \right\}}. \end{aligned}$$

Therefore, the likelihood function can be derived as

$$L(\mu_1, \mu_2, \sigma_1, \sigma_2, \eta, \theta) = \prod_{i=1}^n f(T_i, 1)^{\delta_i} f(T_i, 2)^{\delta_i^*} S_{T_{i_1}, T_{i_2}}(T_i, T_i)^{1-\delta_i-\delta_i^*}.$$

The log-likelihood function can be re-expressed as

$$\begin{aligned} \ell(\mu_1, \mu_2, \sigma_1, \sigma_2, \eta, \theta) &= - \left\{ m \log \sigma_1 + m^* \log \sigma_2 + (\theta+1) \frac{m\mu_1}{\sigma_1} + (\theta+1) \frac{m^*\mu_2}{\sigma_2} \right\} \\ &\quad + \sum_{i=1}^n \left\{ \delta_i \left(\frac{\theta+1}{\sigma_1} - 1 \right) + \delta_i^* \left(\frac{\theta+1}{\sigma_2} - 1 \right) \right\} \log T_i \\ &\quad - \frac{\theta}{\theta+1} \sum_{i=1}^n (\delta_i + \delta_i^*) \log \left[\sum_{k=1}^2 \left\{ \frac{T_i}{\exp(\mu_k)} \right\}^{\frac{\theta+1}{\sigma_k}} \right] \\ &\quad - \sum_{i=1}^n \left(\delta_i + \delta_i^* + \frac{1}{\eta} \right) \log \left(1 + \eta \left[\sum_{k=1}^2 \left\{ \frac{T_i}{\exp(\mu_k)} \right\}^{\frac{\theta+1}{\sigma_k}} \right]^{\frac{1}{\theta+1}} \right), \end{aligned}$$

where $m = \sum_{i=1}^n \delta_i$ and $m^* = \sum_{i=1}^n \delta_i^*$. Under some identifying restriction $\theta = \omega(\eta)$, we obtain the maximum likelihood estimator (MLE) by maximizing $\ell(\mu_1, \mu_2, \sigma_1, \sigma_2, \eta, \theta = \omega(\eta))$ over the ranges of parameters: $\mu_1 \in \mathbb{R}$, $\mu_2 \in \mathbb{R}$, $\sigma_1 \in (0, \infty)$, $\sigma_2 \in (0, \infty)$, $\theta \in [0, \infty)$ and $\eta \in (0, \infty)$.

Similarly, for the AFT model of Equation (4), the log-likelihood is given by

$$\begin{aligned}
& \ell(\beta_{10}, \beta_{20}, \beta_{11}, \beta_{21}, \sigma_1, \sigma_2, \eta, \theta) \\
&= - \left\{ m \log \sigma_1 + m^* \log \sigma_2 + (\theta + 1) \frac{(m\beta_{10} + m_x \beta_{11})}{\sigma_1} + (\theta + 1) \frac{(m^* \beta_{20} + m_x^* \beta_{21})}{\sigma_2} \right\} \\
&+ \sum_{i=1}^n \left\{ \delta_i \left(\frac{\theta + 1}{\sigma_1} - 1 \right) + \delta_i^* \left(\frac{\theta + 1}{\sigma_2} - 1 \right) \right\} \log T_i \\
&- \frac{\theta}{\theta + 1} \sum_{i=1}^n (\delta_i + \delta_i^*) \log \left[\sum_{k=1}^2 \left\{ \frac{T_i}{\exp(\beta_{k0} + \beta_{k1} X_i)} \right\}^{\frac{\theta + 1}{\sigma_k}} \right] \\
&- \sum_{i=1}^n \left(\delta_i + \delta_i^* + \frac{1}{\eta} \right) \log \left(1 + \eta \left[\sum_{k=1}^2 \left\{ \frac{T_i}{\exp(\beta_{k0} + \beta_{k1} X_i)} \right\}^{\frac{\theta + 1}{\sigma_k}} \right]^{\frac{1}{\theta + 1}} \right),
\end{aligned}$$

where $m = \sum_{i=1}^n \delta_i$, $m^* = \sum_{i=1}^n \delta_i^*$, $m_x = \sum_{i=1}^n \delta_i X_i$ and $m_x^* = \sum_{i=1}^n \delta_i^* X_i$. Under some identifying restriction $\theta = w(\eta)$, we obtain the MLE by numerically maximizing $\ell(\beta_{10}, \beta_{20}, \beta_{11}, \beta_{21}, \sigma_1, \sigma_2, \eta, \theta = w(\eta))$ over the ranges $\beta_{10} \in \mathbf{R}$, $\beta_{20} \in \mathbf{R}$, $\beta_{11} \in \mathbf{R}$, $\beta_{21} \in \mathbf{R}$, $\sigma_1 \in (0, \infty)$, $\sigma_2 \in (0, \infty)$, $\theta \in [0, \infty)$ and $\eta \in (0, \infty)$.

Standard errors (SEs) and confidence intervals (CIs) of all parameters are obtained from the Hessian matrix (the second derivatives of the log-likelihood) evaluated at the values of the MLE. The delta method can be applied to compute the SE and CI for a differentiable function of parameters, such as the marginal quantile. Supplementary Materials give the computational algorithms for maximizing the log-likelihood, and obtaining SEs and CIs. However, end-users might simply apply our R package (Section 3) without going through the technical details.

2.5 Model diagnostic

As the proposed methods rely on several distributional assumptions, we propose a model diagnostic method for model (3) for a given dataset. For competing risks data, the usual QQ plots for the marginal distributions give biased results since failure times are subject to dependent censoring by competing risks. To develop a proper diagnostic plot, we follow the existing ideas ^[20,22,26] of checking the adequacy of the “sub-distribution functions”, defined as $F(t, j) = \int_0^t f(x, j)dx$ for $j = 1$ and 2, where $f(t, j)$ is the sub-density function defined previously.

Our method consists of three steps: **(i)** parametric estimation of $F(t, j)$, **(ii)** nonparametric estimation of $F(t, j)$, **(iii)** comparison of the parametric and nonparametric estimators. If the two estimators are nearly equivalent, we conclude that model (3) is adequate.

(i) Parametric estimation of sub-distribution functions

Under model (3) and by the formula of $f(t, j)$, the parametric estimator of the sub-distribution function (for Event $j=1$ or 2) is derived as

$$F_{\hat{\Phi}}(t, j) = \int_0^t \left(1 + \hat{\eta} \left[\left\{ \frac{s}{\exp(\hat{\mu}_1)} \right\}^{\frac{\theta+1}{\hat{\sigma}_1}} + \left\{ \frac{s}{\exp(\hat{\mu}_2)} \right\}^{\frac{\theta+1}{\hat{\sigma}_2}} \right]^{\frac{1}{\theta+1}} \right)^{-\frac{1+\hat{\eta}}{\hat{\eta}}} \times \left[\left\{ \frac{s}{\exp(\hat{\mu}_1)} \right\}^{\frac{\theta+1}{\hat{\sigma}_1}} + \left\{ \frac{s}{\exp(\hat{\mu}_2)} \right\}^{\frac{\theta+1}{\hat{\sigma}_2}} \right]^{\frac{1}{\theta+1}-1} \frac{s^{\frac{\theta+1}{\hat{\sigma}_j}-1}}{\hat{\sigma}_j \exp \left\{ \hat{\mu}_j \left(\frac{\theta+1}{\hat{\sigma}_j} \right) \right\}} ds.$$

A numerical integration method is required since the integral does not have a closed-form. For instance, one can apply an R function *integrate(.)*.

(ii) Nonparametric estimation of sub-distribution functions

For nonparametric estimation, let $T_{(1)} < T_{(2)} < \dots < T_{(k)}$ be distinct uncensored times. Then, the

nonparametric estimators of sub-distribution functions ^[26] are

$$\widehat{F}(t, j) = \sum_{i: T_{(i)} \leq t} \widehat{S}(t) \frac{d_{ij}}{n_i}, \quad j = 1, 2,$$

where $n_i = \sum_{j=1}^n \mathbf{I}(T_j \geq T_{(i)})$, $d_i = \sum_{j=1}^n \mathbf{I}(T_j = T_{(i)})$, $d_{i1} = \sum_{j=1}^n \delta_i \mathbf{I}(T_j = T_{(i)})$,

$$d_{i2} = \sum_{j=1}^n \delta_i^* \mathbf{I}(T_j = T_{(i)}), \quad \text{and} \quad \widehat{S}(t) = \prod_{i: T_{(i)} \leq t} (1 - d_i / n_i).$$

(iii) The Cramér-von Mises type statistic

As a measure of fit, we define the statistic

$$CvM = \sum_{i=1}^n \delta_i \{ F_{\Phi}(T_i, 1) - \widehat{F}(T_i, 1) \}^2 + \sum_{i=1}^n \delta_i^* \{ F_{\Phi}(T_i, 2) - \widehat{F}(T_i, 2) \}^2 .$$

A large value of CvM means a lack of fit for model (3). In practice, a graphical diagnostic for the value of CvM is more helpful than looking at the value of CvM . We suggest plotting $F_{\Phi}(t, j)$ and $\widehat{F}(t, j)$ on the same graph for each j . The value of CvM may be used to select a link function. However, the link function chosen according to CvM is usually equivalent to the link function chosen by the likelihood criterion. Since the latter criterion is easier to compute, we suggest a likelihood-based method for selecting the link function with CvM as a reference value. We will further demonstrate this graphical model diagnostic through the data analysis (Section 5).

3. SOFTWARE

To facilitate the use of our proposed methods, we developed an R package “gammaGumbel”. The package consists of nine R functions. The PDF package manual (in Supplementary Materials) includes the examples for applying the package to both real data and simulated data. Below, we explain the structure of the nine R functions.

The first three functions *gammaGumbel.MLE(.)*, *gammaGumbel.MLE1(.)*, and *gammaGumbel.MLE2(.)*, can compute the MLE, SE, and 95% CI. The three functions correspond to the three identifying assumptions ($\theta = \text{known}$, $\theta = \eta$ and $\theta = \eta/2$). The next three functions: *gammaGumbel.CvM(.)*, *gammaGumbel.CvM1(.)*, and *gammaGumbel.CvM2(.)*, can perform the model diagnostic methods, again under the three identifying assumptions. The last three functions *gammaGumbel.reg(.)*, *gammaGumbel.reg1(.)*, and *gammaGumbel.reg2(.)* can compute the MLE, SE and 95% CI under the AFT model under the three identifying assumptions. Initial values for computing the MLE are chosen automatically except for the variance parameter of the frailty term that must be specified by users (default is $\eta^{(0)} = 1$).

All the subsequent numerical studies are performed using the R package “gammaGumbel”.

4. SIMULATIONS

We conducted extensive simulation studies to analyze the performance of the proposed methods under various parameter settings, censoring percentages, and sample sizes.

4.1 Simulations under the model (3)

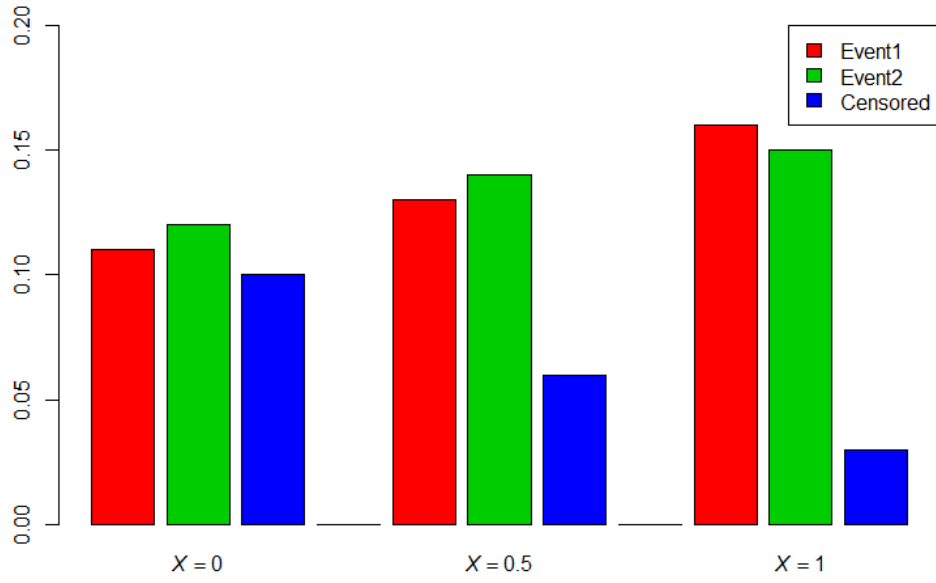
We generated failure times (T_{i1}, T_{i2}) 's with sample sizes $n=100, 200, 400, 800,$ and 1600 from model (3) under some parameters. We also generated independent censoring times C_i 's from a uniform distribution that gives 20% (light censoring) and 60% censoring (heavy censoring) percentage. We then formed triplets $(T_i, \delta_i, \delta_i^*)$'s, where $T_i = \min(T_{i1}, T_{i2}, C_i)$, $\delta_i = \mathbf{I}(T_i = T_{i1})$

and $\delta_i^* = \mathbf{I}(T_i = T_{i2})$. Using the triplets of size n , we computed the MLE by applying our R package “gammaGumbel” (Section 3). Here, the MLE is computed under an identifiable assumption ($\theta=6$, $\theta=\eta$ or $\theta=\eta/2$). We replicated 500 times and summarized the results. Supplementary Material includes the simulation results (in Tables S1-S6) and the discussion of the results. Overall, we found that all the unknown parameters in the model (3) were estimated accurately by the MLE, and the performance of the SE and 95%CI were satisfactory..

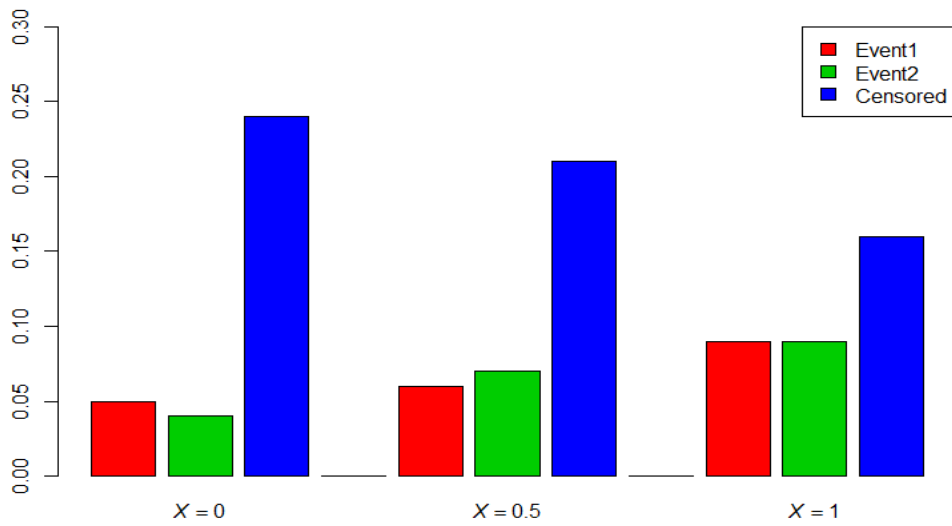
4.2 Simulations under the AFT model (4)

We randomly select X_i 's from one of three design levels $(0, 0.5, 1)$. We then generated (T_{i1}, T_{i2}) 's of size $n=200, 400, 800$, and 1600 from the model (4) under some parameters. We also set the independent Type I censoring time $C_i = c$ that gives 20% (light censoring) or 60% censoring (heavy censoring) percentage. In the case of 60% censoring, the percentages of observing Events 1 and 2 are both 20%. In the case of 20% censoring, the percentage of observing Events 1 and 2 are both 40%. Figure 2 shows the percentage of observing Events 1 and 2 in the three designed levels of X_i . We see that the level $X_i = 0$ produces the highest censoring percentage while the level $X_i = 1$ produces the lowest censoring percentage.

After the failure and censoring times were generated, we computed the observed values $T_i = \min(T_{i1}, T_{i2}, C_i)$, $\delta_i = \mathbf{I}(T_i = T_{i1})$, $\delta_i^* = \mathbf{I}(T_i = T_{i2})$, and X_i . Based on the data $\{(T_i, \delta_i, \delta_i^*, X_i) : i = 1, \dots, n\}$, we computed the MLE by applying our R package “gammaGumbel” (Section 3). The MLE was computed under an identifiability assumption ($\theta=6$, $\theta=\eta$ or $\theta=\eta/2$). We replicated 500 times and summarized the results. Supplementary Material includes the simulation results (in Tables S7-S12). Overall, we found that all the unknown parameters in the model (4) were estimated accurately by the MLE, and the performance of the SE and 95%CI were satisfactory.



(a) 20% censoring (light censoring)



(b) 60% censoring (heavy censoring)

FIGURE 2: The proportions of events and censoring for the simulation settings.

5. REAL DATA ANALYSIS

In this section, we analyzed the ARC-1 VHF radio dataset from Mendenhall and Hader ^[1]. For these data, the main interest lies in the estimation of the distribution of the failure time of radio transmitters used in an airline company. The analysis had to handle a competing risk due to the apparent failure erroneously reported by operators. The second interest is the estimation of the dependence between the real failure and apparent failure. Two sources of dependence are suspected: one due to frailty (unobserved operating conditions), and the other is due to the unit-level dependence structure (the apparent failure may have some negative impact on the lifetime of the radios). The proposed methods are particularly helpful to account for the two types of dependence.

5.1 The ARC-1 VHF radio data

The dataset includes $n = 369$ radios for the times (hours) to failure taken from the ARC-1 VHF communication transmitter receivers of an airline company. ARC-1 is a kind of radio and VHF means “very high frequency”. The failed radios were removed from the aircraft for maintenance. The maintenance center then classifies the failure into two failure types: the confirmed failure (Event 1) and unconfirmed failure (Event 2), as described in Figure 3.

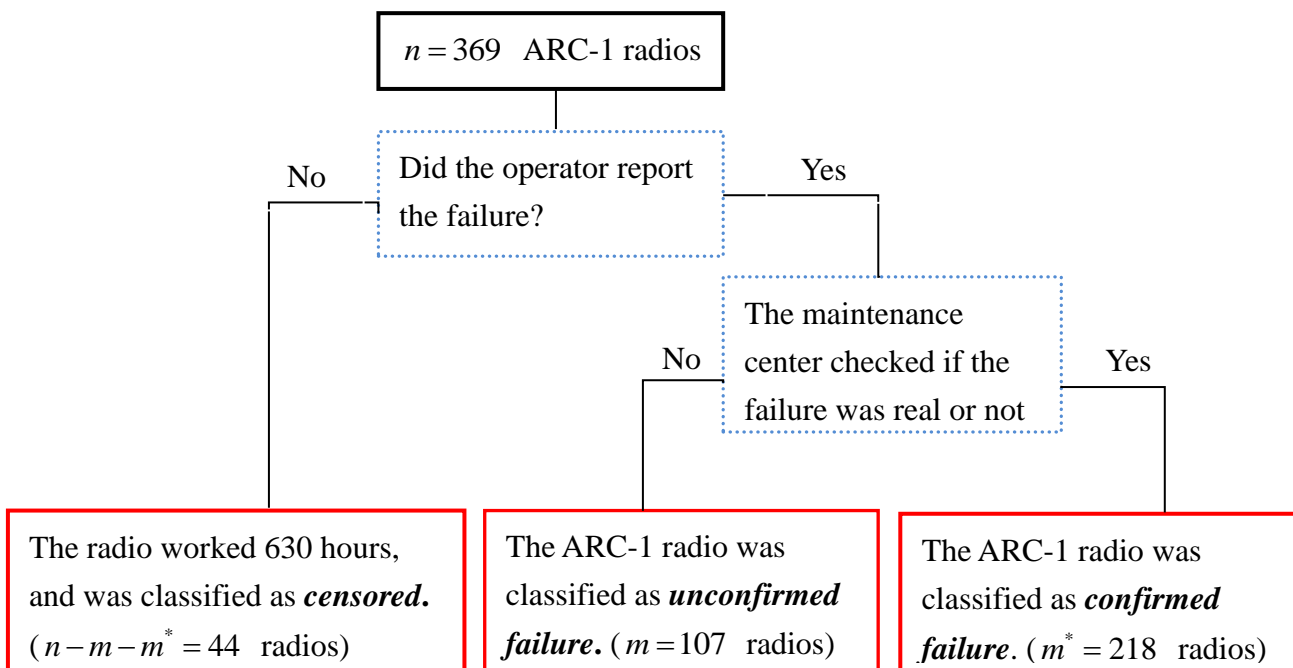


FIGURE 3: The ARC-1 VHF radio dataset from Mendenhall and Hader ^[1].

We set $\delta_i = 1$ for confirmed failure, and set $\delta_i^* = 1$ for unconfirmed failure. The radios that had operated 630 hours were censored at 630 hours due to the policy of airline to remove radios. We set $\delta_i = \delta_i^* = 0$ for these censored ratios. The data include $m = \sum_{i=1}^n \delta_i = 218$ confirmed failures, $m^* = \sum_{i=1}^n \delta_i^* = 107$ unconfirmed failures, and $n - m - m^* = 44$ censored ratios (Figure 3). The sample means of the confirmed and unconfirmed failures (in hours) are

$$\bar{T}_1 = \frac{1}{m} \sum_{i=1}^n T_i \delta_i = 229.61, \quad \bar{T}_2 = \frac{1}{m^*} \sum_{i=1}^n T_i \delta_i^* = 191.20,$$

and their sample SDs are,

$$s_1 = \sqrt{\frac{\sum_{i=1}^n \delta_i (T_i - \bar{T}_1)^2}{m-1}} = 165.08, \quad s_2 = \sqrt{\frac{\sum_{i=1}^n \delta_i^* (T_i - \bar{T}_2)^2}{m^*-1}} = 141.60.$$

5.2 Fitted results of the proposed model

We fitted the ARC-1 data to model (3) and computed the MLE for $\Phi \equiv (\mu_1, \mu_2, \sigma_1, \sigma_2, \eta)$ under a restriction of θ . We considered seven different models of restricting θ , and see how the MLE changes across different restrictions (sensitivity analysis). In particular, five assumed copula parameters $\theta \in (0, 0.33, 1, 3, 33.33)$, and two link functions $\theta = \eta$ and $\theta = \eta/2$ were examined (Table 1). The value $\theta = 0$ gives the conditional independence model of Liu ^[27]. The value $\theta = 33.33$ gives strong dependence between Events 1 and 2 (Kendall's tau: $\tau_\theta = 0.98$; $\tau_\theta^Z = 0.97$).

Table 1 shows that the MLEs for $(\mu_1, \mu_2, \sigma_1, \sigma_2, \eta)$ are similar among different models. However, Table 1 also shows that the estimates for the p -th quantiles of Events 1 and 2 can have practical difference between $\theta = 0$ and $\theta = 33.33$. For instance, the 0.01-th quantile for confirmed failure time (Event 1) is 14.8 (95%CI: 8.7-20.8) under $\theta = 0$, while it is 9.4 (95%CI: 5.3-13.6) under $\theta = 33.33$. Hence, if one had ignored the dependence by fitting $\theta = 0$, the quantile would be over-estimated. To see why the p -th quantile changes abruptly as θ changes, we recall

$$t_{p,j} = \exp(\mu_j) \left\{ \frac{(1-p)^{-\eta} - 1}{\eta} \right\}^{\sigma_j}.$$

Hence, the quantile is greatly affected by μ_j through $\exp(\cdot)$. We also see from Table 1 that μ_j decreases monotonically when θ increases.

[Table 1 here]

We chose the best way of restricting the value of θ among the seven different ways. Table 1 shows that the choice of $\theta=33.33$ gives the best result (both the highest maximized log-likelihood and the smallest CvM statistics). By noting that all the values of $\theta \geq 35.9$ give lower maximized log-likelihood values, $\theta=33.33$ can be regarded as the best value among all the values of θ that we have exhaustively examined. Indeed, Figure 4 suggests that model (3) with $\theta=33.33$ fits the data quite well, since the parametric and nonparametric estimators of the sub-distribution functions are nearly identical. The sub-distribution functions estimated under $\theta=0$ show only slightly poorer fit ($CvM=0.0591$) than the best value $\theta=33.33$ ($CvM=0.0581$). This small difference in fits for very different parameters is related to the general issue of non-identifiability of competing risks models.

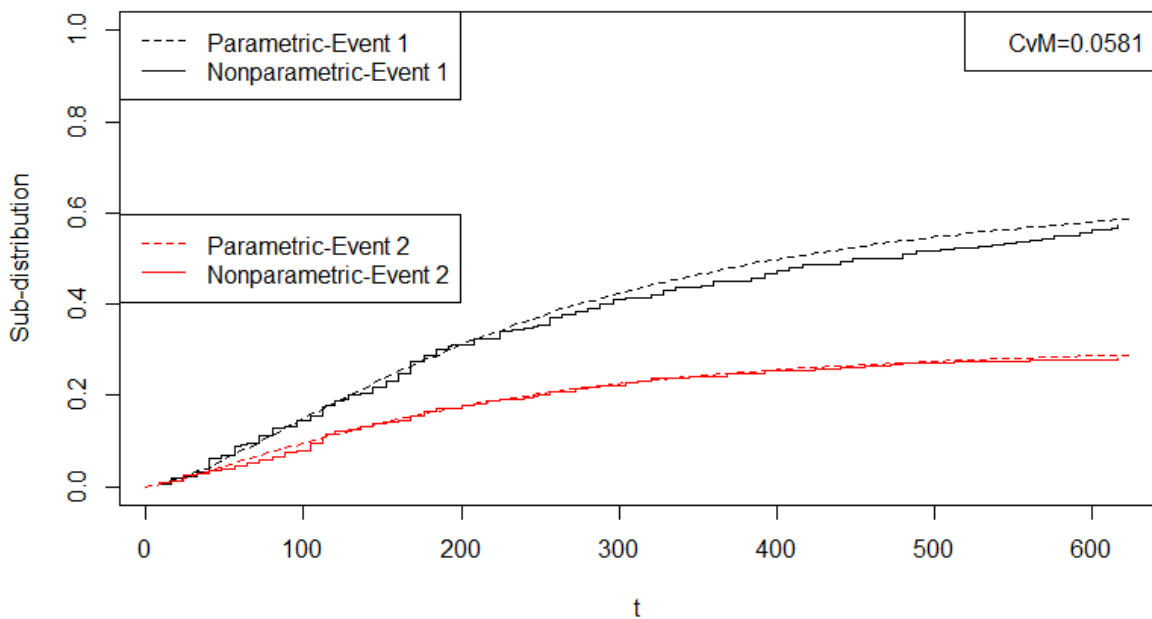


FIGURE 4: Model diagnostic plots using the ARC-1 data (under $\theta=33.33$). The two estimators of the sub-distribution function (parametric estimator vs. nonparametric estimator) are displayed.

Once the value $\theta = 33.33$ is fixed, we examine the reliability of the radios. The 0.01-th quantile for confirmed failure time is estimated as $t_{0.01,1} = 9.4$ (95% CI: 5.3-13.6). Hence, the radios work for more than 10 hours with 99%. This estimate is reasonable since there are only two confirmed failures found before 10 hours in the 369 samples. The 0.1-th quantile for confirmed failure time is $t_{0.1,1} = 49.9$ (95% CI: 40.3-59.6). Some maintenance may be required after 50 hours as there is 10% chance of failure. The median of the confirmed failure time is estimated as $t_{0.5,1} = 212$ (95% CI: 188-237). Thus, around half of the radios would fail after 210 hours.

The heterogeneity variance is estimated as $\eta = 0.61$ (95% CI: 0.29-1.31). Since the lower confidence limit is greater than zero, there exists the heterogeneity among the units. This is a natural conclusion since the data do not include any covariate to explain the heterogeneity. This finding indicates the presence of covariates (operating conditions) that influence the frailty of the units.

The value $\theta = 33.33$ expresses the strong positive dependence between unconfirmed failure and confirmed failure. This may imply that the radios with unconfirmed failure may have some negative impacts on the radios, leading to the eventual failure of the unit. Thus, the unconfirmed failure may predict the confirmed failure. If the unconfirmed failure was simply due to a mistake of operators (e.g., units were mistakenly switched off), the two types of failure would be independent.

5.3 Comparison with other models

We tried to fit the ARC-1 data to other available models for competing risks data. However, we experienced difficulties in fitting various other existing models to the dataset for several reasons. First, the MLE computed under some models did not converge, e.g., the bivariate Pareto model ^[9]. This seriously reduced our choices of models. Second, some models that assume ties $\Pr(T_{i1} = T_{i2}) > 0$, e.g. Marshall-Olkin type bivariate Weibull model ^[4], are theoretically not suitable for the data.

For our numerical comparison, we chose the models satisfying the following conditions: (i) the fitting algorithm is available to users, (ii) the MLE converges for our dataset, (iii) Kendall's tau is available from the MLE. Accordingly, we choose two types of the Frank copula models proposed by Shih et al. ^[10], which were implemented in an R package *Bivariate.Pareto*. These models use the Pareto lifetime distribution for their marginal models.

Table 2 shows the results of fitting the selected models to the data. We see that the proposed method gives the best fit to the data in terms of AIC given by $AIC=2(DF-\log L)$, where DF =Degrees of freedom (the number of parameters) and $\log L$ =maximized log-likelihood. The smaller value gives the better fit. An interesting phenomenon is that negative dependence was detected under the Frank copula models. This means that the positive or negative dependence structure is highly sensitive to the chosen model. Again, this phenomenon is related to the general difficulty of estimating dependence structure from competing risks data ^[40].

6. CONCLUSION

This paper proposes a new statistical model to the analysis of bivariate competing risks data. The main idea of the proposed model is to relax the conditional independence assumption of the frailty model of Liu ^[27] by using the Gumbel copula function. The model is practical in the sense that the survival function and Kendall's tau remain tractable forms compared to the original model of Liu. While there are a number of survival models available for competing risks data, the proposed model differs from none of them. In fact, our model comprises a broad class by having a copula parameter θ and a frailty parameter η that are flexibly linked by a function $\theta = \omega(\eta)$. Our model reduces to the frailty model of Liu by choosing the link function $\omega(\eta)=0$. Other link functions can produce a variety of different models.

The proposed model focuses on the Gumbel copula for mathematical convenience as well as its practical relevance. The Gumbel copula is appropriate in the presence of stronger dependence in short failure times than long failure times. In the context of the example with confirmed and

apparent failure times, permitting for strong dependence in short failure times is important, because short time to apparent failure can then be used as a signal of confirmed failure.

The proposed model could be further extended by replacing the Gumbel copula with other copulas, such as the Clayton copula. Moreover, it is possible to replace the Weibull marginal models to other models, such as the Pareto models. However, these extended models can have complex forms, and it may not be straightforward to extend them to AFT models. In addition, an extension to a model with more than two causes of failure would be interesting. In the presence of more than two competing failure types, however, cause-specific hazard models are workable approaches by avoiding the specifications of the complex models ^[41].

On the basis of the proposed model, we have developed likelihood-based inference methods, including the development of a model-diagnostic procedure and computer software packages. Furthermore, we have extended the proposed model to include a covariate (acceleration factor), which may be useful for analyzing data from industrial life tests. These tools will serve for the applications of the proposed model to the design of accelerated life tests, as discussed by Liu ^[27]. However, the details of this topic are beyond the scope of the paper. Also, the applications of the proposed model to the series system's reliability analysis ^[42] is another issue that we do not cover in this paper, although, these types of topics would be interesting for both theoretical statisticians and engineering researchers. Other future works include the residual life prediction under the proposed model. Readers are referred to the residual life analysis in competing risks ^[43], bivariate survival models ^[44,45], and truncated survival models ^[46]. Of particular interest is the residual lifetime prediction of one event time conditional on the status of the other event as previously studied for semi-competing risks models ^[47,48].

APPENDIX: Proof of Proposition 1

The goal is to show that the parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2, \eta, \theta)$ in model (3) are identifiable from the observed $T_i = \min(T_{i1}, T_{i2})$ and $\delta_i = \mathbf{I}(T_i = T_{i1})$ under the link function $\theta = \omega(\eta)$.

Step 1: Identification of (η, θ) : Let

$$\Lambda(t) \equiv \left[\left\{ \frac{t}{\exp(\mu_1)} \right\}^{\frac{\theta+1}{\sigma_1}} + \left\{ \frac{t}{\exp(\mu_2)} \right\}^{\frac{\theta+1}{\sigma_2}} \right]^{\frac{1}{\theta+1}}.$$

Putting this into model (3), we have

$$S_{T_{i1}, T_{i2}}(t, t) = (1 + \eta \Lambda(t))^{-\frac{1}{\eta}}.$$

Ridder[49] shows that $(\eta, \Lambda(t))$ are nonparametrically identifiable given that the unobserved frailty Z has a finite mean. Thus, θ is also identifiable by the link function $\theta = \omega(\eta)$.

Step 2: Identification of $(\mu_1, \mu_2, \sigma_1, \sigma_2)$: Model (3) can be expressed using a copula form:

$$S_{T_{i1}, T_{i2}}(t_1, t_2) = C_{\theta, \eta}(S_{T_{i1}}(t_1), S_{T_{i2}}(t_2)).$$

where $C_{\theta, \eta}(u_1, u_2) = [1 + \{(u_1^{-\eta} - 1)^{\theta+1} + (u_2^{-\eta} - 1)^{\theta+1}\}^{1/(\theta+1)}]^{-1/\eta}$ is the BB1 copula. Since (η, θ) is identified, the copula is known. Under the assumed copula, the marginal survival distributions

$$S_{T_{ij}}(t) = \left[1 + \eta \left\{ \frac{t}{\exp(\mu_j)} \right\}^{\frac{1}{\sigma_j}} \right]^{-\frac{1}{\eta}}, \quad j = 1, 2,$$

are identifiable by the copula-graphic estimator. Thus, the functions $\Lambda_j(t) \equiv \left\{ \frac{t}{\exp(\mu_j)} \right\}^{1/\sigma_j}$,

$j = 1, 2$, are identifiable as well. Consequently, $(\mu_1, \mu_2, \sigma_1, \sigma_2)$ are identifiable. ■

SUPPLEMENTARY MATERIAL

The following supplementary materials are available from the journal webpage.

- The PDF manual for our R package *gammaGumbel* (the package is available from the corresponding author and/or ResearchGate).
- Supplementary Material, a PDF file including
 - S1. Computation for the MLE
 - S2. Standard errors and confidence intervals
 - S3. Simulation results (Tables S1-S12)
 - S4. Computation for the MLE under the AFT model
 - S5. Standard errors and confidence intervals under the AFT model

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TABLE 1: The parameter estimates (95%CI) for fitting the ARC-1 data to the proposed model.

	$\theta = 0$	$\theta = 0.33$	$\theta = 1$	$\theta = 3$	$\theta = 33.33$	Identity link	Linear link 1
	$\tau_{\theta}^{ Z} = 0$	$\tau_{\theta}^{ Z} = 0.25$	$\tau_{\theta}^{ Z} = 0.5$	$\tau_{\theta}^{ Z} = 0.75$	$\tau_{\theta}^{ Z} = 0.97$	$\theta = \eta$	$\theta = \eta/2$
μ_1	5.70 (5.46-5.94)	5.63 (5.40-5.86)	5.56 (5.34-5.78)	5.50 (5.29-5.71)	5.44 (5.24-5.65)	5.59 (5.33-5.85)	5.63 (5.40-5.86)
μ_2	6.24 (5.87-6.61)	6.04 (5.72-6.37)	5.84 (5.56-6.12)	5.64 (5.40-5.88)	5.46 (5.25-5.67)	5.92 (5.50-6.34)	6.04 (5.72-6.36)
σ_1	0.65 (0.54-0.79)	0.66 (0.56-0.80)	0.68 (0.57-0.81)	0.69 (0.58-0.82)	0.70 (0.58-0.83)	0.67 (0.57-0.79)	0.66 (0.56-0.80)
σ_2	0.76 (0.62-0.94)	0.75 (0.61-0.91)	0.73 (0.60-0.88)	0.71 (0.59-0.86)	0.70 (0.59-0.83)	0.73 (0.60-0.90)	0.75 (0.61-0.91)
η	0.65 (0.31-1.34)	0.64 (0.31-1.33)	0.63 (0.30-1.32)	0.62 (0.30-1.32)	0.61 (0.29-1.31)	0.64 (0.31-1.32)	0.64 (0.31-1.33)
$t_{0.01,1}$	14.8 (8.7-20.8)	13.2 (7.9-18.5)	11.7 (7.0-16.5)	10.5 (6.1-14.8)	9.4 (5.3-13.6)	12.4 (8.0-16.8)	13.2 (8.4-18.0)
$t_{0.1,1}$	70.2 (57.4-83.0)	64.0 (52.8-75.3)	58.4 (48.1-68.7)	53.4 (43.7-63.2)	49.4 (39.9-59.0)	60.8 (50.5-71.1)	64.2 (53.2-75.2)
$t_{0.5,1}$	274 (235-313)	255 (221-288)	237 (208-267)	222 (195-248)	209 (185-233)	245 (206-283)	255 (216-295)
$t_{0.01,2}$	15.4 (7.6-23.2)	13.6 (7.2-20.0)	12.0 (6.6-17.4)	10.6 (6.0-15.2)	9.5 (5.3-13.6)	12.7 (7.5-18.0)	13.7 (7.7-19.7)
$t_{0.1,2}$	94.8 (73.1-116.5)	80.5 (64.3-96.7)	68.2 (55.6-80.9)	57.8 (47.2-68.4)	49.9 (40.3-59.6)	73.3 (58.0-89.0)	80.8 (62.9-98.7)
$t_{0.5,2}$	464 (349-578)	380 (302-458)	310 (259-362)	254 (220-288)	212 (188-237)	338 (240-436)	382 (272-491)
$\ell(\hat{\Phi})$	-2377.591	-2377.577	-2377.562	-2377.547	-2377.534	-2377.569	-2377.577
CvM	0.0591	0.0589	0.0587	0.0585	0.0581	0.0587	0.0589

NOTE: θ =parameter of the Gumbel copula; $\tau_{\theta}^{|Z}$ =Kendall's tau given the frailty term Z .

TABLE 2: The results of fitting different models to the ARC-1 data.

	Frank copula $\theta = -5$; $\tau_\theta = -0.46$	Frank copula with common margins $\hat{\theta} = -2.13$; $\hat{\tau}_\theta = -0.23$	Proposed model $\theta = 33.33$; $\tau_\theta = 0.98$; $\tau_\theta^Z = 0.97$
DF	4	3	5
Likelihood	-2385.506	-2405.108	-2377.534
AIC	4779.012	4816.217	4765.068
R function	<i>MLE.Frank.Pareto(,Theta = -5)</i>	<i>MLE.Frank.Pareto.com()</i>	<i>gammaGumbel.MLE(,theta=32.33)</i>
R package	<i>Bivariate.Pareto</i>	<i>Bivariate.Pareto</i>	<i>gammaGumbel</i>

NOTE: DF=Degrees of freedom (the number of parameters); θ =a fixed parameter of the Frank or Gumbel copula which gives the largest log-likelihood value (i.e., the profile likelihood estimate); $\hat{\theta}$ =an estimated parameter of the Frank copula by the MLE; τ_θ =Kendall's tau, τ_θ^Z =Kendall's tau given the frailty term Z .