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Spatial Dependence in Option Observation Errors*

Torben G. Andersen† Nicola Fusari‡ Viktor Todorov§ Rasmus T. Varneskov¶

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Abstract

In this paper, we develop the first formal nonparametric test for whether the observation errors in option panels display spatial dependence. The panel consists of options with different strikes and tenors written on a given underlying asset. The asymptotic design is of the infill type – the mesh of the strike grid for the observed options shrinks asymptotically to zero, while the set of observation times and tenors for the option panel remains fixed. We propose a Portmanteau test for the null hypothesis of no spatial autocorrelation in the observation error. The test makes use of the smoothness of the true (unobserved) option price as a function of its strike and is robust to the presence of heteroskedasticity of unknown form in the observation error. A Monte Carlo study shows good finite-sample properties of the developed testing procedure and an empirical application to S&P 500 index option data reveals mild spatial dependence in the observation error, which has been declining in recent years.

Keywords: Heteroskedasticity, Infill Asymptotics, Large Data Sets, Nonparametric Inference, Options, Panel Data, Spatial Dependence, Stable Convergence.

JEL classification: C51, C52, G12.

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†Department of Finance, Kellogg School of Management, Northwestern University, Evanston, IL 60208; NBER, Cambridge, MA; and CREATES, Aarhus, Denmark; e-mail: t-andersen@northwestern.edu.

‡The Johns Hopkins University Carey Business School, Baltimore, MD 21202; e-mail: nicola.fusari@jhu.edu.

§Department of Finance, Kellogg School of Management, Northwestern University, Evanston, IL 60208; e-mail: v-todorov@northwestern.edu.

¶Department of Finance, Copenhagen Business School, 2000 Frederiksberg, Denmark; CREATES, Aarhus, Denmark; Multi Assets at Nordea Asset Management, Copenhagen, Denmark; e-mail: rtv.fi@cbs.dk.

1 Introduction

Option prices embed important information about the risks in the underlying asset and the compensation for those risks demanded by investors. Consequently, option data may serve both to complement and enhance inference regarding dynamic asset pricing models from underlying asset returns. The inclusion of option data results in more efficient inference and, if the return data by themselves do not identify all the parameters associated with the risk pricing, the option prices typically resolve the issue, rendering full identification and estimation feasible. For example, in the Black-Scholes model, Black and Scholes (1973), in which the continuously-compounded asset returns are i.i.d. normally distributed, all parameters are identified, yet a single option on the underlying asset suffices to determine the volatility parameter of the model, thus supplementing the standard return-based volatility estimator and enhancing the efficiency of the inference. In contrast, for a more general setting with time-varying volatility, such as the well-known Heston model, Heston (1993), the price of volatility risk cannot be identified from return data alone without auxiliary assumptions. Here, option prices provide the necessary information for both identifying and pricing the model's volatility risk component.

Prior work uses a variety of parametric and nonparametric methods to extract information from options. The earlier empirical option pricing literature typically relies on only a small number of options on the underlying asset to complement the return data, see, e.g., Bates (1996, 2000), Pan (2002), Pastorello et al. (2003), Eraker (2004) and Gagliardini et al. (2011). The reason is twofold. One, there are often limitations in the strike coverage, stemming from a lack of liquidity in the option markets. Two, using a large number of options can be challenging, because option prices usually are known only in semi-closed form, as for the popular exponential-affine model class of Duffie et al. (2000), necessitating numerical techniques for computation. In these circumstances, the asset returns remain the primary source of information, and options are included for estimation of parameters that are not identifiable from return data and/or provide information about state variables, which are latent from the perspective of return data alone, such as stochastic volatility.

Over the last several decades, however, the liquidity in option markets has improved greatly, implying that much more option data may be utilized for inference. Specifically, in our empirical application, using options written on the S&P 500 index over 2007-2014, we have, on average, in excess of 70 end-of-day option quotes with different strikes for each fixed short maturity. Inspired by this trend, Andersen et al. (2015) develop parametric option inference methods in an infill asymptotic setting, for which the cross-sectional dimension of the option data, for a given day and written on the same underlying asset, increases asymptotically while, simultaneously, the mesh of the strike grid shrinks to zero.[1] In such a setup, one may consistently estimate both the parameters and the concurrent state variable realizations from a cross-section of options recorded at a given point in time.

Infill asymptotics are also at work for various nonparametric measures based on the following

[1]Their inference method has been extended in various directions by Andersen et al. (2019a,b), who, however, maintain similar cross-sectional infill asymptotic frameworks.

general option spanning result due to Carr and Madan (2001),

$$\mathbb{E}_t^{\mathbb{Q}}(f(X_{t+\tau})) \;=\; f(F_{t,\tau}) \;+\; e^{r_{t,\tau}\cdot\tau} \int_0^{\infty} O_{t,\tau}(\log(K))\, f^{''}(K)\, dK\,, \tag{1}$$

where f is some twice-continuously differentiable function, $\mathbb{E}^{\mathbb{Q}}(\cdot)$ denotes the expectation under the risk-neutral measure, X_t is the date t price of the underlying asset and $F_{t,\tau}$ is the date t price of a futures contract written on the asset and expiring at $t+\tau$, $r_{t,\tau}$ is the annualized risk-free rate for $[t, t+\tau]$, and $O_{t,\tau}(k)$ denotes the out-of-the-money (OTM) option price written on the underlying asset at time t with strike e^k and expiring at time $t+\tau$. All of these quantities are formally defined in Section 2 below. The integral on the right-hand side of the equation can be approximated via a Riemann sum using the available options at date t, expiring at date $t+\tau$. By supplementing this approximation with $f(F_{t,\tau})$, we readily obtain a nonparametric estimator of $\mathbb{E}_t^{\mathbb{Q}}(f(X_{t+\tau}))$. For this to constitute a consistent estimator, the mesh of the strike grid of available options must shrink asymptotically to zero, i.e., consistency can be established in an infill asymptotic setup. A leading example of this type of nonparametric estimator is the VIX volatility index computed by the Chicago Board of Option Exchange (CBOE). The VIX corresponds to setting $f(x) = \log(x)$ in equation (1), see, e.g., Britten-Jones and Neuberger (2000) and Carr and Wu (2009). Other examples include the risk-neutral moments of Bakshi and Madan (2000) and Bakshi et al. (2003), the tail measures of Bollerslev and Todorov (2011) and Bollerslev et al. (2015), the corridor volatility estimators of Andersen et al. (2015), the characteristic function portfolios used in Qin and Todorov (2019) and Todorov (2019), the divergence portfolio of Schneider and Trojani (2019), etc.

To develop feasible inference procedures based on options, one must take a stand on the nature of the associated observation errors, see Renault (1997) for an early discussion of the subject. Absence of arbitrage implies that the fundamental option value lies within the bid-ask spread posted by the brokers, dealers and trading firms operating as market makers. The liquidity provision offered by market makers induces random variation in their holdings of individual options - reflecting whether customers are buying or selling. In particular, investor groups may follow hedging or trading strategies that involve shifts in positions across a limited segment of the strikes, because the price and payoff of any option is similar to those for options with nearby strikes. This can generate clientele effects, where the market makers experience an unbalanced order flow across a limited set of options. They typically seek to mitigate such exposures through their own hedges but, in particular, out-of-the-money options can be difficult and costly to insulate against abrupt shifts (jumps) in asset prices. Therefore, the size of the spreads across distinct regions of the strike range may well fluctuate with the developing inventory positions and perceptions of the future asset price variation, reflecting compensation demanded by market makers for handling their risk exposures. A natural implication is a certain degree of dependence among mid-quote prices for options with similar strike prices. However, as markets grow increasingly liquid and attract a more diverse set of customers, it is also natural to expect such hedging and trading costs as well as the impact of random demand imbalances to decline.

One extreme approach is to assume that options are observed without error as, for example, in Pan (2002). However, option bid-ask spreads are sizable, see, e.g., Figure 1 in Andersen et al. (2015), suggesting that observation errors are non-negligible. If such errors are acknowledged, consistent estimation of option-related quantities requires the imposition of assumptions regarding the error dependence. The exact conditions will depend on whether the analysis is parametric or nonparametric, and whether the asymptotic design is applied to the time series and/or the cross-section. In particular, for the infill asymptotic setting, the dependence in the observation errors across neighboring strike prices must be limited. Henceforth, we label this feature *spatial dependence*, with the spatial dimension referring to the range of option strikes for a given tenor and date. Most existing studies, e.g., Carr and Wu (2007), Bakshi et al. (2008), Andersen et al. (2015) and Todorov (2019), assume option errors are uncorrelated. A rare exception is Bates (2003), who relies on an asymptotic setting with a fixed set of option strikes per maturity. At each date and tenor, he invokes *perfect* correlation in the option observation error across certain strike ranges. Such errors cannot be mitigated by integrating option prices across strikes and, therefore, cannot be separated from the true latent option prices in an infill asymptotic setting. The only possibility to mitigate such errors is through intertemporal averaging, i.e., by resorting to large T asymptotics, with T representing the time span of the option panel. In other words, for any given date, these errors are indistinguishable from pricing errors and, indeed, Bates labels them "specification errors." Accordingly, for inference procedures exploiting a cross-sectional asymptotic scheme, we deem errors, that are highly persistent across a broad, fixed range of strikes, as "true" pricing errors stemming from model misspecification rather than observation errors.

Even excluding perfect, or very high, spatial error dependence from the category of observation errors we may still, as discussed previously, expect a host of institutional features to generate weak spatial dependence in the observation errors. To formally account for this type of dependence within parametric and nonparametric inference procedures, we need a spatial infill asymptotic counterpart to the long-run variance in the time series setting. No such statistic has been developed in the literature, so a natural first step is to establish whether this type of extension is empirically relevant. The goal of the current paper, therefore, is to develop a nonparametric test for the presence of cross-sectional dependence in the option observation error within an infill asymptotic setting where, for the option prices recorded on a given date with fixed tenor, the mesh of the strike grid shrinks asymptotically to zero. Specifically, we derive consistent estimates for the spatial autocorrelation in the option observation error, and we derive a feasible central limit theory for these autocorrelation coefficients under the null hypothesis of no spatial dependence. This new limit theory enables us to devise a simple Portmanteau test for spatial autocorrelation.

The basic intuition behind our estimates for spatial autocorrelation is as follows. Under general conditions for the data generating process, the true (unobserved) option prices are smooth functions of their strikes. Therefore, by applying a suitable second-order difference to the observed (noisy) option price in the strike domain, we can eliminate the true latent price component and "isolate" the option observation error, subject only to an asymptotically negligible error due to our infill asymptotic setting.

This is analogous to the removal of fixed effects in classical panel data. The resulting estimates of the option observation errors are correlated in the strike dimension, even under the null hypothesis of no spatial error dependence, because a set of adjacent options are employed in estimating the error. However, this dependence arises from an overlap involving only a few options, and is readily accounted for. Therefore, we can test for the presence of spatial error dependence by comparing the sample spatial autocorrelation of the estimated option observation errors with their asymptotic limit under the hypothesis of no spatial dependence.

Our feasible CLT for spatial correlation of the option observation errors is derived under the null hypothesis of no error dependence. The limit distribution is mixed Gaussian due to the heteroskedastic observation errors. The latter feature is important, because the option bid-ask spread displays pronounced heterogeneity and heteroskedasticity, both across the strike range and over time. On the basis of the limit theory we develop, we propose a Portmanteau test, akin to the one of Box and Pierce (1970) and Ljung and Box (1978), for the null hypothesis of no spatial dependence in the observation error. However, because we do not impose an i.i.d. assumption for the process under the null hypothesis, we must scale our test statistic appropriately to account for heteroskedastic observation errors. The limit distribution of our Portmanteau test for spatial dependence is pivotal, but it is not chi-squared because the limiting covariance matrix of the spatial autocorrelation vector is non-diagonal.

Our paper is related to a large econometric literature dealing with spatial dependence. Conley (1999) develops GMM estimation techniques for cross-sectional dependence. Jenish and Prucha (2012) derive general limit results for processes with spatial dependence. Kuersteiner and Prucha (2013) consider estimation for panel models with spatial dependence in the error term. Delgado and Robinson (2015) and Robinson (2008) derive tests for spatial dependence in the error term of regressions. Kelejian and Prucha (2007, 2010) and Robinson and Thawornkaiwong (2012) propose HAC asymptotic variance estimators in settings with spatial dependence. There is also a large literature on inference for spatial autoregressive models, see, e.g., Gupta and Robinson (2015, 2018), Kelejian and Prucha (1999), Lee (2002, 2003, 2004), Lee and Liu (2010), Robinson (2010), Su and Jin (2010) and Xu and Lee (2015). The major distinction between this body of work and our contribution lies in the asymptotic setup. We can only obtain consistent nonparametric estimates for the spatial autocorrelation of the option observation errors within an infill asymptotic design. That is, as we gather more option observations, the distance between consecutive strikes shrinks to zero. This has a number of implications. First, since the latent option price is similar for nearby strikes – a direct consequence of virtually all asset pricing models – it is natural to assume that market participants quote them similarly, implying the existence of spatial dependence in the observation errors, and this effect will diminish as the gap between the option strikes widens. Second, and more importantly, the infill asymptotic setup is crucial for our nonparametric inference procedure. We exploit this feature by differencing consecutive option observations, ordered in terms of their strike prices, to annihilate, asymptotically, the unobserved true option price, generating a "direct" estimator of (differences of) the latent observation error. That is, unlike existing econometric work for regression models, we do not need a model for the latent option price to obtain an estimate for the observation error. This is attractive from an applied point of view, given the extensive evidence for misspecification of standard option pricing models. Finally, unlike most of the work on spatial dependence – with the notable exception of Kuersteiner and Prucha (2013) – our limit distributions are mixed Gaussian. This arises from the generality of allowing for stochastic heteroskedasticity of unknown form. This feature is crucial for the applicability of our analysis because financial data are known to exhibit pronounced conditional heteroskedasticity.

Our paper is also related to the recent work of Dalla et al. (2019) on testing the null hypothesis of white noise for time series exhibiting heteroskedasticity and, possibly, general forms of serial dependence. In particular, they show that the sample autocorrelations may have a limiting distribution that is non-standard for general white noise time series. As a result, the conventional tests of Box and Pierce (1970) and Ljung and Box (1978) for testing for presence of autocorrelation may no longer be valid if the process under the null hypothesis is a white noise, but not an i.i.d., sequence. Dalla et al. (2019) use their limit results to design a test for white noise, with chi-squared limiting distribution, which works in general settings. As in their case, we allow for general forms of heteroskedasticity in the option observation error. Moreover, and similarly to them, the sample autocorrelations of the estimated option observation error have a limiting covariance matrix that is not diagonal. The determining factors for the latter, however, differ. In our case, it is due to the differencing of the observed option price, which is done to account for the true, unknown, and latent option price. In Dalla et al. (2019), the limiting covariance of the sample autocorrelation vector might be non-diagonal due to general forms of dependence in the underlying (white noise) time series.

Upon implementing our new testing procedures for options written on the S&P 500 index, we find evidence for correlation in the option observation error across strikes. This dependence vanishes quickly, as the gap between the option strikes increases. In addition, the spatial dependence in the observation errors exhibits a strong time series and maturity pattern. In particular, it is weakest for the short-dated options and substantially stronger for more long-dated ones. In addition, we find that the spatial dependence for the short-dated options has declined over time across our sample period 2007-2014. This is consistent with the fact that short-dated options are relatively more liquid, and this liquidity has increased notably over time, see, e.g., Andersen et al. (2017).

The rest of the paper is organized as follows. We first introduce our setup and the option observation scheme in Section 2. We next state our assumptions in Section 3. Section 4 provides our nonparametric estimators for spatial dependence in the observation error, and Section 5 presents the associated feasible limit theory for the estimators and related tests for the presence of spatial dependence in the observation error. Section 6 contains a Monte Carlo study of the finite sample properties of the proposed estimator and its associated test. Section 7 provides an empirical application of the developed inference procedures on options written on the S&P 500 index. Finally, Section 8 concludes. The proofs of the asymptotic results are relegated to Section 9.

2 Setting and Option Observation Scheme

Our inference is based on options written on an underlying asset whose price is denoted by X. The asset and option price series are defined on the probability space $\big(\Omega^{(0)}, \mathcal{F}^{(0)}, \mathbb{P}^{(0)}\big)$, with $(\mathcal{F}_t^{(0)})_{t\geq 0}$ being the associated filtration. We rely on "plain-vanilla" European call and put options. A call option gives the owner the right to buy the underlying asset at a pre-specified strike price on the future maturity date of the option, while the put option, similarly, gives the owner the right to sell at the strike price at maturity. For each strike, we use either a call or a put price, since the put-call parity relation, which holds in the absence of arbitrage, implies an exact linear functional relationship between these option prices and the spot price of the underlying asset. Specifically, for each strike, we rely on the so-called out-of-the-money (OTM) option price – the cheaper of the call and the put for the given strike – which would be worth zero, if the option were to expire today. We denote OTM option prices by $O_{t,\tau}(k)$, where t is the time the option prices are recorded, τ is the tenor, or time to expiration, and k is their log-strike price. The OTM option price, $O_{t,\tau}(k)$, is a call if $k > \log(F_{t,\tau})$, and a put, if $k \leq \log(F_{t,\tau})$, where $F_{t,\tau}$ is the time t futures price of the asset with expiration date $t+\tau$.

Our data consist of options observed at time $t \in \mathbb{T}$ and expiring at time $t+\tau$, for $\tau > 0$ and $\tau \in \mathcal{T}_t$. Here, $\mathbb{T}$ denotes a finite set of observation times and, for each $t \in \mathbb{T}$, $\mathcal{T}_t$ represents the set of option tenors available at that time. For each pair (t,τ), we observe N_t^τ options with log-strikes given by,

$$\underline{k}_{t,\tau} \;\equiv\; k_{t,\tau}(1) \;<\; k_{t,\tau}(2) \;<\; \cdots k_{t,\tau}(N_t^\tau) \;\equiv\; \overline{k}_{t,\tau}\,. \tag{2}$$

The gap between the log-strikes is denoted $\Delta_{t,\tau}(i) = k_{t,\tau}(i) - k_{t,\tau}(i-1)$, for $i = 2, ..., N_t^\tau$. The log-strike grid need not be equidistant, i.e., $\Delta_{t,\tau}(i)$ may differ across i's. The asymptotic theory developed below is of infill type, i.e., we keep the observation times, option tenors and associated strike range fixed, while the mesh of the log-strike grid, $\sup_{i=2,\dots,N_t^\tau} \Delta_{t,\tau}(i)$, shrinks towards zero, as $N_t^\tau \to \infty$.

Following common practice, we convert the option price into a Black-Scholes Implied Volatility, or BSIV. The BSIV corresponding to $O_{t,\tau}(k)$ is labeled $\kappa_{t,\tau}(k)$. We note that $\kappa_{t,\tau}(k)$ is a (known) nonlinear transformation of $\frac{e^{r_{t,\tau}}\, O_{t,\tau}(k)}{X_t}$, where $r_{t,\tau}$ is the continuously-compounded risk-free interest rate over $[t, t+\tau]$, and $r_{t,\tau}$ and X_t are observed. We stress that we do not assume the Black-Scholes model is true, but simply use BSIV as a convenient way of quoting the observed option prices.[2]

Finally, we allow for observation error, i.e., instead of observing $\kappa_{t,\tau}(k_{t,\tau}(j))$ directly, we observe,

$$\widehat{\kappa}_{t,\tau}(k_{t,\tau}(j)) \;=\; \kappa_{t,\tau}(k_{t,\tau}(j)) \,+\, \epsilon_{t,\tau}^N(j), \qquad \epsilon_{t,\tau}^N(j) \;=\; \sigma_{t,\tau}(k_{t,\tau}(j))\, \zeta_{t,\tau}^N(j), \quad j = 1, ..., N_t^\tau\,, \tag{3}$$

where $\sigma_{t,\tau}(k)$ is an $\mathcal{F}_t^{(0)}$-adapted process, the sequence $\{\zeta_{t,\tau}^N(j)\}_{j\geq 1}$ is defined on a space $\Omega^{(1)} = \bigtimes_{t\in\mathbb{T}} \bigtimes_{\tau\in\mathcal{T}_t} \mathbb{R}^{\mathbb{R}}$, and N refers to the cross-sectional infill sampling scheme. The space $\Omega^{(1)}$ is equipped with the product Borel σ-field $\mathcal{F}^{(1)}$, and transition probability $\mathbb{P}^{(1)}(\omega^{(0)}, d\omega^{(1)})$ from the original probability

[2] Similarly, coupon bonds are often quoted in terms of their annualized yield to maturity without the need to reference any formal term structure or bond pricing model.

space $\Omega^{(0)}$, on which X is defined, to $\Omega^{(1)}$. We further define,

$$\Omega \;=\; \Omega^{(0)} \times \Omega^{(1)}, \quad \mathcal{F} \;=\; \mathcal{F}^{(0)} \times \mathcal{F}^{(1)}, \quad \mathbb{P}(d\omega^{(0)}, d\omega^{(1)}) \;=\; \mathbb{P}^{(0)}(d\omega^{(0)})\, \mathbb{P}^{(1)}(\omega^{(0)}, d\omega^{(1)})\,.$$

The observation error consists of two parts. The first component, $\sigma_{t,\tau}(k_{t,\tau}(j))$, is $\mathcal{F}_t^{(0)}$- adapted and captures the $\mathcal{F}_t^{(0)}$- conditional heteroskedasticity of the observation error. The second component, $\zeta_{t,\tau}^N$, is assumed independent of $\mathcal{F}^{(0)}$ and has mean zero, but may be spatially dependent in the strike domain. Specifically, for every $t \in \mathbb{T}$ and $\tau \in \mathcal{T}_t$, the sequence $\{\zeta_{t,\tau}^N(j)\}_{j=1,\ldots,N_t^\tau}$ is a stationary process, when viewed as a random function of j. Furthermore, the observation errors $\{\zeta_{t,\tau}^N(j)\}_{j=1,\ldots,N_t^\tau}$ are independent across different tenors and days, i.e., the errors are independent for $(t,\tau) \neq (t',\tau')$.

Since $\{\zeta_{t,\tau}^N(j)\}_{j=1,\ldots,N_t^\tau}$ is stationary, we have, for any pair (t,τ), that $\mathrm{Cov}(\zeta_{t,\tau}^N(j), \zeta_{t,\tau}^N(j-h))$ depends on h, but not j, for $h, j \in \mathbb{N}$. The autocovariance function of $\{\zeta_{t,\tau}^N(j)\}_{j=1,\ldots,N_t^\tau}$ is given as,

$$\gamma_{t,\tau}(h) = \mathrm{Cov}(\zeta_{t,\tau}^N(j), \zeta_{t,\tau}^N(j-h)), \quad h, j \in \mathbb{N}. \tag{4}$$

Stationarity of the observation error component $\zeta_{t,\tau}^N(j)$ implies that its covariance structure depends only on the distance between the strikes (in terms of number of strikes on the observed strike grid), not on the location of the points within the strike domain. This assumption may be weakened by allowing the dependence in the observation error to be a function of the given region of the strike domain. We also note that, in this asymptotic design, for any two fixed points in the strike domain, the dependence between the observation errors of the corresponding option prices converges to zero, as the mesh of the strike grid shrinks. Of course, in practice, we have a fixed strike grid, and the asymptotic setup serves only as an approximation to this configuration for a fixed N_t^τ. The situation is analogous to the modeling of dependence in the microstructure noise, contaminating asset prices at high frequency, see, e.g., Jacod et al. (2017) and Varneskov (2017).

In the current setting, our primary objective is to design a nonparametric test for the null hypothesis of $\gamma_{t,\tau}(h)$ being identically equal to zero for any integer $h \neq 0$, i.e., that the observation errors feature no spatial correlation.

3 Assumptions

Our theoretical results require assumptions for the true latent option price, the observation scheme and the observation error. These assumptions are stated below.

A1. *We have*

$$|\kappa_{t,\tau}(k_1) - \kappa_{t,\tau}(k_2)| \;\leq\; C_t\, |k_1 - k_2|, \qquad k_1, k_2 \in [\underline{k}_{t,\tau}, \overline{k}_{t,\tau}], \quad t \in \mathbb{T}, \quad \tau \in \mathcal{T}_t\,, \tag{5}$$

where C_t is a finite-valued $\mathcal{F}_t^{(0)}$- adapted random variable.

A2. *The log-strike grids* $\{k_{t,\tau}(j))\}_{j=1}^{N_t^\tau}$ *are* $\mathcal{F}_t^{(0)}$*-adapted, and we have,*

$$\underline{C}_t\,\Delta \;\leq\; \Delta_{t,\tau}(j) \;\leq\; \overline{C}_t\,\Delta, \quad j=1,...,N_t^\tau, \quad t\in\mathbb{T}, \quad \tau\in\mathcal{T}_t, \quad \text{as } \Delta\downarrow 0\,, \tag{6}$$

where $0<\underline{C}_t\leq\overline{C}_t$ *are finite-valued* $\mathcal{F}_t^{(0)}$*-adapted random variables, and* Δ *is a deterministic sequence. In addition, for some arbitrary small* $\zeta>0$ *as well as* $t\in\mathbb{T}$*, and* $\tau\in\mathcal{T}_t$*, as* $\Delta\downarrow 0$*,*

$$\sup_{j=2,\ldots,N_t^\tau}\left|\frac{\Delta_{t,\tau}(j)}{\Delta} - \psi_{t,\tau}^{-1}(k_{t,\tau}(j))\right| \;=\; o_p(\Delta^{1/2}), \qquad \left|\,N_t^\tau\Delta - \vartheta_{t,\tau}^{-1}\,\right| \;=\; o_p(\Delta^{1/2}), \tag{7}$$

where $\vartheta_{t,\tau}$ *and* $\psi_{t,\tau}(k)$ *are* $\mathcal{F}_t^{(0)}$*-adapted variables and functions, respectively, that are bounded from above and below for all* $t\in\mathbb{T}$ *and* $\tau\in\mathcal{T}_t$*, and* $\psi_{t,\tau}(k)$ *is continuously differentiable in* k*.*

A3. *The function* $\sigma_{t,\tau}(k)$ *is continuously differentiable in* k*, for* $t\in\mathbb{T}$ *and* $\tau\in\mathcal{T}_t$*. The sequences* $\{\zeta_{t,\tau}^N(j)\}_{j=1,\ldots,N_t^\tau}$ *are defined on* $\Omega^{(1)}$*, are independent from* $\mathcal{F}^{(0)}$ *and for different pairs* (t,τ)*. Moreover,* $\mathbb{E}(\zeta_{t,\tau}^N(j))=0$*,* $\mathbb{E}(\zeta_{t,\tau}^N(j)^2)=1$*, and* $\mathbb{E}(|\zeta_{t,\tau}^N(j)|^{4+\varsigma})<\infty$*, for some* $\varsigma>0$*.*

A4. *For every* (t,τ) *and* $N_t^\tau\in\mathbb{N}_+$*, we have* $\zeta_{t,\tau}^N(j)=\sum_{g=-\infty}^{\infty}\phi_g\,\nu_{t,\tau}(j-g)$*, where the sequence* $\{\nu_{t,\tau}(g)\}_{g\in\mathbb{Z}}$ *is defined on* $\Omega^{(1)}$*, is independent of* $\mathcal{F}^{(0)}$*, is i.i.d. across* t*,* τ *and* j*, and has moments* $\mathbb{E}(\nu_{t,\tau}(j))=0$*,* $\mathbb{E}(\nu_{t,\tau}^2(j))=1$ *and* $\mathbb{E}(\nu_{t,\tau}^4(j))=\eta<\infty$*. Finally,* $\sum_{g=-\infty}^{\infty}|\phi_g|<\infty$*.*

We briefly discuss these assumptions. Assumption A1 imposes Lipschitz continuity on the true option price. Note that C_t in equation (5) is a random variable and, therefore, A1 is satisfied, whenever the density of the conditional (risk-neutral) return distribution is continuous. Assumption A2 concerns the asymptotic behavior of the strike grid. In stating this assumption, we introduce a reference deterministic sequence $\Delta\to 0$, which may be interpreted as the "average mesh" of the log-strike grid. We note, however, that Δ does not appear in the feasible versions of the limit results, we develop below. Assumption A2 allows for a non-equidistant strike grid, and the function $\psi_{t,\tau}(k)$ controls the sampling frequency in the strike domain (for given t and τ) relative to the benchmark grid. Similarly, $\vartheta_{t,\tau}$ captures the relative sampling frequency across maturities on a given day. Assumptions A3 and A4 concern the observation error. In Assumption A3, we assume the error is $\mathcal{F}^{(0)}$- conditionally centered at zero and that $\mathcal{F}^{(0)}$- conditional moments above four exist. In Assumption A4, we require that $\zeta_{t,\tau}^N(j)$ is an infinite moving average, viewed as a process in j, with absolutely summable coefficients. This assumption can likely be relaxed to cover all stationary processes, but such an extension seems inconsequential from a practical point of view, as measurement errors in derivatives data, typically, have limited spatial dependence, readily accommodated through Assumption A4.

We note that Assumption A4 implies the dependence between two observation errors on the observable strike grid depends not on the distance between strikes, but on the number of available strikes between them. When the strike grid is irregular, the dependence between observation errors with the same distance in terms of strikes can differ due to the different denseness of the strike grid in separate regions of the strike domain. A possible extension of Assumption A4, in this case, would be, similarly to Robinson (2011), to let the moving average coefficients ϕ_g in $\zeta_{t,\tau}^N(j) = \sum_{g=-\infty}^{\infty} \phi_g \, \nu_{t,\tau}(j-g)$ depend also on j. This extension adds a layer of complexity to the analysis since stationarity of $\zeta_{t,\tau}$ is no longer preserved. We do not pursue such a generalization of Assumption A4 here because the strike grids are approximately equidistant for the option panels typically available.

4 Nonparametric Recovery of Spatial Dependence in the Error

We are now ready to develop our inference techniques for measuring the spatial dependence in the observation error. We do not observe the true (theoretical) option price directly, and we prefer to avoid imposing potentially misspecified parametric assumptions for it. Therefore, we instead exploit the infill sampling scheme along with Assumption A1 ensuring smoothness of the true option price $\kappa_{t,\tau}(k)$. In particular, we rely on the following estimate of the option error,

$$\begin{aligned} \widehat{\epsilon}_{t,\tau}^N(j) &= \sqrt{\frac{2}{3}}\left(\widehat{\kappa}_{t,\tau}(k_{t,\tau}(j)) - \frac{1}{2}\Big(\widehat{\kappa}_{t,\tau}(k_{t,\tau}(j-1)) + \widehat{\kappa}_{t,\tau}(k_{t,\tau}(j+1))\Big)\right), \quad j = 1, ..., N_t^\tau - 1, \\ &\simeq \sqrt{\frac{2}{3}}\left(\epsilon_{t,\tau}^N(k_{t,\tau}(j)) - \frac{1}{2}\Big(\epsilon_{t,\tau}^N(k_{t,\tau}(j-1)) + \epsilon_{t,\tau}^N(k_{t,\tau}(j+1))\Big)\right), \end{aligned} \tag{8}$$

where the approximation above makes use of Assumption A1, with the precise statement of this type of relation provided in the proofs. The scaling by $\sqrt{2/3}$ in equation (8) is just a normalization of $\widehat{\epsilon}_{t,\tau}^N(j)$ ensuring that we have $\mathrm{Var}(\widehat{\epsilon}_{t,\tau}^N(j)|\mathcal{F}^{(0)}) \simeq \sigma_{t,\tau}^2(k_{t,\tau}(j))$ in the leading case of no spatial dependence in the option observation error. Of course, $\widehat{\epsilon}_{t,\tau}^N(j)$ is not a consistent estimator for the observation error at a given point in the strike domain. It is rather a consistent estimate for the second-order difference of the observation error (viewed as a function of the strike of the option it is attached to) for a given strike. Nonetheless, cross-sectional averages involving $\widehat{\epsilon}_{t,\tau}^N(j)$ can form the basis for consistent estimation of the spatial dependence in the observation error. Specifically, using $\widehat{\epsilon}_{t,\tau}^N(j)$, we form estimators of the covariance in the option error, across the strikes, via,

$$\widehat{\chi}_{t,\tau}(h) = \frac{1}{N_t^\tau - h - 2} \sum_{j=h+2}^{N_t^\tau - 1} \widehat{\epsilon}_{t,\tau}^N(j)\, \widehat{\epsilon}_{t,\tau}^N(j-h), \quad h \in \mathbb{N}. \tag{9}$$

In the next section, we establish the following convergence in probability,

$$\begin{aligned} \widehat{\chi}_{t,\tau}(h) \xrightarrow{\mathbb{P}} \frac{2}{3}&\left(\frac{3}{2}\gamma_{t,\tau}(h) - \gamma_{t,\tau}(h+1) - \gamma_{t,\tau}(h-1) + \frac{1}{4}\gamma_{t,\tau}(h-2) + \frac{1}{4}\gamma_{t,\tau}(h+2)\right) \\ &\times \vartheta_{t,\tau} \int_{\underline{k}(t,\tau)}^{\overline{k}(t,\tau)} \psi_{t,\tau}(k)\, \sigma_{t,\tau}^2(k)\, dk, \quad h \in \mathbb{N}. \end{aligned} \tag{10}$$

The autocovariances $\gamma_{t,\tau}(h \pm 1)$ and $\gamma_{t,\tau}(h \pm 2)$ appearing in the probability limit of $\widehat{\chi}_{t,\tau}(h)$ reflect the second-order differencing of the option price in the construction of $\widehat{\epsilon}^N_{t,\tau}(j)$. The quantities $\vartheta_{t,\tau}$ and $\psi_{t,\tau}(k)$ naturally impact the limiting value, because they capture the relative denseness of the strike grid, as formalized through Assumption A2. Generally, $\vartheta_{t,\tau}$ and $\int_{\underline{k}(t,\tau)}^{\overline{k}(t,\tau)} \psi_{t,\tau}(k)\, \sigma^2_{t,\tau}(k)\, dk$ are random, as they may differ from one day to another and across different option tenors. Nevertheless, the following correlation coefficients have scale-free and nonrandom probability limits,

$$\widehat{\rho}_{t,\tau}(h) \;=\; \frac{\widehat{\chi}_{t,\tau}(h)}{\widehat{\chi}_{t,\tau}(0)}, \tag{11}$$

with

$$\widehat{\rho}_{t,\tau}(h) \;\xrightarrow{\mathbb{P}}\; \overline{\rho}_{t,\tau}(h) \;\equiv\; \frac{\frac{3}{2}\,\gamma_{t,\tau}(h) - \gamma_{t,\tau}(h+1) - \gamma_{t,\tau}(h-1) + \frac{1}{4}\,\gamma_{t,\tau}(h-2) + \frac{1}{4}\,\gamma_{t,\tau}(h+2)}{\frac{3}{2}\,\gamma_{t,\tau}(0) - 2\,\gamma_{t,\tau}(1) + \frac{1}{2}\,\gamma_{t,\tau}(2)}, \tag{12}$$

where $\frac{3}{2}\gamma_{t,\tau}(0) - 2\gamma_{t,\tau}(1) + \frac{1}{2}\gamma_{t,\tau}(2)$ equals $\mathrm{Var}\left(\epsilon^N_{t,\tau}(j) - 0.5\epsilon^N_{t,\tau}(j-1) - 0.5\epsilon^N_{t,\tau}(j+1) | \mathcal{F}^{(0)}\right) > 0$ up to a proportionality factor given by an $\mathcal{F}^{(0)}$- measurable random variable.

The second-order differencing of the option price induces spatial dependence in the estimate $\widehat{\epsilon}^N_{t,\tau}(j)$, even if the option observation error is independent across strikes. Specifically, under the null hypothesis of no spatial $\mathcal{F}^{(0)}$- conditional dependence in the observation errors, $\overline{\rho}_{t,\tau}(h)$ takes the form,

$$\rho^{\mathrm{ind}}(h) \;=\; \begin{cases} -\frac{2}{3}, & \text{if } h = \pm 1, \\ \frac{1}{6}, & \text{if } h = \pm 2, \\ 0, & \text{if } |h| > 2. \end{cases} \tag{13}$$

Hence, deviations of $\overline{\rho}_{t,\tau}(h)$ from $\rho^{\mathrm{ind}}(h)$ are indicative of spatial dependence. Given the normalizations in Assumptions A3 and A4, the null hypothesis of no spatial dependence implies that $\phi(L) = \sum_{g=-\infty}^{\infty} \phi_g L^g = \phi_0 = 1$. Consequently, we may characterize our null hypothesis as follows,

$$\mathcal{H}_0: \quad \phi(L) \,=\, 1 \qquad \Longleftrightarrow \qquad \overline{\rho}_{t,\tau}(h) \,=\, \rho^{\mathrm{ind}}(h), \text{ for } h \in \mathbb{N}\,. \tag{14}$$

Remark 1. *In the estimation strategy outlined above, we draw inference for the dependence in the option observation errors based on the second-order difference of the latter. This is carried out to account for the unknown $\kappa_{t,\tau}(k)$ in the observed option price but, naturally, leads to a loss of information when compared to the infeasible case, where inference is based directly on the option observation errors. Our choice of second-order differencing of the observed options implies that we rely on a linear approximation for the true option price between the two surrouding observations in the strike domain.*

Alternatively, inference may be drawn based on first-order differencing, which will lead to a smaller loss of information about the option errors relative to second-order. However, it also implies that a larger bias will impact the estimation, stemming from the changes to the true option price as a function of its strike. This feature arises since the first-order differencing of observed options rests on an approximation that the true option price is constant over two consecutive strikes.

Remark 2. *An alternative to the differencing approach adopted in this paper is to construct a kernel-based estimate of the true option price at a given strike from neighboring option prices. When the estimate is subtracted from the observed option price, we may work, asymptotically, as if the option error is directly observable since the estimation error for the kernel-based estimate of the true option price will be of higher order (due to the local averaging).*

However, for such an inference strategy to work well in practice, i.e., to generate an inconsequential bias from the estimation of the true option price, we require the latter to be approximately constant over the local window from which it is recovered. Empirically relevant option pricing models, including the one used in our Monte Carlo experiments, typically generate a lot of movement in the true option price across nearby strikes (particularly around the money) given the strike grids available in existing derivatives markets. For this reason, we rely on the second-difference approach developed above.

5 Inference for the Nonparametric Estimators

This section provides theoretical results regarding the inference for the spatial autocorrelation in the option observation error, and it introduces our Portmanteau test for spatial dependence.

5.1 Limit Theory

In order to design a test for spatial dependence, it is necessary to derive limit theory for $\widehat{\rho}_{t,\tau}(h)$. To this end, we first present a consistency result for the spatial autocorrelation estimator.

Theorem 1. *Under Assumptions A1-A4, it follows that,*

$$\widehat{\rho}_{t,\tau}(h) \xrightarrow{\mathbb{P}} \overline{\rho}_{t,\tau}(h), \qquad \forall h \in \mathbb{N}, \tag{15}$$

for t and τ fixed, and $\Delta \to 0$.

We next present a Central Limit Theorem (CLT) associated with this convergence in probability under the null hypothesis of no spatial dependence in the observation error, i.e., when the observation error is $\mathcal{F}^{(0)}$- conditionally independent across strikes.

Theorem 2. *Under Assumptions A1-A4 with $\phi(L) = \sum_{g=-\infty}^{\infty} \phi_g L^g = 1$, it follows that,*

$$\frac{1}{\sqrt{\Delta}} \left(\{\widehat{\rho}_{t,\tau}(h)\}_{h=1,\dots,H} - \{\rho^{\text{ind}}(h)\}_{h=1,\dots,H} \right) \xrightarrow{\mathcal{L}-s} (\boldsymbol{\mathcal{M}}_{t,\tau}^{H,\text{ind}})^{1/2} \times \mathbf{Y}_{t,\tau}, \tag{16}$$

for t and τ fixed, and $\Delta \to 0$. $\mathbf{Y}_{t,\tau}$ is an $H \times 1$ vector of standard Gaussian random variables, defined on an extension of the original probability space, independent from $\mathcal{F}$. The asymptotic covariance matrix $\boldsymbol{\mathcal{M}}_{t,\tau}^{H,\text{ind}}$ is provided in equation (43).

The (random) covariance matrix $\boldsymbol{\mathcal{M}}^{H,\mathrm{ind}}_{t,\tau}$ depends on the $\mathcal{F}^{(0)}$- conditional volatility of the observation error as well as limiting features of the sampling scheme ($\psi_{t,\tau}(k)$ and $\vartheta_{t,\tau}$) through the ratio,

$$\frac{\mathcal{Q}^4_{t,\tau}}{(\mathcal{Q}^2_{t,\tau})^2} = \frac{\vartheta^2_{t,\tau}\int_{\underline{k}(t,\tau)}^{\overline{k}(t,\tau)}\psi_{t,\tau}(k)\sigma^4_{t,\tau}(k)dk}{\left(\vartheta_{t,\tau}\int_{\underline{k}(t,\tau)}^{\overline{k}(t,\tau)}\psi_{t,\tau}(k)\sigma^2_{t,\tau}(k)dk\right)^2} = \frac{\int_{\underline{k}(t,\tau)}^{\overline{k}(t,\tau)}\psi_{t,\tau}(k)\sigma^4_{t,\tau}(k)dk}{\left(\int_{\underline{k}(t,\tau)}^{\overline{k}(t,\tau)}\psi_{t,\tau}(k)\sigma^2_{t,\tau}(k)dk\right)^2}. \tag{17}$$

This term does not feature in the limiting distribution of estimators for the autocorrelation of stationary time series with conditionally homoskedastic errors, see, e.g., Theorem 7.2.1 in Brockwell and Davis (1991). It appears in the asymptotic variance $\boldsymbol{\mathcal{M}}^{H,\mathrm{ind}}_{t,\tau}$, because the option observation errors display $\mathcal{F}^{(0)}$- conditional heteroskedasticity. This feature is important, as the magnitude of the observation error varies with the volatility of the underlying asset, and fluctuates significantly across time.

5.2 Towards Feasible Inference and Testing

The fact that $\boldsymbol{\mathcal{M}}^{H,\mathrm{ind}}_{t,\tau}$ is random implies that the limiting distribution for the autocorrelation estimator is mixed Gaussian. Feasible implementation of the CLT in Theorem 2 requires that we construct a consistent estimate of $\boldsymbol{\mathcal{M}}^{H,\mathrm{ind}}_{t,\tau}$ and utilize the fact that this convergence result holds stably. The latter allows to use Slutsky's theorem, even if the convergence in probability is to a random variable.[3] To estimate $\boldsymbol{\mathcal{M}}^{H,\mathrm{ind}}_{t,\tau}$, we must estimate the ratio in equation (17). This is carried out under $\mathcal{H}_0$ of no spatial dependence in the observation error. For the denominator, we may simply use $\widehat{\chi}_{t,\tau}(0)$. For the numerator, we make use of the following class of estimators, as a function of h,

$$\widehat{\mathcal{L}}_{t,\tau}(h) \equiv \frac{1}{\Delta\,(N^\tau_t - h - 2)}\,\frac{1}{N^\tau_t - h - 2}\sum_{j=h+2}^{N^\tau_t - 1}(\widehat{\epsilon}^N_{t,\tau}(j))^2\,(\widehat{\epsilon}^N_{t,\tau}(j-h))^2. \tag{18}$$

The intuition behind this estimator is as follows. First, if $h = 0$, then for any $k \in \mathbb{Z}$, and under $\mathcal{H}_0$,

$$\begin{aligned}\frac{1}{N^\tau_t - h - 2}\sum_{j=2}^{N^\tau_t - 1}(\widehat{\epsilon}^N_{t,\tau}(j))^4 &\xrightarrow{\mathbb{P}} \mathbb{E}\left(\left(\zeta^N_{t,\tau}(k) - \frac{1}{2}\,\zeta^N_{t,\tau}(k-1) - \frac{1}{2}\zeta^N_{t,\tau}(k+1)\right)^4 \,\middle|\, \mathcal{F}^{(0)}\right)\\ &\quad\times\ \vartheta_{t,\tau}\int_{\underline{k}(t,\tau)}^{\overline{k}(t,\tau)}\psi_{t,\tau}(k)\,\sigma^4_{t,\tau}(k)\,dk\,,\end{aligned}$$

and from Assumptions A3 and A4, we note that, under $\mathcal{H}_0$, the random variables $\zeta^N_{t,\tau}(j)$ are i.i.d. The constant $\mathbb{E}\left(\left(\zeta^N_{t,\tau}(j) - 0.5\zeta^N_{t,\tau}(j-1) - 0.5\zeta^N_{t,\tau}(j+1)\right)^4 \,|\, \mathcal{F}^{(0)}\right)$ depends on the unknown fourth and cross-product moment of $\zeta^N_{t,\tau}(j)$. To avoid these terms, we consider products of second powers of $\widehat{\epsilon}^N_{t,\tau}(j)$ for different values of j, and exploit the smoothness of $\sigma_{t,\tau}(k)$ from Assumption A3, plus the fact that $\frac{2}{3}\,\mathbb{E}\left(\left(\zeta^N_{t,\tau}(j) - \frac{1}{2}\zeta^N_{t,\tau}(j-1) - \frac{1}{2}\zeta^N_{t,\tau}(j+1)\right)^2 \,|\, \mathcal{F}^{(0)}\right) = 1$, due to $\zeta^N_{t,\tau}(j)$ having unit $\mathcal{F}^{(0)}$- conditional

[3] In Slutsky's theorem, $X_n \xrightarrow{\mathcal{L}} X$ and $Y_n \xrightarrow{\mathbb{P}} Y$ implies $X_nY_n \xrightarrow{\mathcal{L}} XY$, provided Y is non-random. If $X_n \xrightarrow{\mathcal{L}-s} X$, then the Slutsky theorem can be extended to the case where Y is random.

variance by Assumption A3. Specifically, under $\mathcal{H}_0$,

$$\frac{1}{N_t^\tau - h - 2} \sum_{j=h+2}^{N_t^\tau - 1} (\widehat{\epsilon}_{t,\tau}^N(j))^2 \, (\widehat{\epsilon}_{t,\tau}^N(j-h))^2 \;\xrightarrow{\mathbb{P}}\; \vartheta_{t,\tau} \int_{\underline{k}(t,\tau)}^{\overline{k}(t,\tau)} \psi_{t,\tau}(k)\, \sigma_{t,\tau}^4(k)\, dk, \quad h \geq 2. \tag{19}$$

By imposing $h \geq 2$, we avoid overlap between the options entering into $\widehat{\epsilon}_{t,\tau}^N(j)$ and $\widehat{\epsilon}_{t,\tau}^N(j-h)$. Hence, they are $\mathcal{F}^{(0)}$-conditionally independent under $\mathcal{H}_0$. Now, equation (19) follows by applying the conditional moment results for $\widehat{\chi}_{t,\tau}(h)$ in Lemmas 3 and 6 as well as Assumption A2 to show $1/(\Delta(N_t^\tau - h - 2)) \xrightarrow{\mathbb{P}} \vartheta_{t,\tau}$. Thus, by collecting results, it follows that, under $\mathcal{H}_0$,

$$\widehat{\chi}_{t,\tau}(0) \;\xrightarrow{\mathbb{P}}\; \mathcal{Q}_{t,\tau}^2\,, \qquad\qquad \widehat{\mathcal{L}}_{t,\tau}(2) \;\xrightarrow{\mathbb{P}}\; \mathcal{Q}_{t,\tau}^4\,, \tag{20}$$

and, hence, the ratio in equation (17) may be estimated via $\widehat{\mathcal{L}}_{t,\tau}(2)/\widehat{\chi}_{t,\tau}^2(0)$. Utilizing this relation, we define $\widehat{\boldsymbol{\mathcal{M}}}_{t,\tau}^{H,\mathrm{ind}} \equiv \{\widehat{\mathcal{M}}_{t,\tau}^{\mathrm{ind}}(h,g)\}_{h,g=1}^{H}$, with elements given by,

$$\widehat{\mathcal{M}}_{t,\tau}^{\,\mathrm{ind}}(h,g) \;\equiv\; \mathcal{W}^{\,\mathrm{ind}}(h,g) \times \frac{\widehat{\mathcal{L}}_{t,\tau}(2)}{\widehat{\chi}_{t,\tau}^2(0)}\,, \tag{21}$$

where $\mathcal{W}^{\,\mathrm{ind}}(h,g)$ is a matrix of constants, provided in equation (43) of the proofs, and $\widehat{\boldsymbol{\mathcal{M}}}_{t,\tau}^{H,\mathrm{ind}}$ is our consistent estimator of $\boldsymbol{\mathcal{M}}_{t,\tau}^{H,\mathrm{ind}}$ under $\mathcal{H}_0$ of no spatial dependence in the observation error.

5.3 Bias Correction

Before stating the feasible CLT, we bias-correct our estimator. Specifically, $\widehat{\rho}_{t,\tau}(h)$ is biased due to the variability of $\widehat{\chi}_{t,\tau}(0)$ and $\widehat{\chi}_{t,\tau}(h)$ as well as the nonlinear transformation of these quantities in forming $\widehat{\rho}_{t,\tau}(h)$. Although the bias is asymptotically negligible and has no effect on the CLT, it affects the estimation in small samples. Hence, we construct the bias-corrected estimator of $\overline{\rho}_{t,\tau}(h)$,

$$\widehat{\rho}_{t,\tau}^{\mathrm{BC}}(h) \;\equiv\; \widehat{\rho}_{t,\tau}(h) \;-\; \frac{\widehat{\chi}_{t,\tau}(h)}{\widehat{\chi}_{t,\tau}(0)^3}\, \widehat{\mathrm{Acov}}_{t,\tau}(0,0) \;+\; \frac{1}{\widehat{\chi}_{t,\tau}(0)^2}\, \widehat{\mathrm{Acov}}_{t,\tau}(0,h)\,, \tag{22}$$

where $\widehat{\mathrm{Acov}}_{t,\tau}(h,g) \equiv \widehat{\mathrm{Acov}}(\widehat{\chi}_{t,\tau}(h), \widehat{\chi}_{t,\tau}(g))$ is a consistent estimator of the asymptotic covariance between $\widehat{\chi}_{t,\tau}(h)$ and $\widehat{\chi}_{t,\tau}(g)$, defined as,

$$\begin{aligned} \Delta^{-1}\, \widehat{\mathrm{Acov}}_{t,\tau}(h,g) \;=\; & \Big(\widehat{\eta}_{t,\tau} - 3 \times \widehat{\mathcal{L}}_{t,\tau}(2)\Big)\, \rho^{\mathrm{ind}}(h)\, \rho^{\mathrm{ind}}(g) \\ & +\; \widehat{\mathcal{L}}_{t,\tau}(2) \sum_{s=-\infty}^{\infty} \Big\{ \rho^{\mathrm{ind}}(s)\, \rho^{\mathrm{ind}}(s+h-g) \;+\; \rho^{\mathrm{ind}}(s+h)\, \rho^{\mathrm{ind}}(s-g) \Big\}\,, \end{aligned} \tag{23}$$

and for which the fourth moment of the observation error is estimated via,

$$\widehat{\eta}_{t,\tau} \;\equiv\; \frac{9}{5} \left(\widehat{\mathcal{L}}_{t,\tau}(0) \;-\; \frac{14}{9}\, \widehat{\mathcal{L}}_{t,\tau}(2) \right). \tag{24}$$

5.4 The Feasible CLT

We are now in a position to state a feasible CLT result. To this end, let $\mathbf{0}_H$ and $\boldsymbol{I}_H$ be an H-dimensional vector of zeroes and an $H \times H$ identity matrix, respectively.

Corollary 1. *Under the conditions of Theorem 2, it follows that,*

$$\frac{1}{\sqrt{\Delta}} \left(\widehat{\boldsymbol{\mathcal{M}}}_{t,\tau}^{H,\text{ind}} \right)^{-1/2} \left(\{\widehat{\rho}_{t,\tau}^{\text{BC}}(h)\}_{h=1,\dots,H} - \{\rho^{\text{ind}}(h)\}_{h=1,\dots,H} \right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbf{0}_H, \boldsymbol{I}_H\right). \tag{25}$$

Importantly, Corollary 1 shows that the bias-correction has no first-order impact on the central limit theorem and that the consistent estimator of the conditional asymptotic covariance matrix, in conjunction with the scale $1/\sqrt{\Delta}$, normalizes the statistic. Moreover, despite the reference mesh of the strike grid Δ appearing, when defining the bias correction and the conditional variance estimator, it cancels from the relevant expressions when constructing test statistics using equation (25). To see this, note that $\widehat{\mathcal{L}}_{t,\tau}(h)$ depends on Δ^{-1}, thus vanishing when computing $\widehat{\text{Acov}}_{t,\tau}(h,g)$ in equation (22) as well as when using the unnormalized variance estimator, $\Delta\, \widehat{\boldsymbol{\mathcal{M}}}_{t,\tau}^{H,\text{ind}}$. Hence, for feasible inference, Δ is not needed. It merely represents a convenient way of conveying the convergence rate for our estimator.

5.5 Tests for Spatial Dependence in the Observation Error

This section leverages the (feasible) limit theorem and presents our test for the null hypothesis of no spatial dependence in the option observation error. Given the limit result in Corollary 1, a natural candidate test for this hypothesis, based on the first H autocorrelations, is given by,

$$\frac{1}{\Delta} \left(\{\widehat{\rho}_{t,\tau}^{\text{BC}}(h)\}_{h=1}^{H} - \{\rho^{\text{ind}}(h)\}_{h=1}^{H} \right)' \left(\widehat{\boldsymbol{\mathcal{M}}}_{t,\tau}^{H,\text{ind}} \right)^{-1} \left(\{\widehat{\rho}_{t,\tau}^{\text{BC}}(h)\}_{h=1}^{H} - \{\rho^{\text{ind}}(h)\}_{h=1}^{H} \right). \tag{26}$$

This self-normalized test statistic may be considered an option analogue of the corresponding statistic in Dalla et al. (2019) for examining the null hypothesis of white noise in time series, that can exhibit heteroskedasticity and general forms of dependence. Importantly, from Corollary 1, equation (26) converges in law to a $\chi^2(H)$ distribution, rendering it readily implementable in practice.

However, the finite sample performance of a test based on equation (26) depends on the sample correlations between the elements of the vector $\{\widehat{\rho}_{t,\tau}^{\text{BC}}(h)\}_{h=1,\dots,H}$ being well approximated by their asymptotic counterparts. In particular, for the relatively small sample sizes available in a typical application to option data, including the sample of S&P 500 options considered below, this type of approximation may be inadequate due to a non-trivial impact of higher-order biases.

As a result, we suggest working with an alternative test, that does not depend on the cross-products $\widehat{\rho}_{t,\tau}^{\text{BC}}(h)\, \widehat{\rho}_{t,\tau}^{\text{BC}}(g)$, for $h \neq g$. Our test can be viewed as a heteroskedasticity-adjusted Box and Pierce test. That is, it is defined as the sum of squared differences between the estimated spatial autocorrelations $\widehat{\rho}_{t,\tau}^{\text{BC}}(h)$ and their asymptotic limits under the null hypothesis of no error dependence, adjusted by a scaling factor to account for heteroskedasticity in the series. More specifically, the proposed test takes the following form,

$$\widehat{Q}_{t,\tau}^{H} \;=\; \frac{1}{\Delta}\,\frac{\widehat{\chi}_{t,\tau}^{2}(0)}{\widehat{\mathcal{L}}_{t,\tau}(2)}\,\sum_{h=1}^{H}\left(\,\widehat{\rho}_{t,\tau}^{\,\mathrm{BC}}(h)-\rho^{\mathrm{ind}}(h)\,\right)^{2}, \tag{27}$$

where $\rho^{\mathrm{ind}}(h)$, defined in equation (13), equals the probability limit of $\widehat{\rho}_{t,\tau}^{\,\mathrm{BC}}(h)$ in the absence of $\mathcal{F}^{(0)}$-conditional spatial dependence in the observation error. By Corollary 1 and equation (21),

$$\widehat{Q}_{t,\tau}^{H} \;\xrightarrow{\mathcal{L}}\; \boldsymbol{Y}'\,\boldsymbol{\mathcal{W}}^{H,\mathrm{ind}}\,\boldsymbol{Y}, \tag{28}$$

where $\boldsymbol{Y}$ is an $H\times 1$ vector of standard Gaussian random variables and $\boldsymbol{\mathcal{W}}^{H,\mathrm{ind}} \equiv \{\mathcal{W}^{H,\mathrm{ind}}(h,g)\}_{h,g=1}^{H}$ with $\mathcal{W}^{\mathrm{ind}}(h,g)$ being a matrix of constants, provided in equation (43). The limiting distribution of $\widehat{Q}_{t,\tau}^{H}$ is time invariant and known. We denote its $(1-\vartheta)$'th quantile by,

$$q_H(\vartheta) = \{x \in \mathbb{R}_+ \cup 0 : \mathbb{P}(\boldsymbol{Y}'\,\boldsymbol{\mathcal{W}}^{H,\mathrm{ind}}\,\boldsymbol{Y} > x) \geq \vartheta\},$$

which can be easily computed via simulation. We have the following formal size and power result.

Theorem 3. *Under Assumptions A1-A4, it follows that for t and τ fixed, as $\Delta \downarrow 0$,*

(a) If $\mathcal{H}_0 : \phi(L) = 1$ holds, $\quad \mathbb{P}\big(\widehat{Q}_{t,\tau}^{H} > q_H(\vartheta) | \mathcal{F}^{(0)}\big) \;\to\; \vartheta.$

(b) If $\mathcal{H}_A : \phi(L) \neq 1$ holds, $\exists H \geq 1$ sufficiently large, so that $\;\mathbb{P}\big(\widehat{Q}_{t,\tau}^{H} > q_H(\vartheta) | \mathcal{F}^{(0)}\big) \to 1.$

Implementation of the test (27) requires a choice of H, which involves a size-power tradeoff. Specifically, from equation (13), we have non-trivial theoretical predictions for the first two autocorrelations under $\mathcal{H}_0$. Hence, to ensure proper size of the test, we must choose $H \geq 2$. However, the addition of, possibly many, insignificant higher-order autocorrelations will reduce test power. Consequently, we may want to restrict H to only exceed two slightly. We implement the test with $H = 4$ in a numerical exercise below and find that it has excellent finite sample properties in empirically realistic settings.

5.6 Extensions

The estimation precision and the power of the spatial dependence test may be increased by pooling option data across tenors and dates, leading to the following set of correlation coefficients,

$$\widehat{\rho}_{\mathbb{T},\mathcal{T}}(h) \;=\; \frac{\sum_{t\in\mathbb{T}}\sum_{\tau\in\mathcal{T}_t}\,\widehat{\chi}_{t,\tau}(h)}{\sum_{t\in\mathbb{T}}\sum_{\tau\in\mathcal{T}_t}\,\widehat{\chi}_{t,\tau}(0)}\,. \tag{29}$$

Then, to draw feasible inference, we utilize the fact that, per Assumption A4, we have $\mathcal{F}^{(0)}$- conditional independence between observation errors for options with different tenors and/or recorded at different times. Hence, we may define the aggregated bias-corrected estimators analogously,

$$\widehat{\rho}^{\mathrm{BC}}_{\mathbb{T},\mathcal{T}}(h) \;\equiv\; \widehat{\rho}_{\mathbb{T},\mathcal{T}}(h) \;-\; \frac{\sum_{t\in\mathbb{T}}\sum_{\tau\in\mathcal{T}_t}\widehat{\chi}_{t,\tau}(h)}{\left(\sum_{t\in\mathbb{T}}\sum_{\tau\in\mathcal{T}_t}\widehat{\chi}_{t,\tau}(0)\right)^3}\sum_{t\in\mathbb{T}}\sum_{\tau\in\mathcal{T}_t}\widehat{\mathrm{Acov}}_{t,\tau}(0,0)$$
$$\;+\; \frac{1}{\left(\sum_{t\in\mathbb{T}}\sum_{\tau\in\mathcal{T}_t}\widehat{\chi}_{t,\tau}(0)\right)^2}\sum_{t\in\mathbb{T}}\sum_{\tau\in\mathcal{T}_t}\widehat{\mathrm{Acov}}_{t,\tau}(0,h), \tag{30}$$

and the associated estimate of the asymptotic variance by,

$$\widehat{\boldsymbol{\mathcal{M}}}^{H,\mathrm{ind}}_{\mathbb{T},\mathcal{T}} \;=\; \boldsymbol{\mathcal{W}}^{H,\mathrm{ind}} \times \frac{\sum_{t\in\mathbb{T}}\sum_{\tau\in\mathcal{T}_t}\widehat{\mathcal{L}}_{t,\tau}(2)}{\left(\sum_{t\in\mathbb{T}}\sum_{\tau\in\mathcal{T}_t}\widehat{\chi}_{t,\tau}(0)\right)^2}, \quad \boldsymbol{\mathcal{W}}^{H,\mathrm{ind}} \equiv \{\mathcal{W}^{H,\mathrm{ind}}(h,g)\}_{h,g=1}^{H}. \tag{31}$$

With this notation, applying Corollary 1 and exploiting the $\mathcal{F}^{(0)}$-conditional independence in the observation errors across time and tenors, we have, under no spatial dependence in observation errors,

$$\frac{1}{\sqrt{\Delta}}\left(\widehat{\boldsymbol{\mathcal{M}}}^{H,\mathrm{ind}}_{\mathbb{T},\mathcal{T}}\right)^{-1/2}\left(\{\widehat{\rho}^{\mathrm{BC}}_{\mathbb{T},\mathcal{T}}(h)\}_{h=1,\ldots,H} - \{\rho^{\mathrm{ind}}(h)\}_{h=1,\ldots,H}\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbf{0}_H, \boldsymbol{I}_H\right). \tag{32}$$

Finally, we may similarly extend our test statistic as,

$$\widehat{Q}^{H}_{\mathbb{T},\mathcal{T}} \;=\; \frac{1}{\Delta}\,\frac{\left(\sum_{t\in\mathbb{T}}\sum_{\tau\in\mathcal{T}_t}\widehat{\chi}_{t,\tau}(0)\right)^2}{\sum_{t\in\mathbb{T}}\sum_{\tau\in\mathcal{T}_t}\widehat{\mathcal{L}}_{t,\tau}(2)}\sum_{h=1}^{H}\left(\widehat{\rho}^{\mathrm{BC}}_{\mathbb{T},\mathcal{T}}(h) - \rho^{\mathrm{ind}}(h)\right)^2, \tag{33}$$

with its limit distribution being the same as that of $\widehat{Q}^{H}_{t,\tau}$.

Remark 3. *Our pooling together of cross-sections of options across different time periods and tenors is based on the premise that the associated errors are uncorrelated. Following Dalla et al. (2019), we can also construct a cross-correlation test of such a hypothesis.*

6 Monte Carlo Study

This section provides simulation evidence focused on the finite-sample performance of the spatial correlation estimator defined in equation (30) and the Portmanteau test statistic in equation (33).

6.1 Setup

The numerical experiments rely on the double-jump stochastic volatility model introduced by Duffie et al. (2000) and studied empirically in Broadie et al. (2007). Specifically, the risk-neutral dynamics of the stock price in this model obey the stochastic differential equations,

$$\frac{dX_t}{X_{t-}} = (r_t - \delta_t)dt + \sqrt{V_t}\,dW_t + dL_{x,t}\,, \qquad dV_t \;=\; \kappa_d\left(\overline{v} - V_t\right)dt + \sigma_d\sqrt{V_t}\,dB_t + dL_{v,t}\,, \tag{34}$$

where (W_t, B_t) is a two-dimensional Brownian motion with $\text{corr}(B_t, W_t) = \rho_d$, $(L_{x,t}, L_{v,t})$ is a bivariate compound Poisson process with intensity λ_j, and $L_{x,t}$ is demeaned, making it a jump martingale. The marginal distribution of volatility jumps, Z_v, is exponential with mean μ_v, while jumps in the log-price, $\log(Z_x + 1)$, are Gaussian with mean $\mu_x + \rho_j Z_v$ and standard deviation σ_x, conditional on Z_v.

Although our inference procedure only requires a characterization of the data generating process under the risk-neutral measure, the dynamics of the state variables must be generated from the true (statistical) probability measure in the simulation experiment. For simplicity, we assume X belongs to the same parametric model class under the statistical ($\mathbb{P}$) and risk-neutral ($\mathbb{Q}$) measures. In addition, following Broadie et al. (2007), we impose $\rho_j = 0$. Finally, for convenience, we also let $r_t = \delta_t = 0$. Table 1 reports the full set of parameter values used in our numerical experiments.

Under $\mathbb{P}$				Under $\mathbb{Q}$			
Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
ρ_d	−0.4600	λ_j	1.0080	ρ_d	−0.4600	λ_j	1.0080
$\overline{v}$	0.0144	μ_x	−0.0284	$\overline{v}$	0.0144	μ_x	−0.0501
κ_d	4.0320	σ_x	0.0490	κ_d	4.0320	σ_x	0.0751
σ_d	0.2000	μ_v	0.0315	σ_d	0.2000	μ_v	0.0930

Table 1: *Parameter Setting for the Numerical Experiments*

In the simulations, we price options on an equispaced log-moneyness grid, covering the range $[-4, 1] \cdot \sigma^{\text{ATM}}\sqrt{\tau}$, where σ^{ATM} is the at-the-money Black-Scholes implied volatility (ATM-BSIV) on a given day. This results in a time-varying range of moneyness, depending on the level of volatility, which mimics the corresponding trait in actual option data. For the volatility of the option error, we follow Andersen et al. (2015) and set $\sigma_{t,\tau}(k) = 0.5\,\psi_k\,\kappa_{t,\tau}(k)/Q_{0.995}$, with ψ_k denoting the estimate from a kernel regression of the relative bid-ask spreads of SPX options used in our empirical application against their volatility-adjusted log-strike, and $Q_{0.995}$ signifying the 0.995-quantile of the standard normal distribution. This specification of $\sigma_{t,\tau}(k)$ allows for significant time-variation of the (conditional) noise variance, depending on both the level of volatility and the moneyness. The errors $\{\zeta^N_{t,\tau}(j)\}_{j=1,\dots,N^\tau_t}$ are generated as jointly normally distributed variables with mean zero and variance one. When studying dependence under the null hypothesis, they are independent. In addition, we consider two alternative specifications that accommodate spatial dependence in the observation errors,

- **Alternative (A1)**: For $\forall(t,\tau)$, $\zeta^N_{t,\tau}(j)$ is an MA(1) process with moving-average parameter 0.5.
- **Alternative (A2)**: For $\forall(t,\tau)$, $\zeta^N_{t,\tau}(j)$ is an AR(1) process with autoregressive parameter 0.5,

Finally, we consider three different scenarios for the option observation scheme,

- **Scenario 1**: $|\mathbb{T}| = 1$, $|\mathcal{T}_t| = 1$, $N_t^{\tau} = 500$ and $\tau = 15$ days,
- **Scenario 2**: $|\mathbb{T}| = 1$, $|\mathcal{T}_t| = 1$, $N_t^{\tau} = 50$ and $\tau = 15$ days,
- **Scenario 3**: $\mathbb{T}$ consists of 21 consecutive days, $|\mathcal{T}_t| = 1$, $N_t^{\tau} = 50$ and $\tau = 15$ days.

6.2 Simulation Results

We start by assessing the finite-sample performance of the feasible CLT for $\widehat{\rho}^{\mathrm{BC}}_{\mathbb{T},\mathcal{T}}(h)$ under the null hypothesis. Figures 1, 2, and 3 show the histograms of $(\widehat{\rho}^{\mathrm{BC}}_{\mathbb{T},\mathcal{T}}(h) - \rho^{\mathrm{ind}}(h)))/\sqrt{\Delta\,\mathcal{M}^{H,\mathrm{ind}}_{\mathbb{T},\mathcal{T}}(h,h)}$, for $h = 1, \cdots, 4$ on the basis of 1,000 Monte Carlo replications under the null hypothesis of no cross-sectional dependence in the observation error. Figure 1 shows that the feasible CLT for $\widehat{\rho}^{\mathrm{BC}}_{\mathbb{T},\mathcal{T}}(h)$ works very well in the setup of Scenario 1, for which we have a lot of option data on a single day. For Scenario 2, Figure 2 reveals a slight upward bias at lag 1 and a downward bias at lag 2. Relative to Scenario 1, Scenario 2 has a much coarser strike grid, so the quality of our asymptotic expansions worsens somewhat, as the true option price displays more variability across adjacent strikes, rendering the underlying approximation in equation (8) less accurate. Finally, Figure 3 shows there is an upward bias at lag 2 under Scenario 3, contrary to a slight downward bias at that lag under Scenario 2. Scenarios 2 and 3 have the same denseness of the strike grid, but Scenario 3 utilizes data across multiple days, corresponding to different realizations of the latent stochastic volatility, leading to additional variation in the true option price. Hence, the pooling of option data across volatility regimes will naturally generate a different behavior for the correlation coefficients under Scenarios 2 and 3.

Overall, the finite-sample performance of our feasible inference for the spatial autocorrelations is very good. This conclusion is further corroborated by comparing empirical coverage rates for the confidence intervals of these quantities, reported in Table 2, with their nominal levels. In all the cases we consider, the empirical coverage rates are very close to their nominal counterparts.

We next explore the finite-sample behavior of our Portmanteau test for cross-sectional dependence in the observation error. We set $H = 4$ in our implementation, i.e., we use the first four autocorrelations. The results are reported in Table 2. In all three scenarios, the rejection rates of the test under the null hypothesis are very close to the nominal size. The test appears only mildly undersized for the 5% and 1% size under Scenario 3. In terms of power, not surprisingly, Scenario 2 is the weakest. This scenario relies on a single cross-section of 50 options on one day. For the identical denseness of the strike grid, we gain significantly more power in detecting spatial dependence in the observation errors by pooling data across days, as evident from the results for Scenario 3. Of course, by pooling option data across days, we can make inference only about the average correlations across the days used in

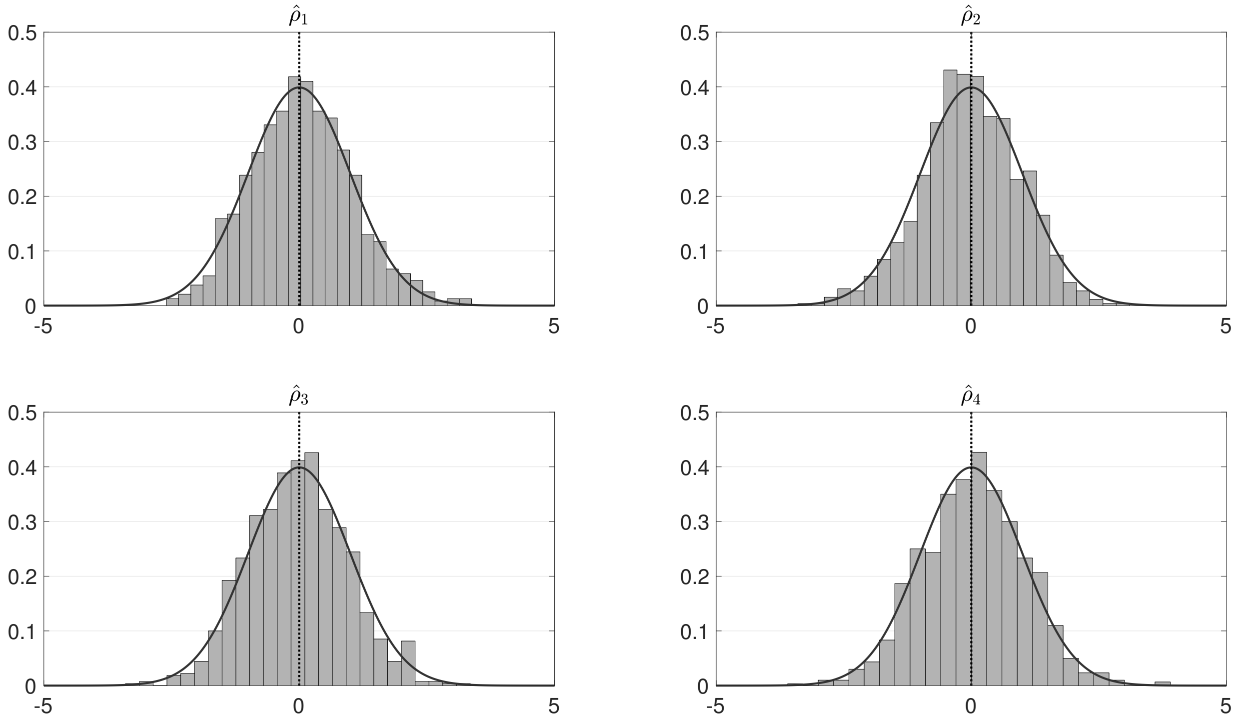

Figure 1: *Finite-sample distribution of* $(\widehat{\rho}^{\mathrm{BC}}_{\mathbb{T},\mathcal{T}}(h) - \rho^{\mathrm{ind}}(h))/\sqrt{\Delta\,\widehat{\mathcal{M}}^{H,\mathrm{ind}}_{\mathbb{T},\mathcal{T}}(h,h)}$ *under* $\mathcal{H}_0$ *for Scenario 1.* The solid line corresponds to the pdf of the standard normal distribution, while the dashed vertical line originates at zero. The histograms are computed from 1,000 Monte Carlo replications.

the test. This seems, however, to constitute a reasonable compromise in settings for which we may not have enough option data available each day to obtain sufficient test power.

Finally, we repeated the above Monte Carlo study with the identical setup, except that we replaced $\tau = 15$ days with the longer maturities of $\tau = 90$ and $\tau = 250$ days. The performance of the test for the longer-dated options is very similar to the one for the short-dated options reported in Table 2. These results are omitted for brevity.

7 Empirical Application

We now apply our newly developed inference techniques to study the dependence in observation errors for options written on the S&P 500 index. These options are European style and trade on the CBOE with ticker SPX. The data consist of closing bid and ask quotes and are obtained from OptionMetrics. We take the mid-quote as our observed option price. The sample covers January 2007 – December 2014. We apply standard filters. First, we retain only options with a tenor below one year, as longer maturity contracts are less liquid. We do, however, retain very short-maturity options, as they become

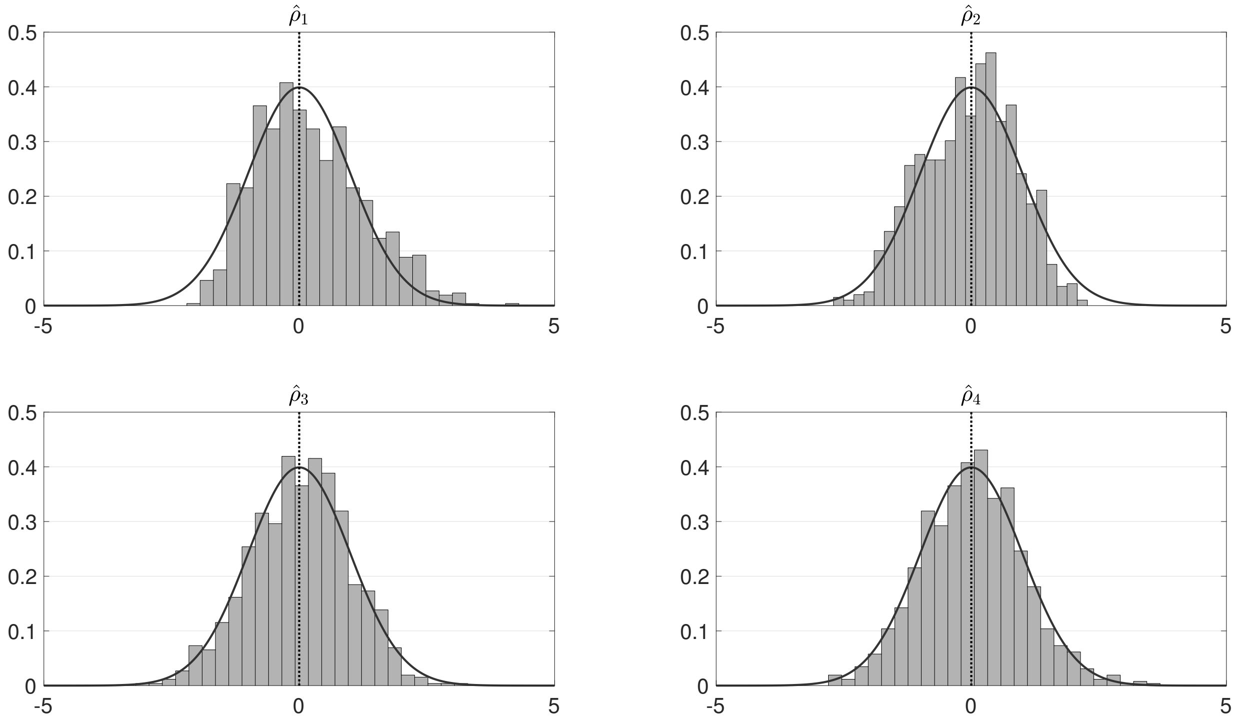

Figure 2: *Finite-sample distribution of* $(\widehat{\rho}^{\mathrm{BC}}_{\mathbb{T},\mathcal{T}}(h) - \rho^{\mathrm{ind}}(h))/\sqrt{\Delta\,\widehat{\mathcal{M}}^{H,\mathrm{ind}}_{\mathbb{T},\mathcal{T}}(h,h)}$ *under* $\mathcal{H}_0$ *for Scenario 2.* The solid line corresponds to the pdf of the standard normal distribution while the dashed vertical line originates at zero. The histograms are computed from $1,000$ Monte Carlo replications.

very liquid in the recent part of the sample, see, e.g., Andersen et al. (2017). Second, for each day and maturity, we compute the implied futures price on the underlying index using put-call parity. For this purpose, we retain only cross-sections with, at least, five put-call contracts with the same strike price. We then extract the futures price from the full set of put-call pairs with identical strikes.

Table 3 provides summary statistics for our final dataset. Importantly, the moneyness range, expressed as standard deviations from the futures price, covers on average the interval $[-5, 2]$ and the average number of available options per maturity is between 37 and 96. This resembles the setup for our numerical experiments in Section 6.

In our empirical application, we pool all options within a month, corresponding to the sampling scheme described by Scenario 3 in Section 6. Moreover, we explore short- ($\tau \leq 31$), medium- ($31 < \tau \leq 180$), and longer-dated ($180 < \tau \leq 365$) options separately to examine features of the option surface. This categorization of the maturities is standard within the broad option pricing literature; see, e.g., Christoffersen et al. (2012) and Babaoglu et al. (2017), among many others.

The results from our test for spatial dependence in the option observation errors are reported in the last panel of Table 3. We draw several conclusions. First, there is evidence of spatial dependence in

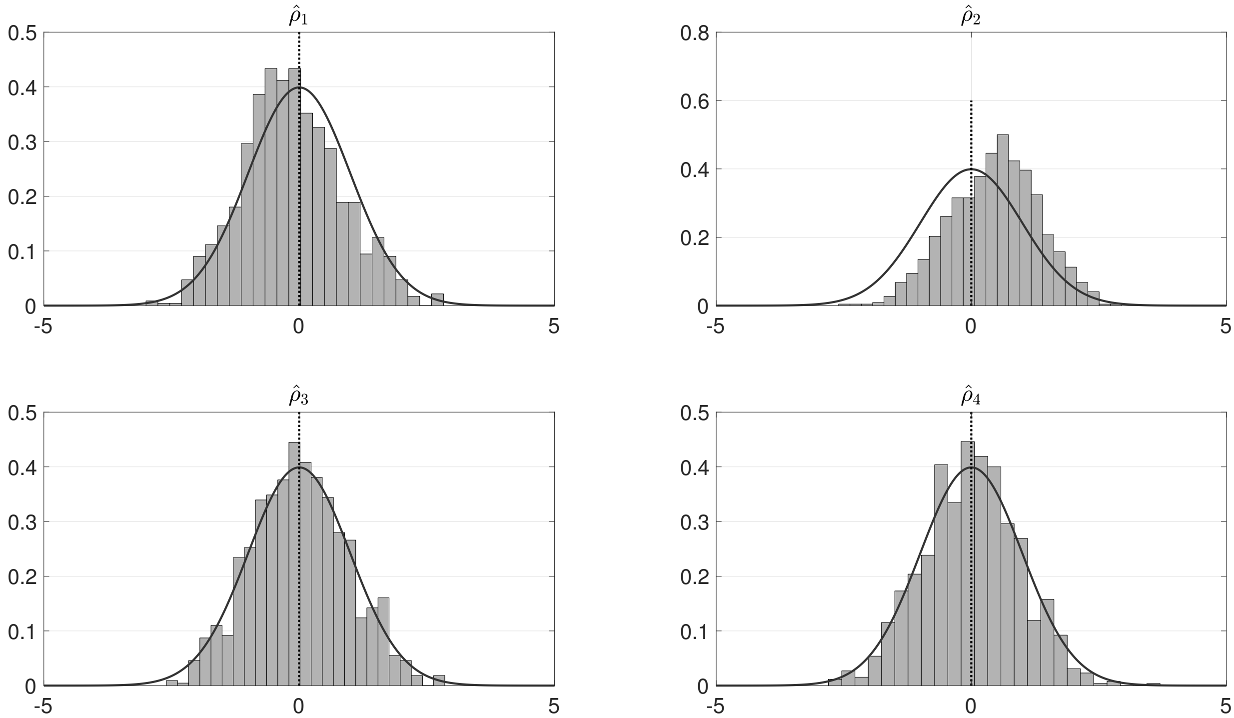

Figure 3: Finite-sample distribution of $(\widehat{\rho}^{\mathrm{BC}}_{\mathbb{T},\mathcal{T}}(h) - \rho^{\mathrm{ind}}(h))/\sqrt{\Delta\,\widehat{\mathcal{M}}^{H,\mathrm{ind}}_{\mathbb{T},\mathcal{T}}(h,h)}$ under $\mathcal{H}_0$ for Scenario 3. The solid line corresponds to the pdf of the standard normal distribution while the dashed vertical line originates at zero. The histograms are computed from $1,000$ Monte Carlo replications.

the option observation error. It is weakest for the short-dated options and strongest for the long-dated ones. Second, the dependence has declined over time for short- and medium-dated options, while it remains largely unchanged for longer-dated options. These patterns are visualized in Figure 4 which plots the time series of the test statistics along with the corresponding 5% critical values.

To further assess the evolution of the spatial dependence in the option observation error over time, Figure 5 depicts the first two autocorrelation coefficients centered around their value under the null hypothesis of spatial error independence, $\widehat{\rho}_{t,\tau}(1) - \rho^{\mathrm{ind}}(1)$ and $\widehat{\rho}_{t,\tau}(2) - \rho^{\mathrm{ind}}(2)$, as well as an estimate for the average (across strikes) absolute value of the option observation error. Consistent with our evidence from the test reported above, the spatial dependence in the option observation error has declined markedly over time for the short- and medium-maturity options, while it appears roughly constant for the ones with long tenor.

To provide a context for the reported figures we note that, in the case of spatial MA(1) dynamics for the observation errors with MA coefficient of θ, we have,

$$\overline{\rho}_{t,\tau}(1) \;=\; \frac{7\overline{\theta} - 4}{6 - 8\overline{\theta}} \quad \text{and} \quad \overline{\rho}_{t,\tau}(2) \;=\; \frac{1 - 4\overline{\theta}}{6 - 8\overline{\theta}}\,, \qquad \text{where} \quad \overline{\theta} \;=\; \frac{\theta}{\theta^2 + 1}\,.$$

	Scenario 1			Scenario 2			Scenario 3		
	90%	95%	99%	90%	95%	99%	90%	95%	99%
Under the Null									
$\hat{\rho}_1$	90.80	94.90	99.20	87.90	93.20	98.20	90.30	95.80	99.20
$\hat{\rho}_2$	91.00	95.70	99.10	94.00	98.30	99.90	91.30	96.00	99.80
$\hat{\rho}_3$	91.40	95.20	99.30	91.00	96.20	99.50	91.60	97.20	99.60
$\hat{\rho}_4$	91.20	95.50	98.50	90.80	95.10	98.60	92.20	96.80	99.50
Test Size	8.70	4.80	1.00	9.70	4.30	0.40	9.40	3.10	0.50
Under the Alternative									
Power A1 (MA 0.5)	99.70	99.40	95.90	35.30	16.70	1.70	100.00	100.00	100.00
Power A2 (AR 0.5)	49.40	32.20	8.10	15.10	7.00	0.70	83.00	66.30	31.90

Table 2: *Simulation results for short-maturity options.* Rows 1-4 provide coverage rates for two-sided confidence intervals for the autocorrelation coefficients under the hypothesis of no spatial dependence in the observation errors. Rows 5-7 give rejection rates of the non-central Chi-square test for no spatial dependence in the option errors, under the null hypothesis (test size), and the alternatives A1 and A2 (test power), respectively.

Therefore, the reported autocorrelations for the short-maturity options are consistent with an MA(1) model with θ around 0.45 for the first half of the sample and θ around only 0.03 for the second half, reflecting the significant decline in the persistence of the observation error. In contrast, the evidence for the long-dated options cannot be rationalized with an MA(1) model, as the latter is incompatible with a large value for $\widehat{\rho}_{t,\tau}(1) - \rho^{\text{ind}}(1)$ and, simultaneously, a small value for $\widehat{\rho}_{t,\tau}(2) - \rho^{\text{ind}}(2)$. Hence, the dependence structure for these options is more complex. We further note the fairly dramatic drop over time in the size of the observation error for short- and, to a lesser extent, medium-dated options. A possible explanation is the strong improvement in the liquidity of the market for shorter dated options over the latter part of our sample. As discussed in Section 1, improved liquidity should lower the exposure of the market makers to inventory risk and–all else equal–reduce the size of the bid-ask spreads. Consistent with this prediction, Figure 6 documents a sizable drop in the spreads for the short-dated options over the sample period, while the decline is more modest for the medium-dated options, and barely noticeable for the longer-dated ones.

8 Conclusion

In this paper, we develop the first procedure for nonparametric inference regarding spatial dependence of option observation errors. It is based on a panel of option prices with different strikes and tenors written on a given underlying asset. The asymptotic design is of the infill type – the set of times and tenors at which the options are observed remains fixed, while the strike grid mesh becomes finer. In

	2007-2010	2011-2014	2007-2014
Q05 moneyness short	-5.85	-7.06	-6.45
Q05 moneyness medium	-5.11	-7.05	-6.07
Q05 moneyness long	-4.06	-6.29	-5.17
Q95 moneyness short	2.19	1.86	2.03
Q95 moneyness medium	2.00	1.70	1.85
Q95 moneyness long	1.68	1.44	1.56
# maturities short	21.49	62.52	41.78
# maturities medium	88.57	137.00	112.53
# maturities long	65.91	87.50	76.59
# options short	74.03	79.46	78.05
# options medium	65.60	90.50	80.59
# options long	36.98	49.71	44.17
Test rejection rate short (%)	38.30	19.57	29.03
Test rejection rate medium (%)	97.87	78.26	88.17
Test rejection rate long (%)	93.62	97.83	95.70

Table 3: *S&P 500 index option data description and test results.* The table reports: (i) the time series average of the 5th and 95th monthly quantiles of the moneyness range, measured in units of Black-Scholes Implied Volatility from at-the-money, in rows 1-3 and 4-6, respectively; (ii) the average number of maturities per month in rows 7-9; (iii) the average number of options per maturity in rows 10-12; and (iv) the 5% rejection rates for our Portmanteau test in rows 13-15. Short, medium, and long refer to options with maturities less than 31, between 31 and 180, and more than 180 calendar days, respectively.

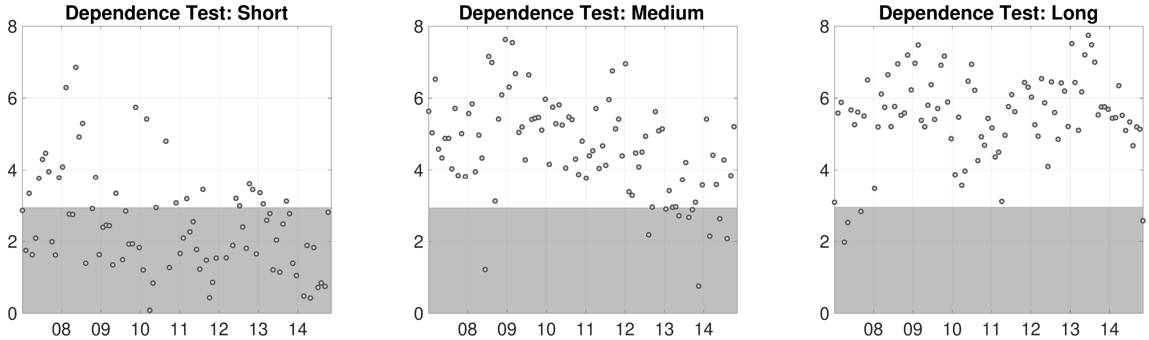

Figure 4: *Tests for spatial dependence in observation error for S&P 500 index options.* The shaded area represents the acceptance region of the test at the 5% critical level, while dots indicate the value of the test statistic. The left panel corresponds to options with maturity up to 31 calendar days, the middle panel to options with maturities of 31-180 days, and the right panel to options with maturities of 181-365 days. The value of the test and the test critical level are reported on a logarithmic scale.

this setup, we develop estimators for the spatial autocorrelation of the observation error by using a second-order difference of the option price as a function of its strike. We derive a feasible CLT for the spatial autocorrelations, with the limit distribution being mixed Gaussian. Exploiting the newly developed limit theory, we propose a Portmanteau test for the presence of cross-sectional correlation in the option observation errors. The proposed inference procedures are evaluated on simulated data and used to study the spatial dependence in the observation errors for S&P 500 index options. We find evidence of limited dependence, but also note that the spatial dependence at short tenors has declined sharply in recent years, as the liquidity of trading in short-dated options has improved.

9 Proofs

This section contains definitions of various quantities given in the main text as well as the proofs of the theoretical results. Before proceeding, however, let us introduce some general notation to be used throughout the proofs. First, denote by K a generic constant, which may take different values from line to line and from (in)equality to (in)equality. Similarly, we let K_t denote a generic finite-valued and $\mathcal{F}_t^{(0)}$-adapted random variable that can take different values in different places. Finally, we will be using shorthand notation $\epsilon_{t,\tau}^N(k_{t,\tau}(j)) \equiv \epsilon_{t,\tau}^N(j)$ and,

$$\bar{\epsilon}_{t,\tau}^N(j) \equiv \sqrt{\frac{2}{3}}\left(\epsilon_{t,\tau}^N(j) - \left(\frac{\epsilon_{t,\tau}^N(j-1) + \epsilon_{t,\tau}^N(j+1)}{2}\right)\right),$$

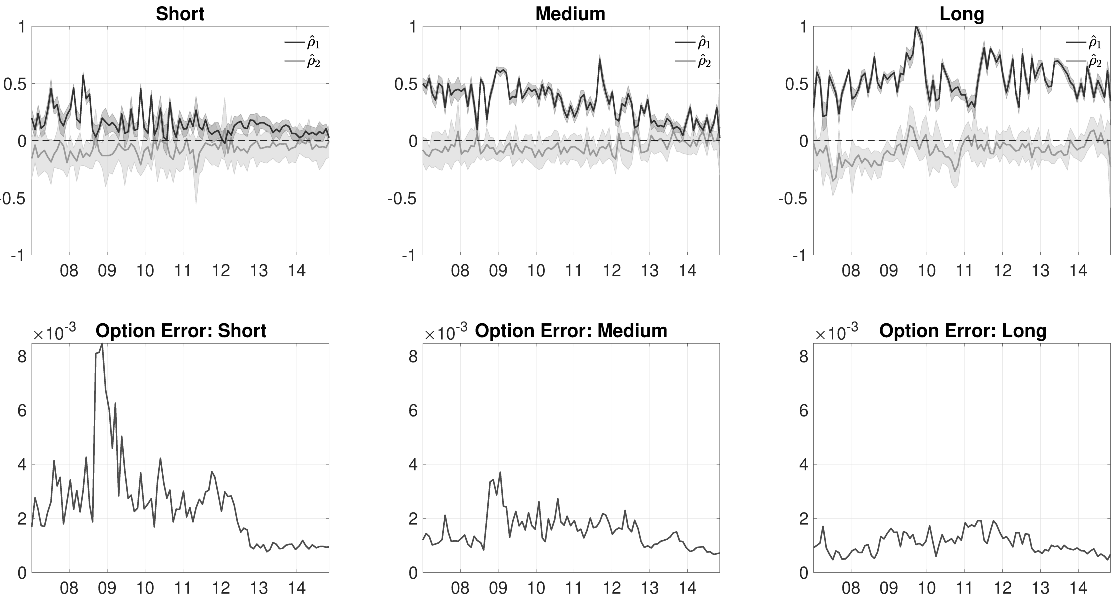

Figure 5: *Autocorrelation coefficients and size of observation errors for S&P 500 index options.* The top panels display the differences $\widehat{\rho}_{t,\tau}(1) - \rho^{\text{ind}}(1)$ and $\widehat{\rho}_{t,\tau}(2) - \rho^{\text{ind}}(2)$. The bottom panels display the average absolute value of the option observation errors across strikes. The left column covers options with maturity up to 31 calendar days, the middle one covers options with maturities of 31-180 days, and the right panel covers options with maturity of 181-365 days.

$$\bar{\chi}_{t,\tau}(h) \equiv \frac{1}{N_t^\tau - h - 2} \sum_{j=h+2}^{N_t^\tau - 1} \bar{\epsilon}_{t,\tau}^N(j)\bar{\epsilon}_{t,\tau}^N(j-h), \quad \widehat{\varpi}_{t,\tau}(h) \equiv \frac{1}{N_t^\tau - h - 2} \sum_{j=h+2}^{N_t^\tau - 1} \epsilon_{t,\tau}^N(j)\epsilon_{t,\tau}^N(j-h),$$

$$\bar{\gamma}_{t,\tau}(h) \equiv \frac{2}{3}\left(\frac{3}{2}\gamma_{t,\tau}(h) - \gamma_{t,\tau}(h+1) - \gamma_{t,\tau}(h-1) + \frac{1}{4}\gamma_{t,\tau}(h-2) + \frac{1}{4}\gamma_{t,\tau}(h+2)\right),$$

$$\tilde{\epsilon}_{t,\tau}^N(j) \equiv \sqrt{\frac{2}{3}}\bar{\zeta}_{t,\tau}^N(j)\sigma_{t,\tau}(j), \quad \bar{\zeta}_{t,\tau}^N(j) \equiv \zeta_{t,\tau}^N(j) - \left(\frac{\zeta_{t,\tau}^N(j-1) + \zeta_{t,\tau}^N(j+1)}{2}\right),$$

$$\tilde{\chi}_{t,\tau}(h) \equiv \frac{1}{N_t^\tau - h - 2} \sum_{j=h+2}^{N_t^\tau - 1} \tilde{\epsilon}_{t,\tau}^N(j)\tilde{\epsilon}_{t,\tau}^N(j-h), \qquad h \in \mathbb{N},$$

$$\mathcal{Q}_{t,\tau}^2 \equiv \vartheta_{t,\tau} \int_{\underline{k}(t,\tau)}^{\overline{k}(t,\tau)} \psi_{t,\tau}(k)\sigma_{t,\tau}^2(k)dk, \quad \mathcal{Q}_{t,\tau}^4 \equiv \vartheta_{t,\tau}^2 \int_{\underline{k}(t,\tau)}^{\overline{k}(t,\tau)} \psi_{t,\tau}(k)\sigma_{t,\tau}^4(k)dk.$$

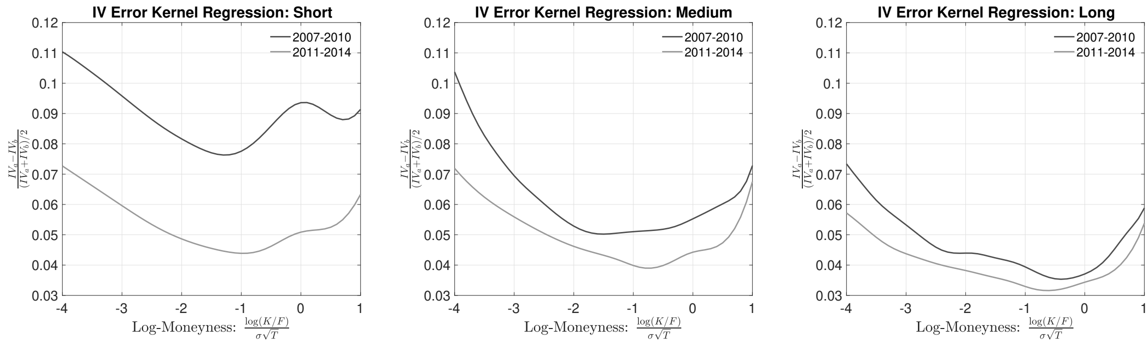

Figure 6: *Average bid-ask spreads for S&P 500 index options.* From left to right, the panels display the average size of the percentage bid-ask spread for short-, medium- and long-dated options, respectively, as a function of log-moneyness. The bid-ask spread is measured in units of implied volatility, with IV_a and IV_b being the Black-Scholes Implied Volatility computed at the ask and bid prices. Each panel displays the results for 2007-2010 and 2011-2014 separately.

9.1 Moment Definitions

For any $h, g \in \mathbb{N}$, define the $\mathcal{F}^{(0)}$-conditional asymptotic covariance between $\widehat{\varpi}_{t,\tau}(h)$ and $\widehat{\varpi}_{t,\tau}(g)$,

$$\mathcal{C}_{t,\tau}(h,g) \equiv \mathcal{V}_{t,\tau}(h,g) \times \mathcal{Q}^4_{t,\tau}, \quad \text{with} \tag{35}$$
$$\mathcal{V}_{t,\tau}(h,g) \equiv (\eta-3)\gamma_{t,\tau}(h)\gamma_{t,\tau}(g) + \sum_{s=-\infty}^{\infty}\big\{\gamma_{t,\tau}(s)\gamma_{t,\tau}(s+h-g) + \gamma_{t,\tau}(s+h)\gamma_{t,\tau}(s-g)\big\}.$$

Note that the structure of (35) is reminiscent of the corresponding asymptotic covariance for dependent, yet stationary processes, given in Proposition 7.3.4 of Brockwell and Davis (1991). The main differences between their expression and the asymptotic covariance in (35) is that we estimate spatial, not time series, autocovariances, the variances of the individual observations are $\mathcal{F}^{(0)}$-conditionally heteroskedastic, and the option "error" observations, whose dependence we are interested in estimating and testing for, are measured with "noise"; namely, the latent BSIVs.

For the asymptotic covariance between $\widehat{\chi}_{t,\tau}(h)$ and $\widehat{\chi}_{t,\tau}(g)$, define $\boldsymbol{a} \equiv (3/2, -1-1, 1/4, 1/4)'$,

$$\widehat{\boldsymbol{\varpi}}_{t,\tau}(h) = (\widehat{\varpi}_{t,\tau}(h), \widehat{\varpi}_{t,\tau}(h-1), \widehat{\varpi}_{t,\tau}(h+1), \widehat{\varpi}_{t,\tau}(h-2), \widehat{\varpi}_{t,\tau}(h+2))', \tag{36}$$

as well as the 5×5 matrix,

$$\bar{\boldsymbol{\mathcal{B}}}_{t,\tau}(h,g) \equiv \tag{37}$$

$$\begin{pmatrix} \mathcal{V}_{t,\tau}(h,g) & \mathcal{V}_{t,\tau}(h,g-1) & \mathcal{V}_{t,\tau}(h,g+1) & \mathcal{V}_{t,\tau}(h,g-2) & \mathcal{V}_{t,\tau}(h,g+2) \\ \mathcal{V}_{t,\tau}(h-1,g) & \mathcal{V}_{t,\tau}(h-1,g-1) & \mathcal{V}_{t,\tau}(h-1,g+1) & \mathcal{V}_{t,\tau}(h-1,g-2) & \mathcal{V}_{t,\tau}(h-1,g+2) \\ \mathcal{V}_{t,\tau}(h+1,g) & \mathcal{V}_{t,\tau}(h+1,g-1) & \mathcal{V}_{t,\tau}(h+1,g+1) & \mathcal{V}_{t,\tau}(h+1,g-2) & \mathcal{V}_{t,\tau}(h+1,g+2) \\ \mathcal{V}_{t,\tau}(h-2,g) & \mathcal{V}_{t,\tau}(h-2,g-1) & \mathcal{V}_{t,\tau}(h-2,g+1) & \mathcal{V}_{t,\tau}(h-2,g-2) & \mathcal{V}_{t,\tau}(h-2,g+2) \\ \mathcal{V}_{t,\tau}(h+2,g) & \mathcal{V}_{t,\tau}(h+2,g-1) & \mathcal{V}_{t,\tau}(h+2,g+1) & \mathcal{V}_{t,\tau}(h+2,g-2) & \mathcal{V}_{t,\tau}(h+2,g+2) \end{pmatrix}.$$

Moreover, let us define the estimated version of $\mathcal{V}_{t,\tau}(h,g)$, as

$$\bar{\mathcal{V}}_{t,\tau}(h,g) \equiv (\eta-3)\bar{\gamma}_{t,\tau}(h)\bar{\gamma}_{t,\tau}(g) + \sum_{s=-\infty}^{\infty} \big\{\bar{\gamma}_{t,\tau}(s)\bar{\gamma}_{t,\tau}(s+h-g) + \bar{\gamma}_{t,\tau}(s+h)\bar{\gamma}_{t,\tau}(s-g)\big\}. \tag{38}$$

By noting that $\sum_{s=-\infty}^{\infty} \gamma_{t,\tau}(s)\gamma_{t,\tau}(s+h-g) = \sum_{s=-\infty}^{\infty} \gamma_{t,\tau}(s+g)\gamma_{t,\tau}(s+h)$ as well as the composite variance $\boldsymbol{a}'\bar{\boldsymbol{\mathcal{B}}}_{t,\tau}(h,g)\boldsymbol{a}$ being linear and symmetric in its leads, lags and loadings, it follows that the expressions $\bar{\mathcal{V}}_{t,\tau}(h,g)$ and $\boldsymbol{a}'\bar{\boldsymbol{\mathcal{B}}}_{t,\tau}(h,g)\boldsymbol{a}$ are exactly equivalent, implying that the $\mathcal{F}^{(0)}$-conditional asymptotic covariance may be defined as

$$\bar{\mathcal{C}}_{t,\tau}(h,g) = \bar{\mathcal{V}}_{t,\tau}(h,g) \times \mathcal{Q}^4_{t,\tau}. \tag{39}$$

Next, to define and derive the $\mathcal{F}^{(0)}$-conditional asymptotic covariance matrix between the corresponding correlation estimates, let

$$\boldsymbol{D}^H_{t,\tau} \equiv \left(\bar{\gamma}_{t,\tau}(0) \times \mathcal{Q}^2_{t,\tau}\right)^{-1} \begin{pmatrix} -\bar{\rho}_{t,\tau}(1) & 1 & 0 & \dots & 0 \\ -\bar{\rho}_{t,\tau}(2) & 0 & 1 & & 0 \\ \vdots & & & \ddots & \\ -\bar{\rho}_{t,\tau}(H) & 0 & 1 & \dots & 1 \end{pmatrix} \tag{40}$$

be an $H \times (H+1)$ matrix of partial derivatives, with $\bar{\rho}_{t,\tau}(h)$, $h = 1, \dots, H$, defined in (12). Hence, we may, then, define the $H \times H$ covariance matrix $\boldsymbol{\mathcal{M}}^H_{t,\tau} \equiv \{\mathcal{M}_{t,\tau}(h,g)\}^H_{h,g=1}$, with elements given by,

$$\mathcal{M}_{t,\tau}(h,g) \equiv \mathcal{W}_{t,\tau}(h,g) \times \frac{\mathcal{Q}^4_{t,\tau}}{(\mathcal{Q}^2_{t,\tau})^2}, \quad \text{with} \tag{41}$$

$$\begin{aligned} \mathcal{W}_{t,\tau}(h,g) \equiv \sum_{k=-\infty}^{\infty} \Big[& \bar{\rho}_{t,\tau}(k)\bar{\rho}_{t,\tau}(k+h-g) + \bar{\rho}_{t,\tau}(k+h)\bar{\rho}_{t,\tau}(k-g) + 2\bar{\rho}_{t,\tau}(h)\bar{\rho}_{t,\tau}(g)\bar{\rho}^2_{t,\tau}(k) \\ & - 2\bar{\rho}_{t,\tau}(h)\bar{\rho}_{t,\tau}(k)\bar{\rho}_{t,\tau}(k-g) - 2\bar{\rho}_{t,\tau}(g)\bar{\rho}_{t,\tau}(k)\bar{\rho}_{t,\tau}(k+h) \Big]. \end{aligned} \tag{42}$$

When invoking the null hypothesis, $\mathcal{H}_0 : \phi(L) = 1$, the asymptotic variance of the correlation estimates may be written as $\boldsymbol{\mathcal{M}}^{H,\text{ind}}_{t,\tau} \equiv \{\mathcal{M}^{\text{ind}}_{t,\tau}(h,g)\}^{H}_{h,g=1}$, with elements defined as

$$\mathcal{M}^{\text{ind}}_{t,\tau}(h,g) \equiv \mathcal{W}^{\text{ind}}(h,g) \times \frac{\mathcal{Q}^4_{t,\tau}}{(\mathcal{Q}^2_{t,\tau})^2}, \quad \text{with} \tag{43}$$

$$\mathcal{W}^{\text{ind}}(h,g) \equiv \sum_{k=-\infty}^{\infty} \Bigg[\rho^{\text{ind}}(k)\rho^{\text{ind}}(k+h-g) + \rho^{\text{ind}}(k+h)\rho^{\text{ind}}(k-g) + 2\rho^{\text{ind}}(h)\rho^{\text{ind}}(g)(\rho^{\text{ind}}(k))^2 - 2\rho^{\text{ind}}(h)\rho^{\text{ind}}(k)\rho^{\text{ind}}(k-g) - 2\rho^{\text{ind}}(g)\rho^{\text{ind}}(k)\rho^{\text{ind}}(k+h)\Bigg]. \tag{44}$$

9.2 Technical Lemmas

Lemma 1. *Under Assumptions A1-A4. Then, for some $\mathcal{F}^{(0)}_t$-adapted random variable K_t,*

(a) $\big|\widehat{\epsilon}^N_{t,\tau}(j) - \bar{\epsilon}^N_{t,\tau}(j)\big| \le K_t\Delta$, *for* $j = 2, \ldots, N^\tau_t$.

(b) $|\widehat{\chi}_{t,\tau}(h) - \bar{\chi}_{t,\tau}(h)| \le O_p(\Delta)$.

(c) $|\bar{\epsilon}^N_{t,\tau}(j) - \tilde{\epsilon}^N_{t,\tau}(j)| \le K_t\Delta(|\zeta^N_{t,\tau}(j-1)| + |\zeta^N_{t,\tau}(j+1)|)$, *for* $j = 2, \ldots, N^\tau_t$.

Proof. **For (a)**. First, let us make the decomposition,

$$\begin{aligned}
|\widehat{\epsilon}^N_{t,\tau}(j) - \bar{\epsilon}^N_{t,\tau}(j)| &= \sqrt{\frac{2}{3}}\left|\kappa_{t,\tau}(k_{t,\tau}(j)) - \left(\frac{\kappa_{t,\tau}(k_{t,\tau}(j-1)) + \kappa_{t,\tau}(k_{t,\tau}(j+1))}{2}\right)\right| \\
&\le K_t\Bigg(|k_{t,\tau}(j) - k_{t,\tau}(j-1)| + |k_{t,\tau}(j) - k_{t,\tau}(j+1)|\Bigg) \\
&\le K_t\Bigg(\Delta_{t,\tau}(j) + \Delta_{t,\tau}(j+1)\Bigg), \qquad j = 2, \ldots, N^\tau_t - 1,
\end{aligned} \tag{45}$$

using Assumptions A1. The result, then, follows by Assumption A2, equation (6).

For (b). First, by addition and subtraction as well as the triangle inequality,

$$|\widehat{\chi}_{t,\tau}(h) - \bar{\chi}_{t,\tau}(h)| \le \frac{K}{N^\tau_t - h - 2}\sum_{j=h+2}^{N^\tau_t - 1}\Bigg(\big|\widehat{\epsilon}^N_{t,\tau}(j) - \bar{\epsilon}^N_{t,\tau}(j)\big|\big|\widehat{\epsilon}^N_{t,\tau}(j-h)\big| + \big|\bar{\epsilon}^N_{t,\tau}(j)\big|\big|\widehat{\epsilon}^N_{t,\tau}(j-h) - \bar{\epsilon}^N_{t,\tau}(j-h)\big|\Bigg). \tag{46}$$

The result, then, follows as by **(a)** since $\mathbb{E}\big|\bar{\epsilon}^N_{t,\tau}(j)\big| < K$, $j = 1, \ldots, N^\tau_t$, by Assumption A4.

For (c). First, let us decompose $\bar{\epsilon}^N_{t,\tau}(j) - \tilde{\epsilon}^N_{t,\tau}(j)$ for $j = 2, \ldots, N^\tau_t$,

$$\big|\bar{\epsilon}^N_{t,\tau}(j) - \tilde{\epsilon}^N_{t,\tau}(j)\big| \le \frac{1}{\sqrt{6}}\big|\big(\sigma_{t,\tau}(j) - \sigma_{t,\tau}(j-1)\big)\zeta_{t,\tau}(j-1)\big| + \frac{1}{\sqrt{6}}\big|\big(\sigma_{t,\tau}(j) - \sigma_{t,\tau}(j+1)\big)\zeta_{t,\tau}(j+1)\big|.$$

The result, then, follows by Assumption A2, equation (6) and Assumption A3 (specifically, the continuous differentiability of $\sigma_{t,\tau}(k)$ as a function of k). ☐

Lemma 2. *Under Assumptions A1-A4. Then, for $h \in \mathbb{N}$ and some $\mathcal{F}_t^{(0)}$-adapted variable K_t,*

(a) $\sup_{j=h,\ldots,N_t^\tau} \big|\sigma_{t,\tau}(j) - \sigma_{t,\tau}(j-h)\big| \leq K_t|h|\Delta$.

(b) $\sup_{j=h,\ldots,N_t^\tau} \big|\sigma_{t,\tau}^2(j) - \sigma_{t,\tau}^2(j-h)\big| \leq K_t|h|\Delta$.

(c) $\big|\frac{1}{N_t^\tau - h - 2}\sum_{j=h+2}^{N_t^\tau - 1}\sigma_{t,\tau}^2(j) - \mathcal{Q}_{t,\tau}^2\big| \leq o_p(\Delta^{1/2})$ *and* $\big|\frac{1}{N_t^\tau - h - 2}\sum_{j=h+2}^{N_t^\tau - 1}\sigma_{t,\tau}^4(j) - \vartheta_{t,\tau}^{-1}\mathcal{Q}_{t,\tau}^4\big| \leq o_p(\Delta^{1/2})$.

Proof. **For (a)**. First, by the triangle inequality,

$$\big|\sigma_{t,\tau}(j) - \sigma_{t,\tau}(j-h)\big| \leq \sum_{g=0}^{h-1}\big|\sigma_{t,\tau}(j-g) - \sigma_{t,\tau}(j-g-1)\big| \tag{47}$$

Next, since $\sigma_{t,\tau}(k)$ is continuously differentiable in k by Assumption A3, we may use the mean-value theorem in conjunction with Assumptions A1-A2, as for Lemma 1(a), to show

$$|\sigma_{t,\tau}(j) - \sigma_{t,\tau}(j-1)| \leq K_t\Delta_{t,\tau}(j), \qquad j = 2,\ldots,N_t^\tau. \tag{48}$$

Hence, since $\psi_{t,\tau}(k)$ is $\mathcal{F}_t^{(0)}$-adapted, finite-valued and continuously differentiable in $k \in [\underline{k}_{t,\tau}, \bar{k}_{t,\tau}]$ by Assumption A2, we may combine the latter with (48) and A3 to establish the stated bound.

For (b). First, by addition and subtraction and the triangle inequality,

$$\big|\sigma_{t,\tau}^2(j) - \sigma_{t,\tau}^2(j-h)\big| \leq \sigma_{t,\tau}(j)\big|\sigma_{t,\tau}(j) - \sigma_{t,\tau}(j-h)\big| + \sigma_{t,\tau}(j-h)\big|\sigma_{t,\tau}(j) - \sigma_{t,\tau}(j-h)\big|. \tag{49}$$

The results, then, follows by Assumptions A2-A3 in conjunction with the bound in **(a)**.

For (c). We only establish the first convergence below as the second follows by identical arguments. First, write

$$\frac{\Delta}{\Delta(N_t^\tau - h - 2)}\sum_{j=h+2}^{N_t^\tau - 1}\sigma_{t,\tau}^2(j) - \mathcal{Q}_{t,\tau}^2 = \mathcal{E}_{t,\tau,1} + \mathcal{E}_{t,\tau,2} + \mathcal{E}_{t,\tau,3}, \tag{50}$$

where the three error terms are defined as:

$$\begin{aligned}
\mathcal{E}_{t,\tau,1} &\equiv \Delta\left(\frac{1}{\Delta(N_t^\tau - h - 2)} - \vartheta_{t,\tau}\right)\sum_{j=h+2}^{N_t^\tau - 1}\sigma_{t,\tau}^2(j),\\
\mathcal{E}_{t,\tau,2} &\equiv \vartheta_{t,\tau}\sum_{j=h+2}^{N_t^\tau - 1}\sigma_{t,\tau}^2(j)\Delta_{t,\tau}(j)\left(\frac{\Delta}{\Delta_{t,\tau}(j)} - \psi_{t,\tau}(k_{t,\tau}(j))\right),\\
\mathcal{E}_{t,\tau,3} &\equiv \vartheta_{t,\tau}\sum_{j=h+2}^{N_t^\tau - 1}\psi_{t,\tau}(k_{t,\tau}(j))\sigma_{t,\tau}^2(j)\Delta_{t,\tau}(j) - \mathcal{Q}_{t,\tau}^2.
\end{aligned}$$

By Assumptions A2 and A3 and the continuous mapping theorem, we have $|\mathcal{E}_{t,\tau,1}| \leq o_p(\Delta^{1/2})$ and

$|\mathcal{E}_{t,\tau,2}| \leq o_p(\Delta^{1/2})$. Next, by continuous differentiability of $\sigma_{t,\tau}(k)$ and $\psi_{t,\tau}(k)$ in k, uniformly on the interval finite-valued interval $k_{t,\tau}(j) \in [\underline{k}_{t,\tau}, \bar{k}_{t,\tau}]$, it follows that

$$\left|\psi_{t,\tau}(k_{t,\tau}(j))\sigma^2_{t,\tau}(j)\Delta_{t,\tau}(j) - \int_{(j-1)\Delta_{t,\tau}(j)}^{j\Delta_{t,\tau}(j)} \psi_{t,\tau}(k)\sigma^2_{t,\tau}(k)dk\right| \leq O_p(\Delta^2_{t,\tau}(j)), \tag{51}$$

implying that, by invoking Assumption A2, $|\mathcal{E}_{t,\tau,3}| \leq O_p(\Delta)$. Hence, by combining the three bounds with the triangle inequality and (50), this provides the stated bound in the lemma. □

Lemma 3. *Under Assumptions A1-A4,*

(a) $\left|\mathbb{E}\big(\widehat{\varpi}_{t,\tau}(h)|\mathcal{F}^{(0)}\big) - \gamma_{t,\tau}(h)\mathcal{Q}^2_{t,\tau}\right| \leq o_p(\Delta^{1/2})$, *for* $h \in \mathbb{N}$.

(b) $\Delta^{-1}\mathbb{COV}\big(\widehat{\varpi}_{t,\tau}(h), \widehat{\varpi}_{t,\tau}(g)|\mathcal{F}^{(0)}\big) \xrightarrow{\mathbb{P}} \mathcal{C}_{t,\tau}(h,g)$, *for* $h, g \in \mathbb{N}$.

(c) $\left|\mathbb{E}\big(\bar{\chi}_{t,\tau}(h)|\mathcal{F}^{(0)}\big) - \bar{\gamma}_{t,\tau}(h)\mathcal{Q}^2_{t,\tau}\right| \leq o_p(\Delta^{1/2})$, *for* $h \in \mathbb{N}$.

(d) $\Delta^{-1}\mathbb{COV}\big(\bar{\chi}_{t,\tau}(h), \bar{\chi}_{t,\tau}(g)|\mathcal{F}^{(0)}\big) \xrightarrow{\mathbb{P}} \bar{\mathcal{C}}_{t,\tau}(h,g)$, *for* $h, g \in \mathbb{N}$.

Proof. **For (a).** By Assumption A3 and stationarity of $\zeta^N_{t,\tau}(j)$, we may write

$$\mathbb{E}\big(\widehat{\varpi}_{t,\tau}(h)|\mathcal{F}^{(0)}\big) = \frac{\gamma_{t,\tau}(h)}{N^\tau_t - h - 2}\sum_{j=h+2}^{N^\tau_t - 1}\sigma_{t,\tau}(j)\sigma_{t,\tau}(j-h).$$

The result, then, follows by applying Lemmas 2(a) and 2(c).

For (b). Before proceeding, define $z_{t,\tau}(j) \equiv \sigma_{t,\tau}(j)\nu_{t,\tau}(j)$ and observe that,

$$\mathbb{E}\big(z_{t,\tau}(j_1)z_{t,\tau}(j_2)z_{t,\tau}(j_3)z_{t,\tau}(j_4)|\mathcal{F}^{(0)}\big) = \begin{cases} \eta\sigma^4_{t,\tau}(j_1) & \text{if } j_1 = j_2 = j_3 = j_4 \\ \sigma^2_{t,\tau}(j_1)\sigma^2_{t,\tau}(j_3) & \text{if } j_1 = j_2 \neq j_3 = j_4 \\ 0 & \text{if } j_1 \neq j_2, j_1 \neq j_3, j_1 \neq j_4 \end{cases} \tag{52}$$

by independence of $\nu_{t,\tau}(j)$ from $\mathcal{F}^{(0)}$ as well as the spatial properties of $\nu_{t,\tau}(j)$ by Assumptions A3 and A4. Moreover, by standard quadrature distribution expectations,

$$\begin{aligned}
&\mathbb{E}\big(\epsilon^N_{t,\tau}(j)\epsilon^N_{t,\tau}(j-h)\epsilon^N_{t,\tau}(i)\epsilon^N_{t,\tau}(i-g)|\mathcal{F}^{(0)}\big) = \mathbb{E}\big(\epsilon^N_{t,\tau}(j)\epsilon^N_{t,\tau}(j-h)|\mathcal{F}^{(0)}\big) \times \mathbb{E}\big(\epsilon^N_{t,\tau}(i)\epsilon^N_{t,\tau}(i-g)|\mathcal{F}^{(0)}\big) \\
&\quad + \mathbb{E}\big(\epsilon^N_{t,\tau}(j)\epsilon^N_{t,\tau}(i)|\mathcal{F}^{(0)}\big) \times \mathbb{E}\big(\epsilon^N_{t,\tau}(j-h)\epsilon^N_{t,\tau}(i-g)|\mathcal{F}^{(0)}\big) \\
&\quad + \mathbb{E}\big(\epsilon^N_{t,\tau}(j)\epsilon^N_{t,\tau}(i-g)|\mathcal{F}^{(0)}\big) \times \mathbb{E}\big(\epsilon^N_{t,\tau}(j-h)\epsilon^N_{t,\tau}(i)|\mathcal{F}^{(0)}\big) + \mathcal{K}_4(i-j, h, g-h|\mathcal{F}^{(0)}),
\end{aligned}$$

with $\mathcal{K}_4(i-j, h, g-h|\mathcal{F}^{(0)})$ being the fourth conditional cumulant. Hence, by applying these decompositions and conditional expectation results, we may write

$$\mathbb{E}\big(\widehat{\varpi}_{t,\tau}(h)\widehat{\varpi}_{t,\tau}(g)|\mathcal{F}^{(0)}\big) = \mathcal{A}_{t,\tau}(h,g,1) + \mathcal{A}_{t,\tau}(h,g,2) + \mathcal{A}_{t,\tau}(h,g,3) + \mathcal{A}_{t,\tau}(h,g,4), \tag{53}$$

whose components, with $N^{\tau}_{t,h} \equiv N^{\tau}_t - h - 2$ and $N^{\tau}_{t,g} \equiv N^{\tau}_t - g - 2$, are defined as follows:

$$\begin{aligned}
\mathcal{A}_{t,\tau}(h,g,1) &\equiv \mathbb{E}\big(\widehat{\varpi}_{t,\tau}(h)|\mathcal{F}^{(0)}\big) \times \mathbb{E}\big(\widehat{\varpi}_{t,\tau}(g)|\mathcal{F}^{(0)}\big),\\
\mathcal{A}_{t,\tau}(h,g,2) &\equiv \frac{1}{N^{\tau}_{t,h}}\frac{1}{N^{\tau}_{t,g}}\sum_{j=h+2}^{N^{\tau}_t-1}\sum_{i=g+2}^{N^{\tau}_t-1}\mathbb{E}\big(\epsilon^{N}_{t,\tau}(j)\epsilon^{N}_{t,\tau}(i)|\mathcal{F}^{(0)}\big)\times\mathbb{E}\big(\epsilon^{N}_{t,\tau}(j-h)\epsilon^{N}_{t,\tau}(i-g)|\mathcal{F}^{(0)}\big),\\
\mathcal{A}_{t,\tau}(h,g,3) &\equiv \frac{1}{N^{\tau}_{t,h}}\frac{1}{N^{\tau}_{t,g}}\sum_{j=h+2}^{N^{\tau}_t-1}\sum_{i=g+2}^{N^{\tau}_t-1}\mathbb{E}\big(\epsilon^{N}_{t,\tau}(j)\epsilon^{N}_{t,\tau}(i-g)|\mathcal{F}^{(0)}\big)\times\mathbb{E}\big(\epsilon^{N}_{t,\tau}(j-h)\epsilon^{N}_{t,\tau}(i)|\mathcal{F}^{(0)}\big),\\
\mathcal{A}_{t,\tau}(h,g,4) &\equiv \frac{1}{N^{\tau}_{t,h}}\frac{1}{N^{\tau}_{t,g}}\sum_{j=h+2}^{N^{\tau}_t-1}\sum_{i=g+2}^{N^{\tau}_t-1}\mathcal{K}_4(i-j,h,g-h|\mathcal{F}^{(0)}).
\end{aligned}$$

Now, as $\mathbb{COV}\big(\widehat{\varpi}_{t,\tau}(h),\widehat{\varpi}_{t,\tau}(g)|\mathcal{F}^{(0)}\big) = \mathbb{E}\big(\widehat{\varpi}_{t,\tau}(h)\widehat{\varpi}_{t,\tau}(g)|\mathcal{F}^{(0)}\big) - \mathcal{A}_{t,\tau}(h,g,1)$, it suffices to analyze the three remaining terms. First, for $\mathcal{A}_{t,\tau}(h,g,2)$, using Assumption A3, it follows that

$$\mathcal{A}_{t,\tau}(h,g,2) = \frac{1}{N^{\tau}_{t,h}}\frac{1}{N^{\tau}_{t,g}}\sum_{j=h+2}^{N^{\tau}_t-1}\sum_{i=g+2}^{N^{\tau}_t-1}\gamma_{t,\tau}(i-j)\gamma_{t,\tau}(i-g-j+h)\sigma_{t,\tau}(j)\sigma_{t,\tau}(i)\sigma_{t,\tau}(j-h)\sigma_{t,\tau}(i-g). \tag{54}$$

Next, by a change of variable $i-j=s$, Assumption A2 and Lemmas 2(a)-2(b),

$$\begin{aligned}
&\left|\mathcal{A}_{t,\tau}(h,g,2) - \frac{1}{N^{\tau}_{t,h}N^{\tau}_{t,g}}\sum_{j=h+2}^{N^{\tau}_t-1}\sigma^4_{t,\tau}(j)\sum_{|s|\le N^{\tau}_t-1}\left(1-\frac{|s|}{N^{\tau}_t-1}\right)\gamma_{t,\tau}(s)\gamma_{t,\tau}(s-g+h)\right| \qquad (55)\\
&\le \frac{1}{N^{\tau}_{t,h}N^{\tau}_{t,g}}\sum_{j=h+2}^{N^{\tau}_t-1}\sigma^4_{t,\tau}(j)\sum_{|s|\le N^{\tau}_t-1}\left(1-\frac{|s|}{N^{\tau}_t-1}\right)|\gamma_{t,\tau}(s)\gamma_{t,\tau}(s-g+h)|\Big(|s|+|h|+|s-g|\Big)K_t\Delta\\
&\le \frac{K_t}{N^{\tau}_{t,h}N^{\tau}_{t,g}}\sum_{j=h+2}^{N^{\tau}_t-1}\sigma^4_{t,\tau}(j)\sum_{|s|\le N^{\tau}_t-1}|s|\Delta\left(1-\frac{|s|}{N^{\tau}_t-1}\right)|\gamma_{t,\tau}(s)\gamma_{t,\tau}(s-g+h)|\,.
\end{aligned}$$

Hence, when taking limits over j and using Lemma 2(c) in conjunction with absolutely summability of the dependence parameters in Assumption A4, the dominated convergence theorem, the uniform convergence conditions for the sampling scheme in A2 and the continuous mapping theorem,

$$\frac{\Delta^{-1}}{N^{\tau}_{t,h}N^{\tau}_{t,g}}\sum_{j=h+2}^{N^{\tau}_t-1}\sigma^4_{t,\tau}(j)\sum_{|s|\le N^{\tau}_t-1}\left(1-\frac{|s|}{N^{\tau}_t-1}\right)\gamma_{t,\tau}(s)\gamma_{t,\tau}(s-g+h)$$

$$\xrightarrow{\mathbb{P}} \mathcal{Q}_{t,\tau}^4 \sum_{s=-\infty}^{\infty} \gamma_{t,\tau}(s)\gamma_{t,\tau}(s-g+h), \tag{56}$$

and, by another application of the dominated convergence theorem and Assumptions A2 and A4,

$$\frac{\Delta^{-1}K_t}{N_{t,h}^\tau N_{t,g}^\tau} \sum_{j=h+2}^{N_t^\tau-1} \sigma_{t,\tau}^4(j) \sum_{|s|\leq N_t^\tau-1} |s|\Delta\left(1-\frac{|s|}{N_t^\tau-1}\right)\gamma_{t,\tau}(s)\gamma_{t,\tau}(s-g+h) \xrightarrow{\mathbb{P}} 0. \tag{57}$$

As a result, we have

$$\Delta^{-1}\mathcal{A}_{t,\tau}(h,g,2) \xrightarrow{\mathbb{P}} \mathcal{Q}_{t,\tau}^4 \sum_{s=-\infty}^{\infty} \gamma_{t,\tau}(s)\gamma_{t,\tau}(s-g+h), \tag{58}$$

and, by the same arguments, $\Delta^{-1}\mathcal{A}_{t,\tau}(h,g,3) \xrightarrow{\mathbb{P}} \mathcal{Q}_{t,\tau}^4 \sum_{s=-\infty}^{\infty}\gamma_{t,\tau}(s-g)\gamma_{t,\tau}(s+h)$. Next, for the last term in the covariance decomposition, $\mathcal{A}_{t,\tau}(h,g,4)$, we may similarly write

$$\mathcal{A}_{t,\tau}(h,g,4) = \frac{\eta-3}{N_{t,h}^\tau N_{t,g}^\tau} \sum_{j=h+2}^{N_t^\tau-1}\sum_{i=g+2}^{N_t^\tau-1} \sigma_{t,\tau}^4(j) \sum_{z=-\infty}^{\infty} \phi_z\phi_{z-h}\phi_{z+i-j}\phi_{z+i-j-g}. \tag{59}$$

Hence, by the same arguments used to establish (56) and (57),

$$\Delta^{-1}\mathcal{A}_{t,\tau}(h,g,4) \xrightarrow{\mathbb{P}} (\eta-3)\mathcal{Q}_{t,\tau}^4 \sum_{s=-\infty}^{\infty}\sum_{z=-\infty}^{\infty} \phi_z\phi_{z-h}\phi_{z+s}\phi_{z+s-g} = (\eta-3)\mathcal{Q}_{t,\tau}^4\gamma_{t,\tau}(h)\gamma_{t,\tau}(g). \tag{60}$$

The covariance result follows by combining limits for $\mathcal{A}_{t,\tau}(h,g,2)$, $\mathcal{A}_{t,\tau}(h,g,3)$ and $\mathcal{A}_{t,\tau}(h,g,4)$.

For (c). First, note that by collection terms and Assumptions A2-A4, we have

$$\begin{aligned}\bar{\chi}_{t,\tau}(h) &\equiv \frac{1}{N_t^\tau-h-2}\sum_{j=h+2}^{N_t^\tau-1} \bar{\epsilon}_{t,\tau}^N(j)\bar{\epsilon}_{t,\tau}^N(j-h) \\ &= \frac{2}{3}\left(\frac{3}{2}\widehat{\varpi}_{t,\tau}(h) - \widehat{\varpi}_{t,\tau}(h-1) - \widehat{\varpi}_{t,\tau}(h+1) + \frac{1}{4}\widehat{\varpi}_{t,\tau}(h+2) + \frac{1}{4}\widehat{\varpi}_{t,\tau}(h-2)\right) + O_p(\Delta),\end{aligned} \tag{61}$$

uniformly. The result, thus, follows by repeated application of **(a)**.

For (d). First, write (61) in vector form, $\bar{\chi}_{t,\tau}(h) = \frac{2}{3}\boldsymbol{a}'\widehat{\boldsymbol{\varpi}}_{t,\tau}(h) + O_p(\Delta)$, with the 5×1 vectors $\boldsymbol{a}$ and $\widehat{\boldsymbol{\varpi}}_{t,\tau}(h)$ defined in Section 9.1. Hence, we can rewrite the conditional covariance as,

$$\Delta^{-1}\mathbb{COV}\big(\bar{\chi}_{t,\tau}(h), \bar{\chi}_{t,\tau}(g)|\mathcal{F}^{(0)}\big) = \Delta^{-1}(2/3)^2\boldsymbol{a}'\mathbb{COV}\big(\widehat{\boldsymbol{\varpi}}_{t,\tau}(h), \widehat{\boldsymbol{\varpi}}_{t,\tau}(g)'|\mathcal{F}^{(0)}\big)\boldsymbol{a} + O_p(\Delta^{1/2}). \tag{62}$$

The result, thus, follows by applying **(b)** to each element of $\mathbb{COV}\big(\widehat{\boldsymbol{\varpi}}_{t,\tau}(h), \widehat{\boldsymbol{\varpi}}_{t,\tau}(g)'|\mathcal{F}^{(0)}\big)$ as well as the variance definitions in (37)-(39), concluding the proof. □

Lemma 4. *Suppose Assumptions A1-A4 hold. Moreover, let $\boldsymbol{f}(\cdot)$ be a function mapping from $\mathbb{R}^{H+1}$*

to $\mathbb{R}^H$, defined by

$$\boldsymbol{f}([x_0, x_1, \ldots, x_H]') = [x_1/x_0, \ldots, x_H/x_0]', \quad x_0 \neq 0. \tag{63}$$

Then, for $\bar{\boldsymbol{\xi}}_{t,\tau}(h,H) \equiv (\bar{\xi}_{t,\tau}(h), \bar{\xi}_{t,\tau}(h+1), \ldots, \bar{\xi}_{t,\tau}(H))'$, $\xi = \{\chi, \gamma\}$, write $\boldsymbol{f}(\bar{\boldsymbol{\gamma}}_{t,\tau}(0,H))$ and let $\boldsymbol{D}^H_{t,\tau}$ be its associated $H \times (H+1)$ matrix of partial derivatives defined in (40)*, it follows:*

$$\Delta^{-1}\boldsymbol{D}^H_{t,\tau}\mathbb{COV}\big(\bar{\boldsymbol{\chi}}_{t,\tau}(0,H), \bar{\boldsymbol{\chi}}_{t,\tau}(0,H)'|\mathcal{F}^{(0)}\big)(\boldsymbol{D}^H_{t,\tau})' \xrightarrow{\mathbb{P}} \boldsymbol{\mathcal{M}}^H_{t,\tau}.$$

Proof. The result follows by Lemma 3(d), by applying the definitions (37)-(39), observing that

$$\begin{aligned}\mathcal{M}_{t,\tau}(h,g) = &\frac{\mathcal{Q}^4_{t,\tau}}{(\bar{\gamma}(0)\mathcal{Q}^2_{t,\tau})^2} \\ &\times \big(\bar{\mathcal{C}}_{t,\tau}(h,g) - \bar{\rho}_{t,\tau}(h)\bar{\mathcal{C}}_{t,\tau}(0,g) - \bar{\rho}_{t,\tau}(g)\bar{\mathcal{C}}_{t,\tau}(h,0) + \bar{\rho}_{t,\tau}(h)\bar{\rho}_{t,\tau}(g)\bar{\mathcal{C}}_{t,\tau}(0,0)\big)\end{aligned} \tag{64}$$

and by rewriting the expression on correlation form, noticing that the cumulants cancel. □

Lemma 5. *Suppose Assumptions A1-A4 hold with $\phi(L) = 1$. Then, for some $H \in \mathbb{N}$,*

(a) $\{\Delta^{-1/2}\left(\widehat{\varpi}_{t,\tau}(h) - \gamma_{t,\tau}(h)\mathcal{Q}^2_{t,\tau}\right)\}_{h=1,\ldots,H} \xrightarrow{\mathcal{L}-s} \{Y_{t,\tau}(h)\}_{h=1,\ldots,H}$*,*

(b) $\{\Delta^{-1/2}\left(\bar{\chi}_{t,\tau}(h) - \bar{\gamma}_{t,\tau}(h)\mathcal{Q}^2_{t,\tau}\right)\}_{h=1,\ldots,H} \xrightarrow{\mathcal{L}-s} \{\bar{Y}_{t,\tau}(h)\}_{h=1,\ldots,H}$*,*

where $\{Y_{t,\tau}(h)\}_{h=1,\ldots,H}$ and $\{\bar{Y}_{t,\tau}(h)\}_{h=1,\ldots,H}$ are defined on an extension of the original probability space, and are $\mathcal{F}$-conditionally zero-mean $H \times 1$ Gaussian vectors with $\mathbb{E}(Y_{t,\tau}(h)Y_{t,\tau}(g)|\mathcal{F}) = \mathcal{C}_{t,\tau}(h,g)$ and $\mathbb{E}(\bar{Y}_{t,\tau}(h)\bar{Y}_{t,\tau}(g)|\mathcal{F}) = \bar{\mathcal{C}}_{t,\tau}(h,g)$, for $h, g = 1, ..., H$.

Proof. **For (a).** First, let us denote

$$\mathcal{Z}^N_{t,\tau}(h) \equiv \Delta^{-1/2}(\widehat{\varpi}_{t,\tau}(h) - \gamma_{t,\tau}(h)\mathcal{Q}^2_{t,\tau}), \quad h = 1, ..., H.$$

Then, given the product structure of $\mathcal{F}$, we need to prove

$$\mathbb{E}\left(Zg(\{\mathcal{Z}^N_{t,\tau}(h)\}_{h=1,\ldots,H})\right) \longrightarrow \mathbb{E}\left(Zg(\{Y_{t,\tau}(h)\}_{h=1,\ldots,H})\right), \tag{65}$$

for a continuous and bounded function $g : \mathbb{R}^H \to \mathbb{R}$ and Y being a bounded random variable, which is either $\mathcal{F}^{(0)}$-adapted or is a function of a fixed number of the error terms $\{\epsilon_{t,\tau}(j)\}_{j=1,\ldots,N^\tau_t}$. Given the $\mathcal{F}^{(0)}$-conditional H-dependence of the summands in $\widehat{\varpi}_{t,\tau}(h)$ (recall, we are in the case $\phi(L) = 1$), this convergence will follow if we can establish the following, stronger, result

$$\{\Delta^{-1/2}\left(\widehat{\varpi}_{t,\tau}(h) - \gamma_{t,\tau}(h)\mathcal{Q}^2_{t,\tau}\right)\}_{h=1,\ldots,H} \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} \{Y_{t,\tau}(h)\}_{h=1,\ldots,H}, \tag{66}$$

where $\mathcal{L}|\mathcal{F}^{(0)}$ means convergence in probability of the conditional probability laws when the latter are considered as random variables taking values in the space of probability measures equipped with the weak topology. For this, it suffices to show that every subsequence of $\{\mathcal{Z}_{t,\tau}^N(h)\}_{h=1,\ldots,H}$ has a further subsequence, which converges to $\{Y_{t,\tau}(h)\}_{h=1,\ldots,H}$ for every $\omega^{(0)} \in \mathcal{F}^{(0)}$. Since $\mathcal{F}^{(0)}$-conditionally, the summands in $\widehat{\varpi}_{t,\tau}(h)$ are H-dependent, we may apply a CLT for H-dependent sequences, see e.g., Berk (1973). Hence, taking into account that $\mathcal{Z}_{t,\tau}^N(h) - \mathbb{E}(\mathcal{Z}_{t,\tau}^N(h)|\mathcal{F}^{(0)}) = o_p(\sqrt{\Delta})$ by Lemma 5(a), the proof will be completed by showing:

$$\begin{cases} \mathbb{E}(\mathcal{Z}_{t,\tau}^N(h)\mathcal{Z}_{t,\tau}^N(g)|\mathcal{F}^{(0)}) - \mathbb{E}(\mathcal{Z}_{t,\tau}^N(h)|\mathcal{F}^{(0)})\mathbb{E}(\mathcal{Z}_{t,\tau}^N(g)|\mathcal{F}^{(0)}) \xrightarrow{\mathbb{P}} \mathcal{C}_{t,\tau}(h,g), \;\; h,g = 1,\ldots,H, \\ \sum_{h=1}^H \mathbb{E}(|\mathcal{Z}_{t,\tau}^N(h)|^{2+\iota}|\mathcal{F}^{(0)}) \xrightarrow{\mathbb{P}} 0, \quad \text{for some } \iota > 0. \end{cases} \tag{67}$$

First, since the spatially (and time-) varying volatility $\sigma_{t,\tau}(j)$, $j = 1,\ldots,N_t^\tau$, is $\mathcal{F}_t^{(0)}$-adapted and the respective sequences of observation errors $\epsilon_{t,\tau}^N(j)$, $j = 1,\ldots,N_t^\tau$, are independent for $(t,\tau) \neq (t',\tau')$, the first result in (67) follows by successive conditioning in conjunction with the conditional moment results in Lemmas 3(a) and 3(b). Next, for some $\mathcal{F}_t^{(0)}$-adapted bounded constant $K_t > 0$,

$$\begin{aligned} \mathbb{E}(|\mathcal{Z}_{t,\tau}^N(h)|^{2+\iota}|\mathcal{F}^{(0)}) &\leq \frac{K_t\Delta^{-1-\iota/2}}{(N_t^\tau)^{2+\iota}} \sum_{j=h+2}^{N_t^\tau-1} \mathbb{E}\left(|\zeta_{t,\tau}^N(j)\zeta_{t,\tau}^N(j-h)|^{2+\iota}\right) |\sigma_{t,\tau}(j)\sigma_{t,\tau}(j-h)|^{2+\iota} \\ &\leq \frac{K_t\Delta^{-1-\iota/2}}{(N_t^\tau)^{1+\iota}} \frac{1}{N_t^\tau} \sum_{j=h+2}^{N_t^\tau-1} |\sigma_{t,\tau}(j)\sigma_{t,\tau}(j-h)|^{2+\iota}, \end{aligned}$$

using the triangle inequality for the first inequality, and the fact that $\mathbb{E}\left(|\zeta_{t,\tau}^N(j)|^{4+\varsigma}|\mathcal{F}^{(0)}\right)$ is bounded by a finite positive constant, for some $\varsigma > 0$, by Assumption A3 for the second inequality. Next, by applying Lemma 2(a) in conjunction with the triangle inequality,

$$|\sigma_{t,\tau}(j)\sigma_{t,\tau}(j-h)|^{2+\iota} \leq K_t|\sigma_{t,\tau}(j)|^{2+\iota}(|\sigma_{t,\tau}(j)|^{2+\iota} + (|h|\Delta)^{2+\iota}).$$

Moreover, since we may write $|h| \leq N_t^\tau$ for all finite samples, without loss of generality, it follows that $(|h|\Delta)^{2+\iota} \leq K_t + o_p(\Delta^{1/2})$. Hence, it suffices to establish that the sum $(N_t^\tau)^{-1}\sum_{j=h+2}^{N_t^\tau-1}|\sigma_{t,\tau}(j)|^{4+2\iota} = O_p(1)$ since the corresponding stochastic bound for the second term in the decomposition will, then, follow by the dominated convergence theorem. Now, since $\sigma_{t,\tau}(j)$ is continuously differentiable, we may use the same arguments as for Lemma 2(c) to show

$$(N_t^\tau)^{-1}\sum_{j=h+2}^{N_t^\tau-1}|\sigma_{t,\tau}(j)|^{4+2\iota} \xrightarrow{\mathbb{P}} \vartheta_{t,\tau}\int_{\underline{k}(t,\tau)}^{\overline{k}(t,\tau)} \psi_{t,\tau}(k)\sigma_{t,\tau}^{4+2\iota}(k)dk.$$

Hence, by combining results, this shows $\mathbb{E}(|\mathcal{Z}_{t,\tau}^N(h)|^{2+\iota}|\mathcal{F}^{(0)}) \leq O_p((\Delta N_t^\tau)^{-1-\iota/2}/(N_t^\tau)^{\iota/2})$, which, by invoking Assumption A2 and the continuous mapping theorem, provides the final result in (67) and thereby **(a)**.

(b) follows by applying the decomposition in (61), the moment results of Lemma 3(c) and (d) in conjunction with successive conditioning, as for **(a)**, the stable central limit theory in **(a)** for each $h \in \mathbb{N}$ and a stable Cramér-Wold theorem, see, e.g., (Varneskov, 2017, Lemmas C.1(d)-(e)). $\square$

Lemma 6. *Under the conditions of Theorem 2, then,*

(a) $\widehat{\mathcal{L}}_{t,\tau}(h) \xrightarrow{\mathbb{P}} \mathcal{Q}_{t,\tau}^4$, *for some finite integer* $h \geq 2$.

(b) $\widehat{\mathcal{L}}_{t,\tau}(0) \xrightarrow{\mathbb{P}} ((5/9)\eta + 14/9)\,\mathcal{Q}_{t,\tau}^4$.

(c) $\widehat{\eta}_{t,\tau} \xrightarrow{\mathbb{P}} \eta\mathcal{Q}_{t,\tau}^4$.

Proof. **For (a)**. First, by addition and subtraction of $\tilde{\epsilon}_{t,\tau}^N(j)$ in (18) together with the triangle inequality,

$$\left|\widehat{\mathcal{L}}_{t,\tau}(h) - \widetilde{\mathcal{L}}_{t,\tau}(h)\right| \leq \frac{1}{\Delta(N_t^\tau - h - 2)^2}\sum_{j=h+2}^{N_t^\tau - 1}\left|(\widehat{\epsilon}_{t,\tau}^N(j))^2 - (\widetilde{\epsilon}_{t,\tau}^N(j))^2\right|(\widehat{\epsilon}_{t,\tau}^N(j-h))^2$$
$$+\frac{1}{\Delta(N_t^\tau - h - 2)^2}\sum_{j=h+2}^{N_t^\tau - 1}(\widetilde{\epsilon}_{t,\tau}^N(j))^2\left|(\widehat{\epsilon}_{t,\tau}^N(j-h))^2 - (\widetilde{\epsilon}_{t,\tau}^N(j-h))^2\right|, \qquad (68)$$

where $\widetilde{\mathcal{L}}_{t,\tau}(h) \equiv \frac{1}{\Delta(N_t^\tau - h-2)}\frac{1}{N_t^\tau - h - 2}\sum_{j=h+2}^{N_t^\tau - 1}(\widetilde{\epsilon}_{t,\tau}^N(j))^2(\widetilde{\epsilon}_{t,\tau}^N(j-h))^2$. Next, using addition and subtraction, once again, we have

$$\left|(\widehat{\epsilon}_{t,\tau}^N(j))^2 - (\widetilde{\epsilon}_{t,\tau}^N(j))^2\right| \leq 2|\widetilde{\epsilon}_{t,\tau}^N(j)|\left|\widehat{\epsilon}_{t,\tau}^N(j) - \widetilde{\epsilon}_{t,\tau}^N(j)\right| + 2\left|\widehat{\epsilon}_{t,\tau}^N(j) - \widetilde{\epsilon}_{t,\tau}^N(j)\right|^2, \qquad (69)$$

as in (49). From here, we can make use of Lemma 1(a) and Lemma 1(c), the $\mathcal{F}^{(0)}$-conditional independence of $\zeta_{t,\tau}^N(j)$ and $\zeta_{t,\tau}^N(j')$ for $j \neq j'$, to conclude that $|\widehat{\mathcal{L}}_{t,\tau}(h) - \widetilde{\mathcal{L}}_{t,\tau}(h)| \leq O_p(\Delta)$. Hence, we may analyze $\widetilde{\mathcal{L}}_{t,\tau}(h)$ henceforth.

Next, to establish the stated convergence result, we have, for $h \geq 2$ and under $\mathcal{H}_0$,

$$\mathbb{E}\left(\widetilde{\mathcal{L}}_{t,\tau}(h)|\mathcal{F}^{(0)}\right) = \frac{(2/3)^2}{\Delta(N_t^\tau - h - 2)^2}\sum_{j=h+2}^{N_t^\tau - 1}\mathbb{E}\big(\bar{\zeta}_{t,\tau}^N(j)^2\big)\mathbb{E}\big(\bar{\zeta}_{t,\tau}^N(j-h)^2\big)\sigma_{t,\tau}^2(j)\sigma_{t,\tau}^2(j-h)$$
$$= \frac{(\bar{\gamma}_{t,\tau}(0))^2}{\Delta(N_t^\tau - h - 2)}\frac{1}{N_t^\tau - h - 2}\sum_{j=h+2}^{N_t^\tau - 1}\sigma_{t,\tau}^2(j)\sigma_{t,\tau}^2(j-h) \xrightarrow{\mathbb{P}} \mathcal{Q}_{t,\tau}^4,$$

using $\bar{\gamma}_{t,\tau}(0)=1$ under $\mathcal{H}_0$, independence of $\bar{\zeta}^N_{t,\tau}(j)$ and $\bar{\zeta}^N_{t,\tau}(j-h)$ for $h\geq 2$, and Assumption A2 for $\Delta(N^\tau_t-h-2)$, Lemma 3(c) and the continuous mapping theorem. Moreover, by the same arguments as well as $N^\tau_t/(N^\tau_t-h-2)\leq K_t$, we have

$$\begin{aligned}\mathbb{V}\left(\widetilde{\mathcal{L}}_{t,\tau}(h)|\mathcal{F}^{(0)}\right) &\leq \left(\frac{K_t^2}{(\Delta N_t^\tau)N_t^\tau}\right)^2 \sum_{j=h+2}^{N_t^\tau-1}\sum_{i=h+2}^{N_t^\tau-1} \mathbb{E}\big(\widetilde{\epsilon}^N_{t,\tau}(j)^2\widetilde{\epsilon}^N_{t,\tau}(j-h)^2\widetilde{\epsilon}^N_{t,\tau}(i)^2\widetilde{\epsilon}^N_{t,\tau}(i-h)^2|\mathcal{F}^{(0)}\big)\\ &\leq 3\left(\frac{K_t^2}{(\Delta N_t^\tau)N_t^\tau}\right)^2 \sum_{j=h+2}^{N_t^\tau-1} \mathbb{E}\big(\widetilde{\epsilon}^N_{t,\tau}(j)^4|\mathcal{F}^{(0)}\big)\mathbb{E}\big(\widetilde{\epsilon}^N_{t,\tau}(j-h)^4|\mathcal{F}^{(0)}\big)\end{aligned}$$

using, again, independence between $\bar{\zeta}^N_{t,\tau}(j)$ and $\bar{\zeta}^N_{t,\tau}(j-h)$ for $h\geq 2$, in conjunction with the Cauchy-Schwarz inequality. Hence, since $\sigma_{t,\tau}(j)$ is continuously differentiable by Assumption 3, the integration bounds, $k\in[\underline{k}_{t,\tau},\bar{k}_{t,\tau}]$ are $\mathcal{F}^{(0)}_t$-adapted and finite valued by Assumption A2, $\bar{\zeta}^N_{t,\tau}$ have a finite fourth moment by Assumption A3-A4, and since the same arguments for $\mathbb{E}(\widetilde{\mathcal{L}}_{t,\tau}(h)|\mathcal{F}^{(0)})$ apply, it follows that $\mathbb{V}(\widetilde{\mathcal{L}}_{t,\tau}(h)|\mathcal{F}^{(0)})=O_p(\Delta)$, thereby showing convergence of $\widetilde{\mathcal{L}}_{t,\tau}(h)$ in $\mathbb{L}^2$.

For (b). By the same arguments provided for **(a)**, it follows that $|\widehat{\mathcal{L}}_{t,\tau}(0)-\widetilde{\mathcal{L}}_{t,\tau}(0)|\leq O_p(\Delta)$ where, as above, $\widetilde{\mathcal{L}}_{t,\tau}(0)\equiv\frac{1}{\Delta(N^\tau_t-2)}\frac{1}{N^\tau_t-2}\sum_{j=2}^{N^\tau_t-1}\widetilde{\epsilon}^N_{t,\tau}(j)^4$, and we may, thus, analyze $\widetilde{\mathcal{L}}_{t,\tau}(0)$ henceforth.

Next, to establish the stated convergence result under $\mathcal{H}_0$, note first that

$$\begin{aligned}\widetilde{\epsilon}^N_{t,\tau}(j)^2=\frac{2}{3}\Big(&\zeta^N_{t,\tau}(j)^2-\zeta^N_{t,\tau}(j)\zeta^N_{t,\tau}(j-1)-\zeta^N_{t,\tau}(j)\zeta^N_{t,\tau}(j+1)\\ &+\zeta^N_{t,\tau}(j-1)^2/4+\zeta^N_{t,\tau}(j+1)^2/4+\zeta^N_{t,\tau}(j-1)\zeta^N_{t,\tau}(j+1)/2\Big)\sigma^2_{t,\tau}(j)\end{aligned}\tag{70}$$

such that, by utilizing (52) and independence of $\zeta^N_{t,\tau}(j)$ across j, we may write

$$\begin{aligned}\mathbb{E}\left(\widetilde{\epsilon}^N_{t,\tau}(j)^4|\mathcal{F}^{(0)}\right)&=(2/3)^2\Big(\mathbb{E}\left(\zeta^N_{t,\tau}(j)^4\right)+\mathbb{E}\left(\zeta^N_{t,\tau}(j-1)^4\right)/4^2+\mathbb{E}\left(\zeta^N_{t,\tau}(j+1)^4\right)/4^2+\\ &\qquad+(3/2)\mathbb{E}\left(\zeta^N_{t,\tau}(j)^2\right)\mathbb{E}\left(\zeta^N_{t,\tau}(j-1)^2\right)+(3/2)\mathbb{E}\left(\zeta^N_{t,\tau}(j)^2\right)\mathbb{E}\left(\zeta^N_{t,\tau}(j+1)^2\right)\\ &\qquad+\mathbb{E}\left(\zeta^N_{t,\tau}(j)^2\right)\mathbb{E}\left(\zeta^N_{t,\tau}(j+1)^2\right)/2\Big)\sigma^4_{t,\tau}(j)\\ &=(2/3)^2\left((5/4)\eta+7/2\right)\sigma^4_{t,\tau}(j)=\left((5/9)\eta+14/9\right)\sigma^4_{t,\tau}(j).\end{aligned}\tag{71}$$

Using this decomposition in conjunction with Assumption A2 for $\Delta(N^\tau_t-h-2)$, Lemma 3(c) and the continuous mapping theorem, this provides

$$\mathbb{E}\left(\widetilde{\mathcal{L}}_{t,\tau}(0)\big|\mathcal{F}^{(0)}\right)\xrightarrow{\mathbb{P}}\left((5/9)\eta+14/9\right)\mathcal{Q}^4_{t,\tau}.\tag{72}$$

Now, by noticing that $\widetilde{\epsilon}^N_{t,\tau}(j)$ and $\widetilde{\epsilon}^N_{t,\tau}(j')$ are $\mathcal{F}^{(0)}$-conditionally independent whenever $|j-j'|>1$, making use of Burkholder-Davis-Gundy inequality, for some $\iota\in(0,\varsigma/4)$, with ς being the constant in

Assumption A3, we have,

$$
\begin{aligned}
&\mathbb{E}\left(\left|\widetilde{\mathcal{L}}_{t,\tau}(0)-\mathbb{E}\left(\widetilde{\mathcal{L}}_{t,\tau}(0)\big|\mathcal{F}^{(0)}\right)\right|^{1+\iota}\bigg|\mathcal{F}^{(0)}\right)\\
&\leq \frac{K_t}{(\Delta N_t^\tau)^{1+\iota}}\frac{1}{(N_t^\tau)^{1+\iota}}\mathbb{E}\left(\sum_{j=2}^{N_t^\tau-1}\left|(\widetilde{\epsilon}_{t,\tau}^N(j))^4-\mathbb{E}\left((\widetilde{\epsilon}_{t,\tau}^N(j))^4\big|\mathcal{F}^{(0)}\right)\right|^{1+\iota}\bigg|\mathcal{F}^{(0)}\right) \qquad (73)\\
&\leq \frac{K_t}{(\Delta N_t^\tau)^{1+\iota}}\frac{1}{(N_t^\tau)^{\iota}}\sup_{k\in[\underline{k}_{t,\tau},\overline{k}_{t,\tau}]}(|\sigma_{t,\tau}(k)|^{4+4\iota}\vee 1),
\end{aligned}
$$

where we, additionally, utilized the boundedness of the $(4+\varsigma)$th moment of $\zeta_{t,\tau}^N(j)$ from Assumption A3. Since $\sigma_{t,\tau}(j)$ is continuously differentiable, $\sup_{k\in[\underline{k}_{t,\tau},\overline{k}_{t,\tau}]}|\sigma_{t,\tau}(k)|^{4+4\iota}$ is finite almost surely. Moreover, by Assumption A2, we have $\Delta N_t^\tau = O_p(1)$. Therefore, altogether, we get

$$
\widetilde{\mathcal{L}}_{t,\tau}(0)-\mathbb{E}\left(\widetilde{\mathcal{L}}_{t,\tau}(0)\big|\mathcal{F}^{(0)}\right)=o_p(1).
$$

By applying this bound together with the convergence result in (72), this provides **(b)**.

For (c). It readily follows by **(a)** and **(b)**. □

9.3 Proof of Theorem 1

First, by Lemmas 1(b), 3(c), 3(d) and the law of iterated expectation, $\widehat{\chi}_{t,\tau}(h)$ converges to $\bar{\gamma}_{t,\tau}(h)\mathcal{Q}_{t,\tau}^2$ in $\mathbb{L}^2$. The consistency result for $\hat{\rho}_{t,\tau}(h)$, thus, follows by the continuous mapping theorem. □

9.4 Proof of Theorem 2

The stable CLT readily follows by combining the delta method with the moment result in Lemma 4, the stable central limit theorem for autocovariances in Lemmas 5(b) and Slutsky's theorem. □

9.5 Proof of Corollary 1

The feasible limit result follows by combining the stable central limit theory in Theorem 2 with Theorem 1, Lemma 6 and the continuous mapping theorem to establish consistency of the asymptotic covariance matrix and bias correction, respectively, and Slutsky's theorem. □

9.6 Proof of Theorem 3

For (a). First, by Theorem 1, Lemma 6 and the continuous mapping theorem, we have, under the null hypothesis $\mathcal{H}_0$, $\widehat{\chi}_{t,\tau}^2/(0)\widehat{\mathcal{L}}_{t\tau}(2)\xrightarrow{\mathbb{P}}(\mathcal{Q}_{t,\tau}^2)^2/\mathcal{Q}_{t,\tau}^4$, thus eliminating the heteroskedasticity of the statistic. Hence, the result follows by combining Theorem 2, Corollary 1 and Slutsky's theorem.

For (b). By Theorem 1, we have $\widehat{\rho}_{t,\tau}(h) \xrightarrow{\mathbb{P}} \bar{\rho}_{t,\tau}(h)$, $\forall h \in \mathbb{N}$. Hence, the proof proceeds in three steps. First, we show $|\widehat{\rho}^{\text{BC}}_{t,\tau}(h) - \widehat{\rho}_{t,\tau}(h)| \leq O_p(\Delta)$, $\forall h \in \mathbb{N}$. Second, we establish $\widehat{\chi}^2_{t,\tau}(0)/\widehat{\mathcal{L}}_{t\tau}(2) = O_p(1)$. Finally, we show that when $\mathcal{H}_A$ holds, $\bar{\rho}_{t,\tau}(h) \neq \rho^{\text{ind}}(h)$ for some $h \in 1, \ldots, H$, for sufficiently high H.

For the first part, since $\widehat{\chi}^2_{t,\tau}(h) \xrightarrow{\mathbb{P}} \bar{\gamma}_{t,\tau}(h)\mathcal{Q}^2_{t,\tau}$, $\forall h \in \mathbb{N}$, by Lemmas 1(b), 3(c)-(d), it suffices to study the properties of $\widehat{\mathcal{L}}_{t,\tau}(0)$ and $\widehat{\mathcal{L}}_{t,\tau}(2)$ under $\mathcal{H}_A$. As for Lemma 6(a) and 6(b), this simplifies by the bounds $|\widehat{\mathcal{L}}_{t,\tau}(0) - \widetilde{\mathcal{L}}_{t,\tau}(0)| \leq O_p(\Delta)$ and $|\widehat{\mathcal{L}}_{t,\tau}(2) - \widetilde{\mathcal{L}}_{t,\tau}(2)| \leq O_p(\Delta)$ such that we may analyze,

$$\widetilde{\mathcal{L}}_{t,\tau}(0) \equiv \frac{1}{\Delta(N^\tau_t - 4)^2} \sum_{j=4}^{N^\tau_t - 1} (\widetilde{\epsilon}^N_{t,\tau}(j))^4, \quad \widetilde{\mathcal{L}}_{t,\tau}(2) \equiv \frac{1}{\Delta(N^\tau_t - 4)^2} \sum_{j=4}^{N^\tau_t - 1} (\widetilde{\epsilon}^N_{t,\tau}(j))^2(\widetilde{\epsilon}^N_{t,\tau}(j-2))^2,$$

in the following. By Assumption A3 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathbb{E}\left(|\widetilde{\mathcal{L}}_{t,\tau}(2)|\big|\mathcal{F}^{(0)}\right) &= \frac{(2/3)^2}{\Delta(N^\tau_t - 4)^2} \sum_{j=4}^{N^\tau_t - 1} \mathbb{E}\left((\bar{\zeta}^N_{t,\tau}(j))^2(\bar{\zeta}^N_{t,\tau}(j-2))^2\right) \sigma^2_{t,\tau}(j)\sigma^2_{t,\tau}(j-2) \\
&\leq \frac{(2/3)^2}{\Delta(N^\tau_t - 4)^2} \sum_{j=4}^{N^\tau_t - 1} \sqrt{\mathbb{E}\left((\bar{\zeta}^N_{t,\tau}(j))^4\right)\mathbb{E}\left((\bar{\zeta}^N_{t,\tau}(j-2))^4\right)}\sigma^2_{t,\tau}(j)\sigma^2_{t,\tau}(j-2) \\
&\leq \frac{(2/3)^2 K}{\Delta(N^\tau_t - 4)^2} \sum_{j=4}^{N^\tau_t - 1} \sigma^2_{t,\tau}(j)\sigma^2_{t,\tau}(j-2)
\end{aligned}$$

Next, by Assumption A2, the continuous mapping theorem, Lemmas 2(b)-(c) and the dominated convergence theorem, we have $\Delta^{-1}(N^\tau_t - 4)^{-2} \sum_{j=4}^{N^\tau_t - 1} \sigma^2_{t,\tau}(j)\sigma^2_{t,\tau}(j-2) \xrightarrow{\mathbb{P}} \mathcal{Q}^4_{t,h}$. Hence, this implies that $\widetilde{\mathcal{L}}_{t,\tau}(2) \leq O_p(1)$, uniformly, and similar arguments provide $|\widetilde{\mathcal{L}}_{t,\tau}(0)| \leq O_p(1)$. Together with consistency of $\widehat{\chi}^2_{t,\tau}(h)$, $\forall h \in \mathbb{N}$, and the continuous mapping theorem, this establishes that the bias correction in $\widehat{\rho}^{\text{BC}}_{t,\tau}(h)$ is, uniformly, $O_p(\Delta)$, thus providing the first result.

For the second part, we may use the same arguments for $\widehat{\chi}^2_{t,\tau}(h)$ and $\widetilde{\mathcal{L}}_{t,\tau}(2)$ with the continuous mapping theorem to show $\widehat{\chi}^2_{t,\tau}(0)/\widehat{\mathcal{L}}_{t\tau}(2) \leq O_p(1)$, uniformly, under the alternative $\mathcal{H}_A$.

For the third part, note that this is equivalent to establishing that, if

$$\bar{\gamma}_{t,\tau}(h) = \begin{cases} 1, & \text{if } h = 0, \\ -\frac{2}{3}, & \text{if } h = \pm 1, \\ \frac{1}{6}, & \text{if } h = \pm 2, \\ 0, & \text{if } |h| > 2, \end{cases} \tag{74}$$

this necessarily implies $\gamma_{t,\tau}(h) = 0$ for $|h| > 0$. In the following, without loss of generality, we will show this assuming that $\gamma_{t,\tau}(h) = \gamma_{t,\tau}(-h)$, as the arguments apply correspondingly to positive and negative lags. Moreover, define $\theta(L) \equiv \frac{2}{3}(3/2 - (L + L^{-1}) + (L^2 + L^{-2})/4) = \frac{2}{3}(L-1)^4/(4L^2)$, that is, the lag polynomial describing the autocovariance structure of $\bar{\gamma}_{t,\tau}(h)$ in terms of $\gamma_{t,\tau}(h)$. Next, let us define $\gamma_{t,\tau,k}(h) = (L-1)^k \gamma_{t,\tau}(h)$. Then, using (74), $(L-1)^4 \gamma_{t,\tau}(h) = 0$, for $|h| \geq 5$, which further implies that $(L-1)\gamma_{t,\tau,3}(h) = 0$ and, thus, $\gamma_{t,\tau,3}(h) = \gamma_{t,\tau,3}(h-1)$. Moreover, by utilizing absolute summability of the autocovariances, $\gamma_{t,\tau}(h)$, in Assumptions A3-A4, we must have $\gamma_{t,\tau,3}(h) = \gamma_{t,\tau,3}(h-1) = 0$ for $|h| \geq 5$, that is, $\gamma_{t,\tau,3}(h)$ for $|h| \geq 4$. We may, thus, continue and use similar deductions to show that $\gamma_{t,\tau,3}(h) = 0$ implies $(L-1)\gamma_{t,\tau,2}(h) = 0$ for $|h| \geq 4$, further providing $\gamma_{t,\tau,2}(h) = 0$ for $|h| \geq 3$, $\gamma_{t,\tau,1}(h) = 0$ for $|h| \geq 2$, and $\gamma_{t,\tau}(h) = 0$ for $|h| \geq 1$. Hence, under $\mathcal{H}_A$, $\exists H \geq 1$ that is sufficiently high such that $\bar{\rho}_{t,\tau}(h) \neq \rho^{\text{ind}}(h)$ for some $h \in 1, \ldots, H$. Together with the first and second parts, this shows that $\hat{q}_h(\vartheta) = O_p(1)$ and $\widehat{Q}^H_{t,\tau} = O_p(\Delta^{-1})$, thus providing the final result. □

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