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Necessity of the terminal condition in the infinite horizon dynamic optimization problems with unbounded payoff☆

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ABSTRACT

In this paper, we prove the necessity of a terminal condition for a solution of the Bellman Equation to be the value function in dynamic optimization problems with unbounded payoffs. We also state the weakest sufficient condition, which can be applied in a large class of problems, including economic growth, resource extraction, or human behaviour during an epidemic. We illustrate the results by examples, including simple linear–quadratic problems and problems of resource extraction, with multiple solutions to the Bellman Equation or the maximizer of the right hand side of the Bellman Equation with the actual value function being the worst control instead of being optimal.

1. Introduction

The theory of discrete time dynamic optimization problems with the infinite horizon, started by Bellman (1957), Blackwell (1965), Stokey, Lucas, and Prescott (1989) and Strauch (1966), is far from complete, especially for unbounded payoffs.

In this paper, we formulate a dynamic optimization problem in a very general form without a priori topological assumptions about the payoff, the state dynamics, and the sets of available control parameters. We consider the global feedback solution as well as a restriction of the problem with a fixed set of possible initial conditions. We formulate the *weakest possible terminal condition* concerning the limit behaviour of the value function along every admissible trajectory for each of those two problems. This terminal condition constitutes a part of a *sufficient condition* for optimality and we prove that this terminal condition is *necessary* under quite weak assumptions about the dynamic optimization problem considered. The main focus of the paper is on the *necessity* of appropriate terminal conditions for a standard feedback optimal control problem and its restrictions. To the best of our knowledge, the necessity of such terminal conditions has never been proven before. By sufficiency, these terminal conditions can be used to choose the value function among multiple solutions of the Bellman equation, which implies finding an optimal control whenever it exists, while by the necessity, it can be used to exclude spurious candidates for the value function or prove suboptimality of a control. As a side effect, we obtain uniqueness, as well as, under an additional assumption, existence, and uniqueness of a solution of the Bellman equation with the terminal condition.

One of the starting points of this paper is a side effect of the game analysed in Singh and Wiszniewszka-Matyszkiel (2018), in which it has been noticed that not checking the terminal condition can lead to the derivation of false value functions and false optima even in a constrained linear–quadratic problem, regarded as well examined. What is more impressive, the false value function in that case is more plausible than the actual one and the model has a clear economic interpretation.

Applications Problems with unbounded payoffs, including payoffs that can have singularities and be equal to $-\infty$ at the states

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which are not viable, are extensively used in ecological and economic applications, so there is a need to have appropriate tools to solve such problems correctly.

One of classes of such applications is Fish Wars models — models of extraction of common or interrelated renewable resources by at least two independent agents with logarithmic current payoffs and depletion possible, started by Levhari and Mirman (1980): a short and far from complete selection of the papers in which Fish Wars has been further studied is Breton and Keoula (2012, 2014), Cave (1987), Fischer and Mirman (1992, 1996), Kwon (2006),Doyen, Cissé, Sanz, Blanchard, and Pereau (2018), Górniewicz and Wiszniewska-Matyszkiel (2018), Nowak (2006, 2008), Dutta and Sundaram (1993), Mazalov and Rettieva (2009, 2010a, 2010b), Okuguchi (1981), Rettieva (2012, 2014), Breton, Dahmouni, and Zaccour (2019), Wiszniewska-Matyszkiel (2005).

Other important applications are economic growth models. This is the field which, throughout the last three decades, has been developed to a large extent together with the theory of the unbounded infinite horizon dynamic optimization problems: e.g., Stokey et al. (1989) illustrate their unbounded payoffs section by an economic growth model with the logarithmic payoff, in Le Van and Morhaim (2002), the economic growth model is the main motivation, Kamihigashi and Roy (2007) study such a model with the production function which is nonconvex and only upper semicontinuous while the current payoff unbounded, Becker, Bosi, Le Van, and Seegmuller (2015) study the Ramsey equilibrium model with heterogeneous agents in which they prove the existence of the equilibrium and non-existence of rational bubbles.

Theory When the theory of deterministic problems is concerned, most of the current theoretical papers concentrate on the existence and uniqueness of the solution of the Bellman Equation in some classes of continuous functions and the procedure of calculating the value function by iterations using strong topological assumptions guaranteeing existence and uniqueness: (Hosoya & Yao, 2013; Le Van & Morhaim, 2002; Matkowski & Nowak, 2011; Martins-da Rocha & Vailakis, 2010), (Rincón-Zapatero & Rodríguez-Palmero, 2003, 2009). Existence without uniqueness, again with strong assumptions, encompassing nonpositivity besides assumptions of topological character, is examined in Keerthi and Gilbert (1985) and Guo, Hernández-del Valle, and Hernández-Lerma (2010, 2011). Besides, the rule of choice from multiple solutions of the Bellman equation is proposed in Guo et al. (2010, 2011). The problem what kind of inaccuracies in calculation or computation of the value function have no influence on the optimal solution along the corresponding trajectory given a certain subset of initial conditions, is considered in Wiszniewska-Matyszkiel and Singh (2020) also with a generalization to dynamic games.

In fact, no topological assumptions are needed for the necessity of the Bellman equation, called "principle of optimality" (Stokey et al., 1989 and Kamihigashi, 2008). Besides, sufficient conditions for the value function can be formulated without them (Stokey et al., 1989, Wiszniewska-Matyszkiel, 2011, Kamihigashi, 2014a, 2014b). The same applies to an optimal control (which has been proved by Stokey et al., 1989 and generalized by Wiszniewska-Matyszkiel, 2011). Nevertheless, to the best of our knowledge, the necessity of a terminal condition for the value function has never been considered in this context. The applicability of the results of this paper can be compared to the results of the previous works by the fact that the sufficient condition for the value function of Stokey et al. (1989) (Theorem 4.3) does not hold in the Fish Wars problems even if the model is constrained to make the current payoff always finite. Another type of sufficient condition results is in Kamihigashi (2014a) (further discussed in Kamihigashi, 2014b), who proves the existence and uniqueness of the value function in an interval of functions $[v_-, v_+]$ with $v_- \leq v_+$ for which instead of the Bellman equation two analogous inequalities with opposite signs hold and which fulfil some terminal conditions. We prove that those conditions are implied by Terminal condition 1 of this paper. Moreover, Kamihigashi (2014a) proves that the value function can be obtained from v_- as a limit of iterations. Necessity of the terminal conditions has not been considered in this context, either. A more detailed description of results of Kamihigashi (2014a) is in Section 6.

A parallel approach to that considered in this paper and the aforementioned papers is the approach based on a discrete time quasi-equivalent of the infinite horizon Pontryagin maximum principle or Euler equation — extensively studied in Aseev, Krastanov, and Veliov (2017), Blot and Hayek (2014) or Brunovský and Holecyová (2018). That approach is applicable with open loop controls, i.e. controls that are functions of time only. It is worth mentioning that, although equivalent in usual dynamic optimization problems, in dynamic games, it usually leads to different solutions than the approach based on the Bellman Equation (see e.g. a discussion in Wiszniewska-Matyszkiel, 2014). In the context of open loop controls, a kind of terminal condition different from the one considered in this paper is the *transversality condition* whose necessity under some convexity/concavity and differentiability assumptions is examined in e.g. Kamihigashi (2002), related to the open loop approach. Transversality, however, is a kind of a terminal condition that is fulfilled along the optimal trajectory only. It requires not only continuity, but also differentiability of the current payoff function.

Plan The paper is constructed as follows: The problem is formulated in Section 2, in Section 3 the weakest sufficient condition is cited from the literature and slightly weakened, and examples in Section 4 illustrate that each part of the terminal condition is essential. Section 5 is devoted to necessary conditions with Section 5.1 illustrating by an example that the assumption is essential. Section 6 contains solutions, by the methods derived in this paper, of problems from the literature that either have not got complete proofs or which can be proven using our result more easily or without additional restrictions of the problems, and comparison of the results of this paper to related earlier papers. Section 7 discusses problems in which discontinuities are inherent, so, for which most of the methods from the literature, i.e. all the methods based on strong topological assumptions on the components of the model, fail. For clarity of exposition, all the not-immediate proofs are in the Appendix.

2. The problem

We consider a discrete time infinite horizon dynamic maximization problem with the *time set* being the set of nonnegative integers $\mathbb{N}$, the *set of states* $\mathbb{X}$, the *set of control parameters* $\mathbb{U}$, the *function describing the dynamics of the state variable* $f : \mathbb{X} \times \mathbb{U} \times \mathbb{N} \to \mathbb{X}$ (in some applications called the *regeneration function*) and the *current payoff function* $g : \mathbb{X} \times \mathbb{U} \times \mathbb{N} \to \mathbb{R} \cup \{-\infty\}$, with the discount factor $\delta \in (0, 1]$. We use the term *payoff* for the objective function to make it clear that we consider the *maximization*. Additionally, there are state-dependent constraints on controls given by a multivalued correspondence called the *available control correspondence* $D : \mathbb{X} \times \mathbb{N} \multimap \mathbb{U}$. More generally, we also consider f with a bigger co-domain: a set $\tilde{\mathbb{X}} \supseteq \mathbb{X}$, $f : \mathbb{X} \times \mathbb{U} \times \mathbb{N} \to \tilde{\mathbb{X}}$ with $f(x, u, t) \in \mathbb{X}$ for all $u \in D(x, t)$. In the latter case, we do not have to specify $\tilde{\mathbb{X}}$, since it does not influence the results.

The dynamic optimization problem (P).

Given initial $\bar{t}$ and $\bar{x}$, we maximize over the set of functions $U : \mathbb{X} \times \mathbb{N} \to \mathbb{U}$ with $U(x, t) \in D(x, t)$, called *feedback controls*, with the set of feedback controls denoted by $\mathcal{U}$, the *payoff function* $J : \mathbb{X} \times \mathbb{N} \times \mathcal{U} \to \bar{\mathbb{R}}$ defined by

$$J(\bar{x}, \bar{t}, U) = \sum_{t=\bar{t}}^{\infty} g(X(t), U(X(t), t), t)\delta^{t-\bar{t}}, \tag{1}$$

where the *trajectory* of the state variable $X : \mathbb{N} \to \mathbb{X}$ is defined by

$$X(t+1) = f(X(t), U(X(t), t), t) \text{ with the initial condition } X(\bar{t}) = \bar{x}. \tag{2}$$

General assumption We assume that J is always well defined.

We are especially interested in controls which are optimal whatever the initial condition $(\bar{x}, \bar{t})$ is. So, the optimal control is $\overline{U}$ such that $J(x, t, \overline{U}) = \sup_{U\in\mathcal{U}} J(x, t, U)$ for all $(x, t) \in \mathbb{X} \times \mathbb{N}$.

The *trajectory* X given by (2) is called *corresponding to* U. If we want to emphasize its dependence on the control, we write X^U, while if we want to emphasize also its dependence on the initial condition, we write $X^U_{\bar{x},\bar{t}}$.

We denote the set of all trajectories by $\mathcal{X}$.

Dependence of g on time is related to any dependence on time not reflected by discounting, e.g. seasonality. Besides, direct dependence of both f and g on time appears quite often while looking for Nash equilibria in dynamic games, i.e. control problems with many controllers (called players), in which at the first stage of calculations, for each player, a dynamic optimization problem is solved given controls (called strategies) of the other players, which may be dependent directly on time. This may happen at equilibrium even if there is no direct dependence on time in the formulation of the game (see e.g. duopolistic market games in which interlaced advertising is obtained (Wiszniewska-Matyszkiel, 2008); which reflects the behaviour of firms at some real markets).

We are going to examine various issues related to the *value function* $\bar{V} : \mathbb{X} \times \mathbb{N} \to \bar{\mathbb{R}}$ defined by

$$\bar{V}(x, t) := \sup_{U\in\mathcal{U}} J(x, t, U). \tag{3}$$

Usually, in papers on dynamic optimization with feedback controls, the *Bellman Equation*

$$V(x, t) = \sup_{u\in D(x,t)} g(x, u, t) + \delta\, V(f(x, u, t), t+1), \tag{4}$$

and the condition (we call it the *Bellman Inclusion* for easier reference)

$$U(x, t) \in \operatorname*{Argmax}_{u\in D(x,t)} g(x, u, t) + \delta V(f(x, u, t), t+1) \tag{5}$$

are examined.

In this paper, we especially focus on *terminal conditions*.

We start from a weak terminal condition from Wiszniewska-Matyszkiel (2011).

Terminal condition 1. ***(i)*** *For every trajectory* X, $\limsup_{t\to\infty} V(X(t), t)\, \delta^t \le 0$ *and*

(ii) *for every trajectory* X, *if* $\limsup_{t\to\infty} V(X(t), t)\, \delta^t < 0$, *then* $J(x, t, U) = -\infty$ *for every* U *such that* $X = X^U_{x,t}$.

In physical problems with viability, Terminal condition 1 states that the upper limit of the discounted value function is at most zero for every state trajectory and it can be only negative for trajectories which are not viable (by which we understand payoff $-\infty$).

Notational simplification If the functions f and g and the correspondence D are not dependent on time, then the value function is obviously independent of time. In such a case we are especially interested in U that are stationary. Then, by a slight abuse of notation, we skip the t argument in all these functions.

Most of the literature concern this stationary case. On the other hand, each dynamic optimization problem **(P)** may be represented in this way by extending the state space by including the time variable as an additional coordinate of the state space (as in the proof of Theorem 2 in the Appendix).

Since the initial condition is fixed in many applications, we shall also consider the restricted problem, assuming that the initial condition for the state is in some $\mathbb{X}_0 \subseteq \mathbb{X}$, possibly a singleton.

Then, instead of considering the whole state space, it is enough to consider the *reachable set* (to be more specific — the *set of reachable state-time pairs*):

$$\begin{aligned}\mathrm{REACH}(\mathbb{X}_0, t_0) = \{&(x, t) \in \mathbb{X} \times \{t_0, t_0+1, \dots\} : \\ &x = X^U_{\bar{x},t_0}(t) \text{ for some } U \in \mathcal{U},\ \bar{x} \in \mathbb{X}_0\}.\end{aligned} \tag{6}$$

This results in the restricted problem $\mathbf{(P)}^{\mathrm{REACH}(\mathbb{X}_0,t_0)}$ in which we restrict ourselves to looking for the optimal control defined only on the set $\mathrm{REACH}(\mathbb{X}_0, t_0)$.

We denote the value function for this problems by $V^{\mathrm{REACH}(\mathbb{X}_0,t_0)}$.

Remark 1. Obviously, $\bar{V}|_{\mathrm{REACH}(\mathbb{X}_0,t_0)} = V^{\mathrm{REACH}(\mathbb{X}_0,t_0)}$, if $\bar{U}$ is an optimal control for **(P)**, then $\bar{U}|_{\mathrm{REACH}(\mathbb{X}_0,t_0)}$ is an optimal control for $\mathbf{(P)}^{\mathrm{REACH}(\mathbb{X}_0,t_0)}$. Nevertheless, the existence of an optimal control for $\mathbf{(P)}^{\mathrm{REACH}(\mathbb{X}_0,t_0)}$ does not imply it can be extended to an optimal control for **(P)**.

3. The weakest sufficient conditions and their consequences

We start by citing a theorem stating a very weak sufficient condition for the optimal solution and the value function being a version of Theorem 1 of Wiszniewska-Matyszkiel (2011), designed especially for the infinite horizon dynamic optimization problems with unbounded payoffs like those considered in the Fish Wars stream of papers. Subsequently, we shall prove that, under quite general assumptions, it is the weakest possible sufficient condition and, moreover, we can derive some uniqueness conclusions from it.

Theorem 1. *Assume that a function* $V : \mathbb{X} \times \mathbb{N} \to \bar{\mathbb{R}}$ *fulfils Terminal condition 1 and for every* $x \in \mathbb{X}$ *and every* $t \in \mathbb{N}$, *it fulfils the Bellman equation (4).*

(a) Then V *is the value function of the dynamic optimization problem.*

(b) Moreover, if for every $x \in \mathbb{X}$ *and every* $t \in \mathbb{N}$, *a control function* $\bar{U}$ *fulfils the Bellman Inclusion (5), then* $\bar{U}$ *is an optimal control.*

Corollary 1. *There exists at most one solution of the Bellman Equation in the class of functions fulfilling Terminal condition 1.*

Proof. The value function exists and it is unique by its definition, the rest is immediate by Theorem 1. □

As a consequence of Theorem 1, we can state the implication for the restricted problem $\mathbf{(P)}^{\mathrm{REACH}(\mathbb{X}_0,t_0)}$.

First, we rewrite the terminal condition to a weaker version. Its interpretation remains almost the same, with "every trajectory" restricted to trajectories and payoffs with the initial condition restricted to $\mathrm{REACH}(\mathbb{X}_0, t_0)$.

Terminal condition 2. ***(i)*** *for every $\bar{t} \geq t_0$ and every trajectory X with the initial condition $X(\bar{t}) \in \mathrm{REACH}(\mathbb{X}_0, t_0)$, $\limsup_{t\to\infty} V(X(t), t)\, \delta^t \leq 0$ and*

(ii) *For every $\bar{t} \geq t_0$ and every trajectory X with the initial condition $(X(\bar{t}), \bar{t}) \in \mathrm{REACH}(\mathbb{X}_0, t_0)$, if $\limsup_{t\to\infty} V(X(t), t)\, \delta^t < 0$, then for every $(x, t) \in \mathrm{REACH}(\mathbb{X}_0, t_0)$, $J(x, t, U) = -\infty$ for every U such that $X = X^U_{x,t}$.*

Theorem 2. *Fix any $\mathbb{X}_0 \subseteq \mathbb{X}$ and $t_0 \in \mathbb{N}$.*

Assume that a function $V : \mathrm{REACH}(\mathbb{X}_0, t_0) \times \mathbb{N} \to \mathbb{R}$ fulfils Terminal condition 2 and for every $(x, t) \in \mathrm{REACH}(\mathbb{X}_0, t_0)$, it fulfils the Bellman equation (4).

(a) Then $V = V^{\mathrm{REACH}(\mathbb{X}_0,t_0)}$.
(b) Moreover, if for every $(x, t) \in \mathrm{REACH}(\mathbb{X}_0, t_0)$, a control function $\bar{U}$ fulfils the Bellman Inclusion (5), then $\bar{U}$ is an optimal control of $(\mathbf{P})^{\mathrm{REACH}(\mathbb{X}_0,t_0)}$.

4. Examples showing the importance of assumptions

Here, we present examples showing that neglecting checking the terminal condition may lead to finding a false optimal control and/or a false value function even in very simple problems with obvious applications.

We start from two linear–quadratic dynamic optimization problems: with or without constraints.

Example 1. Consider $\mathbb{X} = \mathbb{U} = \mathbb{R}_+$, $f(x, u) = (1 + \xi)x - u$, $g(x, u) = (A - \frac{Bu}{2})u$, for some $A, B, \xi > 0$, $D(x) = [0, (1 + \xi)x]$ and $\delta = \frac{1}{1+\xi}$.

This problem can be interpreted as a renewable resource extraction problem e.g. a fishery extraction problem. Then each time instant corresponds to one year, starting just after the closed season related to spawning and before hatching of the new generation of fish, x denotes the biomass or an approximate number of fish measured at the beginning of this period, ξ is the natural net growth rate of fish biomass, while the constraint $u \leq (1+\xi)x$ means that the maximal catch is equal to all fish including the current year's offspring. The current payoff represents selling at the market price given by a linear demand and quadratic cost. The discount factor is equal to $\frac{1}{1+\xi}$, which may be regarded as a kind of *golden rule* (in a trivial form, since the growth rate is constant). This model is a generalization of the optimal control version of a dynamic game considered in Singh and Wiszniewska-Matyszkiel (2019), Singh and Wiszniewszka-Matyszkiel (2018) modelling exploitation of a common divided fishery and a "pathological" limit case of a dynamic optimization problem considered in Singh and Wiszniewska-Matyszkiel (2020) with $\delta < \frac{1}{1+\xi}$.

Proposition 1. *(a) The value function equals*

$$\bar{V}_1(x) := \begin{cases} \hat{g} \cdot x + \frac{\hat{h}}{2} \cdot x^2 & \text{if } x \in (0, \tilde{x}), \\ \tilde{k} & \text{otherwise}, \end{cases}$$

for $\tilde{x} = \frac{\hat{u}}{\xi}$, $\hat{u} = \frac{A}{B}$, $\hat{h} = -B\xi\,(1+\xi)$, $\hat{g} = A(1+\xi)$, and $\tilde{k} = \frac{A^2(1+\xi)}{2B\xi}$, while the unique optimal control is $\bar{U}_1(x) := \begin{cases} \xi x, & \text{for } x \in (0, \tilde{x}), \\ \hat{u} & \text{otherwise}. \end{cases}$

(b) The Bellman Equation has also a quadratic solution $V^{\text{false}} = \hat{g} \cdot x + \frac{\hat{h}}{2} \cdot x^2$. The solution of the Bellman Inclusion (5) with V^{false} is $U^{\text{false}} = \xi x$ and it results in a constant trajectory.

(c) V^{false} fulfils Terminal condition 1(i), but it does not fulfil Terminal condition 1(ii) (see Fig. 1).

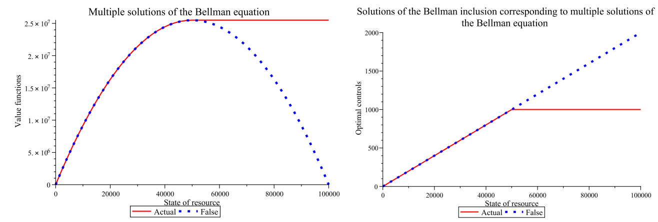

Fig. 1. Two solutions of the Bellman Equation (left) and the corresponding solutions of the Bellman Inclusion (right) for $n = 4$, $A = 1000$, $B = 1$, $\xi = 0.02$ for Example 1. The red solid line denotes the actual value function or optimal control, the blue dotted line their spurious counterparts.

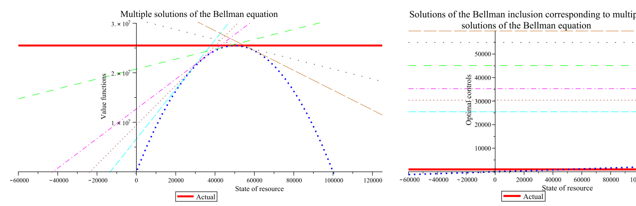

Fig. 2. Multiple solutions of the Bellman Equation (left) and the corresponding solutions of the Bellman Inclusion for $n = 4$, $A = 1000$, $B = 1$, $\xi = 0.02$ for Example 2. The red solid line denotes the actual value function or optimal control.

It is worth emphasizing that the standard procedure for solving LQ dynamic optimization problem is by assuming a quadratic form of the value function and, consequently, a linear optimal control. So, if checking the terminal condition is skipped, the unique quadratic solution of the Bellman Equation may be, in such a case, treated as the value function while the unique solution of the Bellman Inclusion, which is linear and leads to a constant trajectory, as an optimal control. So, this example is very important to illustrate the danger of such an error even in such a simple problem.

Next, we consider a modification of Example 1 in which we skip the constraints on the state variable and control, to show that this nonuniqueness can happen in quite standard unconstrained linear–quadratic problems.

Example 2. Consider $\mathbb{X} = \mathbb{U} = \mathbb{R}$, $f(x, u) = (1 + \xi)x - u$, $g(x, u) = (A - \frac{Bu}{2})u$, for some $A, B, \xi > 0$, $D(x) = \mathbb{U}$ and $\delta = \frac{1}{1+\xi}$.

Proposition 2. *(a) The value function is $\bar{V}_2(x) = \tilde{k}$, while the unique optimal control is $\bar{U}_2(x) = \hat{u}$ for $\tilde{k}$ and $\hat{u}$ from Proposition 1.*

(b) The Bellman Equation has a continuum of at most quadratic solutions, which yields a continuum of affine solutions of the Bellman Inclusion.

(c) Each of those solutions of the Bellman Equation besides the actual value function violates Terminal condition 1(i) or (ii), and, if $\mathbb{X}_0 \neq \{0\}$, Terminal condition 2(i) or (ii) (see Fig. 2).

Example 3. Consider exploitation of a non-renewable resource without discounting and with strictly increasing strictly concave payoff: $\mathbb{X} = \mathbb{U} = [0, 1]$ and $g(x, u) = \ln(u + 1)$, $f(x, u) = x - u$, $D(x) = [0, x]$ and $\delta = 1$.

Proposition 3. *(a) The function $\bar{V}_3(x) = x$ fulfils the Bellman Equation and the control $U \equiv 0$ is the only solution of the Bellman Inclusion (5) with $\bar{V}_3$, but 0 is not the optimal control, but the worst control.*

(b) Terminal conditions 1(i) and 2(i) are not fulfilled, while 1(ii) and Terminal conditions 2(ii) are fulfilled.

5. Necessity of the terminal condition

Examples 1–3 show that if at least one part of Terminal condition 1 is not fulfilled by a solution V of the Bellman equation (4), then it may happen that V is not the value function and/or the solution of the Bellman Inclusion (5) with V is not the optimal control. Similarly, if at least one part of Terminal condition 2 is not fulfilled by a solution V of the Bellman equation (4), then it may happen that V is not the value function even for the restricted problem and/or the solution of the Bellman Inclusion (5) with V is not the optimal control for the restricted problem.

So, an obvious implication is a question about the necessity of the terminal conditions.

The "principle of optimality" i.e., the necessity of the Bellman equation (4) and Bellman Inclusion (5) in a dynamic optimization problem for unbounded payoffs, has been proven by Stokey et al. (1989) when the current payoffs are finite and the payoff is well defined — Theorems 4.2 and 4.4, respectively. The problem has been further discussed by Kamihigashi (2008), proving the Bellman Equation is necessary whenever its rhs is well defined, i.e. in the case when current payoffs can attain $-\infty$, the second term cannot be $+\infty$ (Kamihigashi, 2008 Theorem 2). Our extension in this direction, concerning that the Bellman Inclusion is necessary for the case out of the scope of the Stokey et al. (1989) results is simple, but we state it, together with proof, for easy reference.

On the contrary, to the best of our knowledge, the necessity of Terminal conditions 1 and 2 has not been analysed before.

Theorem 3. *(a) Assume that the right hand side of the Bellman equation (4) is well defined for some (x, t). Then the value function $\bar{V}$ fulfils the Bellman equation (4) for this (x, t), while every optimal control fulfils the Bellman Inclusion (5) for this (x, t).*

(b) The value function fulfils Terminal condition 1(ii).

(c) Assume that

$$\begin{aligned}&\forall(\bar{x}, t_0) \in \mathbb{X} \times \mathbb{N},\ \epsilon > 0\ \exists T\ \forall t > T,\ U \in \mathcal{U},\ x \in \mathbb{X}\\ &\quad \text{with } (x, t) \in \mathrm{REACH}(\{\bar{x}\}, t_0),\ J(x, t, U)\delta^{t-t_0} \le \epsilon.\end{aligned} \tag{7}$$

Then the value function fulfils Terminal condition 1(i).

Condition (7) means that given an initial condition, there is a convergent to zero common upper bound on "tails" of series defining payoffs for every admissible trajectory with this initial condition. If the current payoffs along such trajectories are bounded from above by a constant, then it is enough to have $\delta < 1$, if the current payoffs along such trajectories are bounded from above by an at most exponential function of time, this requires that δ is sufficiently small. For $\delta = 1$ sufficiently fast convergence of a common upper bound of current payoffs along such trajectories to zero is required. Condition (7) is a weaker version of "tail insensitivity" assumption of Le Van and Morhaim (2002) and we are going to further weaken it in Theorem 4.

Corollary 2. *(a) If the payoff is always greater than $-\infty$, then the value function fulfils $\limsup_{t\to\infty} V(X(t))\delta^t \ge 0$.*

(b) If the payoff is always greater than $-\infty$ and Condition (7) holds, then $\limsup_{t\to\infty} V(X(t))\delta^t = 0$.

So, if we return to the physical interpretation with viability, Corollary 2(b) is translated to: "if Condition (7) holds in a problem in which all the trajectories are viable, then the upper limit of the discounted value function along every admissible trajectory is zero".

Corollary 3. *If Condition (7) is fulfilled, then the Bellman equation with Terminal condition 1 has a unique solution.*

Theorem 4. *(a) Assume that the right hand side of the Bellman Equation for $V = V^{\mathrm{REACH}(\mathbb{X}_0,t_0)}$ is well defined on $\mathrm{REACH}(\mathbb{X}_0, t_0)$.*

Then V^{REACH} fulfils the Bellman equation (4), while every optimal control of $\mathbf{(P)}^{\mathrm{REACH}}$ fulfils the Bellman Inclusion (5) on the set $\mathrm{REACH}(\mathbb{X}_0, t_0)$, respectively.

(b) $V^{\mathrm{REACH}(\mathbb{X}_0,t_0)}$ fulfils condition Terminal condition 2(ii).

(c) Assume that

$$\begin{aligned}&\forall\bar{x} \in \mathbb{X}_0,\ \epsilon > 0\ \ \exists T\ \forall t > T,\ U \in \mathcal{U},\ x \in \mathbb{X}\\ &\text{with } (x, t) \in \mathrm{REACH}(\bar{x}, t_0),\ J(x, t, U)\delta^{t-t_0} \le \epsilon.\end{aligned} \tag{8}$$

Then $V^{\mathrm{REACH}(\mathbb{X}_0,t_0)}$ fulfils Terminal conditions 2(i).

Proof. Rewrite the proof of Theorem 3 for the problem redefined as in the proof of Theorem 2. □

In some of earlier works, the following condition is considered as a part of the necessary condition for an optimal control $\bar{U}$.

$$\begin{aligned}&\text{If } \bar{U} \text{ is the optimal control and } \bar{V} \text{ the value function}\\ &\qquad\text{then for each } (\bar{x}, \bar{t}),\ \lim_{t\to\infty} \bar{\delta}^t V(X^{\bar{U}}_{\bar{x},\bar{t}}(t), t) = 0.\end{aligned} \tag{9}$$

In our model, in which the payoff does not have to be finite, this does not have to hold. The class of problems in which it cannot hold encompasses e.g. all nontrivial problems related to viability, whenever there is an initial condition for which there is no viable trajectory originating from it (e.g. the depletion state 0 in Fish Wars). On the other hand, for all (x, t) for which there exists a viable trajectory originating from (x, t) and the plus infinity payoff cannot be reached, then, obviously, Eq. (9) is fulfilled even in our larger class of problems, which is immediate as a consequence of Terminal condition 1(ii) and the fact that the payoff is well defined.

5.1. Showing importance of the assumption with an example

While Terminal conditions 1(ii) and 2(ii) are necessary without any additional assumptions, we have proven Terminal conditions 1(i) and 2(i) are necessary under Assumptions (7) and (8), respectively. In this section, we illustrate by an example in which Assumptions (7) and (8) are not fulfilled that the value function does not have to fulfil Terminal conditions 1(i) and 2(i). In this case, the solution of the Bellman Inclusion with the actual value function is not the optimal control.

Proposition 4. *Consider the dynamic optimization problem from Example 3.*

(a) The function $\bar{V}_3$ from Proposition 3 is the value function.

(b) Assumption (7) and Assumption (8) for $\mathbb{X}_0 \ne \{0\}$ do not hold.

Corollary 4. *Not checking Terminal conditions 1(i), although a solution of the Bellman equation (4) V is indeed the value function, may result in the Bellman Inclusion (5) with V returning the worst control instead of the optimal control. This is possible only when Assumption (7) does not hold.*

Proof. Immediate by Propositions 3, 4 and Theorem 3. □

6. Analysis of the usefulness of results by examples from the literature and comparison to other non-topological results

We start the analysis from a simplified version of the Fish Wars example from Levhari and Mirman (1980). A similar analysis can be repeated as a proof in most of the papers further developing their model, described in the introduction, and it completes their proofs without a need to impose additional constraints on the model.

Example 4. Consider a fishery with one species of fish, with the state denoting the biomass $\mathbb{X} = [0,1]$, the set of control parameters (fishing) $\mathbb{U} = [0,1]$, $D(x) = [0,x]$, the dynamics given by $f(x,u) = (x-u)^\alpha$ for some $\alpha \in (0,1)$ and the payoff $g(x,u) = \ln u$ with $\ln 0$ understood as $-\infty$, $\delta < 1$.

Proposition 5. *(a) The value function is $\bar{V}_4(x) = A\ln x + B$, while the unique optimal control is $\bar{U}_4(x) = ax$, with $A = \frac{1}{1-\delta\alpha} > 0$ and $a = \frac{1}{1+\delta\alpha A}$.*

(b) The terminal condition of Stokey et al. (1989)

$$\limsup_{t\to\infty} \delta^t \bar{V}_4(X(t)) = 0 \text{ for every admissible } X \tag{10}$$

as well as finiteness of lifetime utility assumption

$$J(\bar{x}, \bar{t}, U) \in \mathbb{R} \text{ for every admissible control } U \tag{11}$$

used in e.g. Kamihigashi (2005) does not hold *in this model and they* do not hold if we modify *the state set to* $(0,1]$ *and* $D(x)$ *to* $(0,x)$ *to guarantee finite current payoffs.*

We present the proof here to show how the problem is usually solved by the undetermined coefficients method and how easily the proof can be completed by checking the Terminal condition 1.

Proof. (a) Substituting $\bar{V}_4$ and $\bar{U}_4$ to the Bellman Equation and Bellman Inclusion yields unique constants A, B, and a (we skip the exact value of B as less interesting and influencing neither the optimal control nor the terminal condition).

Since $A\ln x$ is always nonpositive, Terminal condition 1(i) is trivially fulfilled.

Assume that for some $X(t)$, $\limsup_{t\to\infty} \delta^t(A\ln(X(t)) + B) < 0$. So, $\limsup_{t\to\infty} \delta^t \ln(X(t)) < 0$. Since $U(x) \le x$, $J(X(\bar{t}), \bar{t}, U) = \sum_{t=\bar{t}}^{\infty} \ln(U(X(t)))\delta^{t-\bar{t}} \le \delta^{-\bar{t}} \sum_{t=\bar{t}}^{\infty} \ln(X(t))\delta^t = -\infty$. So, Terminal condition 1(ii) is also fulfilled. So, by Theorem 1, $\bar{V}$ is the value function while $\bar{U}$ the optimal control, while by Theorem 3, there is no other optimal control.

(b) In the original example, every trajectory with $X(t) = 0$ for some t violates (10), while every control resulting in such X violates (11).

In the modified problem, define $\bar{X}(t) = \exp\left(-\delta^{-t}\right)$. $\ln \bar{X}(t)\delta^t = -1$, so, $\limsup_{t\to\infty} \delta^t(A\ln(\bar{X}(t)) + B) = -A < 0$. $\bar{X}$ is a trajectory: it corresponds to any control with $U(\bar{X}(t), t) = \bar{X}(t) - (\bar{X}(t+1))^{\frac{1}{\alpha}} \in (0, \bar{X}(t))$ with the initial condition $\bar{x} = \frac{1}{e}$. So, $\bar{X}$ violates (10). Since $U(\bar{X}(t), t) < \bar{X}(t)$, $\delta^t \ln(U(\bar{X}(t), t)) \to -\infty$, and U violates (11). □

The next example is the "pathological" case from Rincón-Zapatero and Rodríguez-Palmero (2003) Example 5. We show that Theorem 1 can be used to find the value function among multiple solutions of the Bellman Equation, and/or Theorem 3 can be used to reject a solution which is not the value function.

Example 5. Let $\mathbb{X} = \mathbb{U} = \mathbb{R}_+$, $D(x) = [0, 2x]$, $g(x,u) = -2x + u$, $f(x,u) = u$ and $\delta \in (\frac{1}{2}, 1)$.

Proposition 6. *The Bellman Equation has two linear solutions: $\bar{V}_5(x) \equiv 0$, which is the value function, and $\bar{V}_6(x) = -2x$, which does not fulfil the necessary Terminal condition 1(ii). The optimal control is $\bar{U}_5(x) = 2x$.*

Proof. We find the two solutions of the Bellman Equation by the undetermined coefficient method within the class of functions of the form $Ax + B$.

$\bar{V}_5$ obviously fulfils Terminal condition 1, so, by Theorem 1, it is the value function.

To reject $\bar{V}_6$, we note that the trajectory $X(t) = x_0 2^t$ is an admissible trajectory corresponding to $U(x) = 2x$. Since $\delta > \frac{1}{2}$, $\limsup_{t\to\infty} -2X(t)\delta^t = -\infty < 0$ whenever $x_0 \neq 0$, while $J(x_0, 0, U) = 0$. So, Terminal condition 1(ii) does not hold. □

Comparison with Kamihigashi's results. Next, we present the related results of Kamihigashi (2014a): Theorem 2.1, rewritten to the notation of this paper. It is stated for the case when all the functions and correspondences are independent of time.

Theorem 5. *Let B denote the Bellman operator on the set $\mathbb{V}$ of all functions $v : \mathbb{X} \to [-\infty, +\infty)$ defined by $(Bv)(x) = \sup_{u\in D(x)} g(x,u) + \delta v(f(x,u))$. Assume there exist two functions $v_-, v_+ : \mathbb{X} \to [-\infty, +\infty)$, $v_- \le v_+$ (in this context, $\le$ is the partial ordering given by inequality for all $x \in \mathbb{X}$) with $Bv_- \ge v_-$ and $Bv_+ \le v_+$ which fulfil the following terminal conditions*

$$\liminf_{t\to\infty} \delta^t v_-(X(t)) \ge 0 \text{ for every } U \text{ with finite } J(x,U); \tag{12}$$

$$\limsup_{t\to\infty} \delta^t v_+(X(t)) \le 0 \text{ for every trajectory } X. \tag{13}$$

Then the Bellman operator has a unique fixed point in the interval $[v_-, v_+]$, which is the value function $\bar{v}$ and for all $x \in \mathbb{X}$, $B^t v_-(x) \nearrow \bar{v}(x)$.

At the first sight, Conditions (12)–(13) seem weaker than Terminal condition 1. We shall prove, that they are not.

Proposition 7. *Conditions (12)–(13) with $v_- \le v_+$ imply that both v_- and v_+ fulfil Terminal condition 1.*

Proof. Take any control U which yields payoff greater than $-\infty$ and $X = X_x^U$. Then, by $v_- \le v_+$ and Eq. (12)–(13),

$0 \le \liminf_{t\to\infty} \delta^t v_-(X(t)) \le \liminf_{t\to\infty} \delta^t v_+(X(t)) \le \limsup_{t\to\infty} \delta^t v_+(X(t)) \le 0$ and

$0 \le \liminf_{t\to\infty} \delta^t v_-(X(t)) \le \limsup_{t\to\infty} \delta^t v_-(X(t)) \le \limsup_{t\to\infty} \delta^t v_+(X(t)) \le 0$. So, both $\delta^t v_-(X(t))$ and $\delta^t v_+(X(t))$ converge to zero.

By $v_- \le v_+$ and Eq. (13), for a control U yielding payoff $-\infty$, $\limsup_{t\to\infty} \delta^t v_-(X_x^U(t)) \le 0$.

So, both functions are assumed to fulfil Terminal condition 1(i). We rewrite Terminal condition 1(ii) in an equivalent form "If $J(x,U) > -\infty$, then $\limsup_{t\to\infty} \delta^t \bar{v}(X_x^U(t)) \ge 0$", which is fulfilled by both functions, since, as we have proven, they converge to zero for trajectories corresponding to such controls. □

So, the terminal condition which was a part of the sufficient condition from Theorem 1 and, under some additional assumption, a necessary condition for the solution of the Bellman equation to be the value function, is indirectly assumed for a function v_- which is used to calculate the value function by iterations. An interesting question is whether Terminal condition 1 is necessary also for the function which can be used for calculating the value function by iterations. It is worth adding that Theorem 5 has been proven in a more general context of controls for which the infinite series defining J does not have to be well defined. In such a case, lim sup or lim inf criterion was considered in the definition of the payoff J. Checking the necessity of an analogue of the Terminal condition 1 in such a generalized approach may be an interesting continuation of this paper.

Although the results of this paper have the mentioned above hidden common part with Kamihigashi (2014a), the approach of both papers differs substantially. The main focus of this paper is on the necessity of the terminal condition and a sufficient condition that is easy to use, as we can see in Examples 1–5.

This approach works immediately whenever multiple solutions of the Bellman equations can be calculated. By the necessity, not only is it going to work when the actual value function is within the set of calculated solutions, but also when it is not — to immediately reject the spurious value functions, as it has been done in Proposition 2 in which the only solution of the Bellman equation in the class of quadratic functions, natural in the linear–quadratic context of Example 1, is rejected. Such rejection has been the starting point to look for the actual value function, which in this case is less regular. The main effort of the procedure based on Theorem 2.1 of Kamihigashi (2014a) is related to finding two constraint functions v_- and v_+ to guarantee existence and uniqueness, and afterwards, calculating the value function as the pointwise limit of $B^t v_-$. Finding the constraint functions is immediate when the current payoff function is bounded, otherwise, it may be complicated.

7. Discontinuous problems

Let us emphasize again that the continuity or even semi-continuity of the current payoff and the available control correspondence with respect to the state-control pair is not required for the necessity and sufficiency results considered in this paper. Most of papers on the infinite horizon assume continuity of f, g and D. Exemptions are Kamihigashi and Roy (2007), in which only upper semi-continuity of f is required, and the order-theoretic results described in the introduction.

Discontinuity of the components appears obviously in many applications: ecological or epidemiological problems with regulations, e.g. environmental levies or fines for exceeding epidemiological social distancing constraints — prevalent during current COVID-19 epidemics; economic problems with quality or energy efficiency classes; profit optimization with the cost of switching on an additional production line; manipulating customers' opinions on the internet by a firm or fuel consumption in a car with switching gears.

Another class of dynamic optimization problems in which discontinuity may appear naturally, even in initially continuous problems, are dynamic or multistage games. To calculate a feedback Nash equilibrium one has to solve a dynamic optimization problem with feedback controls (*feedback strategies* in the game-theoretic language) of the other players treated as parameters. Given discontinuous strategies of the other players, the current payoff, the function determining the next stage state, and the available control correspondence of this parametrized optimization become discontinuous. It turns out that this concerns not only linear problems, in which bang–bang solutions are natural but even constrained linear–quadratic problems with concave payoffs with obvious economic interpretation, like Singh and Wiszniewska-Matyszkiel (2019), studying an extension of the problem from Example 1 to a dynamic game, in which instead of one owner of the fishery, there are two co-owners, each of them fishing in his/her own Exclusive Economic Zone with fish dispersing equally over the whole fishery during the closed season. In Singh and Wiszniewska-Matyszkiel (2019), it has been proven that all the symmetric feedback Nash equilibria are discontinuous with respect to the state variable and the resulting players' current payoff functions given strategies of the others in some of those equilibria are not even upper-semicontinuous in the state variable.

For Theorems 1–4, the resulting discontinuities of both types, including lack of upper-semicontinuity, do not cause any problems.

8. Conclusions and further research

In this paper, we have presented a study of infinite horizon deterministic optimal control problems with unbounded payoffs concentrated on the sufficient and necessary condition, with the focus on appropriate terminal conditions and their necessity. Without continuity and compactness assumptions, these necessity and sufficiency results can be used in a large class of problems, they make the selection from multiple solutions from the Bellman equation easy and they are a natural tool to conclude about the value function and optimal controls after reasoning based on the undetermined coefficient method to solve the Bellman Equation. We have illustrated the essentiality of the terminal conditions by examples of problems with obvious applications in which we have obtained either multiple solutions of the Bellman equation or the fact that the unique maximizer of its right hand side with the actual value function is the worst control. We have shown the applicability of the results by solving some incompletely solved problems from the literature. As a side effect, we have proven the uniqueness of the solution of the Bellman equation with Terminal condition 1 as well as the existence and uniqueness under an additional assumption.

Since the sufficient condition has been proven to be necessary under very weak assumptions, it seems that further generalizations in the class of deterministic problems of the very general form studied in this paper cannot be proven. However, those results can be generalized to stochastic problems, considered by e.g. Cruz-Suárez, Ilhuicatzi-Roldán, and Montes-de Oca (2014), Feinberg, Jaśkiewicz, and Nowak (2020), Jaśkiewicz and Nowak (2011), Jaśkiewicz and Nowak (2011), Matkowski and Nowak (2011). Another interesting extension of theoretical results contained in this paper may be by considering generalized discounting, like in e.g. Jaśkiewicz, Matkowski, and Nowak (2014), or more generally, recursive utility like in e.g. Le Van and Vailakis (2005) and Rincón-Zapatero and Rodríguez-Palmero (2007) or an attempt to analyse a Bellman-type approach to quasi-hyperbolic discounting as considered in Balbus, Reffett, and Woźny (2015, 2018).

Appendix. Proofs of results

In this Appendix, we prove the non-immediate results.

Proof of Theorem 2. In order not to write the elaborate proof like the proof of Theorem 1 from scratch, we first re-state the problem to make it not directly dependent on time. To do this, we extend the state space.

The new state space is first $\tilde{\mathbb{X}} = \mathbb{N} \times \mathbb{X}$ (we denote the first coordinate by $\tilde{x}_0$, while the normal state variable by $\tilde{x}_{\sim 0}$), the new current payoff $\tilde{g}(\tilde{x}, u) = g(\tilde{x}_{\sim 0}, u, \tilde{x}_0)$, the new function $\tilde{f}(\tilde{x}, u) = \begin{bmatrix} \tilde{x}_0 + 1 \\ f(\tilde{x}_{\sim 0}, u, \tilde{x}_0) \end{bmatrix}$, $\tilde{D}(\tilde{x}) = D(\tilde{x}_{\sim 0}, \tilde{x}_0)$.

Next, we rewrite Theorem 1 for the new problem in the time-independent way with the state set $\mathrm{REACH}(\mathbb{X}_0, t_0)$ instead of $\tilde{\mathbb{X}}$. This can be done since $\mathrm{REACH}(\mathbb{X}_0, t_0)$ is an invariant set of the new dynamics resulting from any choice of U, so it may be treated as a modified state space in a modified dynamic optimization problem.

To complete the proof, we return to the original formulation. □

Next, we prove Proposition 1. Although the proof can be just by checking the sufficient condition of Theorem 1 — we present the proof of Proposition 1 in a way in which the solutions are derived by the Ansatz method.

Proof of Proposition 1. First, let us assume that the value function is of the form $V(x) = k + gx + \frac{h}{2}x^2$. We look for a solution of the Bellman equation (4) in this class of functions.

Afterwards, we find u maximizing the right hand side of the Bellman Equation over the set of available decisions.

We check the first order condition for the internal u from the Bellman Inclusion (5) and get the value of u as follows

$$u = \frac{(hx - A)(1+\xi) + g}{(h - B(1+\xi))}. \tag{A.1}$$

Finally, we substitute the optimal u to the Bellman equation (4), which allows us to calculate the constants for which this equation is fulfilled. In this way, we obtain three sets of values of unknowns as follows:

$$k = 0,\ g = \hat{g},\ h = \hat{h}; \tag{A.2a}$$

$$k = \tilde{k},\ g, h = 0; \tag{A.2b}$$

$$h = 0,\ \text{arbitrary } g \neq 0, k(g) = \frac{(A - \delta g)^2}{2B(1-\delta)}. \tag{A.2c}$$

Since $h \leq 0$ for all such sets of constants, u defined by Eq. (A.1), if $u \in [0, (1+\xi)x]$, is the global maximizer and it is unique. We consider the following cases.

case 1. The values of unknowns k, g and h are as in (A.2a), which yields V^{false}.

The candidate for an optimal control, in this case, is equal to ξx, which, obviously, is less than $(1+\xi)x$ and the maximized function is strictly concave. So, it defines the unique maximizer for case 1 and $f(x, \xi x) \equiv x$.

This ends the proof of (b).

case 2. The values of unknowns k and h are as in (A.2b). Then, we have $\bar{V}_2(x) = \tilde{k}$ and the Bellman equation (4) has the form $\bar{V}_2(x) = \sup_{u\in[0,(1+\xi)x]} g(x, u) + \delta\tilde{k}$.

Therefore, a candidate for an optimal strategy of each player is independent of x and equal to $\bar{U}_2 \equiv \hat{u}$.

Note that for x close to 0, $\hat{u} > (1+\xi)x$, so, $\hat{u}$ is not admissible and $\tilde{k} \neq g(x, (1+\xi)x) + \delta\tilde{k}$. So, by Theorem 3, $\bar{V}_2(x)$ cannot be the value function.

case 3. Consider a combination of case 1 and case 2.

The only continuous combination of $\bar{V}^{\text{false}}$ and $\bar{V}_2$ with $\bar{V}(0) = 0$ is $\bar{V}_1$.

First, note that this $\bar{V}_1$ is differentiable and concave. The corresponding candidate for the optimal profile is $\bar{U}_1(x)$.

After derivation of the candidates for the value function and the optimal control, we have to prove that the Bellman Equation and Bellman Inclusion are really fulfilled by the piecewise defined functions.

The set of u for which $f(x, u) \leq \tilde{x}$, is denoted by $S_{\text{I}}(x)$, while the set of the remaining u by $S_{\text{II}}(x)$. $S_{\text{I}}(x)$ is always nonempty.

If for some x, $S_{\text{II}}(x) = \emptyset$, which may hold only for $x \leq \tilde{x}$, then for this x, the Bellman equation (4) reduces to $\bar{V}^{\text{false}}(x) = \sup_{u\in[0,(1+\xi)x]} g(x, u) + \delta\bar{V}^{\text{false}}(f(x, u))$ and the supremum in this case is attained at the zero derivative point ξx, so Eq. (5) is fulfilled (which we have already checked during calculation of coefficients in case 1).

So, consider $S_{\text{II}}(x) \neq \emptyset$. In this case, both $S_{\text{I}}(x)$ and $S_{\text{II}}(x)$ are non-empty. This situation can be decomposed into two cases.

(I) For $x \leq \tilde{x}$, the Bellman equation (4) can be rewritten as $\bar{V}^{\text{false}}(x) = \sup_{u\in[0,(1+\xi)x]} g(x, u) + \delta\bar{V}_1(f(x, u)) = \max\{\sup_{u\in S_{\text{I}}(x)} g(x, u) + \delta\bar{V}^{\text{false}}(f(x, u)), \sup_{u\in S_{\text{II}}(x)} g(x, u) + \delta\bar{V}_2(f(x, u))\}$. Since $\sup_{u\in S_{\text{I}}(x)} g(x, u) + \delta\bar{V}^{\text{false}}(f(x, u))$ is attained at the zero derivative point $\xi x \in S_{\text{I}}(x)$, ξx maximizes strictly concave $g(x, u) + \delta\bar{V}_1(f(x, u))$. So, the Bellman equation (4) is fulfilled and $\bar{U}_1(x)$ fulfils the Bellman Inclusion (5) for this x.

(II) If $x > \tilde{x}$, then the Bellman equation (4) can be rewritten as $\bar{V}_2(x) = \sup_{u\in[0,(1+\xi)x]} g(x, u) + \delta\bar{V}_1(f(x, u)) = \max\{\sup_{u\in S_{\text{I}}(x)} g(x, u) + \delta\bar{V}^{\text{false}}(f(x, u)), \sup_{u\in S_{\text{II}}(x)} g(x, u) + \delta\bar{V}_2(f(x, u))\}$. First, let us consider optimization over $S_{\text{II}}(x)$. In this case, $\hat{u} \in S_{\text{II}}(x)$.

Since $\hat{u}$ is a zero derivative point of strictly concave $g(x, u) + \delta\bar{V}_1(f(x, u))$, $\hat{u}$ is the maximum. Again, the supremum is attained at $\bar{U}_1(x)$, and Eq. (5) is fulfilled.

Therefore, in case 3, the Bellman equation (4) as well as the Bellman Inclusion (5) is fulfilled. Terminal condition 1 is obvious, since $\bar{V}_1$ is bounded, $\lim_{t\to\infty} \delta^t V_1(X(t)) = 0$ for every X. So, the function $\bar{V}_1$ is the value function, while $\bar{U}_1(x)$ is the optimal control.

The optimal control is unique, since Eq. (5) is necessary for a control to be optimal by Theorem 3, which ends the proof of (a).

(c) V^{false} does not fulfil Terminal condition 1(ii), since $\lim_{t\to+\infty} V^{\text{false}}(X^U(t))\delta^t = -\infty$ for X^U being the trajectory corresponding to the profile $U \equiv 0$ with a nonzero initial condition. Terminal condition 1(i) is fulfilled, since V^{false} is bounded from above. □

Proof of Proposition 2. (b) Similarly to the proof of Proposition 1, we solve the Bellman Equation assuming a quadratic value function and we obtain solutions given by Eq. (A.1)–(A.2c). Besides the three cases which appeared in the proof of Proposition 1, we consider additionally the case when the values of unknowns are as in (A.2c). In this case, the maximized function at the right hand side of the Bellman Equation is strictly concave in the control parameter and there are no constraints, so, the supremum is attained at the zero derivative point, constant. Another solution is V^{false} with U^{false}.

(a) The function $\bar{V}_2$ is a solution of the Bellman equation (4), while the control $\bar{U}_2$ is a solution of the Bellman Inclusion (5) and $\bar{V}_2$ fulfils Terminal condition 1, so, the sufficient condition is fulfilled.

(c) For V^{false}, it has been checked in the proof of Proposition 1. For $h = 0, g \neq 0$, $\lim_{t\to+\infty}(gX^0_{\bar{x},\bar{t}}(t) + k(g))\delta^t = g\bar{x}$, which is nonzero for $\bar{x} \neq 0$. Since the initial condition $\bar{x}$ is arbitrary, both Terminal conditions 1(i) and (ii) are violated.

In the restricted case with $\mathbb{X}_0 \neq \{0\}$, depending on g and signs of initial conditions in $\mathbb{X}_0$, Terminal condition 2(i) or (ii) is violated. □

Proof of Proposition 3. (a) $\text{Argmax}_{u\in[0,x]} \ln(u+1) + x - u = \{0\}$. So, 0 fulfils the Bellman Inclusion (5) with $\bar{V}_3$.

$\max_{u\in[0,x]} \ln(u+1) + x - u = \ln 1 + x = x$. So, $\bar{V}_3$ fulfils the Bellman equation (4).

0 results in payoff 0, while payoffs are nonnegative. So, 0 is the worst control.

(b) Terminal conditions 1(i) and 2(i) are violated by the trajectory corresponding to the zero control, since $\bar{V}(X(t)) = \bar{x}$. Terminal conditions 1(ii) and 2(ii) are trivially fulfilled, since $\bar{V}$ is nonnegative. □

Proof of Theorem 3. (a) The proof of (a) follows similar lines to the proofs of Theorems 4.2 and 4.4 of Stokey et al. (1989).

Consider a pair (x, t) for which the right hand side of the Bellman Equation is well defined. If $g(x, u, t)$ is finite for all u, then it is exactly the result of Stokey et al. (1989).

The only difference is the case when $g(x, u, t)$ is equal to $-\infty$.

If this does not hold for all u, then the supremum is not $-\infty$ and the result is also immediate. The opposite situation may happen only in the case when the value function at the resulting state at the next stage is less than $+\infty$ since otherwise, the right hand side of the Bellman Equation is not well defined. If this holds for all currently admissible controls, then the value function at (x, t) is $-\infty$ and every control parameter fulfils the Bellman Inclusion, otherwise, it does not influence the right hand side of the Bellman Equation. So, the Bellman Equation and the Bellman Inclusion are fulfilled also at such (x, t).

(b) Assume that there exist $U \in \mathcal{U}$ and $(\bar{x}, t_0) \in \mathbb{X} \times \mathbb{N}$ such that for $X = X^U_{\bar{x},t_0}$, $\limsup_{t\to\infty} \bar{V}(X(t), t)\delta^t < 0$ and $J(\bar{x}, t_0, U) > -\infty$. So, $\exists\, t_k \to \infty$ such that $\bar{V}(X(t_k), t_k)\delta^{t_k-t_0} \to \eta \in [-\infty, 0)$.

$\bar{V}(X(t_k), t_k) \geq J(X(t_k), t_k, U)$ implies $\delta^{t_k-t_0}\bar{V}(X(t_k), t_k) \geq \delta^{t_k-t_0}J(X(t_k), t_k, U)$. Since $J(\bar{x}, t_0, U)$ is well defined and greater than $-\infty$, $\delta^{t_k-t_0}J(X(t_k), t_k, U)$ is either convergent to 0 or to $+\infty$. So, the left hand side converges to $\eta < 0$, while the right hand side converges to a non-negative limit — zero or $+\infty$, which is a contradiction.

(c) Assume that there exist $U \in \mathcal{U}$ and $(\bar{x}, t_0) \in \mathbb{X} \times \mathbb{N}$ such that for $X = X^U_{\bar{x},t_0}$, $\limsup_{t\to\infty} \bar{V}(X(t), t)\delta^t > 0$.

So, there exists a sequence $t_k \to \infty$ such that $\lim_{k\to\infty} \bar{V}(X(t_k), t_k)\delta^{t_k-t_0} = \eta > 0$. Consequently, $\bar{V}(X(t_k), t_k)\delta^{t_k-t_0} \neq -\infty$ for large k—$\bar{V}(X(t_k), t_k)$ is finite or $+\infty$.

Consider $\lim_{k\to\infty} \bar{V}(X(t_k), t_k)\delta^{t_k-t_0}$ finite. By this and Eq. (3), $\forall \epsilon_1 > 0\, \exists \bar{k} \in \mathbb{N}\, \forall k > \bar{k}$, $\exists U^{\epsilon_1,k}$ such that $J\left(X(t_k), t_k, U^{\epsilon_1,k}\right) \geq \bar{V}(X(t_k), t_k) - \epsilon_1$. So, $\forall \epsilon_2 > 0, \exists \bar{k} \in \mathbb{N}\, \forall k > \bar{k}, J\left(X(t_k), t_k, U^{\epsilon_1,k}\right)\delta^{t_k-t_0} \geq \eta - \epsilon_2$, which contradicts Assumption (7).

Next, assume that $\lim_{k\to\infty} \bar{V}(X(t_k), t_k)\delta^{t_k-t_0} = +\infty$. Then, by Eq. (3), $\forall M > 0 \exists \bar{k} \in \mathbb{N}\, \forall k > \bar{k} \exists U^M \in \mathcal{U}$ such that $J\left(X(t_k), t_k, U^M\right)\delta^{t_k-t_0} \geq M$. Thus, Assumption (7) cannot be fulfilled. □

Proof of Proposition 4. (a) $\bar{V}_3$ is the value function as the limit of the value functions at the initial time for the finite horizon truncations of the problem, i.e. the problems restricted to finite horizon T, in which the payoff is

$$J^T(\bar{x}, \bar{t}, U) = \sum_{t=\bar{t}}^{T} g(X(t), U(X(t), t), t)\delta^{t-\bar{t}}. \tag{A.3}$$

The optimum of the truncated problem can be calculated from the finite horizon Bellman equation and Bellman inclusion, but in this specific case it is easier to use the Jensen inequality by strict concavity of the current payoff function. We get $U^T(x, t) = \frac{x}{T+1-t}$ for $t \leq T$, zero otherwise, which implies that the value function for the truncated problem is $V^T(x, t) = (T + 1 - t)\ln\left(\frac{x}{T+1-t} + 1\right)$. So, for all t, $\lim_{T\to\infty} V^T(x, t) = \bar{V}_3(x)$. Next, we prove that $\bar{V}_3 = \bar{V}$.

By the definition, $\bar{V}(x, t) \geq J(x, t, U^T) = V^T(x, t)$ for every (x, t). So, if we take the limit with respect to T, we get $\bar{V} \geq \bar{V}_3$.

To prove the converse inequality, take any (x, t) and a control U^ϵ such that $\bar{V}(x, t) \leq J(x, t, U^\epsilon) + \epsilon$. Since the series in (1) is absolutely convergent, for every $\epsilon_1 > 0$, there exists $\bar{T}$ such that for $T > \bar{T}$, $J(x, t, U^\epsilon) \leq J^T(x, t, U^\epsilon) + \epsilon_1 \leq V^T(x, t) + \epsilon_1$.

So, $\bar{V}(x, t) \leq V^T(x, t) + \epsilon + \epsilon_1$. Taking the limit ends the proof of (a).

(b) Consider a sequence of the controls U_t which are 0 besides time t, at which they are equal to x. $J(x, t, U_t) = \ln(x + 1)$. So, Assumptions (7) and (8) for $\mathbb{X}_0 \neq \{0\}$ are not fulfilled. □

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