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Inference for Local Distributions at High Sampling Frequencies: A Bootstrap Approach*

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September 4, 2019

Abstract

We study inference for the local innovations of Itô semimartingales. Specifically, we construct a resampling procedure for the empirical CDF of high-frequency innovations that have been standardized using a nonparametric estimate of its stochastic scale (volatility) and truncated to rid the effect of “large” jumps. Our locally dependent wild bootstrap (LDWB) accommodates issues related to the stochastic scale and jumps as well as account for a special block-wise dependence structure induced by sampling errors. We show that the LDWB replicates first and second-order limit theory from the usual empirical process and the stochastic scale estimate, respectively, in addition to an asymptotic bias. Moreover, we design the LDWB sufficiently general to establish asymptotic equivalence between it and a nonparametric local block bootstrap, also introduced here, up to second-order distribution theory. Finally, we introduce LDWB-aided Kolmogorov-Smirnov tests for local Gaussianity as well as local von-Mises statistics, with and without bootstrap inference, and establish their asymptotic validity using the second-order distribution theory. The finite sample performance of CLT and LDWB-aided local Gaussianity tests are assessed in a simulation study and an empirical application. Whereas the CLT test is oversized, even in large samples, the size of the LDWB tests is accurate, even in small samples. The empirical analysis verifies this pattern, in addition to providing new insights about the fine scale distributional properties of innovations to equity indices, commodities and exchange rates.

Keywords: Bootstrap inference, High-frequency data, Itô semimartingales, Kolmogorov-Smirnov test, Stable processes, von-Mises statistics.

JEL classification: C12, C14, C15, G1

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1 Introduction

Itô semimartingales comprise an important class of continuous time processes that are widely used in finance and economics, among others, to describe the evolution of financial asset prices, exchange rates, interest rates, commodities, asset return volatility, derivatives prices, volume of trades, innovations in aggregate consumption as well as network traffic. This broad class of processes include jump-diffusions as the, unequivocally, most commonly adopted subclass of models across a variety of applications, see e.g., Andersen & Benzoni (2012) and many references therein. This subclass characterizes the innovations to the process of interest as a stochastic differential equation of the form,

$$dZ_t = \alpha_t dt + \sigma_t dW_t + dY_t, \quad (1)$$

where the drift α_t and volatility σ_t are processes with càdlàg paths, W_t is a standard Brownian motion and Y_t is an Itô semimartingale of the pure-jump type (formal assumptions are given below). The model in (1) allows Z_t to follow a drift, subject to innovations of the mixed Gaussian type (σ_t being the stochastic mixing scale) and display larger, and more infrequent, jumps. Moreover, by allowing for correlation between the increments $d\sigma_t$ and dZ_t , the model can capture leverage and volatility feedback effects, working through either continuous or discontinuous jump channels. Importantly, despite allowing for general continuous and discontinuous sample paths as well as correlation between the various components of the model, the specification (1) is consistent with no arbitrage in financial markets, e.g., Back (1991) and Delbaen & Schachermayer (1994).

While the behavior of (1) may be very complex at longer time horizons, its structure simplifies considerably at high sampling frequencies. To see this, suppose that t is restricted to the interval $[0, 1]$, and we consider its (infill) asymptotic behavior for some shrinking time interval from t to $t + sh$ with $h \rightarrow 0$, then the Brownian motion will dominate the drift and jump components, provided that the stochastic scale, σ_t , is non-vanishing. That is, for fixed $0 \leq t < s \leq 1$,

$$h^{-1/2} \frac{Z_{t+sh} - Z_t}{\sigma_t} \xrightarrow{d} W'_{t+s} - W'_s, \quad \text{as } h \rightarrow 0, \quad (2)$$

where W'_t is a standard Brownian motion (again, technical details are given below). Hence, (2) highlights that the model in (1) is locally mean-zero and mixed Gaussian with stochastic scale, illustrating that the model is capable of generating fat-tail returns, even at high-sampling frequencies, through the mixture-of-distributions effect, e.g., Clark (1973). Moreover, it makes strong predictions about the local distributional properties of the *standardized* innovations; namely Gaussianity. This assumption is fundamental, not only when describing the dynamics of asset and state variables (such as those listed above) as well as when pricing derivatives, but also for many multivariate problems where correlations are critical to the analysis, e.g., portfolio allocation. The intricate relation (2) facilitates testing of this fundamental assumption. This feature is important, especially since local Gaussianity rules out another class of Itô semimartingales of the pure-jump type, which has recently been demonstrated,

using different nonparametric techniques, to provide accurate descriptions of the local distributional properties of various assets at high sampling frequencies, see e.g., Todorov & Tauchen (2011b), Jing, Kong & Liu (2012), Andersen, Bondarenko, Todorov & Tauchen (2015), and Hounyo & Varneskov (2017). Such pure-jump semimartingales may be characterized similarly to (1) with, however, the Brownian increments replaced by a Lévy jump process of infinite variation. Under general conditions, the latter can be shown to be locally equivalent to a stable process, S_t , with activity (or tail) index $1 < \beta < 2$, e.g., Todorov & Tauchen (2012). As is illustrated by Figure 1, a stable distribution with activity index lower than the Gaussian boundary case $\beta = 2$ is characterized by having fatter, possibly asymmetric, tails and larger excess kurtosis than a comparable Gaussian distribution. Hence, the selection between modeling paradigms – jump-diffusions and pure-jump semimartingales – amounts to testing whether the local (and standardized) increments at high sampling frequencies are better described by a Gaussian or stable distribution with $1 < \beta < 2$. If these increments, indeed, are stable, this alters the way we need to study a plethora of economic phenomena. Specifically, rather than relying on jump-diffusions, this suggests the use of parametric models with stable innovations, or other pure-jump Lévy processes of infinite variation, to capture the dynamics of asset returns, as in, e.g., Mandelbrot (1961, 1963), Fama (1963), Fama & Roll (1968) and, more recently, Carr, Geman, Madan & Yor (2002) and Kelly & Jiang (2014); to price derivatives such as options, as seen in, e.g., Carr & Wu (2003, 2004) and Andersen, Fusari, Todorov & Varneskov (2018); as well as to model return volatility, e.g., Barndorff-Nielsen & Shephard (2001), Carr, Geman, Madan & Yor (2003) and Todorov, Tauchen & Gryniv (2014); network traffic, e.g., Mikosch, Resnik, Rootzen & Stegeman (2002); and electricity prices, e.g., Klüppelberg, Meyer-Brandt & Schmidt (2010). Moreover, Aït-Sahalia & Jacod (2009) and Todorov & Tauchen (2011a), among others, show that the magnitude of β is essential for the estimation of, and inference on, risk measures such as power variation.¹

In this paper, we seek to draw inference on the local distributional properties of Z_t . This is particularly challenging in the present setting vis-a-vis (1), since Z_t , in addition to exhibiting local distributional properties that may either be Gaussian or stable, can have a stochastic drift, α_t , a stochastic scale, σ_t , and “residual” jumps, Y_t . Specifically, we consider bootstrap inference based on the empirical CDF statistic by Todorov & Tauchen (2014), which, asymptotically, recover the distribution of the locally leading term, W_t or S_t , by nonparametrically standardizing and truncating high-frequency increments of Z_t in its construction. While the standardization and truncation alleviate estimation errors generated by σ_t and Y_t , respectively, to recover information about the locally leading term, they show that a bias-correction is generally needed and develop asymptotic central limit theory for the first-order estimation error – its empirical process – as well as the (higher-order) estimation

¹When the local innovations have $1 < \beta < 2$, rather than $\beta = 2$, this introduces a β -dependent scale for the power variation measures to be asymptotically bounded. Heuristically, since the leading component is $O_p(n^{-1/\beta})$, realized variance will sum up $O_p(n^{-2/\beta})$ elements, requiring a re-scaling with $n^{\beta/2-1}$ for the measure to be $O_p(1)$. If the process additionally exhibit finite variation jumps, Y_t , and the statistic is re-scaled, this complicates detection, estimation and interpretation of the latter as their respective squared sum is $O_p(1)$, before scaling. Aït-Sahalia & Jacod (2009) and Todorov & Tauchen (2011a) provide detailed analyses of power variations in such settings, which remain useful risk measures that need, however, to be estimated, and interpreted, with care, requiring knowledge of β .

sampling error that arises from having replaced the unobservable scale, σ_t , with a nonparametric estimate. This second-order distribution theory is utilized in designing a Kolmogorov-Smirnov (KS) test for local Gaussianity.² However, despite having strong theoretical appeal, the KS test performs unstably, and often unsatisfactory, in their finite sample Monte Carlo study, rejecting (wrongfully) either 10.3% or 32.8% of the times when the nominal size is 1% or 5%, respectively, thus highlighting the need for improved inference and testing procedures. The bootstrap represents a natural alternative to inference based on central limit theory. However, in addition to facing the same challenges as the original statistics in Todorov & Tauchen (2014), the bootstrap procedures, we seek to develop, will not only need to replicate the first-order limit theory, but also account for the bias and the second-order distribution. This combination of issues is unprecedented in the bootstrap literature.

There are alternative ways to test for local Gaussianity in the present setting, e.g., by testing whether the activity index, β , is strictly less than two; see, among others, Aït-Sahalia & Jacod (2009, 2010), Todorov & Tauchen (2011a), Hounyo & Varneskov (2017) and references therein. Todorov & Tauchen (2014, Table 2), however, show that tests based on the empirical CDF is (much) more powerful in finite samples than corresponding tests based on β estimates. Hence, when applied to testing, our new bootstrap seeks to improve upon an already powerful framework.

The first contribution of the paper is to provide a locally dependent wild bootstrap (LDWB) procedure that enables inference on the local distribution of the leading term in Z_t . To this end, following to the discussion above, we accommodate issues related to the stochastic scale and residual jumps as well as design the bootstrap to account for a special block-wise dependence structure created by the sampling errors that arise when replacing σ_t with a nonparametric estimate. By accounting for such features, we show that the LDWB is not only asymptotically valid for the first-order distributional properties, the empirical process limit, but also for the second-order distribution *and* it accommodates the bias. Second, we show that our LDWB framework is sufficiently general to nest a nonparametric local block bootstrap (NLBB), also developed in this paper, thereby establishing asymptotic equivalence between two separate bootstrap paradigms up to a second-order distribution, in a general semimartingale setting. Third, we utilize the bootstrap in designing new Kolmogorov-Smirnov tests for local Gaussianity and establishes their asymptotic properties. Fourth, we design von-Mises statistics based on nonparametrically standardized and truncated high-frequency returns, provide a LDWB procedure for such and establish their asymptotic properties. Both are new to high-frequency financial econometrics and both rely on the second-order distribution theory for the LDWB.

The theoretical contributions of this paper represent advances for two different literatures; bootstrap inference for empirical processes and high-frequency econometric inference and hypothesis testing. First, our bootstrap is related to the dependent wild bootstrap procedures in Shao (2010) and Doukhan, Lang, Leucht & Neumann (2015), who consider inference on the time series mean of a stationary dependent process and its empirical process, respectively, as well as the block bootstraps in, among

²Strictly speaking, the test is for local mixed Gaussianity of the increments, dZ_t , or, asymptotically equivalently, local Gaussianity of the standardized and truncated increments. We will explicate when necessary.

others, Bickel & Freedman (1981), Bühlmann (1994), and Naik-Nimbalkar & Rajarshi (1994), who consider inference for empirical processes in either i.i.d. or stationary and dependent settings. In particular, and relative to previous bootstraps, the LDWB accommodate non-stationarities in the increments of the observed process through σ_t and Y_t , a special block-wise dependence structure, arising from nonparametric standardization errors, and, finally, it asymptotically replicates first *and* second-order distribution theory. Moreover, we specify the external random variables in the design sufficiently general to establish asymptotic equivalence of the LDWB and NLBB, up to second-order distribution theory. Such results are hitherto not available in the bootstrap literature, even under simplifying assumptions for Z_t . In relation to the high-frequency financial econometrics literature, we provide new (bootstrap) inference techniques for local distributions in infill asymptotic settings, introduce local von-Mises statistics to the literature (with and without the bootstrap), as well as provide new nonparametric bootstrap-aided tests for local Gaussianity. The contributions closest to ours are Todorov & Tauchen (2014), as explained above, and the bootstraps for power variations in Gonçalves & Meddahi (2009), Hounyo & Varneskov (2017), Hounyo (2019), and Dovonon, Gonçalves, Hounyo & Meddahi (2019), who consider either local Gaussian or local stable settings. It is important to note, however, that direct adaptations of their bootstrap designs will result in inference procedures that lose all dependence on the original data; see Remark 3 and Appendix B below.³

In addition to the theoretical contributions, we examine the finite sample properties of Kolmogorov-Smirnov tests for local Gaussianity based on either the central limit theory (CLT) in Todorov & Tauchen (2014), the LDWB or the NLBB via simulations. Consistent with the results in Todorov & Tauchen (2014), we find severe size distortions for the CLT-based test, even in large samples. In contrast, the LDWB-aided test enjoys accurate size, even in small samples, and good power properties. In fact, its power is similar to an (infeasible) size-adjusted CLT test. The NLBB performs similarly to the LDWB, albeit with slightly worse size properties, showing the benefits of our general bootstrap framework. To illustrate the usefulness of the procedure, we test for local Gaussianity in high-frequency futures data on three different asset classes; equity indices, foreign exchange rates and commodities. Interestingly, we find that the high-frequency innovations to equity indices and commodities are well-described as (mixed) Gaussian. In contrast, we strongly reject local Gaussianity for the exchange rate series, which, on the other hand, are better described as locally stable with tail index in the 1.80 to 1.90 range. Moreover, we verify the size results from the simulation study; the CLT rejects uniformly more often than the LDWB-aided test for equity indices and commodities.

The paper proceeds as follows. Section 2 introduces the semimartingale framework, the statistics of interest and reviews some critical results. Section 3 introduces the locally dependent wild bootstrap

³Our paper is also related to Andersen, Bollerslev & Dobrev (2007), who consider testing for Gaussianity using daily, or other sufficiently sparse returns, that are standardized by realized measures constructed from high-frequency data. Using a sequential procedure, they accommodate jumps and stochastic volatility in their design. However, their testing framework is based on a long time span of “standardized” returns over fixed time spans as well as high-frequency observations, whereas we exclusively rely on the latter. Hence, our time interval is generally shrinking and we utilize local spot volatility measure. Both are crucial for the feasible limit theory and facilitates testing without a long time span of data. Finally, and importantly, we carry out testing using bootstrap procedures.

procedure and establishes its asymptotic properties as well as its equivalence to the nonparametric local block bootstrap. Section 4 provides new bootstrap-aided Kolmogorov-Smirnov tests for local Gaussianity and local von-Mises statistics. Section 5 contains the simulation study, and Section 6 provides the empirical analysis. Finally, Section 7 concludes. Appendices A-C have additional assumptions, theory, proofs, technical results and implementation details.

2 A General Semimartingale Framework

This section introduces a general class of semimartingales, the formal assumptions for the theoretical analysis as well as provides examples of such processes in applied work. Moreover, we define the empirical statistics of interest for the bootstrap analysis in the remainder of the paper.

2.1 Setup and Assumptions

Suppose the process Z is defined on a filtered probability space, $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, where the information filtration $(\mathcal{F}_t) \subseteq \mathcal{F}$ is an increasing family of σ -fields satisfying \mathbb{P} -completeness and right continuity. Specifically, assume that Z obeys a semimartingale process that generalizes (1) and has the following dynamics

$$dZ_t = \alpha_t dt + \sigma_t dS_t + dY_t, \quad 0 \leq t \leq 1, \quad (3)$$

where α_t and σ_t are (\mathcal{F}_t) -adapted processes with càdlàg paths, Y_t is a pure-jump process of finite variation, and S_t is a stable process with stability index $1 < \beta \leq 2$, whose (log-)characteristic function is defined as

$$\ln \mathbb{E} [e^{iuS_t}] = -t|cu|^\beta (1 - i\gamma \operatorname{sign}(u) \tan(\pi\beta/2)), \quad (4)$$

where $\gamma \in [-1, 1]$ controls its skewness. We have depicted the density of S_t for various choices of the activity index β and skewness γ in Figure 1, illustrating how the two affect the skewness, kurtosis and, in particular, the tails of the density. Note that for $\beta = 2$ and $c = 1/2$, the semimartingale process in (3) reduces to the jump-diffusion model in (1). When $1 < \beta < 2$, on the other hand, Z_t is a pure-jump semimartingale of infinite variation for which the innovations to S_t still dominate the drift and “residual” jump process, Y_t , at fine time scales. That is, under the regularity conditions to be outlined below, we have $h^{-1/\beta}(Z_{t+sh} - Z_t)/\sigma_t \xrightarrow{d} S'_{t+s} - S'_t$ as $h \rightarrow 0$ with convergence holding under the Skorokhod topology on the space of càdlàg functions, where S'_t is a Lévy process with a distribution identical to the one implied by (4). Yet, despite similar scaling properties, the fine scale behavior generated by (4) allows for much richer dynamics relative to a standard Gaussian.

Before proceeding to the assumptions, let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and (E, \mathcal{E}) denote an auxiliary Polish space, defined in addition to the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Moreover, going forward, we write $S_t = W_t$ when $\beta = 2$ and $c = 1/2$ when emphasizing the model being a jump-diffusion.

Assumption 1. Z_t satisfies (3) with the following conditions on its components:

(a) The process Y_t obeys

$$Y_t = \int_0^t \int_E \delta^Y(s, x) \mu(ds, dx),$$

where $\mu(ds, dx)$ is a Poisson measure on the space $\mathbb{R}_+ \times E$, which is characterized by the Lévy measure $\nu(dx)$, and $\delta^Y(t, x)$ is some predictable function on $\Omega \times \mathbb{R}_+ \times E$.

(b) $|\sigma_t|^{-1}$ and $|\sigma_{t-}|^{-1}$ are strictly positive.

(c) σ_t is a semimartingale process of the form

$$\sigma_t = \sigma_0 + \int_0^t \tilde{\alpha}_u du + \int_0^t \tilde{\sigma}_{u-} dS_u + \int_0^t \tilde{\sigma}'_u dW'_u + \int_0^t \int_E \delta^\sigma(s, x) \mu(ds, dx),$$

where W'_t is a standard Brownian motion independent of S_t , irrespective of β ; the triplet $\tilde{\alpha}_t$, $\tilde{\sigma}_t$ and $\tilde{\sigma}'_t$ are processes with càdlàg paths; and $\delta^\sigma(t, x)$ is some predictable function on $\Omega \times \mathbb{R}_+ \times E$. Moreover, $\tilde{\sigma}_t$ and $\tilde{\sigma}'_t$ are both Itô semimartingales with càdlàg paths and whose jumps being integrals of some predictable functions, $\delta^{\tilde{\sigma}}(t, x)$ and $\delta^{\tilde{\sigma}'}(t, x)$, with respect to $\mu(ds, dx)$.

(d) There exists a sequence of stopping times T_p , increasing to infinity, and, for each p , a non-negative function $\phi_p(x)$ on (E, \mathcal{E}) satisfying $\nu(x : \phi_p(x) \neq 0) < \infty$ such that, for $t \leq T_p$,

$$|\delta^Y(t, x)| \wedge 1 + |\delta^\sigma(t, x)| \wedge 1 + |\delta^{\tilde{\sigma}}(t, x)| \wedge 1 + |\delta^{\tilde{\sigma}'}(t, x)| \wedge 1 \leq \phi_p(x).$$

Assumption 1 deserves a few comments. First, the regularity conditions are similar to those imposed by Todorov & Tauchen (2014, Assumption B). The only two differences are that we refrain from imposing $S_t = W_t$ here, but will rather make this restriction later, and we allow increments of S_t to enter σ_t in 1(c). The main motivation behind for these minor departures is that we wish to state a set of unified conditions under which all subsequent asymptotic results hold, in conjunction with restrictions on β . Todorov & Tauchen (2014) use two separate sets of regularity conditions for their consistency and central limit theory (CLT) analysis. Since we are mainly concerned with inference using bootstrap methods, we will invoke the stronger of those assumptions from the outset.⁴

Second, the conditions in Assumption 1 are very mild. The Itô semimartingale condition on the stochastic scale σ_t is satisfied in most applications. Moreover, there are no restrictions on the dependence between the residual jumps, Y_t , and the triplet $(S_t, \alpha_t, \sigma_t)'$. This implies that Z_t not necessarily inherits the tail properties of S_t at *all* frequencies and may be driven by a tempered stable process, which can display tail behavior that is very different from that of a stable process.

Third, as examples of jump-diffusion models are plentiful in the literature (see the introduction for references), we end this subsection by providing examples of models that obeys a subclass of the

⁴Note that the conditions in Assumption 1 are very similar to those in related work, e.g., Aït-Sahalia & Jacod (2010), Todorov & Tauchen (2011a), Hounyo & Varneskov (2017) and references therein.

models in (3) with $1 < \beta < 2$, and which have been successfully applied to describe processes in economics and finance, thus providing powerful alternatives to models with local Gaussianity.

Example 1. *Barndorff-Nielsen & Shephard (2001) introduce a non-Gaussian OU process for volatility, and Todorov et al. (2014) considers an exponential version of the model, thus accommodating a broader class of leading jump processes. For illustration, let μ and κ be positive constants, then the exponential version may be written as $\ln \sigma_t = \mu + V_t$ with $dV_t = -\kappa + dL_t$ where the driving Lévy process, L_t , behaves locally (as $h \rightarrow 0$) like a stable process with characteristic function (4).*

Example 2. *Let $S_t = S_t(\beta, \gamma)$ be a β -stable random variable with skewness γ . Moreover, let r and q be the risk-free and dividend rate, respectively, and let μ be a convexity adjustment, then the log-stable option pricing model by Carr & Wu (2003) is defined as $dZ_t = (r - q + \mu)dt + \sigma dS_t$.*

Example 3. *The framework in (3) and (4) accommodate a general class of time changed stochastic processes. Specifically, we can write $Z_t = X_{z_t}$ where X_t is a Lévy process and z_t is an increasing process with càdlàg paths. In such a setting, Monroe (1978) shows that all semimartingale processes may be written as a time-changed Brownian motion, and Sato (1999) that Lévy processes subordinated by a positive Lévy process yields new Lévy processes. Wu (2008) give several practical examples of such processes, and Clark (1973) and Ané & Geman (2000), among others, use the time-change framework to jointly model the number of trades, transaction times and asset returns.*

2.2 Empirical Statistics of Interest

First, let Z_t be observed at an equidistant time grid $t_i \in [0, 1]$, for $i = 0, \dots, n$, and write the high-frequency increments as $\Delta_i^n Z = Z_{t_i} - Z_{t_{i-1}}$.⁵ Next, divide the fixed time interval into blocks, each of which containing k_n increments with $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$. For each block, we compute an estimate of the spot variation σ_t^2 by means of the local bipower variation statistic⁶,

$$\widehat{V}_{n,j} = \frac{\pi}{2} \frac{n}{k_n - 1} \sum_{i=(j-1)k_n+2}^{jk_n} |\Delta_{i-1}^n Z| |\Delta_i^n Z|, \quad j = 1, \dots, \lfloor n/k_n \rfloor. \quad (5)$$

Despite $\widehat{V}_{n,j}$ being consistent for σ_t^2 , we will need to use a modified estimator to scale the high-frequency increments and forming the empirical CDF. Specifically, as we need independence between

⁵The definitions of the high-frequency statistics, including notation, follow Todorov & Tauchen (2014) closely.

⁶Todorov & Tauchen (2014) also analyzes the subsequent empirical CDF statistic using a localized truncated realized variance estimator of the spot variation, generating a similar rate of convergence, but smaller bias and variance. Our locally dependent wild bootstrap will, with minor modifications, apply to this design as well.

the i th increment $\Delta_i^n Z$ in the numerator and the denominator, we will exclude said increment as

$$\widehat{V}_{n,j}(i) = \begin{cases} \frac{k_n-1}{k_n-3} \widehat{V}_{n,j} - \frac{\pi}{2} \frac{n}{k_n-3} |\Delta_i^n Z| |\Delta_{i+1}^n Z| & \text{for } i = (j-1)k_n + 1; \\ \frac{k_n-1}{k_n-3} \widehat{V}_{n,j} - \frac{\pi}{2} \frac{n}{k_n-3} (|\Delta_{i-1}^n Z| |\Delta_i^n Z| + |\Delta_i^n Z| |\Delta_{i+1}^n Z|), & \\ \text{for } i = (j-1)k_n + 2, \dots, jk_n - 1; \\ \frac{k_n-1}{k_n-3} \widehat{V}_{n,j} - \frac{\pi}{2} \frac{n}{k_n-3} |\Delta_{i-1}^n Z| |\Delta_i^n Z|, & \text{for } i = jk_n. \end{cases} \quad (6)$$

Now, to form the empirical CDF and devise its feasible CLT, Todorov & Tauchen (2014) selects only the first m_n increments on each block and require $1 > m_n/k_n \rightarrow 0$ as $n \rightarrow \infty$. Intuitively, this is to ensure that the estimation errors from $\widehat{V}_{n,j}(i)$, both its finite sample bias and variance, vanish sufficiently fast upon averaging relative to the contribution of each block to the empirical CDF. This implies that the total number of increments used for estimation is given by

$$N_n(\alpha, \varpi) = \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} 1 \left\{ \frac{\sqrt{n} |\Delta_i^n Z|}{\sqrt{\widehat{V}_{n,j}}} \leq \alpha n^{1/2-\varpi} \right\}, \quad (7)$$

where $\alpha > 0$ and $\varpi \in (0, 1/2)$, and that the empirical CDF is formed as,

$$\widehat{F}_n(\tau) = \frac{1}{N_n(\alpha, \varpi)} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} 1 \left\{ \frac{\sqrt{n} \Delta_i^n Z}{\sqrt{\widehat{V}_{n,j}(i)}} \leq \tau \right\} 1 \left\{ \frac{\sqrt{n} |\Delta_i^n Z|}{\sqrt{\widehat{V}_{n,j}}} \leq \alpha n^{1/2-\varpi} \right\}. \quad (8)$$

In addition to standardizing the increments by the stochastic scale, the empirical CDF in (8) truncates the increments of Z_t to reduce the impact of larger jumps on $\widehat{F}_n(\tau)$.⁷ While the latter is strictly not needed to obtain consistency and CLT for the latter, the truncation serves to reduce the higher-order bias in $\widehat{F}_n(\tau)$ due to jumps. Additionally, note that $N_n(\alpha, \varpi)/(\lfloor n/k_n \rfloor m_n) \xrightarrow{\mathbb{P}} 1$. Finally, before stating the asymptotic results due to Todorov & Tauchen (2014), the following assumption collects rate conditions on the tuning parameters determining the block sizes, k_n and m_n .⁸

Assumption 2. m_n and k_n satisfy either of the following two conditions as $n \rightarrow \infty$,

- (a) $k_n \asymp n^q$, for some $q \in (0, 1)$ and $m_n \rightarrow \infty$;
- (b) $k_n \asymp n^q$, for some $q \in (0, 1/2)$ and $m_n/k_n \rightarrow 0$ such that $(nm_n)/k_n^3 \rightarrow \lambda \geq 0$.

⁷Lee & Mykland (2012) utilize a related vanishing block-length design to test for finite activity jumps, Y_t , in a Brownian semimartingale contaminated by microstructure noise. Specifically, they use averaged (log-)returns on shrinking blocks and standardize these with a “global” variance estimate, relying on constant volatility. Whereas the blocks alleviate noise in their case, we require shrinking blocks to estimate the time-varying spot variation. Moreover, since m_n observations is used on each block, rather than one, this increases the rate of convergence of the empirical CDF with $\sqrt{m_n}$.

⁸All asymptotic results for empirical CDF statistics, and their bootstrap analogs, are stated as uniformly in τ over compact subsets of \mathbb{R} , that is, equipped with a uniform topology in the spirit of van der Vaart (1998, Chapter 19). In Todorov & Tauchen (2014), this is referred to this as converging locally uniformly in τ .

Lemma 1. *If Assumptions 1 and 2(a) hold, then, uniformly in τ over compact subsets of \mathbb{R} ,*

$$\widehat{F}_n(\tau) \xrightarrow{\mathbb{P}} F_\beta(\tau),$$

where $F_\beta(\tau)$ is the CDF of $\sqrt{2/\pi} \frac{S_1}{\mathbb{E}|S_1|}$ and S_1 is the value of the β -stable process S_t at $t = 1$. In particular, $F_2(\tau)$ equals the CDF of a standard Gaussian random variable $\Phi(\tau)$.

Lemma 2. *If Assumptions 1 and 2(b) hold, and let $S_t = W_t$, i.e., Z_t be a jump-diffusion, then, uniformly in τ over compact subsets of \mathbb{R} ,*

$$\widehat{F}_n(\tau) - \Phi(\tau) = \widehat{H}_{n,1}(\tau) + \widehat{H}_{n,2}(\tau) + H_3(\tau)/k_n + o_p(1/k_n)$$

where $\sqrt{[n/k_n]m_n}(\widehat{H}_{n,1}(\tau), \sqrt{k_n/m_n}\widehat{H}_{n,2}(\tau)) \xrightarrow{d} (H_1(\tau), H_2(\tau))$ with $H_1(\tau)$ and $H_2(\tau)$ being two mean-zero independent Gaussian processes with covariance functions,

$$\begin{aligned} \text{Cov}[H_1(\tau_1), H_1(\tau_2)] &= \Phi(\tau_1 \wedge \tau_2) - \Phi(\tau_1)\Phi(\tau_2) \\ \text{Cov}[H_2(\tau_1), H_2(\tau_2)] &= \left(\frac{\tau_1 \Phi'(\tau_1)}{2} \frac{\tau_2 \Phi'(\tau_2)}{2} \right) \left(\left(\frac{\pi}{2} \right)^2 + \pi - 3 \right), \end{aligned} \quad (9)$$

for $\tau_1, \tau_2 \in \mathbb{R}$. Finally,

$$H_3(\tau) = \frac{\tau^2 \Phi''(\tau) - \tau \Phi'(\tau)}{8} \left(\left(\frac{\pi}{2} \right)^2 + \pi - 3 \right). \quad (10)$$

Lemma 1 shows that the CDF of standardized increments $\Delta_i^n Z$ may be estimated consistently, as long as their fine scale behavior belongs to the class of stable processes described by (4). Lemma 2 improves this result for the jump-diffusion model, showing that a CLT holds with a rate of convergence that may be arbitrarily close to \sqrt{n} , depending on m_n and k_n . The limiting distribution, however, are affected by the nonparametric standardization of the increments. For specificity, whereas $H_1(\tau)$ is well-known from Donsker's theorem for empirical processes, e.g., van der Vaart (1998), the additional components $H_2(\tau)$ and $H_3(\tau)$ are lower-order estimation errors and an asymptotic bias, respectively, induced by use of the estimate $\widehat{V}_{n,j}$ rather than the latent σ_t . Importantly, all components of the limit depend only on τ , not on σ_t , meaning that this result is amenable to feasible inference.

By utilizing their CLT result in Lemma 2, Todorov & Tauchen (2014) design a Kolmogorov-Smirnov-type test for local Gaussianity of Z_t , i.e., $\mathcal{H}_0 : S_t = W_t$, as

$$\widehat{\text{KS}}_n(\mathcal{A}) = \sup_{\tau \in \mathcal{A}} \sqrt{N_n(\alpha, \varpi)} \left| \widehat{F}_n(\tau) - \Phi(\tau) \right|, \quad (11)$$

where $\mathcal{A} \subset \mathbb{R} \setminus 0$ denotes a finite union of compact sets with positive Lebesgue measure. The critical region of the test is $C_n(\theta, \mathcal{A}) = \{\widehat{\text{KS}}_n(\mathcal{A}) > q_n(\theta, \mathcal{A})\}$, where $\theta \in (0, 1)$, and $q_n(\theta, \mathcal{A})$ is the $(1 - \theta)$ th quantile of

$$\sup_{\tau \in \mathcal{A}} \left| H_1(\tau) + \sqrt{\frac{m_n}{k_n}} H_2(\tau) + \sqrt{\frac{m_n}{k_n}} \frac{\sqrt{n}}{k_n} H_3(\tau) \right|.$$

The test for local Gaussianity, $\widehat{\text{KS}}_n(\mathcal{A})$, similarly to the empirical CDF, contain two additional terms compared with standard Kolmogorov-Smirnov distribution testing, $H_2(\tau)$ and $H_3(\tau)$, arising from the use of nonparametric, and noisy, estimates of the stochastic scale, σ_t , when standardizing the increments. Whereas the second term is of strictly lower order by $m_n/k_n \rightarrow 0$ as $n \rightarrow \infty$, the bias term has first-order impact since $\sqrt{(m_n n)/k_n^3} \rightarrow \sqrt{\lambda} \geq 0$. Hence, not only does the limit theory in Lemma 2 aid the correction of systematic testing errors by accounting for $H_3(\tau)$, the explicit utilization of higher-order asymptotic theory through $H_2(\tau)$ may generate improved testing properties in finite samples, where the ratio m_n/k_n can be non-trivial. However, when gauging the size results for their test in Todorov & Tauchen (2014, Table 1), $\widehat{\text{KS}}_n(\mathcal{A})$ is seen to be very sensitive to block size k_n and may display large distortions, e.g., rejecting either 10.3% or 32.8% of the times when the nominal size is 1% or 5%, respectively, highlighting the need for improved testing procedures.

In what follows, we will study inference for the empirical CDF as well as testing for local Gaussianity using bootstrap methods to restore the size properties of such tests. However, Lemma 2 shows that this is particularly challenging in the present setting since such bootstrap procedures need not only to replicate the first-order distribution theory, reflected by $H_1(\tau)$, asymptotically, but also to account for the asymptotic bias, $H_3(\tau)$, as well as to replicate the higher-order limit theory, $H_2(\tau)$.

Remark 1. *The central limit theory is provided on compact sets of τ , $\mathcal{A} \subset \mathbb{R} \setminus 0$, since the error in the estimation of the CDF for $\tau \rightarrow \pm\infty$ due to large jumps is affected by truncation. As a result, the bootstrap methods, we develop below, will similarly apply to the set \mathcal{A} .*

Remark 2. *Market microstructure noise is a concern when sampling the observations at very high frequencies, for example, more frequently than every minute or every 15 ticks, see, e.g., Hansen & Lunde (2006) and Bandi & Russell (2008). Suppose, in this case, that the observed increments decompose $\Delta_i^n \tilde{Z} = \Delta_i^n Z + \Delta_i^n \mathcal{N}$ where \mathcal{N}_{t_i} , $i = 1, \dots, n$ are i.i.d. random variables, defined on a product extension of the original probability space and are independent of the filtration \mathcal{F} . Then, Todorov & Tauchen (2014) shows that the empirical CDF converges to the CDF of standardized noise increments, which differs from $\Phi(\tau)$, thus providing a different violation of local Gaussianity. We will detail the potential impact of noise in our empirical application below.*

3 Bootstrapping the Empirical CDF at High Frequency

In this section, we introduce a new and general resampling procedure - the locally dependent wild bootstrap - to draw inference on the empirical CDF in (8) as well as for testing whether Z_t is better described by a jump-diffusion model (1) against the alternative in (3) with $1 < \beta < 2$, that is, to test local Gaussianity against distributions with fatter tails and, possibly, skewness. Specifically, the bootstrap resamples centered, standardized and dependent observations using a (possibly, dependent) external random variable. We establish the asymptotic properties of the procedure as well as discuss the similarities and differences between related bootstrap procedures in the classical time series and empirical process literature, in particular, a nonparametric local block bootstrap.

3.1 Bootstrap Notation

As is standard in the bootstrap literature, \mathbb{P}^* , \mathbb{E}^* and \mathbb{V}^* denote the probability measure, expected value and variance, respectively, induced by the resampling and is, thus, conditional on a realization of the original time series. For any bootstrap statistic $Z_n^* \equiv Z_n^*(\cdot, \omega)$ and any (measurable) set A , we write $\mathbb{P}^*(Z_n^* \in A) = \mathbb{P}^*(Z_n^*(\cdot, \omega) \in A) = \Pr(Z_n^*(\cdot, \omega) \in A | \mathcal{X}_n)$, where \mathcal{X}_n denotes the observed sample. Moreover, we say $Z_n^* \xrightarrow{\mathbb{P}^*} 0$ in probability- \mathbb{P} (or $Z_n^* = o_p^*(1)$ in probability- \mathbb{P}) if for any $\varepsilon > 0$, $\delta > 0$, $\lim_{n \rightarrow \infty} \mathbb{P}[\mathbb{P}^*(|Z_n^*| > \delta) > \varepsilon] = 0$. Similarly, $Z_n^* = O_p^*(1)$ in probability- \mathbb{P} if for all $\varepsilon > 0$ there exists an $M_\varepsilon < \infty$ such that $\lim_{n \rightarrow \infty} \mathbb{P}[\mathbb{P}^*(|Z_n^*| > M_\varepsilon) > \varepsilon] = 0$. Finally, for a sequence of random variables (or vectors) Z_n^* , a definition of weak convergence (convergence in distribution) in probability- \mathbb{P} is needed. Hence, we write $Z_n^* \xrightarrow{d^*} Z$ as $n \rightarrow \infty$, if, conditional on the sample, Z_n^* converges weakly to Z under \mathbb{P}^* , for all samples contained in a set with probability- \mathbb{P} approaching one.

3.2 The Local Dependent Wild Bootstrap for the Empirical CDF

The framework in Section 2 presents several challenges that are unprecedented in the bootstrap literature, e.g., the combination of the general class processes in (3), the infill asymptotic setting and the need for replication of higher-order central limit theory. To overcome such challenges, we design a new and general resampling procedure - the locally dependent wild bootstrap (LDWB) - which is inspired by the dependent wild bootstraps (DWBs) in Shao (2010) and Doukhan et al. (2015), but, as will be detailed below, differs in subtle, yet important ways, to remain valid in the present setting.

First, let us define

$$X_{(j-1)k_n+i} \equiv 1 \left\{ \frac{\sqrt{n}\Delta_{(j-1)k_n+i}^n Z}{\sqrt{\widehat{V}_{n,j}((j-1)k_n+i)}} \leq \tau \right\} 1 \left\{ \frac{\sqrt{n}|\Delta_{(j-1)k_n+i}^n Z|}{\sqrt{\widehat{V}_{n,j}}} \leq \alpha n^{1/2-\varpi} \right\}, \quad (12)$$

for $j = 1, \dots, \lfloor n/k_n \rfloor$ and $i = 1, \dots, m_n$, and use this to write

$$\widetilde{F}_n(\tau) = \frac{N_n(\alpha, \varpi)}{\lfloor n/k_n \rfloor m_n} \widehat{F}_n(\tau) = \frac{1}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=1}^{m_n} X_{(j-1)k_n+i}. \quad (13)$$

The random increments and empirical CDF in (12) and (13), respectively, illustrate the differences between the present bootstrap setting and the corresponding in Doukhan et al. (2015), who also consider DWB inference for empirical processes. In our case, the problem is more challenging due to the distributional properties of (3) may differ at coarse and fine time scales, depending on S_t and Y_t , which necessitates an infill asymptotic approach to estimation and the identification of the locally dominant stochastic component, S_t . Moreover, the process (3) is allowed to have a stochastic scale (volatility, if Gaussian), time-varying drift and display jumps, in contrast with the stationarity requirement for the data generating process in Doukhan et al. (2015, Assumption A1). Third, the nonparametric standardization of the increments in (12) creates a nonlinear m_n -dependence within blocks (that is,

across i), which impact the bootstrap design as well as its asymptotic theory. In particular, and as highlighted by Lemma 2, our local DWB need not only to replicate the first-order asymptotic theory, it needs to account for an asymptotic bias as well as to replicate the higher-order limit theory, generated by the nonparametric estimates of the stochastic scale used for the standardization.

Specifically, our LDWB resamples the centered, locally (and nonparametrically) standardized and truncated increments in (12) as follows

$$X_{(j-1)k_n+i}^* = \tilde{F}_n(\tau) + \frac{N_n(\alpha, \varpi)}{\lfloor n/k_n \rfloor m_n} \left(X_{(j-1)k_n+i} - \tilde{F}_n(\tau) \right) v_{(j-1)k_n+i}^*, \quad (14)$$

where v_i^* , $i = 1, \dots, n$, is a sequence of external random variables subject to mild regularity conditions, which are formalized below. The bootstrap variables in (14) may, then, be utilized in designing a new inference procedure for the empirical CDF at high (i.e., infill) sampling frequencies as

$$\begin{aligned} \hat{F}_{W,n}^*(\tau) &= \frac{1}{N_n(\alpha, \varpi)} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=1}^{m_n} X_{(j-1)k_n+i}^* \\ &= \hat{F}_n(\tau) + \frac{1}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=1}^{m_n} \left(X_{(j-1)k_n+i} - \tilde{F}_n(\tau) \right) v_{(j-1)k_n+i}^*. \end{aligned} \quad (15)$$

The LDWB decomposes into the empirical CDF, $\hat{F}_n(\tau)$, capturing the “mean” of the bootstrap statistic and an “innovation” aimed at capturing its distribution. The asymptotic properties of (15), however, depend crucially on v_i^* , and we impose the following, general, conditions:

Assumption DWB. *Define sets of observations $\mathcal{M}_{n,j} = \{(j-1)k_n + i; i = 1, \dots, m_n\}$ for the blocks $j = 1, \dots, \lfloor n/k_n \rfloor$. The sequence of random variables v_i^* , $i = 1, \dots, n$, is independent of the observed sample path \mathcal{X}_n . Moreover, it is either stationary or block-stationary, that is, with the random variables being the same within a block, $\mathcal{M}_{n,j}$, and stationary across blocks $\mathcal{M}_{n,j}$ and $\mathcal{M}_{n,j'}$ where $j \neq j'$. Finally, the sequence satisfies the following regularity conditions:*

- (a) $\mathbb{E}[v_i^*] = 0$, $\mathbb{V}[v_i^*] \rightarrow 1$ and $\mathbb{E}[|v_i^*|^4] < \infty$.
- (b) When the sequence v_i^* is stationary, $\text{Cov}(v_i^*, v_g^*) \rightarrow C_{i,g}$ for $i \neq g$ where $C_{i,g} \geq 0$ is a nonrandom constant. When v_i^* is block-stationary, $\text{Cov}(v_i^*, v_g^*) \rightarrow 1$ for $i, g \in \mathcal{M}_{n,j}$ and $\text{Cov}(v_i^*, v_g^*) \rightarrow \tilde{C}_{i,g}$ for $i \in \mathcal{M}_{n,j}$, $g \in \mathcal{M}_{n,j'}$ and $j \neq j'$, where $\tilde{C}_{i,g} \geq 0$ is a nonrandom constants.
- (c) v_i^* is b_n -dependent with $\sum_{r=1}^{\lfloor n/k_n \rfloor m_n} \text{Cov}(v_1^*, v_r^*) = O(b_n)$ for some $b_n/m_n \rightarrow \rho \geq 0$ as $n \rightarrow \infty$.

Together with the decompositions in (14) and (15), Assumption DWB highlight some important features of the LDWB. First, the centering of the external random variable in the resampling implies that $\mathbb{E}^*[\hat{F}_{W,n}^*(\tau)] = \hat{F}_n(\tau)$, that is, the LDWB implicitly corrects for the asymptotic bias in the empirical CDF. Second, time series dependence in $X_{(j-1)k_n+i}$ plays a different role in our setting

compared with in Shao (2010) and Doukhan et al. (2015). Whereas they seek to replicate a lead-lag covariance structure of the observations, needing a condition of the form $\text{Cov}(v_s^*, v_r^*) \rightarrow 1$ as $n \rightarrow \infty$, dependence in the present setting is created by the unwarranted estimation errors in $\widehat{V}_{n,j}$, which are perfectly dependent within a given block $j = 1, \dots, \lfloor n/k_n \rfloor$, but independent across blocks, generating a tradeoff between the rate of convergence, the asymptotic bias and the impact from the higher-order distribution. This particular within-block across-blocks structure explains why we allow for both stationary and “block stationary” external random variables, which can be designed to replicate such dependencies. Third, it is important to note that the leading impact from these estimation errors are generated by Brownian increments (see Lemmas A.1-A.2 in the appendix), which have trivial lead-lag dependence. Hence, we accommodate $\text{Cov}(v_i^*, v_j^*) \rightarrow C_{i,j}$ where $C_{i,j} \geq 0$ is a generic nonrandom constant as well as dependence that does not match the blocks, i.e., the case $b_n/m_n \rightarrow 0$. In fact, in the infill asymptotic limit, since the stochastic scale in (3) is approximately constant over a block, we allow $v_i^* \sim \text{i.i.d.}(0, 1)$, subject to a bounded fourth moment. Fourth, we need to impose an upper bound on the dependence, $b_n/m_n \rightarrow \varrho \geq 0$, since b_n controls the asymptotic order of the “noise” coming from the nonparametric estimator $\widehat{V}_{n,j}$ in the resampling, similarly to m_n in the original statistic. Fifth, despite it strictly not being needed to replicate the distribution theory in Lemma 2, it may be preferable designing the bootstrap with $\text{Cov}(v_i^*, v_j^*) \rightarrow 1$ and $b_n \asymp m_n$, since, e.g., Lemma A.2(b) in the appendix shows that this would aid the replication of higher-order covariance from the second-order distribution term, $H_2(\tau)$. Finally, whereas Doukhan et al. (2015) require v_i^* to be Gaussian, we avoid parameterizing its distribution. This is critical for the (asymptotic) analysis of the similarities between the LDWB and a local nonparametric block bootstrap in the next section.

These features of the resampling, in conjunction with the standardization and truncation of the increments in (12) allow us to accommodate the array of additional challenges in the present setting and replicate the asymptotic inference of the bias-corrected empirical process,

$$\widehat{\mathcal{G}}_n(\tau) = \sqrt{N_n(\alpha, \varpi)} \left(\widehat{F}_n(\tau) - \Phi(\tau) - H_3(\tau)/k_n \right), \quad \text{i.e.} \quad \mathcal{G}_n(\tau) \equiv H_1(\tau) + \sqrt{\frac{m_n}{k_n}} H_2(\tau), \quad (16)$$

using the LDWB in (15), hence, up to second order. This is formalized in the following theorem.

Theorem 1. *Suppose the conditions of Lemma 2 as well as Assumption DWB hold. Then, uniformly in τ over compact subsets of \mathbb{R} , it follows, as $n \rightarrow \infty$, that*

$$(a) \quad \widehat{\mathcal{F}}_{W,n}^*(\tau) \equiv \sqrt{N_n(\alpha, \varpi)} (\widehat{F}_{W,n}^*(\tau) - \widehat{F}_n(\tau)) = (\widehat{H}_{n,1}^*(\tau) + \widehat{H}_{n,2}^*(\tau)) + o_p^*(1), \text{ where, with the two mean-zero Gaussian functions } H_1(\tau) \text{ and } H_2(\tau) \text{ being defined as in Lemma 2,}$$

$$\sqrt{\lfloor n/k_n \rfloor m_n} \left(\widehat{H}_{n,1}^*(\tau), \sqrt{k_n/m_n} \widehat{H}_{n,2}^*(\tau) \right) \xrightarrow{d^*} (H_1(\tau), H_2(\tau)), \quad \text{in probability-}\mathbb{P}.$$

$$(b) \quad \sup_{x \in \mathbb{R}} \left| \mathbb{P}^* \left(\widehat{\mathcal{F}}_{W,n}^*(\tau) \leq x \right) - \mathbb{P} \left(\widehat{\mathcal{G}}_n(\tau) \leq x \right) \right| \xrightarrow{\mathbb{P}} 0.$$

Theorem 1 demonstrates that our LDWB for nonparametrically standardized and truncated in-

crements replicates the asymptotic distribution of the bias-corrected empirical CDF statistic up to second order. Not only is this feature achieved in the general setting (3), allowing for time-varying drift, stochastic volatility and jumps in the underlying process of interest as well as mild conditions on the external random variables, the central limit theory goes well-beyond the corresponding results for the respective DWBs in Shao (2010) and Doukhan et al. (2015), who provide first-order limits, which, in our setting, is equivalent to establishing $\hat{\mathcal{F}}_{W,n}^*(\tau) \xrightarrow{d^*} H_1(\tau)$, in probability- \mathbb{P} . Similar comments apply to classical results in the bootstrap literature for empirical processes, e.g., for the i.i.d setup in Bickel & Freedman (1981) as well as for block bootstrap methods applied to stationary and dependent processes in Bühlmann (1994) and Naik-Nimbalkar & Rajarshi (1994). Hence, both the LDWB procedure and its second-order asymptotic theory are new to the bootstrap literature. Furthermore, we formally analyze the similarities between block bootstrap methods and the LDWB in the next section. Finally, and as indicated in Section 2.2, the replication of second-order limit theory is very important in the present setting, as it alleviates the inference errors due to the use of a nonparametric spot volatility estimator $\hat{V}_{n,j}$, converging at a slower rate $n^{1/4}$, instead of the latent σ_t .

Remark 3. *Assumption DWB accommodates local Gaussian resampling, that is, $v_i^* \sim N(0, 1)$. However, it is important to note that this bootstrap, using (14), is distinct from the local Gaussian bootstrap for power variation statistics in Hounyo (2019), who resamples the increments $\Delta_i^n Z$ and establishes third-order refinements in a Brownian semimartingale setting. In fact, in Appendix B, we show that the use of this “standard” local Gaussian resampling scheme looses all dependence on the original data in the present setting and, thus, no longer provide bootstrap inference for the empirical CDF, but rather can be interpreted as a simulation based inference procedure.*

Remark 4. *In addition to local Gaussian resampling, several locally dependent processes satisfies Assumption DWB. Two examples, following Shao (2010) and Doukhan et al. (2015), see also the bootstraps in Leucht & Neumann (2013) and Smeekes & Urbain (2014), are autoregressive (AR) and moving average (MA) processes, defined for $i = 1, \dots, n$ as*

$$v_i^* = e^{-1/b_n} v_{i-1}^* + \xi_i \quad \text{and} \quad v_i^* = \varsigma_i + \dots \varsigma_{i-b_n+1},$$

respectively, with $\xi_i \sim N(0, 1 - e^{-2/b_n})$ and $\varsigma_i \sim N(0, 1/b_n)$ are both i.i.d. The finite sample properties of both resampling procedures are examined in the simulation study.

3.3 The Local DWB vs The Nonparametric Block Bootstrap

Whereas DWB procedures are relatively new to the resampling literature, starting with Shao (2010), block bootstrap methods for dependent processes have been actively researched since the seminal contributions by Carlstein (1986), Kunsch (1989) and Liu & Singh (1992), who study various time series problems, and by Bühlmann (1994) and Naik-Nimbalkar & Rajarshi (1994), who consider inference for empirical processes. Hence, as a natural alternative to the LDWB, and inspired by the extant literature,

we propose a nonparametric local block bootstrap (NLBB). Moreover, we will formally show that our LDWB is sufficiently general to nest the NLBB, thus providing a theoretical link between the two separate strands of the resampling literature, in a general setting.

First, for the design of the NLBB, we, once again, utilize that the original time series, $X_{(j-1)k_n+i}$ with $j = 1, \dots, \lfloor n/k_n \rfloor$ and $i = 1, \dots, m_n$, has a special nonlinear block-dependence structure across i for a given j , generated by the standardization with the nonparametric estimate $\hat{V}_{n,j}$. To this end, define a sequence of blocks $B_j = \{X_{(j-1)k_n+i}; i = 1, \dots, m_n\}$ for $j = 1, \dots, \lfloor n/k_n \rfloor$, then our proposed resampling procedure draws $\lfloor n/k_n \rfloor$ blocks randomly with replacement and patches them together to form a bootstrap series, inspired by, e.g., the non-overlapping block bootstrap in Carlstein (1986). The resampling, thus, preserves the m_n -dependence within each block as well as the asymptotic independence between blocks. To formalize the discussion, let \mathcal{I}_j , $j = 1, \dots, \lfloor n/k_n \rfloor$, be i.i.d random variables distributed uniformly on $\{1, \dots, \lfloor n/k_n \rfloor\}$, then we may write

$$X_{(j-1)k_n+i}^\diamond \equiv X_{(\mathcal{I}_j-1)k_n+i}, \quad i = 1, \dots, m_n \quad \text{and} \quad j = 1, \dots, \lfloor n/k_n \rfloor, \quad (17)$$

and use these to define block bootstrap (BB) versions of $\tilde{F}_n(\tau)$ and $\hat{F}_n(\tau)$ as

$$\tilde{F}_{BB,n}^*(\tau) = \frac{1}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=1}^{m_n} X_{(j-1)k_n+i}^\diamond, \quad \hat{F}_{BB,n}^*(\tau) = \frac{\lfloor n/k_n \rfloor m_n}{N_n(\alpha, \varpi)} \tilde{F}_{BB,n}^*(\tau), \quad (18)$$

respectively. Next, let $p_n = \lfloor n/k_n \rfloor$ be the number of blocks, then it is important to note that representation (18) may equivalently be written using a sequence of multinomial random variables with probability $1/p_n$ and number of trials p_n , defined as $\zeta_{p_n,j}^\diamond$, $j = 1, \dots, p_n$. Specifically,

$$\tilde{F}_{BB,n}^*(\tau) = \frac{1}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \zeta_{p_n,j}^\diamond \sum_{i=1}^{m_n} X_{(j-1)k_n+i} \quad (19)$$

where $\zeta_{p_n,j}^\diamond$ signify the number of times the j th block, B_j , has been (re-)drawn randomly from the total set of blocks. By the properties of multinomial random variables, it follows that

$$\mathbb{E}[\zeta_{p_n,j}^\diamond] = 1, \quad \mathbb{V}[\zeta_{p_n,j}^\diamond] = 1 - 1/p_n, \quad \text{Cov}(\zeta_{p_n,j}, \zeta_{p_n,i}) = -1/p_n \quad \text{when} \quad i \neq j,$$

and, importantly, that $\sum_{j=1}^{p_n} \zeta_{p_n,j}^\diamond = p_n$. Now, by utilizing these properties and defining the external random variable $v_{(j-1)k_n+i}^\diamond = \zeta_{p_n,j}^\diamond - 1$ for $i = 1, \dots, m_n$ across blocks $j = 1, \dots, \lfloor n/k_n \rfloor$, we may rewrite the representation (19) using addition and subtraction as

$$\tilde{F}_{BB,n}^*(\tau) = \tilde{F}_n(\tau) + \frac{1}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=1}^{m_n} \left(X_{(j-1)k_n+i} - \tilde{F}_n(\tau) \right) v_{(j-1)k_n+i}^\diamond, \quad (20)$$

thus on the same form as the LDWB in (15). Indeed, the following lemma establishes that the sequence

of random variables $v_{(j-1)k_n+i}^\diamond$ satisfy the regularity conditions imposed in Assumption DWB.

Lemma 3. *Let the blocks $\mathcal{M}_{n,j}$, $j = 1, \dots, \lfloor n/k_n \rfloor$, be defined as in Assumption DWB, then the sequence of observations $v_{(j-1)k_n+i}^\diamond$, with $i = 1, \dots, m_n$ and $j = 1, \dots, \lfloor n/k_n \rfloor$, is block stationary and satisfy,*

- (a) $\mathbb{E}[v_i^\diamond] = 0$, $\mathbb{V}[v_i^\diamond] = 1 - 1/p_n$ and $\mathbb{E}[|v_i^\diamond|^4] < \infty$.
- (b) $\text{Cov}(v_i^\diamond, v_g^\diamond) = 1 - 1/p_n$ for $i, g \in \mathcal{M}_{n,j}$.
- (c) $\text{Cov}(v_i^\diamond, v_g^\diamond) = -1/p_n$ for $i \in \mathcal{M}_{n,j}$, $g \in \mathcal{M}_{n,j'}$ and $j \neq j'$.
- (d) v_i^\diamond is m_n -dependent with $\sum_{i=1}^{\lfloor n/k_n \rfloor m_n} \text{Cov}(v_1^\diamond, v_i^\diamond) = o(m_n)$.

Hence, Theorem 1 and Lemma 3 may be combined to show:

Corollary 1. *Uniformly in τ over compact subsets of \mathbb{R} , it follows, as $n \rightarrow \infty$, that*

- (a) $\hat{\mathcal{F}}_{BB,n}^*(\tau) \equiv \sqrt{N_n(\alpha, \varpi)}(\hat{F}_{BB,n}^*(\tau) - \hat{F}_n(\tau)) = (\hat{H}_{n,1}^*(\tau) + \hat{H}_{n,2}^*(\tau)) + o_p^*(1)$, where, with the two mean-zero Gaussian functions $H_1(\tau)$ and $H_2(\tau)$ being defined as in Lemma 2,

$$\sqrt{\lfloor n/k_n \rfloor m_n} \left(\hat{H}_{n,1}^*(\tau), \sqrt{k_n/m_n} \hat{H}_{n,2}^*(\tau) \right) \xrightarrow{d^*} (H_1(\tau), H_2(\tau)), \quad \text{in probability-}\mathbb{P}.$$

- (b) $\sup_{x \in \mathbb{R}} \left| \mathbb{P}^* \left(\hat{\mathcal{F}}_{BB,n}^*(\tau) \leq x \right) - \mathbb{P} \left(\hat{\mathcal{G}}_n(\tau) \leq x \right) \right| \xrightarrow{\mathbb{P}} 0$.

Lemma 3 and Corollary 1 are intriguing, demonstrating that the general class of LDWBs nests the NLBB and, consequently, that the latter also replicates the second-order distribution theory for the empirical CDF. The nesting result is related to prior results on the exchangeability of weighted bootstraps for the empirical process. Specifically, in a setting with i.i.d. observations, Præstgaard & Wellner (1993) show that the seminal nonparametric i.i.d. bootstrap by Efron (1979) is nested within a general class weighted bootstraps for empirical processes that replicates the asymptotic distribution of a Brownian bridge (that is, of $H_1(\tau)$). Moreover, Shao (2010) establishes that the bias and variance of the DWB for long-run variance estimation are second-order equivalent to those for the tapered block bootstrap of Paparoditis & Politis (2001, 2002), whose properties are generally favorable to those of moving block bootstraps, e.g., Kunsch (1989) and Liu & Singh (1992). Hence, our result provides additional insights into the relation between bootstrap paradigms. First, the LDWB can be interpreted as a generally weighted bootstrap, with Assumption DWB providing sufficient conditions on the weights. Second, the asymptotic equivalence between DWBs and BBs hold for the general class of processes (3), hence not confined to i.i.d. observations as assumed by prior studies, and it holds for both first and second-order central limit theory. Both dimensions significantly generalizes existing discussions in Præstgaard & Wellner (1993) and Shao (2010). Finally, the second-order replication of the central limit theory generalizes prior first-order results for BBs in Carlstein (1986) as well as for empirical processes in Bühlmann (1994) and Naik-Nimbalkar & Rajarshi (1994).

Remark 5. *The NLBB is designed using non-overlapping blocks, as in Carlstein (1986), due to natural block-dependence of the nonparametrically standardized data. It may be feasible to consider moving blocks as well, if the differential dependence within a (moving) block is accounted for, e.g., by an additional external variable. The design of such a resampling procedure, including its (asymptotic) relation to the LDWB, is not straightforward and we leave it for further research.*

Remark 6. *The representation in NLBB is reminiscent of the blockwise wild bootstrap for spectral testing of white noise against serial dependence in Shao (2011). Specifically, using a similar block structure as above, Shao (2011) proposes to use an external i.i.d. variable with $\mathbb{E}[u_i^*] = 0$, $\mathbb{V}[u_i^*] = 1$ as well as $\mathbb{E}[|u_i^*|^4] < \infty$. Hence, by the same arguments provided for the NLBB, one can show that the blockwise bootstrap is, similarly, nested within the LDWB class in the present, general, setting.*

4 Testing for Local Gaussianity at High Frequencies

This section introduces new bootstrap-aided tests for local Gaussianity of (3). First, and similarly to Todorov & Tauchen (2014), we provide a LDWB Kolmogorov-Smirnov (KS) test. Second, we propose new Cramér-von Mises (CM) statistics for the empirical CDF at high frequencies and provide associated tests for local Gaussianity, based on either the limit theory in Lemma 2 or the LDWB. The introduction of CM-based tests is motivated, in part, by Shapiro & Wilk (1965), Shapiro, Wilk & Chen (1968) and Stephens (1974), who show that the former enjoys non-trivial power advantages over KS procedures when testing for Gaussianity in many (albeit, more traditional) settings.

4.1 Bootstrap Kolmogorov-Smirnov Testing

In analogy with the KS test in (11), we define a LDWB version of the test statistic and the corresponding critical region of the bootstrap test by

$$\text{KS}_n^*(\mathcal{A}) = \sup_{\tau \in \mathcal{A}} \sqrt{N_n(\alpha, \varpi)} \left| \widehat{F}_{W,n}^*(\tau) - \widehat{F}_n(\tau) \right|, \quad C_n^*(\mathcal{A}) = \left\{ \widehat{\text{KS}}_n(\mathcal{A}) > q_n^*(\theta, \mathcal{A}) \right\}, \quad (21)$$

respectively, where, again, $\theta \in (0, 1)$, $\mathcal{A} \subset \mathbb{R} \setminus 0$ is a finite union of compact sets with positive Lebesgue measure and $q_n^*(\theta, \mathcal{A})$ is the $(1 - \theta)$ th quantile of the LDWB distribution,

$$\sup_{\tau \in \mathcal{A}} \left| \frac{\sqrt{N_n(\alpha, \varpi)}}{[n/k_n] m_n} \sum_{j=1}^{[n/k_n]} \sum_{i=1}^{m_n} \left(X_{(j-1)k_n+i} - \widetilde{F}_n(\tau) \right) v_{(j-1)k_n+i}^* \right|.$$

The validity of the LDWB-aided KS test follows directly from Lemma 2 and Theorem 1:

Theorem 2. *Suppose the regularity conditions for Theorem 1 hold. Moreover, define the quantile function $q_n^*(\theta, \mathcal{A}) = \inf\{x \in \mathcal{A} : \mathbb{P}^*(\text{KS}_n^*(\mathcal{A}) > x) \geq \theta\}$. Then, for any compact subset $\mathcal{A} \subset \mathbb{R} \setminus 0$ with positive Lebesgue measure, it follows that $\mathbb{P}(\widehat{\text{KS}}_n(\mathcal{A}) > q_n^*(\theta, \mathcal{A})) \rightarrow \theta$ as $n \rightarrow \infty$.*

4.2 von Mises Statistics and Testing

Let $\ell : \mathbb{R}^2 \rightarrow \mathbb{R}$ denote a measurable function, whose double integral is assumed to exist, then we may write general von-Mises (V -)statistics for a CDF statistic $\mathcal{C} \in \{\widehat{F}_n - H_3/k_n, \Phi\}$, that is, either the bias-corrected empirical CDF or its limit under the null hypothesis, as

$$\mathcal{V}_\ell(\mathcal{C}, \mathcal{A}) = \int_{\tau_1 \in \mathcal{A}} \int_{\tau_2 \in \mathcal{A}} \ell(\tau_1, \tau_2) d\mathcal{C}(\tau_1) d\mathcal{C}(\tau_2), \quad (22)$$

where, unlike standard V -statistics, we restrict integration to the compact set $\mathcal{A} \subset \mathbb{R} \setminus 0$, again, to avoid the truncation of big jumps affecting the central limit theory. Now, let us further impose:

Assumption 3. ℓ is continuous, bounded and symmetric in its arguments $\ell(\tau_1, \tau_2) = \ell(\tau_2, \tau_1)$. Moreover, let $\ell, \ell_\Phi(\cdot) = \int_{\tau_2 \in \mathcal{A}} \ell(\cdot, \tau_2) d\Phi(\tau_2)$, and $\ell(\tau_1, \cdot)$ have bounded variation.

Let $h(\tau-)$ denote the limit from the left of a function h at a point τ , then by Lemma 2 and Assumption 3, we may invoke Beutner & Zähle (2014, Lemmas 3.4 and 3.6) to decompose

$$\begin{aligned} \mathcal{V}_\ell(\widehat{F}_n - H_3/k_n, \mathcal{A}) - \mathcal{V}_\ell(\Phi, \mathcal{A}) &= -2 \int_{\tau_1 \in \mathcal{A}} (\widehat{F}_n - \Phi - H_3/k_n)(\tau_1-) d\ell_\Phi(\tau_1) \\ &+ \int_{\tau_1 \in \mathcal{A}} \int_{\tau_2 \in \mathcal{A}} (\widehat{F}_n - \Phi - H_3/k_n)(\tau_1-) (\widehat{F}_n - \Phi - H_3/k_n)(\tau_2-) d\ell(\tau_1, \tau_2) \equiv \mathcal{V}_{\ell, N}(\mathcal{A}) + \mathcal{V}_{\ell, D}(\mathcal{A}), \end{aligned}$$

whose parts are typically labeled non-degenerate and degenerate ($\ell_\Phi(\tau_1) \equiv 0$), respectively. Examples of non-degenerate V -statistics are Gini's mean difference and CDF-based variance estimation. Notice, however, that under $\mathcal{H}_0 : S_t = W_t$, the infill asymptotic limit of standardized increments (2) are standard Gaussian, subject to estimation errors from the nonparametric stochastic scale, or a mean-zero stable process with characteristic function (4) under the alternative, making testing of such features less interesting.⁹ Hence, we focus on the degenerate part, $\mathcal{V}_{\ell, D}(\mathcal{A})$, for which we can construct tests of the local (again, infill asymptotic) distributional properties of the increments dZ_t .

Theorem 3. Suppose the conditions of Lemma 2 and Assumption 3 hold. Then, for any compact subset $\mathcal{A} \subset \mathbb{R} \setminus 0$ with positive Lebesgue measure,

$$N_n(\alpha, \varpi) \mathcal{V}_{\ell, D}(\mathcal{A}) \xrightarrow{d} \int_{\tau_1 \in \mathcal{A}} \int_{\tau_2 \in \mathcal{A}} \mathcal{G}_n(\tau_1) \mathcal{G}_n(\tau_2) d\ell(\tau_1, \tau_2).$$

The general result for V -statistics in Theorem 3 goes beyond the asymptotic analysis of the empirical CDF in Todorov & Tauchen (2014) and facilitates general test statistics of the L_2 -type to examine local distributional properties of dZ_t . In particular, the asymptotic result allows us to introduce a new class of weighted and bias-corrected Cramér-von Mises tests for \mathcal{H}_0 ,

$$\widehat{\text{CM}}_n(k, \mathcal{A}) = N_n(\alpha, \varpi) \int_{\tau \in \mathcal{A}} k(\tau) (\widehat{F}_n(\tau) - \Phi(\tau) - H_3(\tau)/k_n)^2 d\Phi(\tau), \quad (23)$$

⁹In fact, we cannot recover the drift of (3) in an infill asymptotic setting, see Jacod (2012).

for any measurable weight function $k : \mathbb{R} \rightarrow \mathbb{R}_+$, nesting the classical Cramér-von Mises and Anderson-Darling weights with $k(\tau) = 1$ and $k(\tau) = 1/(\Phi(\tau)(1 - \Phi(\tau)))$, respectively. Now, by applying the result in Theorem 3, $\widehat{\text{CM}}_n(k, \mathcal{A}) \xrightarrow{d} \text{CM}_n(k, \mathcal{A})$ where $\text{CM}_n(k, \mathcal{A}) = \int_{\tau \in \mathcal{A}} k(\tau) \mathcal{G}_n(\tau)^2 d\Phi(\tau)$.¹⁰ Similarly to the definitions for the KS test, let $Q_n(\theta, k, \mathcal{A})$ be the $(1 - \theta)$ th quantile of $\text{CM}_n(k, \mathcal{A})$ for $\theta \in (0, 1)$, then Lemma 2 and Theorem 3 establish validity of the class of CM tests in (23):

Corollary 2. *Suppose the regularity conditions for Theorem 3 hold. Moreover, define the quantile function $Q_n(\theta, k, \mathcal{A}) = \inf\{x \in \mathcal{A} : \mathbb{P}(\text{CM}_n(k, \mathcal{A}) > x) \geq \theta\}$. Then, for any compact subset $\mathcal{A} \subset \mathbb{R} \setminus 0$ with positive Lebesgue measure, it follows that $\mathbb{P}(\widehat{\text{CM}}_n(k, \mathcal{A}) > Q_n(\theta, k, \mathcal{A})) \rightarrow \theta$ as $n \rightarrow \infty$.*

The class of bias-corrected CM tests in (23) differs, as for the KS test in (11), from standard CM testing by, among others, the contributions of the terms $H_2(\tau)$ and $H_3(\tau)$ arising from the use of nonparametric, and noisy, estimates of the stochastic scale when standardizing the increments as well as the truncation of large jumps in the increments, impacting the integration range.

Remark 7. *Although not pursued here, and as discussed in Arcones & Giné (1992) and Beutner & Zähle (2014), the statistic $\widehat{S}_n(\mathcal{A}) = \int_{\tau \in \mathcal{A}} (\widehat{F}_n(-\tau) - (1 - \widehat{F}_n(\tau)))^2 d\tau$ may be used to test symmetry of the null distribution. If combined with the CM test in (23), $\widehat{S}_n(\mathcal{A})$ will reveal whether the alternative distribution if \mathcal{H}_0 is rejected, that is, a local stable, has asymmetric tails.*

4.3 Bootstrap von Mises Statistics and Testing

The asymptotic distribution in Theorem 3 may be analytically intractable for several choices of kernel function, ℓ , making inference and testing, e.g., using Corollary 2 hard in practice. However, such difficulties may readily be circumvented using the LDWB. Specifically, let

$$\mathcal{V}_{\ell, D}^*(\mathcal{A}) = \int_{\tau_1 \in \mathcal{A}} \int_{\tau_2 \in \mathcal{A}} (\widehat{F}_{W, n}^* - \widehat{F}_n)(\tau_1 -) (\widehat{F}_{W, n}^* - \widehat{F}_n)(\tau_2 -) d\ell(\tau_1, \tau_2), \quad (24)$$

be general bootstrapped V -statistics, and

$$\text{CM}_n^*(k, \mathcal{A}) = N_n(\alpha, \varpi) \int_{\tau \in \mathcal{A}} k(\tau) (\widehat{F}_{W, n}^*(\tau) - \widehat{F}_n(\tau))^2 d\Phi(\tau), \quad (25)$$

the corresponding bootstrap CM test. Moreover, let $Q_n^*(\theta, k, \mathcal{A})$ be the $(1 - \theta)$ th quantile of,

$$\int_{\tau \in \mathcal{A}} k(\tau) \left(\frac{\sqrt{N_n(\alpha, \varpi)}}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=1}^{m_n} \left(X_{(j-1)k_n+i} - \widetilde{F}_n(\tau) \right) v_{(j-1)k_n+i}^* \right)^2 d\Phi(\tau).$$

The validity of the LDWB statistics in (24) and (25), then, follows by Lemma 2, Theorem 1 in conjunction with the same arguments provided for Theorem 3 and Corollary 2:

¹⁰As explained in Beutner & Zähle (2014, Example 3.13), the CM test can be viewed as V -statistic with kernel function defined by, $\ell(\tau_1, \tau_2) = \int_{\tau \in \mathcal{A}} k(\tau) (1\{\tau_1 \leq \tau < \infty\} - \Phi(\tau) - H_3(\tau)/k_n)(1\{\tau_2 \leq \tau < \infty\} - \Phi(\tau) - H_3(\tau)/k_n) d\Phi(\tau)$.

Theorem 4. *Suppose the conditions of Theorem 3 hold. Then, for any compact subset $\mathcal{A} \subset \mathbb{R} \setminus 0$ with positive Lebesgue measure,*

$$N_n(\alpha, \varpi) \mathcal{V}_{\ell, D}^*(\mathcal{A}) \xrightarrow{d^*} \int_{\tau_1 \in \mathcal{A}} \int_{\tau_2 \in \mathcal{A}} \mathcal{G}_n(\tau_1) \mathcal{G}_n(\tau_2) d\ell(\tau_1, \tau_2), \quad \text{in probability-}\mathbb{P}.$$

Corollary 3. *Suppose the regularity conditions for Theorem 4 hold. Moreover, define the quantile function $Q_n^*(\theta, k, \mathcal{A}) = \inf\{x \in \mathcal{A} : \mathbb{P}^*(\text{CM}_n^*(k, \mathcal{A}) > x) \geq \theta\}$. Then, for any compact subset $\mathcal{A} \subset \mathbb{R} \setminus 0$ with positive Lebesgue measure, it follows that $\mathbb{P}(\widehat{\text{CM}}_n(k, \mathcal{A}) > Q_n^*(\theta, k, \mathcal{A})) \rightarrow \theta$ as $n \rightarrow \infty$.*

5 Simulation Study

In this section, we assess the relative finite sample properties of the Kolmogorov-Smirnov (KS) tests for local Gaussianity, $\mathcal{H}_0 : S_t = W_t$, based on the CLT in Todorov & Tauchen (2014) as well as our bootstrap aided-versions. Specifically, we study whether the LDWB or the NLBB can alleviate the previously reported (severe) finite sample size distortions that characterizes CLT-based test.

5.1 Simulation Setup

The data is simulated to match a standard 6.5-hour trading day and with the trading window normalized to the unit interval, $t \in [0, 1]$, making 1 second correspond to an increment of size $1/23400$. In particular, we consider four different data generating processes (DGPs) in the simulations; two under \mathcal{H}_0 and two under the alternative where S_t is a time-changed tempered stable process (\mathcal{H}_1), allowing us to study the size and power properties of the proposed testing procedures. Specifically, for DGPs under \mathcal{H}_0 , let

$$dZ_t = adt + \sigma_t dW_t + dY_t, \quad dY_t = \int_{\mathbb{R}} k_0 x \mu(dt, dx), \quad (26)$$

where the stochastic scale, σ_t , is assumed to follow a two-factor model,

$$\begin{aligned} \sigma_t &= \text{sexp}(b_0 + b_1 \tilde{\tau}_{1,t} + b_2 \tilde{\tau}_{2,t}) \quad \text{where} \quad d\tilde{\tau}_{1,t} = a_1 \tilde{\tau}_{1,t} dt + dB_{1,t}, \\ d\tilde{\tau}_{2,t} &= a_2 \tilde{\tau}_{2,t} dt + (1 + \phi \tilde{\tau}_{2,t}) dB_{2,t}, \quad \text{Corr}(B_{1,t}, W_t) = \rho_1, \quad \text{Corr}(B_{2,t}, W_t) = \rho_2, \end{aligned} \quad (27)$$

and both $B_{1,t}$ and $B_{2,t}$ are standard Brownian motions, following, e.g., Chernov, Gallant, Ghysels & Tauchen (2003) and Huang & Tauchen (2005).¹¹ The stochastic scale (or volatility) has two driving sources of uncertainty, two standard Brownian motions, which are correlated with W_t , thereby accommodating leverage effects. We fix the parameters in (26) and (27) as in Huang & Tauchen (2005), that is, $\alpha = 0.03$, $b_0 = -1.2$, $b_1 = 0.04$, $b_2 = 1.5$, $a_1 = -0.00137$, $a_2 = -1.386$, $\phi = 0.25$, as well as the correlation coefficients $\rho_1 = \rho_2 = -0.3$. Moreover, the two volatility factors are initialized at the onset of each “trading day” by randomly drawing the most persistent factor from its unconditional distribution,

¹¹The function “sexp” is, following the literature, defined as an exponential with a linear growth function splined in at high values of its argument: $\text{sexp}(x) = \exp(x)$ if $x \leq x_0$ and $\text{sexp}(x) = \frac{\exp(x_0)}{\sqrt{x_0}} \sqrt{x_0 - x_0^2 + x^2}$ if $x > x_0$ with $x_0 = \ln(1.5)$.

$\tilde{\tau}_{1,0} \sim N(0, 1/(2a_1))$, and by letting the strongly mean-reverting factor, $\tilde{\tau}_{2,t}$, start at zero. The two DGPs under \mathcal{H}_0 , capturing the size of the tests, differ with respect to the specification of the “residual” jump process in (26). In particular, Y_t is assumed to obey either a symmetric tempered stable process (DGP 1) or a compound Poisson process (DGP 2), which have the following decompositions of their compensators $\nu_t^Y(dx) = dt \otimes \nu^Y(dx)$,

$$\nu^Y(dx) = c_0 \exp(-\lambda_0|x|)|x|^{-(\beta'_0+1)}dx \quad \text{or} \quad \nu^Y(dx) = c_1 \frac{\exp(-x^2/(2\sigma_1^2))}{\sqrt{2\pi}\sigma_1}dx, \quad (28)$$

respectively. For the symmetric tempered stable, $c_0 > 0$, $\lambda_0 > 0$ and $\beta'_0 \in [0, 1)$ measures the degree of jump activity. Moreover, we follow Todorov (2009) and Hounyo & Varneskov (2017) and let $(\beta'_0, k_0, c_0, \lambda_0) = (0.1, 0.0119, 0.125, 0.015)$. This model is calibrated such that the variation of Y_t accounts for 10% of the average quadratic variation of Z_t , reflecting the empirical results in Huang & Tauchen (2005). Similarly, $(c_1, \sigma_1) = (1, 3/2)$ is fixed for the mean-zero, normally distributed, compound Poisson jumps (which have activity index $\beta' = 0$).

Under the alternative hypothesis, we let

$$Z_t = S_{T_t}, \quad \text{with} \quad T_t = \int_0^t \sigma_s^2 ds, \quad (29)$$

where S_t is a symmetric tempered stable martingale with Levy measure $\exp(-0.25|x|)|x|^{-(1.51+1)}$ and for the stochastic time change, T_t , σ_t is specified as in (27). The parameters of S_t are chosen such that it behaves locally like a stable process with $\beta = 1.51$.¹² We either add no residual jumps to the model under the alternative (29) (DGP 3) or compound Poisson jumps as in (28) (DGP 4).

After having simulated Z_{t_i} , we construct equidistant samples $t_i = i/n$ for $i = 0, \dots, n$ and generate returns $\Delta_i^n Z = Z_{t_i} - Z_{t_{i-1}}$. Specifically, we study the performance of the tests for three different samples sizes: $n = \{78, 195, 390\}$, corresponding to sampling every $\{5, 2, 1\}$ minutes, respectively. The tests require the selection of tuning parameters, k_n , m_n , α , ϖ and, specific to the bootstrap tests, the dependence parameter b_n . By Assumption 2(b), we have $\sqrt{n}/k_n \rightarrow \infty$. Hence, similarly to Todorov & Tauchen (2014), we let $\sqrt{n}/k_n = \varrho_1$, with $\varrho_1 = \{1, 5/4\}$, $m_n/k_n = 0.7$ and, for the truncation of the increments, $\alpha = 3$ and $\varpi = 0.49$. For the implementation of the LDWB, we determine dependence of the external random variable through $b_n/m_n = \varrho_2$ with moderate selections $\varrho_2 = \{1/2, 1/3\}$. Using this dependence parameter, we consider four different external random variables:

DWB1: $v_i^* \sim \text{i.i.d. } N(0, 1)$.

¹²Following Todorov et al. (2014), S_t is generated as the difference between two spectrally positive tempered stable processes, which are simulated using the acceptance-rejection algorithm of Baeumer & Meerschaert (2009).

DWB2: The Rademacher (i.e., the two point) distribution: $v_i^* \sim \text{i.i.d.}$ such that

$$v_i^* = \begin{cases} 1 & \text{with probability } \mathcal{P} = 1/2, \\ -1 & \text{with probability } 1 - \mathcal{P}. \end{cases}$$

DWB3: Ornstein-Uhlenbeck process $v_i^* = e^{-1/b_n} v_{i-1}^* + \xi_i$, with $\xi_i \sim \text{i.i.d. } N(0, 1 - e^{-2/b_n})$.

DWB4: Moving average process $v_i^* = \varsigma_i + \dots + \varsigma_{i-b_n+1}$, where $\varsigma_i \sim \text{i.i.d. } N(0, 1/b_n)$.

Note that the four choices of v_i^* are asymptotically valid, satisfying Assumption DWB, and their different dependence structures allow us to assess robustness features of the LDWB.¹³ Moreover, we implement NLBB as a final alternative, which, as explained in Section 3.3, is nested in our LDWB procedure. All KS tests for local Gaussianity are implemented over the set,

$$\mathcal{A} = [Q(0.001) : Q(0.499)] \cup [Q(0.501) : Q(0.999)],$$

where $Q(\theta)$ is the $(1 - \theta)$ th quantile of the standard normal distribution, and adopts a nominal 5% rejection level. Finally, the simulation study is carried out using 999 bootstrap samples for each of the 10,000 Monte Carlo replications. The rejection rates of $\mathcal{H}_0 : S_t = W_t$ are reported in Table 1 for the DGPs 1 and 2 (size) and Table 2 for the DGPs 3 and 4 (power).¹⁴

5.2 Simulation Results

There are several interesting results from Table 1. First, consistent with the evidence in Todorov & Tauchen (2014), we find that the CLT-based KS test is (severely) oversized, especially when the local window for spot volatility estimation is $\varrho_1 = 5/4$. Moreover, for DGP 1 in particular, the size distortions are essentially unaffected by an increase in sample size from $n = 78$ to $n = 360$. Second, the LDWB-aided KS tests have much better size properties for all combinations n , ϱ_1 , ϱ_2 , DGP and external random variables. For example, if considering DGP1, $n = 360$ and $\varrho_1 = 5/4$, the CLT rejects 18.4% of the time, whereas the LDWB1 is very close to the nominal 5% level with a rejection rate of 5.6%. Third, the NLBB performs slightly worse than the LDWBs, in small samples when $\varrho_1 = 1$ and more generally when $\varrho_1 = 5/4$, showing the benefits of our general bootstrap framework.

From the results in Table 2, we observe that all tests have power to reject the null hypothesis when false. Moreover, the rejection properties dramatically improve when the sample size is increased from $n = 78$ to $n = 195$, attaining full power when sampling every minute, i.e., when $n = 390$. The rejection rates for the CLT-based test are slightly higher than the corresponding for bootstrap tests, especially when $n = 78$ and $n = 195$. However, as emphasized by Horowitz & Savin (2000) and Davidson &

¹³The Rademacher distribution, proposed for bootstraps by Liu (1988), is advocated by Davidson & Flachaire (2008) in the context of wild bootstrap inference for regression parameters. We assess its prowess in the case of empirical CDF inference for semimartingales at high sampling frequencies using our LDWB methodology.

¹⁴Implementation details are provided in Appendix C.

MacKinnon (2006), these results are misleading since CLT test suffers from severe size distortions for all sampling frequencies and DGPs considered. Hence, for a more accurate comparison, we also report the size-adjusted power for the CLT-based test.¹⁵ Once the latter is corrected, we observe that its power is very similar to the corresponding for the bootstrap tests, especially when $\rho = 1$.

In general, the simulation results demonstrate the advantages of our bootstrap framework, specifically, by restoring the size properties of tests for local Gaussianity while maintaining excellent finite sample power, being very similar to an (infeasible) size-adjusted CLT test.¹⁶

6 Empirical Analysis

We proceed illustrating the usefulness of our new bootstrap procedure by testing for local Gaussianity in high-frequency (HF) futures data on three different asset classes; equity indices, foreign exchange rates and commodities. For simplicity of exposition, and due to the similarities between the properties of the LDWB tests in Tables 1 and 2, we focus on the differences between the rejection rates of tests based on the CLT from Todorov & Tauchen (2014) and the LDWB1 procedure.¹⁷

6.1 High-frequency Data

We study the null hypothesis, $\mathcal{H}_0 : S_t = W_t$, using high-frequency data from 2010-2013 on eight futures series covering three asset classes. This presents an interesting and diverse sampling period with substantial market turbulence during the first two years, culminating in April-May 2010 with the downgrade of Greece's sovereign debt to junk bond status as well as in August 2011 where stock prices dropped sharply in fear of contagion of the European sovereign debt crises to Italy and Spain, and two calmer years during 2012-2013. Specifically, since, e.g., Andersen et al. (2015) and Hounyo & Varneskov (2017) cannot reject \mathcal{H}_0 for S&P 500 futures, we extend their evidence to two different equity indices, namely the DAX and FTSE 100. Moreover, we consider futures contracts on gold and oil as well as four exchange rates; the Canadian Dollar (CAD), Swiss Franc (CHF), British Pound (GBP), and the Japanese Yen (JPY), all measured against the U.S. Dollar (USD). The series are obtained from Tick Data, include observations from both pit and electronic trading, and are sampled every minute. We use observations from 9.00 to 18.30 CET on the equity futures, 9.00 to 20.00 CET on the two commodities, and from 1.00 to 23.00 CET on the exchange rates since the latter is traded round-the-clock, whereas the trading is sparser in the other contracts outside of regular European and

¹⁵We do not report the size-adjusted power for the bootstrap-based tests for two reasons. First, the size of the tests is already very accurate, alleviating the need for a correction. Second, Horowitz & Savin (2000) and Davidson & MacKinnon (2006, pp. 429-430) argue that it is not straightforward to size-adjust bootstrap tests when assessing power.

¹⁶We have performed a similar exercise for the CLT and bootstrap Cramér von-Mises (CM) tests for local Gaussianity described in Section 4, applying the same set of tuning parameters. The results are qualitatively identical to those reported in Tables 1-2; namely, the LDWB improves the size in all cases. Since the main difference is that bootstrap KS tests have slightly better size and the CM tests slightly better power, the latter is excluded for brevity.

¹⁷As in the simulation study, the tests are implemented with $\varrho_1 = 1$, $m_n/k_n = 0.7$ and 999 bootstrap samples for each series. Not surprisingly, the results for the other bootstrap procedures are similar to those reported for LDWB1. The equivalent results for CM tests (not reported) provide qualitative and quantitatively similar conclusions.

U.S. market hours.¹⁸ Since these futures contracts are very liquid, alleviating concerns about market microstructure noise, we construct series of 1, 2 and 5-minute logarithmic returns. For each series, we report the daily rejection rates of \mathcal{H}_0 by year using a 5% nominal level. The test results are presented in Table 3 (equity indices and commodities) and 4 (exchange rates). We will, however, assess the robustness of our results against noise after presenting the main empirical findings.

6.2 High-frequency Application

From Table 3, we see that \mathcal{H}_0 is rarely rejected for the different combinations of either equity index or commodity and sampling frequency, suggesting that leading term in these assets is a Brownian motion. For the DAX and FTSE 100 indices, this evidence corroborates prior findings for futures contracts on the S&P 500. Interestingly, we find very similar results for gold and oil; the tests fail to reject local Gaussianity. Second, we observe the LDWB1 test to reject uniformly less than the CLT test. For example, for FTSE futures and a 5-minute sampling frequency, the CLT test has an average reject rate of 11.2%, compared to 3.2% for LDWB1. The differences in rejection rates are consistent with the simulation study; our LDWB1 test has an accurate size, whereas the CLT-based test is generally oversized, even in relatively large samples, and the two testing procedures have similar (size-adjusted) power. Third, the conclusions based on 1, 2, and 5-minute are very similar, suggesting that there is no issues with market microstructure noise at these sampling frequencies.¹⁹

When turning to the results for exchange rates in Table 4, they are markedly different from those in Table 3, however, similar across currencies. The tests rarely reject for the 5-minute sampling frequency, but the rejection rates uniformly increase as the sampling frequency increase, rejecting, on average over the whole sample, 91.6%, 98.3%, 67.4%, and 41.8% of the days for 1-minute observations and the CAD, CHF, GBP, and JPY, respectively. This strongly suggests that exchange rates are locally driven by a stable process with $\beta < 2$, not a diffusion. Moreover, the rejection rates differ across the sample, further suggesting that there may be important time-variation in β . To corroborate these findings, we depict estimates of β in Figure 2 for each sampling frequency using the empirical characteristic function approach of Todorov (2015).²⁰ The β estimates are remarkably similar across both currencies and sampling frequencies; being in the 1.80-1.90 range. Moreover, and together with the simulated power results in Table 2, they help to explain the differences in rejection rates across sampling frequencies in Table 4. For sparsely sampled frequencies, our test seemingly lacks the power to reject \mathcal{H}_0 . However, as the power properties improves with sample size, the (bootstrap) tests reject

¹⁸We restrict attention to whole trading days in the Europe and the U.S. Moreover, we have experimented with sampling using different trading hours. The conclusions are qualitatively identical to those presented below.

¹⁹Microstructure noise represents a different rejection of \mathcal{H}_0 , see Todorov & Tauchen (2014, Theorem 2), who consider i.i.d. noise. Hence, if noise affects 1-minute observations, but not more sparsely sampled ones, we would expect the rejection rates to differ, which is not the case in Table 3. Although other noise generating processes may generate different rejection implications, we know from, e.g., Hansen & Lunde (2006) and Barndorff-Nielsen, Hansen, Lunde & Shephard (2009) that if the noise is present at 1-minute frequencies, we would expect it to be well-approximated by i.i.d. dynamics.

²⁰For specificity, for each trading day, we implement the estimator in Todorov (2015, (5.1)) with, in his notation, the tuning parameter selections $p = 0.51$, $u = 0.25$, $v = 0.5$, and $k_n = \{50, 75, 100\}$ for 5, 2, and 1-minute returns. From these, the median estimate for a given calendar month is computed and depicted for all three sampling frequencies.

local Gaussianity almost 100% of the days for the CAD and CHF series.

One may be concerned that the rejection results in Table 4 are due to microstructure noise, not the innovations being locally stable. To combat this point, note first that the β estimates in Figure 2 are analogous to a “volatility signature plot” at each point in time. The remarkably similar estimates across sampling frequencies suggest that noise exert only a smaller impact on the tests, if any.²¹ Second, and more formally, we implement the test for microstructure noise in Ait-Sahalia & Xiu (2019, (36)) and compute the daily rejection rates of \mathcal{H}_0 for all days in the sample, the days where the test fails to reject the null hypothesis of *no noise* and the days where the test rejects the null. The results, presented in Table 5, show that the daily rejections are very similar on test-suggested noise and no-noise days for all combinations of exchange rates and sampling frequencies.²² Hence, despite the test from Ait-Sahalia & Xiu (2019) finding evidence of noise on several trading days for the 1-minute sampling frequency, this does not seem to drive our conclusions. In fact, as the test for noise is developed for Brownian semimartingales, it is unclear how this behaves if the leading term is locally stable.

The combination of results in Tables 2, 4 and 5 as well as the similarity of the β estimates across sampling frequencies in Figure 2 deem the (lack of) power hypothesis for the tests more likely. Of course, to rigorously dismiss the market microstructure noise hypothesis, it would require a noise-robust version of the LDWB1 test. We leave this for further research.²³

In general, we find that equity indices and commodities (gold and oil) are well-described as locally Gaussian, whereas exchange rates are better approximated by locally stable innovations with stability index β in the 1.80-1.90 range. As explained in the introduction, these differences across asset classes hold important implications for model specification, which need to accommodate the distributional implications of the local innovations; risk measures such as power variation, whose estimation and interpretation, as shown by Ait-Sahalia & Jacod (2009) and Todorov & Tauchen (2011a), depend critically on the magnitude of β ; and derivatives pricing as exemplified by the pure-jump option pricing frameworks in Carr & Wu (2003) and Andersen et al. (2018).

7 Conclusion

This paper provides a new inference procedure for the local innovations of Itô semimartingales. Specifically, we construct a resampling procedure for the empirical CDF of high-frequency innovations that have been standardized using a nonparametric estimate of its stochastic scale (volatility) and truncated to rid the effect of “large” and more infrequent jumps. Our locally dependent wild bootstrap

²¹Among others, Jing, Kong & Liu (2011) demonstrate that β estimates diverge to infinity if noise is present

²²We obtain similar results when splitting this exercise by year as well as perform it for equity indices and commodities. All conclusions remain on test-suggested noise and no-noise days. These results are omitted for brevity.

²³The evidence in Table 4 is consistent with the findings in Todorov & Tauchen (2010) and Cont & Mancini (2011), who, using 5-minute observations on the DM-USD exchange rate from the 1990s, argue that exchange rates are locally Gaussian. These test may simply lack the power to reject \mathcal{H}_0 , as for the 5-minute series in Table 4. Moreover, Hounyo & Varneskov (2017) find rejection rates of $\beta = 2$ for currencies to be between 20-56% using a bootstrap-aided realized power variation test. Our results, using the LDWB1, are much stronger, which speaks directly to power differences between tests based on either the empirical CDF and power variation measures, see Todorov & Tauchen (2014, Table 2).

(LDWB) accommodate issues related to the stochastic scale and jumps as well as account for a special block-wise dependence structure induced by sampling errors arising from having replaced the stochastic scale with a nonparametric estimate. We show that the LDWB replicates first and second-order limit theory from the usual empirical process component of the statistic and the stochastic scale estimate, respectively, in addition to an asymptotic bias. Moreover, we design the LDWB sufficiently general to establish asymptotic equivalence between it and a nonparametric local block bootstrap, also introduced here, up to second-order distribution theory, providing new theoretical insights into the relation between bootstrap paradigms. Finally, we introduce LDWB-aided Kolmogorov-Smirnov tests for local Gaussianity as well as local von-Mises statistics, with and without accompanying bootstrap inference, and establish their asymptotic validity using the second-order distribution theory.

The finite sample performance of CLT and LDWB-aided local Gaussianity tests are assessed in a simulation study and an empirical application to high-frequency futures data. Whereas the CLT test is oversized, even in large samples, the size of the LDWB tests are accurate, even in small samples. Moreover, the power of the bootstrap test are similar to an (infeasible) size-adjusted CLT test. The empirical analysis verifies the size pattern, the CLT tests rejects uniformly more often than the LDWB test for assets that are well-described as locally Gaussian such as equity indices and commodities. Moreover, it shows that local Gaussianity is strongly rejected for exchange rate series, which, in contrast, are better described as locally stable with tail index in the 1.80-1.90 range.

Rejection Rates under \mathcal{H}_0								
$\varrho_1 = 1$								
	CLT	NLBB	LDWB1	LDWB2	LDWB3 _{1/2}	LDWB3 _{1/3}	LDWB4 _{1/2}	LDWB4 _{1/3}
DGP 1								
78	16.9	9.8	4.2	4.8	7.3	5.5	4.8	4.3
195	13.6	6.7	3.1	3.4	5.3	4.6	4.0	3.6
390	10.2	5.1	3.0	3.0	4.2	3.7	3.4	3.0
DGP 2								
78	16.5	9.0	4.0	4.8	7.0	5.2	4.4	3.9
195	12.3	6.6	3.5	3.8	5.3	4.8	4.0	3.8
390	9.7	4.3	2.4	2.4	3.7	3.2	2.9	2.6
$\varrho_1 = 5/4$								
	CLT	NLBB	LDWB1	LDWB2	LDWB3 _{1/2}	LDWB3 _{1/3}	LDWB4 _{1/2}	LDWB4 _{1/3}
DGP 1								
78	24.4	12.1	7.2	8.7	11.6	9.1	8.2	7.1
195	18.8	9.5	5.4	5.8	7.9	6.9	6.0	5.6
390	18.4	8.9	5.6	5.8	7.9	6.8	6.4	5.7
DGP 2								
78	23.8	11.9	6.4	7.7	11.4	8.6	7.6	6.3
195	16.7	8.0	4.1	4.5	6.7	5.4	4.7	4.3
390	17.0	8.1	5.0	5.1	7.0	6.1	5.7	5.3

Table 1: Size results. This table provides rejection frequencies of the null hypothesis $\mathcal{H}_0 : S_t = W_t$ for DGPs 1 and 2, sample sizes $n = \{78, 195, 390\}$, as well as eight different tests; CLT, NLBB, LDWB1, LDWB2, LDWB3, and LDWB4. In particular, CLT denotes the Kolmogorov-Smirnov (KS) test in (11), see also Todorov & Tauchen (2014), NLBB is the nonparametric local block bootstrap described in Section 3.3, and LDWB with numbers 1-4 are different implementations of the locally dependent wild bootstrap in Section 3.2, see Theorems 1 and 2 as well as Section 5. The numbers refer to different external random variables: (1) Gaussian; (2) Rademacher; (3) Ornstein-Uhlenbeck and (4) Moving average. For LDWB3 and LDWB4, the subscript refers to $\varrho_2 = \{1/2, 1/3\}$, capturing their dependence structures. The nominal level of the KS tests is 5%. Finally, the exercise is performed for 999 bootstrap samples for every one of the 10,000 Monte Carlo replications.

Rejection Rates under \mathcal{H}_1									
$\varrho_1 = 1$									
	CLT	CLT _A	NLBB	LDWB1	LDWB2	LDWB3 _{1/2}	LDWB3 _{1/3}	LDWB4 _{1/2}	LDWB4 _{1/3}
DGP 3									
78	56.0	35.4	35.2	35.5	38.1	40.0	37.5	35.3	35.4
195	94.7	88.7	82.0	85.6	86.0	85.0	85.1	84.4	84.6
390	99.8	99.5	98.5	99.1	99.1	99.0	98.8	98.8	99.0
DGP 4									
78	57.0	34.8	36.0	36.0	38.8	40.2	38.4	35.8	36.2
195	94.4	88.4	82.5	85.6	86.0	85.3	85.5	84.7	85.0
390	99.8	99.7	98.8	99.1	99.2	99.0	99.1	99.0	99.1
$\varrho_1 = 5/4$									
	CLT	CLT _A	NLBB	LDWB1	LDWB2	LDWB3 _{1/2}	LDWB3 _{1/3}	LDWB4 _{1/2}	LDWB4 _{1/3}
DGP 3									
78	60.7	39.7	37.9	38.7	41.8	43.4	41.3	38.6	38.6
195	91.8	85.9	88.0	81.0	81.5	80.7	81.0	80.0	80.1
390	99.8	99.5	99.0	99.1	99.1	99.0	99.1	99.0	99.0
DGP 4									
78	59.3	38.9	37.7	38.3	40.9	43.0	41.0	40.0	38.2
195	91.0	85.6	77.7	80.1	80.7	80.1	80.0	79.2	79.4
390	99.7	99.5	99.1	99.1	99.1	99.0	99.1	99.0	99.0

Table 2: Power results. This table provides rejection frequencies of the null hypothesis \mathcal{H}_0 for DGPs 3 and 4, sample sizes $n = \{78, 195, 390\}$, as well as eight different tests; CLT, CLT_A, NLBB, LDWB1, LDWB2, LDWB3, and LDWB4. In particular, CLT denotes the Kolmogorov-Smirnov (KS) test in (11), see also Todorov & Tauchen (2014), CLT_A is adjusted for size-distortions, NLBB is the nonparametric local block bootstrap described in Section 3.3, and LDWB with numbers 1-4 are different implementations of the locally dependent wild bootstrap in Section 3.2, see Theorems 1 and 2 as well as Section 5. The numbers refer to different external random variables: (1) Gaussian; (2) Rademacher; (3) Ornstein-Uhlenbeck and (4) Moving average. For LDWB3 and LDWB4, the subscript refers to $\varrho_2 = \{1/2, 1/3\}$, capturing their dependence structures. The nominal level of the KS tests is 5%. Finally, the exercise is performed for 999 bootstrap samples for every one of the 10,000 Monte Carlo replications.

Rejection Rates for Equity Indices and Commodities								
	2010		2011		2012		2013	
	CLT	LDWB1	CLT	LDWB1	CLT	LDWB1	CLT	LDWB1
DAX								
1-min	4.31	1.18	7.03	1.95	4.74	0.40	3.56	0.00
2-min	5.49	0.39	7.81	1.56	5.14	0.40	2.77	0.00
5-min	7.84	0.00	10.94	1.95	6.32	1.98	7.11	0.79
FTSE								
1-min	8.37	2.79	8.43	2.81	5.20	1.60	7.97	1.99
2-min	9.16	1.20	5.62	2.01	8.00	1.60	2.79	0.40
5-min	9.16	3.19	16.06	4.82	7.60	1.20	11.95	3.59
Gold								
1-min	6.59	1.16	5.81	2.33	4.65	0.78	5.43	1.16
2-min	3.10	0.39	2.71	0.39	2.71	0.78	6.98	0.39
5-min	6.98	1.16	6.98	1.55	3.49	0.00	4.26	0.78
Oil								
1-min	5.14	1.19	7.78	3.50	3.88	1.16	4.65	1.55
2-min	3.95	1.19	3.11	0.00	6.98	0.39	3.49	0.00
5-min	11.07	1.98	7.78	0.39	8.91	1.16	8.53	1.16

Table 3: Empirical rejection rates. This table provides rejection frequencies of daily tests of the null hypothesis $\mathcal{H}_0 : S_t = W_t$ for the CLT and LDWB1 testing procedures. In particular, CLT denotes the Kolmogorov-Smirnov (KS) test in (11), see also Todorov & Tauchen (2014), and LDWB1 is the locally dependent wild bootstrap-based test with standard Gaussian external random variables, see Theorems 1 and 2 as well as Section 5. The tests are implemented on high-frequency futures data from both pit and electronic trading for the DAX and FTSE 100 equity indices as well as Gold and Oil futures. Three different sampling frequencies are considered; every 1, 2, and 5 minutes. For the equity index futures, the trading hours are 9.00-18.30 (CET), amounting to sample sizes $n = \{570, 285, 114\}$ for the three sampling frequencies. For the commodity futures, the trading hours are 9.00-20.00 (CET), amounting to sample sizes $n = \{660, 330, 132\}$ for the three sampling frequencies. The nominal level of the KS tests is 5%. Finally, we use 999 replications for the bootstrap resampling, as in the simulation study.

Rejection Rates for Currencies								
	2010		2011		2012		2013	
	CLT	LDWB1	CLT	LDWB1	CLT	LDWB1	CLT	LDWB1
CAD-USD								
1-min	92.31	85.83	91.09	82.59	100	98.77	99.59	99.59
2-min	32.39	16.19	38.46	21.05	69.67	50.00	91.80	84.02
5-min	7.29	1.62	5.25	1.21	6.97	2.46	32.38	13.52
CHF-USD								
1-min	98.79	96.36	99.19	97.57	100	100	99.59	99.18
2-min	61.54	38.87	54.66	31.17	88.93	78.28	86.48	74.59
5-min	6.48	1.21	2.43	1.21	7.79	2.46	10.66	4.51
GBP-USD								
1-min	66.40	44.94	69.23	52.23	95.08	86.48	95.49	86.48
2-min	8.50	4.05	8.50	3.64	29.10	15.57	32.79	15.57
5-min	2.02	0.81	2.43	0.40	3.69	0.00	3.28	0.41
JPY-USD								
1-min	40.49	21.46	56.68	40.08	84.02	73.36	42.31	32.38
2-min	6.07	2.02	14.17	4.45	22.95	9.43	9.43	4.51
5-min	3.24	0.40	4.45	0.81	3.28	0.41	6.56	1.23

Table 4: Empirical rejection rates for currencies. This table provides rejection frequencies of daily tests of the null hypothesis $\mathcal{H}_0 : S_t = W_t$ for the CLT and LDWB1 testing procedures. In particular, CLT denotes the Kolmogorov-Smirnov (KS) test in (11), see also Todorov & Tauchen (2014), and LDWB1 is the locally dependent wild bootstrap-based test with standard Gaussian external random variables, see Theorems 1 and 2 as well as Section 5. The tests are implemented on high-frequency futures data from both pit and electronic trading for four exchange rate futures. Three different sampling frequencies are considered; every 1, 2, and 5 minutes. For the currencies, the trading hours are 1.00-23.00 (CET), amounting to sample sizes $n = \{1320, 660, 264\}$ for the three sampling frequencies. The nominal level of the KS tests is 5%. Finally, we use 999 replications for the bootstrap resampling, as in the simulation study.

Rejection Rates for Currencies: Conditioning on Noise								
	CAD-USD				CHF-USD			
	Full	\mathcal{N}_0	\mathcal{N}_A	\mathcal{N}_A -Freq	Full	\mathcal{N}_0	\mathcal{N}_A	\mathcal{N}_A -Freq
1-min	91.65	96.61	90.43	73.42	98.27	99.21	98.15	72.5
2-min	42.67	43.04	46.56	3.36	55.6	55.89	52.92	4.38
5-min	4.68	4.35	0	1.53	2.34	2.67	0	0.71
	GBP-USD				JPY-USD			
	Full	\mathcal{N}_0	\mathcal{N}_A	\mathcal{N}_A -Freq	Full	\mathcal{N}_0	\mathcal{N}_A	\mathcal{N}_A -Freq
1-min	67.42	68.7	77.23	82.28	41.75	54.78	36.49	70.77
2-min	9.67	9.04	13.19	3.66	5.09	4.91	18.07	2.95
5-min	0.41	0.41	0	1.02	0.71	0.83	0	1.93

Table 5: Empirical rejection rates for currencies: Noise-robustness. This table provides rejection frequencies of daily tests of the null hypothesis $\mathcal{H}_0 : S_t = W_t$ for the LDWB1 testing procedures. In particular, LDWB1 is the locally dependent wild bootstrap-based test with standard Gaussian external random variables, see Theorems 1 and 2 as well as Section 5. The rejection frequency of \mathcal{H}_0 is reported for the full sample covering 2010-2013 (Full) as well as on days where the test of Ait-Sahalia & Xiu (2019, (36)) fails to reject the null hypothesis of *no noise* (\mathcal{N}_0) and days where *no noise* is rejected (\mathcal{N}_A). Finally, \mathcal{N}_A -freq denotes the frequency of days the *no noise* hypothesis is rejected. The tests are implemented on high-frequency futures data from both pit and electronic trading for four exchange rate futures. Three different sampling frequencies are considered; every 1, 2, and 5 minutes. For the currencies, the trading hours are 1.00-23.00 (CET), amounting to sample sizes $n = \{1320, 660, 264\}$ for the three sampling frequencies. The nominal level of the KS tests is 5%. Finally, we use 999 replications for the bootstrap resampling, as in the simulation study.

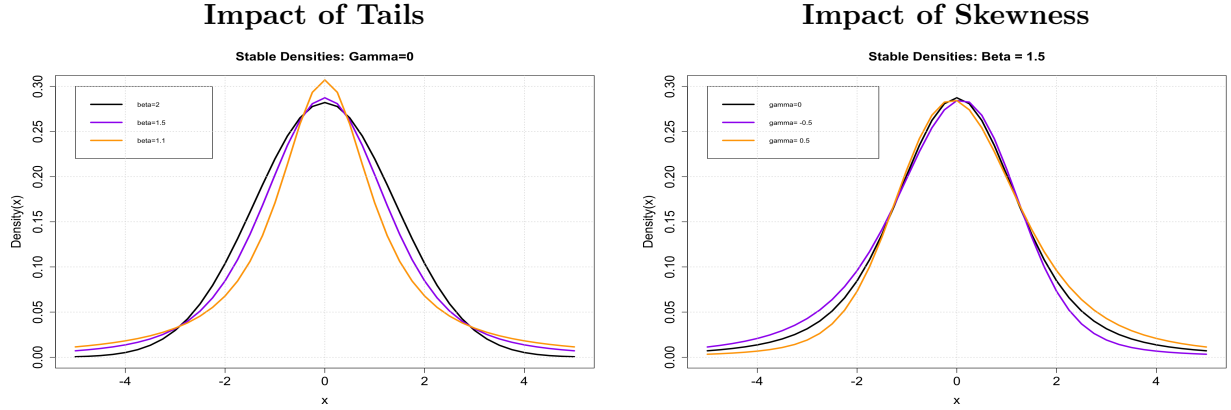


Figure 1: Stable densities. This picture illustrate the density of stable process with different values of their stability and skewness parameters, β and γ , noting that $\beta = 2$ implies a Gaussian variable.

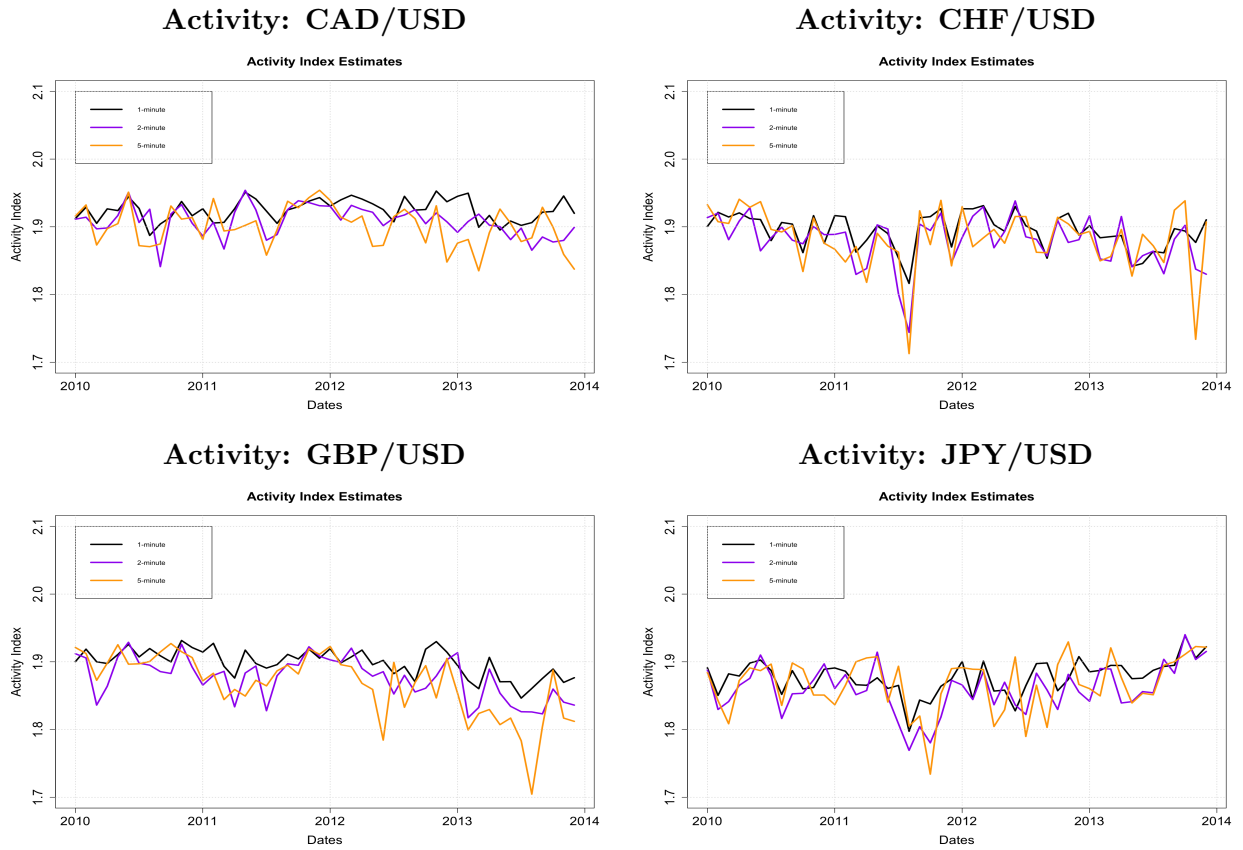


Figure 2: Activity index estimates. This picture depict daily activity index estimates for the four different exchange rates using the empirical characteristic function approach by Todorov (2015). The estimates are provided for three different sampling frequencies; 1-minute (black), 2-minute (purple), and 5-minute (orange). The estimator in Todorov (2015, (5.1)) is implemented with, in his notation, the tuning parameter selections $p = 0.51$, $u = 0.25$, $v = 0.5$, and $k_n = \{50, 75, 100\}$ for 5, 2, and 1-minute returns. From these, the median estimate for a given calendar month in the period 2010-2013 is computed and depicted for all three sampling frequencies.

A Technical Results and Proofs

This section contains additional assumptions and definitions as well as the proofs of the main asymptotic results in the paper. Before proceeding, however, let us introduce some notation. Denote by K a generic constant, which may take different values from line to line or from (in)equality to (in)equality. Moreover, we write $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$ and adopt the following shorthand convention for subscript time indices; $\frac{(j-1)k_n}{n}$ signifies $t_{\frac{(j-1)k_n}{n}}$. Let \circ indicate the hadamard product. Finally, let us write $\mathbb{E}_{i-1}^n[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_{(i-1)/n}]$ and $\mathbb{E}_{i-1}^*[\cdot] = \mathbb{E}_{i-1}^n(\mathbb{E}^*[\cdot | \mathcal{X}_n])$ denote conditional expectations under the physical and bootstrap probability measures, respectively.

A.1 Additional Assumptions

As in Todorov & Tauchen (2014), we shall establish the main Theorem 1 under the following stronger version of Assumption 1, and then rely on a standard localization argument, cf. Jacod & Protter (2012, Lemma 4.4.9), to extend the results to the weaker Assumption 1.

Assumption S1. *In addition to Assumption 1, the following conditions hold:*

- (a) $\alpha_t, \tilde{\alpha}_t, \sigma_t, \sigma_t^{-1}, \tilde{\sigma}_t, \tilde{\sigma}'_t$ and the coefficients of the Itô semimartingale representations of $\tilde{\sigma}_t$ and $\tilde{\sigma}'_t$ are all uniformly bounded on $t \in [0, 1]$.
- (b) For some negative valued function, $\phi(x)$ on the auxiliary Polish space (E, \mathcal{E}) satisfying the regularity conditions $\int_E \nu(x : \phi(x) \neq 0) < \infty$ and $\phi(x) \leq K$,

$$|\delta^Y(t, x)| + |\delta^\sigma(t, x)| + |\delta^{\tilde{\sigma}}(t, x)| + |\delta^{\tilde{\sigma}'}(t, x)| \leq \phi(x). \quad (\text{A.1})$$

A.2 Additional Definitions

We need to introduce several different quantities for the proof of the main Theorem 1. Hence, to improve exposition and ease readability of the latter, we have collected them all in this subsection as well as used the same notation as Todorov & Tauchen (2014) when it is applicable:

- $A_t = \int_0^t \alpha_s ds$ and $B_t = \int_0^t \sigma_s dW_s$. Moreover, for $j = 1, \dots, \lfloor n/k_n \rfloor$, let

$$\tilde{V}_{n,j} = \frac{n}{k_n - 1} \sum_{(j-1)k_n+2}^{jk_n} |\Delta_{i-1}^n B| |\Delta_i^n B|, \quad \bar{V}_{n,j} = \sigma_{\frac{(j-1)k_n}{n}}^2 \frac{\pi n}{2(k_n - 1)} \sum_{i=(j-1)k_n+2}^{jk_n} |\Delta_{i-1}^n W| |\Delta_i^n W|.$$

and define $\tilde{V}_{n,j}(i)$ and $\bar{V}_{n,j}(i)$ analogously, using the same structure as in (6).

- $\tilde{V}_{n,j} - \bar{V}_{n,j} = \sum_{g=1}^4 R_j^{(g)}$ where $\sum_{g=1}^3 R_j^{(g)}$ will not appear explicitly in our derivations below, and we refer to Todorov & Tauchen (2014, (10.4)) for definitions.
- $R_j^{(4)} = \frac{2}{k_n - 1} \sigma_{\frac{(j-1)k_n}{n}} \sum_{(j-1)k_n+2}^{jk_n} \left[\int_{\frac{(j-1)k_n}{n}}^{\frac{i-2}{n}} \tilde{\sigma}_{\frac{(j-1)k_n}{n}} dW_u + \int_{\frac{(j-1)k_n}{n}}^{\frac{i-2}{n}} \tilde{\sigma}'_{\frac{(j-1)k_n}{n}} dW'_u \right].$

- $\tilde{R}_{i,j}^{(4)} = R_j^{(4)} - \frac{2}{k_n-1} \sigma_{\frac{(j-1)k_n}{n}} (jk_n - i - 1) \left[\int_{\frac{i-1}{n}}^{\frac{i}{n}} \tilde{\sigma}_{\frac{(j-1)k_n}{n}} dW_u + \int_{\frac{i-1}{n}}^{\frac{i}{n}} \tilde{\sigma}'_{\frac{(j-1)k_n}{n}} dW'_u \right]$ is the component of $R_j^{(4)}$ that does not contain $\Delta_i^n W$ and $\Delta_i^n W'$ for $i = (j-1)k_n + 1, \dots, jk_n - 2$.
- $R_{i,j}^{(4)}(i)$ and $\tilde{R}_{i,j}^{(4)}(i)$ are the analogous components from $\tilde{V}_{n,j}(i) - \bar{V}_{n,j}(i) = \sum_{g=1}^4 R_j^{(g)}(i)$.

Furthermore, for $i = (j-1)k_n + 1, \dots, (j-1)k_n + m_n$ and $j = 1, \dots, \lfloor n/k_n \rfloor$, define

- $\xi_{n,j}(1) = \frac{\hat{V}_{n,j}(i) - \sigma_{\frac{(j-1)k_n}{n}}^2}{2\sigma_{\frac{(j-1)k_n}{n}}^2}$ and $\xi_{n,j}(2) = \frac{\left(\hat{V}_{n,j}(i) - \sigma_{\frac{(j-1)k_n}{n}}^2\right)^2}{8\sigma_{\frac{(j-1)k_n}{n}}^4}$
- $\tilde{\xi}_{n,i,j}(1) = \frac{\bar{V}_{n,j}(i) + \tilde{R}_{i,j}^{(4)}(i) - \sigma_{\frac{(j-1)k_n}{n}}^2}{2\sigma_{\frac{(j-1)k_n}{n}}^2}$ and $\tilde{\xi}_{n,i,j}(2) = \frac{\left(\bar{V}_{n,j}(i) + \tilde{R}_{i,j}^{(4)}(i) - \sigma_{\frac{(j-1)k_n}{n}}^2\right)^2}{8\sigma_{\frac{(j-1)k_n}{n}}^4}$
- $\bar{\xi}_{n,i,j}(1) = \frac{\bar{V}_{n,j}(i) + R_{i,j}^{(4)} - \sigma_{\frac{(j-1)k_n}{n}}^2}{2\sigma_{\frac{(j-1)k_n}{n}}^2}$ and $\bar{\xi}_{n,i,j}(2) = \frac{\left(\bar{V}_{n,j}(i) + R_{i,j}^{(4)} - \sigma_{\frac{(j-1)k_n}{n}}^2\right)^2}{8\sigma_{\frac{(j-1)k_n}{n}}^4}$
- $\hat{\xi}_{n,j}(1) = \frac{\bar{V}_{n,j}(i) - \sigma_{\frac{(j-1)k_n}{n}}^2}{2\sigma_{\frac{(j-1)k_n}{n}}^2}$ and $\hat{\xi}_{n,j}(2) = \frac{\left(\bar{V}_{n,j}(i) - \sigma_{\frac{(j-1)k_n}{n}}^2\right)^2}{8\sigma_{\frac{(j-1)k_n}{n}}^4}$
- $\xi_{n,i,j}(3) = \frac{\sqrt{n}\Delta_i^n W}{\sigma_{\frac{(j-1)k_n}{n}}} \left[\tilde{\sigma}_{\frac{(j-1)k_n}{n}} \left(W_{\frac{i-1}{n}} - W_{\frac{(j-1)k_n}{n}} \right) + \tilde{\sigma}'_{\frac{(j-1)k_n}{n}} \left(W'_{\frac{i-1}{n}} - W'_{\frac{(j-1)k_n}{n}} \right) \right]$.
- $\xi_{n,i,j}(4) = 1 + \frac{1}{\sigma_{\frac{(j-1)k_n}{n}}} \left[\tilde{\sigma}_{\frac{(j-1)k_n}{n}} \left(W_{\frac{i-1}{n}} - W_{\frac{(j-1)k_n}{n}} \right) + \tilde{\sigma}'_{\frac{(j-1)k_n}{n}} \left(W'_{\frac{i-1}{n}} - W'_{\frac{(j-1)k_n}{n}} \right) \right]$.
- $\chi_{n,i,j}(1) = -\chi_{n,i,j}(1,1) + \chi_{n,i,j}(1,2) - \chi_{n,i,j}(1,3)$, where

$$\chi_{n,i,j}(1,1) = \sqrt{n} \frac{1}{\sigma_{\frac{(j-1)k_n}{n}}} \left(\Delta_i^n A + \Delta_i^n Y + \int_{\frac{i-1}{n}}^{\frac{i}{n}} (\sigma_u - \sigma_{\frac{i-1}{n}}) dW_u \right) 1 \left\{ \frac{\sqrt{n}|\Delta_i^n Z|}{\sqrt{\hat{V}_{n,j}}} \leq \alpha n^{1/2-\varpi} \right\},$$

$$\chi_{n,i,j}(1,2) = (\sqrt{n}\Delta_i^n W + \xi_{n,i,j}(3)) 1 \left\{ \frac{\sqrt{n}|\Delta_i^n Z|}{\sqrt{\hat{V}_{n,j}}} > \alpha n^{1/2-\varpi} \right\},$$

$$\chi_{n,i,j}(1,3) = \left(\frac{\sqrt{n}\Delta_i^n W}{\sigma_{\frac{(j-1)k_n}{n}}} (\sigma_{\frac{i-1}{n}} - \sigma_{\frac{(j-1)k_n}{n}}) - \xi_{n,i,j}(3) \right) 1 \left\{ \frac{\sqrt{n}|\Delta_i^n Z|}{\sqrt{\hat{V}_{n,j}}} \leq \alpha n^{1/2-\varpi} \right\}.$$
- $\chi_{n,i,j}(2) = \left(\frac{\sqrt{\hat{V}_{n,j}(i)}}{\sigma_{\frac{(j-1)k_n}{n}}} - 1 - \xi_{n,j}(1) + \xi_{n,j}(2) \right) + \left(\xi_{n,j}(1) - \xi_{n,j}(2) - \tilde{\xi}_{n,i,j}(1) + \tilde{\xi}_{n,i,j}(2) \right).$

A.3 Proof of Theorem 1

The proof may be divided in two main parts; one establishing the central limit theory for the leading terms and one establishing bounds for lower-order terms. The latter follows along the same lines as Todorov & Tauchen (2014) and we refer to Section A.2 for definitions of corresponding terms. The central limit theory is established through a sequence of auxiliary lemmas in Section A.6. We shall make the references clear when necessary. Without loss of generality, we shall throughout assume that $\tau < 0$, with the equivalent results for $\tau > 0$ following by similar arguments, and $k_n - m_n > 2$, which is no restriction since $m_n \ll k_n$. Finally, we use $\hat{C}_{i,g}$ as unified notation for either $C_{i,g}$ or $\tilde{C}_{i,g}$ when the variables v_i^* are either stationary or block-stationary. As $C_{i,g}$ and $\tilde{C}_{i,g}$ are nonrandom constants, this is without loss of generality in the arguments. Now, let us start by making a decomposition,

$$\hat{\mathcal{F}}_{W,n}^*(\tau) \equiv \sqrt{N_n(\alpha, \varpi)} \left(\hat{F}_{W,n}^*(\tau) - \hat{F}_n(\tau) \right) \equiv \hat{\mathcal{G}}_n^*(\tau) - \hat{R}_n^*(\tau), \quad (\text{A.2})$$

whose parts are defined as,

$$\begin{aligned} \hat{\mathcal{G}}_n^*(\tau) &\equiv \frac{\sqrt{N_n(\alpha, \varpi)}}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=1}^{m_n} (X_{(j-1)k_n+i} - \Phi(\tau)) v_{(j-1)k_n+i}^*, \quad \text{and} \\ \hat{R}_n^*(\tau) &\equiv \left(\tilde{F}_n(\tau) - \Phi(\tau) \right) \frac{\sqrt{N_n(\alpha, \varpi)}}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=1}^{m_n} v_{(j-1)k_n+i}^*. \end{aligned}$$

Since, by Assumption DWB and Lemma 2, it follows that

$$\sup_{\tau \in \mathcal{A}} \left| \hat{R}_n^*(\tau) \right| = \sup_{\tau \in \mathcal{A}} \left| \tilde{F}_n(\tau) - \Phi(\tau) \right| \times \left| \frac{\sqrt{N_n(\alpha, \varpi)}}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=1}^{m_n} v_{(j-1)k_n+i}^* \right| \leq O_p^* \left(\sqrt{\frac{m_n}{N_n(\alpha, \omega)} \frac{b_n}{m_n}} \right),$$

in probability- \mathbb{P} , where, again, $\mathcal{A} \subset \mathbb{R} \setminus 0$ denotes a finite union of compact sets with positive Lebesgue measure, we may analyze the properties of $\hat{\mathcal{G}}_n^*(\tau)$ rather than those of $\hat{\mathcal{F}}_{W,n}^*(\tau)$. Next, recall that the statistics $\hat{\mathcal{G}}_n(\tau)$ and $\mathcal{G}_n(\tau)$ denote the bias-corrected empirical process and second-order asymptotic distribution, respectively, defined as in (16), then we will first show **(a)**,

$$\hat{\mathcal{G}}_n^*(\tau) = \sqrt{N_n(\alpha, \varpi)} \left(\hat{H}_{n,1}^*(\tau) + \hat{H}_{n,2}^*(\tau) \right) + o_p^*(1) \quad (\text{A.3})$$

where, as $n \rightarrow \infty$, $\sqrt{\lfloor n/k_n \rfloor m_n} (\hat{H}_{n,1}^*(\tau), \sqrt{k_n/m_n} \hat{H}_{n,2}^*(\tau)) \xrightarrow{d^*} (H_1(\tau), H_2(\tau))$, in probability- \mathbb{P} , with the functions $H_1(\tau)$ and $H_2(\tau)$ defined as in Lemma 2 and convergence holding uniformly in τ compact subsets of \mathbb{R} , that is, $\tau \in \mathcal{A}$. Once we have obtained this limit result, **(b)** follows by showing,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}^* \left(\hat{\mathcal{G}}_n^*(\tau) \leq x \right) - \mathbb{P} \left(\hat{\mathcal{G}}_n(\tau) \leq x \right) \right| \xrightarrow{\mathbb{P}} 0, \quad (\text{A.4})$$

uniformly for $\tau \in \mathcal{A}$. This simplifies, however, since, under the conditions for Lemma 2, we may apply the functional central limit theorem in the lemma for compact subsets of \mathbb{R} in conjunction with Slutsky's Theorem (as $N_n(\alpha, \varpi)/(\lfloor n/k_n \rfloor m_n) \xrightarrow{\mathbb{P}} 1$) and Polya's Theorem (see, e.g., Bhattacharya & Rao (1986)) to establish

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left(\widehat{\mathcal{G}}_n(\tau) \leq x \right) - \mathbb{P}(\mathcal{G}_n(\tau) \leq x) \right| \xrightarrow{\mathbb{P}} 0. \quad (\text{A.5})$$

Hence, if we can prove (A.3), then, by the same arguments,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left(\widehat{\mathcal{G}}_n^*(\tau) \leq x \right) - \mathbb{P}(\mathcal{G}_n(\tau) \leq x) \right| \xrightarrow{\mathbb{P}} 0, \quad (\text{A.6})$$

and (A.4) follows by the triangle inequality. The proof is, thus, completed by establishing the joint functional central limit theory in (A.3). To this end, let us introduce two quantities,

$$\begin{aligned} \widehat{H}_{n,1}^*(\tau) &\equiv \frac{1}{\lfloor n/m_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} (1 \{ \sqrt{n} \Delta_i^n W \leq \tau \} - \Phi(\tau)) v_i^*, \\ \widehat{H}_{n,2}^*(\tau) &\equiv \frac{\Phi'(\tau)\tau}{\lfloor n/k_n \rfloor} \sum_{j=1}^{\lfloor n/k_n \rfloor} \zeta_{n,j} v_{(j-1)k_n+1}^*, \quad \zeta_{n,j} \equiv \frac{1}{2} \left(\frac{\pi n}{2(k_n-1)} \sum_{i=(j-1)k_n+2}^{jk_n} |\Delta_{i-1}^n W| |\Delta_i^n W| - 1 \right), \end{aligned}$$

and make the decomposition $\widehat{\mathcal{G}}_n^*(\tau) = \widehat{\mathcal{G}}_{n,1}^*(\tau) + \widehat{\mathcal{G}}_{n,2}^*(\tau)$, where

$$\begin{aligned} \widehat{\mathcal{G}}_{n,1}^*(\tau) &\equiv \sqrt{N_n(\alpha, \varpi)} \left(\widehat{H}_{n,1}^*(\tau) + \widehat{H}_{n,2}^*(\tau) \right) \\ \widehat{\mathcal{G}}_{n,2}^*(\tau) &\equiv \frac{\sqrt{N_n(\alpha, \varpi)}}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=1}^{m_n} (X_{(j-1)k_n+i} - \Phi(\tau)) v_{(j-1)k_n+i}^* - \widehat{\mathcal{G}}_{n,1}^*. \end{aligned}$$

From here, the proof proceeds in two main steps:

Step 1: Show that $\sqrt{\lfloor n/k_n \rfloor m_n} (\widehat{H}_{n,1}^*(\tau), \sqrt{k_n/m_n} \widehat{H}_{n,2}^*(\tau)) \xrightarrow{d^*} (H_1(\tau), H_2(\tau))$, in probability- \mathbb{P} , with the joint functional convergence holding uniformly in $\tau \in \mathcal{A}$.

Step 2: Show that $\sup_{\tau \in \mathcal{A}} \left| \widehat{\mathcal{G}}_{n,2}^*(\tau) \right| \xrightarrow{\mathbb{P}^*} 0$, in probability- \mathbb{P} .

First, for **Step 1**, define $\widetilde{\mathcal{G}}_{n,1}^*(\tau) \equiv \sqrt{(\lfloor n/k_n \rfloor m_n)/N_n(\alpha, \varpi)} \widehat{\mathcal{G}}_{n,1}^*(\tau)$. Then, the functional central limit theory for the components of $\widetilde{\mathcal{G}}_{n,1}^*(\tau)$ follows by invoking Lemma A.3 for $\widetilde{\mathcal{G}}_{n,1}^*(\tau)$ and applying this with $\lfloor n/k_n \rfloor m_n/N_n(\alpha, \varpi) \xrightarrow{\mathbb{P}} 1$, the continuous mapping theorem and Slutsky's theorem.

Next, for **Step 2**, write, similarly, $\widetilde{\mathcal{G}}_{n,2}^*(\tau) = \sqrt{(\lfloor n/k_n \rfloor m_n)/N_n(\alpha, \varpi)} \widehat{\mathcal{G}}_{n,2}^*(\tau)$ and further decompose this term $\widetilde{\mathcal{G}}_{n,2}^*(\tau) = \widetilde{\mathcal{G}}_{n,2,1}^*(\tau) - \widetilde{\mathcal{G}}_{n,2,2}^*(\tau) - \widetilde{\mathcal{G}}_{n,1}^*(\tau)$, with $\widetilde{\mathcal{G}}_{n,1}^*(\tau)$ being defined in **Step 1**, and where

$$\widetilde{T}_{n,1}^*(\tau) \equiv \frac{1}{\sqrt{\lfloor n/k_n \rfloor m_n}} \widetilde{\mathcal{G}}_{n,2,1}^*(\tau) \equiv \frac{1}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=1}^{m_n} X_{(j-1)k_n+i} v_{(j-1)k_n+i}^*,$$

$$\tilde{T}_{n,2}^*(\tau) \equiv \frac{1}{\sqrt{[n/k_n]m_n}} \tilde{\mathcal{G}}_{n,2,2}^*(\tau) \equiv \frac{1}{[n/k_n]m_n} \sum_{j=1}^{[n/k_n]} \sum_{i=1}^{m_n} \Phi(\tau) v_{(j-1)k_n+i}^*.$$

From here, let us define and rewrite,

$$\begin{aligned} \hat{T}_n^*(\tau) &\equiv \frac{1}{[n/k_n]m_n} \sum_{j=1}^{[n/k_n]} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} 1 \left\{ \sqrt{n} \frac{\Delta_i^n Z}{\sigma_{(j-1)k_n}} 1 \left\{ \frac{|\Delta_i^n Z|}{\sqrt{\hat{V}_{n,j}}} \leq \alpha n^{-\varpi} \right\} \right. \\ &\quad \left. \leq \tau \frac{\sqrt{\hat{V}_{n,j}(i)}}{\sigma_{\frac{(j-1)k_n}{n}}} - \chi_{n,i,j}(1) - \tau \chi_{n,i,j}(2) \right\} v_i^* \\ &= \frac{1}{[n/k_n]m_n} \sum_{j=1}^{[n/k_n]} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} 1 \left\{ \sqrt{n} \Delta_i^n W \leq \tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(2) - \tilde{\xi}_{n,i,j}(3) \right\} v_i^* \end{aligned}$$

using quantities from Section A.2. We next establish that $\sup_{\tau \in \mathcal{A}} (\tilde{T}_{n,1}^*(\tau) - \hat{T}_n^*(\tau)) = o_p^*(1/k_n)$, in probability- \mathbb{P} , such that we may work with $\hat{T}_n^*(\tau)$ throughout the proof. To this end, write

$$\begin{aligned} \mathbb{E}^* \left[\left| \tilde{T}_{n,1}^*(\tau) - \hat{T}_n^*(\tau) \right| \right] &\leq \frac{1}{[n/k_n]m_n} \sum_{j=1}^{[n/k_n]} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} \mathbb{E} |v_i^*| \\ &\quad \times \left| X_i - 1 \left\{ \sqrt{n} \Delta_i^n W \leq \tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(2) - \tilde{\xi}_{n,i,j}(3) \right\} \right|. \end{aligned} \quad (\text{A.7})$$

Moreover, let η_n be a sequence of positive numbers that only depend on n , then we use the fact that the probability density of standard normal variable is uniformly bounded to write

$$\begin{aligned} \mathbb{E} \left| X_i - 1 \left\{ \sqrt{n} \Delta_i^n W \leq \tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(2) - \tilde{\xi}_{n,i,j}(3) \right\} \right| &\leq \mathbb{P}(|\chi_{n,i,j}(1)| + |\chi_{n,i,j}(2)| > \eta_n) \\ &+ \mathbb{E} \left| \Phi \left(\frac{\tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(2) + \eta_n(1 + |\tau|)}{\tilde{\xi}_{n,i,j}(4)} \right) - \Phi \left(\frac{\tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(2) - \eta_n(1 + |\tau|)}{\tilde{\xi}_{n,i,j}(4)} \right) \right| \\ &\leq K \left(\mathbb{P}(|\chi_{n,i,j}(1)| + |\chi_{n,i,j}(2)| > \eta_n) + \eta_n |\tau| \right), \end{aligned}$$

similarly to the corresponding term in Todorov & Tauchen (2014, Section 10.4.1). Hence, we invoke their bounds in equations (10.14), (10.16)-(10.19), (10.28), (10.29) and (10.31) to show

$$\mathbb{P}(|\chi_{n,i,j}(1)| + |\chi_{n,i,j}(2)| > \eta_n) \leq K \left[\frac{1}{n\eta_n} \vee \frac{1}{\eta_n^p [n^{p/2} \wedge (n/k_n)^p \wedge k_n^{3p/2}]} \vee \left(\frac{k_n}{n} \right)^{\frac{1}{1+\iota}} \frac{1}{\eta_n^\iota} \right], \quad (\text{A.8})$$

for every $p \geq 1$ and arbitrarily small $\iota > 0$. Consequently, by picking $\eta_n \asymp n^{-q-\iota}$, $\iota \in (0, 1/2 - q)$ and combining (A.7) with Assumption DWB and (A.8), $k_n \mathbb{E}^* [\tilde{T}_{n,1}^*(\tau) - \hat{T}_n^*(\tau)] \leq o_p^*(1)$ such that for any

finite union of compact subsets, $\mathcal{A} \subset \mathbb{R} \setminus 0$, we have the required, uniform, result,

$$\sup_{\tau \in \mathcal{A}} \left| \widehat{T}_{n,1}^*(\tau) - \widehat{T}_n^*(\tau) \right| = o_p^*(1/k_n). \quad (\text{A.9})$$

Now, let us make the decomposition,

$$\widehat{T}_n^*(\tau) - \widetilde{T}_{n,2}^*(\tau) = \sum_{i=1}^6 A_{n,i}^*(\tau), \quad (\text{A.10})$$

where $A_{n,1}^*(\tau) \equiv \widehat{H}_{n,1}^*(\tau)$ and the remaining terms are defined analogously as

$$\begin{aligned} A_{n,2}^*(\tau) &\equiv \frac{1}{\lfloor n/k_n \rfloor} \sum_{j=1}^{\lfloor n/k_n \rfloor} \left(\Phi \left(\tau + \tau \bar{\xi}_{n,j}(1) - \tau \bar{\xi}_{n,j}(2) \right) - \Phi(\tau) \right) v_{(j-1)k_n+1}^*, \\ A_{n,3}^*(\tau) &\equiv \frac{1}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} a_{n,i}(\tau) \times v_i^*, \quad \text{where, furthermore,} \\ a_{n,i}(\tau) &\equiv 1 \left\{ \sqrt{n} \Delta_i^n W \leq \frac{\tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(2)}{\xi_{n,i,j}(4)} \right\} - 1 \left\{ \sqrt{n} \Delta_i^n W \leq \tau \right\} \\ &\quad + \Phi(\tau) - \Phi \left(\frac{\tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(2)}{\xi_{n,i,j}(4)} \right), \\ A_{n,4}^*(\tau) &\equiv \frac{1}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} v_i^* \\ &\quad \times \left[\Phi \left(\tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(2) \right) - \Phi \left(\tau + \tau \bar{\xi}_{n,j}(1) - \tau \bar{\xi}_{n,j}(2) \right) \right], \\ A_{n,5}^*(\tau) &\equiv \frac{1}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} v_i^* \\ &\quad \times \left[\Phi \left(\frac{\tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(2)}{\xi_{n,i,j}(4)} \right) - \Phi \left(\tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(2) \right) \right], \\ A_{n,6}^*(\tau) &\equiv \frac{1}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} \left(\Phi \left(\tau + \tau \bar{\xi}_{n,j}(1) - \tau \bar{\xi}_{n,j}(2) \right) - \Phi(\tau) \right) v_i^* - A_{n,2}^*(\tau). \end{aligned}$$

Hence, we need to establish bounds for $A_{n,2}^*(\tau) - \widehat{H}_{n,2}^*(\tau)$, $A_{n,3}^*(\tau)$, $A_{n,4}^*(\tau)$, $A_{n,5}^*(\tau)$ and $A_{n,6}^*(\tau)$. For the first term, apply a second-order Taylor expansion to obtain the leading components,

$$A_{n,2}^*(\tau, 1) = \frac{1}{\lfloor n/k_n \rfloor} \sum_{j=1}^{\lfloor n/k_n \rfloor} \Phi'(\tau) \tau \bar{\xi}_{n,j}(1) v_{(j-1)k_n+1}^*,$$

$$A_{n,2}^*(\tau, 2) = \frac{1}{\lfloor n/k_n \rfloor} \sum_{j=1}^{\lfloor n/k_n \rfloor} \left(\frac{\Phi''(\tau) \tau^2 (\bar{\xi}_{n,j}(1))^2}{2} - \Phi'(\tau) \tau \bar{\xi}_{n,j}(2) \right) v_{(j-1)k_n+1}^*.$$

Then, by applying the stochastic bounds,

$$\begin{cases} \mathbb{E} \left| \bar{V}_{n,j} - \sigma_{\frac{(j-1)k_n}{n}}^2 \right|^p \leq K k_n^{-p/2}, \\ \mathbb{E}_{(j-1)k_n}^n \left[R_j^{(4)} \right] = 0, \quad \mathbb{E} \left| R_j^{(4)} \right|^p \leq K (k_n/n)^{p/2}, \quad \forall p \geq 2, \\ \mathbb{E} \left[\left| R_j^{(4)} - \tilde{R}_{i,j}^{(4)} \right|^p + \left| \tilde{R}_{i,j}^{(4)} - \tilde{R}_{i,j}^{(4)}(i) \right|^p \right] \leq K \left(\frac{1}{\sqrt{n}} \right)^p, \quad \forall p > 0, \end{cases} \quad (\text{A.11})$$

cf. Todorov & Tauchen (2014, Equations (10.25)-(10.26)), with Assumption DWB and the fact the probability density of a standard normal density and Φ'' are uniformly bounded, we have

$$\mathbb{E} \left[\mathbb{E}^* \left| A_{n,2}^*(\tau) - A_{n,2}^*(\tau, 1) - A_{n,2}^*(\tau, 2) \right| \right] \leq K (|\tau|^3 \vee |\tau|^2) \left[\left(\frac{k_n}{n} \right)^{3/2} \vee \left(\frac{1}{k_n} \right)^{3/2} \right], \quad (\text{A.12})$$

and, as a result, $\sup_{\tau \in \mathcal{A}} |A_{n,2}^*(\tau) - A_{n,2}^*(\tau, 1) - A_{n,2}^*(\tau, 2)| = o_p^*(1/k_n)$ in probability- \mathbb{P} , similarly to convergence in (A.9). For the first of the two Taylor expansion terms, write

$$\mathbb{E}^* \left| A_{n,2}^*(\tau, 1) - \hat{H}_{n,2}^*(\tau) \right| \leq \frac{1}{\lfloor n/k_n \rfloor} \sum_{j=1}^{\lfloor n/k_n \rfloor} |\Phi'(\tau) \tau| \times |\bar{\xi}_{n,j}(1) - \zeta_{n,j}| \times \mathbb{E} |v_{(j-1)k_n+1}^*|.$$

Hence, by Assumption DWB and the bounds in (A.11), we have

$$\mathbb{E} |\bar{\xi}_{n,j}(1) - \zeta_{n,j}| \leq K (1/\sqrt{n} \vee k_n/n), \quad (\text{A.13})$$

and, consequently, it follows that $\sup_{\tau \in \mathcal{A}} |A_{n,2}^*(\tau, 1) - \hat{H}_{n,2}^*(\tau)| \leq o_p^*(1/k_n)$, in probability- \mathbb{P} . For the second Taylor expansion term, $A_{n,2}^*(\tau, 2)$, we further decompose it into,

$$\begin{aligned} A_{n,2}^*(\tau, 2) &= A_{n,2}^*(\tau, 2, 1) + A_{n,2}^*(\tau, 2, 2), \quad A_{n,2}^*(\tau, 2, 1) = \frac{H_3(\tau)}{\lfloor n/k_n \rfloor k_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} v_{(j-1)k_n+1}^*, \\ A_{n,2}^*(\tau, 2, 2) &= \frac{1}{\lfloor n/k_n \rfloor} \sum_{j=1}^{\lfloor n/k_n \rfloor} \left(\frac{\Phi''(\tau) \tau^2 (\bar{\xi}_{n,j}(1))^2}{2} - \Phi'(\tau) \tau \bar{\xi}_{n,j}(2) - H_3(\tau)/k_n \right) v_{(j-1)k_n+1}^*. \end{aligned}$$

Now, for the first of these terms, we deduce that

$$|A_{n,2}^*(\tau, 2, 1)| \leq K (|\tau| \vee \tau^2) \times O_p \left(\sqrt{b_n/m_n} / \left(\sqrt{k_n n} \right) \right) \leq K (|\tau| \vee \tau^2) \times O_p \left(1 / \left(\sqrt{k_n n} \right) \right)$$

by computing the mean and variance using Assumption DWB.²⁴ For the second term,

$$\mathbb{E}^* |A_{n,2}^*(\tau, 2, 2)| \leq \frac{1}{\lfloor n/k_n \rfloor} \sum_{j=1}^{\lfloor n/k_n \rfloor} \left| \frac{\Phi''(\tau) \tau^2 (\bar{\xi}_{n,j}(1))^2}{2} - \Phi'(\tau) \tau \bar{\xi}_{n,j}(2) - H_3(\tau)/k_n \right| \times \mathbb{E} |v_{(j-1)k_n+1}^*|.$$

As in (A.13), we may apply (A.11) to show $\mathbb{E} |\bar{\xi}_{n,j}(2) - \hat{\xi}_{n,j}(2)| \leq (1/\sqrt{n} \vee (k_n/n))$. Moreover, since we have the identity $(\bar{\xi}_{n,j}(1))^2/2 = \bar{\xi}_{n,j}(2)$ as well as the approximation (Todorov & Tauchen 2014, p. 1883),

$$\mathbb{E}_{(j-1)k_n}^n [\hat{\xi}_{n,j}(2)] = \frac{1}{8k_n} \left(\left(\frac{\pi}{2} \right)^2 + \pi - 3 \right) + o(1/k_n), \quad (\text{A.14})$$

we may collect asymptotic bounds in conjunction with Assumption DWB and successive conditioning to show $|A_{n,2}^*(\tau, 2, 2)| \leq K(|\tau| \vee \tau^2) O_p^*(1/\sqrt{n} \vee k_n/n) + o_p^*(1/k_n)$, in probability- \mathbb{P} . Hence, by combining results,

$$\sup_{\tau \in \mathcal{A}} |A_{n,2}^*(\tau) - \hat{H}_{n,2}^*(\tau)| = o_p^*(1/k_n), \quad \text{in probability-}\mathbb{P}, \quad (\text{A.15})$$

showing that the remaining asymptotic bias is negligible for the LDWB.

For the next term, $A_{n,3}^*(\tau)$, we have $\mathbb{E}^*[A_{n,3}^*(\tau)] = 0$ by $\mathbb{E}^*[a_{n,i}(\tau)v_i^*] = a_{n,i}(\tau)\mathbb{E}[v_i^*] = 0$. Moreover, we have $\mathbb{E}[a_{n,i}(\tau)] = 0$, $\mathbb{E}[a_{n,i}(\tau)^2] \leq K|\tau|((k_n/n)^{1/2} \vee k_n^{-1/2})$, and $\mathbb{E}[a_{n,i}(\tau)a_{n,g}(\tau)] = 0$ for $|i - g| > k_n$ due to independence of the Brownian increments $\Delta_i^n W$, $\Delta_g^n W$, $\Delta_i^n W'$ and $\Delta_g^n W'$. When $|i - g| \leq k_n$, we follow Todorov & Tauchen (2014) and use that $\xi_{n,i,j}(4)$ is adapted to $\mathcal{F}_{t_{i-1}}$ as well as decompose the $a_{n,g}(\tau)$ component into a part with the i th increment removed from $\tilde{\xi}_{n,g,j}(1)$ and $\tilde{\xi}_{n,g,j}(2)$, denoted by $\bar{a}_{n,g}(\tau)$, and a residual $\tilde{a}_{n,g}(\tau) = a_{n,g}(\tau) - \bar{a}_{n,g}(\tau)$. For these, we have $\mathbb{E}[a_{n,g}(\tau)\bar{a}_{n,g}(\tau)] = 0$ and, by their arguments (cf. pp. 1880-1881), the triangle inequality and Chebyshev's inequality,

$$\mathbb{E} |a_{n,g}(\tau)\tilde{a}_{n,g}(\tau)| \leq K(|\tau| \vee \tau^2) \left(\left(\frac{k_n}{n} \right)^{1-2\iota} \vee \frac{1}{k_n^{1-2\iota}} \right)$$

for some arbitrarily small $\iota > 0$ and sufficiently large n . We apply these results in conjunction with the convergence part of Assumption DWB, $\text{Cov}(v_i^*, v_g^*) \rightarrow \hat{C}_{i,g}$ for all $(i, g) \in 1, \dots, n$ where $\hat{C}_{i,g} \geq 0$ is a nonrandom constant, the triangle inequality as well as the Cauchy-Schwarz inequality to show

$$\begin{aligned} \mathbb{E}^* [A_{n,3}^*(\tau)^2] &= \frac{1}{(\lfloor n/k_n \rfloor m_n)^2} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=(j-1)k_n+1}^{(j-1)+m_n} \sum_{h=1}^{\lfloor n/k_n \rfloor} \sum_{g=(j-1)k_n+1}^{(j-1)+m_n} a_{n,i}(\tau) a_{n,g}(\tau) \text{Cov}(v_i^*, v_g^*) \\ &\leq \frac{K(|\tau| \vee \tau^2)}{(\lfloor n/k_n \rfloor m_n)^2} \sum_{j=1}^{\lfloor n/k_n \rfloor} O_p \left(m_n \left(\sqrt{\frac{k_n}{n}} \vee \frac{1}{\sqrt{k_n}} \right) + m_n^2 \left(\left(\frac{k_n}{n} \right)^{1-2\iota} \vee \frac{1}{k_n^{1-2\iota}} \right) \right) \\ &= K(|\tau| \vee \tau^2) \times O_p \left(\frac{1}{\lfloor n/k_n \rfloor m_n} \left(\frac{1}{k_n} \right)^{1/2} \vee \frac{k_n^{2\iota}}{n} \right), \end{aligned}$$

²⁴To see this, note that $\mathbb{E}[A_{n,2}^*(\tau, 2, 1)] = 0$ and $\text{Cov}(v_r^*, v_s^*) = O_p(b_n/(m_n \lfloor n/k_n \rfloor))$ for integers r and s by Assumption DWB. Since the scale is $1/(\lfloor n/k_n \rfloor k_n)^2$ for the second moment and $m_n/k_n \rightarrow 0$, the bound follows immediately.

which, consequently, provides the bound $\sup_{\tau \in \mathcal{A}} |A_{n,3}^*(\tau)| \leq o_p^*(1/k_n)$, in probability- \mathbb{P} .

Next, for $A_{n,4}^*(\tau)$, write

$$\mathbb{E}^* [|A_{n,4}^*(\tau)|] \leq \frac{1}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{(j-1)k_n+1}^{(j-1)k_n+m_n} \left| \hat{A}_{n,4,i}(\tau) \right| \times \mathbb{E} |v_i^*|, \quad \text{where}$$

$$\hat{A}_{n,4,i}(\tau) \equiv \Phi \left(\tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(2) \right) - \Phi \left(\tau + \tau \tilde{\xi}_{n,j}(1) - \tau \tilde{\xi}_{n,j}(2) \right).$$

We then apply a Taylor expansion, similarly to the one for $A_{n,2}^*(\tau)$, and use the same arguments as in (A.12), (A.13) and for $A_{n,2}^*(\tau)$ to show $|\hat{A}_{n,4,i}(\tau)| \leq K(|\tau| \vee \tau^2) O_p(n^{-1/2} \vee (k_n/n)) + o_p(1/k_n)$. By combining this with Assumption DWB, $\sup_{\tau \in \mathcal{A}} |A_{n,4}^*(\tau)| \leq o_p^*(1/k_n)$ in probability- \mathbb{P} .

For $A_{n,5}^*(\tau)$, we apply a decomposition similarly to one on Todorov & Tauchen (2014, pp. 1881-1882). Hence, by the triangle inequality, $\forall \iota > 0$ and n sufficiently high, write

$$\mathbb{E}^* [|A_{n,5}^*(\tau)|] \leq \frac{1}{\lfloor n/k_n \rfloor m_n} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{(j-1)k_n+1}^{(j-1)k_n+m_n} |b_{n,i}(1) + b_{n,i}(2) + b_{n,i}(3)| \times \mathbb{E} |v_i^*|, \quad (\text{A.16})$$

where $b_{n,i}(\tau, 1)$, $b_{n,i}(\tau, 2)$, and $b_{n,i}(\tau, 3)$ are defined as

$$\begin{aligned} b_{n,i}(\tau, 1) &\equiv \left[\Phi \left(\frac{\tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(2)}{\xi_{n,i,j}(4)} \right) - \Phi \left(\tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(2) \right) \right] \\ &\quad \times 1 \left\{ |\xi_{n,i,j}(4) - 1| \geq (k_n/n)^{1/2-\iota} \right\}, \\ b_{n,i}(\tau, 2) &\equiv \Phi' \left(\tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(1) \right) \left(\tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(1) \right) (\xi_{n,i,j}(4) - 1) \\ &\quad \times 1 \left\{ |\xi_{n,i,j}(4) - 1| < (k_n/n)^{1/2-\iota} \right\}, \\ b_{n,i}(\tau, 3) &\leq K \frac{\left| \tau + \tau \tilde{\xi}_{n,i,j}(1) - \tau \tilde{\xi}_{n,i,j}(1) \right|^2}{(1 - (k_n/n)^{1/2-\iota})^3} |\xi_{n,i,j}(4) - 1|^2, \end{aligned}$$

for an arbitrarily small $\iota > 0$. Moreover, we may readily invoke the following inequalities,

$$\begin{aligned} \mathbb{E} [|b_{n,i}(\tau, 1)| + |b_{n,i}(\tau, 3)|] &\leq K(\tau^2 \vee 1) \frac{k_n}{n}, \\ \mathbb{E} \left| b_{n,i}(\tau, 2) - \Phi'(\tau) \tau (\xi_{n,i,j}(4) - 1) 1 \left\{ |\xi_{n,i,j}(4) - 1| < (k_n/n)^{1/2-\iota} \right\} \right| &\leq K|\tau| \frac{1}{\sqrt{n}}. \end{aligned}$$

By combining these with Assumption DWB and the uniform boundedness of probability density of a standard normal distribution and its derivative, $\mathbb{E}^* [|A_{n,5}^*(\tau)|] \leq K(|\tau| \vee \tau^2) O_p((k_n/n) \vee n^{-1/2})$ such that $\sup_{\tau \in \mathcal{A}} |A_{n,5}^*(\tau)| \leq o_p^*(1/k_n)$, in probability- \mathbb{P} .

For the last term, $A_{n,6}^*(\tau)$, define $B_j(\tau) \equiv \Phi(\tau + \tau \bar{\xi}_{n,j}(1) - \tau \bar{\xi}_{n,j}(2)) - \Phi(\tau)$ and rewrite,

$$A_{n,6}^*(\tau) = \frac{1}{\lfloor n/k_n \rfloor} \sum_{j=1}^{\lfloor n/k_n \rfloor} B_j(\tau) z_{(j-1)k_n+1}^*, \quad \text{where} \quad z_{(j-1)k_n+1}^* = \frac{1}{m_n} \sum_{i=(j-1)k_n+2}^{(j-1)k_n+m_n} v_i^*.$$

Hence, $A_{n,6}^*(\tau)$ has the same form as $A_{n,2}^*(\tau)$ with $z_{(j-1)k_n+1}^*$ in place of $v_{(j-1)k_n+1}^*$ and may be treated in a similar manner. As a result, and analogously to the leading term $\hat{H}_{n,2}^*(\tau)$, define

$$\tilde{H}_{n,2}^*(\tau) \equiv \frac{\Phi'(\tau)\tau}{\lfloor n/k_n \rfloor} \sum_{j=1}^{\lfloor n/k_n \rfloor} \zeta_{n,j} z_{(j-1)k_n+1}^*, \quad (\text{A.17})$$

then, by the arguments provided for (A.15), $\sup_{\tau \in \mathcal{A}} |A_{n,6}^*(\tau) - \tilde{H}_{n,2}^*(\tau)| = o_p^*(1/k_n)$, in probability- \mathbb{P} . Next, $\mathbb{E}^*[\tilde{H}_{n,2}^*(\tau)] = 0$ follows by Assumption DWB and, if additionally using boundedness of the probability density of a standard normal variable and independence of the Brownian increments,

$$\mathbb{E} \left(\mathbb{E}^* \left[(\tilde{H}_{n,2}^*(\tau))^2 \right] \right) \leq \frac{K^2 \tau^2}{(\lfloor n/k_n \rfloor)^2} \sum_{j=1}^{\lfloor n/k_n \rfloor} \mathbb{E} [\zeta_{n,j}^2] \times \mathbb{E} \left[(z_{(j-1)k_n+1}^*)^2 \right], \quad (\text{A.18})$$

for which $\mathbb{E}[\zeta_{n,j}^2] \leq K$ by Lemma A.1(a) and with

$$\mathbb{E} \left[(z_{(j-1)k_n+1}^*)^2 \right] = \frac{1}{m_n^2} \sum_{i=(j-1)k_n+2}^{(j-1)k_n+m_n} \sum_{h=(j-1)k_n+2}^{(j-1)k_n+m_n} \text{Cov}(v_i^*, v_h^*) \leq O \left(\frac{b_n}{\lfloor n/k_n \rfloor m_n} \right). \quad (\text{A.19})$$

Hence, by combining results, $|\tilde{H}_{n,2}^*(\tau)| \leq O_p^*((|\tau| \vee \tau^2)(k_n/n)\sqrt{b_n/m_n}) = o_p^*(1/k_n)$, in probability \mathbb{P} , which, together with the triangle inequality, establishes that $\sup_{\tau \in \mathcal{A}} |A_{n,6}^*(\tau)| = o_p^*(1/k_n)$. Now, by collecting asymptotic bounds for the sequence $\sum_{i=1}^6 A_{n,i}^*(\tau)$ and using them in conjunction with the triangle inequality and (A.9), $\sup_{\tau \in \mathcal{A}} |\hat{\mathcal{G}}_{n,2}^*(\tau)| \leq o_p^*(\sqrt{(nm_n)/k_n^3}) = o_p^*(1)$, in probability- \mathbb{P} . Finally, since we have $N_n(\alpha, \varpi)/(\lfloor n/k_n \rfloor m_n) \xrightarrow{\mathbb{P}} 1$, the corresponding asymptotic bound for $\hat{\mathcal{G}}_{n,2}^*(\tau)$ in **Step 2** follows by an application of the continuous mapping theorem. \square

A.4 Proof of Lemma 3

Apart from the fourth moment result, **(a)-(c)** and block-stationarity follows by the properties of multinomial random variables and by $v_{(j-1)k_n+i}^\diamond$ being constant across $i = 1, \dots, m_n$ for a given $j = 1, \dots, \lfloor n/k_n \rfloor$. For the fourth moment bound, we may use the c_r -inequality to deduce

$$\mathbb{E} [|v_i^\diamond|^4] \leq K_1 \mathbb{E} [|\zeta_i^\diamond|^4] + K_2 < K, \quad (\text{A.20})$$

for constants $K_1 < \infty$ and $K_2 < \infty$, using also the bound $\mathbb{E}[|\zeta_i^\diamond|^4] < 15$, see, e.g., Præstgaard & Wellner (1993, Example 3.2), for the last inequality. Finally, for **(d)**, write

$$\begin{aligned} \sum_{i=1}^{\lfloor n/k_n \rfloor m_n} \text{Cov}(u_1^\diamond, u_i^\diamond) &= \sum_{i=1}^{m_n} \text{Cov}(u_1^\diamond, u_i^\diamond) + \sum_{i=m_n+1}^{\lfloor n/k_n \rfloor m_n} \text{Cov}(u_1^\diamond, u_i^\diamond) \\ &= m_n(1 - 1/p_n) - m_n(p_n - 1)/p_n = o(m_n), \end{aligned}$$

using the variance-covariance properties **(a)**-**(c)**, thus concluding the proof. \square

A.5 Proof of Theorem 3

The result, similarly to Doukhan et al. (2015, Theorem 4.4), follows by Beutner & Zähle (2014, Theorem 3.15(ii)), if we can verify conditions (a)-(c) for the latter. First, for (a), we need to verify the conditions for their Lemmas 3.4 and 3.6.²⁵ Specifically, conditions (a)-(c) of Lemmas 3.4 and 3.6 is satisfied since $\hat{F}_n(\tau)$ is the empirical CDF, $\Phi(\tau)$ is Gaussian, $\tau_1, \tau_2 \in \mathcal{A}$ and by the regularity conditions on the kernel function in Assumption 3. Next, for condition (b) of Beutner & Zähle (2014, Theorem 3.15(ii)), this follows by Lemma 2, Assumption 3 and Beutner & Zähle (2014, Remark 3.16). Finally, condition (c) follows since the functional central limit theory in Lemma 2 holds on compact subsets of \mathbb{R} , and since the limiting distribution, $\mathcal{G}_n(\tau)$, has continuous paths on these subsets.

A.6 Technical Results

Lemma A.1 (Todorov & Tauchen (2014), central limit theory for leading terms.). *Suppose that the regularity conditions of Lemma 2 hold. Moreover, let*

$$\begin{pmatrix} \sqrt{\lfloor n/k_n \rfloor m_n} \hat{H}_{n,1}(\tau) \\ \sqrt{\lfloor n/k_n \rfloor k_n} \hat{H}_{n,2}(\tau) \end{pmatrix} \equiv \sum_{i=1}^{\lfloor n/k_n \rfloor k_n} \begin{pmatrix} \mathcal{Z}_i(1) \\ \frac{\Phi'(\tau)\tau}{2} (\mathcal{Z}_i(2) + \mathcal{Z}_i(3)) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\Phi'(\tau)\tau}{2} \tilde{\mathcal{Z}} \end{pmatrix}, \quad (\text{A.21})$$

where, with $\mathbb{I}_n \equiv \{i = (j-1)k_n + 1, \dots, (j-1)k_n + m_n; j = 1, \dots, \lfloor n/k_n \rfloor\}$, the elements of \mathcal{Z}_i are defined as

$$\mathcal{Z}_i = \begin{pmatrix} \frac{1}{\sqrt{\lfloor n/k_n \rfloor m_n}} [1\{\sqrt{n}\Delta_i^n W \leq \tau\} - \Phi(\tau)] \\ \frac{1}{\sqrt{\lfloor n/k_n \rfloor k_n}} \frac{\pi}{2} |\sqrt{n}\Delta_{i-1}^n W| \left(|\sqrt{n}\Delta_i^n W| - \sqrt{2/\pi} \right) \\ \frac{1}{\sqrt{\lfloor n/k_n \rfloor k_n}} \sqrt{\frac{\pi}{2}} \left(|\sqrt{n}\Delta_i^n W| - \sqrt{2/\pi} \right) \end{pmatrix}, \quad i \in \mathbb{I}_n, \quad (\text{A.22})$$

²⁵This simplifies slightly since we restrict the integration range of the arguments to a compact subset $\mathcal{A} \subset \mathbb{R} \setminus 0$. In particular, conditions (d) of both Lemmas 3.4 and 3.6 requires the integral to be well-behaved as the arguments $\tau_1, \tau_2 \rightarrow \pm\infty$. Hence, as the integration is carried out over $\tau_1, \tau_2 \in \mathcal{A}$, such conditions are avoided here.

and, for $i = 1, \dots, n \setminus \mathbb{I}_n$, \mathbf{Z}_i is defined as above, but with the first element replaced by zero. Finally, the residual term, $\tilde{\mathbf{Z}}$, is defined with $\Delta_0^n W = 0$ as

$$\begin{aligned} \tilde{\mathbf{Z}} &= \frac{-(\pi/2)}{\sqrt{\lfloor n/k_n \rfloor k_n}} \sum_{j=1}^{\lfloor n/k_n \rfloor} \tilde{\mathbf{Z}}_j, \quad \text{where} \\ \tilde{\mathbf{Z}}_j &= \left[|\sqrt{n} \Delta_{(j-1)k_n}^n W| \left(|\sqrt{n} \Delta_{(j-1)k_n+1}^n W| - \sqrt{\frac{2}{\pi}} \right) + \sqrt{\frac{2}{\pi}} \left(|\sqrt{n} \Delta_{jk_n}^n W| - \sqrt{\frac{2}{\pi}} \right) \right]. \end{aligned} \quad (\text{A.23})$$

Then, uniformly in τ over compact subsets of \mathbb{R} , it follows that

$$(a) \quad \mathbb{E}_{i-1}^n[\mathbf{Z}_i] = \mathbf{0}, \quad \sum_{i=1}^{\lfloor n/k_n \rfloor k_n} \mathbb{E}_{i-1}^n[\|\mathbf{Z}_i\|^{2+\iota}] \rightarrow 0, \quad \forall \iota > 0 \text{ and}$$

$$\sum_{i=1}^{\lfloor n/k_n \rfloor k_n} \mathbb{E}_{i-1}^n[\mathbf{Z}_i \mathbf{Z}_i'] \rightarrow \mathbf{C}_{\mathcal{Z}}(\tau), \quad \mathbf{C}_{\mathcal{Z}}(\tau) \equiv \begin{pmatrix} \Phi(\tau)(1 - \Phi(\tau)) & 0 & 0 \\ 0 & (\frac{\pi}{2})^2(1 - \frac{2}{\pi}) & \frac{\pi}{2}(1 - \frac{2}{\pi}) \\ 0 & \frac{\pi}{2}(1 - \frac{2}{\pi}) & \frac{\pi}{2}(1 - \frac{2}{\pi}) \end{pmatrix}.$$

$$(b) \quad \text{Let } H_1(\tau) \text{ and } H_2(\tau) \text{ be defined as in Lemma 2, then } \mathbb{E}[\tilde{\mathbf{Z}}^2] \leq K/k_n \text{ and}$$

$$\sum_{i=1}^{\lfloor n/k_n \rfloor k_n} \begin{pmatrix} \mathbf{Z}_i(1) \\ \frac{\Phi'(\tau)\tau}{2} (\mathbf{Z}_i(2) + \mathbf{Z}_i(3)) \end{pmatrix} \xrightarrow{d} \begin{pmatrix} H_1(\tau) \\ H_2(\tau) \end{pmatrix}.$$

Proof. This follows by the arguments on Todorov & Tauchen (2014, pp. 1883-1884). \square

Lemma A.2 (Block Moments and CLT). *For $i = 1, \dots, \lfloor n/k_n \rfloor k_n$, we let $\bar{v}_i^* = v_{(j-1)k_n+1}^*$ when $i \in (j-1)k_n + 1, \dots, jk_n$ with $j \in 1, \dots, \lfloor n/k_n \rfloor$, and write $\mathbf{Z}_i^*(1) = \mathbf{Z}_i(1)v_i^*$, $\mathbf{Z}_i^*(2) = \mathbf{Z}_i(2)\bar{v}_i^*$ and $\mathbf{Z}_i^*(3) = \mathbf{Z}_i(3)\bar{v}_i^*$, and for which the triplet $\mathbf{Z}_i(1)$, $\mathbf{Z}_i(2)$ and $\mathbf{Z}_i(3)$ are defined as in (A.22). Moreover, these are collected in the vector $\mathbf{Z}_i^* = (\mathbf{Z}_i^*(1), \mathbf{Z}_i^*(2), \mathbf{Z}_i^*(3))'$. Finally, let K_n be a sequence of integers that satisfies $1/K_n + K_n/n \rightarrow 0$ and $k_n/K_n \rightarrow \varrho_k \geq 0$ as $n \rightarrow \infty$, then*

- (a) $\frac{\lfloor n/k_n \rfloor k_n}{K_n} \sum_{i=1}^{K_n} \mathbb{E}_{i-1}^n[\mathbf{Z}_i \mathbf{Z}_i'] \rightarrow \mathbf{C}_{\mathcal{Z}}(\tau)$ and, for $i \neq j$, $\frac{\lfloor n/k_n \rfloor k_n}{K_n} \sum_{i,j=1}^{K_n} \mathbf{Z}_i \mathbf{Z}_j' = o_p(1)$.
- (b) $\frac{\lfloor n/k_n \rfloor k_n}{K_n} \sum_{i=1}^{K_n} \mathbb{E}^*[\mathbf{Z}_i^* (\mathbf{Z}_i^*)'] \xrightarrow{\mathbb{P}} \mathbf{C}_{\mathcal{Z}}(\tau)$ and, for $i \neq j$, $\frac{\lfloor n/k_n \rfloor k_n}{K_n} \sum_{i,j=1}^{K_n} \mathbb{E}^*[\mathbf{Z}_i^* (\mathbf{Z}_j^*)'] = o_p^*(1)$.
- (c) Uniformly in τ over compact subsets of \mathbb{R} ,

$$\sqrt{\frac{\lfloor n/k_n \rfloor k_n}{K_n}} \sum_{i=1}^{K_n} \mathbf{Z}_i^* \xrightarrow{d} N(\mathbf{0}, \mathbf{C}_{\mathcal{Z}}(\tau)).$$

Proof. The first part of (a) follows by changing the scale of \mathbf{Z}_i and using Lemma A.1(a). The second part follows by using the Markov inequality for the martingale difference sequence, \mathbf{Z}_i , $i = 1, \dots, n$, and subsequently the $(2 + \iota)$ -moment result in Lemma A.1(a).

Next, for **(b)**, utilize the decomposition $\mathbb{E}^*[\mathbf{Z}_i^*(\mathbf{Z}_j^*)'] = \mathbf{Z}_i \mathbf{Z}_j' \text{Cov}(v_i^*, v_j^*)$ for all $i, j = 1, \dots, K_n$, which, in conjunction with **(a)** and Assumption DWB, delivers the results.

Last, for **(c)**, and similarly to Todorov & Tauchen (2014, pp. 1883-1884), $\mathbb{E}_{i-1}^n(\mathbb{E}^*[\mathbf{Z}_i^*]) = 0$,

$$\begin{aligned} \frac{\lfloor n/k_n \rfloor k_n}{K_n} \sum_{i=1}^{K_n} \mathbb{E}_{i-1}^n(\mathbb{E}^*[\mathbf{Z}_i^*(\mathbf{Z}_i^*)']) &\rightarrow \mathbf{C}_Z(\tau), \quad \text{and} \\ \left(\frac{\lfloor n/k_n \rfloor k_n}{K_n} \right)^2 \sum_{i=1}^{K_n} \mathbb{E}_{i-1}^n(\mathbb{E}^*[\|\mathbf{Z}_i^*\|^4]) &\leq K \left(\frac{\lfloor n/k_n \rfloor k_n}{K_n} \right)^2 \sum_{i=1}^{K_n} \mathbb{E}_{i-1}^n(\|\mathbf{Z}_i\|^4) \mathbb{E}^*[\|\bar{\mathbf{v}}_i^*\|^4] \rightarrow 0, \end{aligned}$$

with $\bar{\mathbf{v}}_i^* = (v_i^*, \bar{v}_i^*, \bar{v}_i^*)'$, using the same arguments as for **(a)** and **(b)**, Lemma A.1(a) as well as Assumption DWB(a). Together independence of the Brownian increments and successive conditioning under the \mathbb{P}^* and \mathbb{P} measures, we may invoke the central limit theorem for martingale difference sequences, e.g., Hall & Heyde (1980, Chapter 3), to establish the limit result, pointwise in τ .

Finally, to establish uniformity of the CLT in τ over compact subsets $\mathcal{A} \subset \mathbb{R} \setminus 0$, and since the two-dimensional vector $(\mathbf{Z}_i^*(2), \mathbf{Z}_i^*(3))'$ is invariant to τ , it suffices to show *stochastic equicontinuity* for the statistic,

$$\mathcal{O}_n^*(\tau) \equiv \sqrt{\frac{\lfloor n/k_n \rfloor k_n}{K_n}} \sum_{i=1; i \in \mathbb{I}_n}^{K_n} \mathbf{Z}_i^*(1, \tau) = \sqrt{\frac{k_n}{m_n K_n}} \sum_{i=1; i \in \mathbb{I}_n}^{K_n} [1\{\sqrt{n}\Delta_i^n W \leq \tau\} - \Phi(\tau)] v_i^*, \quad (\text{A.24})$$

where we have explicated the dependence of $\mathbf{Z}_i^*(1)$ on τ as $\mathbf{Z}_i^*(1, \tau)$ and of the summation on \mathbb{I}_n , the indicator defined in Lemma A.1. Now, as in the proofs of Gine & Zinn (1990, Theorem 3.1) and Doukhan et al. (2015, Corollary 3.2), this involves showing that for any $\epsilon > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{|\tau_1 - \tau_2| < \delta; \tau_1, \tau_2 \in \mathcal{A}} |\mathcal{O}_n^*(\tau_1) - \mathcal{O}_n^*(\tau_2)| > \epsilon \right) = 0. \quad (\text{A.25})$$

To this end, we may use the Markov inequality and establish the following condition,

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E} \left(\sup_{|\tau_1 - \tau_2| < \delta; \tau_1, \tau_2 \in \mathcal{A}} |\mathcal{O}_n^*(\tau_1) - \mathcal{O}_n^*(\tau_2)| \right) = 0. \quad (\text{A.26})$$

First, define $\mathfrak{F} \equiv \{1\{(\tau_1, \tau_2]\} - \Phi((\tau_1, \tau_2]) : -\infty < \tau_1 < \tau_2 < \infty\}$ and $\mathfrak{F}(\delta') \equiv \{f \in \mathfrak{F} : \int f^2 d\Phi < \delta'\}$ for every $\delta' > 0$. Then, to show (A.26), we first apply Han & Wellner (2019, Proposition 4),

$$\mathbb{E} \left(\sup_{f \in \mathfrak{F}(\delta')} \left| \sum_{i=1; i \in \mathbb{I}_n}^k \vartheta_i f(\sqrt{n}\Delta_i^n W) \right| \right) \leq K \sqrt{\frac{m_n k}{k_n}} \mathfrak{J}_{[]}(\sqrt{\delta'}, \mathfrak{F}, \mathbb{L}_2(\Phi)) \left(1 + \frac{\mathfrak{J}_{[]}(\sqrt{\delta'}, \mathfrak{F}, \mathbb{L}_2(\Phi))}{\delta' \sqrt{\frac{m_n k}{K_n}}} \right),$$

for all $\delta' > 0$ and $k = 1, \dots, K_n$, where $\{\vartheta_i\}_{i=1}^k$ is a sequence of i.i.d. Rademacher variables, independent of W_i , and $\mathfrak{J}_{[]}(\delta', \mathfrak{F}, \mathbb{L}_2(\Phi))$ is the bracketing entropy integral defined in Han & Wellner (2019,

(Eq. (5.2)),

$$\mathfrak{J}_{[]}(\delta', \mathfrak{F}, \mathbb{L}_2(\Phi)) \equiv \int_0^{\delta'} \sqrt{1 + \ln \mathfrak{N}(\varepsilon, \mathfrak{F}, \mathbb{L}_2(\Phi))} d\varepsilon, \quad (\text{A.27})$$

with $\mathfrak{N}(\varepsilon, \mathfrak{F}, \mathbb{L}_2(\Phi))$ being the covering number of \mathfrak{F} . Hence, by invoking the upper bound in Han & Wellner (2019, Theorem 1), cf. the form provided in their Remark 1, we have

$$\begin{aligned} & \mathbb{E} \left(\sup_{f \in \mathfrak{F}(\delta')} \left| \sum_{i=1; i \in \mathbb{I}_n}^{K_n} v_i^* f(\sqrt{n} \Delta_i^n W) \right| \right) \\ & \leq 4K \mathfrak{J}_{[]}(\sqrt{\delta'}, \mathfrak{F}, \mathbb{L}_2(\Phi)) \left(\sqrt{\frac{m_n K_n}{k_n}} + \frac{\mathfrak{J}_{[]}(\sqrt{\delta'}, \mathfrak{F}, \mathbb{L}_2(\Phi))}{\delta'} \right) \int_0^\infty \sqrt{\mathbb{P}(|v_i^*| > s)} ds. \end{aligned}$$

Now, since $\mathbb{E}[|v_i^*|^4] < \infty$ implies $\int_0^\infty \sqrt{\mathbb{P}(|v_i^*| > s)} ds < \infty$, and we can apply the same arguments as in van der Vaart (1998, Example 19.6) to show $\mathfrak{J}_{[]}(\delta', \mathfrak{F}, \mathbb{L}_2(\Phi)) = O(\delta' \sqrt{\ln(1/\delta')})$ as $\delta' \rightarrow 0$, we obtain

$$\lim_{\delta' \rightarrow 0} \limsup_{n \rightarrow \infty} \mathbb{E} \left(\sup_{f \in \mathfrak{F}(\delta')} \left| \sqrt{\frac{k_n}{m_n K_n}} \sum_{i=1; i \in \mathbb{I}_n}^{K_n} v_i^* f(\sqrt{n} \Delta_i^n W) \right| \right) = 0. \quad (\text{A.28})$$

Finally, (A.26) follows from (A.28) since there exists a constant $c > 0$ such that $0 < \tau_2 - \tau_1 < \delta$ further implies that $1\{(\tau_1, \tau_2]\} - \Phi((\tau_1, \tau_2]) \in \mathfrak{F}(c\delta)$, thus establishing stochastic equicontinuity. Consequently, by writing $\mathcal{Z}_i^*(\tau) \equiv \mathcal{Z}_i^*$, the multivariate sequence $\frac{\lfloor n/k_n \rfloor k_n}{K_n} \sum_{i=1}^{K_n} \mathcal{Z}_i^*(\tau)$ is tight in the argument $\tau \in \mathcal{A}$. As a result, the joint functional central limit theory is uniform for $\tau \in \mathcal{A}$ by applying van der Vaart (1998, Theorem 18.14) in conjunction with the Cramér-Wold theorem. \square

Lemma A.3 (DWB central limit theory). *Under the conditions of Theorem 1, then, uniformly in τ over compact subsets of \mathbb{R} ,*

$$\begin{pmatrix} \sqrt{\lfloor n/k_n \rfloor m_n} \hat{H}_{n,1}^*(\tau) \\ \sqrt{\lfloor n/k_n \rfloor k_n} \hat{H}_{n,2}^*(\tau) \end{pmatrix} \xrightarrow{d^*} \begin{pmatrix} H_1(\tau) \\ H_2(\tau) \end{pmatrix},$$

in probability- \mathbb{P} , where $H_1(\tau)$ and $H_2(\tau)$ are defined as in Lemma 2.

Proof. First, make a decomposition similarly to (A.21),

$$\begin{pmatrix} \sqrt{\lfloor n/k_n \rfloor m_n} \hat{H}_{n,1}^*(\tau) \\ \sqrt{\lfloor n/k_n \rfloor k_n} \hat{H}_{n,2}^*(\tau) \end{pmatrix} \equiv \sum_{i=1}^{\lfloor n/k_n \rfloor k_n} \begin{pmatrix} \mathcal{Z}_i^*(1) \\ \frac{\Phi'(\tau)\tau}{2} (\mathcal{Z}_i^*(2) + \mathcal{Z}_i^*(3)) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\Phi'(\tau)\tau}{2} \tilde{\mathcal{Z}}^* \end{pmatrix}, \quad (\text{A.29})$$

where the vector $\mathcal{Z}_i^* = (\mathcal{Z}_i^*(1), \mathcal{Z}_i^*(2), \mathcal{Z}_i^*(3))'$, $i = 1, \dots, n$, are defined as in Lemma A.2, and

$$\tilde{\mathcal{Z}}^* = \frac{-(\pi/2)}{\sqrt{\lfloor n/k_n \rfloor k_n}} \sum_{j=1}^{\lfloor n/k_n \rfloor} \tilde{\mathcal{Z}}_j v_{(j-1)k_n+1}^*.$$

As the residual term has $\mathbb{E}(\mathbb{E}^*[\tilde{\mathcal{Z}}^2]) \leq K/k_n$ by Lemma A.1(b) together with Assumption DWB,

we may focus on the first right-hand-side term in (A.29). Here, since v_i^* in $\mathcal{Z}_i^*(1)$ is b_n -dependent by Assumption DWB, and \bar{v}_i^* in $\mathcal{Z}_i^*(2)$ and $\mathcal{Z}_i^*(3)$ is k_n -dependent, we can adopt a large-block-small-block argument in conjunction with a modified Cramér-Wold device to show

$$\sum_{i=1}^{\lfloor n/k_n \rfloor k_n} \boldsymbol{\lambda}' \mathbf{Z}_i^* \xrightarrow{d^*} \boldsymbol{\lambda}' \mathbf{Z}_\infty(\tau), \quad \text{in probability} - \mathbb{P}, \quad (\text{A.30})$$

where $\boldsymbol{\lambda}$ is contained in a countable dense subset of the unit circle $\mathcal{D} = \{\boldsymbol{\lambda}_k : k \in \mathbb{N}\}$, and the asymptotic distribution $\mathbf{Z}_\infty(\tau) \sim N(\mathbf{0}, \mathbf{C}_\mathcal{Z}(\tau))$ with $\mathbf{C}_\mathcal{Z}(\tau)$ defined as in Lemma A.1(a).²⁶ Hence, define a sequence of integers K_n such that $K_n \rightarrow \infty$ and $K_n/n \rightarrow 0$ as $n \rightarrow \infty$, capturing the “large” block size. Moreover, let $\ell_n = \lfloor n/(K_n + k_n) \rfloor \rightarrow \infty$ be the number of blocks, then we may define:

$$\begin{aligned} \mathcal{L}_r &= \{i \in \mathbb{N} : (r-1)(K_n + k_n) + 1 \leq i \leq r(K_n + k_n) - k_n\}, \quad r = 1, \dots, \ell_n, \quad \text{and} \\ \mathcal{S}_r &= \{i \in \mathbb{N} : r(K_n + k_n) - k_n + 1 \leq i \leq r(K_n + k_n)\}, \quad r = 1, \dots, \ell_n - 1, \end{aligned}$$

as well as $\mathcal{S}_{\ell_n} = \{i \in \mathbb{N} : \ell_n(K_n + k_n) - k_n + 1 \leq i \leq n\}$. Now, conditional on the sample path \mathcal{X}_n , we have that $\mathcal{U}_r^*(\tau) = \sum_{i \in \mathcal{L}_r} \boldsymbol{\lambda}' \mathbf{Z}_i^*$ and $\mathcal{V}_r^*(\tau) = \sum_{i \in \mathcal{S}_r} \boldsymbol{\lambda}' \mathbf{Z}_i^*$ are independent across $r = 1, \dots, \ell_n$ and $r = 1, \dots, \ell_n - 1$ for $\mathcal{U}_r^*(\tau)$ and $\mathcal{V}_r^*(\tau)$, respectively. The proof, thus, proceeds by showing existence of sequences K_n and ℓ_n such that the following conditions hold:

- (i) $\sum_{r=1}^{\ell_n} \mathcal{V}_r^*(\tau) = o_p^*(1)$, in probability- \mathbb{P} , uniformly for τ in a compact subset of \mathbb{R} .
- (ii) $\mathbb{E}^*[\sum_{r=1}^{\ell_n} \mathcal{U}_r^*(\tau)] = 0$ and $\mathbb{E}^*[(\sum_{r=1}^{\ell_n} \mathcal{U}_r^*(\tau))^2] \xrightarrow{\mathbb{P}} \mathbf{C}_\mathcal{Z}(\tau)$.
- (iii) $\mathcal{I}_n^*(\tau, \epsilon) \equiv \sum_{r=1}^{\ell_n} \mathbb{E}^*[(\mathcal{U}_r^*(\tau))^2 1\{|\mathcal{U}_r^*(\tau)| > \epsilon\}] \xrightarrow{\mathbb{P}} 0$, for some $\epsilon > 0$.

In conjunction with independence of $\mathcal{U}_r^*(\tau)$ across blocks $r = 1, \dots, \ell_n$, this suffices to show that the limit theory (A.30) holds pointwise in τ over compact subsets of \mathbb{R} . Then, as tightness of the statistic and the uniform limit result, in τ over compact subsets of \mathbb{R} , follow by the same arguments provided for (A.25) and Lemma A.2(c), this shows the joint functional central limit theory.

First, for (i), we have $\mathbb{E}^*[\mathcal{V}_r^*(\tau)] = 0$ by Assumption DWB. Moreover, for $r = 1, \dots, \ell_n - 1$, it follows that $\mathbb{E}^*[(\mathcal{V}_r^*(\tau))^2] = \sum_{i,j \in \mathcal{S}_r} \boldsymbol{\lambda}' \mathbf{Z}_i \mathbf{Z}_j' \boldsymbol{\lambda} \text{Cov}(v_i^*, v_j^*) = O_p(k_n/n)$ by Lemma A.2(a), uniformly for τ in a compact subset of \mathbb{R} , and $\text{Cov}(v_i^*, v_j^*) \rightarrow \hat{C}_{i,j} \geq 0$ by Assumption DWB. By the same argument, we have $\mathbb{E}^*[(\mathcal{V}_{\ell_n}^*(\tau))^2] = O_p(K_n/n)$. Hence, utilizing independence between the blocks, $\mathcal{V}_r^*(\tau)$, this provides the bound

$$\mathbb{E}^* \left[\left(\sum_{r=1}^{\ell_n} \mathcal{V}_r^*(\tau) \right)^2 \right] = O_p \left(\frac{\ell_n k_n}{n} + \frac{K_n}{n} \right),$$

for which $(\ell_n k_n)/n \asymp k_n/K_n \rightarrow 0$ and $K_n/n \rightarrow 0$ as $n \rightarrow \infty$, thereby showing (i).

²⁶ A similar strategy is adopted for the proof of the corresponding Shao (2010, Theorem 3.1), albeit with subtle and important differences. Specifically, we need to provide both pointwise and uniform limit theory.

Next, (ii) follows by Assumption DWB, Lemma A.2(b) and independence between the blocks in the sequence $\mathcal{U}_r^*(\tau)$, $r = 1, \dots, \ell_n$, under the bootstrap measure.

Last, for the Lindeberg condition in (iii), it suffices to show $\mathbb{E}[\mathcal{I}_n^*(\tau, \epsilon)] \rightarrow 0$. Now, by (block-)stationarity of the bootstrap variables and independence of the Brownian increments in \mathcal{Z}_i ,

$$\begin{aligned} \mathbb{E}[\mathcal{I}_n^*(\tau, \epsilon)] &\leq K \ell_n \mathbb{E}(\mathbb{E}^*[(\mathcal{U}_1^*(\tau))^2 1\{|\mathcal{U}_1^*(\tau)| > \epsilon\}]) \\ &= K \mathbb{E} \left(\mathbb{E}^* \left[\left(\sqrt{\ell_n} \mathcal{U}_1^*(\tau) \right)^2 1\{|\sqrt{\ell_n} \mathcal{U}_1^*(\tau)| > \sqrt{\ell_n} \epsilon\} \right] \right). \end{aligned} \quad (\text{A.31})$$

Hence, it suffices to analyze the properties of $\sqrt{\ell_n} \mathcal{U}_1^*(\tau)$ when expectations are taken under both measures. Indeed, since $\ell_n/(\lfloor n/k_n \rfloor k_n/K_n) \rightarrow 1$ as $n \rightarrow \infty$, the use of the continuous mapping theorem and Slutsky's theorem in combination with Lemma A.2(b)-(c) establish that $\sqrt{\ell_n} \mathcal{U}_1^*(\tau) \xrightarrow{d} \boldsymbol{\lambda}' \mathcal{Z}_\infty(\tau)$ as well as $\mathbb{E}(\mathbb{E}^*[\ell_n(\mathcal{U}_1^*(\tau))^2]) \rightarrow \boldsymbol{\lambda}' \mathcal{C}_{\mathcal{Z}}(\tau) \boldsymbol{\lambda}$ as $n \rightarrow \infty$, uniformly in τ over compact subsets of \mathbb{R} . Hence, these results imply uniform integrability of $(\sqrt{\ell_n} \mathcal{U}_1^*(\tau))^2$, providing

$$\begin{aligned} &\mathbb{E} \left(\mathbb{E}^* \left[(\sqrt{\ell_n} \mathcal{U}_1^*(\tau))^2 1\{|\sqrt{\ell_n} \mathcal{U}_1^*(\tau)| > \sqrt{\ell_n} \epsilon\} \right] \right) \\ &\rightarrow \mathbb{E} \left(\mathbb{E}^* \left[(\boldsymbol{\lambda}' \mathcal{Z}_\infty(\tau))^2 1\{|\boldsymbol{\lambda}' \mathcal{Z}_\infty(\tau)| > \sqrt{\ell_n} \epsilon\} \right] \right) \rightarrow 0 \end{aligned} \quad (\text{A.32})$$

since $\sqrt{\ell_n} \epsilon \rightarrow \infty$ when $n \rightarrow \infty$. This shows (iii), thereby concluding the proof. \square

B Standard Local Gaussian Resampling

To elaborate on Remark 3, we follow Hounyo (2019) and generate the high-frequency innovations as,

$$\Delta_{(j-1)k_n+i}^n Z^* = \sqrt{\frac{\widehat{V}_{n,j}}{n}} u_{i+(j-1)k_n}^*, \quad i = 1, \dots, k_n, \quad j = 1, \dots, \lfloor n/k_n \rfloor, \quad (\text{B.1})$$

where $u_{i+(j-1)k_n}^* \sim \text{i.i.d. } N(0, 1)$ across the (i, j) indices. Using these, the analogous bootstrap spot variation estimator may be decomposed as $\widehat{V}_{n,j}^* = \widehat{V}_{n,j} U_{n,j}^*$, where

$$\widehat{V}_{n,j}^* = \frac{\pi}{2} \frac{n}{k_n - 1} \sum_{i=(j-1)k_n+2}^{jk_n} |\Delta_{i-1}^n Z^*| |\Delta_i^n Z^*|, \quad U_{n,j}^* = \frac{\pi}{2(k_n - 1)} \sum_{i=(j-1)k_n+2}^{jk_n} |u_{i-1}^*| |u_i^*|, \quad (\text{B.2})$$

utilizing that $\widehat{V}_{n,j}$ is constant over i for a given j . Moreover, by first defining $\widehat{V}_{n,j}^*(i)$ similarly to (6), that is, replacing $\Delta_i^n Z$ with $\Delta_i^n Z^*$, and using the definition of $\widehat{V}_{n,j}^*$, the former reduces to

$$\widehat{V}_{n,j}^*(i) = \widehat{V}_{n,j} U_{n,i,j}^*(i),$$

where, by expanding and rewriting the analogue of (6), we have

$$U_{n,i,j}^*(i) = \begin{cases} \frac{\pi}{2} \frac{1}{k_n-3} \left(\sum_{l=(j-1)k_n+2}^{jk_n} |u_{l-1}^*| |u_l^*| - |u_i^*| |u_{i+1}^*| \right) & \text{for } i = (j-1)k_n + 1; \\ \frac{\pi}{2} \frac{1}{k_n-3} \left(\sum_{l=(j-1)k_n+2}^{jk_n} |u_{l-1}^*| |u_l^*| - (|u_{i-1}^*| |u_i^*| + |u_i^*| |u_{i+1}^*|) \right), & \\ \text{for } i = (j-1)k_n + 2, \dots, jk_n - 1; \\ \frac{\pi}{2} \frac{1}{k_n-3} \left(\sum_{l=(j-1)k_n+2}^{jk_n} |u_{l-1}^*| |u_l^*| - |u_{i-1}^*| |u_i^*| \right), & \text{for } i = jk_n. \end{cases} \quad (\text{B.3})$$

Now, it is important to note that both bootstrap spot variation estimators, $\widehat{V}_{n,j}^*$ and $\widehat{V}_{n,j}^*(i)$, respectively, decompose into $\widehat{V}_{n,j}$ and additional terms that exclusively consist of the resampled data. Hence, when forming the bootstrap empirical CDF, the key ratios reduce to

$$\mathcal{R}_{n,i,j}^* = \frac{\sqrt{n}\Delta_i^n Z^*}{\sqrt{\widehat{V}_{n,j}^*}} = \frac{u_i^*}{\sqrt{U_{n,i,j}^*}}, \quad \mathcal{R}_{n,i,j}^*(i) = \frac{\sqrt{n}\Delta_i^n Z^*}{\sqrt{\widehat{V}_{n,j}^*(i)}} = \frac{u_i^*}{\sqrt{U_{n,i,j}^*(i)}} \quad (\text{B.4})$$

with $i = (j-1)k_n + 1, \dots, (j-1)k_n + m_n$. In other words, $\mathcal{R}_{n,i,j}^*$ and $\mathcal{R}_{n,i,j}^*(i)$ no longer depend on the original data. However, since the two ratios preserve the exact dependence structure of the corresponding ratios in empirical CDF, $\widehat{F}_n(\tau)$, the relations in (B.4) can be used to simulate the asymptotic distribution of $\widehat{F}_n(\tau)$ under the null hypothesis $\mathcal{H}_0 : S_t = W_t$, which may generate improvements of the finite sample inference. Hence, if one considers the *resampled* empirical CDF,

$$\widehat{F}_{\mathcal{R},n}^*(\tau) = \frac{1}{N_{\mathcal{R},n}^*(\alpha, \varpi)} \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} 1 \{ \mathcal{R}_{n,i,j}^*(i) \leq \tau \} 1 \{ |\mathcal{R}_{n,i,j}^*| \leq \alpha n^{1/2-\varpi} \}, \quad (\text{B.5})$$

where

$$N_{\mathcal{R},n}^*(\alpha, \varpi) = \sum_{j=1}^{\lfloor n/k_n \rfloor} \sum_{i=(j-1)k_n+1}^{(j-1)k_n+m_n} 1 \{ |\mathcal{R}_{n,i,j}^*| \leq \alpha n^{1/2-\varpi} \}, \quad (\text{B.6})$$

and we redefine $\tilde{u}_i^* = u_i^*/\sqrt{n} \stackrel{d}{=} N(0, \Delta_i^n)$, then this process (and CDF statistic) belong to the general class (3) as a special case with $\sigma_t = 1$, $\alpha_t = 0$, $Y_t = 0$ and $S_t = W_t$ for all $0 \leq t \leq 1$. Hence, the CLT for the Gaussian resampled CDF, $\widehat{F}_{\mathcal{R},n}^*(\tau)$, may be obtained as a corollary to Lemma 2:

Corollary 4. *Suppose that (B.1) holds, then, uniformly in τ over compact subsets of \mathbb{R} ,*

$$\widehat{F}_{\mathcal{R},n}^*(\tau) - \Phi(\tau) = \widehat{H}_{\mathcal{R},n,1}^*(\tau) + \widehat{H}_{\mathcal{R},n,2}^*(\tau) + H_3(\tau)/k_n + o_p(1/k_n)$$

where $\sqrt{[n/k_n]m_n}(\widehat{H}_{\mathcal{R},n,1}^*(\tau), \sqrt{k_n/m_n}\widehat{H}_{\mathcal{R},n,2}^*(\tau)) \xrightarrow{d} (H_{\mathcal{R},1}^*(\tau), H_{\mathcal{R},2}^*(\tau))$ with $H_{\mathcal{R},1}^*$ and $H_{\mathcal{R},2}^*(\tau)$ being two independent Gaussian processes with covariances similar to those for $H_1(\tau)$ and $H_2(\tau)$, respectively, in (9). Finally, $H_3(\tau)$ is defined in (10).

Since the local Gaussian CDF statistic, $\widehat{F}_{\mathcal{R},n}^*(\tau)$, is a special case of the empirical CDF without impact from drift, residual jumps, and stochastic volatility, while *exactly* capturing its dependence structure, one could base inference for $\widehat{F}_n(\tau)$ and its Kolmogorov-Smirnov test, T_n , on the resample distributions $\widehat{F}_{\mathcal{R},n}^*(\tau) - \Phi(\tau)$ and $\sup_{\tau \in \mathcal{A}} \sqrt{N_{\mathcal{R},n}^*(\alpha, \varpi)} |\widehat{F}_{\mathcal{R},n}^*(\tau) - \Phi(\tau)|$. However, as this inference procedure has lost all dependence on the original data, it likely suffers from finite sample distortions similarly to those affecting the asymptotic distribution when the underlying process, indeed, exhibits drift, jumps and stochastic volatility. Hence, we prefer, and recommend, the use of the LDWB inference procedure in Section 3.2, which not only preserves dependence on the original data, it also replicates the second-order asymptotic theory induced by the nonparametric standardization.

C Implementation Details

In this section, we detail how one can implement the proposed bootstrap tests. Let B denote the number of bootstrap replications for each of the M Monte Carlo replications. Then, for a given equidistant partition of the normalized time window $[0, 1]$ with step length $1/n$ do the following:

Algorithm 1: The LDWB and/or the NLBB procedure for hypothesis testing

- Step 1.** Simulate $n+1 \in \mathbb{N}$ points of the process Z_t under investigation (a pure-jump semimartingale, a jump-diffusion or even a jump-diffusion contaminated by noise).
- Step 2.** Compute n intraday returns at an equidistant time grid $t_i \equiv i/n \in [0, 1]$, for $i = 0, \dots, n$, as the innovation $\Delta_i^n Z = Z_{t_i} - Z_{t_{i-1}}$.
- Step 3.** Compute the Kolmogorov-Smirnov statistic,

$$\widehat{\text{KS}}_n(\mathcal{A}) = \sup_{\tau \in \mathcal{A}} \sqrt{N_n(\alpha, \varpi)} \left| \widehat{F}_n(\tau) - \Phi(\tau) \right|,$$

where $N_n(\alpha, \varpi)$ and $\widehat{F}_n(\tau)$ are defined as in (7) and (8), respectively. For the compact set \mathcal{A} , one may, e.g., choose (as in Section 5),

$$\mathcal{A} = [Q(0.001) : Q(0.499)] \cup [Q(0.501) : Q(0.999)], \quad (\text{C.1})$$

where $Q(\theta)$ is the θ -quantile of the standard normal distribution.

- Step 4.** Generate an $m_n \lfloor n/k_n \rfloor$ sequence of external random variables $v_{(j-1)k_n+i}^*$, for running indices $i = 1, \dots, m_n$, $j = 1, \dots, \lfloor n/k_n \rfloor$, which are independent of the observations generated in Step 1 as well as satisfy the conditions of Assumption DWB. As advocated in Section 5.1, one may use the random variables underlying DWB1, DWB2, DWB3 or DWB4.²⁷

²⁷For the NLBB, note that observations can be obtained equivalently by resampling, as in equation (17), or by generating external random variable as follows: $v_{(j-1)k_n+i}^* = \zeta_{p_n, j}^* - 1$ for $i = 1, \dots, m_n$ across blocks $j = 1, \dots, \lfloor n/k_n \rfloor$, where

Step 5. Generate the locally dependent wild bootstrap observations as in (14).

Step 6. Compute the bootstrap Kolmogorov-Smirnov statistic $\text{KS}_n^*(\mathcal{A})$ as in (21). In particular,

$$\text{KS}_n^*(\mathcal{A}) = \sup_{\tau \in \mathcal{A}} \sqrt{N_n(\alpha, \varpi)} \left| \widehat{F}_{W,n}^*(\tau) - \widehat{F}_n(\tau) \right|,$$

where $N_n(\alpha, \varpi)$, $\widehat{F}_n(\tau)$, $\widehat{F}_{W,n}^*(\tau)$ and \mathcal{A} are defined as in (7), (8), (15) and (C.1), respectively.

Step 7. Repeat Steps 4-6 B times and keep the values of $\text{KS}_n^{*(j)}(\mathcal{A})$, $j = 1, \dots, B$, where $\text{KS}_n^{*(j)}(\mathcal{A})$ is given as in Step 6. Then, sort $\text{KS}_n^{*(1)}(\mathcal{A}), \dots, \text{KS}_n^{*(B)}(\mathcal{A})$ ascendingly from the smallest to the largest as $\overline{\text{KS}}_n^{*(1)}(\mathcal{A}), \dots, \overline{\text{KS}}_n^{*(B)}(\mathcal{A})$ such that $\overline{\text{KS}}_n^{*(i)}(\mathcal{A}) < \overline{\text{KS}}_n^{*(j)}(\mathcal{A})$ for all $1 \leq i < j \leq B$.

Step 8. Reject $\mathcal{H}_0 : S_t = W_t$ when $\widehat{\text{KS}}_n(\mathcal{A}) > q_n^*(\alpha, \mathcal{A})$ where $q_n^*(\alpha, \mathcal{A})$ is the α quantile of the bootstrap distribution of $\text{KS}_n^*(\mathcal{A})$. For example, if we let $B = 999$, then the 0.05-th quantile of $\text{KS}_n^*(\mathcal{A})$ is estimated by $\text{KS}_n^{*(a)}(\mathcal{A})$ with $a = 0.05 \times (999 + 1) = 50$.

Step 9. Repeat Steps 1-8 M times to get the size or power of the bootstrap test. In particular, if Z_t is simulated as a jump diffusion, then the size is given by $M^{-1}(\#\{\widehat{\text{KS}}_n(\mathcal{A}) > q_n^*(\alpha, \mathcal{A})\})$.

we let $p_n = \lfloor n/k_n \rfloor$ and $\zeta_{p_n, j}^*$, $j = 1, \dots, p_n$ be a sequence of multinomial random variables with probability $1/p_n$ and number of trials p_n . Section 3.3 provide further details on this equivalence.

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