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Trading volatility: the importance of hedging with the right volatility in the right environment

A theoretical and empirical assessment of the P&L effects when trading volatility with Black-Scholes

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Abstract

Volatility as an asset class provides investors with unique opportunities to reap the volatility risk premium. A popular way to trade this premium is to sell options and delta-hedge them, which is a bet on the implied volatility being higher than the realised volatility. A way to price options is through the Black-Scholes model, which most market practitioners still use to calculate the delta of the option and hedge it to be exposed to volatility.

This thesis seeks to address the effectiveness of trading volatility with the Black-Scholes model through a relaxation of the assumptions pertaining to continuous hedging and constant volatility. To do this, this thesis seeks to first simulate several volatility scenarios to develop expectations on how empirical data should perform before conducting a backtest on delta-hedged options over a 13-year period.

This thesis illustrates the inherent path-dependency in trading volatility arising from dollar gamma exposure and imperfect delta-hedging. Further, this thesis addresses the notion that the delta-hedge can be under- or over-hedged relative to the RV, resulting in hedging errors with large implications for the return of the short volatility trade. This thesis further observes that the performance of delta-hedging strategies depends on the stability of the volatility environment. Lastly, this thesis confirms the notion of left-tail risk inherent in selling options and that volatility can be beneficial to time rather than to be continuously exposed to.

In sum, this thesis acts as a comprehensive guide to understanding the mechanics of trading volatility with delta-hedged options and the mechanics that the drive the profitability of these trades.

Abbreviations

ATM: At the money

BS: Black & Scholes

GBM: Geometric Brownian Motion

ITM: In the money

IV: Implied volatility

Lhs: Left-hand side

LIBOR: London Interbank Overnight Rate

OTM: Out of the money

PDE: Partial differential equation

P&L: Profit and loss

RV: Realised volatility

Rhs: Right-hand side

TV: Trailing volatility

W.r.t.: With respect to

Table of contents

1. Introduction

The simplest form of risk that investors are exposed to is the stock price's fluctuations, also commonly called volatility, which is governed by the stock's direction. Investors can gain exposure to this direction by either being long or short the stock, but this does not provide the investor with exposure to the volatility that is independent of the direction of the stock. The intuitive reason behind the wish to gain exposure to volatility is the same as with stock traders, who have an idea of a stock's potential and wish to profit from this, or bond traders who have an estimate of future interest rate expectations. In the same way, volatility traders have a possible future estimate on the level of volatility and seek to profit from this.

Derivatives are financial instruments whose value is dependent and priced from other assets on the market, hereunder stocks. If investors want to gain exposure to volatility, they can do so by entering into over-the-counter derivatives, such as variance swaps, or by trading delta-hedged options. A way to price options was proposed by Black and Scholes (BS) (1973), who assumed that volatility was constant, and that trading could occur instantaneously in order to price options through a replicating portfolio. Assume a complete market with three assets: a stock, a risk-free asset and an option on the stock. In theory, if the assumptions of BS were true, one could perfectly replicate the payoff of the option by taking dynamic positions in the stock and in the risk-free asset, which ensures that the created portfolio is not dependent on the stock's price. This is called a self-financing portfolio (Hull, 2009).

One could also sell an option such as a call on the stock, hedge it based on a ratio of the option's delta, which is the first derivative of the option w.r.t. to the spot of the underlying, by buying a fractional unit of the underlying equal to the sold call's delta. The position in the underlying is financed by a position in the risk-free asset. Since the underlying has a delta of 1 and one can instantaneously rebalance one's position as the underlying undertakes new values, this hedging strategy is called a delta-hedging strategy, where the payoff is equal to the spread between the implied volatility (IV) used to price the option and the stock's realised volatility (RV) (Ahmad & Wilmott, 2005). Given that transactions cannot occur instantaneously, it is evident that the assumption of continuous trading is violated in practice. Since it is affected by variance as a function of the transaction frequency, which creates a hedging error, this creates path-dependency in the profit and loss (P&L) from trading volatility with options. This finding is illustrated in the thesis as it showcases that an increased hedging frequency reduces the variance in the P&L through a reduced hedging error.

Despite the shortcomings of the Black-Scholes model being widely recognized, it remains the most widely used option pricing model by practitioners; this is in stark contrast to literature, as volatility is not constant, but rather opaque and stochastic (Fleming et al., 1995; Whaley, 1993). Since volatility is not known then, it is not possible to hedge the option perfectly, as argued by Black-Scholes. In accordance with the work by Kurpiel & Roncalli (2011), this thesis showcases how the P&L impact of relaxing the constant volatility assumption and how the subsequent hedging error is governed by the moneyness of the option, the chosen volatility, and the movements of the underlying. This thesis further finds that the volatility environment has a large effect on the return of trading volatility, given that it is more profitable to trade volatility with an IV-based hedge in highly volatile environments relative to the (proxied) RV-based hedge. This finding is based on the argument that IV tends to underestimate true future RV in uncertain times and overestimate in more stable environments, which translates into a need to proxy RV with a trailing historical measure rather than IV. This indicates that there is a left-tail risk to trading volatility given stock market returns, which is why this thesis finds that it can be profitable to time volatility based on whether the options are over or underpriced.

From a practical point of view, the research presented below presents the reader with a much more comprehensive analysis of the P&L effects of hedging with a theoretically wrong model, compared to other papers which either focus solely on simulations or empirical aspects.

2. Research question

The motivation for this thesis is to evaluate the effectiveness of trading volatility through the aforementioned delta-hedged options strategy, given a relaxation of several assumptions inherent in the BS option pricing model, and to understand how trading volatility performs in various volatility environments. This thesis takes a practical approach, given that we are inspired by market practitioners' approach to trading volatility through BS as the most common option pricing formula. The research question that this thesis seeks to answer is the following:

"What is the profit and loss effect of hedging with implied vs. (proxied) realised volatility when relaxing various assumptions of the Black-Scholes and in various volatility environments?"

In addition to the main research question, this thesis seeks to answer a sub-question, which is motivated by a larger probability of negative events vis-à-vis positive events in the stock market's returns. The sub-question is:

"How can exposure in short volatility positions be managed to increase the P&L with an option pricing indicator?"

This thesis delimitates itself to the following parameters:

- We focus only on the performance of delta-hedged options on the S&P500 (SPX), as this index is the most common to trade volatility on and the one upon which most volatility and variance derivatives are priced, such as VIX and index variance swaps.

- We only focus on selling options with three months to maturity because we can take a volatility view three months forward rather than 12 months forward, which other maturities could have provided us with.

- We only sell at-the-money (ATM) options and do not consider the impact of trading volatility through out-of-the-money (OTM) or in-the-money (ITM) options. This is because we want to observe whether options with an observable and sensitive delta allow us to effectively trade volatility through delta-hedging.

- We only focus on delta-hedged puts and calls, not a combination of strategies such as iron condors, straddles or strangles because we want to isolate the singular drivers behind the return and P&L of volatility trading.

- We only trade based on observable data, meaning that we model a proxy for the RV as a 30-day trailing volatility (TV) on the SPX. This is done to further mimic the data available to a trader at the point in time the option is struck and increase the finding's practical implications.

2.1 Thesis structure

This thesis is structured as follows: Chapter 3 presents a literature review of both the theory behind option pricing, how options are used to trade volatility, and previous empirical studies on the effectiveness of trading volatility through delta-hedged options. Chapter 4 delineates the methodology used to analyse the effectiveness of trading volatility with delta-hedged options, with a focus on both simulations and an empirical back test. Chapter 5 presents the results and highlights how the finding from the simulations are related to the empirical observations. Chapter 6 provides a critical reflection and discussion of the results, followed by a brief discussion on alternative ways to trade volatility, before outlining the implications for future research and this thesis' limitations. Chapter 7 will provide an overall conclusion of this thesis.

3. Literature review

This chapter will present different theories and studies which showcase how options are priced on and what factors affect the pricing of options. It will also present how to trade volatility and what implications several model assumptions have on the effectiveness of trading volatility. The purpose of this section is to present research perspectives that can help answer the research question.

3.1 The foundation of option pricing

This section seeks to provide the general relationships between option pricing and the various factors influencing its value. before outlining how these concepts are incorporated into this thesis' model and into the theoretical framework used to answer the research question.

3.1.1 An introduction to option pricing

Option pricing theory is a probabilistic approach that has the aim of calculating the probability that an option will end ITM and assigning a value to that contract. There are multiple ways of pricing options, including binomial option models, Monte-Carlo simulations and the BS option pricing model. This thesis will focus on the BS models and various extensions, as these are considered the most influential on how traders price and hedge derivatives (Hull, 2009).

3.1.2 Towards the Black-Scholes model

One of the most famous equations in financial literature and the backbone of all option pricing literature is the Black-Scholes (1973) partial differential equation (PDE) formulated as:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \tag{1}$$

In equation (1), V is the value of the option as a function of the stock price S and time t, r is the risk-free interest rate and σ is the stock's volatility.

The assumptions behind equation (1) are 1) there exists no arbitrage opportunities 2) the market is frictionless, i.e., transaction costs do not exist, 3) stocks do not pay dividends, 4) the risk-free interest rates and volatility of the underlying are constant and known, 5) returns are log-normally distributed and follow a random walk, 6) options can only be exercised at expiry, i.e., options are European, 7) one can borrow or lend at the risk-free rate, 8) one can purchase a fraction of the stock (including shorting it) and 9) trading can take place continuously.

Equation (1) is in fact the result of applying Itô's lemma (1944), an identity used to compute the derivative of a time-dependent stochastic process to a Geometric Brownian Motion (GBM).

The GBM is based on an Itô process, which is a simple transformation of a Brownian motion. It is a variable X that changes over time with drift μ, which is considered the expected change in X and the diffusion coefficient σ, which is the unexpected change in X as:

$$dX_t = \mu_t dt + \sigma_t dB_t \tag{2}$$

In equation (2), B is a Brownian motion and μ and σ can be either constants or random processes of their own. In particular, μ and σ can depend on *X*. This allows one to add further complexity to the model through e.g., a varying risk-free rate or varying volatility.

If one assumes μ and σ are constant, and B_t is a Brownian motion, the path of the stock price S_t is a GBM if:

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{3}$$

Using Ito's formula, one can verify that:

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2}\sigma^2 t + \sigma B_t\right) \tag{4}$$

And from equation (4), one finds that over a discrete time interval, Δt,:

$$\Delta S = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma \Delta B \tag{5}$$

Equation (5) implies that over a discrete time interval, log-returns are normally distributed with mean $(\mu - \frac{1}{2}\sigma^2)\Delta t$ and variance $\sigma^2 \Delta t$. This further allows one to model the GBM.

To derive the BS PDE, an interest in knowing how the payoff of the option *V* evolves using the GBM arises. Substituting equation (3) into Itô's lemma, one finds that:

$$dV = \left(\frac{\partial V}{\partial t}(S,t) + \mu S \frac{\partial V}{\partial S}(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S,t)\right) dt + \sigma S \frac{\partial V}{\partial S}(S,t) \partial B \tag{6}$$

A central notion within the world of BS (1973) is that a hedged portfolio must grow at the risk-free rate given perfect market assumptions. This implies that the change in value of the portfolio over time must be determined through Δ. Restating equation (6) to that of a hedged portfolio:

$$d(V + \Delta S) = \left(\frac{\partial V}{\partial t}(S,t) + \mu S \frac{\partial V}{\partial S}(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S,t) + \Delta \mu S\right) dt + \Delta S \left(\frac{\partial V}{\partial S} + \Delta\right) dB$$

And by:

$$\Delta = -\frac{\partial V}{\partial S}(S,t)$$

It is evident:

$$d(V + \Delta S) = \left(\frac{\partial V}{\partial t}(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S^2}(S,t)\right) dt \tag{7}$$

In equation (7), a portfolio which grows at the risk-free rate is created through hedging, thus nullifying arbitrage opportunities. From this, it can be stated that:

$$\frac{\partial V}{\partial S}(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}(S,t) = r\left(V - S\frac{\partial V}{\partial S}\right)$$

The expression above can be restated into the original BS equation (1) if the dependence of (S,t) is dropped. Rearranging equation (1):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$$

In the above expression, the first part of the left-hand side is the change in *V* as *t* rises, whereas the second part of the left-hand side is the V's convexity relative to the stock's price. The first part of the right-hand side is the risk-free return from holding the option, i.e., being long, combined with a short position of $\frac{\partial v}{\partial s}$ units in the stock, which is the second part of the right-hand side. Given this set of assumptions, this PDE holds for any type of option as long as its price function is twice differentiable with respect to *S* and once with respect to *t*.

3.1.3 The Black-Scholes model

Every derivative whose value depends on t and S must satisfy the BS PDE. Different derivatives have different values only because of their boundary conditions. Examples of boundary conditions are:

$$Value\ of\ call\ option\ at\ maturity\ is\ \max\,[S_T - K, 0]$$

$$Value\ of\ put\ option\ at\ maturity\ is\ \max\,[K - S_T, 0]$$

From equation (1) to (3), it is evident that the boundary and final conditions for a call can only be given by the solution to the BS equation[1], which is the BS model:

$$c(S,T) = c(S_t, K, T-t, r) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{8}$$

Here:

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{1}{2}s^2}\,ds \tag{9}$$

[1] The solution to the Black-Scholes equation can be found in appendix #1.

In equation (9), the cumulative distribution function of the standard distribution is given by:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{10}$$

Similarly, the value of a European put is

$$p(S,T) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1) \tag{11}$$

This can be proven by using the put-call parity, the call price and the fact that $N(x) + N(-x) = 1$ for any *x*:

$$C_t - P_t = S_t - K \cdot B(t,T) \tag{12}$$

The parity in equation (12) states that a portfolio of a long call and a short put must have the same value as a long position in the forward, with the same strike and expiry as the call and put.

In equations (8) - (12), *S* is the current share price, *K* is the strike price of the option, *(T-t*) is the maturity in years, σ is the annual volatility of stock returns, *r* is the annual continuously compounded risk-free interest rate, $B(t,T)$ is the value of a zero-coupon bond that expires at time *T* and *N(.)* is a cumulative standard normal distribution. Of these variables, only σ is not directly observable and one would typically make an estimate of future volatility and assume it to be constant throughout the period. This estimate can either be based on historical volatility or the market's expectations to the future level of volatility. In practice, the latter is the most common (Hull, 2009).

3.1.4 How to measure volatility

Volatility can be denoted as either IV or RV. RV is defined as the stock price's historical fluctuations. Fluctuation in stock prices can be measured in many ways, including maximum drawdown or beta, but the most common way is through standard deviations of the returns. Here, the returns r_1, r_2, …, r_n are given by:

$$\sigma_r = \sqrt{\frac{af}{n-1}\sum_{t=1}^{n}(r_t - \bar{r})^2} \tag{13}$$

In equation (13), $\bar{r} = \frac{1}{n}\sum_{t=1}^{n} r_t$ is the mean return of the underlying and *af* is the annualization factor. Volatility, both IV and RV, within option pricing is usually based on a unit of time equal to a year, which is the annualization factor corresponding to 252, which refers to the average number of business days (Hull, 2009). A central notion behind equation (13) is that returns are distributed randomly and independent of each other. To be consistent with equation (1), the returns used to calculate the RV are logarithmic:

$$r_t = \ln \frac{P_t}{P_{t-1}} \; and \; \bar{r} = \frac{1}{n}\sum_{t=1}^{n} r_t \tag{14}$$

Options are not written based on the historical volatility of the underlying. Since most parameters are by contractual specification, the rates and futures markets, the (implied) volatility is what is left for the option writer to specify. It can be stated as:

$$\frac{dS_t}{S_t} = \mu dt + \sigma_{implied} dW_t \tag{15}$$

Restated from equation (15), the IV is forward-looking upon which a given option pricing model, primarily the BS model, will return a fair theoretical option value equal to the current market price (Bossu et al., 2005). Given this interpretation of IV, an index of the IV can be constructed, providing a measure of market volatility on which future expectations on IV can be formed. The volatility index VIX, a benchmark for S&P500 Index's volatility, measures the expected annualised change in the S&P500 for the next 30 days. VIX is generally known as the investors' "fear gauge" as it gives an indication of whether investors believe volatility will rise or not over the next 30 days (Whaley, 1993, 2000). The formula for VIX is given by:

$$VIX_{t\to T}^2 \equiv \frac{2R_{f,t}}{T-t}\left\{\int_0^{F_{t,T}} \frac{1}{K^2} put_{t,T}(K)dK + \int_{F_{t,T}}^{\infty} \frac{1}{K^2} call_{t,T}(K)dK\right\}. \tag{16}$$

From equation (16), it is evident that VIX places a relatively higher weight on OTM puts compared to OTM calls. This creates a skew in the distribution to capture the effect of left-tail events, i.e., negative events, which happen relatively more than right-tailed events. To emphasise why VIX is a benchmark for IV, VIX can be interpreted as the square-root of a variance swap's strike, which by mathematical convention is the IV (Demeterfi et al., 1999).

Historically, there has been a spread between the IV and the RV. When entering option contracts, the long position has a limited downside, i.e., the premium paid for the contract, whereas the short position has an unlimited downside (Christensen & Prabhala, 1998; Giese, 2010). Restated, RV is what an option buyer earns, but IV is what an option buyer pays for an option. IV is argued to be the market's best estimate on future volatility. However, previous empirical research differs on whether IV has predictive power over future realised. Most researchers argue that IV is better at estimating future volatility than historical volatility (Okomu & Nilsson, 2013). Yet, recent empirical research also finds that IV tends to overestimate RV in normal times and underestimate volatility in times of crisis and is argued to be inaccurate for risk management purposes (Kownatzki, 2015). The degree of mispricing is likely to have been further influenced in recent years by the large inflow of unsophisticated retail investors affecting demand and an increase in stock-market volatility, confirming the need for option sellers to charge a risk-premium (Andersen et al., 2015).

In Figure 1, the VIX over a two-year period and the 30-day rolling RV (both on the left-hand axis) of the S&P500 Index are plotted against the S&P500 Index (right-hand axis). Here, one can see how the IV spiked during the COVID-19 crisis in March 2020. The TV of the S&P500 has been calculated to showcase how RV is usually lower than IV.

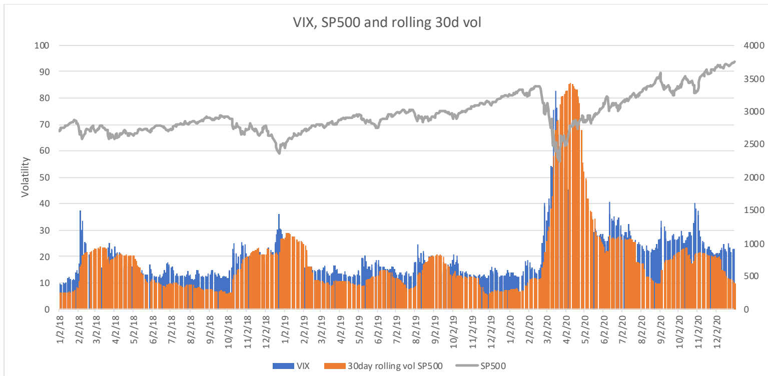


Figure 1: Plot of the VIX (lhs), TV (lhs) and the S&P500 (rhs)

The VIX can be viewed as the market's best guess of future volatility. However, there are also several ways one can forecast volatility using statistical models. The most common way of modelling volatility is through ARCH (Engle, 1982) and GARCH models (Bollerslev, 1986). These models are used to estimate time series regression errors, when the errors experience non-linear properties, meaning that the homoskedasticity assumption is violated. An asset return-series is an example of a time-series with non-linear errors i.e., volatility, which over time is argued to exhibit clustering, mean-reversions, and jumps. These characteristics will be further described in section 3.1.6.

The GARCH model is an extension of the ARCH model, which allows for the conditional variance to change over time as a function of past errors, a longer memory of past observations, and a more flexible lag structure (Bollerslev, 1986). GARCH models also tend to be more parsimonious compared to ARCH models, meaning that it accomplishes a better prediction with fewer variables and avoid overfitting (Brooks, 2014).

Despite the fact that the standard GARCH model can account for volatility clustering, it does not account for the leverage effect (explored in depth in section 3.1.6) nor allow for direct feedback between the conditional variance and the conditional mean. Therefore, many further extensions of the GARCH model have been developed, including the EGARCH model.

The EGARCH model was developed by Nelson (1991) to also account for the leverage effect, which is the empirical observation that positive shocks tend to have less impact on future volatility than negative shocks of the same amount. The need for this extension was confirmed in a study by Chong et al. (1999) where researchers examined the performance of various GARCH models and found that the EGARCH model outperformed other GARCH models when forecasting volatility on the Kuala Lumpur stock exchange. This finding was further supported by Awartani and Corradi (2005) who evaluated forecast performance for the S&P500 index, adjusted for dividends both for one-step ahead and longer forecast horizons. More recently, Kişinbay (2010) found that asymmetric volatility models, including EGARCH models, provide improvements compared to the GARCH model in forecasting at short-to-medium-term horizons.

3.1.5 The Greeks

Equation (8) and (11) are the formulas for valuing European calls and puts. Together, they constitute the famous BS model, but the sensitivity of the value of a portfolio of options to changes in the parameters is not explained. The sensitivity w.r.t. a given parameter, e.g., the stock price, can be found by deriving the partial derivative of the value of the option w.r.t. that parameter. Collectively, all sensitivities are known as "the Greeks" and are illustrated in table 1.

Greek	Derivative	Call	Put
Delta	$\frac{\partial V}{\partial S}$	$N(d_1)$	$-N(-d_1) = N(d_1) - 1$
Gamma	$\frac{\partial^2 V}{\partial S^2}$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$	
Vega	$\frac{\partial V}{\partial \sigma}$	$SN'(d_1)\sqrt{T-t}$	
Theta	$\frac{\partial V}{\partial t}$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(d_2)$
Rho	$\frac{\partial V}{\partial r}$	$K(T-t)e^{-r(T-t)}N(d_2)$	$-K(T-t)e^{-r(T-t)}N(-d_2)$

Table 1: The Greeks and their BS derivations

Delta is the first derivative of the value w.r.t. the stock price. Delta is a linear approximation for the value change in the option if the stock price changes by one unit, and it reflects the position in the underlying an option trader would take to replicate the option i.e., hedge against movements in the stock price. For an option, the delta for long calls or short puts is between 0 and 1, and conversely for short calls and long puts, it is between 0 and -1. The overall portfolio delta can be calculated by summing all the options' delta. The relationship between a call's delta and a put's delta is evidenced in the put-call parity, where a long call and a short put replicate a forward, whose delta is always one. The linear nature of delta means that it works best for smaller price changes (Hull, 2009).

The second derivative of the value w.r.t. the stock price is the gamma. It is defined as the rate of change in the option's delta if the stock price changes by one unit. Gamma has the same value for put and call options. It has a positive value for long positions and conversely, it is negative for short positions. Since options are non-linear, rather than linear instruments, gamma corrects for the convexity in the option's value for which the delta does not account for. A position in the underlying has a gamma of 0, and therefore it has no hedging effect. To hedge the gamma of a portfolio, a position in an option, which is not linearly dependent on the underlying, is needed (Hull, 2009). In Figure 2 (Hull, 2009), two graphs are shown. The graph on the left illustrates the relation between the stock price and gamma, i.e., gamma is the highest when the stock price equals the strike price of the stock; or in other words, when the options are exactly ATM. The graph on the right side illustrates the hedging error which arises from linearly approximating the option value with delta, compared to correcting for convexity with gamma.

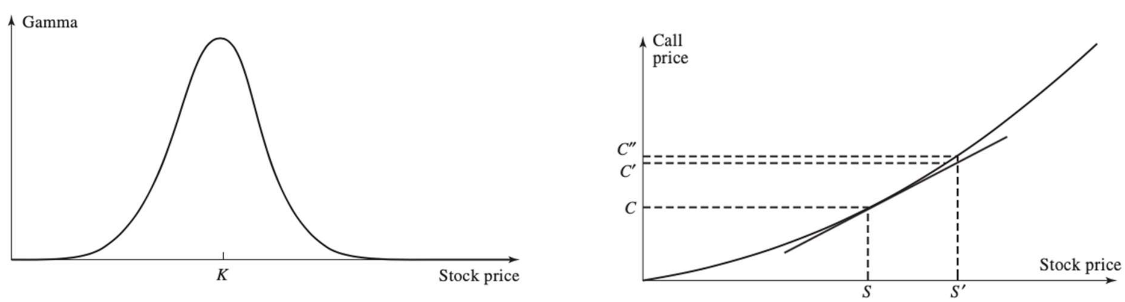


Figure 2: Relationship between gamma and stock price (lhs) hedging error arising from non-linearity in options (rhs).

Vega is the first derivative of the value w.r.t. to the volatility of the stock price (or underlying). Vega is defined as the rate of change of the value if the volatility changes by one unit, and vega is usually denoted in absolute terms. Volatility is assumed constant in the general BS model, but in practice it tends to vary over the lifetime of an option, making it relevant for traders to vega-hedge. Just like with gamma, a position in the underlying has zero vega and cannot be used to hedge. Assuming that all IVs are to change by the same amount during a short period of time, a portfolio can be made vega neutral by using an option dependent on the underlying (Hull, 2009). Vega is positive for a long position and negative for a short position. Both puts and calls increase in value with increases in volatility as both the upside and downside potential increases, which is beneficial for calls (upside) and puts (downside). Longer-dated options usually have a higher vega in absolute terms, which denotes a high sensitivity to changes in the portfolio's value compared to short-dated options, as the longer time to expiry increases the possibility of changes in the volatility.

The passage of time is measured through theta, which is the first derivative of the value w.r.t. the time. Theta can be referred to as the "time decay" of the options. As there is no uncertainty against the passage of time, it does not make sense to hedge in the same way, as is done for the delta (Hull, 2009). Theta is usually negative for long option positions because options lose value with the passage of time, and reversely, it is positive for short option positions. This is evidenced in how the value of an option can be split into two parts: the intrinsic value and the time value. The intrinsic value of an option is the value achieved if the option would be exercised immediately. Assuming no transaction costs, a call with strike 150 on a stock with price 180 would have an intrinsic value of 30. The time value arises from the possibility of waiting for exercise, but *ceteris paribus*, the value decreases as the time to exercise decreases. Figure 3 showcases the time decay of options.

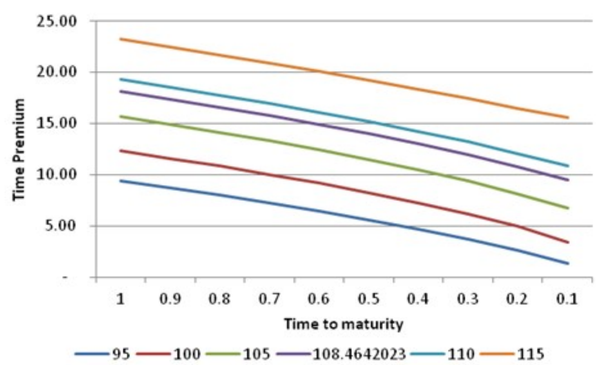


Figure 3: The relationship between option value and the passage of time

Rho measures the rate of change in the portfolio's value w.r.t. changes in the interest rate. Rho is positive for long call and short put positions and negative for short call and long put positions. As with vega, rho is higher for longer-dated options compared to short-dated options. Even though interest rates tend to increase the value of options, rho is not used as extensively as the other Greeks because huge changes or changes in general in the interest rate do not happen as often as changes in the volatility (Hull, 2009).

In an ideal world, traders would try to keep all Greeks equal to zero by hedging frequently. However, a zero gamma and vega are difficult to achieve, as it is difficult in practice to find options that can be traded in the volume required at competitive prices (Hull, 2009). As the delta is only dependent on the underlying, a zero delta is much easier to achieve, which is why it is much more widely used as a hedging tool (ibid).

3.1.6 Challenges & extensions of the Black-Scholes model

The BS model in equation (8) and (11) rests upon several assumptions. Yet, these assumptions are what they are - assumptions. In practice, several of them do not hold. This section will briefly outline the assumptions of the BS model and their inherent challenges.

Continuous trading and transaction costs

It is not possible to trade continuously, and markets are not frictionless as markets have an opening and close and brokers charge transaction fees. Given that transaction costs exist, the perfect replication of the risk-free asset evident in the BS world is eroded, resulting in the preference-independent valuation of options not existing (Barles & Soner, 1998). With transaction costs, it is also evident that continuous trading becomes impossible, further emphasising the point that markets are not frictionless and that the assumptions of the BS model are challenged.

Arbitrage

Building on markets not being frictionless, they are not completely arbitrage-free either. Returning to equation (12), the put-call parity, it is evident that if it does not hold, an arbitrage opportunity arises where one could short the call, short the underlying, buy the put and invest in a zero-coupon bond (Bodie et al., 2008). Through the trades undertaken, this arbitrage opportunity would be eroded, though, which means that the market would revert to being (nearly) arbitrage free.

Dividends

The behaviour of stocks is also constrained by assumptions in the BS model. The model can be extended to include the stocks paying dividends continuously (Merton, 1973):

$$\begin{aligned} c(S,T) &= S_t e^{-q(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2) \\ p(S,T) &= K e^{-r(T-t)} N(-d_2) - S_t e^{-q(T-t)} N(-d_1) \end{aligned} \tag{17}$$

Where q is the continuous dividend yield at time *t*. For discrete dividends:

$$\begin{aligned} c(S,T) &= S_t (1-\delta)^{n(t)} e^{-r(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2) \\ p(S,T) &= K e^{-r(T-t)} N(-d_2) - S_t (1-\delta)^{n(t)} e^{-r(T-t)} N(-d_1) \end{aligned} \tag{18}$$

Where δ is the discrete proportional dividend on the stock and *n(t)* is the number of times the dividend has been paid out at times *t*. This shows that the BS model can be extended to incorporate dividends, both continuously and discretely.

Random walk, log-returns and jumps

For the log-returns, Mandelbrot and Hudson (2004) observed that options on the markets ATM are overpriced compared to options ITM or OTM. This implies that there is a fat tail distribution, which the market assigns to the options relative to the distribution assumed within the BS model. If jumps can occur, the BS model can be extended through a jump diffusion with a Poisson process to account for the discrete jumps in the underlying as the price does not evolve continuously anymore (Merton, 1976). Research indicates that stock market jumps create skewness and kurtosis evidenced in the stock market's return's distribution (Baker et al., 2020; Fortune, 1996; Hanousek et al., 2014). According to Campolongo, Cariboni and Schoutens (2006), the main driver behind uncertainty in estimated option price's results from jumps in the underlying. For models with jumps, the stock price has a drift component, a random component (both much like the GBM) and a jump component, which is new. The stock's evolution is now given by:

$$\frac{dS_T}{S_t} = (r - \lambda k)dt + \sigma dB_t + dJ_t \tag{19}$$

In equation (19), the J_t is a jump process, which has no effect between jumps, but is sudden and can be large. λ is the likelihood of a jump occurring. If it is 1, one expects one jump per *t*. If a jump occurs, it has a certain distribution, based on a mean jump size k. If one assumes a normal distribution with mean 0 for the jump, the process is a standard GBM with a jump component, but otherwise it follows a Poisson process instead.

Volatility

Given that volatility is not directly observable, the BS world just assumes volatility as being known and constant. Yet, volatility has several characteristics and traits of volatility have been suggested in existing literature. First, volatility can cluster, meaning that for certain periods of time, it is high and low for other periods of time (Mandelbrot, 1963). This is due to the manifestation of the absolute returns (or their squares) exhibiting a decaying autocorrelation function (Ding & Granger, 1996). Second, jumps in volatility are rare by nature, but can occur as evidenced during volatility flash crashes (BlackRock, 2010; Todorov & Tauchen, 2011). Thirdly, it can be noted that volatility is a stationary process, i.e., it does diverge to infinity, but rather varies its fluctuations within a fixed range (Tsay, 2010). Fourth, volatility exhibits a mean-reversion in the run (Merville & Pieptea, 1989). Lastly, volatility displays a leverage affect, which means that volatility reacts differently to increases or decreases in the underlying (Aït-Sahalia et al., 2013).

These characteristics together with the volatility's volatility can constitute a volatility smile (Dupire, 2006). The volatility smile has a minimal effect on short-dated options priced with the BS model compared to longer-dated options given the volatility smile is small (Hull, 2009). A volatility smile implies that actual OTM option prices are more expensive than what the BS model values. This is the market accounting for kurtosis as extreme events are more likely than what the assumed normal distributions imply. A skew would mean that extreme downturns are more likely than extreme up movements, which would lead to put options being more expensive. It is evident that volatility is thus stochastic rather than constant as assumed in the BS model. In a stochastic volatility model, the stock price's movement is very similar to the GBM in the BS model:

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t \tag{20}$$

In equation (20), the difference is that the volatility is time varying instead of constant. The volatility dynamic has a drift component $\kappa\left(\sigma_{t,R}^2 - \sigma_t^2\right)dt$, which makes it mean-reverting:

$$d\sigma_t^2 = \kappa(\sigma_{LR}^2 - \sigma_t^2)dt + \gamma\sigma_t dB_t^\sigma \tag{21}$$

In equation (21), if current volatility (σ_t^2) is higher than the long-run volatility mean, (σ_{LR}^2), volatility will be decreasing and if current volatility is lower than the long-run mean, it will be increasing with κ determining the speed of this reversion. If κ is low e.g., 1, then the reversion will be slow, if κ is high e.g., 10, it will be faster. The second term is the noise or the random part, i.e., the volatility of volatility. When volatility is high, the volatility of volatility is also high and vice versa. A classic stochastic volatility model was coined by Heston (1993)

The Brownian motion driving the stock price (B_t) and the Brownian motion driving the volatility B_t^σ are correlated through ρ in equation (22). When ρ is negative, an increase in volatility leads to a drop in the stock path and vice versa (Hull, 2009). Black (1976) argues that stock price movements are negatively correlated with volatility, suggesting a negative ρ, since falling stock prices imply an increased leverage of firms, which entails more uncertainty and hence the stock price volatility tends to rise.

$$B_t = N_1 \,\&B_t^\sigma = \rho N_1 + \sqrt{1-\rho^2}N_2 \tag{22}$$

3.2 Trading volatility

In the classical option pricing model from BS, volatility is a constant and therefore should not be of much interest to option traders. However, as is evident, empirical observations suggest that volatility is far more complicated and the cause for a lot of uncertainty when pricing option (cf. section 3.1.4 and 3.1.6). In short, RV is the amount of noise in the stock price while IV is how the market is pricing this volatility. Since the market does not have perfect knowledge about the future, these two numbers will differ (Ahmad & Wilmott, 2005). This discrepancy is what makes volatility interesting from a trading perspective, which this section will now go on to elaborate upon.

3.2.1 Various ways to trade volatility

Volatility trading is defined as a group of strategies, which enable the trader to profit from the magnitude of price swings of the instrument rather than the direction of such swings (Singh, 2017). The aim is to profit from price movements (or a lack thereof), regardless of whether such movements are increases or decreases in the underlying. The most common ways of trading volatility are to a) go long or short in a straddle b) delta-hedging an option position or c) go long or short in a variance swap.

A straddle

A straddle consists of a call and a put option of the same strike price and maturity, so that the strike price is equal to (or very close to) the current price of the underlying asset. The strike used is typically the fair forward, as this will give somewhat equal portfolio weight to the call and the put at initiation of the trade. In practice, a straddle is by far the most popular type of portfolio to take a speculative position in volatility, as it only requires one to buy (or sell) two options i.e., it has very limited trading costs (Nandi & Waggoner, 2000). If one believes that future RV will be higher than IV, one will take a long position in the straddle, as it can be shown that the higher the volatility of the underlying, the higher the potential payoff from a long position and vice versa (ibid.). Figure 4 shows the payoff profile of a straddle.

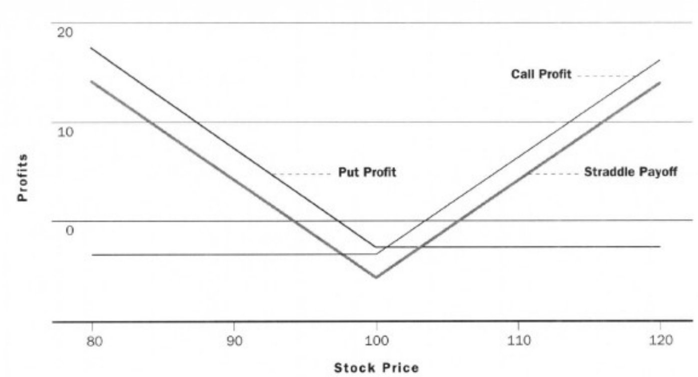


Figure 4: Payoff profile of a straddle

When the straddle is created, Nandi & Waggoner (2000) find that the P&L of this strategy is not very sensitive to changes in the underlying. Further testing this hypothesis over a 10-year period, they find that continuously selling a straddle will yield highly negatively skewed returns, i.e., the probability of large negative returns far exceeds the probability of large positive returns. This is attributed to the behaviour of asset returns, since empirical observations show that the probability of large negative returns exceeds the probability of large positive returns of equal magnitude (ibid.). However, it is only in extreme up or down moves in the underlying, the P&L of the straddle is impacted by the underlying and it is fair to conclude that in general a straddle primarily gain with changes in IV (Carr et al., 2002; Nandi & Waggoner, 2000).

Hedging an option position

Another way to take advantage of IV being different from one's expectations of future RV is to delta-hedge an option (Natenberg, 2012). To hedge an option, one needs to construct a portfolio of other securities with a payoff that exactly matches the payoff of the option. To do so, one needs to be able to buy or sell the underlying asset and borrow or lend at the risk-free rate (Nandi & Waggoner, 2000). A simple strategy to take advantage of IV being lower than ones estimate of future RV comprises a combination of taking a long position in a call and a short position in the underlying or taking a long position in a put option and a long position in the underlying and vice versa (Singh, 2017). This portfolio of an option and a delta-hedged position would yield the following guaranteed expected payoff, where $V(S, T; \sigma)$ is the option value using at maturity and $V(S, t; \bar{\sigma})$ is the option value at time 0 (Ahmad & Wilmott, 2005):

$$P\&L = (S, T; \sigma) - V(S, t; \bar{\sigma})$$

An alternative to trading a single option is to create a portfolio of options e.g., a straddle and delta hedge the portfolio. Delta-hedging a straddle can eliminate the excessive skewness and large drawdowns from continuously selling a straddle position (Nandi & Waggoner, 2000). This argument also holds for delta-hedged portfolios of only puts or calls. Since the delta of a portfolio of options is just the sum of the deltas of all the individual positions (cf. section 3.1.5), the short position in the underlying from the short put can be deducted from the long position in the underlying from the short call, which will lead to a smaller cash position and a lower financing need. If the fair forward is used as the strike, the two positions should more or less cancel each other out at time 0. This is especially relevant in practice if short selling is difficult and/or trading costs are high.

In the BS world, where it is assumed that continuous hedging is possible, volatility is constant and there are no trading costs, the delta-hedging position perfectly offsets the changes in the option value from changes in the underlying, and the trader is able to fully capture the price difference between the option value calculated with IV and the option value calculated with RV.

In practice however, a hedging error is inevitable, as it is argued that the BS assumptions of continuous trading and constant volatility do not hold (cf. section 3.1.6). Even if volatility was estimated perfectly and one could take infinitely small time-intervals, meaning very frequent trading, profits would be eroded away by the high accumulated trading costs (Leland, 1985). Furthermore, the P&L of a delta-hedging strategy will be dependent on choosing the right delta and the path of the underlying, which will be discussed in more detail in section 3.2.2 and forward.

The success of the delta-hedging strategy is heavily dependent on choosing the right delta. Since volatility is an input in calculating delta, the risk-return profile will differ dependent on which volatility you use (Ahmad & Wilmott, 2005). This means that a trader not only realises different P&Ls dependent on whether the hedge is calculated using IV or expected RV, but the strategy also exhibits a substantial volatility model risk. Even if a trader has a correct set of expectations on RV, which are different from implied, and hedges with a delta correctly calculated from the BS model, but the volatility is stochastic, the hedging error will impact the P&L (Carr et al., 2002; Nandi & Waggoner, 2000). If one further relaxes the continuous hedging assumption of the BS model, the P&L of a delta-hedging strategy is said to suffer from path-dependency, which will be elaborated on in more detail in the next section.

Variance swaps

Returning to equation (16) and the notion that VIX is equal to the IV, it is evident that the pricing of variance swaps is essential when trading volatility (Martin, 2017). A variance swap is a derivative which allows a trader to speculate in future RV against IV, where the payoff of a variance swap contract is given by:

$$\left(\sigma_r^2 - K_{vol}^2\right) \cdot Variance\ notional\ (N) \tag{23}$$

In equation (23) σ_r^2 is the realised variance, K_{vol}^2 is the annualised variance delivery price and N is the notional amount per annualised volatility point squared. It is clear that the swap's payoff is linear in its variance, but convex in volatility. Buying a variance swap indicates that one is long in the volatility at the strike level of the variance and vice versa for selling a swap. Yet, since they are convex in volatility, the gains are higher from an increase in volatility than a decrease in volatility (Allen et al., 2006). To price a variance swap or replicate it, a trader can hold a portfolio of vanilla options. The weighting is given by:

$$w(K) = \frac{c}{K^2} \tag{24}$$

In equation (24), the price of a variance swap is the price of a replicating portfolio of options with weighting inversely proportionally to squared strike to create a static gamma. The price can then be thought of as a weighted average of the IV of vanilla options (Bossu et al., 2005).

Variance swaps are used to take a directional bet on volatility while simultaneously circumventing the drawbacks actively managing the delta-hedge of options and the path-dependency inherent in volatility through the dollar gamma. As Allen et al. show (2006), short-selling variance swaps has performed significantly better than short (un)hedged straddles. The attractiveness of using variance swaps to trade volatility stems from two properties: 1) the convexity premium and 2) the way in which the theoretical price of a variance swap is calculated. Variance swaps are therefore path independent.

It is theoretically possible to replicate variance swaps, yet in practice this does not happen given that option far OTM are rarely available due to their illiquidity (Martin, 2017). Given that hedges in practice with near- or ATM options occur, a short-tail risk exposure is created given that the true value of the variance swap is underestimated. The negative volatility risk premium creates a willingness for investors to buy swaps for protection, but there is a large downside to selling variance swaps. This downside can be illustrated through the properties of variance swaps: they cannot be hedged and priced insofar the underlying jumps. Further, the unrealised volatility of the underlying poses a risk to the strike of the variance swap, indicating the variance swaps are not model-independent since the fair movement of the volatility affects the strike and through this, the price (Carr & Lee, 2009; Chriss & Morokoff, 1999).

These risks are the main reason behind why variance swaps are capped to reduce the exposure given that if the underlying goes bankrupt before expiry, the swap's payoff is infinite. Since variance is measured on a close-to-close basis, a market-maker can replicate the variance-swaps payoff through a large enough dynamically hedged delta-portfolio, which is why this master's seeks to explore the viability of trading volatility through delta-hedged options rather than variance swaps (ECB, 2007). Nonetheless, it must be noted that an argument can be made for volatility trading strategies in essence trading variance rather than volatility given variance's additive nature and that the delta-hedged option's P&L depends on the return's square due to the option's convexity, wherein the variance actually captures all IVs at all stock prices (Bennett & Gil, 2012).

3.2.2 Delta-hedging scenarios given different hedging and volatility scenarios

Delta-hedging is a widely used tool by market makers and an integral part of trading volatility through options. However, once one starts relaxing the assumptions of the BS model, it becomes more complicated to hedge without incurring large implications to the P&L of the volatility trading strategy. This is why this thesis will now consider delta-hedging more in detail.

Delta-hedging can occur under three different volatility scenarios, namely whether IV is higher than RV, lower than RV or equal to the RV. The daily option P&L is affected by which volatility scenario the trade occurs within and the P&L can be given by:

$$Daily\ P\&L = \frac{1}{2}\Gamma S^2\left[\left(\frac{\Delta S}{S}\right)^2 - \sigma^2\Delta t\right] \tag{25}$$

In equation (25), under the scenario wherein the IV equals the RV, one would have an expected profit of zero. For the scenario where IV is higher than RV, an option seller would earn money through the time decay of options whereas the option buyer would lose money. The reverse case holds for the seller and buyer if the IV is lower than the RV.

Delta-hedging can occur under unknown or known volatility and under discrete or continuous hedging scenarios (Bennett & Gil, 2012). In a continuous scenario with a known future volatility, the P&L would be path-independent and only hinge on the difference between the paid price for the option and the theoretical fair price, i.e., the volatility spread. If the RV in the continuous scenario is unknown, the delta hedge would be based upon the option's IV, resulting in path-dependence from the market's direction.[2]

Given discrete hedging with a known volatility, no error, which would distort the P&L and create path-dependency, in the pricing of the option occurs. Rather, the P&L is only affected by the distortion created from hedging discretely, which itself is independent of the path, but the overall P&L is affected by path-dependency (Bossu et al., 2005). For unknown volatility, the distortion in the P&L would increase further than in the known scenario given that hedging occurs with a "wrong" delta compared to if the volatility were known.

3.2.3 The P&L effect of path-dependency

The main argument behind delta-hedging in a satisfied BS (1973) world is that a portfolio can be hedged by taking the inverse position of the portfolio's delta in stock as evidenced in equation (7) (cf. section 3.1.2). Through continuous delta-hedging with a known constant volatility, which is the main assumption of the BS model, it is possible to gain exposure to volatility and only the volatility as hedging at a constant known volatility, the profit and loss (P&L) would be independent of the underlying's path and value (Sinclair, 2013).

Yet, by just relaxing either one of those assumptions, as it not is not possible that trading can occur continuously in the real world or that the volatility is always known, would result in a stickiness of the P&L as it then depends on the full path and path's history. This means that the P&L is dependent on both the realised

[2] Even if the volatility were unknown and stochastic, it would have no effect in a continuous delta-hedging scenario, as the payoff of a European option depends only on the final stock price.

path and at what point in time the options are hedged, resulting in both the expected P&L and its variance changing. Relaxing the assumption of continuous trading, but keeping the volatility known and constant, will theoretically illustrate how path-dependency arises.

In practice, the delta-hedge's P&L can be described as:

$$Daily\ P\&L = Gamma\ P\&L + Theta\ P\&L + Vega\ P\&L + Other \tag{26}$$

In equation (26), the three Greek sensitivities are shown and a fourth term, "Other". "Other" encapsulates the P&L from entering and financing the reverse delta position, other (higher order) sensitivities such as Rho or Vomma, the change of rate in the vega w.r.t. to a one unit increase in the volatility, interest rates and expectations on dividends (Bossu et al., 2005). Equation (26) can be rewritten as:

$$Daily\ P\&L = \frac{1}{2}\Gamma(\Delta S)^2 + \Theta(\Delta t) + \Upsilon(\Delta\sigma) + Other$$

As illustrated by Bossu et al. (2005), assuming that IV is constant, the daily P&L of a delta-hedge option can be shown to be:

$$Daily\ P\&L = \frac{1}{2}\Gamma(\Delta S)^2 + \theta(\Delta t) \tag{27}$$

Equation (27) can be viewed as the main point of the BS model as it showcases how option prices change in time relative to convexity. From Bossu et al. (2005), one knows that the relationship between gamma and theta is:

$$\theta \approx -\frac{1}{2}\Gamma S^2\sigma^2 \tag{28}$$

Utilizing the above relationship[3], which in essence is an approximation, and assuming that the risk-free interest is zero, one can substitute equation (28) into equation (27) and factorize S^2, which will give one the daily P&L rewritten as equation (25) shown previously:

$$Daily\ P\&L = \frac{1}{2}\Gamma S^2\left[\left(\frac{\Delta S}{S}\right)^2 - \sigma^2\Delta t\right]$$

From equation (25) above, it is evident that the daily P&L of a delta-hedged portfolio is driven by the spread between the first term of the bracket, which can be interpreted as the RV and the IV (second term of the bracket). A break even occurs when the price's volatility development equals the market's expected future volatility. If all the daily P&L's are summed, equation (25) can be rewritten as:

[3] The derivation of the relationship between theta and gamma can be found in appendix #2.

$$Final\ P\&L = \frac{1}{2}\sum_{t=0}^{n} \gamma_t [r_t^2 - \sigma^2 \Delta t] \quad (29)$$

In equation (29), the delta-hedged P&L in a BS world under discrete hedging is illustrated. Here, it is clear that the P&L is dependent on the dollar gamma's (γ_t) development throughout time. It is also evident the P&L depends on when the realisation of the volatility, i.e., at what points over the path does the volatility increase or decrease. This dependence on gamma is formally known as path dependence. Restated, path-dependency's effect on P&L is evident when the volatility spread is low while the dollar gamma is high or if the volatility spread is high and the dollar is gamma is low (Gatheral, 2006).

To build on this, there is an inverse relationship between (implied) volatility and gamma in general. When the gamma increases, the volatility decreases and vice versa, which is because of the higher sensitivity in the delta given lower volatilities. As noted, there is a negative volatility risk premium, which theoretically can be gained by being short in options and delta-hedging the position. As evidenced with the Greeks (cf. section 3.1.5), a short option position results in an overall negative portfolio gamma. This means that a short position is essentially a bet on the RV being lower than the IV under which the option was written over the option's lifetime, whereas the reverse holds if the position is long.

Delta-hedging a long position, either discretely or continuously, can be thought of as gamma scalping (Bennett & Gil, 2012). Comparing equation (7), the delta-hedge in BS, to equation (25), the overall P&L of an option, illustrates that the overall P&L depends on two terms' size:

1) $\frac{1}{2}\gamma dS^2$: realised gamma term and 2) $\frac{1}{2}\gamma S^2\sigma^2 dt$: the expected gamma term.

As argued, the P&L of any option trade is primarily driven by the volatility spread. Looking at the two terms above, the P&L can also be restated as the spread between options' returns of the theta relative to the gamma, i.e., the first term is what is realised through gamma whereas the second term is what is implied through theta. A short option portfolio entails that the P&L is primarily driven by the options' time decay (Lu, 2015).

3.2.4 The P&L effect of using different volatilities

A prominent piece of work on path-dependency and delta-hedging has been studied by Ahmad and Wilmott (2005). Their work builds on the work of Carr (2005), who derived the profit expression for hedging with different volatilities. Ahmad and Wilmott (2005) look at whether hedging should occur with the RV or IV and given the fulfilment of all BS' assumptions besides continuous trading and no arbitrage, i.e., options can be mispriced in the market. They assume that RV is higher than IV, which a trader can profit from by taking a long position in a call and delta-hedging. The findings suggest that hedging with IV mark-to-market results in a deterministic day-to-day P&L, but the present value of the P&L is path dependent. This means that the P&L is always positive and increasing, but the end-result is random. The maximum achievable P&L is at when the dollar gamma, through the drift in stock, is at its highest, which occurs when the stock is close to ATM, consistent with the theoretical explanation provided by Bossu et al (2005). The mark-to-market profit can be calculated as:

$$dV^i - \Delta^i dS - r\left(V^i - \Delta^i S\right)dt \tag{30}$$

In equation (30), the first term represents the change in option value, the second term is the change in value of the delta-hedge and the final term is the interest from the cash-position. Equation (30) can be rewritten to:

$$\Theta^i dt + \frac{1}{2}\sigma^2 S^2 \Gamma^i dt - r\left(V^i - \Delta^i S\right)dt$$

And further simplified to:

$$\frac{1}{2}(\sigma^2 - \bar{\sigma}^2)S^2\Gamma^i dt \tag{31}$$

In equation (31), there is no random term, which confirms that the profits are deterministic. Furthermore, one will make a profit as long as RV ends up being higher than IV, regardless of whether the RV differs from the original estimate of RV used for calculating the delta.

For RV, the findings suggest that the hedge results in a replication of a correctly priced BS option, which means that on a mark-to-market basis, losses can be incurred, but at expiration, the P&L will be equal to the difference between the market option and the replicated BS option, i.e., the P&L is always known. In conclusion, their findings suggest that the expected P&L does not depend on the volatility input used to hedge. The mark-to-market profit is the same as with the IV, i.e., equation (30):

$$dV^i - \Delta^a dS - r\left(V^i - \Delta^a S\right)dt$$

One knows that a correctly valued option would have the following P&L:

$$dV^a - \Delta^a dS - r(V^a - \Delta^a S)dt = 0 \tag{32}$$

Which allows one to rewrite equation (32) the mark-to-market profit as:

$$\begin{aligned} dV^i - dV^a + r(V^a - \Delta^a S)dt - r\left(V^i - \Delta^a S\right)dt &= dV^i - dV^a - r\left(V^i - V^a\right)dt \\ &= e^{rt} d\left(e^{-rt}\left(V^i - V^a\right)\right) \end{aligned} \tag{33}$$

Taking the present value of equation (33) and using the integral, to get the profit from time 0 to maturity, gives one the equation for total profits:

$$e^{rt_0}\int_{t_0}^{T} d\left(e^{-rt}\left(V^i - V^a\right)\right) = V^a - V^i \tag{34}$$

In equation (34), this confirms that the profit is guaranteed, but how it is achieved will vary. It should be noted that under continuous hedging, the profit will always yield equation (31), but under discrete hedging the final profits can vary slightly.

This study is of significant relevance to this master's thesis, as it showcases how the expected P&L is affected by the type of volatility used. It is therefore of interest to study how the variance in the P&L is affected by the volatility input and implicitly, as suggested by the strategies in using either RV or IV, investigate the ease with which volatility arbitrage can be exploited.

Delta-hedging's effectiveness has empirically been investigated by Clewlow and Hodges (1997). Their work focuses on the optimal delta hedge given the presence of a fixed and proportional transaction cost, which builds on the work by Hodges and Neuberger (1989) who only explored delta-hedging given a proportional transaction cost. Their findings suggest that for hedging strategies, the most important characteristic is hedging based on a band of possible 'incorrect' deltas through IV rather than the 'correct' delta given by actual volatility. This will result in a higher average utility, and through this, a higher average P&L, given the presence of transaction costs. Relating these findings to those of Ahmad and Wilmott (2005), the average P&L differs depending on the delta, but the expected P&L stays the same, consistent with Ahmad and Wilmott's (2005) findings.

Their study is of relevance to this master's thesis as given transaction costs; delta-hedging incurs a penalty for each hedge and the size of the hedging error depends on the frequency of hedging. The hedging error can be minimised by increasing the hedging frequency, as noted by Black-Scholes (1973). Thus, it is of interest to investigate how much the variance in the expected P&L varies given different hedging frequencies, but we do not focus on a transaction perspective, but rather how the hedging frequency minimises the hedging error.

3.2.5 Delta-hedging and model risk

The previous sections have outlined how a trader can take advantage of having a more accurate forecast of future volatility than the market and how relaxing the assumptions of the BS model, to better reflect a real trading scenario, will impact the P&L of an option trading strategy. However, even if the estimate of volatility at maturity is correct, the final option P&L is also affected by the ability to specify the correct volatility model throughout the lifetime of the option i.e., model risk, which will have an impact on the delta position and the ability to accurately hedge the fluctuations in the option price.

Delta-hedging is said to suffer from significant model risk (Carr et al., 2002; Nandi & Waggoner, 2000). To replicate the example from Ahmad and Wilmott (2005), one could hedge with the assumed correct estimate of future RV at every timestep. If the actual volatility process is stochastic, then, whenever the IV differs from the future RV, the IV-based delta will differ from the true model delta. This will cause a hedging error at each intermediate time-steps, which can either positively or negatively impact the P&L, increasing the variance of expected P&L. Even if one could specify a stochastic process for volatility, it is still unlikely that this process will reflect the actual volatility process (ibid.)

However, existing literature suggest that using more complicated models with sophisticated rebalancing schemes does not lead to substantially better results compared to using the classical BS model (Nandi & Waggoner, 2000). This is due to the difficulties in estimating volatility, even with a complex model, due to the characteristics of volatility (cf. section 3.1.6). The same applies to minimizing the mean squared hedging error, which does not substantially improve performance (Primbs & Yamada, 2008).

3.2.6 Delta-hedging in non-constant volatility environments

Once we introduce a stochastic volatility model as described in section 3.1.6, we can no longer hedge the option using only risk-free asset and stock, hence we need an additional option to do so. This is due to there being two sources of uncertainty, the volatility multiplied by the first Brownian motion and the volatility multiplied to the second Brownian motion. This means that hedging with stochastic volatility is further complicated as the option we use to hedge would be dependent on the price of the option we are trying to hedge and vice versa, which implies that there is not a single arbitrage free price anymore (Hull, 2009). Since there is no analytical solution for the price of the option as the conditional distribution of S_T cannot be derived, there is no closed-form solution for delta either.

Failing to correctly specify the underlying model has been shown to lead to model error and the use of an incorrect hedge. In a study by Branger & Schlag (2004) they show that applying a BS delta to a stochastic volatility process will lead to significant hedging error. The sign of the error will depend on the difference between the partial derivative in the Heston model and the partial derivative in the BS model, which is related to the slope of the IV function. Model error is often a problem in empirical studies since the true data generating process is unknown, which is also likely to be an issue for this thesis, since we will apply BS based Greeks to a non-constant underlying volatility.

In a paper by Renault & Touzi (1996) they compare the performance of a delta hedge based on the IV from the BS model and the delta-hedge using the Hull & White (2017) stochastic volatility model. They assume that IV of the options is always higher than the RV, which they assume can be approximated by the Hull & White model. They prove that using IV as a constant, results in a delta-hedging bias, which can be defined as:

$$\Delta_t^{BS}(x, \sigma^i) - \Delta_t^{SV}(x, \sigma) \rightarrow \sigma^i > \sigma$$

They further prove that when $\rho = 0$, using the BS IV to calculate the delta for an ITM option leads to an under-hedged position and the use of BS IV to calculate the delta for an OTM option leads to an over-hedged position. This is evident from the following results:

$$\Delta^{BS}(x, \sigma^i(x, \sigma)) \leq \Delta^{SV}(x, \sigma)$$

$$\Delta^{BS}\big(-x, \sigma^{i}(-x, \sigma)\big) \geq \Delta^{SV}(-x, \sigma)$$

$$\Delta^{BS}\big(0, \sigma^{i}(0, \sigma)\big) \leq \Delta^{SV}(0, \sigma)$$

To fully understand their findings, it is important to note that they assume IV to always be higher than the RV, which is in line with empirical findings (Christensen & Prabhala, 1998). If one where to assume the opposite i.e., RV is higher than IV, the dynamics would likely change, meaning that an ITM option would be over-hedged, and an OTM option would be under-hedged, which this thesis will investigate.

From section 3.1.6, it is evident that ρ has an important impact on the distribution of the stock path and the stock price distribution in a stochastic volatility environment. In a paper by Kurpiel & Roncalli (2011) they find that when correlation ρ is strongly negative, $\rho < -0.5$, using a BS constant IV-based delta leads to a systematically under-hedged position and when correlation is strongly positive $\rho > 0.5$, it will lead to a systematically over-hedged position. They describe this relationship by the following equations:

$$p \to -1 \to \Delta^{BS}\big(x, \sigma^{i}(x, \sigma)\big) \leq \Delta^{SV}$$

$$p \to +1 \to \Delta^{BS}\big(x, \sigma^{i}(x, \sigma)\big) \geq \Delta^{SV}$$

They argue that this difference stems from the fact that when $\rho < 0$, the BS model tends to overestimate the price of OTM call options and underestimate the price of OTM put options. They further argue that this is because when the stock price increases, volatility tends to decrease, making it less likely that really high stock prices will be achieved and when the stock price decreases, volatility tends to increase, making it more likely that really low stock prices will be achieved (ibid.).

Kurpiel & Roncalli (2011) continue to test the above relationships. They find that for the simple delta-hedge, when p = 0, the hedging bias between using the BS constant IV and the stochastic volatility model's RV is rather small. Interestingly, they find that when ρ is extremely high, i.e., close to 1, or extremely low, i.e., close to -1, the BS IV delta-hedge may outperform the stochastic hedge, as some of the over- and under-hedging partly corrects for the lack of a volatility hedge.

They go on to test a delta-vega hedge and finds that there is little difference between using the BS constant IV and the stochastic volatility model's RV. These results are further supported by Bajeux-Besnainou and Rochet (1996) who prove that in a stochastic volatility context with p = 0 the hedging problem can be solved through a delta-vega neutral hedging strategy.

3.2.7 Empirical studies on delta-hedging and the performance of volatility trading strategies

A study by Goyal and Saretto (2009) examined the delta-hedged returns over a 10-year period on the S&P500 index. They find that the performance of a strategy where a call is sold and delta-hedged exhibits lower returns than expected even if the IV was higher than the RV compared to a sold vanilla straddle, which suggests anomalies in the pricing of options. They seek to provide an explanation for this abnormal return by pointing out that a straddle will benefit from a mispricing in both the put and call whereas the delta-hedged call only benefits from the mispriced call. The anomalies in the ineffectiveness of delta-hedging options are further explored by Jones and Shemesh (2011) who find that 'weekend effect', a phenomenon wherein the returns over the weekend are lower compared to the previous week, is persistent in delta-hedged option position and in implied volatilities. Theoretically, they argue that delta-hedged options should not be affected by the effect given that their exposure to it should be eroded through the hedge in the stock, yet, their findings suggest that this is due to the IV having a negative, counterintuitive effect on options over the weekend compared to the general notion of a positive relationship between increasing volatility and option premiums.

These two studies highlight the importance of understanding why option markets and delta-hedging might not behave as implied by the option pricing theory. Thus, it is of interest to further explore the effect that volatility has on options and in the way with which it can be reaped while simultaneously exploring the effect of the used delta's ability to capture volatility.

Within delta-hedging studies, both De Giovanni, Ortobelli and Rachev (2008) and Savickas (2005) explore the effectiveness of delta-hedging given different option pricing models. The results from De Giovanni et al. (2008) suggest that several subordinated models perform better than the original BS model due to their effectiveness at minimising hedging costs. They do not examine the returns associated with delta-hedging whereas Savickas' (2005) suggests that neither the Weibull, BS or gamma model effectively capture the skewness in the distribution of the return.

For this master's thesis, both studies showcase the need to critically evaluate the option pricing model and its inputs as to how each parameter will behave insofar it is changed. Thus, these studies will be used to critically assess and stretch the limitations of the BS model.

In relation to trading volatility, Fieldhouse (2016) argues that markets are not normally distributed as assumed within the BS world, but rather exhibit a left-tail risk, i.e., markets experience larger downturns more often than what is implied by the normal distribution. In addition, he argues that this left-tail risk, if actively traded, should not be hedged, but rather be timed through the use of selling delta-hedged options to trade volatility as the returns are abnormal. This is consistent with the findings of Fleming, Kirby and Ostdiek (2001) who showcase that timing volatility on the S&P500 improves the returns of short-horizon delta-hedged options.

These two studies highlight an important notion that will be explored later on in this master thesis, whether an indicator can be built to time volatility and how timing volatility theoretically could improve the returns of trading volatility with delta-hedged options.

3.3 Theoretical framework

Volatility trading can be undertaken in various forms and the effectiveness of the trade depends on the assumptions made within each model, as illustrated by literature. An argument explaining the divergent result in past studies conducted on volatility trading strategies is that different strategies must be measured on different terms through different models and assumptions. Instead of modelling yet another complicated volatility process, this thesis takes a practical approach and seeks to explore the effectiveness of the BS model given a relaxation of assumptions and how the realised P&L varies from the expected P&L. This will allow us to contribute to the application of the BS model, which is the backbone of most asset option pricing (Hull, 2009).

Inspired by previous research and the outlined literature, this thesis will first outline how path-dependency arises in volatility trading given the inherent dollar gamma exposure in delta-hedged options to the underlying. After this, the thesis will replicate the Ahmad & Wilmott (2005) study to showcase how the expected P&L can be calculated and what dynamics drive the P&L when hedging respectively with the RV and IV. The theoretical findings from these simulations will be combined with previously outlined theory as additional simulations will then be conducted. The subsequent simulations will focus on a simple stochastic volatility process rather than a complex stochastic volatility model, as this allows us to illustrate the same dynamics while still veering within the BS world. This combination will allow us to have certain expectations on what to expect when simulating different volatility environments, hereunder how we wish to contribute to the dynamics explored by Kurpiel & Roncalli (2011) and Renault & Touzi (1996), notably the notion of being over or under-hedged when hedging with an IV-based hedge.

The findings from the simulations allow us to understand the dynamics that may arise when trading volatility with delta-hedging in practice, which is why the thesis will utilize both the theoretical arguments and findings to understand how volatility trading works in practice. Some assumptions from the BS world will still be kept intact, namely that markets are frictionless and that there are no transaction costs. We wish to emphasise how trading volatility performed empirically and not just how it should perform theoretically, which is why this thesis will backtest a volatility trading strategy on the S&P500 index. The goal of this backtest is twofold: 1) to showcase whether the theoretical findings are also evidenced in practice and to be in line with previous studies (De Giovanni et al., 2008; Nandi & Waggoner, 2000), and 2) to provide new avenues for exploring how volatility can be timed and even though past performance is never indictive of future performance, whether there are some learnings from theoretical simulations that can be applied to the empirical part.

3.4 Subset

This literature review has presented several aspects of the foundations of option pricing. It showcased how options were priced based on the BS model and the impact of option Greeks. An exploration of the challenges and extensions of the BS model was also conducted. It has also presented several aspects of how volatility trading can occur and the associated implications of undertaking the different positions. It further provided a review of how options option's P&L is calculated and affected by volatility. An exploration of the model risks inherent in volatility trading models was also explored in combination with an empirical review of volatility trading strategies' effectiveness. In sum, this chapter provides the theoretical framework for answering the research question.

4. Methodology

This thesis aims at investigating the P&L effect of hedging with IV vs. RV when relaxing various assumptions of the BS. It further investigates the performance of a short volatility position in various volatility environments when continuously selling delta-hedged puts and calls and how left-tail risk can be managed by applying a option pricing indicator. First, this thesis will theoretically demonstrate the effects at play and subsequently, it will empirically test the predictions on real option data.

4.1 Option simulations

To test the P&L implications of using various hedging volatilities and relaxing some of the BS assumptions, this thesis will start off by using simulated data. This provides us with a controlled environment where we can better isolate the individual P&L effects, which will help us analyse and understand the empirical results. The simulations will be split into three parts, whereby part 1 focuses on illustrating path-dependency, part 2 focuses on replicating the Ahmad & Wilmott (2005) study, and part 3 focuses on delta-hedging with a simple stochastic volatility process.

4.1.1 Part 1: Illustrating path dependency

First, this thesis will start off by illustrating the path-dependency of the P&L when the continuous hedging assumption is relaxed. This part is mainly motivated by the theoretical findings of Bossu et al. (2005). The primary goal is to showcase how the P&L is affected by the path-dependency incurring through the gamma exposure. Here we assume that the underlying acts according to the BS assumptions and that the stock path follows a GBM, which is defined by:

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

The equation we use to simulate the stock path in RStudio is given by solving the GBM:

$$S_{t+dt} = S_t e^{\left(\mu - \frac{\sigma^2}{2}\right) * dt + \sigma * \varepsilon * \sqrt{dt}}$$

We define the discrete time-interval as *dt*, which governs the simulation frequency, and simulate a random variable N_1 for each path, which is turned into the matrix ε. Due to the random variable-matrix ε, the only source of uncertainty is the path of the underlying asset. The GBM assumes that the parameter $\rho = 0$, meaning that an increase in volatility is not correlated with changes in the stock price.

In accordance with the BS world, we assume that the expected return μ, dividends q, and the volatility σ^{RV} are constant, and that they take the value $\mu = 0.05$, $q = 0$ and $\sigma^r = 20\%$. We further assume that the options are priced using $\sigma^{IV} = 20\%$, i.e., the IV is equal to RV. If we were able to hedge continuously, the P&L here should be equal to the replicating portfolio. We simulate a total of 50 stock paths, whereby we sell a put option for each path with a maturity of 12 months, meaning that $T = \frac{12}{12} = 1$ and delta-hedge to make the position delta-neutral. Yet, we want to model the path-dependency insofar we cannot hedge continuously, which is why we will illustrate path-dependency by hedging 252 times (once a day) and 1008 times (4 times a day) to see how the hedging frequency affects the P&L in a nearly perfect BS world. The strike is the fair forward, calculated as:

$$F_0 = S_o * e^{r*\left(\frac{t}{252}\right)} \tag{35}$$

For each time-step, we simulate the stock price, calculate the option price and delta using the BS equation (cf. section 3.1.2 and 3.1.3), and calculate the daily and cumulative P&L.

4.1.2 Part 2: Replicating the Ahmad & Wilmott (2005) experiment

Secondly, we replicate the Ahmad and Wilmott (2005) hedging experiment, which demonstrate the differences in P&L when hedging with IV relative to RV. Such as in the original experiment and part section 4.1.1, we utilise a GBM to simulate the stock paths in RStudio.

In accordance with the BS world, we assume that the expected return μ, dividends q, and the volatility σ^{RV} are constant and that they take the value $\mu = 0.05$, $q = 0$ and $\sigma^{r} = 20\%$. We further assume that the options are priced using $\sigma^{IV} = 0.3$, which is higher than the RV used to simulate the stock paths.

We simulate a total of 50 stock paths, where we sell a put option for each path with a maturity of 12 months meaning that $T = \frac{12}{12} = 1$ and delta-hedge to make the position delta-neutral. We run the simulation, first hedging with IV, which is the volatility used to price the option, and secondly, hedging with RV, which is the volatility used for the GBM. To demonstrate the importance of continuous hedging, we run the simulations both using 252 time-steps, i.e., re-hedging once a day, and 1008 time-steps, i.e., re-hedging four times a day. The strike is calculated as the fair forward in equation (35) per Nandi and Waggoner (2000).

For each time-step, we simulate the stock price, calculate the option price and delta using the BS equation (cf. section 3.1.2 and 3.1.3), and calculate the daily and cumulative P&L. The P&L from the delta-hedge put option position consists of three components; the change in option value, the change in value of the delta-hedge and the interest paid on the cash-position. The daily P&L is calculated using the methodology from Ahmad & Wilmott (2005), which can be described by the following equations:

$$P\&L_{option,t} = V_t - V_{t-1} * Position$$

$$P\&L_{delta,t} = (S_t - S_{t-1}) * \Delta_{t-1} * Position$$

$$P\&L_{interest,t} = r * dt * (\Delta_{t-1} * S_{t-1} - V_{t-1}) * Position$$

The total P&L at option expiry is simply the sum of all the daily P&Ls.

4.1.3 Part 3: Introducing a simple stochastic volatility process

In the last simulations section, we introduce a simple stochastic element to our volatility process. This is done to assess the P&L implications of relaxing the BS' models constant volatility assumption and the P&L impact of an unanticipated change in volatility, with sudden increases in volatility representing the left-tail risk of selling options (3.2.3 and 3.2.7).

Similar to part 2, we assume that the underlying asset behaves according to the BS assumptions, with the exception of constant volatility, and that the stock path follows a GBM with a minor modification. Instead of the GMB equation as it was stated in part 1, we introduce a volatility vector $\tilde{\sigma}$, which is non-constant and exhibits some stochastic features. The GBM is defined by:

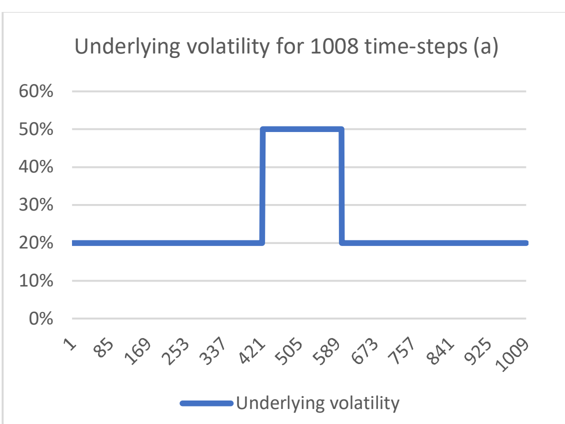
$$dS_t = \mu S_t dt + \tilde{\sigma} S_t dB_t$$

Furthermore, we assume that the expected return μ and dividends q are constant and that they take the value $\mu = 0.05$ and $q = 0$. The hedging frequency is defined as *dt* and this thesis will perform the simulations using a hedging frequency of once per trading day i.e., $dt = 1/252$ and four times per trading day i.e., $dt = 1/1008$. Contrary to the Heston model (1993), where the volatility process is governed by a drift component and a random component (cf. equation (21), also shown below), this thesis takes a more lenient approach and defines volatility as a vector for easy implementation into the GBM. For the hedging frequency of once per trading day, the vector takes the following values (see Figure 5 (b)):

$$\tilde{\sigma}^{RV}_{t=0}, \dots, \tilde{\sigma}^{RV}_{t=100} = 0.2\ , \tilde{\sigma}^{RV}_{t=101}, \dots, \tilde{\sigma}^{RV}_{t=144} = 0.5\ , \tilde{\sigma}^{RV}_{t=145}, \dots, \tilde{\sigma}^{RV}_{t=252} = 0.2$$

For the hedging frequency of four times pr. trading day, the volatility vector takes the values (See Figure 5 (a):

$$\tilde{\sigma}^{RV}_{t=0}, \dots, \tilde{\sigma}^{RV}_{t=424} = 0.2\ , \tilde{\sigma}^{RV}_{t=425}, \dots, \tilde{\sigma}^{RV}_{t=599} = 0.5\ , \tilde{\sigma}^{RV}_{t=600}, \dots, \tilde{\sigma}^{RV}_{t=100} \ = 0.2$$



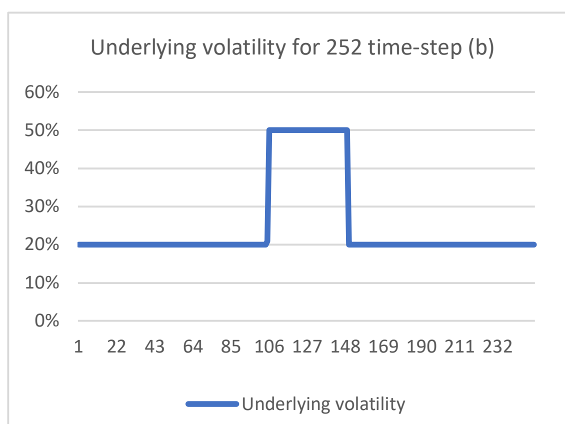


Figure 5: Plot of the underlying volatility used for the modified GBM: (a) is the volatility for 1008 time-steps, (b) is the volatility for 252 time-steps

As researchers, we are aware that this is a rather unusual approach. However, it is argued to capture the same effect as introducing stochastic volatility, but in a much more simplistic manner, making it easier to implement and analyze. Furthermore, it allows us to better simulate an event that exposes our strategy to significant left-tail risk, which is the feature of interest.

$$d\sigma_t^2 = \kappa(\sigma_{LR}^2 - \sigma_t^2)dt + \gamma\sigma_t dB_t^\sigma \quad (21)$$

Although many of the assumptions are similar to part 1, we now have two sources of uncertainty: the underlying and the non-constant volatility. Again, we simulate a total of 50 stock paths, selling one put option for each path with a maturity of 12 months and delta-hedge accordingly. Theoretically, a call should yield the same P&L and return as the put, which holds true when performing the simulations, which is why we only simulate a put. As before, we simulate the stock price, calculate the option price and delta using the BS equation, holding the strike fixed at the fair forward at $t = 0$ (cf. section 3.2.1) and calculate the daily and cumulative P&L for each time-step. The P&L from the delta-hedge put option position consists of three components: the change in option value, the change in value of the delta-hedge, and the interest paid on the cash-position, which is calculated as in section 4.1.2. An example of the RStudio code used can be found in appendix #3.

Since the introduction of the stochastic volatility element is the main interest of this thesis, we run a number of different simulations to better illustrate the various P&L effects we may encounter in the empirical results. First, we simulate the general case, where $\tilde{\sigma}^{RV} = \tilde{\sigma}^{IV}$, to see the effects from a stochastic change in volatility. Secondly, this thesis will compare the scenario where one adjusts the delta at each time-step to the stochastic increase in volatility (see equation (36)) relative to using a constant volatility input into the delta-hedging equation (see equation (37)). More specifically, we maintain $\tilde{\sigma}^{RV}$ as above but introduce a constant hedge volatility $\sigma^\Delta = 20\%$ to see the effect of a hedging error.

$$d_{1,t}^{\tilde{\sigma}} = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\tilde{\sigma}_t{}^2\right)(T-t)}{\tilde{\sigma}_t\sqrt{T-t}} \quad (36) \quad , \quad d_{1,t}^{\sigma} = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma_t^{\Delta^2}\right)(T-t)}{\sigma_t^\Delta\sqrt{T-t}} \quad (37)$$

In equation (36), we are essentially introducing a time-varying volatility, which adds a second uncertain parameter in addition to the underlying, which will in turn lead to a different delta compared to equation (37). This can be seen from equation (38), as d_1 is used as an input to calculate delta.

$$\Delta_{Call} = N(d_1)\ ,\Delta_{Put} = -N(-d_1) = N(d_1) - 1 \quad (38)$$

By introducing a time-varying element in delta and comparing it to the case of using a constant volatility, we are able to evaluate the effect of trying to adjust to an unexpected change in volatility. Relating it to section 3.2.6, we assume that the volatility of the first term is now lower than volatility in the second term and the delta-hedging bias would be defined as:

$$\Delta_t^{BS}\left(x, \sigma_t^{\Delta}\right) - \Delta_t^{SV}(x, \tilde{\sigma}_t^{RV}) \rightarrow \sigma_t^{\Delta} < \tilde{\sigma}_t^{RV}$$

According to Renault & Touzi (1996), since $\sigma_t^{\Delta} < \tilde{\sigma}_t^{RV}$, an ITM option would lead to an over-hedged position, and an OTM option would lead to an under-hedged position, with the P&L implications of this being elaborated upon in the results section.

This thesis will further experiment with the timing and length of a stochastic change in the volatility process by comparing the P&L effects of a short-term increase in volatility relative to a long-term increase in volatility. This is done to simulate a scenario where an increase in volatility could happen, but we do not know for how long; this will help us demonstrate the importance of reducing left-tail risk for our short delta-hedged option strategy. Furthermore, this thesis will introduce a period of lower volatility, to assess the P&L effect of a change in favour of the short volatility trade. Furthermore, we will introduce an increase in volatility without a subsequent decrease, to assess the impact on short-term options, where volatility might not return to the mean before expiry. Finally, we draw inspiration from Ahmad & Wilmott (2005) and introduce an IV vector, which is higher than RV defined as $\tilde{\sigma}^{IV} = (\tilde{\sigma}^{RV} + 0.1)$ in section 5.1.3.3, to analyse how selling an expensive option is affected by unexpected changes in the underlying volatility. For all of the aforementioned simulations, the P&L will be calculated similar to that of section 4.1.2.

4.2 OptionMetrics

This section seeks to outline the data collection process, sample, and limitations of the data used to conduct an empirical backtest of trading volatility on the S&P500 over a 13-year period. The back test will be performed for both one call and one put with around 62 trading days until expiry, i.e., three months until expiry, and delta-hedging with either the IV of the option or the 30-day TV of the underlying. We assume no transaction costs (as per the simulations), but our empirical results incorporate the effect of dividend payments on the S&P500 to replicate the real world more correctly.

4.2.1 Data collection

The empirical backtest results are based on secondary quantitative data gathered through OptionMetrics. OptionMetrics is the most extensive financial data-provider on options, covering all listed options on the major indices and several companies (OptionMetrics, 2021), which is the reason why we have chosen OptionMetrics as the main source for options data. In terms of reliability, as the data-points are measured at the end of the day, our results can be replicated by other studies, which ensures that the empirical results are reliable (Saunders et al., 2009).

By using secondary data, we are able to conduct empirical research which is observational. Since the data collected is over a time-period from the 22nd of January 2007 until the 16th of October 2020, our research is conducted through a longitudinal study. The advantage of a longitudinal study is that it allows us to conduct research where we can, to some extent, establish the cause-and-effect relationship between volatility trading's profitability and the options' Greek. Previous research has considered the effectiveness of different models to trade volatility. The main focus of this thesis' empirical section is to tie together the simulated results with the effectiveness and appropriateness of using the BS model to evaluate and trade volatility.

As we are only selling one pair of options per expiry, we must identify a pair of options sold at the same strike at time t. To pick the most appropriate options that mimic our simulated options' characteristics, we must choose a pair which were struck or trading at the implied fair forward with the smallest delta, as per our simulations. In practice, this is a multistep process. First, we must calculate the fair forward for the specific day. In order to do this, we must first find the spot rate on the S&P500, which we do by pulling its closing price from Yahoo Finance over the entire period. Then we must find the relevant interest rate. The interest rate used to calculate the fair forward (and used within our empirical results when striking options), is the 3M treasury yield quoted on an annual basis, where we pull data for each day over the period from the U.S. Department of the Treasury (USDT, 2021). We then calculate the fair forward for each possible day utilizing equation (35).

Now we are able to identify the options which we will use, leading us to pulling an entire year of traded European options on the S&P500 from OptionMetrics. Starting on the 17th of January 2007, we then select the options that are trading closest to our fair forward, which we equate as the strike as per OptionMetrics conventions too, with an expiry in approximately three months from the date we start on. If multiple pairs of options exist at the same strike, we take the pair of options with the lowest delta, as this equates to being closest to ATM. After having identified the pair of options, we pull their entire end of day characteristics from t until T, such as in the example of the first pair, from the 22nd of January 2007 until the 21st of April 2007. The characteristics of each option are: optionid, date, exercise date, strike, best bid, best offer, implied volatility, delta, gamma.

In order to determine the price of the options, we use the mid-price, given the bid and ask of each option. If an option is missing an IV (and thus delta), we solve this missing observation using BS, given that we know the price. Furthermore, we also pull the 30-day trailing RV, based on calendar days on the S&P500, as an alternative hedging volatility, which we approximate to being close to the true RV in order to have a RV in line with our simulations. In addition to the 30-day TV, we also pull the 90-day trailing RV, which we use to recalculate the price of the option at inception, based on its characteristics, with only the volatility changing (from IV to 90-day TV) to approximate what we expect to earn on each option trade. Finally, we collect data on the trailing 12-month continuous dividend yield of the S&P500 on a monthly basis from the Quandl database for every option (Quandl, 2021). After a pair of options is held until expiry, we repeat the process of gathering the data as per the aforementioned method.

4.2.2 OptionMetrics sample

The final sample is composed of 55 different calls and 55 different puts, all identified as a pair spanning the 22nd of January until the 16th of October 2020; this equates to 3419 unique observations for the calls and 3419 for the puts. For the calls, we had to solve 8 missing volatilities and deltas (primarily towards the end of the option's lifetime) and for the puts 120 missing observations.

Table 2 provides an overview of the summary statistics for the respective calls and puts on the strike, price, best bid, best offer, price, IV, delta, gamma, spot and RV.

Summary statistics for puts

Strike	Price	Best bid	Best offer	IV	Delta	Gamma	Spot	RV
Min. : 825	Min. : 0.00	Min. : 0.0	Min. : 0.00	Min. :0.08	Min. :0.00	Min. :0.0001	Min. : 676.5	Min. :0.034
1st Qu.:1305	1st Qu.: 23.70	1st Qu.: 22.9	1st Qu.: 24.50	1st Qu.:0.13	1st Qu.:0.40	1st Qu.:0.0021	1st Qu.:1319.0	1st Qu.:0.093
Median :1740	Median : 47.80	Median : 47.0	Median : 48.80	Median :0.17	Median :0.54	Median :0.0031	Median :1792.5	Median :0.133
Mean :1865	Mean : 59.76	Mean : 58.8	Mean : 60.72	Mean :0.20	Mean :0.55	Mean :0.0033	Mean :1872.2	Mean :0.168
3rd Qu.:2375	3rd Qu.: 78.58	3rd Qu.: 77.4	3rd Qu.: 79.85	3rd Qu.:0.22	3rd Qu.:0.74	3rd Qu.:0.0041	3rd Qu.:2399.3	3rd Qu.:0.199
Max. :3325	Max. :385.10	Max. :384.8	Max. :385.40	Max. :2.01	Max. :1.00	Max. :0.0286	Max. :3580.8	Max. :0.976

Summary statistics for puts

Strike	Price	Best bid	Best offer	IV	Delta	Gamma	Spot	RV
Min. : 825	Min. : 0.00	Min. : 0.00	Min. : 0.00	Min. :0.068	Min. :-1.0000	Min. :0.0001	Min. : 676.5	Min. :0.034
1st Qu.:1305	1st Qu.: 13.40	1st Qu.: 12.90	1st Qu.: 13.90	1st Qu.:0.13	1st Qu.:-0.5948	1st Qu.:0.0021	1st Qu.:1319.0	1st Qu.:0.093
Median :1740	Median : 34.50	Median : 33.70	Median : 35.30	Median :0.17	Median :-0.4312	Median :0.0031	Median :1792.5	Median :0.133
Mean :1865	Mean : 53.31	Mean : 52.37	Mean : 54.25	Mean :0.21	Mean :-0.4400	Mean :0.0034	Mean :1872.2	Mean :0.168
3rd Qu.:2375	3rd Qu.: 62.77	3rd Qu.: 61.95	3rd Qu.: 63.50	3rd Qu.:0.23	3rd Qu.:-0.2486	3rd Qu.:0.0042	3rd Qu.:2399.3	3rd Qu.:0.199
Max. :3325	Max. :1080.35	Max. :1074.10	Max. :1086.60	Max. :2.84	Max. : 0.0000	Max. :0.0272	Max. :3580.8	Max. :0.976

Table 2: Summary statistics for the OptionMetrics data

From the above table, it is evident that the puts have the highest price, which showcases left-tail risk given that it is more likely that huge economic downturns suddenly occur rather than upswings (cf. section 3.2.7). We also see that both calls and options have IVs over 100% at some points, which is a character trait in the data. As the options get closer to expiry, their IV soars, per the numerical search technique within OptionMetrics, which is the reason why we see IV spiking at 284% for puts and 201% for calls (OptionMetrics, 2021). From the summary statistics, it is also evident that the puts experience a bigger span in their IVs than the calls.

4.2.3 Setting up the empirical test

To test our research question, we set up a BS option pricing model in RStudio to calculate the daily option price, the delta, and the daily P&L from the short volatility trade. The code used can be found in appendix #4. We match the OptionMetrics data with the calculation of the strike, the data on dividends, and the risk-free interest rate for each individual option pair. Subsequently, we run the results for both a short delta-hedged put and a short delta-hedged call, using both an IV-based delta hedge and a 30-day TV-based delta hedge. The 30-day TV used for the delta-hedge is the value at t-1, to ensure that the volatility input chosen would also be available in real-life at the time of the trade. Despite the authors being aware that the t-1 30-day TV is backwards looking and therefore cannot be the true forward-looking volatility, it is argued to be a decent proxy for the RV and implemented due to the observability and large focus on practical implications of this thesis (see section 3.1.4).

For the empirical test, we assume that the position is entered and readjusted at close. This is due to the S&P500 data, provided to us by OptionMetrics and linked to the option price, being adjusted closing prices. For the same reason, we use a re-hedging frequency of once per day, as this is the highest frequency data available to us. As previously mentioned, we use the 3M treasury yield quoted on an annual basis as a proxy for the risk-free rate, and we assume that we can borrow and lend freely at this rate in fractional units. Furthermore, we use the 12-month trailing continuous dividend yield of the S&P500 at option maturity, denoted by T, which we assume is constant over the lifetime of the option, and apply it on a daily basis. Previous research suggests that a forward-looking dividend yield must be calculated by using the following formula (Huang, 2019):

$$q = r - \frac{\log\left(\frac{F}{S_0}\right)}{T}$$

Using a forward-looking estimate is argued to be more correct since it is based on observable information at *t=0*. However, it is merely an estimate based on the assumption made for the parameters, including the risk-free rate, and it is therefore also likely to suffer from imperfections. Since the continuous dividend yield is used to calculate the P&L impact of actual dividend payments, real data is argued to be a good approximation, which is why this thesis opts for the simpler approach.

To calculate the return on the short volatility position, we take the daily P&L relative to the holdings in the margin account, which is made up of the initial margin level adjusted for the proceeds of the short delta-hedged option position at t-1. To calculate the final P&L, we compound the daily returns to get a three-month return equivalent to the holding period. According to the CBOE margin manual, the initial margin requirement of a naked short option position can be calculated using the following equations (Hull, 2009):

$$Call\colon M_t = Max(C_t + \alpha S_t - (K - S_t | K > S_t); C_t + \beta S_t)$$

$$Put\colon M_t = Max(P_t + \alpha S_t - (S_t - K | S_t > K); P_t + \beta K)$$

According to CBOE (2000) & Hull (2009), the values for $\alpha \; and \; \beta$ are dependent on the underlying asset and the type of investor trading the option. For a broad index such as the S&P500, they both approximate $\alpha = 15\% \; and \; \beta = 10\%$.

However, when delta-hedging the short option, some of the exposure to the underlying disappears, which is why the margin requirement is slightly different. Relying on the CBOE margin manual, the initial margin requirement on the delta-hedged short call option is zero for the short option and 50% of the long position in the underlying. The initial margin requirement for delta-hedged short put option is zero for the put option, and the short sale proceeds plus 50% of the position in the underlying. Over the lifetime of the option, margin requirements are likely to change dependent on the position of the underlying. However, this thesis takes a conservative estimate of the initial margin requirement and assume it to be constant over the lifetime of the option at 50% of the initial position in the underlying. The margin requirement will be in addition to the proceeds from the sale of the short option, which will serve as a buffer. Looking at our data, our maximum drawdown over the entire sample period does not come close to this margin requirement assumption and it is therefore argued to be a fair estimate. We calculate the margin requirement for each individual option when it is struck and close the position and margin account when the short delta-hedged option expires, before opening a new option with a new maturity and a new margin requirement level. This means that the margin requirement will differ across the various short delta-hedged options, but since we are only interested in the relative percentage return, this will not influence our results.

The P&L calculation of our short volatility trade consists of four components: the P&L from the change in option value, the P&L from the change in delta-hedging position, the interest on the delta-hedging position, and the dividends on the delta-hedging position. We add these up for every time-step and accumulate them over the lifetime of the option. The P&L follows the methodology from Ahmad & Wilmott (2005) and can be described using the following equations:

$$P\&L_{option,t} = V_t - V_{t-1} * Position$$

$$P\&L_{delta,t} = (S_t - S_{t-1}) * \Delta_{t-1} * Position$$

$$P\&L_{interest,t} = r * dt * (\Delta_{t-1} * S_{t-1} - V_{t-1}) * Position$$

$$P\&L_{dividends,t} = Div * dt * (\Delta_{t-1} * S_{t-1}) * Position$$

4.2.4 Limitations of the dataset

Our data has some limitations to it. First, the options might not exactly be struck three months from expiry, but we assume that the day that we observe them at t_0 is the day that they were struck, which creates some inconsistency in the pricing of the option when we solve for missing inputs. However, with 6838 data points, we argue that this will have limited impact on the final results. Furthermore, we do not include transaction costs at all and consider that we have unlimited ability to borrow and lend at the yield, which we believe to be an approximation of the true yield on the day they were struck. We are aware that this assumption is too strong, but it is done to simplify our results with e.g., trading costs likely to vary dependent on whether the trader is a professional investor or a large financial institution. We also assume that we are able to take fractional units in the underlying to hedge with. Parts of the data input used are deemed the most suitable proxy, but does not constitute the true parameter, which holds true for the 30-day TV and the continuous dividend yield. This can obviously influence our results with especially the 30-day TV as a proxy for the true volatility model being problematic. However, it is important to note that the main focus of this thesis is the practical implication of the results, which is why these proxies were chosen. Furthermore, it is argued that minor differences in either dividends or the daily volatility will have an impact, but given the major limitations of the BS option pricing model, these differences will not blur the generalizability of the results. Lastly, we note that there will be some inconsistencies when we solve the missing data inputs based upon the aforementioned limitations.

4.3 The EGARCH model

This thesis will create a volatility forecast using the EGARCH model, based on the daily closing price for the S&P500 index covering a period of 20 years from 01.01.2000 – 31.12.2020, consisting of a total of 5356 observations. The data is readily available on Yahoo! Finance and is obtained as adjusted closing price. The S&P500 is chosen to match the underlying security of the short options, but it also has great statistical properties due to it being highly liquid and highly diversified. The price series of the S&P500 is converted into weekly data, leaving a total of 1110 observations, which will smoothen out some of the noise from daily fluctuations while allowing for a high degree of model accuracy (Enders, 1995). Since the aim is to forecast volatility in the medium-term, weekly data is argued to be the best fit.

Looking at the time plot of the price of the S&P500 index in Figure 6, it is evident that it exhibits a very clear uptrend, which suggests that the price series is non-stationary. The price series is further tested using the ADF-test, which rejects the null hypothesis of non-stationarity. To account for this, we calculate a log return series from the price series, which will be denoted R_t.

Figure 6: Time plot of S&P500 daily closing prices

To determine the mean equation for the EGARCH model, an ARIMA model is estimated using the auto-ARIMA function in RStudio. Both the AIC and the BIC will be reported, but ultimately the BIC will be used for the selection criteria, as it tends to choose a more parsimonious model, preferring predictive power over goodness of fit (Enders, 1995). The ARIMA model will be estimated using the full data set to capture the structure of the data. After running the auto-ARIMA function, we find that using BIC as the selection criteria recommends an ARIMA(0,0,0) and using AIC as a selection criterion recommends an ARIMA(0,0,4). This result is not surprising, given that the BIC generally is more parsimonious and is argued to prefer predictive power over fit (Enders, 1995). For this reason, this thesis will go with the ARIMA(0,0,0) as the mean equation.

To justify the use of the EGARCH model, we perform a visual inspection of the data followed by an ARCH-test. From the time-plot of the weekly log return series (see Figure 7), it is evident that the return series exhibits some form of time-varying volatility. For the ARCH-test, we use σ_t^2 as a measure for volatility, which will be calculated as R_t^2 since the conditional mean is found to be zero. The ARCH-test of order p can be written as:

$$\sigma_t^2 = \alpha_0 + \alpha_1\sigma_{t-1}^2 + \alpha_2\sigma_{t-p}^2 + \cdots + \alpha_p\sigma_{t-p}^2$$

To determine the number of significant lags, we use the PACF of the model residual and find p = 4. Testing the H_0: $\alpha_1 = \alpha_2 = \cdots = \alpha_4 = 0$ of no volatility clustering we get an LM test statistic of 21.56 compared to the $\chi_{0,95}^2$ critical value with 4 df of 9.49, meaning that we can reject the null of no ARCH effects. This definitively justify the use of a GARCH model.

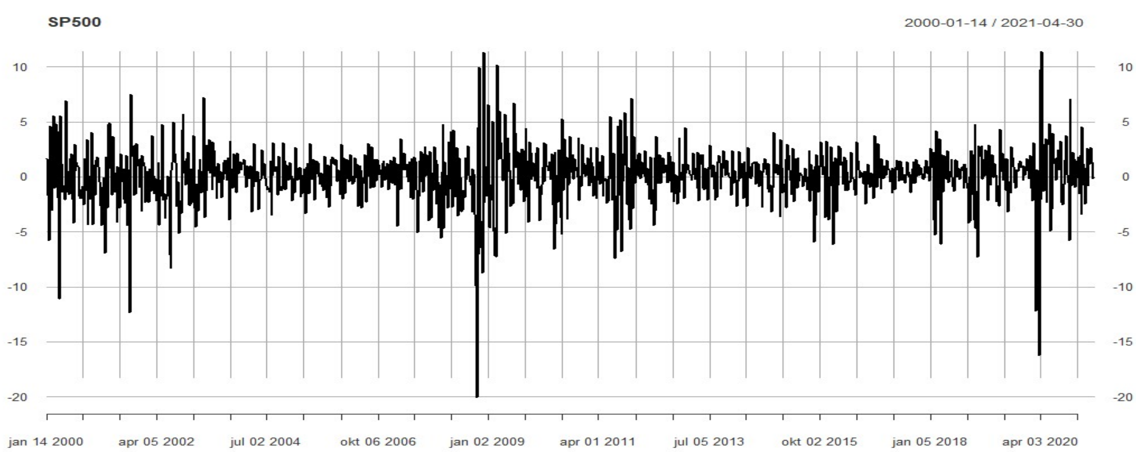


Figure 7: Plot of weekly log returns of the S&P500

Finally, by estimating various GARCH models including the EGARCH model and comparing the BIC and AIC, this thesis finds the EGARCH(1,1) to best fit the data (see table 3), which is the model we will use for forecasting volatility. For the EGARCH(1,1) all coefficients are found to be significant, which indicates that historic volatility has explanatory power on current volatility. The sum of the two coefficients $(\alpha + \beta)$ is less than one, which is required for the process to be mean reverting. The model equation for the conditional variance of the EGARCH(1,1) model is:

$$\log(\sigma_t^2) = \alpha_0 + [\alpha_1 u_{t-1} + \gamma_1(|u_{t-1}| - E|u_{t-1}|)] + \log(\beta\sigma_{t-1}^2) \rightarrow$$

$$\log(\sigma_t^2) = 0.0958 - [0.2222u_{t-1} - 0.2297(|u_{t-1}| - E|u_{t-1}|)] + \log(0.9242\sigma_{t-1}^2)$$

Selection criterion	GARCH(1,1)	GARCH(1,2)	GARCH(2,1)	GARCH(2,2)	EGARCH(1,1)
AIC	4.3012	4.3023	4.3025	4.3041	4.3082
BIC	4.3246	4.3293	4.3295	4.3356	4.3394

Table 3: The AIC and BIC selection criteria for the proposed GARCH and EGARCH models

4.4 The option pricing indicator

For this thesis, the purpose of the EGARCH is to generate a volatility forecast, which can be used to manage the risk of the delta-hedged short delta-hedged option strategy. The volatility forecast will enter as an input into our option pricing indicator that determines how we scale our positioning. The volatility forecast will be performed once for every option at the time the options is struck with a forecast horizon of 13 weeks, which is equivalent to the maturity of the options. The forecast will be recalibrated for every option using all available data from the start of our sample period, denoted as h, until the time the option is struck i.e., the first forecast is performed on the 22nd of January 2007, which is based on the previous $h = 367$ weekly observations. The second forecast is performed on the 23rd of April 2007 and based on $h = 380$ observations and so on and so forth.

To be able to compare the IV, which is noted on an annual basis, with σ_{h+13}, we convert the EGARCH estimates to an annual basis. This is done by multiplying the EGARCH forecast with the number of trading weeks in a year and dividing by 100 to get it in percentage:

$$\sigma^F = \frac{\sigma_{h+13}^2 * \sqrt{52}}{100}$$

The position we take in the short delta-hedged option strategy will be determined by whether the option is categories as overpriced, underpriced or fairly priced. If the spread between IV of the option at time t=0, denoted $\sigma_{t=0}^{IV}$, and the volatility forecast, denoted σ^F, is more than 5%, the option is categories as overpriced i.e., if $5\% < \sigma_{t=0}^{IV} - \sigma^F$. If the spread is $5\% > \sigma_{t=0}^{IV} - \sigma^F > 2\%$, the option will be categories as fairly priced and if the spread $\sigma_{t=0}^{IV} - \sigma^F < 2\%$, the option will be categories as underpriced. Categorizing options as under- over- or fairly priced is similar to the approach taken by Rostan et al. (2020). However, this thesis further introduces a filter of 2% i.e., options where the spread is less than 2% are categories as overpriced, to avoid the influence of forecasting errors and limit the number of risky transactions (Bartels & Lu, 2000).

Since using a GARCH model to forecast volatility is argued to have some explanatory power in identifying systemic mispricing of options and enhance performance of option trading strategies (Bartels & Lu, 2000; Rostan et al., 2020), this thesis will scale the short delta-hedged option strategy up or down dependent on the categorization of the option. We assume that we have an initial fully funded margin account at a brokerage firm of 1. If the option is overpriced, this thesis will sell short delta-hedged option strategy with a factor of 2 i.e. we enter a trade with 100% leverage on the initial fully funded margin account, if the option is fairly priced, this thesis will sell short delta-hedged option strategy with a factor of 1 i.e. we enter the trade with the initial level of funding, and if the option is underpriced, this thesis will sell short delta-hedged option strategy with a factor of 0.5. The reason why we sell despite the EGARCH forecast suggesting that the option is overpriced is due to the previous research suggesting that the EGARCH model is not able to perfectly forecast volatility (ibid.) and that we want to have exposure to the market at all times. Furthermore, scaling down when the options appear underpriced still allows us to reduce our exposure and potentially limit the left-tail risk. When we enter a leveraged trade, we assume no opportunity cost of the initial fully funded margin account and no additional cost or interest on taking on leverage. This is a strong assumption, but deemed appropriate, since the borrowing rate will depend on the type of investor and the objective of this thesis is to demonstrate the effectiveness of the indicator regardless of investor type.

4.5 Subset

This section provided an overview of the approach and model used to answer the research question. We seek to simulate theoretical findings based on a strict relaxation of the BS model, where we investigate the effect of being over and under-hedged, delta-hedging's performance in different volatility environment and how path-dependency and simple stochastic volatility processes affect the possibility to reap the volatility risk premium. Then we transfer these findings into an empirical backtest of delta-hedged puts and calls on the S&P500 before modelling an option pricing indicator to time volatility based on a EGARCH forecast.

5. Results & findings

This section will outline the research question's findings. First, the overall findings will be provided using simulations in three parts, as per the methodology. These findings are based on the relaxation of the BS assumptions. Afterwards, the findings relating to the empirical backtest will be shown. Here, we will provide three case studies for the delta-hedged puts and calls as a way of illustrating different option Greeks. Lastly, we will introduce the option pricing indicator and the performance of the previous delta-hedged calls and puts, if they were to time the volatility by increasing/decreasing the position size instead of just selling one continuously.

5.1 Simulations

The role of option seller is typically filled by financial institutions or market makers, who provide the market with the liquidity needed to function optimally. However, due to the asymmetric payoff structure of a short option, a financial institution selling options takes on a large amount of left-tail risk (Kurpiel & Roncalli, 2011). An important tool to manage this risk is delta-hedging. Under the classical BS assumptions, the delta-hedge would completely offset any gains or losses from the option position and the P&L would be zero (Ahmad & Wilmott, 2005). However, as we start to relax the assumptions, this does not necessarily hold in the real-world. Therefore, this thesis will go on to explore the effects of relaxing various assumptions in a controlled, simulated environment.

5.1.1 Illustrating path-dependency

As is evident from section 3.1.6, empirical evidence suggests that the BS assumptions do not necessarily hold in the real world. By only relaxing the assumption of continuous trading, we would expose our option P&L to some degree of hedging error, which can be seen from the general P&L equation:

$$\frac{1}{2}\left({\sigma^{i}}^{2} - {\sigma^{a}}^{2}\right) S^{2}\Gamma^{i} dt \tag{31}$$

To illustrate this effect, we simulate 50 stock paths, for which we sell a put option with a maturity of 12 months and delta-hedge using a discrete hedging frequency. Since this is a simulated environment, the P&L of the short put and the short call is (almost) exactly the same, which is why we will demonstrate the effect just using a short delta-hedged put. The following parameters have been used for this simulation:

$$S_0 = 100 \qquad K = 105.13 \qquad \mu = 0.05 \qquad \sigma = 0.3 \qquad q = 0 \qquad T = 1$$

The simulation's result can be seen in Figure 8. In Figure 8 (a), we hedge using a discrete rebalancing frequency of once per day and in Figure 8 (b), we use a discrete rebalancing frequency of four times per day, which is argued to be close to continuous hedging. By visually inspecting Figure 8 (a), we find large variation in the P&L, dependent on what path the underlying takes. This is confirmed by the min value being -1.35 and the max value being 3.08. This suggests strong evidence of path-dependency and is in line with the expectation given by equation (31). Furthermore, when we hedge with 252 time-steps, we have a mean value of 0.13 with a standard deviation of 0.70, and when we increase the number of time-steps to 1008, the mean decreases to -0.00025 with a standard deviation of 0.31. This suggests that, as we increase the hedging frequency, we are able to improve the hedging performance and decrease the variance of the final P&L. As we continue to increase the hedging frequency, it will converge towards something similar to continuous hedging, which will ultimately result in a mean and standard deviation very close to 0. This suggests that it is always optimal to hedge using a high rebalancing frequency, assuming no transactions cost.

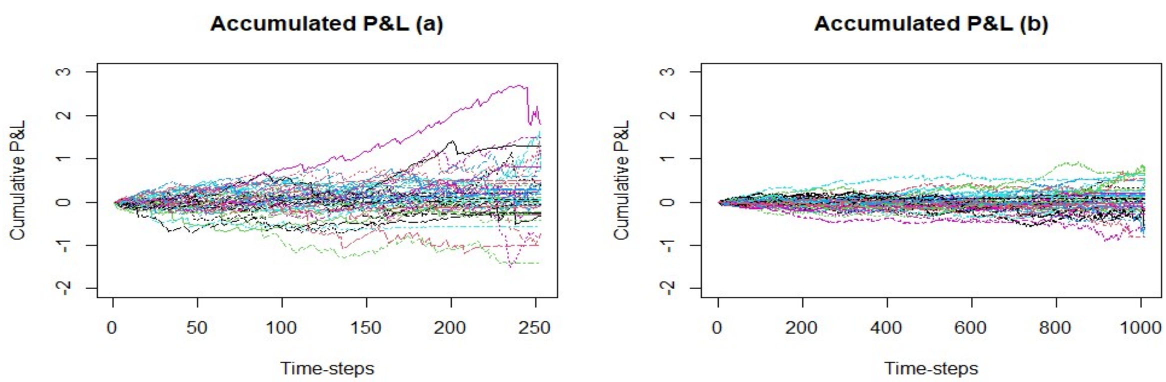


Figure 8: Illustration of path-dependency: (a) the P&L with a hedging frequency of once pr. day (Discrete), (b) the P&L with a hedging frequency of 4 times pr. day (continuous)

As outlined in the literature review, the P&L is primarily driven by the spread between the IV and RV, but the exposure to this is weighted by the option's dollar gamma (Bossu et al., 2005). Dollar gamma is calculated as:

$$\$\Gamma_t = \frac{\Gamma_t S_t^2}{100} \tag{39}$$

This creates a non-constant exposure to the volatility spread, as the dollar gamma evidently changes over the option's lifetime due to the underlying's path. This thesis expects an increase in the absolute dollar gamma to have a negative effect on the P&L. Appendix #5 showcases the dollar-gamma exposure for the simulations in Figure 8. Here, we see that, as the hedging frequency is reduced, the dollar-gamma variance (and fluctuations) in absolute terms increases, leading to a higher degree of path dependency. We also observe that, as the dollar-gamma increases in absolute terms, i.e., gets more negative, the P&L decreases, which is in line with the theoretical findings from Bossu et al. (2005) and equation (27). We clearly see that when hedging with a constant IV, the P&L is driven by the gamma and theta relationship illustrated in appendix #2, and is evident in equation (28), too.

5.1.2 Replicating the Ahmad & Wilmott (2005) experiment

The previous section addresses the hedging error made from using a low rebalancing frequency. However, the primary focus of this thesis is not to sell options to provide liquidity, but to reap the volatility premium. Inspired by the Ahmad and Wilmott (2005) hedge experiment, this section will go on to explore how the P&L of our short delta-hedged option strategy will be affected when the market misprices the option relative to the underlying asset, and how we can best capture this price difference.

The work by Ahmad & Wilmott (2005) looks at a scenario where the IV used to price the option is lower than the RV of the underlying, meaning that one can buy an option that is priced too low. Relating it to our delta-hedged option strategy, this thesis will explore the case where the IV is higher than the RV of the underlying and how we can capture this price difference by selling an option priced too high. As pointed out in section 3.1.4, IVs tend to be slightly higher than RV, as sellers typically charge a premium for bearing the risk. Therefore, this scenario is argued to be fairly realistic and provide us with important insights on how our strategy will perform if we are able to correctly estimate RV.

The expected P&L of taking a long option position and hedging with first IV, and then RV, is derived in section 3.2.4. By rearranging equation (30), we are able to derive the expected P&L from a short put option hedging with IV, which can be seen in equation (40). Just like in the case of the long call, there is no random term in equation (40), which confirms that the profits are deterministic and will be positive as long as RV ends up being lower than IV, which is:

$$\frac{1}{2}\left({\sigma^{i}}^{2} - {\sigma^{a}}^{2}\right) S^{2} \Gamma^{i} dt \tag{40}$$

By rearranging equation (33), we find the expected P&L of a short option hedged with RV, which can be seen in equation (41). Again, this suggests that the final profit is guaranteed, but how it is achieved is random. Under continuous hedging, we would expect the P&L to be exactly the difference shown in equation (41); however, as shown in the previous section, the final P&L can vary under discrete hedging due to the hedging error.

$$e^{rt_0} \int_{t_0}^{T} d\left(e^{-rt}\left(V^{a} - V^{i}\right)\right) = V^{i} - V^{a} \tag{41}$$

To replicate the Ahmad & Wilmott (2005) experiment, and as highlighted in the methodology section, we simulate a total of 50 stock paths using a GBM with a hedging frequency of 1008 time-steps. The following parameters have been used for this simulation:

$$S_0 = 100 \quad K = 105.13 \quad \mu = 0.05 \quad \sigma^{i} = 0.3 \quad \sigma^{a} = 0.2 \quad q = 0 \quad T = 1$$

As it is dependent on the path of the underlying and the dollar gamma, plotting in the values for IV and RV in equation (40), does not give us a specific value. However, we can confirm that it will be positive since IV is higher than RV, which is true because both volatilities are constant. The final P&L will be dependent on the dollar gamma, with the options moving closer to ATM yielding a larger final P&L.

Since the expected P&L for hedging with RV is guaranteed, and assuming that we can hedge continuously, we can calculate the option prices using the parameters above and get the following expected P&L for this simulation:

$$Expected\ P\&L = V^{i} - V^{a} = 11.92 - 7.966 = 3.95$$

The results from our simulations are found in Figure 9. In Figure 9 (a), we observe the smooth P&L paths, which confirms the expected deterministic nature of the P&L when hedging with IV. Hedging with IV results in a mean P&L of 3.92 and a standard deviation of 1.19. The high variance in P&L is also evident from the maximum value of 6.59 and the minimum value of 1.87. This is in line with the prediction of the final P&L being stochastic, as the P&L is dependent on the stochastic process dS_t and we cannot know the exact value of the P&L in advance.

From Figure 9 (b), it is evident that the P&L paths converge towards the expected P&L of 3.95, but how it is achieved appears to be random. The mean value of the P&L is 4.07 and the standard deviation is 0.20. This is slightly higher than what we would expect, but the standard deviation of the P&L is low, which is in line with our predictions. However, as we increase the hedging frequency, the results will converge towards the true theoretical value of the P&L. It is also worth noting that the reason we are able to predict the P&L is because we know the true value of the RV. Using real world data, this might not be possible.

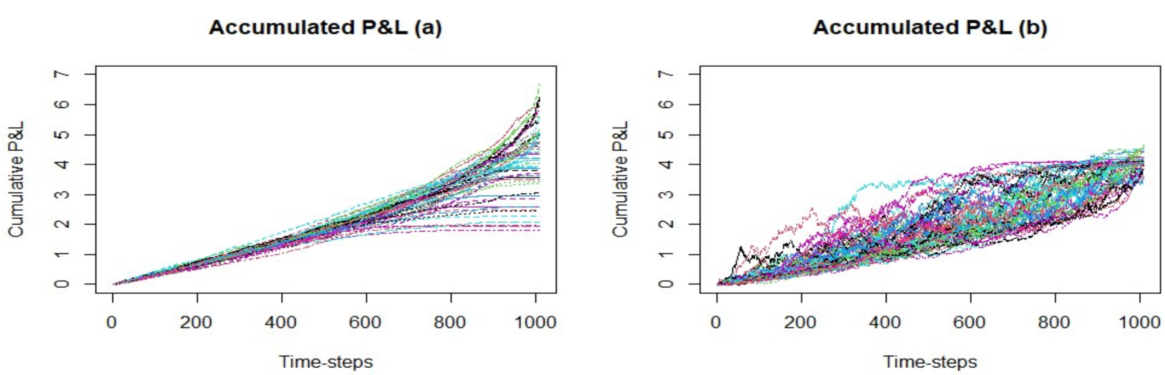


Figure 9: Replication of the Ahmad & Wilmott (2005) hedging experiment, where IV > RV: (a) is hedging with IV, (b) is hedging with RV

When applying our short strategy to real world data, it is also possible that we sell an option using an IV, which ends up being lower than the RV over the period. Using the Ahmad & Wilmott (2005) methodology, we can calculate the expected loss similarly to the P&L equations stated above, as the dynamic works both in a positive and negative direction. If we assume that $\sigma^{i} = 0.2$ and $\sigma^{a} = 0.3$, holding everything else fixed, we expect the

P&L to be strictly negative and deterministic when hedging with IV, and a guaranteed negative P&L when hedging with RV. The simulations can be found in appendix #6 and the results perfectly mirror the results from Figure 2, which is in line with our predictions.

In line with the findings of Ahmad & Wilmott (2005), there are pros and cons of both approaches. From a global risk-management perspective, the variance of the final P&L when hedging with RV is lower and therefore better. However, from a local perspective, the mark-to-market P&L fluctuates more over the lifetime of the options when hedging with RV. The main advantage of hedging with IV is that it is easily observable, unlike an estimate of RV, which requires a trader to have a reliable forecasting model. The observability also ensures that the trade will be profitable as long as one is on the right side of the trade.

5.1.3 Introducing a simple stochastic volatility process

Both section 5.1.1 and 5.1.2 have focused on delta hedging in a constant volatility environment. According to previous empirical research, this does not appear to be a realistic assumption since volatility tends to cluster, exhibit mean-reversion and suffer from the leverage effect (see section 3.1.6). Relaxing the constant volatility assumption is argued to have large implications for the performance of the delta-hedge and the P&L, as it is no longer sufficient to hedge only using the underlying and bonds (Hull, 2009).

Drawing inspiration from the empirical work done by Renault & Touzi (1996) and Kurpiel & Roncalli (2011), who test the performance of delta-hedging in a stochastic volatility environment, this thesis will explore how introducing a stochastic element to the volatility process will affect our ability to delta-hedge using the BS option pricing model. Furthermore, we will relax the assumptions of continuous hedging to better understand the implications for our short delta-hedged option strategy as we move towards real-world data. As outlined in the methodology section, this thesis does not intend to model a full stochastic volatility process, but instead we add a stochastic element to the volatility process, denoted $\tilde{\sigma}$, and use this for the GBM, which assumes that $\rho = 0$ (see Figure 5). This is argued to yield similar results, but in a more parsimonious manner. Since we will not be able to model the true data-generating process when testing the short delta-hedged option strategy empirically, this thesis will not attempt to derive any true model results or true model Greeks for our simulations either. This allows us to focus on the effect of relying on the BS option pricing model when the underlying behaves in a non-constant manner.

5.1.3.1 The general case

In the general case, we adopt a similar approach to that of section 5.1.1 to assess any potential hedging error from using a BS based delta-hedge. We relax the assumption of constant volatility and continuous trading, but all other assumptions of the BS model hold. We assume that $\sigma^i = \tilde{\sigma}$ at every time-step, meaning that we use the realised process to calculate the value of the option and the delta (see Figure 10), and compare the P&L effect of rebalancing the delta-hedge once per day and four times per day. This scenario is equivalent to having full knowledge of the volatility process but while using a simplified model. It is not as unrealistic as one might think, since it can be possible to forecast volatility somewhat accurately on a short-term basis (Kişinbay, 2010), but difficult to specify a forward-looking model based on historical data.

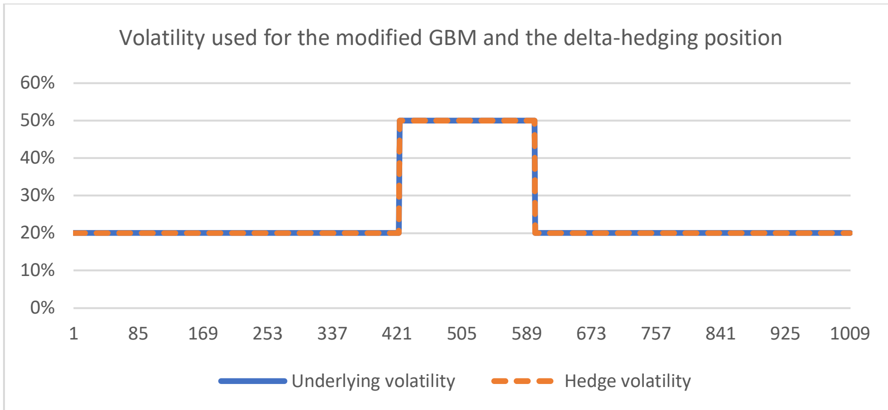


Figure 10: Plot of the underlying volatility and the hedge volatility used for section 5.1.3

We use the following parameters for the simulation, which will be the same for both rebalancing frequencies:

$$S_0 = 100 \quad K = 105.13 \quad \mu = 0.05 \quad \sigma^i = \tilde{\sigma} \quad \sigma^a = \tilde{\sigma} \quad q = 0 \quad T = 1 \quad \rho = 0$$

Expected findings from the simulation

From section 3.2.6, we know that when the underlying volatility process differs from the assumptions made in the BS option pricing model, we will be exposed to model error. This will lead to an inaccurate delta and for a wrong hedging ratio to be applied, which is why we would expect some degree of hedging error (Kurpiel & Roncalli, 2011). However, the hedging error is likely to be minimal, as the stochastic process is very simple and the re-hedging frequency is high, allowing us to quickly adjust to the new volatility level. Since we are selling an option and introducing an intermediate increase in volatility, we would expect to have a negative effect on the P&L, as the overall volatility will be higher than the constant rate assumed when the option was struck.

We know that a perfect hedge would require a position in an additional option, to offset the randomness in the volatility. Since we only hedge using the underlying and bonds, and our delta-hedge does not factor in a second source of randomness, we would expect a drop in P&L from an intermediate increase in volatility and an increase in P&L from an intermediate decrease in volatility. These P&L changes come from the fact that the option value will shift in level from a change in volatility, while the value of our hedging position will not. To show this, we look at the BS formula for calculating the option value:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad , \qquad d_2 = d_1 - \sigma\sqrt{T}$$

$$put = Ke^{-rT}N(-d_2) - S_0N(-d_1)\ , \qquad call = S_0N(d_1) - Ke^{-rT}N(d_2)$$

As σ increases, the denominator of d_1 will increase more than the numerator, making d_1 converge towards zero. As d_1 converges to zero, so will d_2, with the change in d_2 also dependent on the second term, $\sigma\sqrt{T}$. When $S_0 > K$, d_1 is positive, and d_2 will be closer to zero, i.e., less positive than d_1. When $S_0 < K$, d_1 is negative and d_2 will be further from zero i.e., more negative than d_1. This relationship between d_1 and d_2 ensures that, regardless of the moneyness of the option, the change in option value will always be positive.

This thesis would expect the absolute change in option value to be governed by the option's vega, with a higher vega translating to a higher sensitivity to a change in volatility and therefore a higher change in option value. In Figure 11, we see the simulated options' vega at timestep 424, just before the increase in volatility. It is evident that vega is at its highest when the option is close to the strike, meaning that options ATM will exhibit a larger change in value than options ITM or OTM. Therefore, we would expect the increase in option value, and subsequent decrease in option value from changes in volatility at timestep 424 and 599, to be dependent on the spot at those two time-steps. Since we are not hedging away this risk, this thesis expects to observe large variations in the final P&L, as it is likely that the option will not be at the same level of moneyness at timestep 424 and timestep 599. This suggest that the final P&L will be subject to a large degree of path-dependency.

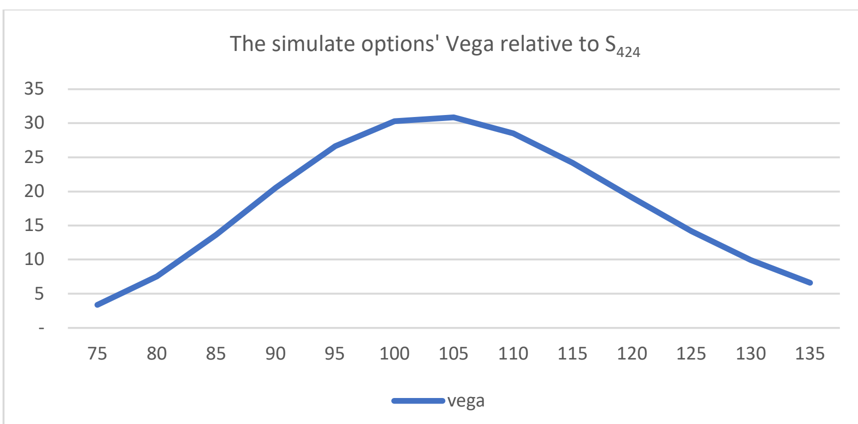


Figure 11: The simulated options' vega at t = 424

Finally, on average this thesis expects to incur a loss from the increase and subsequent decrease in volatility due to the time-decay of the option value. As we increase the number of time-steps with increased volatility, the difference in time to maturity, denoted T, will increase as well. A lower T will lead to a lower option value and a lower absolute change when vega is at its highest, all else equal. The relationship between time to maturity and option value is evident from section 3.1.5 and equations (8) and (11).

The findings from the simulation

Based on an initial visual inspection of the simulated results, we see great similarities between Figure 12 (a) and Figure 12 (b), with both exhibiting a decrease in the P&L at the time-step where volatility increases followed by an increase in the P&L at the time-step where volatility decreases, which is as expected. This is from the option position shifting in value from a change in the volatility parameter, while the hedging position is only adjusting to the change in option value from a change in the underlying. Furthermore, the P&L paths seems to be smoother in Figure 12 (b) relative to Figure 12 (a) as we increase the hedging frequency, which suggests that our results also suffer from path-dependency due to changes in the underlying. Again, this is in accordance with the findings in section 5.1.1 and evident from the fact that the final P&L when hedging over 1008 time-steps is close to the difference between the P&L impact of the increase and subsequent decrease as illustrated in Table 4. Since the observed delta-hedging error appears to be relatively small for Figure 12 (b), we can conclude that the value of the final P&L is to a large extent determined by the magnitude of the change in option value at the time-steps where volatility increase and decrease.

When a re-hedging frequency of four times per day is used, we get a mean P&L of -3.13 with a standard deviation of 2.24. The negative mean P&L is not surprising, since the time to maturity has decreased from 0.59 to 0.41, which, all else equal, should yield a lower absolute change at time-step 599. More specifically, we find the average decrease at time-step 424 to be 8.19 and the average increase at time-step 599 to be 5.04.

Upon further inspection of the simulated data for Figure 12 (b), we find great variation in the final P&L, with the maximum P&L value observed being 0.27 and the minimum being -7.50. This large variation in the final P&L is due to the intermediate increase and subsequent decrease in option value does not necessarily cancel out as they are highly path dependent. Looking at the moneyness of the options at time-step 424 and 599, we find the variation to be much greater at time-step 599. The larger variation in moneyness means that more options have moved further away from the strike at time-step 599, which reduces the increase in option value from a decrease in volatility, and further decreases the final P&L. This is not a surprising result as volatility has increased to 50% for 176 time-steps, which intuitively leads to greater variation in the underlying and more extreme values. The simulated P&L further confirms that the change in option value at the two time-steps where volatility increases is governed by vega i.e., the moneyness of the option. The options ATM are the ones changing the most in value and the options far OTM or far ITM change the least.

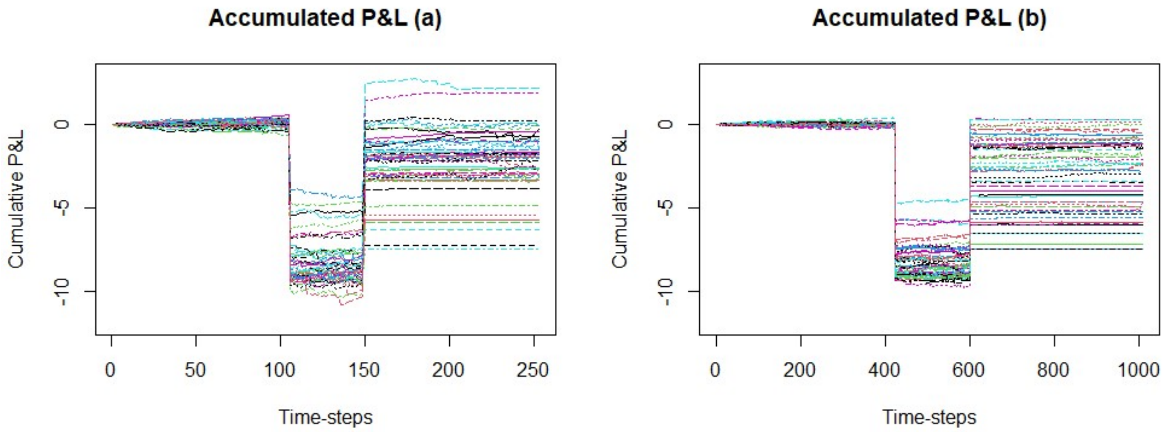


Figure 12: The P&L when introducing the simple stochastic process where $\sigma^i = \tilde{\sigma}$: (a) is hedging discretely, (b) is hedging continuously

For both hedging frequencies we experience a negative final P&L mean, which is in line with our predictions. When rebalancing once per day, we get a mean P&L of -2.39 with a standard deviation of 2.02, and when we rebalance four times per day we get a mean P&L of -3.13 with a standard deviation of 2.24. Interestingly, the results suggest that increasing the hedging frequency, despite the P&L paths being smoother, leads to a lower mean P&L and a higher standard deviation. A small part of the difference could be due to the hedging error from discrete hedging, but looking at the data it appears to be from the underlying taking fewer extreme values at the time-step where volatility increases and when simulating the GBM with fewer time-steps. It is therefore argued to be a feature of the simulated paths rather than an empirically important observation.

P&L impact	Option 1	Option 2	Option 3	Option 4	Option 5	Option 6	Option 7	Option 8	Option 9	Option 10
Decrease at t=424	-8.96	-8.87	-7.16	-9.20	-8.76	-7.75	-9.03	-7.95	-8.01	-7.41
Increase at t=599	-4.97	-7.53	-6.98	-3.50	-4.93	-3.75	-7.73	-7.77	-5.80	-6.29
Difference	-4.00	-1.34	-0.18	-5.70	-3.83	-4.00	-1.30	-0.17	-2.21	-1.12
Final P&L	-4.23	-0.89	-0.03	-5.63	-3.70	-3.97	-1.24	0.11	-2.43	-1.13

P&L impact	Option 11	Option 12	Option 13	Option 14	Option 15	Option 16	Option 17	Option 18	Option 19	Option 20
Decrease at t=424	-9.08	-7.32	-7.86	-7.90	-9.14	-8.51	-5.61	-9.24	-8.92	-9.22
Increase at t=599	-4.90	-7.61	-4.67	-7.63	-3.74	-0.97	-4.42	-7.73	-3.71	-6.25
Difference	-4.17	0.29	-3.19	-0.26	-5.40	-7.54	-1.20	-1.51	-5.21	-2.97
Final P&L	-4.30	0.27	-2.96	-0.55	-4.91	-7.50	-1.51	-2.13	-5.37	-2.36

P&L Impact	Option 21	Option 22	Option 23	Option 24	Option 25	Option 26	Option 27	Option 28	Option 29	Option 30
Decrease at t=424	-9.21	-9.07	-8.83	-5.86	-8.58	-8.79	-8.94	-9.08	-8.96	-6.04
Increase at t=599	-1.96	-4.19	-6.29	-0.67	-7.71	-3.00	-7.82	-6.20	-6.70	-2.03
Difference	-7.25	-4.88	-2.53	-5.18	-0.87	-5.78	-1.12	-2.88	-2.26	-4.01
Final P&L	-7.17	-4.64	-2.39	-5.14	-1.49	-5.86	-0.81	-2.79	-2.30	-3.73

P&L Impact	Option 31	Option 32	Option 33	Option 34	Option 35	Option 36	Option 37	Option 38	Option 39	Option 40
Decrease at t=424	-8.99	-6.88	-8.47	-7.92	-4.73	-8.80	-9.28	-7.99	-8.97	-7.30
Increase at t=599	-3.36	-6.48	-7.77	-5.40	-4.73	-7.69	-5.83	-3.59	-6.58	-2.09
Difference	-5.63	-0.40	-0.71	-2.51	0.00	-1.11	-3.45	-4.40	-2.38	-5.21
Final P&L	-6.02	-0.32	-0.90	-2.93	0.27	-1.16	-3.46	-4.75	-1.88	-5.26

P&L Impact	Option 41	Option 42	Option 43	Option 44	Option 45	Option 46	Option 47	Option 48	Option 49	Option 50
Increase	-8.36	-8.28	-7.65	-6.84	-9.00	-7.54	-9.29	-7.67	-8.24	-7.96
Decrease	-1.63	-2.64	-1.12	-1.95	-6.99	-6.96	-5.51	-6.82	-0.73	-6.56
Difference	-6.73	-5.64	-6.52	-4.89	-2.01	-0.58	-3.78	-0.85	-7.50	-1.40
Final P&L	-6.54	-5.99	-6.57	-4.64	-1.98	-0.64	-3.41	-0.95	-7.48	-1.31

Table 4: Illustrating the impact of vega on the final P&L

Reducing the period of increased volatility

In the simulations depicted in Figure 12, we assume that the increase in volatility is sustained for a period of two months, which has a large negative impact on the P&L. However, if we reduce the period of increased volatility, we expect a better behaving model, since the difference in time to maturity, all else equal, has been reduced. In Figure 13 (a) and (b), the period of increased volatility has been reduced to 22 and 11 trading days, respectively, and we have assumed a re-hedging frequency of 4 times per trading day.

From Figure 13, it is evident that, as we decrease the period of increased volatility, and as a direct consequence decrease the difference in time to maturity, the final P&L comes closer towards zero. More specifically, the mean P&L for 22 trading days of increased volatility is -1.73 with a standard deviation of 1.88, and the mean P&L for 11 trading days of increased volatility is -0.79 with a standard deviation of 1.66. This is a significant improvement from the results above and further suggests that the final P&L is dependent on the length of the shift in volatility level i.e., for how long volatility deviates from the volatility of which the option is struck. However, we continue to observe large variations in the final P&L, which are due to the underlying taking more extreme values and moving further away from the level of moneyness at time-step 424, as volatility increases to 50% for a substantial number of time-steps.

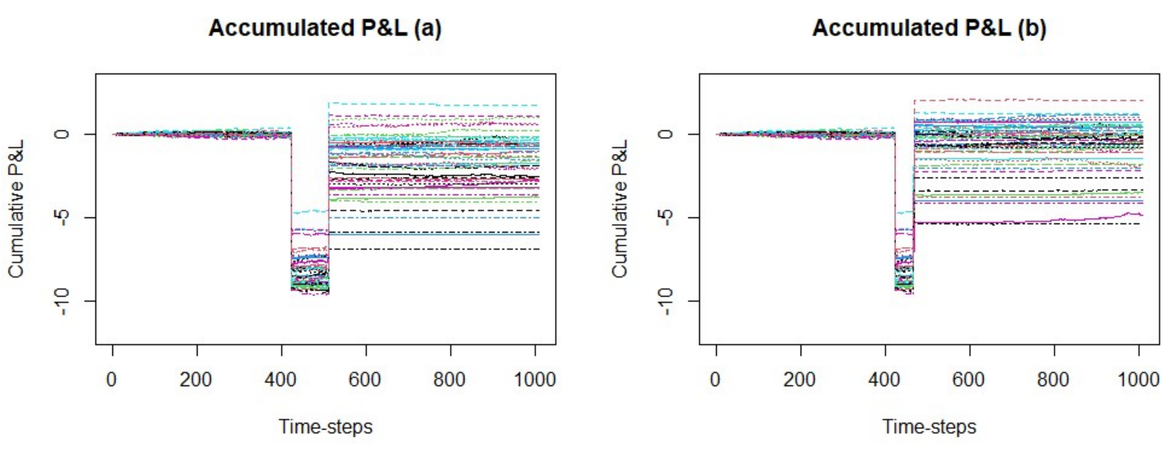


Figure 13: The P&L when reducing the length of the simple stochastic process where $\sigma^i = \tilde{\sigma}$: (a) is 22 trading days, (b) is 11 trading days of increased volatility

Introducing a period of lower volatility

In the above simulations, we only focused on the scenario where volatility increases before returning to the initial value, but for real-world data we are likely also to experience an intermediate decrease in volatility.

In Figure 14, simulated results showing a decrease in volatility from 20% to 10% can be found. These results mirror findings from the previous sections, just as we would expect. When volatility decreases, the option value increases and when volatility subsequently increases, the option value decreases. Similarly to the results above, the increase and subsequent decrease in option value does not cancel out and we have a positive final P&L, with a mean value of 0.53 and a standard deviation of 0.55. The positive P&L comes from the decrease in time to maturity, which leads to the absolute change in option value to be lower at time-step 599 than at time-step 424. However, when we decrease volatility, we intuitively experience less variation in the underlying and therefore less variation in the moneyness of the options at time-step 424 and 599. This results in a much lower variation in the final P&L.

Again, as the length of the shift in volatility level is decreased i.e., how long volatility deviates from the volatility of when the option was struck, the final P&L decreases and the mean converges to zero. This mirrors the results found above and is in line with the predictions outlined earlier.

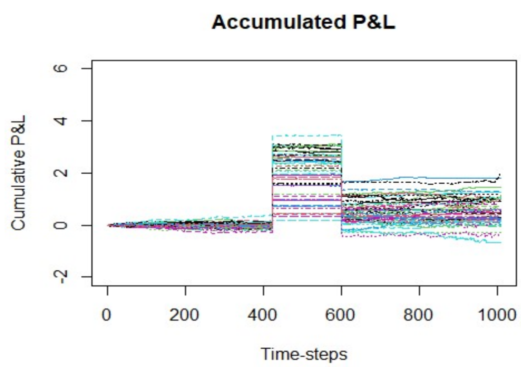


Figure 14: The P&L from introducing a simple stochastic decrease in volatility from 20% to 10% where $\sigma^i = \tilde{\sigma}$

Introducing a stochastic increase in volatility without a subsequent decrease

In our initial simulations, we focused on the scenario where volatility exhibited a sudden, substantial increase before returning to the initial volatility level. This is a likely scenario for an option with a long maturity, as volatility is said to exhibit mean reversion (cf. section 3.1.6). However, for options with shorter maturities or sudden shocks occurring close to option expiry, it is possible that volatility will not return to its initial level before the option expires. This thesis will now go on to explore the P&L impact of such a scenario.

To illustrate the scenario of volatility shifting up in level and remaining elevated throughout the remaining lifetime of the option, we change the underlying volatility path to exhibit an increase in volatility at time-step 424 from 20% to 50%, and remaining elevated from time-step 425 to 1008. As in the previous sections, we use the same volatility to simulate GBM and calculate the delta and option price.

Similar to findings in Figure 12 (b), we see a substantial decrease in the intermediate P&L as a consequence of the increase in volatility at time-step 424. The absolute value of the decrease will be governed by vega and

T at time-step 424. From time-step 425 and onwards, we would expect a steady P&L path, since we are hedging with the correct, constant volatility for the remaining lifetime of the option and use a hedging frequency of four times per day, which is proven in section 5.1.1 to be sufficient to eliminate path-dependency.

From Figure 15 (b), we see the P&L drop at time-step 424, with the drop being exactly equal to that found in Figure 15 (a). Based on further visual inspection, the P&L paths from time-step 425 and onwards appear to be steady and suffer from limited path-dependency. The mean P&L is -8.11, which is very similar to the mean decrease in P&L at time-step 424 of -8.19, and the standard deviation is 1.15. To sum up, the final P&L at option expiry is primarily driven by the intermediate drop in P&L, which in turn is governed by vega and time to maturity.

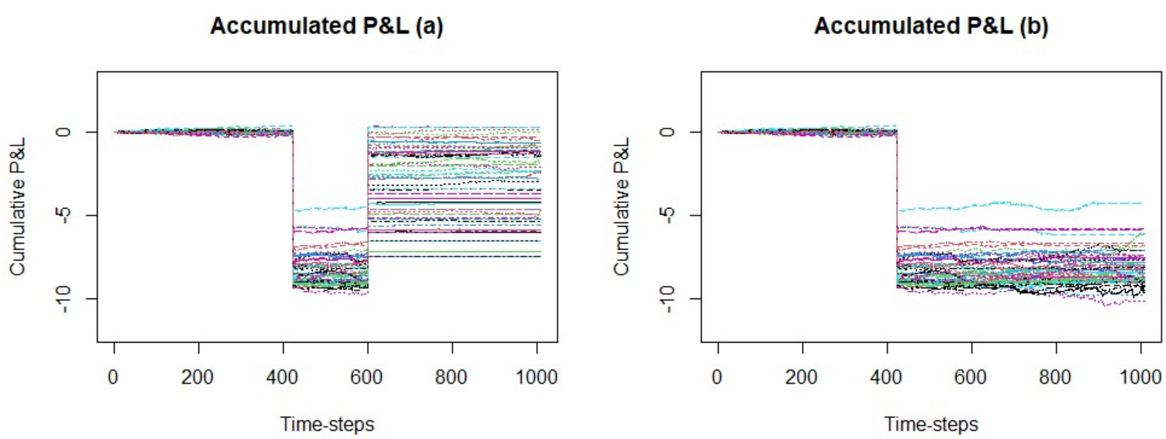


Figure 15: The P&L from the simulated options with a maturity of 12 months where $\sigma^i = \tilde{\sigma}$ for: (a) equivalent to Figure 12 (b), (b) when volatility increase and remains elevated at 50% for the remaining life-time of the option

5.1.3.2 Introducing a constant hedging volatility

In the previous section we had full knowledge of the RV path and could adjust our delta-hedge to the shift in volatility level, but we still suffered exposure to vega. This section will go on to investigate the impact of calculating delta using a constant volatility throughout the lifetime of the option. This setup offers insights into the P&L impact of not having full knowledge over the intermediate RV, the ability to accurately forecast the volatility at t+1, nor the P&L impact of IV being different from the true RV over the period.

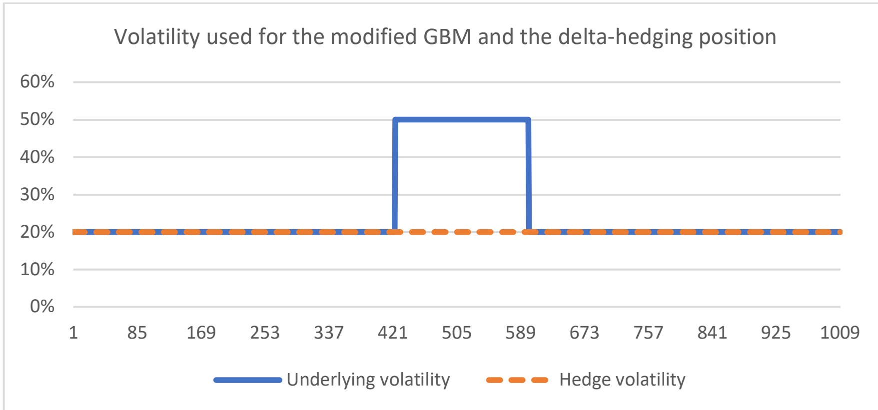


Figure 16: Plot of the underlying volatility and the hedge volatility used for section 5.1.3.2, where we illustrate the P&L effect of using a constant volatility

The RV governing the GBM is the same as in section 5.1.3.1 (see Figure 16) and the constant volatility used for the delta-hedge is denoted σ^{Δ}. Similar to section 5.1.3.1, we relax the assumption of constant volatility and continuous trading while all other assumptions of the BS model hold. We focus on the scenario with 1008 time-steps and compare the P&L effect of using a constant volatility for calculating delta, which is the IV at option issuance, with readjusting our delta-hedge to the RV at every timestep. The parameters used for this simulation is:

$$S_0 = 100 \quad K = 105.13 \quad \mu = 0.05 \quad \sigma^i = \tilde{\sigma} \quad \sigma^a = \tilde{\sigma} \quad \sigma^{\Delta} = 0.2 \quad q = 0 \quad T = 1 \quad \rho = 0$$

Expected findings from the simulation

In this section, we are only changing the volatility input for the delta-calculation thus many of the expectations outlined in section 5.1.3.1 still apply, which is why these will only be covered briefly. Generally, we are aware that our model is exposed to some degree of model error due to the introduction of a stochastic volatility element. This thesis is still short a delta-hedged put option, which is why we would expect an increase in volatility to have a negative effect on the P&L and vice versa. The absolute change in option value from a change in volatility is governed by vega and time to maturity, with a higher vega leading to a larger change and a lower T leading to a smaller change.

In addition to section 5.1.3.1, we now also introduce a substantial delta-hedging bias. We know that the constant volatility of 20% used for calculating the delta is less than the RV, denoted $\tilde{\sigma}$, from time-step 424 to time-step 598. Relating to the results proved by Renault & Touzi (1996), and using the delta equation provided in section 3.1.5, we can define the delta-hedging bias:

$$HB = \Delta_t^{BS}\left(x, \sigma_t^{\Delta}\right) - \Delta_t^{BS}\left(x, \tilde{\sigma}_t^{RV}\right) \rightarrow \sigma_t^{\Delta} < \tilde{\sigma}_t^{RV}$$

$$ITM: \Delta_t^{BS}\left(x, \sigma_t^{\Delta}\right) > \Delta_t^{BS}\left(x, \tilde{\sigma}_t^{RV}\right)$$

$$OTM: \Delta_t^{BS}\left(-x, \sigma_t^{\Delta}\right) < \Delta_t^{BS}(-x, \tilde{\sigma}_t^{RV})$$

$$ATM: \Delta_t^{BS}\left(0, \sigma_t^{\Delta}\right) \cong \Delta_t^{BS}(0, \tilde{\sigma}_t^{RV})$$

This means that a lower volatility input σ_t^{Δ} into the delta equation would lead to a positive delta-hedging bias for an ITM option i.e., it will be over-hedged, a negative delta-hedging bias for an OTM option i.e., it will be under-hedged and close to no bias for an ATM option. Unfortunately, we are not able to derive the true stochastic model and delta value and are therefore not able to compare the hedging bias to the true model delta. However, we saw that the simple BS based delta-hedge did well at hedging changes in the underlying, despite the model not being completely accurate and it is therefore argued to be a good proxy for the true model. For simplicity, we will refer to the delta in section 5.1.3.1 when discussing the hedging bias.

The delta-hedging bias will result in ITM option positions being over-hedged between time-step 424 and 598 and OTM option positions being under-hedged. Therefore, we would expect the following scenarios to play out within this time-frame for the put option: 1) an ITM option where S_0 is decreasing (moving further ITM) would result in a positive P&L effect as the delta-hedging position will increase more than the increase in option value 2) an ITM option where S_0 is increasing would result in a negative P&L effect 3) an OTM option where S_0 is increasing (moving further OTM) would result in a positive P&L effect as the delta-hedging position will decrease less than the decrease in option value and 4) an OTM option where S_0 is decreasing would result in a negative P&L effect 5) an ATM option where the underlying is moving either ITM or OTM would result in a positive P&L effect, but it would be highly dependent on the path of the underlying and how close S_t is to the strike. The effects are outlined in appendix #7 to provide the reader with a better overview.

Finally, when $\rho = 0$ previous empirical research suggests that there is not much difference in the hedging performance of using the true volatility and a constant volatility (Kurpiel & Roncalli, 2011; Poulsen et al., 2009). This is due to the delta-hedging bias being relatively small when the option is close to the money.

The findings from the simulation

In Figure 17 we have summarized the P&L for each stock path for (a) when hedging with the realized volatility $\tilde{\sigma}$, this is similar to Figure 12 (b), and (b) when hedging with the constant volatility σ^{Δ}. Based on an initial visual inspection, it is obvious to see the expected shift in option value at time-step 424 and 598, which is governed by vega and time to maturity. Again, we find the average decrease at time-step 424 to be -8,19 and the average increase at time-step 599 to be 5,04 with both the decrease in T and the greater variation in the moneyness of the options affecting our P&L negatively.

From Figure 17 (b), it is very easy to see the effect from the delta-hedging bias when using a constant volatility from time-step 424 to time-step 598, evident as the large variation in the P&L paths between those two time-steps. Further assessing the data, we find that the results, to a large extend, behave as described above.

However, we do find that the P&L effect from the delta-hedging bias is not only dependent on S_{424} and S_{599}, but also the path of the underlying. More specifically, we find that e.g., ITM options where the stock path moves closer towards the strike during the 176 time-steps before ending up at $S_{424} < S_{599}$, has a lower P&L than if the path of the underlying had decreased in a more consistent manner.

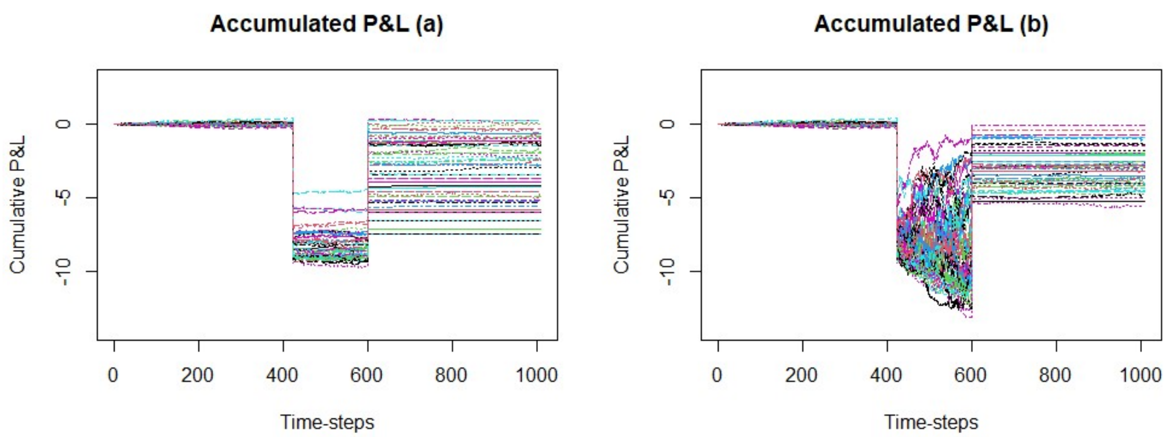


Figure 17: The P&L from simulating a simple stochastic process while heding with a constant volatility: (a) equivalent to Figure 12 (b), (b) where $\sigma^i = \tilde{\sigma}$ but a constant hedge volatility of $\sigma^\Delta = 20\%$ is applied

For Figure 17 (a), we found a mean of -3.13 with a standard deviation of 2.24 and for Figure 17 (b) we find a mean of -3.05 with a standard deviation of 1.38. This suggest that the mean P&L is roughly the same regardless of whether you hedge with the true volatility or a constant volatility equal to IV at option issuance. Our results thereby confirm the findings by Kurpiel & Roncalli (2011) and Poulsen et al. (2009).

Interestingly, hedging with a constant volatility leads to much lower standard deviation. Since we have not changed the moneyness of the options nor the time to maturity, we can confidently conclude that this is due to the P&L effect of the hedging-bias. Further analyzing the data, we find that the P&L effect of the hedging bias works in the opposite direction of the P&L effect from the increased volatility. The intuition behind this result is easy to see. An option ATM at time-step 424, but ITM at time-step 599, will have a large negative P&L due to the increase in option value being substantially larger than the subsequent decrease in option value. However, an option like that will experience a positive P&L effect from the hedging bias as the option moves ITM. The two effects do not entirely cancel out since the latter exhibits path-dependency and the former does not.

Introducing a period of lower volatility

After having found that using a constant volatility for calculating the delta-hedge can have a positive effect on the variation in the P&L, this thesis will now go on to analyze the case where a stochastic decrease in volatility is introduced. In this section, we maintain all the same assumptions and parameters as above, but we introduce a decrease in volatility from 20% to 10% between time-step 424 to time-step 599 instead of an increase from 20% to 50% (See Figure 18).

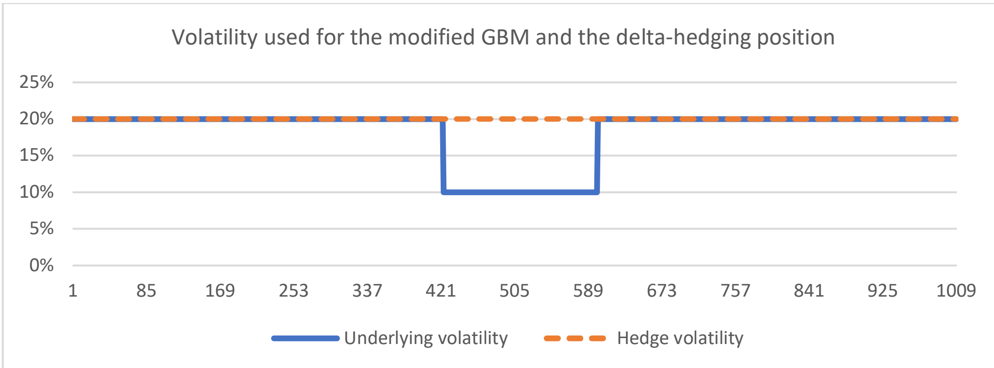


Figure 18: Plot of the underlying volatility and the hedge volatility when introducing a period of lower volatility

When introducing a temporary decrease in volatility, we would expect our results to mirror those found in the previous sections. This includes an increase in the intermediate P&L when volatility decreases followed by a decrease in P&L when volatility increases. Overall, we expect to see a positive final P&L effect due to the difference in time to maturity.

However, the decrease in volatility is likely to reduce the standard deviation of our P&L compared to Figure 17. Intuitively, lower volatility of the underlying leads to lower variation in the moneyness of the options, meaning that the increase and subsequent decrease in option value will be more similar in magnitude. Furthermore, the delta-hedging bias will be opposite of what we found in the previous section, thus ITM options will be under-hedged and OTM options will be over-hedged. Therefore, when the option moves closer to ATM, the delta-hedging bias will affect the P&L positively and when the option moves away from ATM, the delta-hedging bias will have a negative effect on the P&L. Similar to above, when the delta-hedging bias has a positive effect on the P&L, the change in moneyness will have a negative effect on the P&L and the two will almost cancel out.

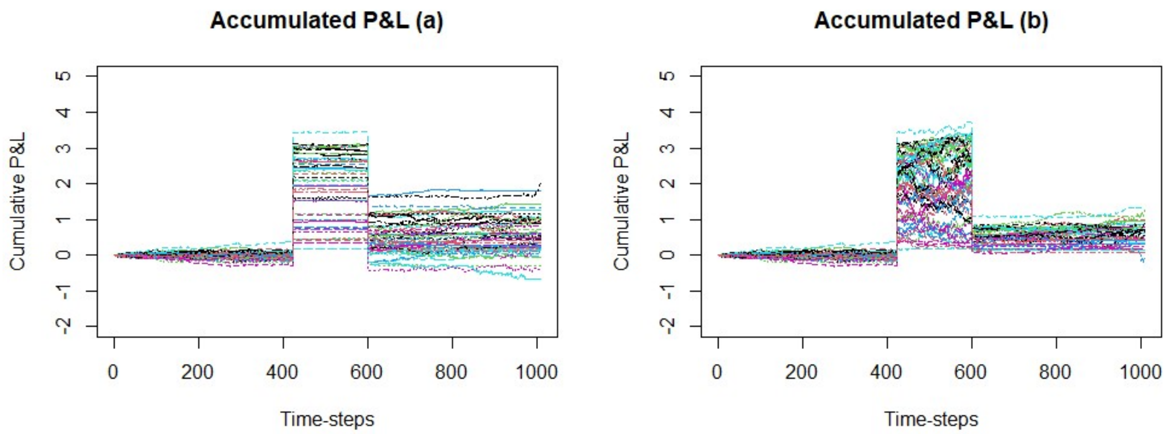


Figure 19: The P&L from a stochastic decrease in volatility for: (a) equivalent to Figure 14 (b) where $\sigma^i = \tilde{\sigma}$ but a constant hedge volatility of $\sigma^\Delta = 20\%$ is applied

From Figure 19 (b), we see that the results generally mirror what we found in the previous sections. The simulations yield a final P&L mean value of 0.56, slightly higher than what we found in Figure 19 (a), and a standard deviation of 0.31, substantially lower than what we found in Figure 19 (a). The lower variation is from the hedging bias cancelling out some of the effect from changes in the moneyness of the option, which is as expected.

Introducing a stochastic increase in volatility without a subsequent decrease

The main finding when using a constant hedge volatility despite the underlying exhibiting stochastic volatility is that it reduces the variation in the final P&L due to the hedging error from our delta-position working in the opposite direction of the P&L impact of vega. However, these results only hold true when RV returns to the initial level. Therefore, this next section will go on to explore the P&L effects from volatility increasing and remaining elevated.

The underlying volatility used for simulating the stock paths and the volatility used to calculate the delta hedge is illustrated in Figure 20, while all other assumptions are similar to that of Figure 17 (b). This means that we would expect to see an initial decrease in P&L at time-step 424 equivalent to that of Figure 17 (b). From Figure 20, it is evident that we will incur a hedging error from timestep 424 until maturity, which will determine whether the P&L will increase or decrease from the shift down in level. Since we no longer observe a decrease in volatility at time-step 599, we expect the P&L from time-step 425 onwards to be highly path-dependent. As previously outlined, the hedging error will result in ITM options being over-hedged and OTM options being under-hedged. The effects are similar to before and can be found in appendix #7. The highly path dependent P&L further suggest that we would observe large variations in the final P&L, but that the mean is likely to be similar to that Figure 15 (b).

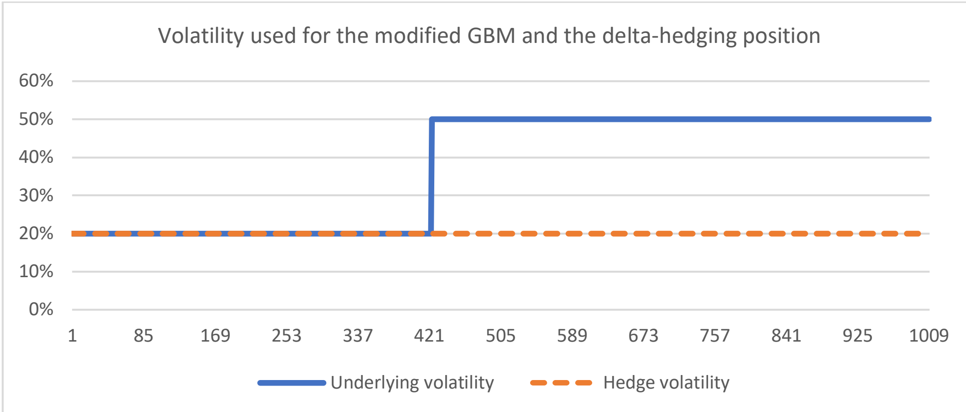


Figure 20: Plot of the underlying volatility and the hedge volatility when introducing an increase without a subsequent decrease

Based on a visual inspection of Figure 21 (b), we find that hedging with a constant volatility results in much more variation in the final P&L and more extreme negative values. Further analyzing the simulated data, we find that the extreme final P&L values are due to the option being ATM at time-step 424, implying a high vega and a large decrease in the P&L, and the option moving around the strike until maturity, where gamma is at its highest i.e., the delta is highly sensitive to changes leading to a large impact from the hedging error. The final P&L values closer to zero are from the option being ITM (OTM) and at time-step 424, implying a low vega resulting in a relatively smaller decrease in P&L and moving further ITM (OTM), which results in a positive P&L effect from the over-hedge (under-hedge). The mean final P&L value is -8.52, which is relatively close to that of Figure 15 (b) of -8.11, and the standard deviation is 5.86. All of the above is in line with our expectations.

These findings add an interesting dimension to our previous results. It still holds true that one can reduce the standard deviation of the final P&L by using a constant hedging volatility under the right circumstance. However, it also exposes the P&L to significant path-dependency and additional risk if the volatility does not return to its initial level before option maturity.

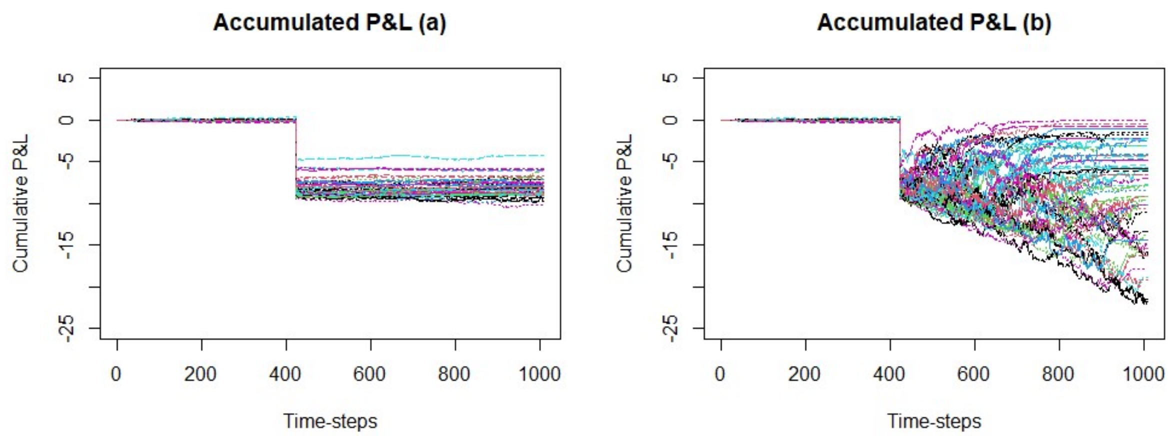


Figure 21: The simulated P&L when volatility shifts up and remains elevated for: (a) when hedging with RV similar to Figure 15 (b), (b) when hedging with a constant volatility parameter of $\sigma^{\Delta} = 20\%$

5.1.3.3 Selling an option with a stochastic element at a higher implied volatility

Building upon the previous sections, this thesis will now go on to explore the effect on a short put option position when IV at the time the option is struck is higher than RV at maturity. In contrast to section 5.1.2, we will relax the assumption of constant volatility and introduce a stochastic element to the volatility process, but still run the simulation hedging with first RV and then IV. This is argued to be an accurate representation of a scenario where a trader has a view on volatility in the medium-term, but the trader is unable to accurately forecast whether this is the peak in the very short-term.

This thesis will simulate this scenario using a RV as in section 5.1.3.1, but IV will always be 10 percentage points higher. We use the following parameters for the simulation:

$$S_0 = 100 \quad K = 105.13 \quad \mu = 0.05 \quad \sigma^i = \tilde{\sigma} + 0.1 \quad \sigma^a = \tilde{\sigma} \quad q = 0 \quad T = 1 \quad \rho = 0$$

Since the results are heavily reliant on the previously outlined dynamics, this thesis will briefly sum up how our expectations change as RV is lower than the IV at which the option is struck. We expect the difference in IV and RV to have a positive P&L effect similar to that of Figure 9 (b) in section 5.1.2. However, the P&L will be negatively impacted by the change in time to maturity between time-step 424 and 599 and the increased variation in moneyness. We further expect large variation in the P&L due to the large variation in moneyness.

From Figure 22 (a), we see that the P&L path exhibits the same random features as described in section 5.1.2, but that the change in volatility level leads to a drop and subsequent jump in the P&L. Further analyzing the data, we find that the P&L paths converge towards the theoretical P&L of $V^i - V^a$, but the final P&L is negatively affected by the difference in the increase and subsequent decrease in option value. The mean P&L value is 0.93 with a standard deviation of 2.25, which is in line with our expectations. From Figure 22 (b), we observe that the P&L is deterministic, but again negatively affected by the difference in the increase and subsequent decrease in option value. The mean value is 0.93 and the standard deviation is 2.78 and we find that the mean value is similar, but the standard deviation is higher when hedging with IV.

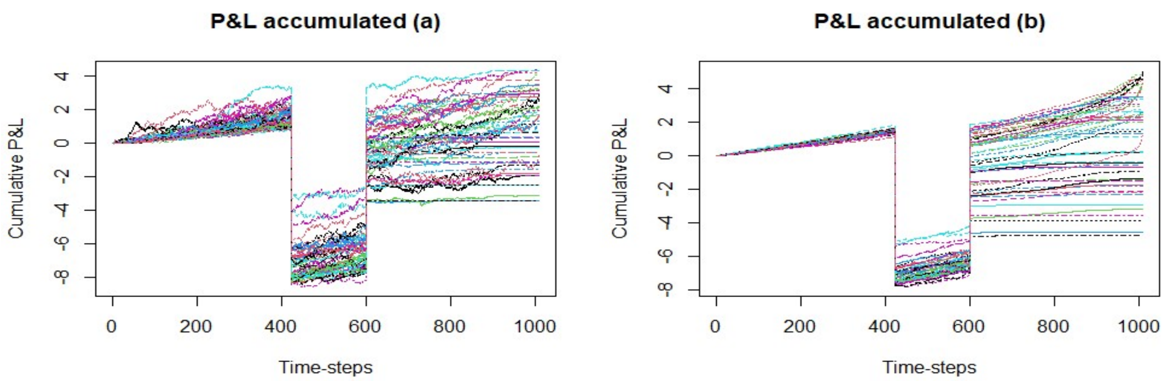


Figure 22: The simulated P&L from selling an option at a higher IV=RV+0.1, hedged with (a) RV and (b) IV

As is evident from these results, if the market is mispricing the option, it appears to be possible to earn positive profits from selling an option and delta-hedging. However, the P&L is highly dependent on the behavior of volatility throughout the life-time of the option. These results warrants caution for volatility traders as the P&L will be impacted by the trader's ability to forecast volatility in the short-term and medium term.

5.1.4 Validity Check of the P&L

To test the validity of our simulation methodology, we perform a simple validity check of the P&L paths. The test is based on the result that the final P&L of long positions should perfectly mirror the final P&L of short positions i.e., the final P&L should have the same absolute value but with opposite sign. The test will be performed once for the results from the Ahmad & Wilmott replication in section 5.1.2 and once for the results from the stochastic volatility, for simplicity. The test on the Ahmad & Wilmott replication is performed with the following parameters for both a long and a short option:

$$S_0 = 100 \quad K = 105.13 \quad \mu = 0.05 \quad \sigma^i = 0.3 \quad \sigma^a = 0.2 \quad q = 0 \quad T = 1$$

The test on the stochastic volatility is performed with the following parameters for both a long and a short option:

$$S_0 = 100 \quad K = 105.13 \quad \mu = 0.05 \quad \sigma^i = \tilde{\sigma} \quad \sigma^a = \tilde{\sigma} \quad q = 0 \quad T = 1 \quad \rho = 0$$

The P&L paths can be found in Figure 23, which confirms that the short option perfectly mirrors the long option. This is further supported by appendix #8, which shows that the absolute difference between the short and the long option for each P&L path is 0. This holds true for both the Ahmad & Wilmott (2005) simulations and the stochastic volatility simulations. A similar approach to that above is taken to test the validity of the empirical results and can be seen in appendix #9.

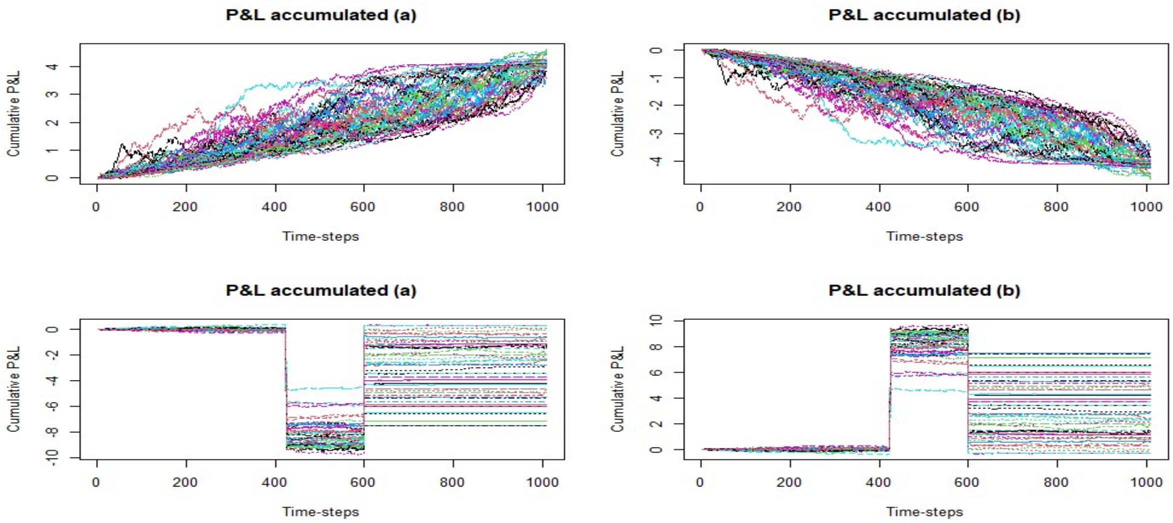


Figure 23: Validity check of the P&L with the first row being the Ahmad & Wilmott replication and the second row being the simple stochastic volatility process: (a) is a short position and (b) is a long position.

5.2 Empirical results

This section seeks to present the empirical observations of trading volatility using delta-hedged short positions in calls and puts on the S&P500 over a timeframe of January 22nd 2007 to 16th October 2020, which encompasses two crises, the financial crisis and the COVID-19 crisis. First, this thesis will analyse the P&L from selling volatility using delta-hedged options with a special focus on relating the empirical results to the findings from our simulations. Finally, this thesis will compare the results of the puts and the calls to determine the optimal volatility trading strategy. Appendix #10 provides an overview of an empirical call's P&L data output taken from RStudio, where "test" is the value calculated with TV and the other is calculated with IV.

5.2.1 Expected findings from the empirical results

Under the classical BS assumptions, the P&L of selling volatility is the difference in option value calculated using IV and option value calculated using RV. As mentioned in the methodology section, this thesis will use the 30-day TV as a proxy for the RV. This is done to improve the practical implementation of our findings, since the 30-day TV can be observed and easily implemented as a hedge volatility. However, as we have clearly demonstrated from the simulated data, once we start relaxing the assumptions about continuous trading and constant volatility, the P&L is also impacted by the path of the underlying and any potential intermediate changes in volatility.

Generally, we would expect the P&L of the delta-hedged short put and call option positions based on empirical data to be dependent on the difference between IV and RV with $\sigma_i > \sigma_r$ having a positive P&L effect and vice versa. The difference between IV at t_0 and 90-day RV at t_T for each option over the 14-year period is found in Figure 24. For the chosen period, 34 of the options have a higher IV and 21 of the options have a higher 90-day trailing RV. The results hold for both the put and the call. This suggest that we should have a positive P&L for the majority of both the call and the put options. We also observe that when 90-day trailing RV is higher than IV it tends to be much higher, as is evident from the large drawdowns around the financial crisis in 2008, illustrated by option 7, and the covid-19 crisis in 2020, illustrated by option 53. However, due to the inevitable hedging error the P&L will be dependent on the gamma, theta and vega and prone to some degree of hedging error, making it difficult to predict the exact final P&L value.

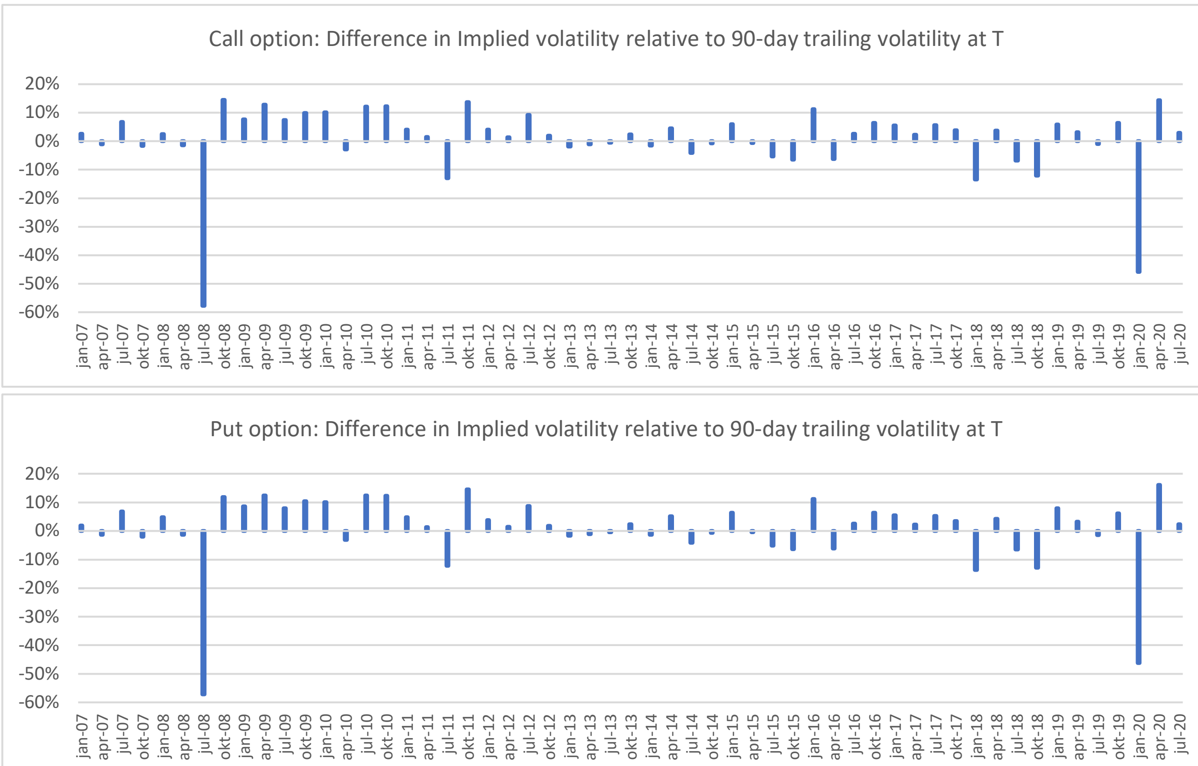


Figure 24: The difference between $\sigma^i_{t=0}$ and σ^r_T for each individual put and call from the OptionMetrics data set

When calculating the strike using a fair forward the strike will be above the spot i.e. all the puts will be ITM and the calls will be OTM, which can cause the IV of the two to differ. Figure 25 is a plot of the difference between the IV at which the call and the put is struck with a positive value meaning that the call is priced with a higher IV than the put. From Figure 25, we see that the call has a higher IV for 25 of the 55 options and the put has a higher IV for the remaining 30. Since we are trading volatility, we would expect the delta-hedged call to perform better than the delta-hedge put, when it is struck with a higher IV and vice versa. This is in contrast to our simulations, where the performance of the put and the call was identical.

Since previous research have found many of the BS assumptions to be too strong for empirical data (cf. section 3.1.6), we would also expect many of the findings from section 5.1.3 to apply to our empirical results. This includes some degree of path-dependency from using a hedging frequency of once pr day, the level of option moneyness at the point in time of changes in volatility, the timing of changes in volatility and the path of the underlying in the inevitable event of hedging error. Finally, it must be noted that empirical data does not fit theoretical option pricing models, particularly not BS, and we are therefore not able to calculate the true Greeks, which will lead to some degree of hedging error blurring our results.

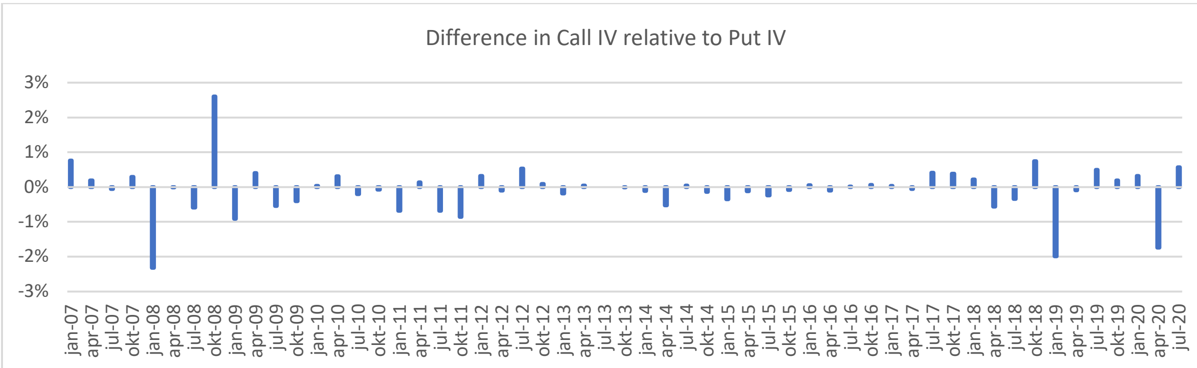


Figure 25: difference between each individual call and put option's $\sigma_{t=0}^{i}$ from the OptionMetrics data

5.2.2 Empirical results for delta-hedged calls

Figure 26 and Table 5 presents the accumulated P&L in percentage terms for continuously selling a delta-hedged ATM call on the SP500 with maturity of three months over a 14-year period and the summary statistics for each call's final P&L. The final accumulated P&L 42.14% if one hedges with the IV of the calls and 43.90% if one hedges with the TV on SP500. This result is generally in line with our expectations since there is a majority of calls with $\sigma_{t=0}^{i} > \sigma_{T}^{r}$, which leads this thesis to expect a positive accumulated P&L.

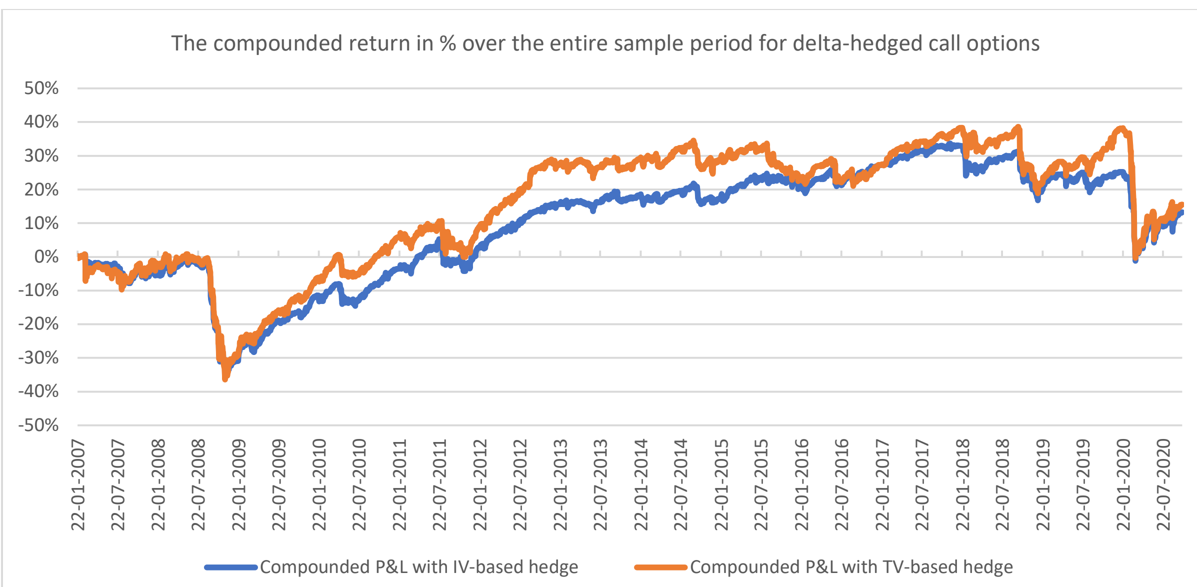


Figure 26: The compounded P&L for continuously selling a delta-hedged call option over the entire sample period when (a) hedging with IV and (b) hedging with 30-day TV

Looking at the calls individually, we get a positive mean return of 0.73% with a standard deviation of 4.21% and a mean return of 0.84% with a standard deviation of 5.66% when hedging with IV and 30-day TV, respectively. it is evident that if one hedges with the IV compared to the 30-day TV, one experiences less variance in the P&L. Relating the statistics to the Ahmad & Wilmott (2005) simulation, we observe the contrary effect compared to our simulation given that we would have expected the standard deviation and variance of the final P&L to be lower from hedging with RV. However, this difference can be explained by the 30-day TV not being a perfect proxy for the true RV.

Hedge-volatility	Com. P&L	Median	Mean	Min	Max	Std. dev.
Implied	42.14%	1.71%	0.73%	-16.31%	7.01%	4.21%
Trailing	43.90%	1.89%	0.84%	-23.26%	9.95%	5.66%

Table 5: Summary statistics of the cumulative P&L over the entire sample period for the call options when hedging with IV and TV, respectively

From Figure 26, we see that the P&L is heavily distorted by two large drawdowns corresponding to the increase in volatility around the financial crisis and the covid-19 crisis. These drawdowns are consistent with Figure 4, which illustrates how 90-day trailing RV ends up being substantially higher than IV for option 7 and 54 leading to a substantial loss. This is very much in line with our expectations and simulated results in section 5.1.3.1, which demonstrate how a large increase in volatility without a subsequent decrease will have a large negative impact on the final P&L. During the financial crisis, the loss on the IV hedged call is 16.31% and the loss on the TV hedged call is 17.77% compared to a loss of 10.42% and 23.26% during the covid-19 crisis. The maximum RV on the SP500 during the COVID-19 crisis was 97.55% on the 6th of April 2020 compared to a peak RV 79.40% on the 16th of October 2008. However, in our data we observe that the difference between IV and 90-day trailing RV is bigger during the financial crisis. This implies that the Financial crises should have a bigger impact on the P&L and it confirms the importance of the time at which the option is struck. We see this for the IV-based hedge, but not the TV-based hedge. The drawdown is likely to be further magnified by discrete hedging, since the delta-hedging position will suffer from path-dependency as demonstrated in section 5.1.

As previously mentioned, we observe that the standard deviation is higher, and the min and max takes more extreme values when hedging with TV (See Table 5). However, we also observe that the median and overall return is higher suggesting that hedging with TV performs better in a more stable environment but performs substantially worse when volatility becomes more extreme. Further analyzing the data, we find that hedging with TV performs better than hedging with IV when RV is decreasing and vice versa. This is primarily due to differences in delta, since there is a difference between IV and TV with the former being more stable during the lifetime of the options. The impact of IV deviating from TV is complicated and dependent on whether IV is higher or lower than the TV, the moneyness of the option and the movement of the underlying.

Relating to previous literature (see section 3.2.6) and the findings from out simulated results (see section 5.1.3.2), when TV is decreasing and IV stay somewhat constant i.e., $\sigma^{IV} > \sigma^{RV}$, the ITM (OTM) options will be under-hedged (over-hedged) when hedging with IV, assuming that TV is a good proxy for the true RV. Since the options have a relatively short maturity, they are likely to be close to ATM, and since volatility and the underlying is negatively correlated, the decrease in volatility will lead to an increase in the underlying and for the option to move from ATM to ITM. When the IV-based delta-hedge position is under-hedged and the underlying is moving further ITM, the option value is increasing, which is negative for a short option P&L, and this increase is not fully compensated for by the delta-hedging position leading to a negative hedging error. However, when TV is increasing to the point where $\sigma^{IV} < \sigma^{RV}$, the OTM (ITM) option will be under-hedged (over-hedged). The increase in volatility will lead to a decrease in the underlying, which will move the option from ATM to OTM. Since the IV-based hedge is under-hedged, the delta-hedging position will decrease less than the decrease in option value leading to a positive P&L effect. To conclude, the constant under-hedge from hedging with IV is the reason for why this hedge performs better when volatility is increasing and worse when volatility is decreasing. This finding is supported by the research by Kurpiel & Roncalli (2011), who finds that when the correlation between RV and the underlying is negative $\rho < 0$, it will lead to a systemically under-hedge position when hedging with IV. It should be noted that this analysis is a generalization with the individual call options likely to exhibit differences in correlation between volatility and the underlying and varying degrees of the effects described above. However, on average, the above analysis is argued to hold.

The difference is partially affected by the selection of calls, as if we had taken other calls with more skewed IV smiles i.e., higher IV, we could have experienced a higher mean and standard deviation than our current (Alexander & Nogueira, 2011). In addition, the P&L path is likely to also suffer from path-dependency due to discrete hedging. Since hedging with TV has a lower delta and therefore a higher sensitivity to the large downward movements in the underlying, path-dependency is likely to have a larger negative effect for hedging with TV. The relationship between IV and TV can be found in Figure 27, which illustrates how IV spike around maturity, but otherwise remains around the same level as the option was struck and TV has fewer spikes, but more variation through the lifetime of the option. These effects will be further discussed and elaborated upon in the subsequent case studies.

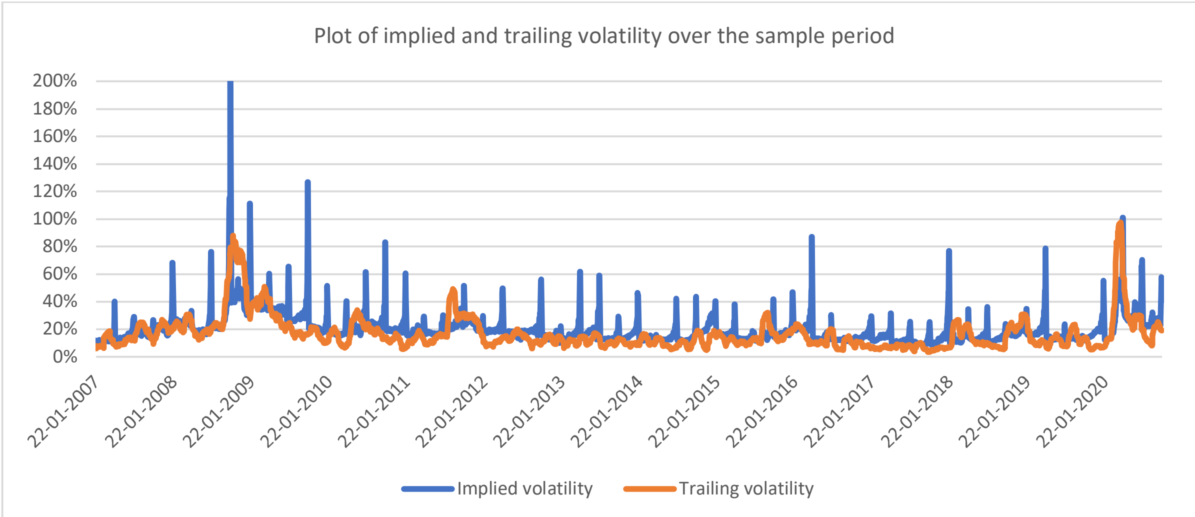


Figure 27: Plot of the call options' IV and the 30-day TV from the OptionMetrics dataset over the entire sample period

Out of the 55 delta-hedged calls, 30 of them end up earning money whereas the remaining 25 of the options lose money. This, in combination with the extremely negative minimum value and relatively moderate maximum value, illustrates the effect of left-tail risk on option trading and how volatility trading through delta-hedging options is imperfect on a discrete basis as one does not perfectly offset the movements in the underlying. Through this, it is evident that one cannot just sell volatility continuously, but must be able to time it to avoid the drawdowns on the P&L.

Case studies

This section will outline three delta-hedged calls which will act as case-studies to showcase what drives the return and understand how different Greeks affect the options. First, we will analyse a short delta-hedged call option where hedging with IV and TV produce negative and positive returns, respectively. Second, we will analyse a delta-hedged call sold prior but expiring during the COVID-19 crash and finally, we will analyse a delta-hedged call sold just after the financial crisis to also understand how different financial environments affect the performance of trading volatility through delta-hedged calls. It should be noted that when we address hedging error, it is the error of and from using an IV-based hedge relative to a hedge based on the proxy for RV.

Case study #1

Figure 28 and 29 shows the first delta-hedged call's IV, 30-day TV and the percentage return over its lifetime and Table 6 and 7 summarises the call's characteristics. It is a delta-hedged call struck on the 22nd of October 2007 with an expiry on the 19th of January 2008. This delta-hedged call is struck just before the worst onset of the financial crisis in October 2008 in a rather stable environment as evidenced by the low 30-day TV. We observe that the 90day RV is 21.85%, which is higher than the IV of 19.99% so we expect to lose money on the delta-hedge of this call, approximately -6.89 in absolute terms, and -1.85% in percentage terms, which is the difference between the observed price at t_0 IV and a theoretical price using the 90d trailing RV to calculate the price at t_0 divided by the constant margin requirement. We note that the return of hedging with TV exhibits a larger standard deviation than hedging with the IV. The skew is positive for hedging with IV whereas it is negative for hedging with TV, implying that the IV-based hedge's returns experience a larger positive distribution relative to the returns from hedging with TV. Yet, the final return from hedging with IV is negative, whereas the return from hedging with TV is positive.

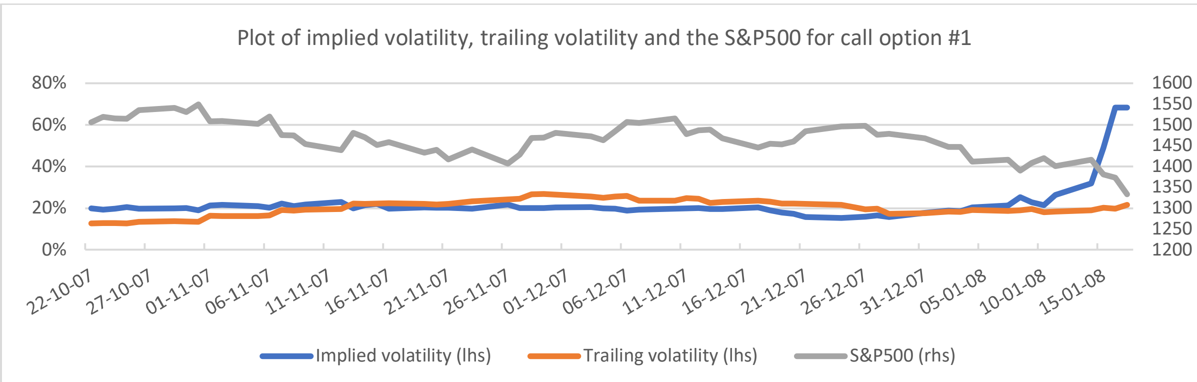


Figure 28: Plot of the IV, TV and the S&P 500 used for the analysis of call #1 from the option is struck until maturity, with volatility in percentages on the lhs and the value of S&P500 in dollars on the rhs

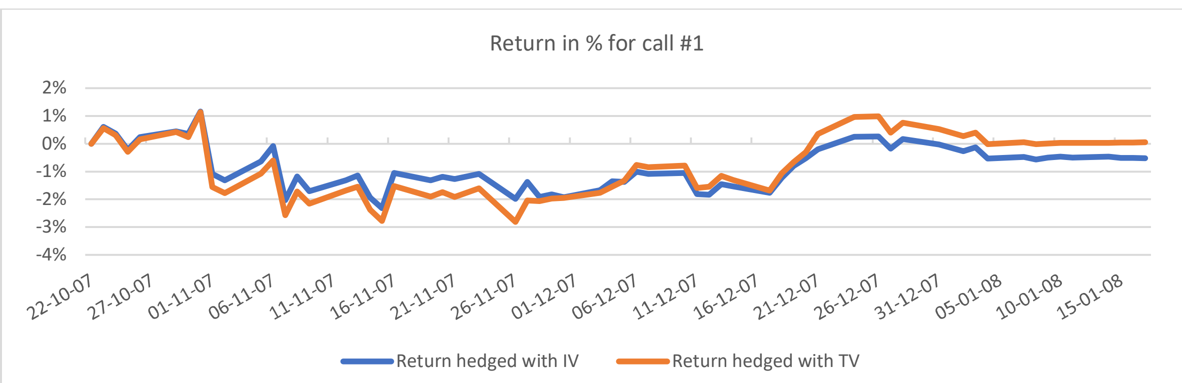


Figure 29: The percentage return from the option is struck until maturity for call #1

Call	IV	TV 30d trailing	RV 90d trailing	Spot	Strike	Risk-free rate	Div.	Price t_0 with IV/RV 90d	Constant margin
Start	19.99%	12.78%	N/A	1506.33	1525	5.02%	2.03%	54.05	371.59.
End	68.23%	21.57%	21.85%	1333.25	1525	5.02%	2.03%	60.94	371.59

Table 6: General information and parameters used for the empirical results of call #1

Call	Final P&L	Skew	Return	Kurtosis	Min	Max	Std	Variance	Exp. returns
IV	-0.83%	0.23	-0.53%	-0.67	-2.32%	1.16%	0.79%	0.01%	-1.85%
TV	-0.79%	-0.04	0.05%	-1.22	-2.81%	1.14%	1.06%	0.01%	-1.85%

Table 7: Return characteristics

The reason for this difference is due to the delta-hedging error occurring from differences in the volatility used to calculate the delta, as evidenced by Renault & Touzi (1996) and Kurpiel & Runcalli (2011). To see this, we rely on the findings from our simulations, which are summarized in appendix #7. Whether the IV-based hedge is over- or under-hedged depends on 1) whether IV > TV and 2) the moneyness of the option. Given that the IV-based hedge is over- or under-hedged, the return effect will be dependent on the movements of the underlying. For this delta-hedged call, there are 3 important moves affecting the return. First, the IV is higher than the 30d TV, which serves as a proxy for the RV, when the option is struck. The spot immediately moves from ATM to ITM and the higher IV causes the option to be under-hedged. As the spot takes a big drop from slightly ITM to OTM, it causes a positive hedging error, which is why the return of the TV hedge quickly becomes lower than return of the IV-based hedge. Secondly, as the underlying continues to move further OTM, TV and IV are relatively close, making the delta hedging positions similar. However, TV eventually starts to increase more than IV and stays higher for the next many time-steps, more specifically from the 13/11/2007 to the 28/12/2007. Since the option is now OTM moving towards ATM, this leads to a negative hedging error, which cancels out the positive return effect from before and it explains why the return is approximately similar on the 17/12/2007 for both the IV-based and TV-based hedges. Finally, as TV is higher than IV and the option is OTM i.e., the option is under-hedged, the underlying makes one last big move from on the 17/12/2007 from OTM towards ATM, leading to a negative hedging error and ultimately a lower final return. These effects are also illustrated in Figure 30 with the evolution of each hedging volatilities' delta. However, the differences in delta are very small in magnitude, which is why the differences in the final return are very small as well. Given that we have a negative delta through the short call, we must take a long position in the stock to be delta-neutral and hedged. Lastly, we can attribute part of the positive performance of the TV to the preference of hedging with TV (as a proxy for the true RV) in rather stable volatility environments relative to hedging with IV. The findings from this trade of volatility through a delta-hedged call highlight the same effects that we see within our simulations, specifically how positions can be under or over-hedged given volatility environments and support previous empirical findings on trading volatility (Kurpiel & Roncalli, 2011) (cf. section 3.2.7).

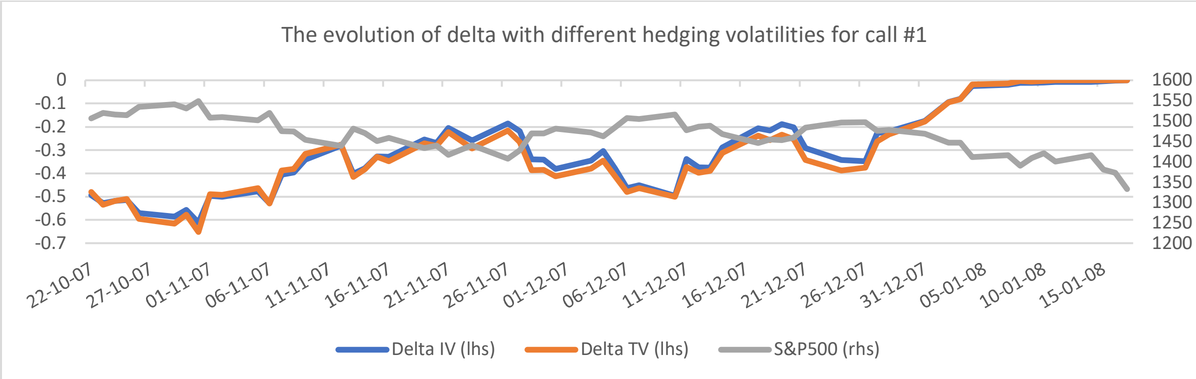


Figure 30: Plot of the deltas used for the analysis of call #1, the lhs is the value of delta and the rhs is the value of the S&P500

Case study #2

Figure 31 showcases the second delta-hedged call's IV and 30-day TV, Figure 32 showcases the return over its lifetime and Table 8 and 9 summarises the delta-hedged call's characteristics. It is a delta-hedged call struck on the 21st of April with an expiry on the 19th of July 2014. This delta-hedged call is struck in quite a stable macroeconomic environment as evidenced by the low IV and 30-day TV. We observe that the 90day RV is 7.54%, which is lower than the IV at inception of 12.49%, leading us to expect a positive P&L at approximately 18.11 in absolute terms and 3.95% in percentage terms. We note that the return of hedging with TV exhibits a larger standard deviation than hedging with IV, just as with the previous delta-hedged call. The skew is negative for the IV-based hedge whereas it is positive for the TV-based hedge, implying that the return from hedging with TV experiences a larger positive distribution relative to the IV-based hedge's returns. This can also be seen from the TV-based hedge outperforming the IV-based hedge. Lastly, the kurtosis is -1.35 for the TV-based hedge, indicating that the distribution of returns is relatively flat.

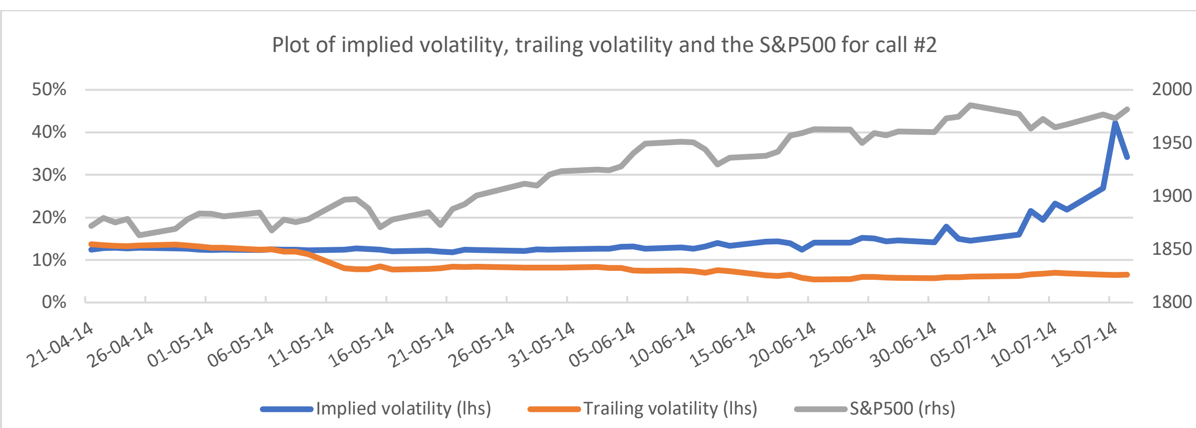


Figure 31: Plot of the IV, TV and the S&P 500 used for the analysis of call #2 from the option is struck until maturity, with volatility in percentages on the lhs and the value of S&P500 in dollars on the rhs

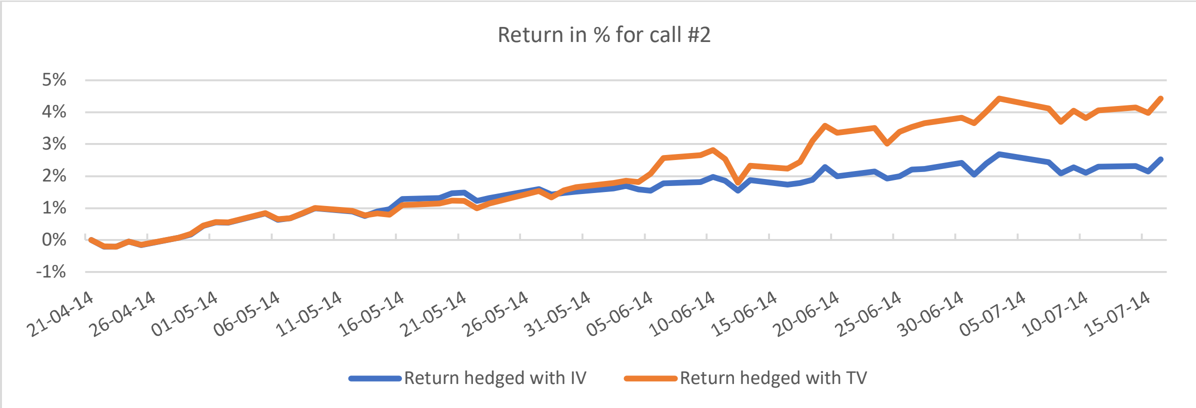


Figure 32: The percentage return from the option is struck until maturity for call #2

With this delta-hedged call, we attribute the majority of the return differences between hedging with IV relative to hedging with TV to the same under- and over-hedged effects as with delta-hedged call#1, and as found in our simulations (cf. appendix #7). We further observe that this delta-hedged call based on its delta profiles confirms the findings of the previous delta-hedged call, namely that hedging with the TV tends to perform better during stable volatility environments given that the IV-based hedge tends to have a negative hedging error vis-à-vis the TV. More specifically, we see from Figure 31, that the IV and TV are very similar until the 9/5-2014 resulting in close to no difference in the return. However, as TV begins to decrease and IV remains somewhat stable, IV becomes significantly greater than TV and with the option being ITM, this results in an under-hedged delta position. As the underlying continues to move further ITM, the IV-based hedge is negatively affected by being under-hedged relative to hedging with TV. This ultimately leads to the large difference in the final return.

Call	IV	TV 30d trailing	RV 90d trailing	Spot	Strike	Rf	Div	Price t_0 with IV/RV 90d	Constant margin
Start	12.49%	13.72%	N/A	1871.89	1870	0.04%	1.91%	54.05	458.75.
End	68.23%	21.57%	21.85%	1958.12	1870	0.04%	1.91%	60.94	458.75

Table 8: General information and parameters used for the empirical results of call #2

Call	Final P&L	Skew	Return	Kurtosis	Min	Max	Std	Variance	Exp. returns
IV	1.46%	-0.60	2.63%	-0.55	-0.21%	2.69%	0.78%	0.01%	3.95%
TV	2.06%	0.13	4.24%	-1.35	-0.20%	4.43%	1.44%	0.02%	3.95%

Table 9: Return characteristics

It can be further illustrated how the path-dependency, specifically within the dollar gamma, affects the return of each delta-hedge. A main driver in the return and return of short options is the gain on the time decay exceeding the dollar gamma loss (cf. appendix #2). This is highlighted in Figure 33 (a). From panel (a), we see that as the spot oscillates near the strike or the volatility - especially the TV - decreases, the absolute dollar gamma increases. This is illustrated in the peak of both the TV-based and IV-based dollar gammas when the spot drops to 1872 as the spot is thus near the strike 1870 at timestep 22 (20th of May 2014). The higher peak in the TV-based dollar gamma relative to the IV-based dollar gamma is due to the drop in TV from 11.38% to 8.05% evidenced in Figure 31 at timestep 16 (13th of May 2014), which the IV does not experience in the same magnitude, since it only decreases from 12.3% to 12.2%. This is in line with the findings that the (dollar) gamma increases as the volatility decreases, which is confirmed by the 3D profile of the IV-based gamma, too.

A decrease in the dollar gamma in absolute terms for a short option seller of the delta-hedged call affects the P&L and therefore return positively, assuming spread between the RV and IV is constant per equation (29) (cf. section 3.2.2). We specifically see this effect in play at timestep 33 (the 5th of June 2014) where the spot increases from 1927.88 to 1940.46, resulting in the dollar gamma decreasing in absolute terms from -137.85 to -124.97 for the IV-based and -149.45 to -103.38 for the TV-based dollar gamma. Here, all else equal, we would expect a better P&L and return performance with the TV-based hedge rather than the IV-based hedge given the larger dollar gamma exposure in absolute terms in the IV-based hedge. The value of the dollar gamma has a negative effect on the return of hedging with IV as the return decreases with 0.04% whereas the return from hedging with TV increases by 0.26%. This is attributed to the dollar gamma exposure of the IV-based delta-hedged, causing it to underperform relative to the TV-based delta-hedge given that we manage to earn more on the time-decay when hedging with TV rather than IV. The findings from this trade of volatility through a delta-hedged call highlight the same effects that we see within our simulations, primarily the findings support previous empirical results on the path-dependency evident within volatility trading (cf. section 3.2.7).

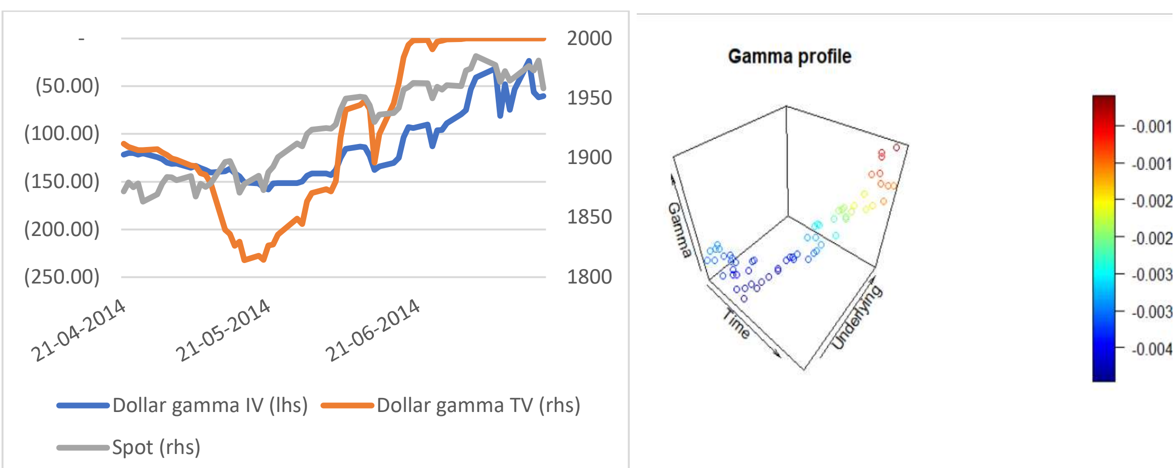


Figure 33: Panel (a) the dollar gamma evolution relative to the spot and panel (b) the 3D gamma profile of the IV's gamma

Case study #3

We observe that the 90day RV is 61.34%, which is higher than the IV at inception of 12.09%, leading us to expect a negative P&L at approximately -312.68 in absolute terms and in percentage terms -37.67%. We note that the return of hedging with TV exhibits a larger standard deviation than hedging with IV, just as with the previous delta-hedged call. This is in line with our findings on how a hedge with a true RV deteriorates in unstable volatility environment, which the COVID-19 crisis can be argued to be. The skew is positive for the IV-based hedge whereas it is negative for the TV-based hedge, implying that the returns from hedging with IV exhibits a larger positive distribution relative to hedging with TV. This can be seen from the IV-based hedge outperforming the TV-based hedge, consistent with the findings of how a true RV-based hedge performs in unstable volatility environments.

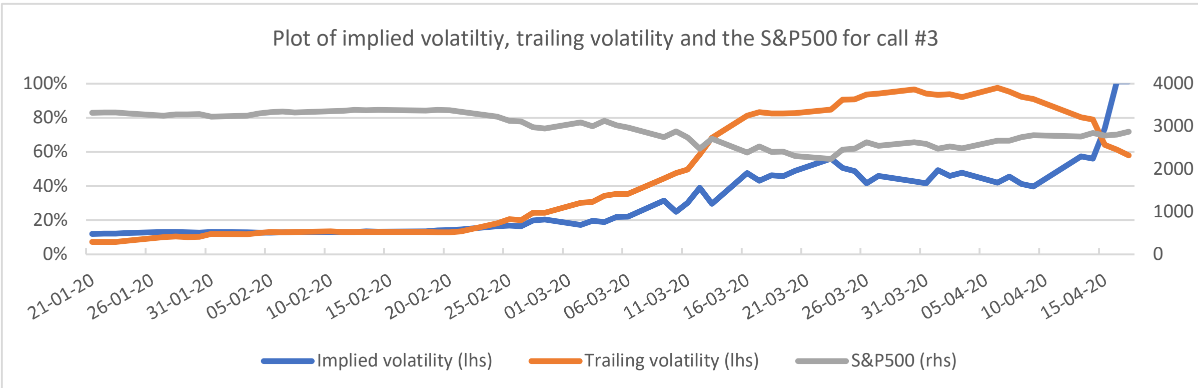


Figure 34: Plot of the IV, TV and the S&P 500 used for the analysis of call #3 from the option is struck until maturity, with volatility in percentages on the lhs and the value of S&P500 in dollars on the rhs

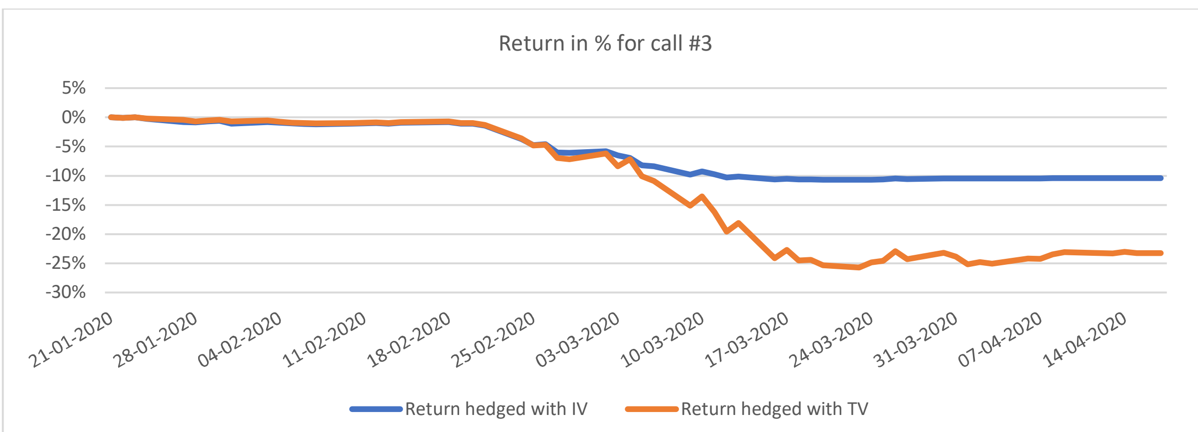


Figure 35: The percentage return from the option is struck until maturity for call #3

To understand how the return deteriorates, Figure 36 illustrates the vega sensitivity at time-step 23 (21st of February 2020) calculated using IV. The 30-day TV is 13.57% and at the next time-step it rises to 18.28%, where the spot decreases from 3337.8 to 3225.9. At this time-step, the returns from hedging with IV and TV are both substantially negatively affected. As in line with the simulations, we see that the vega sensitivity for options is much higher when they are just near or ATM, which is the case here as the delta-hedged calls moves from being ITM to OTM, reflecting a price decrease from 81.55 to 41.5 in the call. As evidenced, a small increase in the volatility thus erodes the return more than if the option was further OTM which can observed by the stabilising trend in the return of hedging with IV in Figure 35 after the vega impact has happened. Vega's attributes act as a big driver behind the negative return for both an IV- and TV-based delta-hedge.

Call	IV	TV 30d trailing	RV 90d trailing	Spot	Strike	Rf	Div	Price t_0 with IV/RV 90d	Constant margin
Start	12.09%	7.31%	N/A	3320.79	3325	1.56%	2.41%	75.55	830.09
End	101.1%	57.93%	61.34%	2874.56	3325	1.56%	2.41%	388.22	830.09

Table 10: General information and parameters used for the empirical results of call #3

Call	Final P&L	Skew	Return	Kurtosis	Min	Max	Std	Variance	Exp. returns
IV	-6.16%	0.24	-10.42%	-1.81	-10.70%	0.00%	4.44%	0.2%	-37.67%
TV	-12.02%	-0.10	-23.26%	-1.85	-25.75%	0.00%	10.61%	1.12%	-37.67%

Table 11: Return characteristics

To understand another driver behind the returns declining, we observe that as the dollar gamma decreases as options get further OTM, the deviations in the returns are not as sensitive to imperfect delta-hedging as if the options were ATM, which is in line with our simulations and notions on vega. Lastly, to understand why the drop is much larger when hedging with 30-day TV compared to hedging with IV, this relates back to the findings by Renault & Touzi (1996) and Kurpiel & Runcalli (2011) and our simulated results. Given that TV > IV and the option moves from OTM to further OTM, the IV-based hedge is severely under-hedged, causing a positive hedging error as the spot decreases. Furthermore, we observe that when the underlying finally starts to increase, it has a more positive impact on the TV hedge, but at this point, the option is so close to maturity and far OTM that both deltas are very close to 0, which reduces this effect. This leads us to experience a much bigger loss when hedging with TV, which is in line with previous findings of the IV-based hedge performing better in more volatile environments. The findings from this trade of volatility through a delta-hedged call further showcase how volatility trading has large downsides for the seller and that wrong volatility bets can be harmful, which is consistent with previous empirical findings and our simulations on volatility environments.

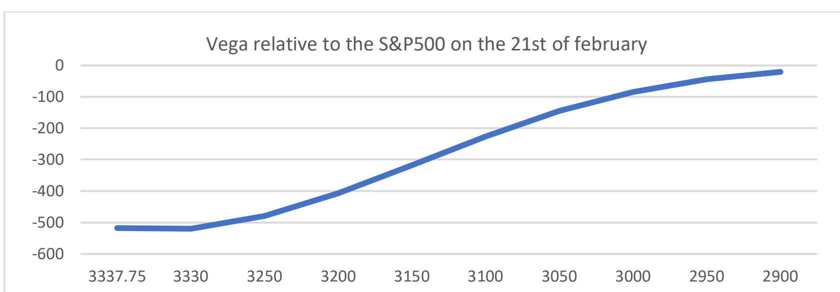


Figure 36: Plot of vega at time-step 23 for call #3

5.2.3 Empirical results for delta-hedged puts

Figure 37 presents the accumulated P&L for selling 55 different delta-hedged ATM put options on the S&P500. The final accumulated P&L is 13.24% if one hedges with the IV of the puts and 15.56% if one hedges with the TV on the SP500. Given that for the majority of the options $\sigma_{t=0}^{i} > \sigma_{T}^{r}$, this thesis would have expected a positive cumulated P&L, which is in line with what we find. When analysing the individual delta-hedged put options, we find that hedging with IV results in a mean return of 0.37% with a standard deviation of 5.13% and hedging with TV results in a mean return of 0.46% with a standard deviation of 5.98%. Again, hedging with TV results in a higher standard deviation. Similar to the calls, this is contrary to the Ahmad & Wilmott simulations where we would have expected the IV-based hedge to have a higher variance than the RV-based hedge and it is likely due to TV not being a perfect proxy for RV.

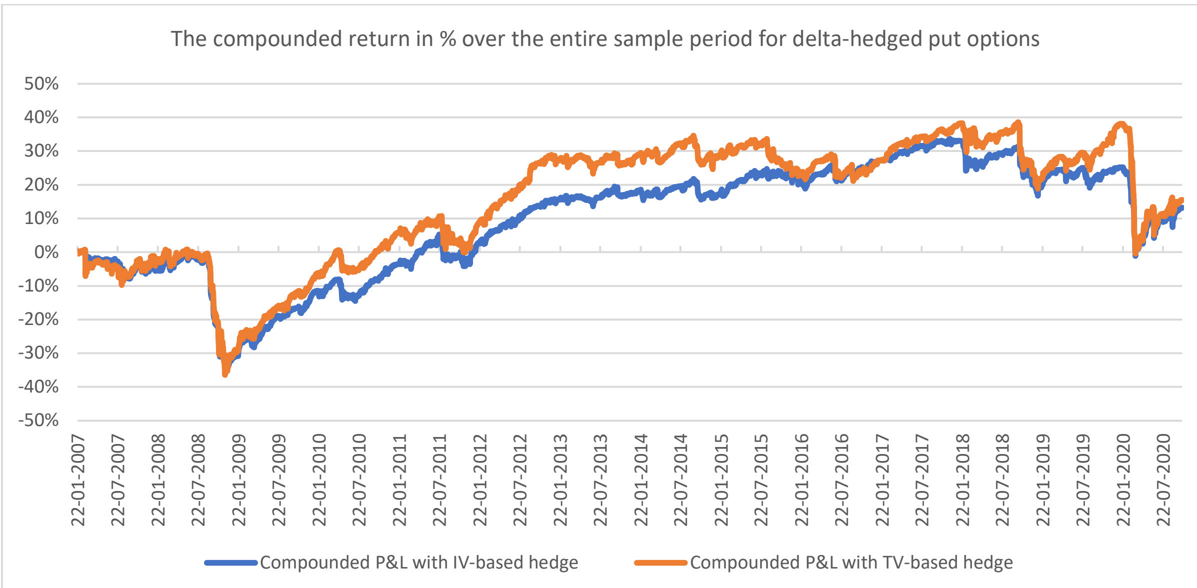


Figure 37: The compounded P&L in percentage returns relative to the capital requirements for continuously selling delta-hedged put options over the entire sample period when (a) hedging with IV and (b) hedging with RV

From the Figure, it is evident that the P&L is heavily impacted by large drawdowns from option 7 and 53 corresponding to the volatility trades made during the financial crisis and the covid-19 crisis. Similarly to the results for the calls, this is due to the RV increasing substantially as illustrated by Figure 24, which is in line with our expectations and simulated results. Given that the Covid-19 crisis had a higher maximum RV than the financial crisis, but the difference in IV and 90-day RV, the final P&L would be very dependent on the path-dependency and the time of which the option is struck. The drawdown from the financial crisis is -20.09% and -19.94% compared to -17.74% and -24.30% for the covid-19 crisis when hedging with IV and TV, respectively. When comparing the drawdown to the maximum P&L value over the entire period of 9.07% and 9.01% for hedging with IV and TV, respectively, these results confirm the massive P&L impact an unexpected increase in volatility can have on the short option P&L.

Hedge-volatility	Acc. P&L	Median	Mean	Min	Max	Std. dev.
Implied	13.24%	1.40%	0.37%	-20.09%	9.07	5.13
Trailing	15.56%	1.39%	0.46%	-24.30%	9.01	5.98

Table 12: Summary statistics of the cumulative P&L over the entire sample period for the put options when hedging with IV and TV, respectively

In contrast to the delta-hedged call options, we find that the maximum value is slightly higher when hedging with IV, but the maximum drawdown during the Covid-19 crisis is higher when hedging with TV. The accumulated return is still slightly higher when hedging with TV, but the standard deviation is also higher. Similar to the analysis of the call P&L, this suggest that hedging with TV performs better in a more stable volatility environment and worse when volatility takes more extreme values. However, the difference for the delta-hedged put options are smaller. Figure 38 contains a plot of IV of the put options relative to TV throughout the trading period. This again confirms that IV fluctuates less than TV during the lifetime of the option but has spikes close to expiry. When selling the put, the delta-hedged position using IV will be over-hedged (under-hedged) when the option is OTM (ITM), when IV is higher than TV, as a proxy for the RV, and the opposite when TV is higher than IV. This is similar to the delta-hedged call option. However, the effect is slightly different since movements in the underlying affects the put differently than the call. Assuming the put is struck close to ATM, when TV is decreasing (increasing) i.e. IV will be higher (lower) than TV, the underlying will increase (decrease) moving the option OTM (ITM). In both of the cases, the put option will be over-hedged, meaning that TV increasing (decreasing) will lead to a negative (positive) hedging error. This again explains why we see TV-based delta-hedge performing better when RV is decreasing and worse when it is increasing. These results are in line with previous literature (Kurpiel & Roncalli, 2011; Renault & Touzi, 1996) and the findings from section 5.1.3.2, which also found great path-dependency, which still holds for our empirical results. Again, this analysis is based on generalized observations with the individual options likely to exhibit varying degrees of these effects. This analysis is to a large extent made possible by volatility and the underlying being negatively correlated, making the path of the underlying more predictable and allowing us to better assess the effects. Once again, part of the difference can be attributed to the selection of options and their volatility smiles given that one could then be over or under-hedged more severely (Alexander & Nogueira, 2011).

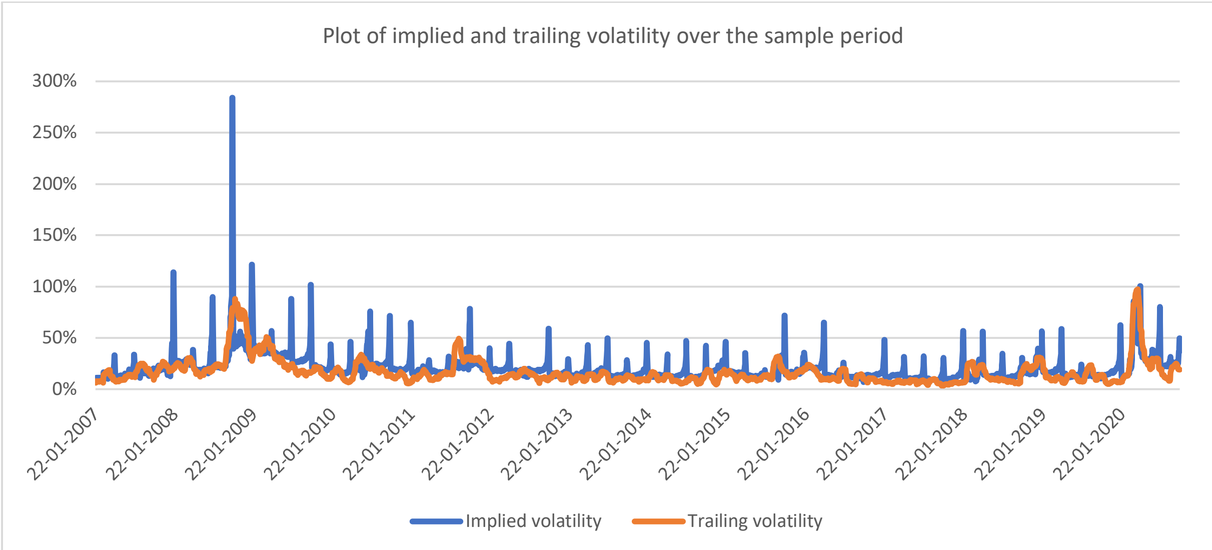


Figure 38: Plot of the put options' IV and the 30-day TV from the OptionMetrics dataset over the entire sample period

Case studies

This section will outline 3 delta-hedged puts which will act as case-studies to showcase what drives the return and understand how different Greeks affect the options. First, we will analyse a delta-hedged put where hedging with IV and TV, respectively, produce positive and negative returns. Secondly, we will analyse a delta-hedged put sold during the financial crisis and finally, we will analyse a delta-hedged put sold just after COVID-19 to better understand how different financial environments affect the performance of trading volatility through short delta-hedged put options. Similar to the case studies of calls, when this thesis refers to over- or under-hedging it is relating to the IV-based hedge.

Case study #1

Figure 40 shows the first delta-hedged put's IV and 30-day TV with the returns over the lifetime of the option being available in Figure 41 and Table 13 and 14 summarise the delta-hedged put's characteristics and returns. It is a delta-hedged put struck on the 21st of July 2008 with an expiry on the 18th of October 2008. We observe that the 90day RV is 50.99%, which is higher than the IV of 22.28%, so we expect approximately to lose -74.64 in absolute terms and in percentage terms -24.44%, which is the difference between the observed price at t_0 IV and a theoretical price using the 90d trailing RV to calculate the price at t_0 divided by the constant margin requirement. We note that the return of hedging with the IV exhibits larger standard deviations compared to the return of hedging with the TV. This is contrary to the expectation of the IV-based hedge performing better than the true RV-based hedge (which we proxy as the TV) in more volatile environments, but we attribute this discrepancy to the limitations of using a 30-day TV as a proxy for the true RV.

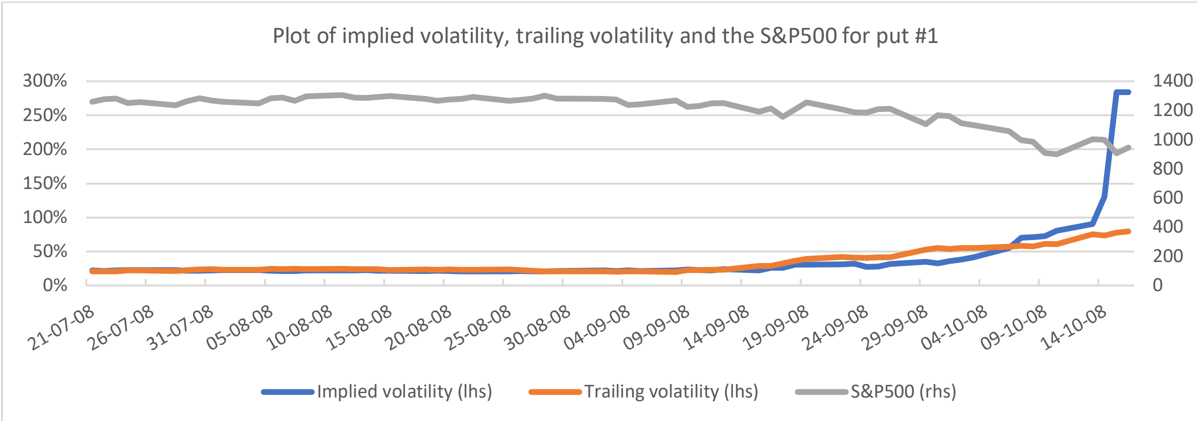


Figure 39: Plot of the IV, TV and the S&P 500 used for the analysis of put #1 from the option is struck until maturity, with volatility in percentages on the lhs and the value of S&P500 in dollars on the rhs

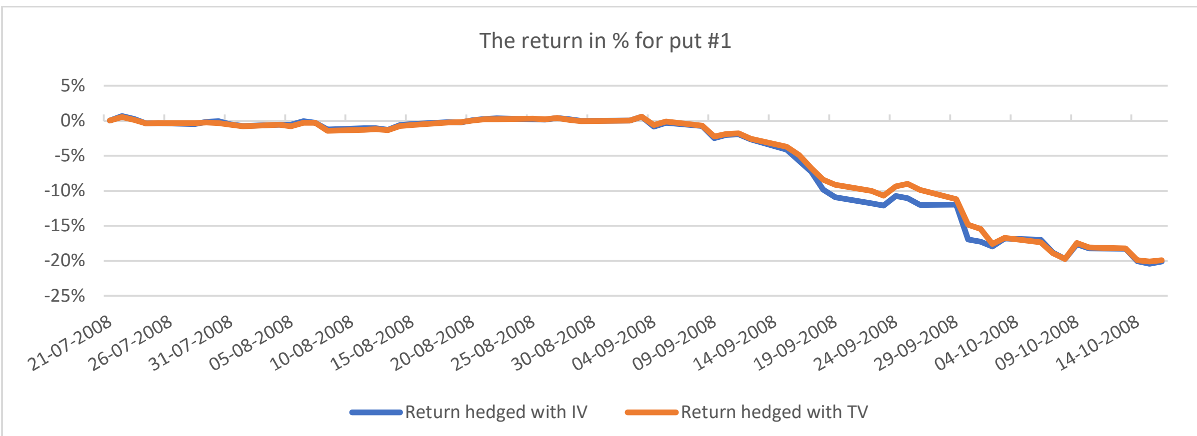


Figure 40: The percentage return from the option is struck until maturity for put #1

Put	IV	TV 30d trailing	RV 90d trailing	Spot	Strike	Rf	Div	Price t_0 with IV/RV 90d	Constant margin
Start	22.28%	20.65%	N/A	1260	1265	1.48%	2.96%	56	305.38
End	284.1%	79.40%	50.99%	946.43	1265	1.48%	2.96%	130.63	305.38

Table 13: General information and parameters used for the empirical results of put #1

Put	Final P&L	Skew	Return	Kurtosis	Min	Max	Std	Variance	Exp. returns
IV	-5.79%	-0.87	-20.09%	-0.93	-20.41%	0.68%	7.48%	0.56%	-24.44%
TV	-5.50%	-0.98	-19.94%	-0.64	-20.10%	0.57%	7.21%	0.52%	-24.44%

Table 14: Return characteristics

From the Figure showcasing volatility, we observe that for nearly 50% of the observed time-steps, our return fluctuates at around 0%, since volatility is rather stable, but as soon as the TV begins to climb and the spot dips, the returns of the volatility trade starts to decrease. The vega exposure is shown in Figure 13 at timestep 35 (8th of September 2008), just before the RV increases from 19.84% to 22.58% and where the spot decreases from 1267.8 to 1224.5. The returns for both the IV-based and TV-based delta-hedges decrease substantially during this time-step from -0.73% and -0.72% to -2.45% and -2.23%, respectively. As in line with the simulations and the findings from the delta-hedged call #1, we see that the vega sensitivity for options is much higher when they are just near or ATM, which is the case here as the delta-hedged puts moves from being ATM to ITM, reflecting a price increase from 34.2 to 59.7. Vega's attributes act as a big driver behind the negative returns of both the IV-based and TV-based hedges. To further emphasise drivers behind returns, we observe that the dollar gamma decreases as the delta-hedged put gets further ITM. This means that as the volatility continue to increase, the P&L becomes less sensitive to imperfect discrete delta-hedging given the reduced dollar gamma and vega exposure. Further, it is interesting to note that the case of being severely under- or over-hedged with the IV-based hedge is not as evident in this delta-hedged put's case given that the IV and TV follow each other quite closely. We observe some differences in IV and TV towards maturity, but at this point, both delta positions are close to -1 and it has little impact on the returns. Had the timing of the increase in TV been different, hedging with IV would likely have had a positive hedging error. This relates back to the findings by Renault & Touzi (1996) and Kurpiel & Runcalli (2011). The findings from this trade of volatility through a delta-hedged put showcase how volatility trading has a large downside for the seller and that wrong volatility bets can be harmful, which is consistent with previous empirical findings and our simulations on volatility environments.

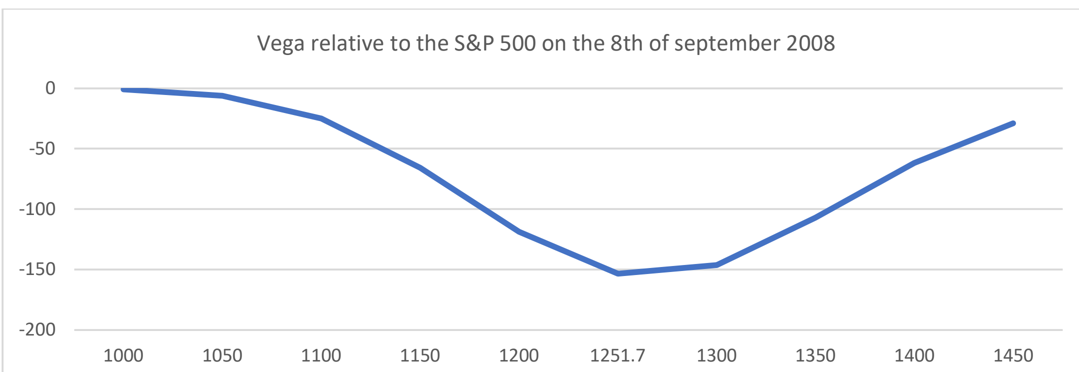


Figure 41: The vega for put #1 at time-step 35

Case study #2

Figure 42 shows the second delta-hedged put's IV, 30-day TV and Figure 43 returns over its lifetime. The delta-hedged put's characteristics returns are summarized in Table 15 and 16. It is a delta-hedged put struck on the 22nd of July 2013 with an expiry on the 19th of October 2013. We observe that the 90day RV is 10.73%, which is lower than the IV of 13.01%, so we expect approximately to earn 7.17 and in percentage terms 1.72%. We note that the return of hedging with the IV exhibits larger standard deviations compared to the return of hedging with the TV, yet the discrepancy is relatively small.

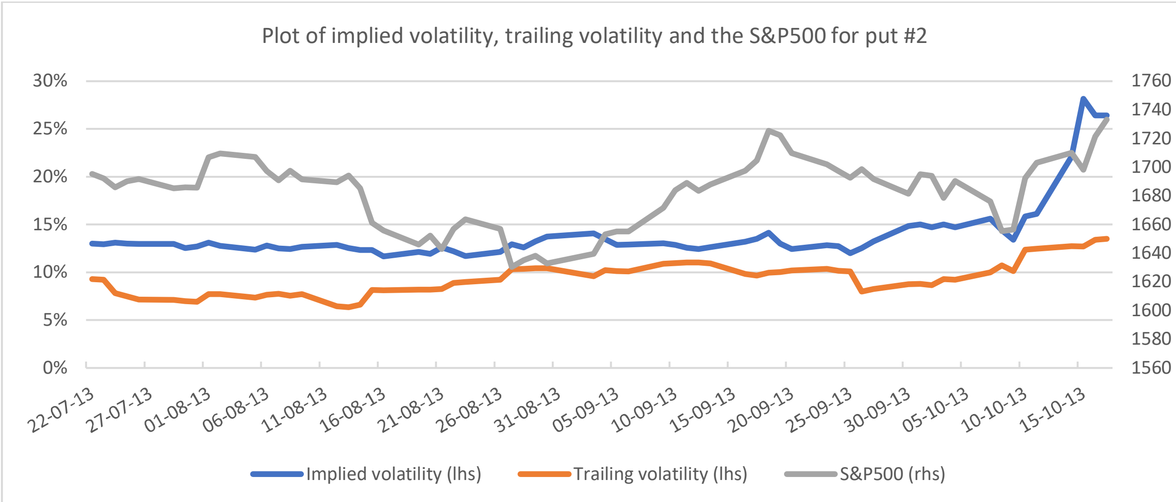


Figure 42: Plot of the IV, TV and the S&P 500 used for the analysis of put #2 from the option is struck until maturity, with volatility in percentages on the lhs and the value of S&P500 in dollars on the rhs

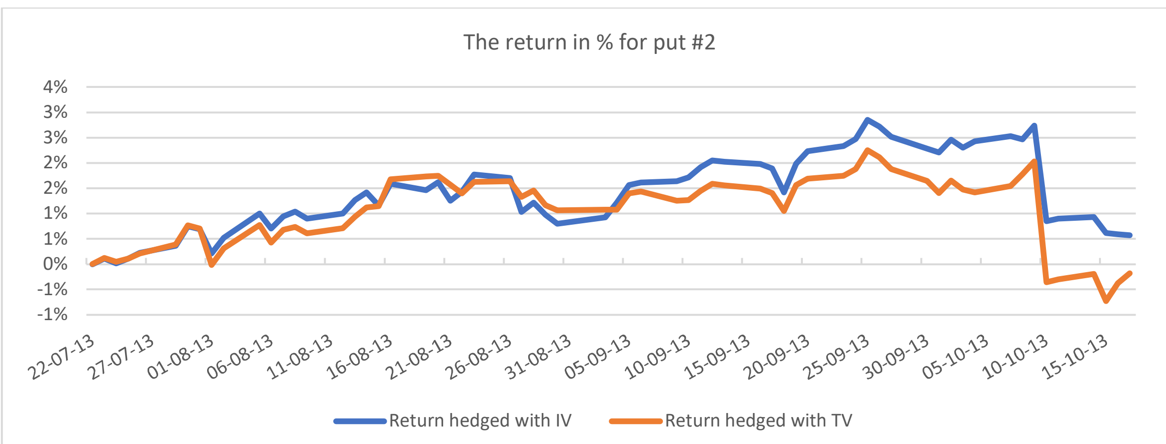


Figure 43: The percentage return from the option is struck until maturity for put #2

What is interesting about this volatility trade is that both the IV-based and TV-based hedges returns were trading positively throughout almost the entire lifetime, up until the 10th of October 2013, when both returns suddenly deteriorate, which is observed in Figure 44. From Figure 44, we observe that the drop in the returns happens at timestep 58 (10th of October 2013) when the spot rises from 1656.4 to 1692.6, causing the value of our puts to decrease from 36.9 to 13.85, but the loss on the hedging position is larger. The effect on the returns is a result of discrete hedging being imperfect given the path-dependency in the underlying. Both hedging deltas are approximately halved in value over this timestep from 0.846 for the IV-based delta and 0.905 for the RV-based delta to respectively 0.475 and 0.471 (see Figure 44). We further note that the dollar gamma of the IV-based delta hedge increases in absolute terms over this time-step from -197.159 to -307.130. This has a negative effect and further confirms the imperfect delta-hedge, with a similar effect observed for the TV-based delta-hedge. These findings highlight how path-dependency affects the ability to trade volatility given that discrete hedging distorts the returns.

Put	IV	TV 30d trailing	RV 90d trailing	Spot	Strike	Rf	Div	Price t_0 with IV/RV 90d	Constant margin
Start	13.01%	9.27%	N/A	1695.53	1690	0.04%	2.01%	43.65	416.45
End	26.42%	13.51%	10.73%	1733.15	1690	0.04%	2.01%	36.48	416.45

Table 15: General information and parameters used for the empirical results of put #2

Put	Final P&L	Skew	Return	Kurtosis	Min	Max	Std	Variance	Exp. returns
IV	1.40%	0.04	0.57%	-0.96	0.00%	2.85%	0.77%	0.01%	1.72%
TV	1.05%	-0.69	-0.18%	-0.49	-0.73%	2.25%	0.72%	0.01%	1.72%

Table 16: Return characteristics

Finally, the return differences between hedging with IV and TV can be attributed to the differences in delta. Since IV > TV throughout the lifetime of the option, the option is under-hedged when ITM and over-hedged when OTM. The main return difference comes from the option moving from ITM towards ATM on 29/8/2013, leading to the option being under-hedged, which results in a positive hedging error. The differences in delta around these time-steps are further illustrated in Figure 44. The findings of the hedge being under-hedged are consistent with the observations of Renault & Touzi (1996) and Kurpiel & Runcalli (2011), and the mechanics of the discrete hedging are also shown and proved in the simulations of this thesis, but we also observe this in other options, e.g., in the aforementioned delta-hedged call#1 (cf. section 5.2.2).

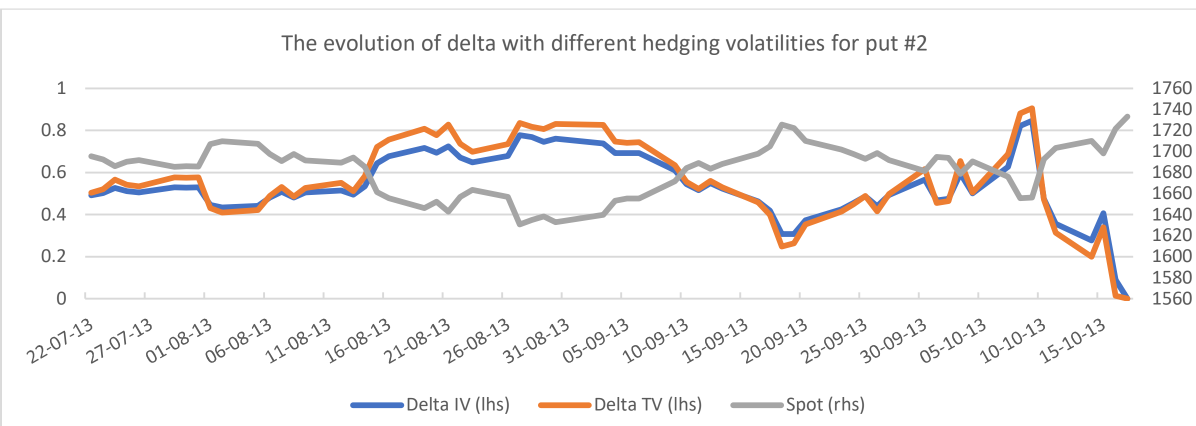


Figure 44: Plot of the deltas used for the analysis of put #2, the lhs is the value of delta and the rhs is the value of the S&P500

Case study #3

Figure 45 shows the last delta-hedged put's IV, 30-day TV, Figure 46 shows the returns over its lifetime and Table 17 and 18 summarise the delta-hedged put's characteristics and returns. It is a delta-hedged put struck on the 20th of July 2020 with an expiry on the 16th of October 2020. We observe that the 90day RV is 17.52%, which is lower than the IV of 21.40%, so we expect approximately to earn 23.81 and in percentage terms

2.93%, which is the difference between the observed price at t_0 IV and a theoretical price using the 90d trailing RV to calculate the price at t_0 divided by the constant margin. We observe that the TV-based hedge performs better in a relatively stable volatility environment compared to IV-based hedge, which is in line with our previous findings and expectations.

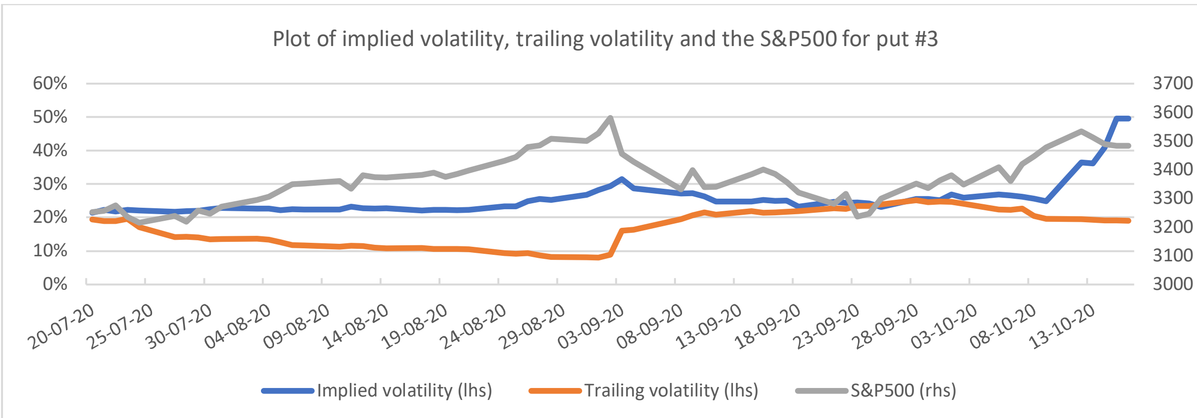


Figure 45: Plot of the IV, TV and the S&P 500 used for the analysis of put #3 from the option is struck until maturity, with volatility in percentages on the lhs and the value of S&P500 in dollars on the rhs

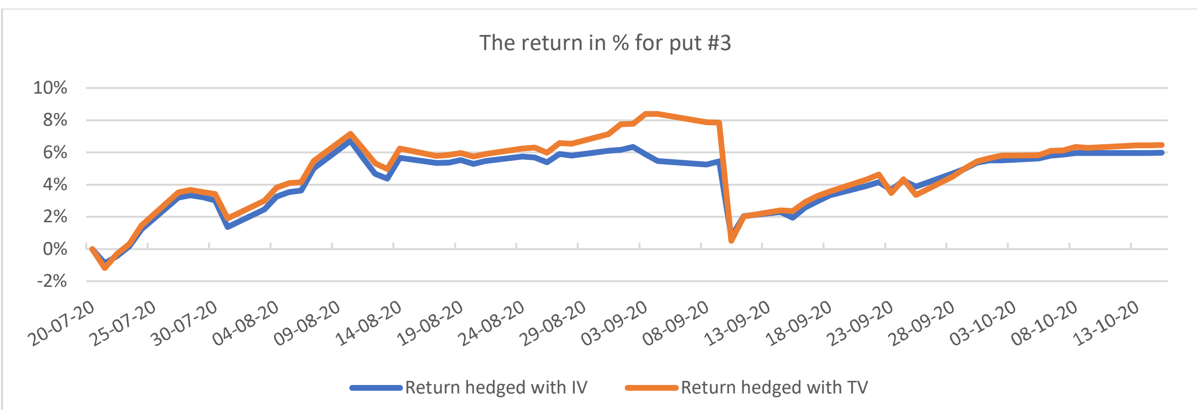


Figure 46: The percentage return from the option is struck until maturity for put #3

With this delta-hedged put, we observe the same attributes as with the previous delta-hedged options regarding the hedging errors impact on the returns. As IV > TV and the option is generally ATM moving towards OTM, the IV-based delta position is over-hedged. More specifically, we observe three important movements of the underlying. The first move is from ATM to OTM and with the IV-based delta being over-hedged this causes a negative hedging error. Secondly, the option moves from OTM towards ATM, which has the opposite effect as before and explains why the returns of the IV-based and TV-based hedges are approximately the same on the 9/9/2020. Finally, the option moves from ATM to OTM, which we would expect to cause a negative hedging error. However, at this point, the IV and TV are very similar and the option is so close to expiry that the delta values are close to 0.

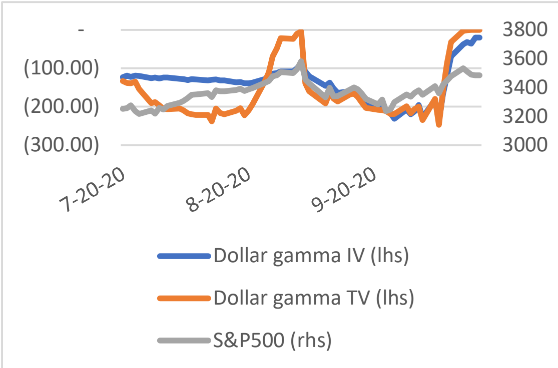
Put	IV	TV 30d trailing	RV 90d trailing	Spot	Strike	Rf	Div	Price t_0 with IV/RV 90d	Constant margin
Start	21.40%	19.43%	N/A	3251.8	3260	0.01%	1.72%	145.9	811.93
End	49.61%	19.02%	17.52%	3483.3	3260	0.01%	1.72%	122.1	811.93

Table 17: General information and parameters used for the empirical results of put #3

Put	Final P&L	Skew	Return	Kurtosis	Min	Max	Std	Variance	Exp. returns
IV	1.38%	-0.07	3.30%	-1.09	-1.96%	3.30%	1.35%	0.02%	2.93%
TV	1.86%	-0.24	3.75%	-1.41	-0.66%	4.39%	1.49%	0.02%	3.43%

Table 18: Return characteristics

To further elaborate on the effects outlined above, Figure 47 illustrates the IV-based and TV-based dollar gamma profiles and a 3D profile of the IV-based gamma. We observe the same effects as with the delta-hedged calls. As volatility increases, the dollar gamma decreases and vice versa, evident in figure (b). An increase in the dollar gamma in absolute terms for a short option seller of the delta-hedged put affects the returns negatively assuming spread between the RV and IV is constant per equation (29). This increase in the dollar gamma in absolute terms is evidenced at timestep 36 (8th of September 2020) where the spot decreases from 3426.96 to 3331.84. The dollar gamma increases in absolute terms from -120.96 to -145.87 for the IV-based hedge and from -159.88 to -191.04 for the TV-based hedge. Assuming the spread in volatility per equation (29) stayed the same, we would expect that the negative returns effect from an IV-based hedge is relatively smaller vis-à-vis the returns effect on the TV-based hedge. We observe that the return for the IV-based hedge increases from -0.44% to 0.46% and the return for the TV-based hedge decreases from 1.17% to 0.85%. This indicates that our expectations are more or less correct, but that the P&L effect when hedging in a discrete environment cannot only be attributed to the dollar gamma exposure, but also to the return from the imperfect delta-hedge, unhedged vega effects and other return effects. The findings from this short volatility trade highlight the same effects as we saw within our simulations, specifically how positions can be under or over-hedged given volatility environments when trading volatility, but primarily the findings support previous empirical results on the path-dependency evident within volatility trading and how gamma exposure drives the return of volatility trading through options (cf. section 3.2.3).



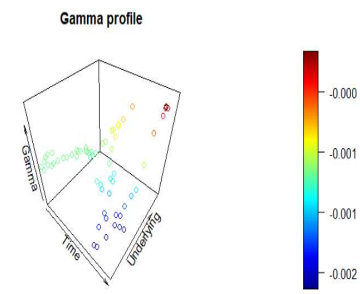


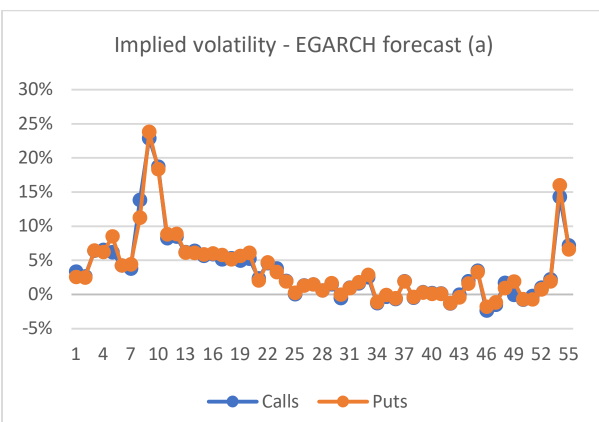
Figure 47: Panel (a) the dollar gamma evolution relative to the spot and panel (b) the 3D gamma profile of the IV's gamma

5.3 The option pricing indicator

The simulated results demonstrate a substantial P&L impact from sudden changes in RV, which was further corroborated by the large drawdowns during periods of increased volatility in our empirical results, indicating a significant left-tail risk for short option strategies. From the perspective of a trader, this suggests great benefit from introducing a volatility timing element to the short option strategy, which will now be explored.

5.3.1 The empirical motivation for the option pricing indicator

Since IV is based on the market price of the option, which is affected by supply and demand, the proposed volatility timing strategy is created around the idea of the market mispricing derivatives. This is primarily due to exchanges overpricing derivatives to cover for unexpected black swan events (Dash et al., 2012) and a large influx of unsophisticated retail investors, with limited option pricing knowledge (Rostan et al., 2020).



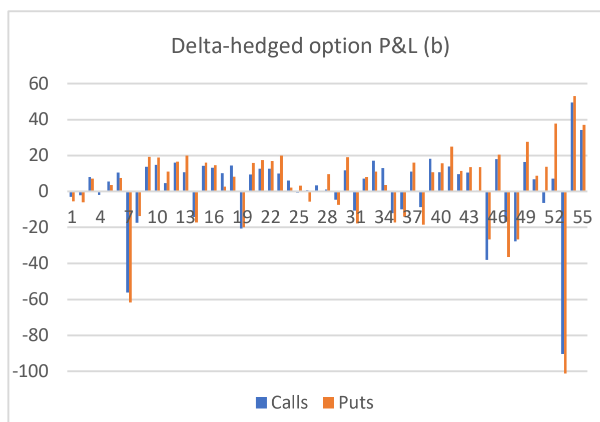


Figure 48: Panel (a) is the difference between IV and the EGARCH forecast used for categorising the options, Panel (b) is the final P&L for each individual short volatility trade

Drawing inspiration from previous empirical studies (Dash et al., 2012; Rostan et al., 2020; Sheu & Wei, 2011), this thesis aims at creating a volatility timing strategy by comparing the IV at $t = 0$ to a 13-weeks

EGARCH forecast of RV to identify over- and underpriced options. If IV is more than 5% larger than the forecast, the option is categories as overpriced and this thesis sells a short delta-hedged option strategy using a factor of 2 i.e. a leverage of 100% of the initial fully funded margin, if IV is between 5% to 2% larger than the forecast, the option is categories as fairly prices and this thesis sells a short option using a factor of 1 and if IV is less than 2% larger than the forecast, the option is considered underpriced and this thesis sells a short delta-hedged option strategy using a factor of 0.5. The empirical justification and further elaboration on this approach is found in the methodology section.

5.3.2 Backtesting the option pricing indicator

Table 19 contains the results from the volatility timing strategy for puts and calls when hedging with both IV and TV. Generally, we find that selling a short delta-hedged option strategy with the option pricing indicator improves the compounded return substantially, it increases the standard deviation slightly, but has little effect on the drawdown and increases the maximum gain. Overall, the option pricing indicator is argued to improve the performance. This is especially true for the put options, where we are able to increase the mean return with a multiplier of 3x, but only increase our standard deviation with a multiplier of 1.2-1.4x. This suggests a much higher Sharpe-ratio and a better risk return trade-off. The same holds for the delta-hedged short call options, although the exhibit a much more modest improvement. We find that when using the option pricing indicator and hedging with IV, the standard deviation remains almost constant, but the mean return increases with a multiplier of 2x. For the TV-based hedge we also increase the mean return with a multiplier of 2x, but also exhibit a relatively large increase in standard deviation. Generally, we would expect a higher standard deviation when returns increase, as the statistic does not differentiate between negative and positive deviations. What is really interesting is the substantial improvement in the risk-return trade-off. We further observe that when introducing the option pricing indicator, we increase the compounded return on all the trading strategies, confirming the effectiveness of our option pricing indicator and its ability to identify options with a positive P&L and scale down the options with an increased risk of a loss. The best performing strategy is the delta-hedged call using the TV, which yields a compounded return of 121.16%, compared to the return of selling a delta-hedged call without the indicator of 43.9%.

Compounded without the option pricing indicator

Type	Hedging volatility	Pricing indicator	Average position size	Compounded return	Median P&L	Mean P&L	Standard deviation	Min P&L	Max P&L
Put	Implied	n.a.	1	13.24%	1.40%	0.37%	5.13%	-20.09%	9.07
Put	Trailing	n.a.	1	15.56%	1.39%	0.46%	5.98%	-24.30%	9.01
Call	Implied	n.a.	1	42.14%	1,71%	0.73%	5.66%	-16.31%	7.01%
Call	Trailing	n.a.	1	43.90%	1.89%	0.84%	4.21%	-23.26%	9.95%

Compounded return using the option pricing indicator

Type	Hedging volatility	Pricing indicator	Average position size	Compounded return	Median P&L	Mean P&L	Standard deviation	Min P&L	Max P&L
Put	Implied	Yes	1.07	60.44%	0.78%	1.12%	7.04%	-23.67%	18.14%
Put	Trailing	Yes	1.07	86.75%	0.81%	1.43%	7.54%	-21.58%	18.02%
Call	Implied	Yes	1.06	105.14%	0.87%	1.49%	5.98%	-16.31%	14.02%
Call	Trailing	Yes	1.06	121.16%	1.11%	1.74%	7.50%	-23.26%	19.90%

Table 19: overview of the short option performance with and without using the option pricing indicator

Looking closer at the data and the performance of our indicator, we find that 18 put options and 17 call options are identified as overpriced (see Table 20). Out of the 18 overpriced put options, 83% ends up with a positive P&L and out of the 17 overpriced call options, 88% ends up with a positive P&L. This holds for both hedging with RV and IV. Comparing this to the other options, we find that only 64% of the underpriced puts and 70% of the underpriced calls ends up with a positive P&L, while 56% of the fairly priced puts and 46% of the fairly priced calls ends up with a positive P&L. It is evident that the option pricing indicator is able to somewhat accurately categories the options in terms of the probability of a positive P&L. Interestingly, the fairly priced have the lowest positive-to-negative ratio.

Category	Put			Call		
	Options in category	Positive P&L	Negative P&L	Options in category	Positive P&L	Negative P&L
Overpriced	**18**	15	3	**17**	15	2
Underpriced	**28**	18	10	**27**	19	8
Fairly priced	**9**	5	4	**11**	5	6

Table 20: overview of the categorization of the options using the option pricing indicator

For further analysis of the different option categorizations, the mean return for the three categories when hedging with IV and RV is found in Table 21. For the put options, we find the mean return of the overpriced options to be large and positive, while the mean of the fairly priced and underpriced options to be negative. This clearly demonstrate the effectiveness of the option pricing indicator and is the main reason for why the delta-hedged put options' compounded return achieve such increase. For the call options, we find both the overpriced and underpriced options to have a positive mean return. This is not surprising, since the delta-hedged call options' compounded return is positive and substantially larger than that of the put options. It is interesting that the fairly priced options have a negative mean return and a large standard deviation. However, the option pricing indicator based on the IV of the call option categories the option sold prior to the covid-19 crash as fairly price, which substantially negatively impact the mean of the fairly priced options. This is in contrast to the indicator based on the puts, which accurately identify the option sold prior to the covid-19 crisis as undervalued.

Category	Hedging volatility	Put			Call		
		Options in category	Mean	Std dev	Options in category	Mean	Std dev
Overpriced	Implied	**18**	4.32%	10.19%	**17**	5.73%	7.66%
Overpriced	Realised	**18**	5.31%	10.57%	**17**	6.99%	8.89%
Underpriced	Implied	**28**	-0.09%	7.19%	**27**	0.25%	0.96%
Underpriced	Realised	**28**	-0.21%	7.45%	**27**	0.36%	1.43%
Fairly priced	Implied	**9**	-1.49%	1.93%	**11**	-2.01%	6.47%
Fairly priced	Realised	**9**	-1.23%	2.73%	**11**	-2.99%	9.16%

Table 21:Overview of the mean return and standard deviation for each category

In section 5.2, we found that hedging with TV performed better exhibiting larger drawdowns, as the hedging error from hedging with IV had a smoothing effect to the P&L. Reducing the drawdowns and amplifying the positive P&L values was a major motivation for creating the option pricing indicator. Since it appears that our option pricing indicator is effective at selecting the options with a positive P&L, we would expect hedging with TV to perform substantially better than hedging with IV once we introduce the volatility timing element. Returning to Table 19, we find that hedging with 30-day TV does lead to a higher return. However, it does still exhibit large drawdowns and a larger standard deviation, which is primarily due to the indicators lack of ability to identify the call option prior to the covid-19 crisis as underpriced, unlike the indicator based on the put.

If we were to base our indicator solely on the spread between IV of the put and the EGARCH forecast and apply it to the delta-hedged short call strategy, we would increase our compounded return when hedging with TV from 121.16% to 137.64% and reduce our maximum drawdown from -23.26% to -17.77%.

5.4 Subset

By combining simulations and empirical testing, this thesis illustrates the substantial left-tail risk of short volatility strategies and how changes in the intermediate volatility can influence the P&L as we relax the assumptions of the BS. With the introduction of stochastic volatility, this thesis demonstrates how differences in IV and RV and the moneyness of the option can cause a delta-hedging error with large implications to the final P&L. This means that the chosen volatility parameter for calculating the delta-hedging position becomes increasingly more important as we no longer have full knowledge of the forward-looking volatility. Over the chosen sample period, a TV-based delta-hedge is found to yield a higher return, but exhibit a higher standard deviation than an IV-based delta-hedge, leading to substantial difference in the delta-hedging position. Furthermore, we observe a negative correlation between the underlying and its volatility, which leads to a systemic under-hedge of the IV-based hedging position. For these same reasons, we find that hedging with IV performs better in more extreme volatility environments and worse in more stable volatility environments. By introducing a simple option pricing indicator, this thesis is able to significantly improve the performance of a short volatility trading strategy. This is primarily due to the indicators ability to identify overpriced options and periods of enhanced risk of increasing volatility.

6. Discussion

This chapter will discuss two main things. First, we will discuss the results and the BS model in depth. For the model, we will primarily be focusing on the assumptions and model error. For the results, we will focus on the implications on literature and as to how our results are affected by our assumptions. We will then also discuss the option pricing indicator and its implications. Secondly, we will discuss alternative ways to trade volatility with VIX derivatives and variance swaps before outlining the limitations of our study and future research paths.

6.1 Critical reflections on the results

This section will outline the impact that the assumptions have on the results and generally discuss our findings, primarily relating to the intact assumptions within BS and the additional assumptions undertaken in our empirical results.

6.1.1. The assumptions of the Black-Scholes

If all assumptions within the BS model were to hold, path-dependency within volatility trading would be completely eroded. Yet, we have shown the implications of just relaxing a few of these assumptions. We will now briefly highlight if the remaining assumptions are a good fit for option pricing and whether alternative models should be used for pricing options rather than the BS assumption.

An inherent assumption within most financial models, and specifically the BS model, is that no arbitrage opportunities exist as apparent by the put-call parity in equation (12) and how the dynamically hedged replicating portfolio of options should mimic the risk-free asset. This is also evident in how option pricing under BS is a martingale process satisfying the following relationship:

$$P_0 = \mathbb{E}\{P_t\}, t \geq 0$$

Given that this martingale process is risk-neutral, it is denoted by a $\mathbb{Q}$ and the evidenced relationship outlined in the equation must be satisfied at all t. $\mathbb{Q}$ encapsulates what is known as the risk-neutral probability and is used to price all derivatives from the assumption that they should be able to be replicated (arbitrage-free) and, subsequently, are risk-neutral. It is primarily used within continuous martingales (Meucci, 2011). The reverse of $\mathbb{Q}$ is $\mathbb{P}$, which stands for the real-world probability rather than the risk-neutral probability. $\mathbb{P}$ is used within portfolio and risk management, primarily within discrete time and is usually estimated rather than known.

Given that both $\mathbb{Q}$ and $\mathbb{P}$ measure risk, one might be inclined to think that they do not interact. Yet, as the main unit of analysis within this thesis is trading volatility through delta-hedged options, we see that both $\mathbb{Q}$ and $\mathbb{P}$

interact with each other. This is evidenced as hedging is a $\mathbb{P}$ concept, but the amount of hedging that one does dependent on the Greeks is derived from the $\mathbb{Q}$ world. The $\mathbb{Q}$ world embedded in the BS model highlights an important, yet highly relevant distinction. Under the BS theory, the market is complete, and the risk-neutral log-normal distribution can be calculated. The BS model is the outcome we achieve by applying the BS theory of complete markets to the valuation of European options. This highlights that their theory and notion of the risk-neutral distribution can be applied multiple places but the formula not so much, illustrating the hedging error arising from hedging with wrong deltas. This notion of the theory being applicable but not the formula is discernible in how American options are priced using numerical methods such as binomial trees or Monte-Carlo simulations rather than the touted BS formula. In relation to our results, we observe the same fact, namely that the hedging error can partially be attributed to the model error in applying a formula to markets wherein the assumptions are not satisfied.

A central notion in the BS world is the assumption of log-distributed stock returns and that the stock follows a random walk. Yet, as seen in our empirical results, it is evident that the returns are not log-normal and that over the timeframe, there is a skewed distribution, primarily in the left tail as there is a possibility of larger changes in the negative space than in the upward space. This observation can also be backed out from a comparison of observed market prices of options at the money, $S \approx K$, relative to the theoretical BS option price, where we observe that the BS formula overprices options relative to the observed market price, which is consistent with the findings of Mandelbrot and Hudson (2004). We see that the general market price is lower than BS' theoretical price, implying that the market price has priced in a possibility of higher tail-risk events than what is assumed within the BS world. This is consistent with the former argument that stock returns have a fat tail distribution rather than a log-normal distribution. Based on this, it is apparent that the BS model then underprices far OTM options (Dunbar, 2016). This implies that there is a model error to using the BS formula when pricing options as it does not correctly account for market mechanisms as demonstrated in the previous paragraph with the properties of $\mathbb{Q}$ and $\mathbb{P}$. This in turn affects the effectiveness of trading volatility as what one really tries to trade is the only unobservable variable within the model, namely volatility, but given that the model misprices options, there is an inherent discrepancy between the traded volatility and the expected volatility which was to be traded. We see this in our simulations and empirical results. Specifically, the Ahmad & Willmott simulations highlight how we exactly can calculate the expected P&L, but the deviations from the approximated P&L in the empirical results showcase the discrepancy in the unobserved volatility being traded vis-à-vis the actual volatility that ends up being traded, which is the hedging error arising from using the wrong delta based on a wrong volatility.

Two more assumptions, which our thesis has disregarded, are the assumptions of no transaction costs and the ability to borrow and lend at the risk-free rate. Given that every market exhibits transaction costs, it is clear that this assumption behind the BS theory and ultimately, the applicability of the model, is violated. Besides positively biasing our results, the presence of transaction costs presents the trader with a trade-off between the hedging frequency relative to the increased transaction costs, as outlined in section 3.1.6. It is quite apparent that if we had modelled transaction costs into our delta-hedging strategy, we would be presented with a cost minimisation problem, wherein we would seek to minimise the amount of costs we would have to occur while increasing the probability of a higher P&L and returns (Davis et al., 1993). This would quite clearly influence our results severely as an optimal hedge ratio would be dependent on whether the costs were either fixed or proportional with each transaction. In relation to our findings, we argue that even though an optimal hedge ratio could be achieved through multiple simulations, the efficiency of this hedge ratio would clearly be distorted given the evidenced under- or over-hedging exposure, which we have observed in empirical volatility trading. This boils down to the fact that with transaction costs, the valuation of an option (and the inherent volatility being traded) is mispriced given the erosion of preference-independence in the BS world. The result of this is an argument that volatility trading with transaction costs boils down to investors' risk profiles in terms of fluctuation hedging errors and frequency, thus clearly influencing the path-dependency inherent in volatility trading, all of which reduces the attractiveness of trading volatility with delta-hedged options. This can also partially explain why delta-hedged straddles rather than either delta-hedged puts or calls are preferred in volatility trading given that one's initial delta-position is lower and insofar the investor finds himself in a rather involatile market, the need for higher delta-hedging frequencies with straddles is reduced vis-à-vis puts or calls, if the underlying is not deviating too far from the strike, *ceteris paribus*.

Pertaining to the element of transaction costs, these are not just relevant when buying the share, but transaction costs would also occur when borrowing or lending at the risk-free rate. In this thesis, we assumed that the risk-free rate equalled the three-month treasury rate in the US, which in itself is a fair assumption when pricing the options but assuming that one can borrow at this rate might not be reasonable when financing the position to buy the underlying. In practice, if one were to borrow at the risk-free rate when delta-hedging options, this would, combined with transaction costs, create a separate position on a debt instrument in combination with the delta-hedged option to trade volatility through perhaps repo-transactions (Fisher, 2002). To replicate the desired risk-free rate, one could first purchase a treasury note or bill with a maturity on *T*-date and enter into a repo-transaction. Here, one would then sell the bond on the settle date and buy it back at a fixed price in the future, just like a forward, wherein the return on this transaction is the repo rate (with the purchased bond being used as collateral, of course). In essence, this transaction becomes risk-free in practice given that it is backed both by the sovereign bond and the counterparty, implying that the repo rate achieved in this transaction can be equivalent to the risk-free rate. Fisher (2002) documents that these repo rates trade in line with treasury bills and notes with small spreads. Yet, this is not the entire picture given that investors face haircuts that result in margin requirements as one cannot borrow the treasury's full nominal value. Haircuts capture the notion of trying to price risk. Relating this to our findings, we would evidently occur transaction costs and face the same dilemma as outlined in the previous paragraph, but in addition, our interest payments would fluctuate more widely as we also would have to post margins, which would reduce our P&L as more costs associated with trading the volatility would be incurred. Further, even if we were to lock in a repo-rate, interest rates on the treasury note or bill can fluctuate over its lifetime, i.e. interest rates are stochastic and not constant as assumed within the BS world. In terms of option pricing, the risk-neutral measure inherent in the BS world would be disregarded and options would have to be priced on forward measures. In terms of our findings then, we argue that if this was the case, the hedging error arising from using a wrong delta would be increased and skewed dependent on the forward measures implemented (Bakshi et al., 1997).

As a final comment on BS model assumptions, extensions of it exist where it has either been modelled to incorporate jump processes as evident by Merton (1976) or more complex stochastic volatility model such as the Heston model (Heston, 1993). Given a jump-process, trading volatility with delta-hedged options is exposed to a jump risk, which cannot be hedged away, and the stochastic volatility process introduces a new uncertain element, namely the volatility's volatility. This documents a known concept, namely the volatility smile, which affects volatility trading given that options' IV differ depending on moneyness, implying that one can take more sophisticated bets on volatility than inherent in the BS assumptions.

Given all the outlined limitations and inherent assumptions, why is the BS formula then the most used model for pricing options and the one from which this thesis has taken its main point of inspiration? The quick answer is that the unobservable IV in BS is the wrong number put into the wrong formula to obtain the right price. This also highlights the most fundamental model error within BS, i.e., volatility must be estimated as it is not widely observed and different methods yield different (incorrect) results. The real answer for why the BS model has been used relates to the argument that the applicability of their formula creates opportunities to understand the dynamics, even if imperfect, of how options are priced and the dynamics that have an effect on the profitability of trading volatility with options. Several extensions and different models exist for pricing options, but most of them have taken some inspiration from the original work by BS. These models rely on different numerical techniques.

In regard to different models, it is evident that each extension or modification of the BS model results in additional uncertainty and thus, replication becomes either 1) impossible as the market is then incomplete if there is only the underlying and a risk-free asset or 2) challenging and more costly to upkeep given multiple hedging dynamics. A possible solution to additional uncertainty introduced by the Heston model is Dupire's equation (1994) which introduces local volatility. This acts as a forward instantaneous volatility surface from which delta-hedging can occur and different studies such as Dumas, Fleming and Whaley (1998) and Hagan, Kumar, Lesniewski and Woodward (2002) show that hedging within this complete market actually performs worse than hedging with a BS delta. In regards to our results, we could model more complex processes to more correctly estimate the volatility process to use a more correct delta and then understand implications within each model. Yet, we would be prone to the same fundamental model error, namely volatility estimations and how these translate into wrong deltas. In the end, the model choice boils down to a trade-off between simplicity and complexity in terms of modelling the real world and option pricing.

6.1.2 Discussion of the empirical results

As previously outlined, the BS model for option pricing is a clearly imperfect model when applied to real-world data but it remains widely used due to a preference for simplicity over complexity. In the absence of a perfect model, the current option pricing methodology applied in practice is argued to be prone to some degree of mispricing. The degree of mispricing is likely to have been further influenced in recent years, by the large inflow of unsophisticated retail investors affecting demand and an increase in stock-market volatility confirming the need for option sellers to charge a risk-premium (Andersen et al., 2015). This hypothesis is confirmed by the empirical results, which clearly demonstrate the substantial left-tail risk exposure of option sellers when applying a simple delta-hedge, which suggests the need for overestimating IV i.e., a smile-effect to cover for such unforeseen events. By achieving substantially higher returns applying an EGARCH model to forecast volatility, this thesis further confirms the presence of mispricing in the derivatives market and the effectiveness of the EGARCH model to identify such, which is in line with the findings of previous literature (Dash et al., 2012; Rostan et al., 2020; Sheu & Wei, 2011).

When discussing the empirical results, one should also consider the assumptions made. The general assumptions for the BS option pricing model discussed in section 6.1.1 also apply to the empirical results. In addition, this section will deep-dive into the specific assumptions made for the empirical results and the practical implication of these and how they impact the performance of the results and the trading strategy. The assumptions deemed the most significant are that of no transaction cost, which was done to better assess the effectiveness of the volatility trade; constant margin requirement, which allowed this thesis to simplify the return calculations; and no restrictions or interest payments on leverage.

Similar to the previous section, a delta-hedged option consists of an option position and a position in the underlying, which primarily adds brokerage fees and bid-ask spreads to the P&L with higher hedging frequencies leading to higher costs. Brokerage fees are typically only relevant at the retail level with the main impact to the P&L being the bid-ask spread (Eraker, 2013). However, if an institutional investor were to hedge with a daily hedging frequency, the cost of doing so would eventually add up highly dependent on the brokerage fees. The bid ask spread is found to have a significant influence on option P&L (ibid.) and since this thesis assumes the fair option price is reflected as the mean of the bid and the ask price, our results are positively biased. To make the results more applicable to a real-life setting, it would be necessary to use bid prices for the short option positions, bid prices for short positions in the underlying and ask prices for long positions in the underlying. However, it is likely that some large institutional investors and market makers are able to trade inside the bid-ask spread that we have available to us through OptionMetrics, slightly reducing the bias. Looking at the volatility timing strategy in particular, the purpose was to limit the position size of the trade when the probability for a loss is higher, which further reduces the impact of the bid-ask spread. Since we found a large positive mean for the overpriced options, adjusting the short volatility trade to the relevant bid-ask spread would not have a negative impact and not change the overall findings of this thesis. Taking the perspective of the retail investor, including transaction costs, paints a different picture. First of all, readjusting the delta-hedging position on a daily basis would accumulate substantial brokerage costs with the final P&L impact being highly dependent on the development in delta. Furthermore, we would incur an additional transaction cost through management fees, which are likely to be in the range of 0.5% – 1% of the delta-hedging position for the position in the underlying on an annual basis if one needs to invest in an ETF to gain exposure to the S&P500 and one time brokerage fees within the same range for the short option positions, since brokerage costs are typically higher for options (Eraker, 2013). Given that the mean return is between 0.36% and 0.84% for the individual short volatility position and 1.12% and 1.74% when using the option pricing indicator, it is possible that this strategy is not profitable for retail investors or alternatively a very small Sharpe-ratio would be achieved, making it an unattractive investment opportunity.

The second assumption made by this thesis is that of a constant margin requirement equal to the funding level. This assumption is made to simplify the return calculation and make the return comparable across time. Generally, when an investor writes a short option, the brokerage firm adjusts the margin requirement relative to the delta-hedging position. Since the delta of a profitable call is likely to go towards zero and the delta of a profitable put is likely to go towards one and vice versa, due to the negative ρ, we would have an asymmetric effect from using an adjustable margin. The margin requirement is further impacted by changes in volatility, with higher margin requirements in periods of high volatility and lower margin requirements in periods of low volatility. This means that margin requirements would become a large issue in periods with high volatility, which is when the P&L of the strategy is at its most negative. However, these are also the periods where the exposure is at its lowest using the option pricing indicator indicator. Overall, using an adjustable margin requirement and the cost associated with doing would limit the attractiveness of some of the option positions, but it is likely to be slightly mitigated by applying the option pricing indicator and again be highly dependent on the type of investor and their access to capital. Furthermore, looking at the maximum drawback, no loss is ever near the assumed margin level further suggesting that such effects will be negligible.

The margin requirements, funding level and access to capital tie into the third major assumption of the empirical results, primarily for the option pricing indicator given that we assume no interest on leverage nor the margin account. Since a margin account can generally be used to trade low-risk assets, an investor would earn interest, either directly from the broker or through an investment in a risk-free asset, positively affecting the P&L. However, since it needs to be a low-risk asset and short-term bond yields are close to 0 for the majority of the trading period, this is argued to have little to no P&L effect making the effect of the assumption made negligible. Furthermore, the broker account would typically yield something close to the risk-free rate minus an additional 20 basis points. More importantly, the volatility timing strategy only works if the investor has access to leverage. This is not an issue for institutional investors, and even some retail investors, but the cost of taking on leverage can vary dependent on the investor type. It is not unlikely that a retail investor pays anything between 5%-10% APR, with such rates potentially eroding any benefits from the strategy. For institutional investors, this rate is likely to be below 1% APR, dependent on the institution and the amount (InteractiveBrokers, 2021). This would have a negative P&L impact, but since the mean of the overpriced positions is between 4% - 7%, using the indicator still offers significant benefits.

Finally, this thesis is aware that the short volatility trade heavily relies on the assumption of lending and borrowing at the risk-free rate when entering the delta-hedging position. From the perspective of a retail investor, this might be unreasonable, but for a large institutional investor or investment bank, implementing the short volatility strategy on a large scale using the three-month LIBOR might not be that far off. Furthermore, one should be aware of any potential liquidity issues if this strategy was to be implemented on a large scale by an institutional investor. By choosing short ATM options on the S&P 500, it is argued that there will be plenty of demand for such options and that it would not impact the market price.

6.1.3 Discussion of the option pricing indicator

In contrast to any existing literature to the knowledge of the authors, this thesis covers volatility timing using an EGARCH model for S&P500 index options over a very long time-period including two major market downturns. This demonstrates the effectiveness of applying the EGARCH model for option trading strategy in both stable and high volatility environments and confirms that the mispricing of options has persisted through time. The ARCH/GARCH volatility model is not a new concept, and their forecasting ability on stock markets has received plenty of attention in the past. This makes the substantial improvement to the P&L as observed by this thesis even more surprising. In addition to confirming that option markets might suffer from some degree of mispricing, the recent influx of retail investors into option markets makes this thesis confident that such mispricing will persist in the short- to medium-term.

Further discussing the results of applying the option pricing indicator, we find that the overpriced options have a large and significant mean, which supports the effectiveness of our option pricing indicator. However, we find mixed results for the underpriced options with calls having a positive mean and puts having a negative mean. However, the mean is relatively close to zero with a very small standard deviation indicating that these options are not profitable, further substantiating the effectiveness of the indicator. Most surprisingly, we find the fairly priced options to have a negative mean for both puts and calls and to be the worst performing (see Table 21). A reason for this is likely the EGARCH model's lack of ability to perfectly hedge volatility. A stylized fact about asset returns is that they will experience excessive volatility from time to time, but volatility tend to return towards the long-run mean (cf. section 3.1.6). Since the EGARCH forecast is always pulled towards the long-run mean, it is likely to perform much better when volatility is at the extremes. This is also what we observe with overpriced options yielding a positive mean and underpriced put options yielding a negative or close to zero mean. Another consideration is the sampling frequency. For simplicity, this thesis decided on a weekly data-frequency to limit the noise around the forecast considering that the forecast would be performed 13 weeks ahead in time. However, some research suggests that using daily returns could potentially yield a better forecast for short time-horizons and using a different hedging frequency could potentially improve the classification of options (Yu & Zivot, 2011). Alternatively, other GARCH models e.g., the GARCH-M model could be considered. However, the scope of this thesis was not to find the best GARCH model, but to test whether an EGARCH model could improve the performance of a short option strategy by creating a option pricing indicator to which we can say yes. Finally, it is reiterated that the family of GARCH models have shortcomings, but they are widely used by academics and practitioners and to a large extent considered the best tool available.

When properly constructed, backtesting is a great way to assess whether a trading strategy has a persistent edge (Long, n.d.). However, one should always be aware of any potential forward-looking bias, data mining, and curve fitting. To limit any potential forward looking bias, this thesis fits the EGARCH model only using S&P500 data up until the point in time where the forecast is made, which is argued to make it an accurate representation of the output one would have received. The main issue in terms of forward-looking bias and data mining is when we set up the trading rules and categorization of the options since we have access to the entire dataset including all options profitability. As researchers, it is a balancing act between looking at the data to find the best performing trading rules, i.e., data mining and letting the trading rules be guided by a theoretical hypothesis. This thesis combined both approaches to create the best-performing option pricing, which was also well-founded in theory and previous empirical research. This thesis drew inspiration from previous empirical studies when deciding to categorize the options into three categories. In addition, this thesis was aware of the imperfect volatility forecast from the EGARCH model and the potential for some degree of forecasting error as proven by previous empirical studies (Rostan et al., 2020). Furthermore, we expected the EGARCH model to perform relatively poorly for options where IV was close to the forecast due to the mean-reversing properties. Therefore, we sat the upper-bound, i.e., the difference in which an option is categorized as overpriced) to 5% and sat the lower bound, i.e., the difference in which an option is categorized as underpriced to be 2% with the latter being in accordance with previous studies (Dash et al., 2012). The upper-bound was defined through data mining i.e., the value that best fits the data. This unavoidably leads to some degree of forward-looking bias, but with no clear theoretical definition of when an option is overpriced, assessing the data is argued to be a meaningful approach. Furthermore, we test the trading rules over a very long time-period and find the specified boundary to yield satisfying results in various volatility environments, suggesting it is good choice.

Since we only perform the forecast on the S&P 500, our trading strategy and option pricing indicator are fitted specifically to this data set, which could indicate some degree of curve-fitting. One way to further substantiate the findings would be to run the same backtesting methodology on NASDAQ-100 options. If the option pricing shows improved performance on this data set as well, the researchers could to a large extent reject the argument of the results being due to noise in the data set as well, we could to a large extent reject the argument of the results being due to noise in the dataset and conclude that the positive performance of the option pricing is caused by mispricing and market behavior. Furthermore, the results and effectiveness of the option pricing indicator is likely to be impacted by the timing at which the option is struck and the chosen maturity. As illustrated in the simulations, the timing of an increase in RV has a significant impact on the P&L, which could further affect the EGARCH model's ability to accurately categorize the options. For the same reason, it would be relevant to test the performance of the option timing indicator across various maturities, moneyness, and time, of which the option is truck before one could safely generalize the said results.

6.1.4 The contribution to literature

In this section, this thesis will discuss the empirical findings in relation to our expectations, the simulated results and previous empirical research. To our knowledge, little research has been done on how hedging with TV as a proxy for RV relative to IV performs when tested on empirical data over a long time-period. Much of the work done on volatility trading is either based on simulations subject to strict assumptions, closely tied to the classical BS model, or based on complicated option pricing models that fit specific historical volatility data with no closed-form solutions or generalizability. In fact, much previous work has been focused on increasingly complicated volatility models calibrated to fit historical vanilla options, stopping short of practical implementation (Ahmad & Wilmott, 2005). Unlike previous literature, this thesis puts a special emphasis on empirical and practical implementation. This is done by relaxing assumptions of the BS model found to be too strong, while continuing to use the BS model for option pricing since it to this day still is the most widely used model by practitioners. While this thesis is fully aware that this is an unusual approach, it allows us to create simulations, which to a large extent can mimic real-life scenarios. Furthermore, it allows the findings of this thesis to serve as guidance to traders and provide insights on the dynamics of real-life volatility trading when utilizing the BS model and all its shortcomings.

An example of the former is the early work by Ahmad & Wilmott (2005) on what volatility to use when calculating the delta-hedging position. The findings are valid in a theoretical setting, assuming constant volatility, with modest empirical implications. This thesis first replicates the simulated results by Ahmad & Wilmott (2005) before building upon their work by further relaxing the assumption of constant volatility and continuous hedging and eventually testing the findings on empirical data from the OptionMetrics database. The work by Ahmad & Wilmott (2005) concludes that the main driver of the P&L is the chosen hedging volatility. This still holds true for the simulations of this thesis and to a large extent for the empirical results.

However, we can further contribute with the impact of the volatility environment, the path of the underlying and substantial intermediate stochastic changes in volatility.

Surprisingly, the empirical results generally support the findings by Ahmad & Wilmott (2005) despite there being large differences in the assumptions made. Similar to their research, this thesis finds that when hedging with TV, the final return comes close to the difference in option price when using IV and 90-day RV divided by the constant margin, while the final return when hedging with IV is to a large extent deterministic and heavily dependent on the movements in the underlying, i.e., path-dependent. This is primarily due to the chosen hedging volatility being the primary driver of the return. Going beyond the paper by Ahmad & Wilmott (2005), the extensive time-period with a wide range of volatility scenarios allowed us to contribute with insights on how hedging with IV relative to hedging with TV as a proxy for RV is performing under different volatility environments. More specifically, this thesis finds that hedging with IV is under-performing when the TV is decreasing and over-performing when TV is increasing. The results further showed that hedging with IV has a much lower standard deviation in the return but yields a lower accumulated return over the chosen time-period. This is in contrast to the paper by Ahmad & Wilmott (2005), who only look at a one-sided mispricing of options. Since the sample period includes both the financial crisis and the COVID-19 crash in a relatively short time frame, it is argued to be heavily exposed to high volatility environments. Therefore, the finding that hedging with IV results in a slightly lower accumulated return over time than hedging with TV should be interpreted lightly. This is due to the previously mentioned overperformance of hedging with IV when TV is increasing. Extending or shortening the time-period i.e., including more low volatility observation so that the two market crashes have a smaller impact or so that the financial crisis is excluded would likely make TV perform relatively better. In addition, the chosen maturity and timing of when the options are sold clearly impacts the results too. This is evidenced in how e.g., the delta-hedged call #3 maturing in April under COVID-19 has a large drawdown. Assumedly if it were sold in February and maturing in May (a skew of one month), this drawdown could have been reduced given that it would then reap the large spread in May which the next delta-hedged call in our sample does.

The paper by Ahmad & Wilmott (2005) assumes that volatility is constant, and one can forecast it accurately, thereby only briefly touching upon the risk of applying a wrong volatility estimate for the delta-hedging position. However, their research on risk is focused on minimum and maximum P&L values and not on the dynamics governing the P&L differences. To further add to existing literature and better elaborate on the governing effects and the risk associated with applying a wrong hedging volatility, this thesis incorporates the work by Kurpiel & Roncalli (2011) on how delta-hedging bias is affecting the return and P&L in stochastic volatility environments. They study the accuracy of the Black Scholes option pricing model when the volatility of asset prices is random by comparing it to the P&L of the Heston option pricing model in a simulated setting. They built upon the findings of Renault & Touzi (1996) and find that using the BS IV, which is higher than the true volatility assumed to be that of the stochastic model, the delta-hedging bias will lead to ITM (OTM) options to be under-hedged (over-hedged). Furthermore, they find that for a strongly negative ρ, the BS IV-based delta hedge leads to a systemically under-hedged position, which can correct partly for the non-hedge of volatility. Despite the fact that this thesis does not compare the results to a Heston option pricing model, these findings are further corroborated by this thesis and are found to hold for both simulated and empirical data, with the IV-based hedge performing better during large increases in IV, when ρ is strongly negative. More specifically, this thesis goes on to show how the difference between IV and RV, the moneyness of the option, and the movement of the underlying determines whether the option is over- or under-hedged and how it impacts the P&L and return.

The combination of extensive simulation work and an empirical backtest on OptionMetrics data is where this thesis really contributes to the research by Kurpiel & Roncalli (2011). Through the empirical backtest, this thesis is able to further substantiate the dynamics of the difference in hedging with IV and TV/RV and the over- and under-hedging on a mark-to-market basis. More specifically, this thesis shows how the hedging bias is dependent on whether IV is higher than TV acting as a proxy for the RV and how this can change over the life-time of the option. This is in contrast to the one-sided approach taken by Kurpiel & Roncalli (2011), only focusing on the scenario where TV is higher than RV in a simulated setting. From a practical point of view, we argue that this thesis presents the reader with a much more comprehensive analysis of the P&L effects of hedging with a theoretically wrong model. However, common for both the paper by Kurpiel & Roncalli (2011) and this thesis is the emphasis on the importance of hedging stochastic volatility. If not hedged properly, it can have a massive negative impact on the P&L, which from a risk-management perspective is especially relevant for financial institutions selling delta-hedged options to provide the market with liquidity.

As briefly mentioned, this thesis does not compare the Black-Scholes IV-based delta-hedge to a stochastic volatility model, as is done by Kurpiel & Roncalli (2011). For a simulation-based paper only, this would have been possible. However, for the results to be consistent across simulated and empirical data and for the previously outlined issues with applying the stochastic volatility model to empirical data, this thesis focuses exclusively on the BS model. Therefore, we compare the BS IV-based delta-hedge to the BS 30-day TV-based hedge acting as a proxy for RV. Since we are introducing a stochastic element to the simulations and know that the assumptions of the BS model are too strong for empirical data, this means that we do not compare the results to the true model. This obviously puts the previously elaborated effects into questions since we cannot guarantee that the results are not occurring due to an anomaly or simply by coincidence. For the simulations, we argue that since we only introduce a simple stochastic element, the BS model is a good proxy for the true model, which is further confirmed by the findings being in line with previous research. For the empirical results, it would be nearly impossible to find a model that would fit the data perfectly and with the focus on practical implementation, the difference between the implied and realised volatility of the BS model is a good choice. Finally, it is argued that since we have the same error across the simulated and empirical results, we achieve consistency and are able to generalize the findings from the simulations to the empirical data.

6.2 Alternative ways of trading volatility

This section will outline and discuss the potential implications of trading volatility through other means, such as either VIX derivatives or variance and volatility swaps rather than through delta-hedged short options.

6.2.1 Trading volatility with VIX derivatives

As the VIX is an index, it is not possible to invest directly in it just as with the S&P500. In order to trade volatility in the VIX, one must replicate the index through derivatives, hereunder VIX futures, options or exchange-traded notes (ETN's) or products (ETP's), such as VXX. As with any derivative, there are both advantages and disadvantages to trading volatility with VIX derivatives relative to delta-hedging sold options.

From our simulations, it is evident that path-dependency affects the effectiveness of trading volatility on the S&P500. If a trader were to utilise VIX-linked futures rather than options, one's exposure to path-dependency in the underlying would be eroded given that the future trades volatility at an absolute level rather than at an absolute strike on the S&P500 in options. Given the absoluteness of trading a specified volatility level, futures on the VIX allow one to take directional bets on the IV inherent in options without paying attention to other factors affecting the pricing of options, such as interest rate levels or dividend payments (Cheng, 2019). The level of volatility being traded is 30-days after the specified settlement date.

As we have outlined throughout this thesis and evidenced in our empirical results, there tends to be a positive spread between IV and RV, which also serves as the main motivation for delta-hedging short options. This spread translates into a skew in the VIX term structure which is positive and implies that throughout the distribution of how market participants view the 30-day forward looking IV, there is a larger possibility of spikes in the VIX occurring (positive events) rather than negative (Christensen & Prabhala, 1998). This makes sense given that the VIX index is on average inversely correlated to the S&P500 as a higher VIX value implies a more volatile environment, wherein volatility is usually associated with unstable and negative returns. In addition, this positive skew implies that the volatility is dependent on the datedness of options. From the VIX term structure then, short-dated options have a lower IV than long-dated options, which has implications on trading volatility with futures given contango and backwardation.

With futures, contango indicates that the delivery price will increase with the time to delivery, implying that the spot is lower than future's strike. In different words, contango is to be thought of as the cost of storing and financing the future, indicating that the forward delivery becomes more costly as time passes. This also presents the first flaw in trading volatility with VIX futures as volatility itself is not directly tradeable, but rather exposure is achieved through a "store and trade" characteristic in options and other derivatives, flawing the pricing of VIX futures and the effectiveness of trading volatility with them (Avellaneda, 2016). VIX futures spend approximately 80% of their lifetime in contango and 20% in backwardation, which is the reverse of contango, i.e., that the strike is lower than the spot (Moran & Liu, 2020).

Trading volatility through VIX futures is then primarily driven by the exposure to short-term volatility given that futures on VIX are usually short-term to be liquid enough to be able to trade them. From the outlined characteristics of VIX and VIX futures, it is evident that VIX future have a negative correlation with the S&P500. If selling futures, the trader enters a long position on the volatility given the inverse relationship and vice versa when buying futures (ibid). From a classical replication approach, volatility trading strategies with VIX futures are systematically grounded in selling VIX futures and taking a long position in a VIX replicating portfolio consisting of options on the S&P500. Yet, this is not entirely possible (though mathematically it should hold), due to several factors. First, to replicate VIX, one must hold several options, which all have high transaction costs associated with them given their low deltas (as they are far OTM). Secondly, the sensitivity in rebalancing due to movements in the S&P500 results in multiple trades during a day, which given discrete hedging and transaction costs is simply not viable as it would eat into the expected returns. Lastly, given that the VIX is based of mid-prices, a discrepancy occurs when constructing a replicating portfolio as one cannot trade at mid-prices, but rather buys at ask prices and sells at bid prices. In addition to this, Asensio (2013) argues that VIX futures should not be used to hedge with but rather diversify a portfolio given that diversification reduces variance in the portfolio whereas hedging protects against losses. Furthermore, it must be highlighted that empirical observations show that colling futures from the first month to the second month creates a performance drag in the long-term on the strategy, indicating that hedging with VIX futures is expensive. All of these arguments highlight that selling VIX futures is only profitable in a bull profit as the volatility exposure is achieved on all strike levels of the volatility, which clearly suffers in volatile bear markets.

Another possible VIX derivative is the exchange-traded note, primarily the VXX (tracks first to three months until delivery), which follows short-term VIX futures. As an index following theoretical future rolls, its inherent characteristics is that the value declines as time passes. Since futures trade at contango 80%, this implies the roll-yield in the VXX is negative, which reflects an underperformance of the VXX vis-à-vis the VIX index and through this, does not provide a true replication of the VIX (Deng et al., 2012). This indicates that the term-structure of the IV in the VIX through the VXX imposes cost on VXX long positions. A different VIX index is the VXN, which tracks the longer-dated VIX futures (between four and seven months until delivery). Given that both track the same underlying, S&P500, arbitrage opportunities occur within the volatility term structure, which can be reaped by shorting different "legs" of the volatility term structure using the ETN's. Yet, since one is trading volatility, which has a characteristic of mean-reversion, VIX ETN's do not perfectly capture VIX and as such, losses can be substantial (such as a flash-crash in volatility), and potential expected returns are also eroded through volatility lag.

What is apparent from this is that when trading volatility through ETN's, several different risks are inherent compared to volatility trading with delta-hedged options. ETN's main risks can be characterized as term structure risk, tracking risk, credit risk and equity hedge risk (Husson & McCann, 2012). The term structure in VIX futures risk is inherent given that when the ETN is bought, the exposure to the term structure is not locked and thus, one must constantly monitor the VXX in order to be able to exit relative to the term structure without incurring huge losses. In continuation of the term structure, given the contango in VIX futures, the term-structure is upwards sloping most of the time, but as backwardation occurs, the term structure rapidly shifts into downwards-sloping structures. Following an investment in commodities, upwards-sloping term structures make sense given storage costs, which we also see within VIX futures, but as the VIX index is entirely synthetic, the VIX future's term structure's upward-sloping trend does not justify why it exhibits time persistence. As changes in the term structure can occur suddenly, as they are related unexpected market outswings, term structure risk is an inherent risk when trading volatility with ETN's (ibid).

Given that the VXX tracks future contract levels, it inherently also exhibits a tracking error risk as it reliant on the market's ability to forecast future VIX levels. Researchers observe that the VXX only predicts around 73% of the RV on the VIX index relative to future prices one month ex ante, indicating that the VXX ETN only indirectly creates exposure to the VIX, making it a flawed volatility instrument (Adrangi et al., 2019). In addition to the tracking risk, the underlying also exhibits credit risk from the issuer (which with the VXX ETN is Barclays). This is evident in how ETN's outstanding and issued by Lehman Brothers eroded in value when the company went bankrupt during the financial crisis (Husson & McCann, 2012).

As for why the VXX is then still traded heavily and used to trade volatility is attributed to argument that trading volatility through the VXX ETN's is to increase portfolio diversification rather than hedging. Yet, Shu and Zhang (2012) argue that the negative correlation between VIX and S&P500 is stronger when the S&P exhibits large intraday movements relative to when small intraday changes happen. With small changes, the VIX does not always move the opposite way of the S&P500, implying that VXX ETN is only effective when hedging extreme movements rather than small movements. This also implies that trading volatility on the VXX ETN is then only profitable given large market outswings in volatile markets rather than stable, involatile markets where delta-hedged options can still be profitable.

Finally, the last way to trade volatility on the VIX index is with pure volatility options, which are not to be confused with trading volatility through delta-hedged options. Volatility options come in several forms, hereunder options on VIX futures where the settlement and price is based on the VIX future, but the strike is the VIX' co-terminal value at settlement. One can also trade options on VXX, but the inherent effect of contango in futures affects the price of the VXX option given the term-structure. This highlights distortions in pricing derivatives, which seek to follow VIX. What is interesting about volatility options on the VIX index through futures is that the futures per their specifications "gravitate" towards the VIX value. As a seller of VIX options then, one is punished, but given that one is long on the VIX, one actually profits from this (Rhoads, 2012). To understand this, remember that option contracts exhibit a time-value component (the intrinsic value relative to the time value of the option) when priced. Given a hypothetical case where the underlying does not exhibit a change over the options lifetime, the only loss in value would be the time value of the option. Taking VIX put options the exhibits a counter-intuitive trait, namely that they have a negative time-value as they are priced based on futures which gravitates towards the value of VIX.

In theory, all VIX derivates should act as a way to gain pure exposure to volatility regardless of the underlying's path. Yet as the discussion has outlined, the pricing of these derivatives and the synthetic traits inherent suggest that VIX derivatives do not work as intended in practice. Thus, these derivatives exhibit naïve characteristics which should be benchmarked against the VIX^2 - SPX^2 spread. In sum, as an asset class, volatility cannot be treated as a buy-and-hold investment through VIX derivatives, but utilising tactical approaches, different approaches to trading volatility through VIX derivatives appear relative to trading volatility with delta-hedged options.

6.2.2 Trading volatility with variance swaps

To briefly repeat from the literature review, variance swaps provide a direct exposure to the underlying's volatility and are independent of the path and can be replicated by a delta-hedged portfolio of options with inverse proportional weight to strike in equation (24). Yet, replication of variance swaps through options is not possible given that not all strikes can be traded and options far OTM are illiquid and not sold often. This implies that only major players are able to trade variance swaps as retail investors are first priced out due to the high notional and that replication is not possible.

What is particularly interesting about variance swaps is that they can be closed before maturity, but this has a caveat - an investor who wishes to exit before the maturity is forced to take into account the volatility's exposure over the remaining lifetime. For swap valued right after being struck, the future expected variance's change is the main factor for the P&L as it is then marked to market (Allen et al., 2006). This remaining exposure decreases with time, linearly to be precise, which in some ways can also be viewed as the forward variance given that pre-closure, volatility has been accrued, but after a closure, it is also the remaining variance which the investor has exposure to.

Variance swaps can be utilised to take advantage of both bull and bear markets. For a bull market, if one expects a correction, one could hedge with put options, but if the market rallies these put option must be re-struck depending on expiry to continue being hedged, which is cost inefficient given transaction costs. Instead, entering a long variance swap provides a protection against this if the correction occurs rather abruptly within any time before settlement. In markets with low volatility expectations, it is extremely profitable to sell variance swaps and this significantly outperforms straddles and delta-hedged straddles given that the spot can move away from the strike, whereby the volatility reaped from the straddles is decreased due to a lower gamma (ibid).

Another way to trade volatility with variance swaps is to conduct roll trades with short variance (Konstantinidi & Skiadopoulos, 2016). What basically occurs within such a trade is that one enters a carry-trade where variance on the underlying index is sold systematically with short dates. The exposure in rolling short variance results in positive P&L from the implied and realised variance spread, but this exhibits large left-tail risks if the volatility spikes, just as with delta-hedged options. This can be viewed as in the same light as with bond indices, where the P&L from coupons and accrued dividends is eroded if the yield rises or defaults. When financing is provided through debt, interest is a necessity and in the same vein, the spread between IV and RV can be viewed as an "equity-insurance" capital provided where rolling the short variance swap captures the "interest payment" on the "equity-insurance".

An additional way in which volatility can be traded through variance swaps is through correlation trading, where being short variance on the index and long variance on its constituents is profitable (Allen et al., 2006). This might be counterintuitive given that one would assume the index variance to be lower in its mean that the basket of stocks because of the diversifying effect inherent in the index. The resulting trade results in a concept called short correlation, which can partially explain why variance swaps are as liquid as they are (ibid). An additional argument when trading correlation in variance and why it is profitable to be short is that option sellers on single stock sell to increase the option premium, but on a general level, there is a larger demand vis-à-vis for protection on the downside risks of the index.

Given that one can also trade forward volatility through variance swaps, it is evident that the P&L from such a position is driven only by the changes in the expectations of the volatility due to the term structure of IV. This means that the exposure to the RV only materialises at the forward date. This mechanic can be thought of in the same vein as how P&L on e.g., Euribor contracts is driven by changes in the expectations in the future rates but settles at the predominant spot rates. Trading variance forward is an appealing proposition given that an investor can achieve the exposure to volatility with incurring the RV downsides that can accrue relative to a long variance contract. Thinking of fixed income again, the forward volatility provides the investor an option to execute a slide and carry trade on the volatility.

Variance swaps can be combined with options to trade specific parts of the volatility term structure given variance swaps exposure to convexity and skew per the pricing functions (cf. section 3.2.1). In order to take advantage of the exposure to convexity can be achieved by being long variance swaps against short delta-hedged straddles, as the residual long convexity is only left because the volatility exposure is hedged out. Yet, as we have seen in our simulations, if the straddles are either far in- or OTM, the exposure to vega and gamma is minimal and through this, the volatility exposure is eroded, implying that straddles need to be resold to keep the volatility exposure hedged. If an investor wants to trade the volatility skew rather than the volatility convexity, this can be achieved by selling naked puts against a long variance contract. As indicated, the skew occurs as volatilities tend to soar on downsides but stay benevolent on upsides (Allen et al. 2006). Considering the combination of a short put and a long variance swap, the P&L at T is given by:

$$N_{vega}\frac{(\sigma - K^2)}{2K} + \frac{N_{Put}}{S_0}\big(P - MAX(S_P - S_T, 0)\big) = N_{vega}\frac{(\sigma - \widetilde{K}(S_T)^2)}{2K}$$

Here, the strike is now adjusted to be reliant on the price of the underlying at maturity and is given by:

$$\widetilde{K}(S_T) = \sqrt{K^2 - \frac{2KN_{Put}}{N_{vega}S_0}(P - MAX(S_P - S_T, 0))}$$

Where it is evident that the put's notional can be equal to the variance's lowest realised possible value, indicating that the skew trade can hedge parts of long put's downsides inherent in variance swaps. An evolvement of this would be conditional variance swaps, which are third generation volatility swaps (ibid).

Returning to second generation volatility such as variance swaps, a key pointer is that they are linear in variance, but convex in volatility. Thus, investors can ask themselves: "*Why not trade volatility instead of realised variance through linear volatility products?"*. As we have shown in the empirical results, path-dependency in options occurs through the gamma exposure. Because variance swaps can be replicated by a portfolio of delta-hedged options, the gamma can be thought of as to be constant with variance swaps. This partially explains why variance swaps dominate the market relative to volatility swaps as volatility swaps cannot be replicated without imposing harsh assumptions on the model (Demetertifi et al., 1999). As a counter argument, it can be argued that the market tries to convene variance swaps as volatility swaps because the notional is in vega instead of variance. From delta-hedging options, we have shown that it is variance that occurs, which is why volatility swaps should be thought of as a derivative to variance, where the payoff is the square-root of the variance swap contract rather.

When selling and delta-hedging variance swaps them, market moves can be exaggerated, *ceteris paribus* (Bennett & Gil, 2012). As the swaps are settled on a close-to-close basis, if an investor wants to delta-hedge his variance swap, he will exaggerate or suppress the market by selling or buying at large amounts. This is illustrated in the following theoretical way: If one is long in a variance swap, it can be hedged by selling a portfolio of options with weights according to equation (24) and delta-hedge at close. Given that the falls, the portfolio of options is now delta negative, indicating that further purchases in the underlying must be taken, which exaggerates the spot's decline. The reverse also holds if an investor is short in a variance swap and the spot increases, then the spot's rise is suppressed through the delta-hedge of the swap given the large notional. Appendix #11 showcases the market mechanic of flows with variance swaps and replicating it.

Variance swaps are nonetheless not risk-free investments given the constant mark-to-market (Martin, 2017). The underlying's unrealised volatility poses a huge risk to variance swaps through the offsetting variance swap's fair value as this creates a possibility of huge losses. In addition to this, variance swaps are not priced on the notion that the underlying can jump, but we know that this can occur, highlighting a jump risk inherent in variance swaps. Based on research by Demeterfi et. al. (1999) and Carr and Lee (2009), it is clear that variance swaps cannot be priced in the case of jumps and are thus not model-independent. As mentioned previously, variance swaps can be replicated by a portfolio of options. Yet, this highlights another risk inherent in variance swaps given that illiquid IV affects the possible replication of variance swaps and thus their effectiveness at trading volatility. These risks highlight why caps are imposed on variance swaps despite the attractiveness of trading volatility with variance swaps given that if the underlying defaults, the payoff (and also loss) is infinite depending on one's position.

6.3 Future research

This thesis has studied the impact of trading volatility with delta-hedged options from a deconstructed BS perspective, which has shed light on how exposure to volatility is both driven by the gamma's exposure to the volatility spread and the notion of being either over or under-hedged with the IV's delta. Further research can be conducted by implementing a transaction cost perspective to analyse how the under/overhedging mechanic is impacted when transaction costs are introduced. This would involve combining the research by Clewlow and Hodges (1997) with our findings to understand how the under/overhedging is affected in the presence of transaction costs and whether there is a benefit to either consistently be over or under-hedged. This is naturally tied into additional research where studies on how the hedging frequency affects the over/under hedging mechanic could be conducted under a transaction cost perspective.

This thesis has primarily studied the effectiveness of trading volatility through ATM delta-hedged options with an exercise date three months from initially being struck. Additional research could be conducted on the effect of trading OTM options (assuming liquidity is sufficient) vis-à-vis ATM options with the same expiry. This could provide insights into whether there are better opportunities to reap the volatility premium as options can be mispriced due to the varying strikes and the embedded volatility smile in option strikes. A natural extension of this research is then to also study the impact of the option's expiry, namely whether it is more effective to trade longer-dated options compared to shorter-dated options, pursuant to one's volatility expectations.

Additionally, research could be undertaken into whether a combination of short and long positions is more profitable than just being either short or long. The aim here would be to see whether a combination of both would provide the trader with a position which is not exposed to the maximum drawdowns evidenced in our short position. Further, one could combine this with a possibility of exploring a mixture of hedging volatilities, i.e., not only hedging with either the RV or IV, but utilise both to gain the highest P&L. This could naturally be tied together with the findings on how volatility trading with RV performs in macroeconomic environments relative to IV.

Lastly, additional research into how to time volatility and a more robust test of our option pricing indicator could be conducted. Here, the goal would be to highlight its effectiveness and test it under different scenarios and assumptions.

6.4 Subset

This section provided a discussion on the reflections that have been made on the findings of this thesis. This part was dedicated to the effect that a relaxation of additional Black-Scholes assumptions would have had on the results, namely transaction costs and interest rate assumptions. We argue that although our results appear to be positively biased, it would not change the generalizability of the findings. More importantly, the chosen sample size is argued to have a large negative impact on the results, due to the inclusion of two high volatility events in a very short time span. Further reflections emphasised the contribution to current literature and how this thesis has helped bridge the gap between simulations and empirical results by taking a simplistic, but practical approach. In addition to this, a part was devoted to discussing the assumptions and validity of the option pricing indicator. Although the EGARCH model has its' imperfections, it is argued that the EGARCH forecast can give an indication of how at-the-money options are priced relative to volatility expectations. Lastly, a critical examination of alternative approaches to trade volatility with VIX derivatives and variance swaps was conducted.

7. Conclusion

Volatility as an asset class provides investors and traders with several opportunities to profit from its inherent characteristics, but there are also large risks associated with trading volatility given that volatility itself is uncertainty. Thus, the purpose of this thesis was to evaluate the effectiveness of trading volatility given various relaxations of the BS model and the performance in various volatility environments.

We first show how the expected P&L can be calculated given continuous hedging in the BS whereafter we relax this assumption and illustrate how the P&L is affected by path-dependency arising from the dollar gamma exposure. We then replicate the study from Ahmad and Wilmott (2005) to show how IV's profit is deterministic whereas the RV's profit is guaranteed, but path independent.

We then model simple stochastic volatility processes where the RV (and sometimes) hedge volatility increases to determine how delta-hedging with a chosen volatility performs in different volatility environment. We here find that the IV can be under or over-hedged relative to the RV and that this can have a negative or positive hedging error dependent on the difference between IV and RV and the moneyness of the options. These findings confirm the work of Kurpiel & Roncalli (2011) and Renault & Touzi (1996).

We then translate our simulated findings into an empirical setting, where we backtest the performance of selling delta-hedged puts and options on the S&P500 over a 13-year period. We find that delta-hedging puts with the IV yields a period return of 13.24% and 15.56% if hedged with the 30-day TV. We find that the performance for calls is better, as delta-hedged calls earn a return 42.14% with the IV and 43.90% with the TV. If we utilise an option pricing indicator to manage left-tail risk and time volatility, we find that the delta-hedged puts provide a return of 60.44% with the IV and 86.75% with the TV. For the calls, we find a period return of 105.14% with the IV and 121.16% for the TV. Through case-studies, we also showcase the dynamics pertinent in the simulations in highlighted delta-hedged calls, where we illustrate the under/overhedge, vega effect and dollar gamma exposure.

Lastly, a practical point of view, this thesis presents the reader with a much more comprehensive analysis of the P&L effects of hedging with a theoretically wrong model compared to other papers which either focus solely on simulations or empirical aspects.

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Appendix

Appendix #1: Solution to the Black-Scholes PDE

Theorem 1: The value of a European call is given by:

$$c(S,t) = c(S(t),K,T-t,r) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \qquad x1$$

Where:

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{1}{2}s^2}ds \qquad x2$$

Where the cumulative distribution function of the standard distribution is given by:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \qquad x3$$

The final condition of a European call is given by:

$$c(S,T) = \max(S-K,0)\,, \text{for all } S \geq 0 \qquad x4$$

The boundary conditions of a European call are given by:

$$c(0,t) = 0, \text{for all } t \geq 0 \qquad x5$$

$$c(S,T) \sim S - Ke^{-r(T-t)}, \text{as } S \to \infty \qquad x6$$

Proof:

Now it must be verified that *c(S,t)* in (x1) satisfies the Black-Scholes equation (1). By noting that $\omega = S \text{ or } t,$ one gets:

$$\frac{\partial N(d_i)}{\partial \omega} = \frac{\partial N(d_i)}{\partial d_i}\frac{\partial d_i}{\partial \omega} = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial d_i}\int_{-\infty}^{d_i} e^{-\frac{s^2}{2}}ds \cdot \frac{\partial d_1}{\partial \omega} = \frac{e^{-\frac{d_i^2}{2}}}{\sqrt{2\pi}}\frac{\partial d_i}{\partial \omega}$$

Checking, one gets that:

$$\frac{\partial d_1}{\partial t} = \frac{d_1}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma} + \frac{\sigma}{2}\right) \text{ and } \frac{\partial d_2}{\partial t} = \frac{d_2}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)$$

Thus, one finds that:

$$\frac{\partial c}{\partial t} = S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial t}$$

$$=\frac{Se^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left[\frac{d_1}{2(T-t)}-\frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)\right]-rKe^{-r(T-t)}N(d_2)$$
$$-\frac{Ke^{-r(T-t)}e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\left[\frac{d_2}{2(T-t)}-\frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\right] \qquad x7$$

And since:

$$\frac{\partial d_i}{\partial S}=\frac{1}{S\sigma\sqrt{T-t}}, i=1,2,$$

One gets:

$$\frac{\partial c}{\partial S}=N(d_1)+S\frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S}-Ke^{-r(T-t)}\frac{\partial N(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S}$$
$$=N(d_1)+\frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{T-t}}-Ke^{-r(T-t)}\frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\frac{1}{S\sigma\sqrt{T-t}} \qquad x8$$

Differentiating again:

$$\frac{\partial^2 c}{\partial S^2}=\frac{e^{-\frac{d_1^2}{2}}}{S\sigma\sqrt{2\pi}\sqrt{T-t}}-\frac{d_1e^{-\frac{d_1^2}{2}}}{S\sigma^2\sqrt{2\pi}(T-t)}+\frac{Ke^{-r(T-t)}e^{-\frac{d_2^2}{2}}}{S^2\sigma\sqrt{2\pi}\sqrt{T-t}}+\frac{Ke^{-r(T-t)}d_2e^{-\frac{d_2^2}{2}}}{S^2\sigma^2\sqrt{2\pi}(T-t)}$$
$$=\frac{2}{S^2\sigma^2}\left\{\frac{Se^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}}\left(\frac{\sigma}{2\sqrt{T-t}}-\frac{d_1}{2(T-t)}\right)+\frac{Ke^{-r(T-t)}e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}}\left(\frac{\sigma}{2\sqrt{T-t}}+\frac{d_2}{2(T-t)}\right)\right\} \qquad x9$$

Substituting (x1), (x7) - (x9) into the left-hands side of the Black-Scholes equation (1), it is evident that the Black-Scholes equation (1) equals 0.

For the boundary condition (x5), under (x3), d_1, $d_2 \to -\infty$ as $S \to 0$, one gets that:

$$\mathrm{N}(-\infty)=0 \text{ and } c(0,t)=0N(-\infty)-\mathrm{Ke}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}N(-\infty)=0$$

For the boundary condition (x6), again it is noted that d_1, $d_2 \to \infty$ as $S \to \infty$, where one gets that:

$$\mathrm{N}(\infty)=1 \text{ and } c(S,t)\to SN(\infty)-\mathrm{Ke}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}N(\infty)\sim S-Ke^{-r(T-t)}$$

Considering the final condition (x4), at $t = T$, if $S > K$, d_1, $d_2 \to \infty$. Thus, $c(S,T)=S-K$. If $S < K$, d_1, $d_2 \to -\infty$, thus $c(S,T)=0$. Lastly, if S = K, by continuity, $c(S,T)=0$.

Q.E.D.

Theorem 2: The value of a European put is defined as:

$$p(S,T)=Ke^{-r(T-t)}N(-d_2)-SN(-d_1) \qquad x10$$

Here, d_1 and d_2 are defined in x3.

Proof:

Instead of verifying equation 1 through the same procedure as under Theorem 1, one can instead derive x10 by using the put-call parity formula stated as:

$$c(S,t) - p(S,t) = S - Ke^{-r(T-t)},$$

Theorem 1 and the identity that$N(d) + N(-d) \equiv 1$ for any d.

Q.E.D.

Theorem 3: Deltas of a European call and European put are given as:

$$\Delta_c(S,t) = N(d_1) \text{ and } \Delta_p(S,t) = N(d_1) - 1 \qquad x11$$

Proof:

$$\Delta_c = N(d_1) + \frac{Ke^{-\frac{1}{2}d_1^2}}{S\sigma\sqrt{2\pi(T-t)}}\left(\frac{S}{K} - e^{-r(T-t)}e^{-\frac{1}{2}d_2^2+\frac{1}{2}d_1^2}\right) = N(d_1)$$

Whereby an equality can be established by:

$$\frac{S}{K} = e^{-r(T-t)}e^{-\frac{1}{2}d_2^2+\frac{1}{2}d_1^2} \Leftrightarrow \ln\left(\frac{S}{K}\right) + r(T-t) = \frac{1}{2}(d_1+d_2)(d_1-d_2),$$

Which is true due to equation (x3) and that $d_2 = d_1 - \sigma\sqrt{T-t}$

By using the put-call parity (7) or reiterating the same process, one is able to find and prove Δ_p.

Q.E.D.

Appendix #2: Derivation of the relationship between gamma and theta

Highlighting the Black-Scholes PDE equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

If each Greek letter is identified corresponding to their partial derivative, the PDE can be rewritten as:

$$r\Pi = \Theta + rS\Delta + \frac{1}{2}\Gamma\sigma^2 S^2$$

Insofar it is delta-hedged, where $\Delta = 0$:

$$\Theta = r\Pi - \frac{1}{2}\Gamma\sigma^2 S^2$$

Given that the interest rate is usually only a few basis points, the right hand side's first term can be negligible and the following relationship is approximated:

$$\Theta \approx -\frac{1}{2}\Gamma\sigma^2 S^2$$

Appendix #3: RStudio Code of the simulations (Can also be found as attachment)

```
library(tidyverse)

library(ggplot2)

library(fOptions)

library(DataCombine)

library(data.table)

#Set up the GBM stock price matrix

set.seed(1234) #makes it reproducible

vecgbm <- function(nsim, tau, mu, sigma, s0) {

  #tau is time steps, nsim is simulations

  dt <- 1/252

  epsilon <- matrix(rnorm(tau*nsim), ncol = nsim, nrow = tau)

  # get GBM and convert to price paths

  gbm <- exp((mu - sigma^2/2) * dt + sigma * epsilon * sqrt(dt))

  gbm <- apply(rbind(rep(s0, nsim), gbm), 2, cumprod)

  return(gbm)

}

gbm <- vecgbm(nsim=50, t=252, mu=0.05, sigma=0.3, s0=100)

#Set up the parameters for the PnL. mu and sigma must match above

Time <- 1

dt <- 1/252

r <- 0.05

sigma <- 0.2

sigmad <- 0.3

t <- seq(0, Time, by = dt)

Maturity <- 252

#Define the fair forward

fairforward <- function(s0,r,t)

{

  s0*exp(r*(t/252))

}

FairForwards <- as.data.frame(fairforward(s0=gbm,r,t=Maturity))

strike <- FairForwards[1,1]
```

```
#Black Scholes function
BS <- function(S, X, r, expiration, sigma, type)
{
  if(type=="c"){
    d1 <- (log(S/X)+(r+sigma^2/2)*(expiration))/(sigma*sqrt(expiration))
    d2 <- d1-sigma*sqrt(expiration)
    value <- (S*pnorm(d1)-X*exp(-r*(expiration))*pnorm(d2))
    return(value)}
    if(type=="p"){
    d1 <- (log(S/X)+(r+sigma^2/2)*(expiration))/(sigma*sqrt(expiration))
    d2 <- d1-sigma*sqrt(expiration)
    value <- (-S*pnorm(-d1)+X*exp(-r*(expiration))*pnorm(-d2))
    return(value)}
}
#delta function
delta <- function(S, X, r, expiration, sigmad, type)
{
  if(type=="c"){
    d1 <- (log(S/X)+(r+sigmad^2/2)*(expiration))/(sigmad*sqrt(expiration))
    value <- (pnorm(d1))
    return(value)}
  if(type=="p"){
    d1 <- (log(S/X)+(r+sigmad^2/2)*(expiration))/(sigmad*sqrt(expiration))
    value <- (pnorm(d1)-1)
    return(value)}
}
#Function for simulating PnL and delta
option_pnl <- function(s, t, type, position, strike, t_purchase, t_sell, sigma, sigmad, r, maturity) {
  op <- data.frame(
    type = type,
    position = position,
    t = t[t_purchase:t_sell],
    s = s[t_purchase:t_sell]
```

```
  )
  # Create price, delta and PnL
  op$price <- BS(type = type, S = op$s, X = strike, expiration = maturity-op$t, r = r, sigma = sigma)
  op$delta <- delta(type = type, S = op$s, X = strike, expiration = maturity-op$t, r = r, sigmad = sigmad)
  op$pnl_delta[1] <- 0
  for(i in 2:nrow(op)) {
    op$pnl_delta[i] <- (op$s[i]-op$s[i-1])*op$delta[i-1]*(-position)
  }
  op$pnl_option[1] <- 0
  for(i in 2:nrow(op)) {
    op$pnl_option[i] <- (op$price[i] - op$price[i-1])*position
  }
  op$interest[1] <- 0
  for(i in 2:nrow(op)) {
    op$interest[i] <- r*dt*(op$delta[i-1]*op$s[i-1]-op$price[i-1])*position
  }
  op$Dailypnl[1] <- 0
  for(i in 2:nrow(op)) {
    op$Dailypnl[i] <- op$pnl_delta[i]+op$pnl_option[i]+op$interest[i]
  }
  op$Totalpnl <- cumsum(op$Dailypnl)
  return(op)
}
#Create the PnL data.frame for each individual option for path 1 at the time
#Path1-10
Put1 <- as.data.frame(option_pnl(s = gbm[,1], t = t, type = "p", position = -1, strike = strike, t_purchase = 1, t_sell = length(t), sigma = sigma, sigmad = sigmad, r = r, maturity = 1))
….
```

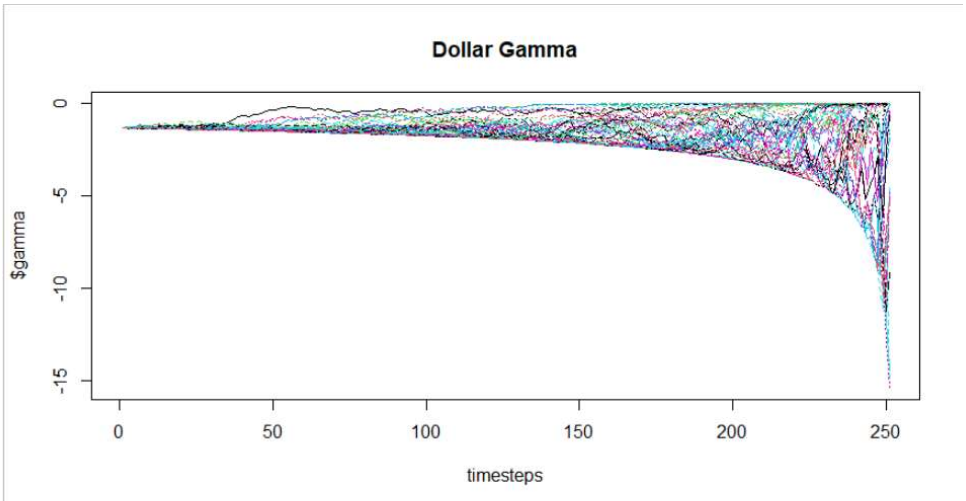
Appendix #4: RStudio Code of the empirical data (Can also be found as attachment)

```
library(tidyverse)
library(ggplot2)
library(fOptions)
library(DataCombine)
library(data.table)

#Import data set
Calls <- read.csv("~/Dropbox/CBS/Finance and Strategic Management/Master's Thesis/OptionMetrics/Calls.csv")
Calls <- read.csv("C:/Users/filip/Dropbox/CBS/Finance and Strategic Management/Master's Thesis/OptionMetrics/Calls.csv")

#delta function
delta <- function(S, X, r, expiration, sigmad, type, div)
{
  if(type=="c"){
    d1 <- (log(S/X)+(r-div+sigmad^2/2)*(expiration))/(sigmad*sqrt(expiration))

    value <- (pnorm(d1))
    return(value)}

  if(type=="p"){
    d1 <- (log(S/X)+(r-div+sigmad^2/2)*(expiration))/(sigmad*sqrt(expiration))

    value <- (pnorm(d1)-1)
    return(value)}
}

#Function for simulating PnL and delta
option_pnl <- function(s, t, type, position, strike, t_purchase, t_sell, sigma, sigmad, r, maturity, div) {
  op <- data.frame(
    type = type,
    position = position,
    t = t[t_purchase:t_sell],
    s = s[t_purchase:t_sell],
    sigma = sigma[t_purchase:t_sell],
    sigmad = sigmad[t_purchase:t_sell]
  )

  # Create price, delta and PnL
  op$price <- Calls$price[1:62]
  op$delta <- Calls$delta[1:62]
  op$deltatest <- delta(type = type, S = op$s, X = strike, expiration = maturity-op$t, r = r, sigmad = sigmad, div)
  op$pnl_deltatest[1] <- 0
  for(i in 2:nrow(op)) {
    op$pnl_deltatest[i] <- (op$s[i]-op$s[i-1])*op$deltatest[i-1]*(-position)
  }
  op$pnl_delta[1] <- 0
  for(i in 2:nrow(op)) {
    op$pnl_delta[i] <- (op$s[i]-op$s[i-1])*op$delta[i-1]*(-position)
  }
```

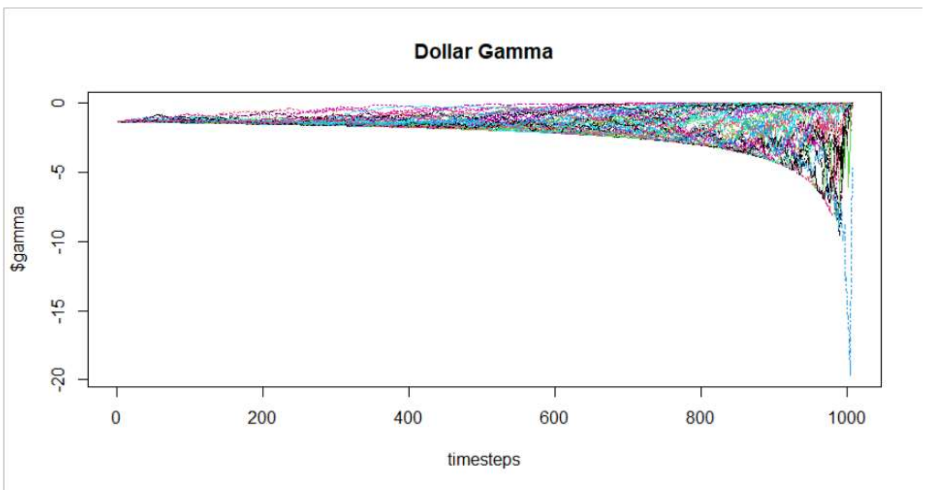
```
  op$pnl_option[1] <- 0
  for(i in 2:nrow(op)) {
    op$pnl_option[i] <- (op$price[i] - op$price[i-1])*position
  }
  op$interesttest[1] <- 0
  for(i in 2:nrow(op)) {
    op$interesttest[i] <- r*dt*(op$deltatest[i-1]*op$s[i-1]-op$price[i-1])*position
  }
  op$dividendstest[1] <- 0
  for(i in 2:nrow(op)) {
    op$dividendstest[i] <- div*dt*(op$deltatest[i-1]*op$s[i-1])*(-position)
  }
  op$interest[1] <- 0
  for(i in 2:nrow(op)) {
    op$interest[i] <- r*dt*(op$delta[i-1]*op$s[i-1]-op$price[i-1])*position
  }
  op$dividends[1] <- 0
  for(i in 2:nrow(op)) {
    op$dividends[i] <- div*dt*(op$delta[i-1]*op$s[i-1])*(-position)
  }
  op$Dailypnl[1] <- 0
  for(i in 2:nrow(op)) {
    op$Dailypnl[i] <- op$pnl_delta[i]+op$pnl_option[i]+op$interest[i]+op$dividends[i]
  }
  op$Dailypnltest[1] <- 0
  for(i in 2:nrow(op)){
    op$Dailypnltest[i] <- op$pnl_deltatest[i]+op$pnl_option[i]+op$interesttest[i]+op$dividendstest[i]
  }
  op$Totalpnl <- cumsum(op$Dailypnl)
  op$Totalpnltest <- cumsum(op$Dailypnltest)
  return(op)
}

#Set up for first call ID 32385331 aka call1
Time <- 61/252
dt <- 1/252
r <- 0.0513 #the rate it was struck with
t <- seq(0, Time, by = dt)
Spot1 <- as.data.frame(Calls[1:62,12])
Strike1 <- as.data.frame(Calls[1:62,5])
IV1 <- as.data.frame(Calls[1:62,9])
IVD1 <- as.data.frame(Calls[1:62,13])
div <- 0.0176

#Create the PnL data.frame for each individual option for SP500
Call1 <- as.data.frame(option_pnl(s = Spot1[,1], t = t, type = "c", position = -1, strike = Strike1[1,1], t_purchase = 1, t_sell =
length(t), sigma = IV1[,1], sigmad = IVD1[,1], r = r, maturity = 1/252*61, div = div))
```

Appendix #5: Dollar gamma exposure for in simulations

#252 timesteps (average variance: 0.76)

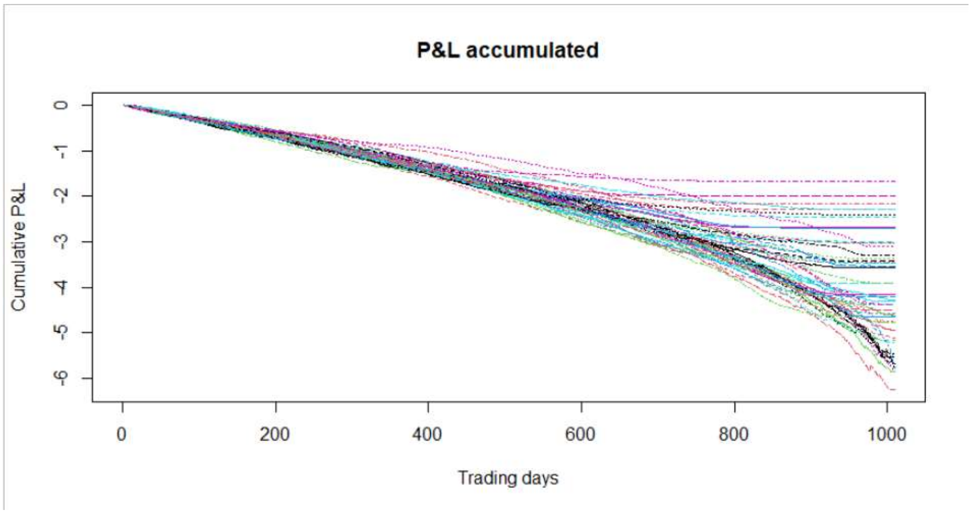


#1008 timesteps (average variance: 0.53)

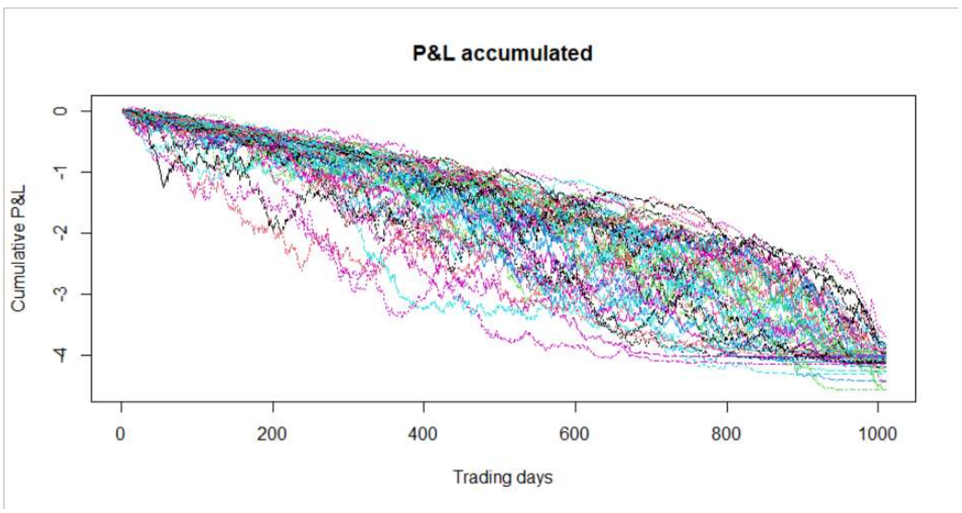


Appendix #6: Wilmott short position replication

IV hedge



RV Hedge



Appendix #7: Over/under-hedged in IV and the effect on the hedging error

if $\sigma^{IV} < \sigma^{RV}$

Option	Money-ness	Hedging error	Underlying (So)	Direction	P&L effect	Comment
Put	ITM	Over-hedged	Increasing	Towards ATM	Negative	Delta position will be more negative than the decrease in option value
Put	ITM	Over-hedged	decreasing	Furhter ITM	Positive	Delta position will increase more than the increase in option value
Put	OTM	Under-hedged	Increasing	Further OTM	Positive	Delta position will lose less than the decrease in option value
Put	OTM	Under-hedged	decreasing	Towards ATM	Negative	Delta position will increase less than the increase in option value
Put	ATM	Little difference	Increasing	Towards OTM	Positive	Once the option starts to move OTM, and continues in that direction, the option will eventually get under-hedged and be similar to row 5
Put	ATM	Little difference	decreasing	Towards ITM	Positive	Once the option starts to move ITM, and continues in that direction, the option will eventually get over-hedged and be similar to row 4
Call	ITM	Over-hedged	Increasing	Furhter ITM	Positive	Delta position will increase more than the increase in option value
Call	ITM	Over-hedged	decreasing	Towards ATM	Negative	Delta position will be more negative than the decrease in option value
Call	OTM	Under-hedged	Increasing	Towards ATM	Negative	Delta position will increase less than the increase in option value
Call	OTM	Under-hedged	decreasing	Further OTM	Positive	Delta position will lose less than the decrease in option value
Call	ATM	Little difference	Increasing	Towards ITM	Positive	Once the option starts to move ITM, and continues in that direction, the option will eventually get over-hedged and be similar to row 9
Call	ATM	Little difference	decreasing	Towards OTM	Positive	Once the option starts to move OTM, and continues in that direction, the option will eventually get under-hedged and be similar to row 12

$if\, \sigma^{IV} > \sigma^{RV}$

Option	Money-ness	Hedging error	Underlying (So)	Direction	P&L effect	Comment
Put	ITM	Under-hedged	Increasing	Towards ATM	Positive	Delta position will be more negative than the decrease in option value
Put	ITM	Under-hedged	decreasing	Furhter ITM	Negative	Delta position will increase more than the increase in option value
Put	OTM	Over-hedged	Increasing	Further OTM	Negative	Delta position will lose less than the decrease in option value
Put	OTM	Over-hedged	decreasing	Towards ATM	Positive	Delta position will increase less than the increase in option value
Put	ATM	Little difference	Increasing	Towards OTM	Negative	Once the option starts to move OTM, and continues in that direction, the option will eventually get under-hedged and be similar to row 5
Put	ATM	Little difference	decreasing	Towards ITM	Negative	Once the option starts to move ITM, and continues in that direction, the option will eventually get over-hedged and be similar to row 4
Call	ITM	Under-hedged	Increasing	Furhter ITM	Negative	Delta position will increase more than the increase in option value
Call	ITM	Under-hedged	decreasing	Towards ATM	Positive	Delta position will be more negative than the decrease in option value
Call	OTM	Over-hedged	Increasing	Towards ATM	Positive	Delta position will increase less than the increase in option value
Call	OTM	Over-hedged	decreasing	Further OTM	Negative	Delta position will lose less than the decrease in option value
Call	ATM	Little difference	Increasing	Towards ITM	Negative	Once the option starts to move ITM, and continues in that direction, the option will eventually get over-hedged and be similar to row 9
Call	ATM	Little difference	decreasing	Towards OTM	Negative	Once the option starts to move OTM, and continues in that direction, the option will eventually get under-hedged and be similar to row 12

Appendix #8: Validity check of the P&L for the simulations

Puts

Path	Short	Long	Abs. Diff.
1	4,05	-4,05	0,00
2	4,19	-4,19	0,00
3	3,92	-3,92	0,00
4	4,18	-4,18	0,00
5	4,26	-4,26	0,00
6	4,03	-4,03	0,00
7	4,20	-4,20	0,00
8	4,09	-4,09	0,00
9	4,12	-4,12	0,00
10	4,42	-4,42	0,00
11	4,11	-4,11	0,00
12	4,03	-4,03	0,00
13	4,28	-4,28	0,00
14	3,94	-3,94	0,00
15	4,27	-4,27	0,00
16	4,17	-4,17	0,00
17	3,95	-3,95	0,00
18	3,73	-3,73	0,00
19	3,84	-3,84	0,00
20	4,45	-4,45	0,00
21	4,51	-4,51	0,00
22	4,19	-4,19	0,00
23	4,30	-4,30	0,00
24	4,12	-4,12	0,00
25	3,93	-3,93	0,00
26	3,95	-3,95	0,00
27	4,65	-4,65	0,00
28	3,82	-3,82	0,00
29	3,94	-3,94	0,00
30	4,09	-4,09	0,00
31	3,84	-3,84	0,00
32	3,95	-3,95	0,00
33	3,93	-3,93	0,00
34	3,60	-3,60	0,00
35	4,11	-4,11	0,00
36	4,23	-4,23	0,00
37	4,09	-4,09	0,00
38	3,52	-3,52	0,00
39	4,57	-4,57	0,00
40	4,04	-4,04	0,00
41	4,06	-4,06	0,00
42	3,87	-3,87	0,00
43	4,04	-4,04	0,00
44	3,97	-3,97	0,00
45	4,14	-4,14	0,00
46	3,99	-3,99	0,00
47	4,53	-4,53	0,00
48	4,00	-4,00	0,00
49	4,12	-4,12	0,00
50	4,04	-4,04	0,00

Calls

Path	Short	Long	Abs. Diff.
1	4,23	-4,23	0,00
2	0,89	-0,89	0,00
3	0,03	-0,03	0,00
4	5,63	-5,63	0,00
5	3,70	-3,70	0,00
6	3,97	-3,97	0,00
7	1,24	-1,24	0,00
8	-0,11	0,11	0,00
9	2,43	-2,43	0,00
10	1,13	-1,13	0,00
11	4,30	-4,30	0,00
12	-0,27	0,27	0,00
13	2,96	-2,96	0,00
14	0,55	-0,55	0,00
15	4,91	-4,91	0,00
16	7,50	-7,50	0,00
17	1,51	-1,51	0,00
18	2,13	-2,13	0,00
19	5,37	-5,37	0,00
20	2,36	-2,36	0,00
21	7,17	-7,17	0,00
22	4,64	-4,64	0,00
23	2,39	-2,39	0,00
24	5,14	-5,14	0,00
25	1,49	-1,49	0,00
26	5,86	-5,86	0,00
27	0,81	-0,81	0,00
28	2,79	-2,79	0,00
29	2,30	-2,30	0,00
30	3,73	-3,73	0,00
31	6,02	-6,02	0,00
32	0,32	-0,32	0,00
33	0,90	-0,90	0,00
34	2,93	-2,93	0,00
35	-0,27	0,27	0,00
36	1,16	-1,16	0,00
37	3,46	-3,46	0,00
38	4,75	-4,75	0,00
39	1,88	-1,88	0,00
40	5,26	-5,26	0,00
41	6,54	-6,54	0,00
42	5,99	-5,99	0,00
43	6,57	-6,57	0,00
44	4,64	-4,64	0,00
45	1,98	-1,98	0,00
46	0,64	-0,64	0,00
47	3,41	-3,41	0,00
48	0,95	-0,95	0,00
49	7,48	-7,48	0,00
50	1,31	-1,31	0,00

Appendix #9: Validity check of the P&L for the empirical parts

Call	Short IV return	Long IV return	Abs. Diff
1	-0.77%	0.77%	0
2	-0.58%	0.58%	0
3	1.95%	-1.95%	0
4	-0.53%	0.53%	0
5	1.67%	-1.67%	0
6	3.11%	-3.11%	0
7	-16.31%	16.31%	0
8	-7.04%	7.04%	0
9	6.98%	-6.98%	0
10	7.01%	-7.01%	0
11	1.91%	-1.91%	0
12	6.05%	-6.05%	0
13	3.75%	-3.75%	0
14	-4.93%	4.93%	0
15	5.49%	-5.49%	0
16	4.54%	-4.54%	0
17	3.18%	-3.18%	0
18	4.42%	-4.42%	0
19	-6.22%	6.22%	0
20	3.01%	-3.01%	0
21	4.05%	-4.05%	0
22	3.91%	-3.91%	0
23	3.02%	-3.02%	0
24	1.73%	-1.73%	0
25	-0.20%	0.20%	0
26	0.19%	-0.19%	0
27	0.82%	-0.82%	0
28	0.27%	-0.27%	0
29	-1.04%	1.04%	0
30	2.63%	-2.63%	0
31	-2.19%	2.19%	0
32	1.48%	-1.48%	0
33	3.44%	-3.44%	0
34	2.55%	-2.55%	0
35	-2.31%	2.31%	0
36	-1.95%	1.95%	0
37	2.31%	-2.31%	0
38	-1.70%	1.70%	0
39	3.36%	-3.36%	0
40	2.03%	-2.03%	0
41	2.46%	-2.46%	0
42	1.65%	-1.65%	0
43	1.71%	-1.71%	0
44	0.08%	-0.08%	0
45	-5.37%	5.37%	0
46	2.73%	-2.73%	0
47	-2.45%	2.45%	0
48	-3.98%	3.98%	0
49	2.58%	-2.58%	0
50	0.93%	-0.93%	0
51	-0.90%	0.90%	0
52	0.95%	-0.95%	0
53	-10.42%	10.42%	0
54	6.99%	-6.99%	0
55	4.26%	-4.26%	0

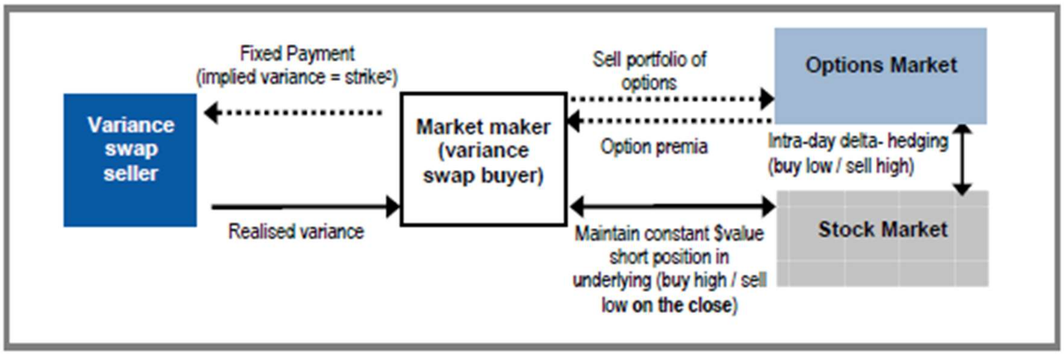
Put	Short IV return	Long IV return	Abs. Diff
1	-1.99%	1.99%	0
2	-0.81%	0.81%	0
3	-0.54%	0.54%	0
4	-2.18%	2.18%	0
5	2.71%	-2.71%	0
6	0.92%	-0.92%	0
7	-20.09%	20.09%	0
8	-11.84%	11.84%	0
9	7.28%	-7.28%	0
10	9.07%	-9.07%	0
11	3.23%	-3.23%	0
12	6.20%	-6.20%	0
13	3.65%	-3.65%	0
14	-5.30%	5.30%	0
15	5.72%	-5.72%	0
16	4.98%	-4.98%	0
17	3.75%	-3.75%	0
18	4.23%	-4.23%	0
19	-4.94%	4.94%	0
20	3.54%	-3.54%	0
21	3.84%	-3.84%	0
22	3.81%	-3.81%	0
23	2.85%	-2.85%	0
24	1.56%	-1.56%	0
25	0.70%	-0.70%	0
26	-0.28%	0.28%	0
27	0.57%	-0.57%	0
28	1.20%	-1.20%	0
29	-1.66%	1.66%	0
30	2.88%	-2.88%	0
31	-2.59%	2.59%	0
32	0.71%	-0.71%	0
33	3.30%	-3.30%	0
34	1.59%	-1.59%	0
35	0.62%	-0.62%	0
36	-2.72%	2.72%	0
37	2.22%	-2.22%	0
38	-1.37%	1.37%	0
39	3.05%	-3.05%	0
40	1.69%	-1.69%	0
41	1.89%	-1.89%	0
42	1.40%	-1.40%	0
43	1.10%	-1.10%	0
44	-0.01%	0.01%	0
45	-5.25%	5.25%	0
46	2.67%	-2.67%	0
47	-2.67%	2.67%	0
48	-4.70%	4.70%	0
49	3.77%	-3.77%	0
50	0.39%	-0.39%	0
51	-1.42%	1.42%	0
52	1.61%	-1.61%	0
53	-17.41%	17.41%	0
54	5.98%	-5.98%	0
55	3.30%	-3.30%	0

Appendix #10: Data and P&L calculation for the first empirical call[4]

	type	position	t	s	sigma	sigmad	price	delta	deltatest	pnl_deltatest	pnl_delta	pnl_option	interesttest	dividendstest	interest	dividends	Dailypnl	Dailypnltest	Totalpnl	Totalpnltest
1	c	-1.00	0.00	1422.95	0.12	0.06	37.00	0.56	0.59	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	c	-1.00	0.00	1427.99	0.11	0.06	38.50	0.58	0.64	2.99	2.80	-1.50	-0.16	0.06	-0.15	0.06	1.20	1.38	1.20	1.38
3	c	-1.00	0.01	1440.13	0.11	0.07	45.10	0.64	0.72	7.74	7.07	-6.60	-0.18	0.06	-0.16	0.06	0.36	1.03	1.56	2.41
4	c	-1.00	0.01	1423.90	0.11	0.08	35.40	0.56	0.58	-11.74	-10.44	9.70	-0.20	0.07	-0.18	0.06	-0.86	-2.17	0.71	0.24
5	c	-1.00	0.02	1422.18	0.11	0.07	33.90	0.55	0.57	-1.00	-0.96	1.50	-0.16	0.06	-0.16	0.06	0.44	0.40	1.14	0.64
6	c	-1.00	0.02	1420.62	0.11	0.07	32.90	0.54	0.56	-0.89	-0.86	1.00	-0.16	0.06	-0.15	0.05	0.04	0.01	1.18	0.65
7	c	-1.00	0.02	1428.82	0.11	0.08	36.80	0.59	0.62	4.57	4.44	-3.90	-0.15	0.06	-0.15	0.05	0.45	0.57	1.63	1.21
8	c	-1.00	0.03	1438.24	0.11	0.08	42.05	0.64	0.69	5.83	5.52	-5.25	-0.17	0.06	-0.16	0.06	0.16	0.47	1.79	1.68
9	c	-1.00	0.03	1445.94	0.11	0.08	47.40	0.67	0.74	5.28	4.89	-5.35	-0.19	0.07	-0.18	0.06	-0.57	-0.20	1.22	1.49
10	c	-1.00	0.04	1448.39	0.11	0.08	48.70	0.68	0.76	1.81	1.64	-1.30	-0.21	0.07	-0.19	0.07	0.22	0.38	1.44	1.86
11	c	-1.00	0.04	1446.99	0.12	0.07	49.10	0.67	0.75	-1.06	-0.96	-0.40	-0.21	0.08	-0.19	0.07	-1.48	-1.59	-0.04	0.27
12	c	-1.00	0.04	1448.00	0.12	0.07	48.60	0.68	0.77	0.76	0.67	0.50	-0.21	0.08	-0.19	0.07	1.05	1.13	1.02	1.39
13	c	-1.00	0.05	1450.02	0.12	0.07	50.00	0.68	0.78	1.55	1.37	-1.40	-0.22	0.08	-0.19	0.07	-0.15	0.01	0.86	1.40
14	c	-1.00	0.05	1448.31	0.12	0.07	48.60	0.68	0.77	-1.33	-1.17	1.40	-0.22	0.08	-0.19	0.07	0.11	-0.08	0.97	1.33
15	c	-1.00	0.06	1438.06	0.12	0.08	41.50	0.63	0.68	-7.89	-6.94	7.10	-0.22	0.08	-0.19	0.07	0.04	-0.92	1.01	0.40
16	c	-1.00	0.06	1433.37	0.12	0.08	37.90	0.60	0.65	-3.21	-2.94	3.60	-0.19	0.07	-0.18	0.06	0.55	0.27	1.55	0.67
17	c	-1.00	0.06	1444.26	0.12	0.08	44.10	0.66	0.73	7.08	6.56	-6.20	-0.18	0.07	-0.17	0.06	0.25	0.76	1.81	1.44
18	c	-1.00	0.07	1455.30	0.11	0.08	51.00	0.72	0.79	8.02	7.30	-6.90	-0.20	0.07	-0.19	0.07	0.28	0.99	2.09	2.43
19	c	-1.00	0.07	1456.81	0.11	0.08	51.20	0.73	0.80	1.20	1.09	-0.20	-0.22	0.08	-0.20	0.07	0.76	0.85	2.85	3.29
20	c	-1.00	0.08	1455.54	0.11	0.08	50.30	0.72	0.80	-1.02	-0.93	0.90	-0.23	0.08	-0.21	0.07	-0.16	-0.27	2.68	3.02
21	c	-1.00	0.08	1459.68	0.11	0.08	52.20	0.75	0.82	3.30	3.00	-1.90	-0.23	0.08	-0.20	0.07	0.97	1.26	3.65	4.28
22	c	-1.00	0.08	1457.63	0.11	0.08	50.70	0.74	0.82	-1.68	-1.54	1.50	-0.23	0.08	-0.21	0.08	-0.17	-0.33	3.48	3.94
23	c	-1.00	0.09	1456.38	0.11	0.08	49.60	0.73	0.81	-1.02	-0.92	1.10	-0.23	0.08	-0.21	0.08	0.05	-0.07	3.53	3.87
24	c	-1.00	0.09	1451.19	0.12	0.07	45.70	0.70	0.79	-4.22	-3.79	3.90	-0.23	0.08	-0.21	0.07	-0.03	-0.47	3.50	3.40
25	c	-1.00	0.10	1449.37	0.12	0.06	45.20	0.69	0.82	-1.44	-1.28	0.50	-0.23	0.08	-0.20	0.07	-0.91	-1.09	2.59	2.31
26	c	-1.00	0.10	1399.04	0.14	0.14	20.40	0.41	0.41	-41.33	-34.49	24.80	-0.23	0.08	-0.19	0.07	-9.82	-16.68	-7.23	-14.37
27	c	-1.00	0.10	1406.82	0.13	0.14	22.35	0.44	0.45	3.18	3.15	-1.95	-0.11	0.04	-0.11	0.04	1.13	1.16	-6.10	-13.21
28	c	-1.00	0.11	1403.17	0.13	0.14	20.00	0.42	0.43	-1.64	-1.62	2.35	-0.12	0.04	-0.12	0.04	0.65	0.63	-5.44	-12.58
29	c	-1.00	0.11	1387.17	0.13	0.14	14.35	0.33	0.34	-6.82	-6.70	5.65	-0.12	0.04	-0.12	0.04	-1.13	-1.24	-6.57	-13.82
30	c	-1.00	0.12	1374.12	0.14	0.14	10.10	0.26	0.28	-4.44	-4.32	4.25	-0.09	0.03	-0.09	0.03	-0.13	-0.25	-6.70	-14.07
31	c	-1.00	0.12	1395.41	0.12	0.15	13.90	0.35	0.39	5.86	5.52	-3.80	-0.07	0.03	-0.07	0.02	1.67	2.01	-5.02	-12.05
32	c	-1.00	0.12	1391.97	0.12	0.15	12.00	0.33	0.37	-1.33	-1.22	1.90	-0.11	0.04	-0.10	0.03	0.62	0.50	-4.40	-11.56
33	c	-1.00	0.13	1401.89	0.12	0.16	15.30	0.39	0.42	3.64	3.23	-3.30	-0.10	0.04	-0.09	0.03	-0.12	0.28	-4.53	-11.28
34	c	-1.00	0.13	1402.84	0.12	0.16	15.10	0.39	0.42	0.40	0.37	0.20	-0.12	0.04	-0.11	0.04	0.50	0.52	-4.03	-10.76
35	c	-1.00	0.13	1406.60	0.11	0.16	14.60	0.40	0.44	1.58	1.47	0.50	-0.12	0.04	-0.11	0.04	1.90	2.00	-2.13	-8.76
36	c	-1.00	0.14	1377.95	0.13	0.17	7.85	0.24	0.30	-12.59	-11.60	6.75	-0.12	0.04	-0.11	0.04	-4.92	-5.92	-7.05	-14.67
37	c	-1.00	0.14	1387.17	0.13	0.17	9.50	0.28	0.34	2.77	2.20	-1.65	-0.08	0.03	-0.07	0.02	0.51	1.07	-6.54	-13.61
38	c	-1.00	0.15	1392.28	0.12	0.17	9.30	0.30	0.36	1.75	1.45	0.20	-0.09	0.03	-0.08	0.03	1.60	1.89	-4.94	-11.72
39	c	-1.00	0.15	1386.95	0.12	0.17	8.10	0.26	0.33	-1.93	-1.58	1.20	-0.10	0.04	-0.08	0.03	-0.44	-0.79	-5.38	-12.51
40	c	-1.00	0.15	1402.06	0.12	0.18	11.65	0.36	0.41	4.93	4.00	-3.55	-0.09	0.03	-0.07	0.03	0.40	1.33	-4.97	-11.19
41	c	-1.00	0.16	1410.94	0.11	0.18	13.40	0.42	0.45	3.64	3.17	-1.75	-0.11	0.04	-0.10	0.03	1.36	1.82	-3.62	-9.37
42	c	-1.00	0.16	1435.04	0.11	0.18	25.75	0.63	0.58	10.95	10.06	-12.35	-0.13	0.04	-0.12	0.04	-2.37	-1.49	-5.98	-10.85
43	c	-1.00	0.17	1434.54	0.12	0.18	26.05	0.62	0.58	-0.29	-0.31	-0.30	-0.17	0.06	-0.18	0.06	-0.73	-0.70	-6.71	-11.55
44	c	-1.00	0.17	1436.11	0.12	0.18	27.10	0.63	0.59	0.92	0.97	-1.05	-0.16	0.06	-0.17	0.06	-0.19	-0.24	-6.91	-11.80
45	c	-1.00	0.17	1437.50	0.11	0.19	25.20	0.66	0.60	0.82	0.87	1.90	-0.17	0.06	-0.18	0.06	2.66	2.62	-4.25	-9.18
46	c	-1.00	0.18	1428.61	0.13	0.18	22.35	0.56	0.55	-5.33	-5.83	2.85	-0.17	0.06	-0.19	0.07	-3.10	-2.59	-7.35	-11.77
47	c	-1.00	0.18	1417.23	0.14	0.18	17.10	0.47	0.48	-6.25	-6.40	5.25	-0.16	0.05	-0.16	0.06	-1.25	-1.10	-8.60	-12.87
48	c	-1.00	0.19	1422.53	0.13	0.14	17.65	0.51	0.51	2.53	2.47	-0.55	-0.13	0.05	-0.13	0.05	1.84	1.90	-6.76	-10.98
49	c	-1.00	0.19	1420.86	0.13	0.14	16.60	0.49	0.49	-0.85	-0.85	1.05	-0.14	0.05	-0.14	0.05	0.11	0.11	-6.65	-10.87
50	c	-1.00	0.19	1424.55	0.13	0.13	16.60	0.52	0.52	1.81	1.82	0.00	-0.14	0.05	-0.14	0.05	1.73	1.72	-4.92	-9.15
51	c	-1.00	0.20	1437.77	0.13	0.13	23.75	0.66	0.65	6.92	6.92	-7.15	-0.15	0.05	-0.15	0.05	-0.32	-0.33	-5.25	-9.47
52	c	-1.00	0.20	1439.37	0.12	0.13	23.40	0.69	0.68	1.04	1.05	0.35	-0.19	0.07	-0.19	0.07	1.28	1.27	-3.97	-8.20
53	c	-1.00	0.21	1443.76	0.12	0.12	26.60	0.73	0.74	2.97	3.02	-3.20	-0.19	0.07	-0.20	0.07	-0.31	-0.35	-4.28	-8.55
54	c	-1.00	0.21	1444.61	0.14	0.12	26.50	0.74	0.75	0.63	0.62	0.10	-0.21	0.07	-0.21	0.07	0.58	0.59	-3.70	-7.96
55	c	-1.00	0.21	1448.39	0.12	0.12	27.00	0.83	0.81	2.85	2.78	-0.50	-0.22	0.08	-0.21	0.07	2.14	2.21	-1.55	-5.75
56	c	-1.00	0.22	1438.87	0.15	0.12	21.30	0.69	0.71	-7.71	-7.90	5.70	-0.23	0.08	-0.24	0.08	-2.35	-2.16	-3.91	-7.91
57	c	-1.00	0.22	1447.80	0.12	0.09	25.70	0.84	0.89	6.38	6.14	-4.40	-0.20	0.07	-0.20	0.07	1.62	1.84	-2.29	-6.07
58	c	-1.00	0.23	1452.85	0.13	0.09	30.10	0.88	0.95	4.51	4.26	-4.40	-0.26	0.09	-0.24	0.09	-0.30	-0.05	-2.59	-6.12
59	c	-1.00	0.23	1468.33	0.17	0.10	44.05	0.98	1.00	14.78	13.64	-13.95	-0.28	0.10	-0.25	0.09	-0.48	0.65	-3.07	-5.47
60	c	-1.00	0.23	1471.48	0.27	0.10	47.40	0.95	1.00	3.14	3.08	-3.35	-0.29	0.10	-0.28	0.10	-0.45	-0.39	-3.52	-5.86
61	c	-1.00	0.24	1472.50	0.40	0.09	48.40	0.94	1.00	1.02	0.97	-1.00	-0.29	0.10	-0.27	0.10	-0.21	-0.17	-3.73	-6.03
62	c	-1.00	0.24	1470.73	0.40	0.09	45.73	1.00	1.00	-1.77	-1.67	2.67	-0.29	0.10	-0.27	0.10	0.83	0.71	-2.90	-5.32

[4] The columns with “name of output”+”test” e.g. “deltatest” is the output calculated using TV and the columns without e.g. “delta” is the output calculated using IV

Appendix #11: Market mechanics of a variance swap



Appendix #12: List of figures

Appendix #13: List of tables