Optimal Portfolio Choice in the Presence of Illiquid Assets

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Abstract

We propose two models of illiquidity to determine how an investor behaves optimally in the presence of an illiquid asset, which can only be traded infrequently and on random dates. The investor is less willing to invest in the illiquid asset compared to a situation where all assets are liquid, and his optimal amount invested in the illiquid asset is almost half of what it would be if the asset was liquid. The presence of illiquidity also distorts the investor's willingness to take gambles in his liquid wealth, because he can only meet his immediate obligations with liquid wealth. Together with the fact that the next trading opportunity is random, the investor is less willing to invest in liquid risky asset. We find that investors with shorter investment horizons will never invest any amount in the illiquid asset, and risk-averse investors will need an investment horizon of at least 5 years before they are willing to buy any amount of an illiquid asset that is trade-able on average every year. There is very little difference between long-term investors, as an investor with a investment horizon of 10 year will allocate 1% less in the illiquid asset than an investor with infinite investment horizon. We conclude that the degree of liquidity is an important determinant of the behaviour of the investor. The more illiquid the asset is, the less willing the investor is to buy it, and therefore the more he will need to be compensated in order to invest in it.

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1 Introduction

In this paper, we seek to derive a financial model that shows how an utility maximizing investor behaves optimally in the presence illiquidity. We define an illiquid asset as asset which can not be traded frequently, but instead only at random and unknown trading dates, i.e only at what we refer to as a "liquidity event" . We model the liquidity event with an independent and identically distributed Poisson process with some intensity λ. The randomness of the liquidity event might seem niche at first, but it is, in fact, a very common thing that assets have unknown trading dates. Some examples of assets not having a known trade date are private equity, venture capital, large real estate, Structured products, infrastructure projects and much more.

Illiquidity is a very relevant phenomena, and it comes in many shapes and form in various different asset markets. Illiquid asset classes range over a big spectrum. Assets with varying degree of illiquidity are more common than pure liquid assets or quasi-liquid assets, where an asset is more illiquid the less frequent it is traded. From Table 1 below, a list of various asset classes and the typical time between transaction can be seen. It can be seen that one of the most liquid asset classes is Public equities, which can be traded every second. The more illiquid assets are OTC product, real estate, private equity and Art. In general, Illiquid asset classes are large. For example, Residential housing has a typical time between transactions of 4-5 years, But ranges from months to years (Hansen 1998). The estimated size of the residential real estate market is \$16 trillion, which is compareable to the market capitalization of NYSE and NASDAQ of approximately \$17 trillion[2]. Kaplan and Violante (2010) show that individuals have the majority of their net wealth in illiquid assets, with roughly 81% of households portfolio tied up in illiquid positions. Illiquid assets, in general, has had an increasing presence in investors portfolios. Pension funds, for example, had an average illiquid asset holdings of 5% of their total portfolio in 1995, which has increased to close to 20% in 2010, as reported in the "Global Pension Asset Study 2011" by Tower Watson.

Asset class	**Typical time between transaction**
Public Equities	Within Seconds
OTC equities	Within days, but many assets over a week
Private equity	The median investment duration is 4 years
Residential Housing	4-5 years
Institutional real estate	8-11 years
Institutional infrastructure	50-60 years
Art	40-70 years

Table 1: Holding Periods of Various Asset Classes. Source: Ang et al. (2014).

[2]Public equities can be traded quasi continuous, as the typical time between trades are less than 1-2 seconds, i.e it is a very liquid asset market

Since Illiquidity is such a big part of various Asset Markets, it is safe to say one would be interested in how an investor behaves optimally in the presence of such an asset. Does illiquidity have any significant effect on the decision of the investor? If so, to what degree does the investor's portfolio choice change? What effect does the intensity of liquidity have on the investors portfolio choice? How much does the investor need to be compensated in order to be willing to buy the illiquid asset as if it was liquid? In other words, what is the quantity of a so-called liquidity premium? These are some of the questions we will try to answer.

1.1 Overview of the Paper

In this paper, we will first consider an asset pricing model with infinite time horizon. That is, we consider an investor who has the choice between consuming or investing at each time period, and his investment horizon is infinite. With this setup, we then derive the so called Hamilton-Jacobian-Bellman (HJB) equation, which we solve numerically to get an approximation of the investors value function, i.e the function that shows how well the investor is off for different values of fraction of total wealth invested in the illiquid asset. The value function is essential in deciphering the behaviour of the investor, as it tells us what the optimal allocation in liquid and illiquid wealth is. We find out that not only does illiquidity distort the investor's allocation in the illiquid asset, but also in the liquid risky asset. The main reason for this, is that liquid and illiquid wealth are imperfect substitutes - That is, the investor can only meet his obligation with liquid wealth, and not illiquid wealth. In our model, the investor's only obligation is his consumption, which can be interpreted as spendings, payments and general obligations. If the investor's liquid wealth drops low enough, the investor can not meet his obligations before the next liquidity event. This coupled with the fact that the investor does not know when he will be able to trade the illiquid asset makes the investor behave in an extremely risk-averse manner compared to a stiuation where all assets are liquid. As such, the investor prefers to take fewer risks in both his liquid and illiquid wealth, and thereby reduce his allocation in both liquid risky asset and illiquid risky asset, in order to reduce the chance of being in a scenario with zero liquid wealth.

We also consider a generalized version of the model, where we allow the investor to have finite investment horizon. This does not change the main result, i.e distortion of investment in both risky liquid asset and risky illiquid asset allocation, but it does show us how an investors behaves optimally with different investment horizons. Noticeably, the investor will never hold any amount of the illiquid asset for dates closer to the terminal date. This, of course, depends on the intensity of liquidity, i.e how often the illiquid asset is trade-able on average. With finite time horizon, our approach in estimating the value function will be similar to the one used in the infinite horizon model. That is, we derive the HJB equation and approximate the value function numerically from the HJB equation. However, with finite time horizon the numerical approximation becomes much more complicated. We will study this closer later.

As explained, we will consider two theoretical models with illiquidity. one with infinite time horizon and one with finite time horizon. Both illiquidity models we consider will be a generalization of the Merton Model, and they will be compared to the Merton Model thoroughly throughout the paper. Hence, the Merton Model will be used as a benchmark to quantify the effect of illiquidity. We derive both a result for the behaviour of an investor with infinite time horizon and finite time horizon with the Merton Model.

1.2 Our approach compared to other

Illiquidity and its effects has been studied thoroughly in the literature, with different approaches and different underlying assumptions. We compare our approach to earlier approaches in the literature. Our notion of liquidity is identical to the one studied in Ang et al. 2014, and most of our theory with regards to liquidity will be based on this. But our notion of liquidity is also conceptually different from other two common ways done earlier in the literature. The first commonly studied concept is that liquidity is costly. That is, every asset can be traded as if they were liquid, but each asset requires a premium paid to be traded. depending on the size of the premium, some assets are more liquid than others, i.e an asset with a very small premium is a liquid asset and an asset with a very high premium is an illiquid asset (See Grossman and Laroque 1990, Vayanos 1998). This notion of liquidity does not take into account the uncertain waiting time, which is the case for many assets. Our results imply that the uncertain waiting time has an important effect on the investor's portfolio choice. The second type of liquidity studied is where the illiquid asset can be traded at known dates (e.g., Kahl et al. 2003). In other words, the investor knows when the asset will be trade-able again. The difference from our notion of liquidity is that the investor can pick his portfolio and consumption choices with the deterministic trading date in mind, i.e the investor can account for the illiquidity, where as in our case the investor can not account for the illiquidity, due to the stochastic nature we impose. As such, we would expect any effect of illiquidity to be magnified with our setup compared to the one with deterministic trading dates.

2 Discrete Time Framework

While the model we will use in our main analysis will be in continuous time, we will start by deriving some results in discrete time. The reason for this is that a continuous time framework can intuitively be viewed as the limit of a discrete time frame work, when the length between periods go towards 0. As such, some of the results in continuous time can easily be derived and motivated by simply taking the discrete time corresponding result and let the length between periods go towards 0.

We consider an individual living over the the period $[0;T]$ for $T > 0$. We further assume that the individual can rebalance his portfolio at any date $t_i = i \cdot \Delta t$, where $t_0 = 0$, $t_1 = t_0 + \Delta t$, $t_2 = t_1 + \Delta t$,..., $t_N = t_{N-1} + \Delta t = T$. We assume there are $n+1$ assets in the economy, where n of the assets are risky and 1 of them is a risk-free asset. The risk-free asset has a rate of return of r_t, such that the return over a period $[t, t+\Delta t]$ is $r_t \Delta t$. Let $\mathcal{T} = \{t_0, t_1, ..., t_{N-1}\}$ be the set consisting of date, where the investor can rebalance his portfolio. Let P_t^0 denote the price of the risk-free rate at time t, and let $\boldsymbol{P}_t = (P_t^1, ..., P_t^n)^T$ denote the vector of prices of the n risky assets at time t. Lastly let $\boldsymbol{R}_{t+\Delta t} = (R_{t+\Delta t}^1, ..., R_{t+\Delta t}^n)^T$ be a vector of returns where each entrance corresponds to each of the risky assets, such that $R_{t+\Delta t}^i = \frac{P_{t+\Delta t}^i - P_t^i}{P_t^i}$. The return vector $\boldsymbol{R}_{t+\Delta t}$ is not necessarily known at time t. Other than choosing how much to invest in the assets, the investor also chooses how much to consume c_t at any given period t, such that his consumption over a period $[t, t+\Delta t]$ is $c_t \Delta t$.

With the above in mind, we will now try to specify the dynamics of the investors wealth over time. The wealth of the investor at some date $t \in \mathcal{T}$ will be given by:

$$W_t = \sum_{i=0}^{n} M_{t-\Delta t} P_t^i$$

Where $M_{t-\Delta t}^i$ is the amount invested in asset i the previous period. This formula for wealth at time t imposes a natural restriction on how much the investor can consume at time t. We have:

$$c_t \Delta t \leq \sum_{i=0}^{n} M_{t-\Delta t} P_t^i - \sum_{i=0}^{n} M_t P_t^i = \sum_{i=0}^{n} (M_{t-\Delta t} - M_t) P_t^i$$

That is, the investor can not consume more than what is left from his investment decision at time t. As the investor is utility maximizing, he will consume everything not used for investing. Thus, we have that the above equation is binding:

$$c_t \Delta t = \sum_{i=0}^{n} (M_{t-\Delta t} - M_t) P_t^i$$

Using this, we can calculate:

$$
\begin{aligned}
W_{t+\Delta t} - W_t &= \sum_{i=0}^{n} M_t P_{t+\Delta t}^i - \sum_{i=0}^{n} M_{t-\Delta t} P_t^i \\
&= \sum_{i=0}^{n} M_t P_{t+\Delta t}^i - \sum_{i=0}^{n} M_{t-\Delta t} P_t^i - \sum_{i=0}^{n} M_t P_t^i + \sum_{i=0}^{n} M_t P_t^i \\
&= \sum_{i=0}^{n} M_t (P_{t+\Delta t}^i - P_t^i) - \sum_{i=0}^{n} (M_t - M_{t-\Delta t}) P_t^i \\
&= \sum_{i=0}^{n} M_t (P_{t+\Delta t}^i - P_t^i) - c_t \Delta t
\end{aligned}
$$

Let $\theta_t^i = M_t^i P_t^i$ denote the amount invested in asset i at time t, and further let $\boldsymbol{\theta}_t = (\theta_t^1, ..., \theta_t^n)^T$. Rewriting the above in terms of returns, we get:

$$
W_{t+\Delta t} - W_t = \theta_t^0 r_t \Delta t + \boldsymbol{\theta}_t^T \boldsymbol{R}_{t+\Delta t} - c_t \Delta t \tag{1}
$$

We will now decompose the returns into a stochastic term and a deterministic term, i.e we have:

$$
\boldsymbol{R}_{t+\Delta t} = \boldsymbol{\mu}_t \Delta t + \underline{\underline{\sigma}}_t \epsilon_{t+\Delta t} \sqrt{\Delta t}
$$

Where $\boldsymbol{\mu}_t$ is a vector of expected rates of return,$\epsilon_{t+\Delta t}$ is a stochastic variable representing the shock to the economy from time t to time $t + \Delta t$ with mean 0 and variance of 1, and $\underline{\underline{\sigma}}_t$ is the volatility matrix determining how the assets are affected by the shocks. Notice that the shock, $\epsilon_{t+\Delta t}$, is not known until $t + \Delta t$. Inserting this in the above wealth dynamics, we get:

$$
\begin{aligned}
W_{t+\Delta t} - W_t &= \theta_t^0 r_t \Delta t + \boldsymbol{\theta}_t^T \boldsymbol{R}_{t+\Delta t} - c_t \Delta t \\
&= \theta_t^0 r_t \Delta t + \boldsymbol{\theta}_t^T (\boldsymbol{\mu}_t \Delta t + \underline{\underline{\sigma}}_t \epsilon_{t+\Delta t} \sqrt{\Delta t}) - c_t \Delta t \\
&= (\theta_t^0 r_t + \boldsymbol{\theta}_t^T \boldsymbol{\mu}_t - c_t) \Delta t + \underline{\underline{\sigma}}_t \epsilon_{t+\Delta t} \sqrt{\Delta t}
\end{aligned} \tag{2}
$$

The above gives a nice, intuitive representation of the dynamics of the investors wealth over time. It says that the change in the investors wealth over a time period Δt is equal to the average return he gets over that period from the risk-free asset and his risky portfolio minus what he consumes. Lastly, there is also a stochastic term affecting the investors change in wealth over the period to represent the idiosyncratic and risky nature of investing. This equation can also easily be converted to continuous time, which we will do later on when we introduce The Merton Model.

2.1 Behaviour of the investor

So far, we have only derived a budget constraint on the investors wealth without minding how the investor behaves or what his preferences are. First, let us define $\boldsymbol{\pi}_t = (\pi_t^1, ..., \pi_t^n)^T$, where $\pi_t^i = \frac{\theta_t^i}{W_t - c_t \Delta t}$ for $i = 1, .., n$. Hence, $\boldsymbol{\pi}_t$ is the vector of weights for the risky assets of the portfolio. We assume that the investor maximizes his life-time utility at any given date, and that his lifetime utility in discrete time is given by:

$$U(c_{t_0}, ..., c_{t_N}) = \sum_{j=0}^{N} e^{-\beta t_j} u(c_{t_j}) \Delta t$$

Where $u(c_{t_j})$ is the utility gained from consumption c_{t_j} at time t_j. The investor maximizes the above, or more correctly„ and β is the subjective discount factor of the investor. he maximizes the expectation of the above as seen from time t, since his future wealth is unknown. The maximum obtained expected utility at time $t = i\Delta t \in \mathcal{T}$ is thus given by:

$$F_t = \max_{(c_{t_j}, \boldsymbol{\pi}_{t_j})_{j=i}^N} E_t \left[\sum_{j=i}^{N} e^{-\beta(t_j - t)} u(c_{t_j}) \Delta t \right]$$

The subscript on the expectation denotes that the expectation is taken conditional on information available at time t, i.e we have that $E[X_{t+1}|X_t] := E_t[X_{t+1}]$. The above is also referred to as the indirect utility of the investor, as F_t is the highest attainable expected life-time utility the investor can derive from his current wealth level. The careful reader would notice that such a maximum does not always exist, and while that is true, for our purpose, we will assume that such a maximum always exist[3]. The indirect utility will be the main component in our derivations when introducing an illiquid asset in a continuous time setting, and we will now derive an important simplification of the indirect utility. consider F_t at some time t. We can thus rewrite:

$$\begin{aligned}
F_t &= \max_{(c_{t_j}, \boldsymbol{\pi}_{t_j})_{j=i}^N} E_t \left[\sum_{j=i}^{N} e^{-\beta(t_j - t)} u(c_{t_j}) \Delta t \right] \\
&= \max_{(c_{t_j}, \boldsymbol{\pi}_{t_j})_{j=i}^N} E_t \left[e^{-\beta(t-t)} u(c_t) \Delta t + \sum_{j=i+1}^{N} e^{-\beta(t_j - t)} u(c_{t_j}) \Delta t \right] \\
&= \max_{(c_{t_j}, \boldsymbol{\pi}_{t_j})_{j=i}^N} E_t \left[u(c_t) \Delta t + e^{-\beta \Delta t} \sum_{j=i+1}^{N} e^{-\beta(t_j - (t - \beta \Delta t))} u(c_{t_j}) \Delta t \right] \\
&= \max_{(c_{t_j}, \boldsymbol{\pi}_{t_j})_{j=i}^N} E_t \left[E_{t+\Delta t} \left[u(c_t) \Delta t + e^{-\beta \Delta t} \sum_{j=i+1}^{N} e^{-\beta(t_j - (t - \beta \Delta t))} u(c_{t_j}) \Delta t \right] \right] \\
&= \max_{(c_{t_j}, \boldsymbol{\pi}_{t_j})_{j=i}^N} E_t \left[u(c_t) \Delta t + e^{-\beta \Delta t} E_{t+\Delta t} \left[\sum_{j=i+1}^{N} e^{-\beta(t_j - (t - \beta \Delta t))} u(c_{t_j}) \Delta t \right] \right] \\
&= \max_{c_t, \boldsymbol{\pi}_t} E_t \left[u(c_t) \Delta t + e^{-\beta \Delta t} \max_{(c_{t_j}, \boldsymbol{\pi}_{t_j})_{j=i+1}^N} E_{t+\Delta t} \left[\sum_{j=i+1}^{N} e^{-\beta(t_j - (t - \beta \Delta t))} u(c_{t_j}) \Delta t \right] \right] \\
&= \max_{c_t, \boldsymbol{\pi}_t} E_t \left[u(c_t) \Delta t + e^{-\beta \Delta t} F_{t+\Delta t} \right]
\end{aligned}$$

[3]This assumption is not as restrictive as it might seem. We could easily do all the derivations with supremum instead of a maximum. With this assumption, we do not need to go into any arguments with supremum.

First equality comes from the definition of indirect utility, second equality comes from splitting the sum, third equality comes from adding 0 in power of e, fourth equality comes from law of iterated expectations (that is for $t < t'$, we have $E_t[X] = E_t[E_{t'}[X]]$), fifth equality comes from linearity of expectation and the fact that $u(c_t)\Delta t$ is not stochastic at time $t + \Delta t$, sixth equality comes from splitting the maximization problem in two, and finally the last equality comes from the definition of indirect utility. In total, we have:

$$F_t = \max_{c_t,\boldsymbol{\pi}_t} E_t\left[u(c_t)\Delta t + e^{-\beta\Delta t}F_{t+\Delta t}\right] = \max_{c_t,\boldsymbol{\pi}_t}\left(u(c_t)\Delta t + e^{-\beta\Delta t}E_t\left[F_{t+\Delta t}\right]\right) \tag{3}$$

This is called the **Bellman Equation**. The Bellman equation is central in understanding how an investor decides on an optimal investment and consumption scheme. The investors decision at time t can be split in two: (1) The amount to consume and invest at time t, and (2) the amount to invest and consume in all future periods. In other words, the investor makes a utility maximizing decision on consumption and portfolio under the assumption that he behaves optimally in all future periods.

3 The Merton Model

The merton model is the baseline model, which we will expand to account for liquidity. In this section, we will briefly review the model. The model is a continuous-time model, and as such, we will be in a continuous time setup. As mentioned earlier, a continuous time model can be seen as the limit of a discrete-time model where the length between periods go towards 0. Therefore, we can use many of the derivations we derived earlier in discrete time by simply taking the limit. While this approach is quite heuristic and a bit informal, it holds the advantage that we do not need to dive deep into the continuous time arguments (although very interesting), such as sigma-algebra and measurability of sets, and still make use of the intuition and precision of a continuous time model.

Consider an agent, which maximizes his expected lifetime utility of consumption with the utility given by Constant Relative Risk Aversion (CRRA):

$$u(C) = \frac{C^{1-\gamma}}{1-\gamma} \tag{4}$$

Where C is consumption. We Assume that the investor has infinite time horizon. We further assume constant investment opportunities, i.e the short-term interest rate r, expected rates of return $\boldsymbol{\mu}$, and the volatility matrix $\underline{\underline{\sigma}}$ are all assumed to be constant. We further assume that the volatility matrix is non-singular, i.e it is invertible. Thus, the investors indirect utility will be given by:

$$F(W,t) = \max_{c_t,\pi_t} E_t\left[\int_t^{\infty} e^{-\beta(s-t)}\frac{c_s^{1-\gamma}}{1-\gamma}ds\right]$$

In order to get as general of a result as possible, we will first consider an investor with finite time horizon, and later return to an investor with infinite time horizon. Thus, we first consider an investor with the following indirect utility:

$$F(W,t) = \max_{c_t,\pi_t} E_t\left[\int_t^T e^{-\beta(s-t)}\frac{c_s^{1-\gamma}}{1-\gamma}ds\right] \tag{5}$$

Taking the limit of (2) for $\Delta t \to 0$ and assuming that $\epsilon_{t+\Delta t}$ is multivariate standard normally distributed, we get that the dynamics of wealth in continuous time is given by:

$$dW_t = (\theta_t^0 r + \boldsymbol{\theta}_t^T \boldsymbol{\mu} - c_t)dt + \underline{\underline{\sigma}} d\boldsymbol{z}_t$$

Where dz_t is an n-dimensional standard brownian motion. The amount invested in the risk-free asset is the total wealth minus what is invested in the risky asset:

$$\theta_t^0 = W_t - \boldsymbol{\theta}_t^T \mathbf{1}$$

where $\mathbf{1}$ is an n dimensional vector with 1 in each entrance. Now defining the following n-dimensional process:

$$\boldsymbol{\lambda} = \underline{\underline{\sigma}}^{-1}(\boldsymbol{\mu} - r\mathbf{1})$$

$$\Updownarrow$$

$$\boldsymbol{\mu} = \underline{\underline{\sigma}}\boldsymbol{\lambda} + r\mathbf{1}$$

$\boldsymbol{\lambda}$ is called the vector of market prices of risk, as it measures the excess rate of return per units of standard deviation for each asset. Inserting these results in the dynamics of wealth, we get:

$$dW_t = (W_t r - \boldsymbol{\theta}_t^T \mathbf{1} r + \boldsymbol{\theta}_t^T(\underline{\underline{\sigma}}\boldsymbol{\lambda} + r\mathbf{1}) - c_t)dt + \underline{\underline{\sigma}} d\boldsymbol{z}_t$$

$$\Updownarrow$$

$$dW_t = (W_t r + \boldsymbol{\theta}_t^T \underline{\underline{\sigma}}\boldsymbol{\lambda} - c_t)dt + \underline{\underline{\sigma}} d\boldsymbol{z}_t$$

And in terms of portfolio weights, $\boldsymbol{\pi}_t = (\pi_t^1, ..., \pi_t^n)$, the above can be rewritten to:

$$dW_t = (W_t[r + \boldsymbol{\pi}_t^T \underline{\underline{\sigma}}\boldsymbol{\lambda}] - c_t)dt + W_t \boldsymbol{\pi}_t^T \underline{\underline{\sigma}} dz_t \tag{6}$$

, Where we have that $\boldsymbol{\pi}_t$ is a vector of portfolio weights for the n risky assets, $\underline{\underline{\sigma}}$ is an $n \times n$ volatility matrix such that $\underline{\underline{\sigma}} \times \underline{\underline{\sigma}}^T$ is the variance covarince matrix and $\boldsymbol{\lambda}$ is the market price of risk as described above. To find a solution to the investors problem, we use a dynamic programming approach.

3.1 Dynamic Programming Approach

In order to continue with the derivations of the Merton Model, we will briefly introduce dynamic programming and the Hamilton-Jacobian-Bellman (HJB) equation in this section. Notice that we will motivate the HJB equation in a general setting, and not under the assumption of constant investment opportunities as we assumed earlier. The reason for this is so the derivations can be used in the widest possible application. Thus, we allow for r_t, $\boldsymbol{\mu}_t$ and $\underline{\underline{\sigma}}_t$ to vary with time. Our main goal is to solve the investor's problem with regards to consumption strategy and investment strategy. To solve the investor's problem, that is maximizing his expected infinite continuous sum of utility, we will use an approach called dynamic programming. This way of solving the investor's maximization problem in continuous-time was first done in Merton (1969, 1971). The approach requires a so-called state variable X_t to exist (possibly multi-dimensional), such that this variable follows a Markov process. The idea is then, that we assume the state variable captures all the variation of the risk-free return, expected return of the assets and the volatility of the assets. In other words, we can write:

$$r_t = r(X_t), \quad \boldsymbol{\mu}_t = \boldsymbol{\mu}(X_t, t), \quad \underline{\underline{\sigma}}_t = \underline{\underline{\sigma}}(X_t, t)$$

With this assumption, we can through the state variable derive a highly non-linear second-order partial differential equation called the Hamilton-Jacobi-Bellman (HJB) equation, and use the HJB equation (under some technical assumptions) to derive the optimal investment strategies and indirect utility of the investor. We will now go through all the derivations. For simplicity, assume that we are in a 1-dimensional setting, such that we have 1 risky asset[4]. Let θ_t be the fraction of wealth invested in the risky asset. Thus, we have that the multi-dimensional notation from earlier can be rewritten to:

$$\underline{\underline{\sigma}}_t = \sigma_t \in \mathbb{R} \quad \boldsymbol{\lambda}_t = \frac{\mu_t - r_t}{\sigma_t} \in \mathbb{R}, \quad \boldsymbol{\pi}_t = \theta_t \in \mathbb{R}, \quad r_t = 1 - \theta_t \in \mathbb{R}$$

So we can rewrite the dynamics of wealth from (6) to:

$$\frac{dW_t}{W_t} = (r + (\mu - r_t)\theta_t - c_t)dt + \theta_t \sigma dZ_t^1$$

where Z_t^1 is a 1-dimensional standard brownian motion. Now let the state variable X_t be given by:

$$\frac{dX_t}{X_t} = m(X_t)dt + v_1 dZ_t^1 + v_2 dZ_t^2$$

Where Z_t^2 is also a standard brownian motion independent of Z_t^1 and $v_1, v_2 \in \mathbb{R}$. From the Bellman equation in (3), we have:

$$F_t = \max_{C_t, \theta_t} \Big[u(c_t)\Delta t + e^{-\beta \Delta t} E_t[F_{t+\Delta t}] \Big]$$

[4] The derivations for mutli-dimensional setting is indentical to the 1-dimensional setting, but as our analysis, when introducing liquidity, is in the 1-dimensional setting, we will try to stay consistent with that.

If we multiply with $e^{\beta\Delta t}$, we get:

$$F_t e^{\beta\Delta t} = \max_{C_t,\theta_t}\left[e^{\beta\Delta t}u(c_t)\Delta t + E_t[F_{t+\Delta t}]\right]$$

Subtracting $F_t(W,x)$ and deviding with Δt on both sides, we get:

$$F_t \cdot \left(\frac{e^{\beta\Delta t}-1}{\Delta t}\right) = \max_{C_t,\theta}\left[e^{\beta\Delta t}u(c_t) + \frac{E_t[F_{t+\Delta t}]-F_t}{\Delta t}\right] \tag{7}$$

Letting $\Delta t \to 0$, then by l'Hôspital's rule we have:

$$\lim_{\Delta t\to 0}\frac{e^{\beta\Delta t}-1}{\Delta t} = \lim_{\Delta t\to 0}\frac{\beta e^{\beta\Delta t}}{1} = \beta$$

And furthermore when $\Delta t \to 0$, then per the definition of the drift of a process, we have that:

$$\lim_{\Delta t\to 0}\frac{E_t[F_{t+\Delta t}]-F_t}{\Delta t} \to drift(F_t)$$

This comes from the fact that $E_t[F_{t+\Delta t}] - F_t$ is the expected change in F_t over a small periode Δt. dividing this number with Δt yields the expected change over 1 periode of time, which is exactly the drift of the F_t process.
Using Ito's Lemma on $F_t = F(W,X,t)$, we see that:

$$drift(F_t) = \frac{\partial F}{\partial t} + F_W W(r_t + (\mu_t - r_t)\theta_t - c_t) + F_X m(X) + \frac{1}{2}F_{WW}W^2\theta_t^2\sigma_t^2 + \frac{1}{2}F_{XX}(X^2v_1^2 + X^2v_2^2)$$
$$+ F_{WX}W_tX_t\theta\sigma_t v_1$$

, Where F_W, F_X, F_{WW} and F_{WX} denotes the derivatives, such that $F_W = \frac{\partial F}{\partial W}$ and similar for the others. Using these facts, and taking the limit of (7) when $\Delta t \to 0$, we get:

$$\beta F_t(W,X) = \max_{c,\theta}\left[u(c) + \frac{\partial F}{\partial t} + F_W W(r_t + (\mu_t - r_t)\theta_t - c) + F_X m(X) + \frac{1}{2}F_{WW}W^2\theta^2\sigma_t^2 \right.$$
$$\left. + \frac{1}{2}F_{XX}(X^2v_1^2 + X^2v_2^2) + F_{WX}W_tX_t\theta\sigma_t v_1\right] \tag{8}$$

This is called the Hamilton-Jacobi-Bellman (HJB) equation corresponding to the above stated optimization problem. The above maximization can be conveniently rewritten as:

$$\beta F_t(W,X) = \max_{c,\theta}\left[L^c F_t(W,X) + L^\theta F_t(W,X) + \frac{\partial F}{\partial t} + rWF_W + F_X m(X) + \frac{1}{2}F_{XX}(X^2v_1^2 + X^2v_2^2)\right] \tag{9}$$

Where

$$L^c F_t(W,X) = \max_c\left[u(c) - cF_W\right]$$

$$L^\theta F_t(W,X) = \max_\theta\left[F_W W(\mu - r_t)\theta + \frac{1}{2}F_{WW}W^2\theta^2\sigma^2 + F_{WX}W_tX_t\theta\sigma_t v_1\right]$$

Equivalently, one can derive the HJB equation for a multi-dimensional setup in exactly the same way as the one-dimensional setup. The multi-dimensional HJB is given by:

$$\beta F_t(W,X) = \max_{c,\boldsymbol{\pi}} \left[L^c F_t(W,X) + L^{\boldsymbol{\pi}} F_t(W,X) + \frac{\partial F}{\partial t} + r_t W F_W + F_X m(X) + \frac{1}{2} X^2 F_{XX}(W,X)(\boldsymbol{v}_1^T \boldsymbol{v}_1 + v_2^2) \right] \tag{10}$$

Where

$$L^c F_t(W,X) = \max_c \left[u(c) - c F_W \right]$$

$$L^{\boldsymbol{\pi}} F_t(W,X) = \max_{\boldsymbol{\pi} \in \mathbb{R}^n} \left[W F_W(W,X) \boldsymbol{\pi}^T \underline{\underline{\sigma}}_t \boldsymbol{\lambda}_t + \frac{1}{2} F_{WW} W^2 \boldsymbol{\pi}^T \underline{\underline{\sigma}}_t \underline{\underline{\sigma}}_t^T \boldsymbol{\pi} + F_{WX} W \boldsymbol{\pi}^T \underline{\underline{\sigma}}_t \boldsymbol{v} \right]$$

In total, we have derived a highly non-linear equation, but this equation in itself does not appear to be the same as the investors maximization problem in (5), but the two are, in fact, closely linked. It can be shown that if you can solve the above HJB equation and find a feasible consumption scheme c^* and an feasible investment portfolio $\boldsymbol{\pi}^*$, then this solution to the HJB equation is equal to the investors indirect utility, and the feasible consumption and investment scheme for the HJB equation are optimal for the investor as well. There are, of course, some technical assumptions, but we will not go into details with them here. We will simply use the fact that if we can solve the HJB equation, and thereby find a feasible consumption and investment strategy, then we have solved the investors problem. This is called **The Verification Theorem**. For a precise statement and proof of the theorem, see Øksendal (2003). We will continue below with the derivation of the Merton Model, and apply the verification theorem directly.

3.2 Merton Model - Continued

With constant investment opportunities, the HJB equation can be rewritten to:

$$\beta F(W,t) = L^c F(W,t) + L^{\boldsymbol{\pi}} F(W,t) + \frac{\partial F}{\partial t}(W,t) + r W F_W(W,t)$$

where

$$L^c F(W,t) = \max_c \left[u(c) - c F_W(W,t) \right] \tag{11}$$

$$L^{\boldsymbol{\pi}} F(W,t) = \max_{\boldsymbol{\pi}} \left[W F_W(W,t) \boldsymbol{\pi}^T \underline{\underline{\sigma}} \boldsymbol{\lambda} + \frac{1}{2} F_{WW}(W,t) W^2 \boldsymbol{\pi}^T \underline{\underline{\sigma \sigma}}^T \boldsymbol{\pi} \right] \tag{12}$$

This comes from the fact that the volatility matrix, market price of risk and returns are constant, which implies that the state variable will have no impact on them, and thereby no impact on the value function. Hence, all derivatives with respect to the state variable X_t will be 0. Since the utility function is concave, and $F_W(W,t)$ is a constant in consumption, we can solve the maximization problem of (11) by taking the first order condition. Taking the first order condition for (11) gives us:

$$u'(c) = F_W(W,t) \tag{13}$$

This is the so-called envelope condition. Intuitively, the condition states that the utility gained from consuming 1 extra unit of wealth today should equal the utility from investing 1 extra unit of wealth optimally, and consuming it optimally at a later date. If that is not the case, then an investor can get a positive utility gain from reallocating wealth between consumption and investment. For example, if $u'(c) < F_W(W,t)$, then the amount invested should increase and consumption today should decrease. Let I denote the inverse of $u'(c)$. Taking the inverse of both side in the above equation, we get:

$$C(W,t) := c_t = I(F_W(W_t,t))$$

substituting this optimal consumption scheme in to (11), we get:

$$L^c F(W,t) = u(I(F_W(W_t,t))) - I(F_W(W_t,t))F_W(W,t)$$

Similarly, taking the FOC in (12), we get:

$$WF_W(W,t)\underline{\underline{\sigma}}\boldsymbol{\lambda} + F_{WW}(W,t)W^2\underline{\underline{\sigma\sigma}}^T\boldsymbol{\pi} = 0$$

$$\Updownarrow$$

$$\boldsymbol{\pi} = -\frac{WF_W(W,t)}{F_{WW}(W,t)W^2}(\underline{\underline{\sigma}}^T)^{-1}\boldsymbol{\lambda}$$

Inserting this optimal solution in to (12), we get:

$$\begin{aligned}
L^{\boldsymbol{\pi}}F(W,t) =& -WF_W(W,t)\frac{WF_W(W,t)}{F_{WW}(W,t)W^2}\boldsymbol{\lambda}^T\underline{\underline{\sigma}}^{-1}\underline{\underline{\sigma}}\boldsymbol{\lambda} + \frac{1}{2}F_{WW}(W,t)W^2\frac{WF_W(W,t)}{F_{WW}(W,t)W^2}\boldsymbol{\lambda}^T\underline{\underline{\sigma}}^{-1}\underline{\underline{\sigma\sigma}}^T\boldsymbol{\pi} \\
&= -WF_W(W,t)\frac{WF_W(W,t)}{F_{WW}(W,t)W^2}\boldsymbol{\lambda}^T\boldsymbol{\lambda} + \frac{1}{2}WF_W(W,t)\boldsymbol{\lambda}^T\underline{\underline{\sigma}}^T\frac{WF_W(W,t)}{F_{WW}(W,t)W^2}(\underline{\underline{\sigma}}^T)^{-1}\boldsymbol{\lambda} \\
&= -WF_W(W,t)\frac{WF_W(W,t)}{F_{WW}(W,t)W^2}||\boldsymbol{\lambda}||^2 + \frac{1}{2}WF_W(W,t)\frac{WF_W(W,t)}{F_{WW}(W,t)W^2}||\boldsymbol{\lambda}||^2 \\
&= -\frac{1}{2}WF_W(W,t)\frac{WF_W(W,t)}{F_{WW}(W,t)W^2}||\boldsymbol{\lambda}||^2
\end{aligned}$$

Where the second equality comes from Transpose and inverse rules, third equality comes from definition of the $\mathbb{R}^n$ product norm and last equality comes from simplifying. Thus, the HJB equation can be rewritten to:

$$\begin{aligned}
\beta F(W,t) =& u(I(F_W(W,t))) - I(F_W(W,t))F_W(W,t) - \frac{1}{2}WF_W(W,t)\frac{WF_W(W,t)}{F_{WW}(W,t)W^2}||\boldsymbol{\lambda}||^2 + \frac{\partial F}{\partial t}(W,t) \\
&+ rWF_W(W,t)
\end{aligned} \tag{14}$$

To simplify the HJB equation further, we calculate the marginal utility function from (4):

$$u'(c) = c^{-\gamma}$$

$$\Updownarrow$$

$$c = u'(c)^{-\frac{1}{\gamma}}$$

Thus the inverse of the above function evaluated at some point $a \in \mathbb{R}$ will be given by:

$$I(a) = a^{-\frac{1}{\gamma}}$$

We can further calculate:

$$u(I(a)) = \frac{I(a)^{1-\gamma}}{1-\gamma} = \frac{(a^{-\frac{1}{\gamma}})^{1-\gamma}}{1-\gamma} = \frac{a^{1-\frac{1}{\gamma}}}{1-\gamma}$$

and thereby, we have:

$$u(I(a)) - aI(a) = \frac{a^{1-\frac{1}{\gamma}}}{1-\gamma} - aa^{-\frac{1}{\gamma}} = \frac{a^{1-\frac{1}{\gamma}}}{1-\gamma} - aa^{-\frac{1}{\gamma}} = a^{1-\frac{1}{\gamma}}\left(\frac{1}{1-\gamma} - 1\right)$$
$$= a^{1-\frac{1}{\gamma}}\frac{\gamma}{1-\gamma}$$

With this the first two terms in (14) will be given by:

$$u(I(F_W(W_t,t))) - I(F_W(W,t))F_W(W,t) = F_W(W,t)^{1-\frac{1}{\gamma}}\frac{\gamma}{1-\gamma} \tag{15}$$

Now let $k \in \mathbb{R}$. For a given level of wealth W and an optimal consumption plan c^*, we will assume that if we multiply the wealth with some constant k, then they optimal consumption plan for $k \cdot W$ will be given by $k \cdot c^*$. Intuitively, this means that a doubling in wealth will result in a doubling in optimal consumption. The linear nature of this assumption stems from how the wealth dynamics in (6) has been defined. In other words, to offset an increase in W, a proportional decrease in consumption would be needed. Per this assumption, we see from our definition of $F(W,t)$ that:

$$F(kW,t) = \max_{c_t,\pi_t} E_t\left[\int_0^T e^{-\beta(s-t)}\frac{(kc_s)^{1-\gamma}}{1-\gamma}ds\right] = k^{1-\gamma}\max_{c_t,\pi_t} E_t\left[\int_0^T e^{-\beta(s-t)}\frac{(c_s)^{1-\gamma}}{1-\gamma}ds\right] = k^{1-\gamma}F(W,t) \tag{16}$$

$$\Updownarrow$$

$$F(W,t) = \frac{1}{k^{1-\gamma}}F(kW,t)$$

Thus, letting $k = \frac{1}{W}$, we get:

$$F(W,t) = W^{1-\gamma}F(1,t) = W^{1-\gamma}F(1,t)\frac{1-\gamma}{1-\gamma} = \frac{W^{1-\gamma}H(t)^\gamma}{1-\gamma}$$

where we define $H(t)^\gamma = (1-\gamma)F(1,t)$. Calculating the derivatives of F, we get:

$$F_W = W^{-\gamma}H(t)^\gamma, \quad F_{WW} = -\gamma W^{-\gamma-1}H(t)^\gamma$$

$$\frac{\partial F}{\partial t}(W,t) = W^{1-\gamma}\gamma H(t)^{\gamma-1}\frac{H'(t)}{1-\gamma}$$

Inserting these derivatives in our HJB equation together with the equation for $F(W,t)$ and (15), we get:

$$\beta\frac{W^{1-\gamma}H(t)^{\gamma}}{1-\gamma} = F_W(W,t)^{1-\frac{1}{\gamma}}\frac{\gamma}{1-\gamma} - \frac{1}{2}WW^{-\gamma}H(t)^{\gamma}\frac{WW^{-\gamma}H(t)^{\gamma}}{-\gamma W^{-\gamma-1}H(t)^{\gamma}W^2}||\boldsymbol{\lambda}||^2 + W^{1-\gamma}\gamma H(t)^{\gamma-1}\frac{H'(t)}{1-\gamma}$$
$$+ rWW^{-\gamma}H(t)^{\gamma}$$

$$\Updownarrow$$

$$0 = -\beta\frac{W^{1-\gamma}H(t)^{\gamma}}{1-\gamma} + (W^{-\gamma}H(t)^{\gamma})^{1-\frac{1}{\gamma}}\frac{\gamma}{1-\gamma} - \frac{1}{2}W^{-\gamma}H(t)^{\gamma}\frac{W^{-\gamma}H(t)^{\gamma}}{-W^{-\gamma-1}H(t)^{\gamma}\gamma}||\boldsymbol{\lambda}||^2$$
$$+ W^{1-\gamma}\gamma H(t)^{\gamma-1}\frac{H'(t)}{1-\gamma} + rWW^{-\gamma}H(t)^{\gamma}$$

$$\Updownarrow$$

$$0 = \Big(-\beta\frac{H(t)^{\gamma}}{1-\gamma} + H(t)^{\gamma-1}\frac{\gamma}{1-\gamma} + \frac{1}{2\gamma}H(t)^{\gamma}||\boldsymbol{\lambda}||^2 + \gamma H(t)^{\gamma-1}\frac{H'(t)}{1-\gamma} + rH(t)^{\gamma}\Big)W^{1-\gamma}$$

$$\Updownarrow$$

$$0 = \Big(\Big(\frac{-\beta}{1-\gamma} + \frac{1}{2\gamma}||\boldsymbol{\lambda}||^2 + r\Big)H(t) + \gamma\frac{H'(t)}{1-\gamma} + \frac{\gamma}{1-\gamma}\Big)W^{1-\gamma}H(t)^{\gamma-1}$$

We see that the above equation has to hold for any $t \in [0,T)$ and any $W \geq 0$. If we make a reasonable assumption that $\exists t \in [0,T) : H(t) \geq 0$, then the only way the above can be 0 is if:

$$\Big(\frac{-\beta}{1-\gamma} + \frac{1}{2\gamma}||\boldsymbol{\lambda}||^2 + r\Big)H(t) + \gamma\frac{H'(t)}{1-\gamma} + \frac{\gamma}{1-\gamma} = 0 \tag{17}$$

$$\Updownarrow$$

$$H'(t) = A \cdot H(t) - 1 \tag{18}$$

where $A = \frac{\beta + r(\gamma-1)}{\gamma} + \frac{1}{2}\frac{\gamma-1}{\gamma^2}||\boldsymbol{\lambda}||^2$. This is a well known Ordinary Differential Equation with explicit solution. The solution to the given ODE is given by:

$$H(t) = \frac{1}{A}\left(1 - e^{-A(T-t)}\right)$$

This can quickly be verified by calculating:

$$H'(t) = -e^{-A(T-t)} = 1 - e^{-A(T-t)} - 1 = A\frac{1}{A}(1 - e^{-A(T-t)}) - 1 = AH(t) - 1$$

Hence, we have derived a formal solution to the investors initial optimization problem. Inserting the solution in the optimal portfolio derived from earlier, we can further calculate the explicit composition of the portfolio:

$$\boldsymbol{\pi}^* = -\frac{F_W(W,t)}{F_{WW}(W,t)W}(\underline{\underline{\sigma}}^T)^{-1}\boldsymbol{\lambda} = -\frac{W^{-\gamma}H(t)^{\gamma}}{-\gamma W^{-\gamma-1}H(t)^{\gamma}W}(\underline{\underline{\sigma}}^T)^{-1}\boldsymbol{\lambda} = \frac{1}{\gamma}(\underline{\underline{\sigma}}^T)^{-1}\boldsymbol{\lambda} \tag{19}$$

Notice that in order to get the solution for an investor with infinite time horizon, we can simply let $T \to \infty$, which yields:

$$H(t) = \frac{1}{A}$$

In conclusion, we have derived a theoretical expression for the value of function of the investor, and his associated optimal investment scheme.

3.3 Merton model with Two Risky Assets

In this section we will go through the results for the Merton model with two risky assets, where we assume a set of specific parameter values. We will use the results from this section to compare with the case where one of the assets is illiquid, so we can see what effect illiquidity has on the investors optimal decision. Thus, assume we have two risky asset and one risk-free asset. We assume, for simplicity, that the two asset are uncorrelated. Assume the following parameter values:

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 0.15 & 0 \\ 0 & 0.15 \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 0.12 \\ 0.12 \end{pmatrix}$$

$$\gamma = 6, \quad \beta = 0.1, \quad r = 0.04, \rho = 0$$

The above parameter values are the same we will be using later on, when we introduce illiquidity. Inserting the above values in (19), we get:

$$\boldsymbol{\pi}^* = \frac{1}{6} \begin{pmatrix} 0.15 & 0 \\ 0 & 0.15 \end{pmatrix}^{-1} \begin{pmatrix} 0.15 & 0 \\ 0 & 0.15 \end{pmatrix}^{-1} \left(\begin{pmatrix} 0.12 \\ 0.12 \end{pmatrix} - \begin{pmatrix} 0.04 \\ 0.04 \end{pmatrix} \right) = \begin{pmatrix} 0.5926 \\ 0.5926 \end{pmatrix}$$

With the given parameters, the optimal portfolio weights are given by roughly 0.6 in the Merton two asset case.

4 Model with Illiquid Asset and Infinite Time Horizon

We will, in this section ,go through a similar asset pricing model as the Merton Model with two risky assets, but instead of assuming both assets can be traded continuously, we will assume that one of the assets is an illiquid asset. We define an illiquid asset as an asset which can not be traded on a continuous basis, and the next day, which the asset can be traded again, is unknown and random. We further impose that the investor can only meet his obligations through his liquid wealth, that is everything of value he has except the illiquid wealth.

There are three assets in the economy, one risk-free bond B, a liquid risky asset S and an illiquid risky asset P. The dynamics of the risk-free bond is given by:

$$dB_t = rB_t dt \tag{20}$$

The price of the liquid risky asset follows a geometric Brownian motion with drift μ and volatility σ:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t^1 \tag{21}$$

Where Z_t^1 is a standard Brownian motion. The illiquid risky asset also follows a geometric Brownian motion with drift ν and volatility ϕ. Furthermore, we allow for correlation between the risky liquid asset and the illiquid risky asset through a correlation parameter ρ:

$$\frac{dP_t}{P_t} = \nu dt + \phi\rho dZ_t^1 + \phi\sqrt{1-\rho^2}dZ_t^2 \tag{22}$$

Where Z_t^2 is a standard Brownian motion independent of Z_t^1. We differentiate the illiquid risky asset P from the others by allowing it only to be traded at a stochastic time τ. The way we model the timing of trading date τ is through a poisson process N_t with intensity λ. The parameter λ captures the severity of illiquid asset. The expected time before the illiquid asset can be traded again is given by $1/\lambda$. Thus, the investor can freely trade the illiquid asset at price P_t when the Poisson process hits, but not at any other time period.

We denote the investor's liquid wealth at any time t with W_t, and his illiquid wealth with X_t. Since the only liquid assets he can invest in is the risk-free bond and the illiquid risky asset, then the dynamics of his liquid wealth will be composed of those two minus what he consumes over an instant. Let θ_t denote the amount the investor has invested in the risky liquid asset at time t and let c_t denote the fraction of liquid wealth consumed. Thus, we have:

$$\frac{dW_t}{W_t} = \frac{dB_t}{B_t}(1-\theta_t) + \frac{dS_t}{S_t}\theta_t - \frac{C_T}{W_t}dt = rdt(1-\theta_t) + (\mu dt + \sigma dZ_t^1)\theta_t - c_t dt \tag{23}$$

rearranging the above, we get:

$$\frac{dW_t}{W_t} = (r + (\mu - r)\theta_t - c_t)dt + \theta_t\sigma dZ_t^1 \tag{24}$$

Similarly, the dynamics of the illiquid wealth, X_t, is simply given by the dynamics of the what is invested in the risky illiquid asset:

$$\frac{dX_t}{X_t} = \frac{dP_t}{P_t} = \nu dt + \phi\rho dZ_t^1 + \phi\sqrt{1-\rho^2}dZ_t^2 \tag{25}$$

To impose transfer of wealth from liquid to illiquid and vice versa, the investor can transfer an amount dI_τ from his liquid wealth to illiquid, when the Poisson process hits. Thus, the above two types of wealth can be rewritten to:

$$\frac{dW_t}{W_t} = (r + (\mu - r)\theta_t - c_t)dt + \theta_t\sigma dZ_t^1 - \frac{dI_t}{W_t} \tag{26}$$

$$\frac{dX_t}{X_t} = m(X_t)dt + v_1 dZ_t^1 + v_2 dZ_t^2 + \frac{dI_t}{X_t} \tag{27}$$

Where $dI_t = 0$ except for when $t = \tau$. These are the two processes we will primarily work with, and later on when we derive a HJB equation, we will use the illiquid wealth as a state variable. We will further assume that there is some incentive to invest in the illiquid asset, i.e we assume that:

$$\frac{\nu - r}{\phi} \geq \frac{\mu - r}{\sigma}$$

So the illiquid asset has at least as high of a Sharpe ratio as the liquid risky asset.

As in the Merton model, we assume that the investor is CRRA utility maximizer, and that he maximizes the expected utility with infinite time horizon. Let C_t denote the investors consumption at time t. Thus, the investors indirect utility is given by:

$$F(W, X, t) = \max_{C_t, \pi_t} E_t \left[\int_t^{\infty} e^{-\beta(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right] \tag{28}$$

Where β is the subjective discount factor and γ is the relative risk-aversion. We will, interchangeably, also call the indirect utility F for the value function. We assume that $\gamma > 1$. For simplicity, we assume that the investor is standing at time $t = 0$. Hence, we can drop the "t" from our notation, and reduce the above to:

$$F(W, X) = \max_{c_t, \pi_t} E_t \left[\int_0^{\infty} e^{-\beta \cdot s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right] \tag{29}$$

The assumption of infinite time horizon is conservative, as any effect of illiquidity will be much greater with finite time horizon. The intuition here is, that an investor with finite time horizon would be less willing to buy illiquid assets, as the poisson process might not necessarily hit during his lifetime, where an investor with infinite time horizon would care less about whether or not the Poisson process hits before a specified time period, but if it hits at some point.

4.1 Solving the Investor's Problem

As mentioned earlier, the investor performs the maximization:

$$F(W, X) = \max_{c_t, \pi_t} E_t \left[\int_0^{\infty} e^{-\beta \cdot s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right] \tag{30}$$

Subject to the dynamics of liquid and illiquid wealth given in (26) and (27) respectively. Identical to the derivations done in (16), it can be shown that with illiquid wealth as well F is homogeneous of degree $1-\gamma$. Thus, we can rewrite:

$$F(W, X, t) = (W + X)^{1-\gamma} H(x) \tag{31}$$

Where $x = \frac{X}{W+X}$ (the fraction of total wealh in illiquid assets), and $H(x) = F(1, x)$. The rewriting of the indirect utility function is key in our simulation of the optimum, as we now only need to consider a function H with one unknown variable x, as oppose to a function F with two unknowns W and X.

When the Poisson process hits, the investor can rebalance his portfolio, such that the value function will potentially jump discretely. Denote this new value function as F^*. At the arrival of the Poisson process, we characterize the new value function F^* as:

$$F^*(W_t, X_t) = \max_{I \in [-X_t, W_t)} F(W_t - I, X_t + I) \tag{32}$$

We notice that $F^* \geq F$, since we have that:

$$F^*(W_t, X_t) = \max_{I \in [-X_t, W_t)} F(W_t - I, X_t + I) \geq F(W_t, X_t) \tag{33}$$

Intuitively, the investor rebalances optimally, and if he can not get a higher value function by changing consumption or his portfolio, he will simply keep the same consumption and portfolio. In other words, he will only rebalance if he can get a higher value function. Thus, the total jump at time τ will be given by $F^* - F$. We can now derive the first import result of this paper.

Proposition 1

Assume that the investor's problem is given by:

$$F(W, X) = \max_{c_t, \pi_t} E_t \left[\int_0^\infty e^{-\beta \cdot s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right] \tag{34}$$

Furthermore, assume that the investor must satisfy the budget constraints (26) and (27). Then the function $H(x)$ is characterized by the following:

$$0 = \max_{c,\theta} \left[\frac{1}{1-\gamma} c^{1-\gamma} (1-x)^{1-\gamma} - \beta H(x) + \lambda (H^* - H(x)) + H(x) A(x, c, \theta) + H'(x) B(x, c, \theta) \right.$$
$$\left. + \frac{1}{2} H''(x) C(x, c, \theta) \right]$$

Where $H^* = \max_x H(x)$, and we have that:

$$A(x, c, \theta) = (1-\gamma) \left(r + (1-x) \left([(\mu - r)\theta] - c \right) + x(\nu - r) - \frac{1}{2} \gamma \left(\sigma^2 \theta^2 (1-x)^2 + \phi^2 x^2 + 2x(1-x)\phi\theta\sigma\rho \right) \right)$$

$$B(x, c, \theta) = x(1-x) \left(- \left([r + (\mu - r)\theta] - c \right) + \gamma (1-x) \sigma^2 \theta^2 + \nu + \gamma x \phi^2 + \gamma (2x - 1) \phi\theta\sigma\rho \right)$$

$$C(x, c, \theta) = x^2 (1-x)^2 \left(\sigma^2 \theta^2 + \phi^2 - 2\phi\theta\sigma\rho \right)$$

Proof:

Wé can prove the statement by using the theory for the HJB equation derived earlier. Writing the HJB equation associated with $F = F(W, X, t)$, as derived in the dynamic programming section and expressed in (8), we get:

$$\beta F(W, X, t) = \max_{c,\theta} \left[\frac{1}{1-\gamma} (cW)^{1-\gamma} + \lambda (F^* - F) + F_W(W, X, t) W \left([r + (\mu - r)\theta] - c \right) \right.$$
$$+ \frac{1}{2} F_{WW}(W, X, t) W^2 \sigma^2 \theta^2 + F_X(W, X, t) X \nu + \frac{1}{2} F_{XX}(W, X, t) (\phi^2 X^2 \rho^2$$
$$\left. + \sqrt{1-\rho^2}^2 X^2 \phi^2) + F_{WX}(W, X, t) W X \phi\theta\sigma\rho \right]$$

Notice that there is a slight difference between how the HJB is formulated in (8) and the above equation. First, in the equation above, we have that $\frac{\partial F}{\partial t} = 0$. The reason for this, is that we have an investor with infinite time horizon. With infinite time horizon, the value function does not change if the start period is changed, i.e everything else equal, it does not matter if the investor starts at time t or t', his optimal utility will be the same with infinite time horizon, which in turn means that $\frac{\partial F}{\partial t} = 0$. Second, we have a term $\lambda(F^* - F)$ in the above equation, which is not in (8). This comes from the fact that the HJB equation consists of two parts. It consists of the utility gained today, and the drift of the value function, i.e the expected value gained over a period. $\lambda(F^* - F)$ is the expected amount that will be gained over a period from the possibility of rebalancing the portfolio, and therefore it is included in the drift term of the HJB equation.

Since we have that $F(W, X) = (W + X)^{1-\gamma} H(\frac{X}{W+X})$, we can calculate all the associated derivatives in the above HJB equation. A standard verification argument gets the desired result, i.e calculating all the associated derivative, and inserting them in the HJB equation, we can simplify, isolate and reduce the HJB equation until we get to the desired form. Thus, we calculate:

$$\begin{aligned} F_W(W, X, t) &= (1-\gamma)(W+X)^{-\gamma} H(\frac{X}{W+X}) - (W+X)^{1-\gamma} H'(\frac{X}{W+X}) \frac{X}{(W+X)^2} \\ &= (1-\gamma)(W+X)^{-\gamma} H(\frac{X}{W+X}) - H'(\frac{X}{W+X}) \frac{X}{(W+X)^{1+\gamma}} \end{aligned}$$

$$\begin{aligned} F_{WW}(W, X, t) = &- (1-\gamma)\gamma(W+X)^{-\gamma-1} H(\frac{X}{W+X}) - (1-\gamma)(W+X)^{-\gamma} H'(\frac{X}{W+X}) \frac{X}{(W+X)^2} \\ &+ H''(\frac{X}{W+X}) \frac{X}{(W+X)^2} \frac{X}{(W+X)^{1+\gamma}} + (1+\gamma) H'(\frac{X}{W+X}) \frac{X}{(W+X)^{2+\gamma}} \\ = &- (1-\gamma)\gamma(W+X)^{-\gamma-1} H(\frac{X}{W+X}) + H''(\frac{X}{W+X}) \frac{X^2}{(W+X)^{3+\gamma}} \\ &+ 2\gamma H'(\frac{X}{W+X}) \frac{X}{(W+X)^{2+\gamma}} \end{aligned}$$

$$\begin{aligned} F_X(W, X, t) = &(1-\gamma)(W+X)^{-\gamma} H(\frac{X}{W+X}) + (W+X)^{1-\gamma} H'(\frac{X}{W+X}) \left(\frac{1}{W+X} - \frac{X}{(W+X)^2} \right) \\ = &(1-\gamma)(W+X)^{-\gamma} H(\frac{X}{W+X}) + H'(\frac{X}{W+X}) \left(\frac{1}{(W+X)^{\gamma}} - \frac{X}{(W+X)^{1+\gamma}} \right) \end{aligned}$$

$$
\begin{aligned}
F_{XX} =& -(1-\gamma)\gamma(W+X)^{-\gamma-1}H(\frac{X}{W+X}) + (1-\gamma)(W+X)^{-\gamma}H'(\frac{X}{W+X})\left(\frac{1}{W+X} - \frac{X}{(W+X)^2}\right) \\
&+ \left(\frac{1}{W+X} - \frac{X}{(W+X)^2}\right) H''(\frac{X}{W+X})\left(\frac{1}{(W+X)^{\gamma}} - \frac{X}{(W+X)^{1+\gamma}}\right) \\
&+ H'(\frac{X}{W+X})\left(\frac{-\gamma}{(W+X)^{\gamma+1}} - \frac{1}{(W+X)^{1+\gamma}} + (1+\gamma)\frac{X}{(W+X)^{2+\gamma}}\right) \\
=& -(1-\gamma)\gamma(W+X)^{-\gamma-1}H(\frac{X}{W+X}) + 2H'(\frac{X}{W+X})\left(-\gamma\frac{1}{(W+X)^{1+\gamma}} + \gamma\frac{X}{(W+X)^{2+\gamma}}\right) \\
&+ \left(\frac{1}{W+X} - \frac{X}{(W+X)^2}\right) H''(\frac{X}{W+X})\left(\frac{1}{(W+X)^{\gamma}} - \frac{X}{(W+X)^{1+\gamma}}\right)
\end{aligned}
$$

$$
\begin{aligned}
F_{WX} =& -(1-\gamma)\gamma(W+X)^{-\gamma-1}H(\frac{X}{W+X}) + (1-\gamma)(W+X)^{-\gamma}H'(\frac{X}{W+X})\left(\frac{1}{W+X} - \frac{X}{(W+X)^2}\right) \\
&- H''(\frac{X}{W+X})\left(\frac{1}{W+X} - \frac{X}{(W+X)^2}\right)\frac{X}{(W+X)^{1+\gamma}} \\
&- H'(\frac{X}{W+X})\left(\frac{1}{(W+X)^{1+\gamma}} - (1+\gamma)\frac{X}{(W+X)^{2+\gamma}}\right) \\
=& -(1-\gamma)\gamma(W+X)^{-\gamma}H(\frac{X}{W+X}) - \gamma(W+X)^{-\gamma}H'(\frac{X}{W+X})\left(\frac{1}{W+X} - 2\frac{X}{(W+X)^2}\right) \\
&- H''(\frac{X}{W+X})\left(\frac{1}{W+X} - \frac{X}{(W+X)^2}\right)\frac{X}{(W+X)^{1+\gamma}}
\end{aligned}
$$

With the above, we can calculate the following expressions, which the HJB equation consists of:

$$
\begin{aligned}
&F_W(W,X,t)W\left([r+(\mu-r)\theta]-c\right) \\
&= \left((1-\gamma)(W+X)^{-\gamma}H(\frac{X}{W+X}) - H'(\frac{X}{W+X})\frac{X}{(W+X)^{1+\gamma}}\right)W\left([r+(\mu-r)\theta]-c\right) \\
&= (1-\gamma)(W+X)^{-\gamma}H(\frac{X}{W+X})W\left([r+(\mu-r)\theta]-c\right) \\
&\quad - H'(\frac{X}{W+X})\frac{X}{(W+X)^{1+\gamma}}\left(W[r+(\mu-r)\theta]-c\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}F_{WW}(W,X,t)W^2\sigma^2\theta^2 = \frac{1}{2}\Bigg(& -(1-\gamma)\gamma(W+X)^{-\gamma-1}H(\frac{X}{W+X}) + H''(\frac{X}{W+X})\frac{X^2}{(W+X)^{3+\gamma}} \\
&+ 2\gamma H'(\frac{X}{W+X})\frac{X}{(W+X)^{2+\gamma}}\Bigg)W^2\sigma^2\theta^2
\end{aligned}
$$

$$
F_X(W,X,t)X\nu = (1-\gamma)(W+X)^{-\gamma}H(\frac{X}{W+X})X\nu + (W+X)^{1-\gamma}H'(\frac{X}{W+X})\left(\frac{1}{W+X} - \frac{X}{(W+X)^2}\right)X\nu
$$

$$
\begin{aligned}
\frac{1}{2}F_{XX}(W,X,t)(\phi^2X^2\rho^2+\sqrt{1-\rho^2}^2X^2\phi^2) = &\frac{1}{2}F_{XX}(W,X,t)X^2\phi^2 \\
= &-\frac{1}{2}(1-\gamma)\gamma(W+X)^{-\gamma-1}H(\frac{X}{W+X})X^2\phi^2 \\
&+H'(\frac{X}{W+X})\left(-\gamma\frac{1}{(W+X)^{1+\gamma}}+\gamma\frac{X}{(W+X)^{2+\gamma}}\right)X^2\phi^2 \\
&+\frac{1}{2}\left(\frac{1}{W+X}-\frac{X}{(W+X)^2}\right)H''(\frac{X}{W+X}) \\
&\cdot\left(\frac{1}{(W+X)^{\gamma}}-\frac{X}{(W+X)^{1+\gamma}}\right)X^2\phi^2
\end{aligned}
$$

Since we have an investor with infinite time horizon, it holds that $\frac{\partial F}{\partial t}=0$.

Inserting all the above results in the HJB equation, we get:

$$
\begin{aligned}
0 = \max_{c,\theta}\Bigg[&\frac{1}{1-\gamma}(cW)^{1-\gamma}+\lambda(F^*-F)-\beta F \\
&+\left((1-\gamma)(W+X)^{-\gamma}H(\frac{X}{W+X})-H'(\frac{X}{W+X})\frac{X}{(W+X)^{1+\gamma}}\right)W\left([r+(\mu-r)\theta]-c\right) \\
&+\frac{1}{2}\left(-(1-\gamma)\gamma(W+X)^{-\gamma-1}H(\frac{X}{W+X})+H''(\frac{X}{W+X})\frac{X^2}{(W+X)^{3+\gamma}}+2\gamma H'(\frac{X}{W+X})\frac{X}{(W+X)^{2+\gamma}}\right) \\
&\cdot W^2\sigma^2\theta^2+\left((1-\gamma)(W+X)^{-\gamma}H(\frac{X}{W+X})+H'(\frac{X}{W+X})\left(\frac{1}{(W+X)^{\gamma}}-\frac{X}{(W+X)^{1+\gamma}}\right)\right)X\nu \\
&-\frac{1}{2}(1-\gamma)\gamma(W+X)^{-\gamma-1}H(\frac{X}{W+X})X^2\phi^2+H'(\frac{X}{W+X})\left(-\gamma\frac{1}{(W+X)^{1+\gamma}}+\gamma\frac{X}{(W+X)^{2+\gamma}}\right)X^2\phi^2 \\
&+\frac{1}{2}\left(\frac{1}{W+X}-\frac{X}{(W+X)^2}\right)H''(\frac{X}{W+X})\left(\frac{1}{(W+X)^{\gamma}}-\frac{X}{(W+X)^{1+\gamma}}\right)X^2\phi^2 \\
&+\Big(-(1-\gamma)\gamma(W+X)^{-\gamma-1}H(\frac{X}{W+X})+H'(\frac{X}{W+X})\left(\frac{-\gamma}{(W+X)^{1+\gamma}}+2\gamma\frac{X}{(W+X)^{2+\gamma}}\right) \\
&-H''(\frac{X}{W+X})\left(\frac{1}{W+X}-\frac{X}{(W+X)^2}\right)\frac{X}{(W+X)^{1+\gamma}}\Big)WX\phi\theta\sigma\rho\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \max_{c,\theta} \Bigg[\frac{1}{1-\gamma}(cW)^{1-\gamma} + \lambda(F^* - F) + \frac{\partial F}{\partial t} - \beta F \\
&\quad + (1-\gamma)(W+X)^{-\gamma} H(\frac{X}{W+X}) \Bigg(W \left([r + (\mu - r)\theta] - c \right) - \frac{1}{2} W^2 \sigma^2 \theta^2 \gamma (W+X)^{-1} + X\nu \\
&\quad - \frac{1}{2} X^2 \phi^2 \gamma (W+X)^{-1} - (W+X)^{-1} W X \gamma \phi \theta \sigma \rho \Bigg) \\
&\quad + H'(\frac{X}{W+X}) \Bigg(\frac{X}{(W+X)^{1+\gamma}} W \left([r + (\mu - r)\theta] - c \right) + \gamma \frac{X}{(W+X)^{2+\gamma}} W^2 \sigma^2 \theta^2 \\
&\quad + \left(\frac{1}{(W+X)^{\gamma}} - \frac{X}{(W+X)^{1+\gamma}} \right) X\nu + \left(-\gamma \frac{1}{(W+X)^{1+\gamma}} + \gamma \frac{X}{(W+X)^{2+\gamma}} \right) X^2 \phi^2 \\
&\quad + \left(\frac{-\gamma}{(W+X)^{1+\gamma}} + 2\gamma \frac{X}{(W+X)^{2+\gamma}} \right) W X \phi \theta \sigma \rho \Bigg) \\
&\quad + H''(\frac{X}{W+X}) \Bigg(\frac{1}{2} \frac{X^2}{(W+X)^{3+\gamma}} W^2 \sigma^2 \theta^2 + \frac{1}{2} \left(\frac{1}{W+X} - \frac{X}{(W+X)^2} \right) \left(\frac{1}{(W+X)^{\gamma}} - \frac{X}{(W+X)^{1+\gamma}} \right) \\
&\quad \cdot X^2 \phi^2 - \left(\frac{1}{W+X} - \frac{X}{(W+X)^2} \right) \frac{X}{(W+X)^{1+\gamma}} W X \phi \theta \sigma \rho \Bigg) \Bigg] \\
&= \max_{c,\theta} \Bigg[\frac{1}{1-\gamma}(cW)^{1-\gamma} + \lambda(F^* - F) + \frac{\partial F}{\partial t} - \beta F + (1-\gamma) H(\frac{X}{W+X}) \frac{W}{W+X} \left([r + (\mu - r)\theta] - c \right) \\
&\quad - \frac{1}{2} W^2 \sigma^2 \theta^2 \gamma (W+X)^{-2} + (W+X)^{-1} X\nu - \frac{1}{2} X^2 \phi^2 \gamma (W+X)^{-2} - (W+X)^{-2} W X \gamma \phi \theta \sigma \rho \\
&\quad + H'(\frac{X}{W+X}) \Bigg(\frac{X}{(W+X)^2} W \left([r + (\mu - r)\theta] - c \right) + \gamma \frac{X}{(W+X)^3} W^2 \sigma^2 \theta^2 \\
&\quad + \left(1 - \frac{X}{(W+X)^1} \right) \frac{X}{W+X} \nu + \left(-\gamma \frac{1}{(W+X)^1} + \gamma \frac{X}{(W+X)^2} \right) \frac{X^2}{W+X} \phi^2 \\
&\quad + \left(\frac{-\gamma}{(W+X)^1} + 2\gamma \frac{X}{(W+X)^2} \right) \frac{WX}{W+X} \phi \theta \sigma \rho \Bigg) + H''(\frac{X}{W+X}) \Bigg(\frac{1}{2} \frac{X^2}{(W+X)^4} W^2 \sigma^2 \theta^2 \\
&\quad + \frac{1}{2} \left(\frac{1}{W+X} - \frac{X}{(W+X)^2} \right) \left(1 - \frac{X}{(W+X)^1} \right) \frac{X^2}{W+X} \phi^2 - \left(\frac{1}{W+X} - \frac{X}{(W+X)^2} \right) \\
&\quad \cdot \frac{X}{(W+X)^2} W X \phi \theta \sigma \rho \Bigg) \Bigg]
\end{aligned}
$$

Now let $x = \frac{X}{W+X}$. We notice that $1 - x = \frac{W+X}{W+X} - \frac{X}{W+X} = \frac{W}{W+X}$. We can thus rewrite the above to:

$$
\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{1}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} - \beta H(x) + \lambda(H^* - H(x)) \\
& + (1-\gamma) H(\frac{X}{W+X}) \left((1-x)\left([r+(\mu-r)\theta] - c\right)\right. \\
& \left. - \frac{1}{2}\sigma^2\theta^2\gamma(1-x)^2 + x\nu - \frac{1}{2}\gamma\phi^2x^2 - x(1-x)\gamma\phi\theta\sigma\rho\right) \\
& + H'(\frac{X}{W+X})\Bigg(- x(1-x)\left([r+(\mu-r)\theta] - c\right) + \gamma x(1-x)^2\sigma^2\theta^2 \\
& + x(1-x)\nu + \gamma x^2(1-x)\phi^2 + \gamma x(1-x)(2x-1)\phi\theta\sigma\rho\Bigg) \\
& + H''(\frac{X}{W+X})\left(\frac{1}{2}x^2(1-x)^2\sigma^2\theta^2 + \frac{1}{2}x^2(1-x)^2\phi^2 - x^2(1-x)^2\phi\theta\sigma\rho\right)\Bigg]
\end{aligned}
$$

rearranging and simplifying the above yields:

$$
\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{1}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} - \beta H(x) + \lambda(H^* - H(x)) + H(x)A(x,c,\theta) + H'(x)B(x,c,\theta) \\
& + \frac{1}{2}H''(x)C(x,c,\theta)\Bigg]
\end{aligned}
$$

Where we have that:

$$
\begin{aligned}
A(x,c,\theta) = (1-\gamma)\Big(& r + (1-x)\left([(\mu-r)\theta] - c\right) + x(\nu - r) - \frac{1}{2}\gamma(\sigma^2\theta^2(1-x)^2 + \phi^2x^2 \\
& + 2x(1-x)\phi\theta\sigma\rho)\Big)
\end{aligned}
$$

$$
B(x,c,\theta) = x(1-x)\Bigg(- \left([r+(\mu-r)\theta] - c\right) + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x\phi^2 + \gamma(2x-1)\phi\theta\sigma\rho\Bigg)
$$

$$
C(x,c,\theta) = x^2(1-x)^2\left(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho\right)
$$

This concludes the proof. □

4.2 Numerical Estimation of the HJB Equation

As mentioned earlier, we wish to estimate the HJB equation numerically. The reason is that it is very hard to get an explicit solution to a partial differential equation such as the HJB equation (although not impossible). The reason why it is so hard, is that you basically have to guess the solution in order to solve it. Although one can come up with a very qualified guess, the exact solution can be very far off. We will now describe how we solve the HJB equation numerically. From proposition 1, we have that:

$$
\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{1}{1-\gamma} c^{1-\gamma} (1-x)^{1-\gamma} - \beta H(x) + \lambda (H^* - H(x)) \\
& + H(x)(1-\gamma) \left(r + (1-x) \left([(\mu - r)\theta] - c \right) + x(\nu - r) - \frac{1}{2}\gamma \left(\sigma^2 \theta^2 (1-x)^2 + \phi^2 x^2 + 2x(1-x)\phi\theta\sigma\rho \right) \right) \\
& + H'(x)x(1-x) \Bigg(-([r + (\mu - r)\theta] - c) + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x \phi^2 + \gamma(2x-1)\phi\theta\sigma\rho \Bigg) \\
& + \frac{1}{2} H''(x) x^2 (1-x)^2 \Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho \Bigg) \Bigg]
\end{aligned}
$$

To solve the HJB equation numerically, we follow the method outlined in Kushner and Dupuis (1992). First, we use a discrete state space for the input variables with a discrete step of $h = 1/100$. Hence we have:

$$
x, c, \theta \in \left\{ 0, \frac{1}{100}, \frac{2}{100}, \frac{3}{100}, ..., 1 \right\}
$$

Now let x_n denote the value of x at grid point n, such that $x_1 = 0, x_2 = \frac{1}{100}, x_n = \frac{n}{100} ..., x_{100} = 1$. We separate the positive and negative part of the coefficient for the first derivative in order to ensure that the implied probabilities we get are positive[5]. Thus, we rewrite the above to:

$$
\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{1}{1-\gamma} c^{1-\gamma} (1-x)^{1-\gamma} - \beta H(x) + \lambda (H^* - H(x)) \\
& + H(x)(1-\gamma) \left(r + (1-x) \left([(\mu - r)\theta] - c \right) + x(\nu - r) - \frac{1}{2}\gamma \left(\sigma^2 \theta^2 (1-x)^2 + \phi^2 x^2 + 2x(1-x)\phi\theta\sigma\rho \right) \right) \\
& + H'_+(x)x(1-x) \Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x \phi^2 \Bigg) \\
& + H'_-(x)x(1-x) \Bigg(-([r + (\mu - r)\theta) + \gamma(2x-1)\phi\theta\sigma\rho \Bigg) \\
& + \frac{1}{2} H''(x) x^2 (1-x)^2 \Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho \Bigg) \Bigg]
\end{aligned}
$$

Notice, the above is under the assumption that $\mu \geq r$, i.e the expected return of the liquid risky asset is greater than the risk-free asset. With a discrete state space and with $h = 1/100$, we use various differencing methods to estimate the derivatives. We use the following approximations:

$$
H'_+(x) = \frac{H_{n+1} - H_n}{h}
$$

$$
H'_-(x) = \frac{H_n - H_{n-1}}{h}
$$

$$
H''(x) = \frac{H_{n+1} + H_{n-1} - 2H_n}{h^2}
$$

[5]This is mainly done to ensure stability when the algorithm converges, and it does not have any large implication on the actual converged result

The first two can be derived from standard differentiation theory. The third one comes from Taylor expanding $H(x+h)$ and $H(x-h)$ around x, and solving two equations with two unknowns for $H''(x)$. Inserting these estimated values in the HJB equation, and isolating for H_n, we get:

$$
\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{1}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} - \beta H(x) + \lambda(H^* - H(x)) \\
& + H_n(1-\gamma)\Bigg(r + (1-x)\left([(\mu - r)\theta] - c\right) + x(\nu - r) - \frac{1}{2}\gamma \Big(\sigma^2\theta^2(1-x)^2 \\
& + \phi^2 x^2 + 2x(1-x)\phi\theta\sigma\rho \Big) \Bigg) \\
& + \frac{H_{n+1} - H_n}{h} x(1-x) \Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x \phi^2 \Bigg) \\
& + \frac{H_n - H_{n-1}}{h} x(1-x) \Bigg(-\left([r + (\mu - r)\theta]\right) + \gamma(2x-1)\phi\theta\sigma\rho \Bigg) \\
& + \frac{1}{2} \frac{H_{n+1} + H_{n-1} - 2H_n}{h^2} x^2(1-x)^2 \Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho \Bigg) \Bigg]
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{h}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} + h\lambda H^* \\
& + h \cdot H_n \Bigg(-\lambda - \beta + (1-\gamma)\Bigg(r + (1-x)\left([(\mu - r)\theta] - c\right) + x(\nu - r) \\
& - \frac{1}{2}\gamma \Big(\sigma^2\theta^2(1-x)^2 + \phi^2 x^2 + 2x(1-x)\phi\theta\sigma\rho \Big) \Bigg) \Bigg) \\
& + (H_{n+1} - H_n) x(1-x) \Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x \phi^2 \Bigg) \\
& + (H_n - H_{n-1}) x(1-x) \Bigg(-\left([r + (\mu - r)\theta]\right) + \gamma(2x-1)\phi\theta\sigma\rho \Bigg) \\
& + \frac{1}{2} \frac{H_{n+1} + H_{n-1} - 2H_n}{h} x^2(1-x)^2 \Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho \Bigg) \Bigg]
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{h^2}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} + h^2 \lambda H^* \\
& + h^2 \cdot H_n \Bigg(-\lambda - \beta + (1-\gamma)\Bigg(r + (1-x)\left([(\mu - r)\theta] - c\right) + x(\nu - r) \\
& - \frac{1}{2}\gamma \left(\sigma^2\theta^2(1-x)^2 + \phi^2 x^2 + 2x(1-x)\phi\theta\sigma\rho \right) \Bigg) \Bigg) \\
& + (H_{n+1} - H_n) h x (1-x) \Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x \phi^2 \Bigg) \\
& + (H_n - H_{n-1}) h x (1-x) \Bigg(-([r + (\mu - r)\theta]) + \gamma(2x-1)\phi\theta\sigma\rho \Bigg) \\
& + \frac{1}{2}(H_{n+1} + H_{n-1} - 2H_n) x^2 (1-x)^2 \Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho \Bigg) \Bigg]
\end{aligned}
$$

We now define the following constants:

$$
\begin{aligned}
A = & \Bigg(-\lambda - \beta + (1-\gamma)\Bigg(r + (1-x)\left([(\mu - r)\theta] - c\right) + x(\nu - r) \\
& \quad - \frac{1}{2}\gamma \left(\sigma^2\theta^2(1-x)^2 + \phi^2 x^2 + 2x(1-x)\phi\theta\sigma\rho \right) \Bigg) \Bigg) \\
B = & \Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x \phi^2 \Bigg) \\
C = & \Bigg(-([r + (\mu - r)\theta]) + \gamma(2x-1)\phi\theta\sigma\rho \Bigg) \\
D = & \Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho \Bigg)
\end{aligned}
$$

Rewriting the HJB equation with these constants, we get:

$$
\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{h^2}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} + h^2 \lambda H^* + h^2 \cdot H_n \cdot A + (H_{n+1} - H_n) h B + (H_n - H_{n-1}) h C \\
& + \frac{1}{2}(H_{n+1} + H_{n-1} - 2H_n) D \Bigg]
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{h^2}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} + h^2 \lambda H^* + H_n \cdot (h^2 \cdot A - hB + hC - D) + H_{n+1} h B - H_{n-1} h C \\
& + \frac{1}{2}(H_{n+1} + H_{n-1}) D \Bigg]
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{aligned}
-H_n \cdot (h^2 \cdot A - hB + hC - D) = \max_{c,\theta} \Bigg[& \frac{h^2}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} + h^2 \lambda H^* + H_{n+1} h B - H_{n-1} h C \\
& + \frac{1}{2}(H_{n+1} + H_{n-1}) D \Bigg]
\end{aligned}
$$

$$\Updownarrow$$

$$-H_n \cdot (h^2 \cdot A - hB + hC - D) = \max_{c,\theta} \Bigg[\frac{h^2}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} + h^2 \lambda H^* + H_{n+1}(hB + \frac{1}{2}D) + H_{n-1}(-hC + \frac{1}{2}D) \Bigg]$$

$$\Updownarrow$$

$$H_n = \max_{c,\theta} \Bigg[\frac{h^2}{-h^2 \cdot A + hB - hC + D} \left(\frac{1}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} + \lambda H^* \right) + \frac{1}{-h^2 \cdot A + hB - hC + D} H_{n+1}(hB + \frac{1}{2}D) + \frac{1}{-h^2 \cdot A + hB - hC + D} H_{n-1}(-hC + \frac{1}{2}D) \Bigg]$$

Once again, define the following variables to make future calculations more simple:

$$C_1 := B - C = x(1-x)\Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x \phi^2 + ([r + (\mu - r)\theta]) - \gamma(2x-1)\phi\theta\sigma\rho \Bigg)$$

$$C_2 := -A = \Bigg(\lambda + \beta - (1-\gamma)\Bigg(r + (1-x)\left([(\mu - r)\theta] - c\right) + x(\nu - r) - \frac{1}{2}\gamma \left(\sigma^2\theta^2(1-x)^2 + \phi^2 x^2 + 2x(1-x)\phi\theta\sigma\rho \right) \Bigg)\Bigg)$$

$$C_3 := D = x^2(1-x)^2\Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho \Bigg)$$

Inserting the above definitions in the derived expression for H_n, we get:

$$H_n = \max_{c,\theta} \Bigg[\frac{h^2}{h^2 \cdot C_2 + hC_1 + C_3} \left(\frac{1}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} + \lambda H^* \right) + \frac{1}{h^2 \cdot C_2 + hC_1 + C_3} H_{n+1} \Bigg(hx(1-x)(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x \phi^2) + \frac{1}{2}x^2(1-x)^2(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho) \Bigg) + \frac{1}{h^2 \cdot C_2 + hC_1 + C_3} H_{n-1} \Bigg(-hx(1-x)(-([r + (\mu - r)\theta]) + \gamma(2x-1)\phi\theta\sigma\rho) + \frac{1}{2}x^2(1-x)^2(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho) \Bigg) \Bigg]$$

Now define:

$$\Delta t_n(c,\theta) := \frac{h^2}{h^2 \cdot C_2 + hC_1 + C_3}$$

$$p_n^u(c,\theta) := \frac{hx(1-x)(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x \phi^2) + \frac{1}{2}x^2(1-x)^2(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho)}{h^2} \Delta t_n(c,\theta)$$

$$p_n^d(c,\theta) = \frac{-hx(1-x)(-([r + (\mu - r)\theta]) + \gamma(2x-1)\phi\theta\sigma\rho) + \frac{1}{2}x^2(1-x)^2(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho)}{h^2} \Delta t_n(c,\theta)$$

Inserting these values in the expression for H_n, we get:

$$H_n = \max_{c,\theta}\left[\Delta t_n(c,\theta)\left(\frac{1}{1-\gamma}c^{1-\gamma}(1-x)^{1-\gamma}+\lambda H^*\right)+H_{n+1}p_n^u(c,\theta)+H_{n-1}p_n^d(c,\theta)\right] \tag{35}$$

The numerical algorithm iterates over steps i, where we start at step $i = 0$. At $i = 0$, initiate the algorithm with an initial guess of H^0, such that we guess H_n^0 for each $n \in \left\{0, \frac{1}{100}, \frac{2}{100}, \frac{3}{100}, ..., 1\right\}$.For each iteration step i, do the following:

1. compute the optimal rebalancing utility:

$$H^* = \max_n H_n^i$$

2. Given H^i, compute the optimal policies for step i+1 at each grid point $n \in \left\{0, \frac{1}{100}, \frac{2}{100}, \frac{3}{100}, ..., 1\right\}$ based on:

$$c_n^{i+1} = \arg\max_c\left[\Delta t_n(c,\theta)\left(\frac{1}{1-\gamma}c^{1-\gamma}(1-x)^{1-\gamma}+\lambda H^*\right)+H_{n+1}^i p_n^u(c,\theta)+H_{n-1}^i p_n^d(c,\theta)\right]$$

$$\theta_n^{i+1} = \arg\max_c\left[\Delta t_n(c,\theta)\left(\frac{1}{1-\gamma}c^{1-\gamma}(1-x)^{1-\gamma}+\lambda H^*\right)+H_{n+1}^i p_n^u(c,\theta)+H_{n-1}^i p_n^d(c,\theta)\right]$$

3. Given consumption and fraction invested in liquid risky asset for step $i+1$, c_n^{i+1} and θ_n^{i+1} for each n, we can now compute H^{i+1} based on:

$$H_n^{i+1} = \Delta t_n(c,\theta)\left(\frac{1}{1-\gamma}c^{1-\gamma}(1-x)^{1-\gamma}+\lambda H^*\right)+H_{n+1}^i p_n^u(c,\theta)+H_{n-1}^i p_n^d(c,\theta)$$

4. Repeat step 1-3 until H^i converges.

5. You should now have a function H^i over the grid with an associated maximum x^*.

Notice, that the algorithm is not feasible for end-points of the grid, that is $n = 0$ and $n = 100$. We will now see how we can handle this case.

4.3 Handling the end-points of the grid

The careful reader would notice that our algorithm is not feasible for when $n = 0$ and $n = 100$. The reason for this, is that in step 3, we interpolate between H_{n-1} and H_{n+1}. So if $n = 0$ or $n = 100$, then we would be outside of the grid. To handle this issue, we assume that the value function is linear at the end points. With this, we can use the following approximations of the relevant derivatives for when $n = 0$:

$$H'_+(x) = \frac{H_{n+1} - H_n}{h}$$

$$H'_-(x) = \frac{H_{n+1} - H_n}{h}$$

$$H''(x) = 0$$

And for $n = 100$, we get:

$$H'_+(x) = \frac{H_n - H_{n-1}}{h}$$

$$H'_-(x) = \frac{H_n - H_{n-1}}{h}$$

$$H''(x) = 0$$

Thus, for $n = 0$, The HJB equation can be rewritten to:

$$\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{h^2}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} + h^2 \lambda H^* \\
& + h^2 \cdot H_n \Bigg(- \lambda - \beta + (1-\gamma)\Bigg(r + (1-x)\left([(\mu - r)\theta] - c\right) + x(\nu - r) \\
& - \frac{1}{2}\gamma\left(\sigma^2\theta^2(1-x)^2 + \phi^2 x^2 + 2x(1-x)\phi\theta\sigma\rho\right)\Bigg)\Bigg) \\
& + (H_{n+1} - H_n) h x (1-x)\Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x \phi^2 \Bigg) \\
& + (H_{n+1} - H_n) h x (1-x)\Bigg(- ([r + (\mu - r)\theta]) + \gamma(2x-1)\phi\theta\sigma\rho \Bigg) \\
& + \frac{1}{2} 0 x^2 (1-x)^2 \Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho \Bigg) \Bigg]
\end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{h^2}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} + h^2 \lambda H^* \\
& + h^2 \cdot H_n \Bigg(- \lambda - \beta + (1-\gamma)\Bigg(r + (1-x)\left([(\mu - r)\theta] - c\right) + x(\nu - r) \\
& - \frac{1}{2}\gamma\left(\sigma^2\theta^2(1-x)^2 + \phi^2 x^2 + 2x(1-x)\phi\theta\sigma\rho\right)\Bigg)\Bigg) \\
& + (H_{n+1} - H_n) h x (1-x)\Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x \phi^2 - ([r + (\mu - r)\theta]) \\
& + \gamma(2x-1)\phi\theta\sigma\rho \Bigg) \Bigg]
\end{aligned}$$

$$\Updownarrow$$

$$0 = \max_{c,\theta}\left[\frac{h^2}{1-\gamma}c^{1-\gamma}(1-x)^{1-\gamma} + h^2\lambda H^* + h^2\cdot H_n M + (H_{n+1}-H_n)hx(1-x)N\right]$$

where

$$M = \Bigg(-\lambda - \beta + (1-\gamma)\Bigg(r + (1-x)\left([(\mu-r)\theta] - c\right) + x(\nu - r)$$
$$- \frac{1}{2}\gamma\left(\sigma^2\theta^2(1-x)^2 + \phi^2x^2 + 2x(1-x)\phi\theta\sigma\rho\right)\Bigg)\Bigg)$$
$$N = \Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x\phi^2 - ([r + (\mu - r)\theta]) + \gamma(2x-1)\phi\theta\sigma\rho \Bigg)$$

Thus we get:

$$H_n(-h^2M + hx(1-x)N) = \max_{c,\theta}\left[\frac{h^2}{1-\gamma}c^{1-\gamma}(1-x)^{1-\gamma} + h^2\lambda H^* + H_{n+1}hx(1-x)\right]$$
$$\Updownarrow$$
$$H_n = \max_{c,\theta}\left[\frac{1}{(-h^2M + hx(1-x)N)}\left(\frac{h^2}{1-\gamma}c^{1-\gamma}(1-x)^{1-\gamma} + h^2\lambda H^* + H_{n+1}hx(1-x)\right)\right]$$

Similarly, we can calculate for $n = 100$, and get:

$$H_n = \max_{c,\theta}\left[\frac{1}{(-h^2M - hx(1-x)N)}\left(\frac{h^2}{1-\gamma}c^{1-\gamma}(1-x)^{1-\gamma} + h^2\lambda H^* - H_{n-1}hx(1-x)\right)\right]$$

4.4 Problems with Numerical Approximations

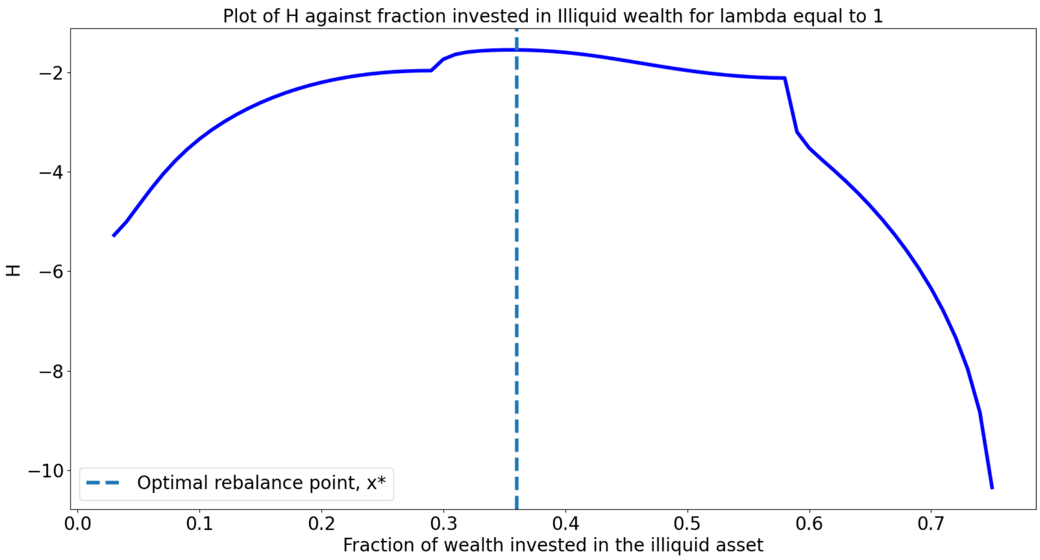
While we have derived a theoretical pleasing way of approximating the value function, the method is not without issues. Numerical approximation is not an exact science, and it is not even guaranteed that the function converges, let alone converging to something meaningful. Implementing such an algorithm is both time consuming and difficult, as you have multiple variables that needs to be "aligned" in order for the algorithm to converge properly. There are a couple of issues related to our numerical approximation approach. First, we have to make a guess of H^* in order to initiate the algorithm. This is in it self not an issues, if the guess is reasonable and close to the actual value function. But more often than not, the guess will be such that the algorithm diverges to either ∞ for every points in the grid, or it diverges to $-\infty$ for every points in the grid. The curvature of the function is also important. If the guess consists of a function that is too steep at some points, or being monotonous wrong (that is, increase when it should be decreasing and vice versa), then the numerical approach has a tendency to explode in one of the end points, and since each point affects the value of adjacent points in the next iteration of the algorithm, the whole approximation can converge to something horribly wrong after a couple of iterations. For these reasons, the initial guess is very important for the result, and one is often led to do a trial-and-error approach in order to get a good initial guess. This can be very time consuming - especially if the algorithm takes a lot of time - and in our case it took days before we got to a reasonable initial guess. Second, The program can become computationally very heavy very quickly, if one is not careful. One can easily fall in the "loop" trap, and make a for loop into a for loop into a for loop and so on, as this approach seems most appropriate and intuitive. This is, however, very ineffecient code wise, and it can easily overload the program, making it take several hours - which is bad, since bugfixing and adjustment of initiation variables and analysis in general becomes very slow and time consuming. As such, we used a lot of time to make the code efficient and be smart with how it was implemented, namely by using vectorized packages and efficient matrix calculations. Overall, the whole process of implementing the algorithm, fixing bugs, adjusting variables and making the code efficient is a slow and time consuming process, but a necessary one to yield useful results.

Another difficulty with approximating non-linear second order partial differential equations such as the HJB equation, is that we have to approximate derivatives. approximating derivatives, especially second order derivatives, can lead to some very unexpected results, as we are dividing with a very small number. This is most impacting when an error has been made in the previous iteration of the algorithm. Any mistakes made i estimating H_n will be magnified by the derivative, as we are dividing with a very small number. This leads to the issue of picking h. While a smaller h is desirable, since it corresponds to a better approximation, it does magnify the issue explained above. A smaller h also makes the algorithm computationally heavier, as the number of points in the grid becomes larger. As such, we set $h = 1/100$, i.e not too small so it is computationally feasible, and not too big, so the approximation is reasonably correct.

4.5 Results of numerical estimation of the HJB

In our numerical solution, we take conservative parameter values. We set $\mu = 0.12$ and $\sigma = 0.15$ and we set the risk-free rate $r = 0.04$. We also set $\nu = \mu = 0.12$ and $\phi = 0.15$, i.e the same values as for the liquid risky asset. We do this in order to isolate the effect of illiquidity, and avoid any "noise" from other parameters We work with the case of the investor being risk-averse and let $\gamma = 6$, which corresponds to a holding of 60% equity and 40% bond for many institutional investors. These will be the parameter values used throughout the entire paper, unless otherwise stated. We do not, at any point, hold λ constant, as we want to see what happens with the investors behaviour when the intensity of illiquidity changes.

Figure 1



In figure 1, a plot of H against the fraction of wealth invested in illiquid asset, that is $x = \frac{X}{W+X}$, can be seen for $\lambda = 1$[6]. In the merton two-asset model derived earlier, we saw that the optimal fraction of total wealth invested in the second risky asset was 0.6, which is not the case when the second risky asset is illiquid. We see that the presence of illiquidity distorts the optimal investment in illiquid asset from $x = 0.6$ to $x = 0.37$. The explanation here is straightforward and intuitive, as the pressence of illiquidity makes the second asset less attractive. The investor can only meet his obligations (consumption in our model) with liquid wealth, except at time $t = \tau$, and therefore there is an extra risk associated with investing in the illiquid risky asset. The ratio between liquid and illiquid wealth holds the importance here. Since the illiquid asset cannot be traded, the ratio of liquid to illiquid wealth is not under the investor's control, except for trading dates. As such, the investor does not want to invest as much as in the Merton case, as that would yield a large portion of his wealth outside of his control. The ability to optimally rebalance is only available at liuidity events, i.e on average every year.

In Figure 2 and Figure 3, we see what happens when we vary λ from the initial value of 1. In Figure 2, we see what happens with the value function H as $\lambda \to 0$. We see that the value function starts exploding around $x = 0$. This means that for λ sufficiently close to 0, the optimal fraction of total wealth invested in illiquid asset is 0. The intuition is that when $\lambda \to 0$, then the average period between dates where the illiquid asset can be traded will approach $\lim_{\lambda \to 0} \frac{1}{\lambda} = \infty$. Hence, the investor finds the illiquid asset less attractive, as there will be less trading opportunities. And when λ approaches 0, the investor will invest

[6] $\lambda = 1$ corresponds to the illiquid asset being tradeable in expectation every $1/\lambda = 1$ year

0 in the illiquid asset, because his opportunity to trade it will approach "never" - even when the investor has infinite time horizon. The investor will, in fact, act as if the illiquid asset does not exist, and we will be back to a case where the investor chooses optimally between 1 risk-free asset and 1 risky asset, i.e exactly the same as the Merton 1 asset case.

In figure 3, we see what happens to the value function H, when λ gets bigger. We see that the value function smooths out, and it takes a maximum close to $x = 0.6$, which corresponds to the Merton two asset case. The reason for this, is that when λ gets bigger, the average time between periods where the illiquid asset can be rebalanced goes towards 0, i.e the illiquid asset can be traded more frequent. The illiquid asset approaches a state where it can be continuously traded, and thereby the optimal fraction invested in this asset will correspond to the Merton case. The maximum λ value we did the model for was $\lambda = 100$, because the algorithm became very unstable for greater values of λ. That is to say, it often diverged towards infinity, and on rare occasions it converged.

Figure 2

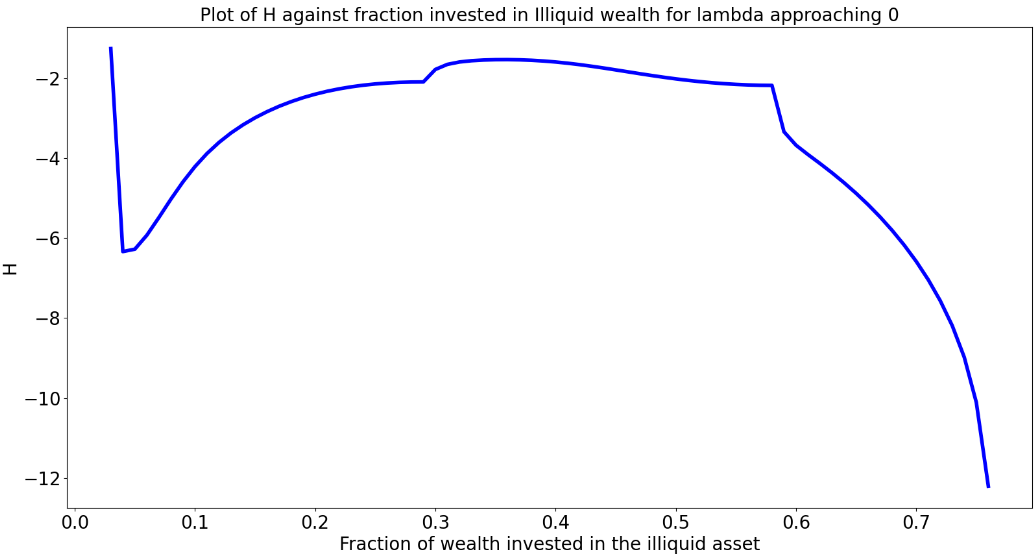
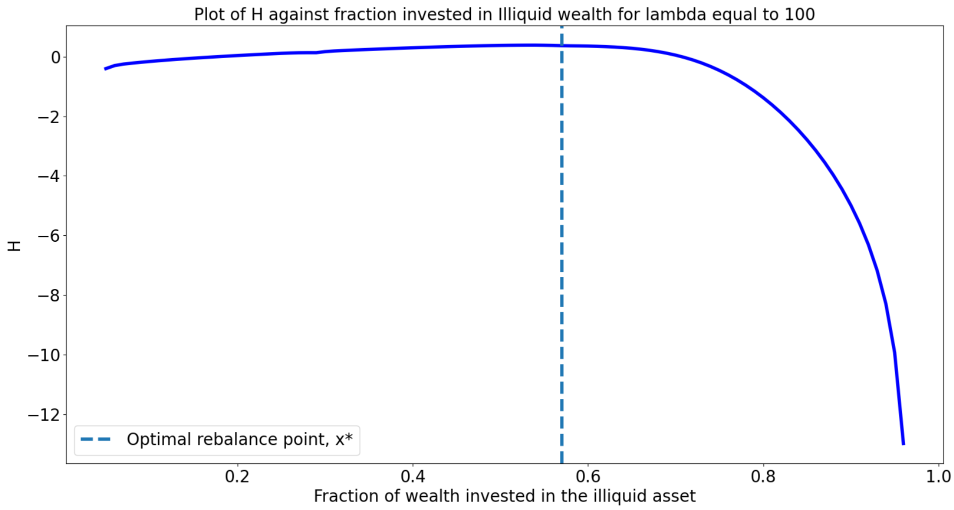


Figure 3



4.6 Illiquidity effect on Liquid Asset Holdings

We will now see how illiquidity affects the investors optimal holdings of the liquid risky asset. To do this, we will compare the behaviour of an investor in the Merton model with the behaviour of an investor in the illiquidity baseline model. The Relative Risk Aversion (RRA) with respect to wealth W for a utility maximizer is be defined as following:

$$RRA = W\frac{F_{WW}}{F_W}$$

Relative Risk Aversion is measurement to capture the curvature of the investors preferences, i.e how willing is an investor to take gambles in wealth W. As such, we can use this measure to capture how an investors willingness to take gambles, and thereby willingness to invest in the liquid risky asset, changes in the presence of illiquidity. Calculating the RRA of a Merton investor by using the derivatives found in the Merton model derivation section, we get:

$$RRA_{Merton} = \gamma \tag{36}$$

As we have seperated the wealth in a liquid and illiquid part, the value function from a merton investor perspective will be given by:

$$F(W, X, t) = \frac{(W+X)^{1-\gamma}\frac{1}{A^{\gamma}}}{1-\gamma}$$

And thereby the relevant derivatives to calculate the RRA will be given by:

$$F_W = (1-\gamma)(W+X)^{-\gamma}\frac{\frac{1}{A^{\gamma}}}{1-\gamma}$$

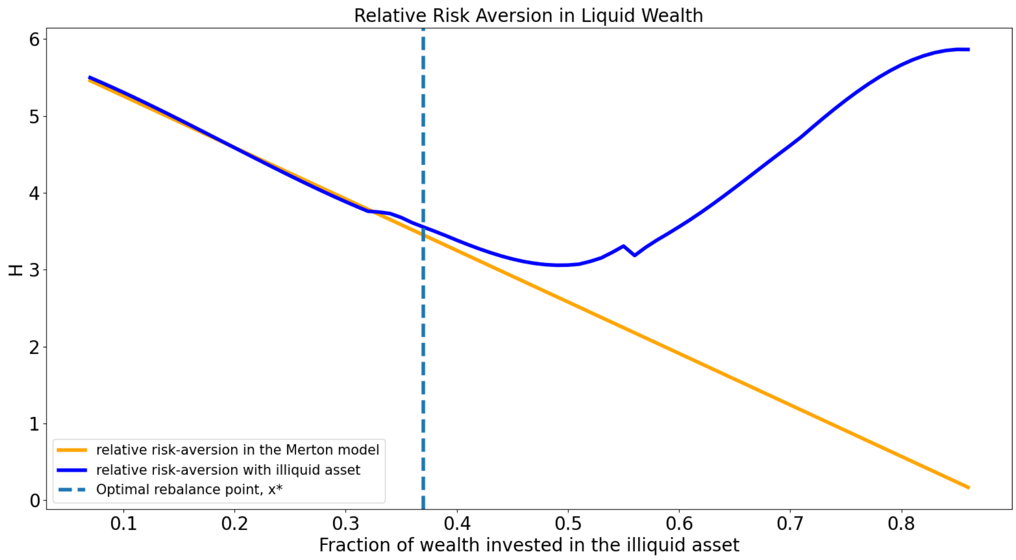
$$F_{WW} = -\gamma(1-\gamma)(W+X)^{-\gamma-1}\frac{\frac{1}{A^{\gamma}}}{1-\gamma}$$

Thus, the Relative Risk Aversion in liquid wealth will be given by:

$$RRA_{Merton} - W\frac{F_{WW}}{F_W} = -W\frac{-\gamma(W+X)^{-\gamma-1}\frac{1}{A^\gamma}}{(W+X)^{-\gamma}\frac{1}{A^\gamma}} = \gamma(1-x)$$

Similarly, we can calculate the RRA for an investor acting in the pressence of illiquidity by using the derivatives derived in the proof of proposition 1. A plot of the two Relative Risk Aversion's as a function of x can be seen in figure 4. As oppose to the Relative Risk Aversion of the merton investor, which is a linear decreasing function in fraction of wealth invested in the illiquid asset, the relative risk aversion of the investor facing illiquidity increases non-linearly with x from around 0.4. For small values of allocation to the illiquid asset, x, the two types of wealth are viewed as perfect substitutes, and the two types of investors behave in a similar fashion. As the allocation in liquid wealth, W, decreases in the investors total wealth $W + X$, the investor's aversion towards gambles in the liquid wealth, W, decreases as well. However, when the investors liquid wealth becomes sufficiently low, the investor aversion towards gambles in liquid wealth increases heavily, as the liquid wealth is no longer viewed as a substitute for illiquid wealth. The reason for this, is that the investor can only meet his obligations with liquid wealth, and as such, if he already has a lot of illiquid wealth relative to his total wealth, he will be more "protective" of his liquid wealth. This shows that for suffiently large illiquid wealth, the investor will invest less in the risky liquid asset, because he becomes more risk-averse as with liquid wealth, as it is the only reliable means of consumption for him.

Figure 4



In total, we see that illiquidity, isolated and in itself, distorts both the allocation in the illiquid asset and the allocation in the risky liquid asset.

5 Model with Illiquidity and Finite Time Horizon

We will now extend the model presented earlier, and generalize it to a setting that allows for the investor to have a finite time horizon. All of our assumptions with regards to the investor and the investor's budget constraint will be identical to the previous section, with the only difference being that the investor now has an investment horizon spanning over $[0, T]$, instead of $[0, \infty)$. Thus, the wealth dynamics of the investor is still given by:

$$\frac{dW_t}{W_t} = (r + (\mu - r)\theta_t - c_t)dt + \theta_t \sigma dZ_t^1 - \frac{dI_t}{W_t} \tag{37}$$

$$\frac{dX_t}{X_t} = m(X_t)dt + v_1 dZ_t^1 + v_2 dZ_t^2 + \frac{dI_t}{X_t} \tag{38}$$

Where X_t is the illiquid wealth and W_t is the liquid wealth, as derived in (26) and (27). The method we will use is also identical to the one used earlier, where we assumed that the investor had an infinite time horizon. That is, we will first derive an expression for the HJB equation. Given the HJB equation, we will then estimate the investor's value function numerically. The main difference in our methodology from earlier is that we will have a slightly different HJB equation and the numerical estimation method will be vastly different. The method we will use is called **The implicit finite difference approach**, as described in Munk: *Fixed Income Modelling*, Oxford University Press, 2011. Estimating a model where the investor has finite time horizon is generally computationally heavier than estimating a model where the investor has infinite time horizon. The reason is that we have to calculate the investor's preferences at any time period t before the end date, whereas with infinite time horizon, we only have to estimate the preferences at one period. For example, if we are interested in the investor's preferences 30 years before the end date, and we work with an interval of 1 month, the total time of computing will be roughly $12 \cdot 30 = 360$ times longer than if we had a model with infinite horizon. The derivative $\frac{\partial F}{\partial t}$ will no longer be 0, which also further complicates both the derivation of the value function, and the numerical computation. We will look at this later on.

With finte time horizon, we assume that the investor has an investment horizon spanning over $[0, T]$, and he performs the following maximization problem:

$$F(W, X) = \max_{c_t, \pi_t} E_t \left[\int_0^T e^{-\beta \cdot s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right] \tag{39}$$

We further assume that at time T, he consumes all the wealth he is able to consume, that is all his liquid wealth[7]. Thus, his terminal utility will be given by:

$$F(W,X,T) = \frac{((W+X)(1-x))^{1-\gamma}}{1-\gamma} \tag{40}$$

With the terminal condition, we will now describe how the HJB equation looks. The difference between the HJB equation derived in proposition 1 and the one with finite time horizon is minimal, although not insignificant at all. It is identical to the HJB equation derived in proposition 1, but only with $\frac{\partial H}{\partial t}$ added to the right-side of the equation, such that we get:

$$0 = \max_{c,\theta}\left[\frac{1}{1-\gamma}c^{1-\gamma}(1-x)^{1-\gamma} + \frac{\partial H}{\partial t} - \beta H(x) + \lambda(H^* - H(x)) + H(x)A(x,c,\theta) + H'(x)B(x,c,\theta) \right.$$
$$\left. + \frac{1}{2}H''(x)C(x,c,\theta)\right]$$

To get to the above form, simply derive the HJB equation again, but where $\frac{\partial F}{\partial t} = 0$ is not assumed. While this extra addition of $\frac{\partial H}{\partial t}$ seems simple at first, it heavily complicates the process of estimating the HJB equation numerically. We now have an extra derivative to estimate. Not only that, the derivative is with respect to a different variable than x, which adds an extra dimension in the numerical optimization problem. We will see later how we deal with this.

To solve the above stated HJB equation numerically, we will transform it to a sequence of differenced equations which can be solved iteratively starting from the known terminal utility (40). To do this, we first assume that x can only take a finite number of values between 0 and 1 (discretization of the state space), such that we have:

$$x_{min} \equiv 0 = x_0, x_1, x_2, ..., x_{N-1}, x_N = 1 \equiv x_{max}$$

Where $x_{n+1} - x_n = h = \frac{1}{100}$. We further assume that the time variable can only take the following values:

$$0, \Delta t, 2\Delta t, ..., T$$

We set $\Delta t = 1/12$, i.e corresponding to 1 month in a year and the highest T we consider is 30 years. Thus, the state space in our numerical solution is given by the lattice:

$$\{x_0, x_1, x_2, ..., x_{N-1}, x_N\} \times \{0, \Delta t, 2\Delta t, ..., T\}$$

The Value function H in the lattice note (n,t) is denotedby $H_{n,t}$, and it corresponds to the x-value x_n and time step $\Delta t \cdot t$. As in the section with infinite time horizon, we use the following estimators for the derivatives of H with respect to x:

$$H'_+(x_n) = \frac{H_{n+1} - H_n}{h}$$

[7] He could technically consume all of his wealth if the Poisson process hits at time T, but in that case he would simply sell all the illiquid wealth and convert it to liquid, and we would technically still have him consuming all his liquid wealth.

$$H'_-(x_n) = \frac{H_n - H_{n-1}}{h}$$
$$H''(x_n) = \frac{H_{n+1} + H_{n-1} - 2H_n}{h^2}$$

Where we have separated the first order derivative of H into a positive and a negative part, such that the HJB equation is given by:

$$\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{1}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} + \frac{\partial H}{\partial t} - \beta H(x) + \lambda(H^* - H(x)) \\
& + H(x)(1-\gamma)\left(r + (1-x)\left([(\mu - r)\theta] - c\right) + x(\nu - r) - \frac{1}{2}\gamma\left(\sigma^2\theta^2(1-x)^2 + \phi^2 x^2 + 2x(1-x)\phi\theta\sigma\rho\right)\right) \\
& + H'_+(x)x(1-x)\Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x\phi^2\Bigg) \\
& + H'_-(x)x(1-x)\Bigg(- ([r + (\mu - r)\theta) + \gamma(2x-1)\phi\theta\sigma\rho\Bigg) \\
& + \frac{1}{2}H''(x)x^2(1-x)^2\Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho\Bigg)\Bigg]
\end{aligned}$$

To approximate the derivative with respect to time, we use the implicit finite first difference approach, and thereby the approximation:

$$\frac{\partial H}{\partial t}(x_n, t) \approx \frac{H_{n,t+1} - H_n}{\Delta t}$$

Inserting the relevant differencing approximations in the HJB equation, we get:

$$\begin{aligned}
0 = \max_{c,\theta} \Bigg[& \frac{1}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} + \frac{H_{n,t+1} - H_{n,t}}{\Delta t} - \beta H_{n,t} + \lambda(H^* - H_{n,t}) \\
& + H_{n,t}(1-\gamma)\left(r + (1-x)\left([(\mu - r)\theta] - c\right) + x(\nu - r) - \frac{1}{2}\gamma\left(\sigma^2\theta^2(1-x)^2 + \phi^2 x^2 + 2x(1-x)\phi\theta\sigma\rho\right)\right) \\
& + \frac{H_{n+1,t} - H_{n,t}}{h}x(1-x)\Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x\phi^2\Bigg) \\
& + \frac{H_{n,t} - H_{n-1,t}}{h}x(1-x)\Bigg(- ([r + (\mu - r)\theta) + \gamma(2x-1)\phi\theta\sigma\rho\Bigg) \\
& + \frac{1}{2}\frac{H_{n+1,t} + H_{n-1,t} - 2H_{n,t}}{h^2}x^2(1-x)^2\Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho\Bigg)\Bigg]
\end{aligned}$$

Isolating $H_{n,t+1}$, we get:

$$\begin{aligned}
H_{n,t+1} = -\max_{c,\theta} \Bigg[& H_{n,t} + \Delta t\frac{1}{1-\gamma} c^{1-\gamma}(1-x)^{1-\gamma} - \Delta t\beta H_{n,t} + \Delta t\lambda(H^* - H_{n,t}) \\
& + \Delta t H_{n,t}(1-\gamma)\left(r + (1-x)\left([(\mu - r)\theta] - c\right) + x(\nu - r) - \frac{1}{2}\gamma\left(\sigma^2\theta^2(1-x)^2 + \phi^2 x^2 + 2x(1-x)\phi\theta\sigma\rho\right)\right) \\
& + \Delta t\frac{H_{n+1,t} - H_{n,t}}{h}x(1-x)\Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x\phi^2\Bigg) \\
& + \Delta t\frac{H_{n,t} - H_{n-1,t}}{h}x(1-x)\Bigg(- ([r + (\mu - r)\theta) + \gamma(2x-1)\phi\theta\sigma\rho\Bigg) \\
& + \Delta t\frac{1}{2}\frac{H_{n+1,t} + H_{n-1,t} - 2H_{n,t}}{h^2}x^2(1-x)^2\Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho\Bigg)\Bigg]
\end{aligned}$$

Which we can further reduce to:

$$H_{n,t+1} = -\max_{c,\theta}\Bigg[H_{n,t}(1-\Delta t\beta) + \Delta t\frac{1}{1-\gamma}c^{1-\gamma}(1-x)^{1-\gamma} + \Delta t\lambda(H^* - H_{n,t})$$
$$+ \Delta t H_{n,t}(1-\gamma)\left(r + (1-x)\left([(\mu-r)\theta] - c\right) + x(\nu - r) - \frac{1}{2}\gamma\left(\sigma^2\theta^2(1-x)^2 + \phi^2x^2 + 2x(1-x)\phi\theta\sigma\rho\right)\right)$$
$$+ (H_{n+1,t} - H_{n,t})\frac{\Delta t}{h}x(1-x)\Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x\phi^2\Bigg)$$
$$+ (H_{n,t} - H_{n-1,t})\Delta t\frac{1}{h}x(1-x)\Bigg(- ([r + (\mu-r)\theta) + \gamma(2x-1)\phi\theta\sigma\rho\Bigg)$$
$$+ (H_{n+1,t} + H_{n-1,t} - 2H_{n,t})\Delta t\frac{1}{2}\frac{1}{h^2}x^2(1-x)^2\Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho\Bigg)\Bigg]$$

$$\Updownarrow$$

$$H_{n,t+1} = -\max_{c,\theta}\Bigg[H_{n,t}\Bigg((1-\Delta t\beta) + \Delta t(1-\gamma)\Bigg(r + (1-x)\left([(\mu-r)\theta] - c\right) + x(\nu-r)$$
$$- \frac{1}{2}\gamma\left(\sigma^2\theta^2(1-x)^2 + \phi^2x^2 + 2x(1-x)\phi\theta\sigma\rho\right)\Bigg) - \frac{\Delta t}{h}x(1-x)\Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x\phi^2\Bigg) - \Delta t\lambda$$
$$+ \Delta t\frac{1}{h}x(1-x)\Bigg(- ([r + (\mu-r)\theta) + \gamma(2x-1)\phi\theta\sigma\rho\Bigg) - 2\Delta t\frac{1}{2}\frac{1}{h^2}x^2(1-x)^2\Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho\Bigg)\Bigg)$$
$$+ \Delta t\frac{1}{1-\gamma}c^{1-\gamma}(1-x)^{1-\gamma} + \Delta t\lambda H^*$$
$$+ H_{n+1,t}\frac{\Delta t}{h}x(1-x)\Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x\phi^2 + \frac{1}{2}\frac{1}{h}x(1-x)\Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho\Bigg)\Bigg)$$
$$+ H_{n-1,t}\Bigg(\Delta t\frac{1}{2}\frac{1}{h^2}x^2(1-x)^2\Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho\Bigg) - \Delta t\frac{1}{h}x(1-x)\Bigg(- [r + (\mu-r)\theta] + \gamma(2x-1)\phi\theta\sigma\rho\Bigg)\Bigg)\Bigg]$$

And lastly, we have that $H_{n,t+1}$ is given by:

$$H_{n,t+1} = -\max_{c,\theta}\left[H_{n,t}A + \Delta t\frac{1}{1-\gamma}c^{1-\gamma}(1-x)^{1-\gamma} + \Delta t\lambda H^* + H_{n+1,t}B + H_{n-1,t}C\right] \quad (41)$$

where:

$$A = (1-\Delta t\beta) + \Delta t(1-\gamma)\Bigg(r + (1-x)\left([(\mu-r)\theta] - c\right) + x(\nu-r)$$
$$- \frac{1}{2}\gamma\left(\sigma^2\theta^2(1-x)^2 + \phi^2x^2 + 2x(1-x)\phi\theta\sigma\rho\right)\Bigg) - \frac{\Delta t}{h}x(1-x)\Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x\phi^2\Bigg)$$
$$- \Delta t\lambda + \Delta t\frac{1}{h}x(1-x)\Bigg(- ([r + (\mu-r)\theta) + \gamma(2x-1)\phi\theta\sigma\rho\Bigg)$$
$$- 2\Delta t\frac{1}{2}\frac{1}{h^2}x^2(1-x)^2\Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho\Bigg)$$

$$B = \frac{\Delta t}{h}x(1-x)\Bigg(c + \gamma(1-x)\sigma^2\theta^2 + \nu + \gamma x\phi^2 + \frac{1}{2}\frac{1}{h}x(1-x)\Bigg(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho\Bigg)\Bigg)$$

$$C = \Delta t \frac{1}{2}\frac{1}{h^2} x^2(1-x)^2\Big(\sigma^2\theta^2 + \phi^2 - 2\phi\theta\sigma\rho\Big) - \Delta t \frac{1}{h} x(1-x)\Big(-[r + (\mu - r)\theta] + \gamma(2x-1)\phi\theta\sigma\rho\Big)$$

With the above form of $H_{n,t+1}$, we can now calculate the value of H for each point in the lattice. We start backwards with the known terminal utility condition. From terminal condition in (40), we have:

$$F(W, X, T) = \frac{((W+X)(1-x))^{1-\gamma}}{1-\gamma}$$

$$\Updownarrow$$

$$H(x, T) = \frac{(1-x)^{1-\gamma}}{1-\gamma}$$

And thereby, $H_{n,T}$ will be given by the following for every $n \in \{0, ..., N\}$:

$$H_{n,T} = \frac{(1-x_n)^{1-\gamma}}{1-\gamma} \tag{42}$$

With this and (41) , we can find a solution for $H_{n,t}$ for every node in the lattice. Suppose $H_{n,t+1}$ is known for every $n \in 0, ..., N$, and suppose c and θ are picked such that the maximization in (41) is solved, we can reduce it to the following:

$$H_{n,t+1} = -H_{n,t}A - \Delta t \frac{1}{1-\gamma} c^{1-\gamma}(1-x_n)^{1-\gamma} - \Delta t \lambda H^* - H_{n+1,t}B - H_{n-1,t}C$$

Since (41) has to hold for every $n = 1, ..., N-1$, we have a system of linked equations with the unknown function $H_{n,t}$. More precisely, we have $N-1$ equations and $N+1$ unknowns $H_{0,t}, ..., H_{N,t}$. Furthermore, if we handle the end points of the lattice, and add equations on the form:

$$H_{0,t+1} = -H_{0,t}A - \Delta t \frac{1}{1-\gamma} c^{1-\gamma}(1-x_0)^{1-\gamma} - \Delta t \lambda H^* - H_{0+1,t}B$$

$$H_{N,t+1} = -H_{N,t}A - \Delta t \frac{1}{1-\gamma} c^{1-\gamma}(1-x_N)^{1-\gamma} - \Delta t \lambda H^* - H_{N-1,t}C$$

We have a full system of linear equations, which we can solve for each time periode t. The below linear system corresponds to what we have to solve at each time period.

$$\begin{pmatrix} A_{0,t} & B_{0,t} & 0 & 0 & 0 & ... & 0 \\ C_{1,t} & A_{1,t} & B_{1,t} & 0 & 0 & ... & 0 \\ 0 & C_{2,t} & A_{2,t} & B_{2,t} & 0 & ... & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & ... & 0 & 0 & C_{N-1,t} & A_{N-1,t} & B_{N-1,t} \\ 0 & ... & 0 & 0 & 0 & C_{N,t} & A_{N,t} \end{pmatrix} \begin{pmatrix} H_{0,t} \\ H_{1,t} \\ H_{2,t} \\ \vdots \\ H_{N-1,t} \\ H_{N,t} \end{pmatrix} = \begin{pmatrix} d_{0,t+1} \\ d_{1,t+1} \\ d_{2,t+1} \\ \vdots \\ d_{N-1,t+1} \\ d_{N,t+1} \end{pmatrix} \tag{43}$$

Where we define $d_{n,t} = H_{n,t+1} + \Delta t \frac{1}{1-\gamma} c^{1-\gamma}(1-x_n)^{1-\gamma} + \Delta t \lambda H_t^*$.

With the above characterisation of the solution to the HJB over a lattice, we can now describe how to solve it numerically. The algorithm goes as following:

1. Initiate the algorithm with the terminal condition. That is, calculate $H_{n,T}$ for every n as described in (42).

2. Guess on a value of the optimal H^*_{T-1} to initiate the calculation of $H_{n,T-1} \forall n \in \{0, ..., N\}$ from (5).

3. Given H^*_{T-1}, then for every combination of c and θ solve the linear system below for $H_{0,T-1}, ..., H_{N,T-1}$:

$$\begin{pmatrix} A_{0,T-1} & B_{0,T-1} & 0 & 0 & 0 & ... & 0 \\ C_{1,T-1} & A_{1,T-1} & B_{1,T-1} & 0 & 0 & ... & 0 \\ 0 & C_{2,T-1} & A_{2,T-1} & B_{2,T-1} & 0 & ... & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & ... & 0 & 0 & C_{N-1,T-1} & A_{N-1,T-1} & B_{N-1,T-1} \\ 0 & ... & 0 & 0 & 0 & C_{N,T-1} & A_{N,T-1} \end{pmatrix} \begin{pmatrix} H_{0,T-1} \\ H_{1,T-1} \\ H_{2,T-1} \\ \vdots \\ H_{N-1,T-1} \\ H_{N,T-1} \end{pmatrix} = \begin{pmatrix} d_{0,T} \\ d_{1,T} \\ d_{2,T} \\ \vdots \\ d_{N-1,T} \\ d_{N,T} \end{pmatrix}$$

You should now have a series of values $H_{0,T-1}, ..., H_{N,T-1}$ over the lattice $\{0, ..., N\} \times \{T-1\}$ for each combination of c and θ.

4. Pick the combination of c and θ which solves:

$$\max_{c,\theta} \left[-H_{n,T-1}A_{n,T-1} - \Delta t \frac{1}{1-\gamma} c^{1-\gamma}(1-x_n)^{1-\gamma} - \Delta t \lambda H^*_{T-1} - H_{n+1,t}B - H_{n-1,t}C_{n,T-1} \right]$$

Denote the optimal values by c^* and θ^*.

5. From the series of values $H_{0,T-1}, ..., H_{N,T-1}$ corresponding to c^* and θ^*, let $H^*_{T-1} = max\{H_{0,T-1}, ..., H_{N,T-1}\}$.

6. Repeat step 3-5 for the new H^*_{T-1} until convergence of $\arg\max_x H^*_{T-1}$.

7. With step 1-6, you now have a characterization of the value function H over the given lattice at time $T-1$. To calculate the value function at time $T-2$, redo step 1-6 with the converged H^*_{T-1} as the terminal utility, and replace T with $T-1$. Repeat this until you get a characterization of the value function enough time before the end date as desired.

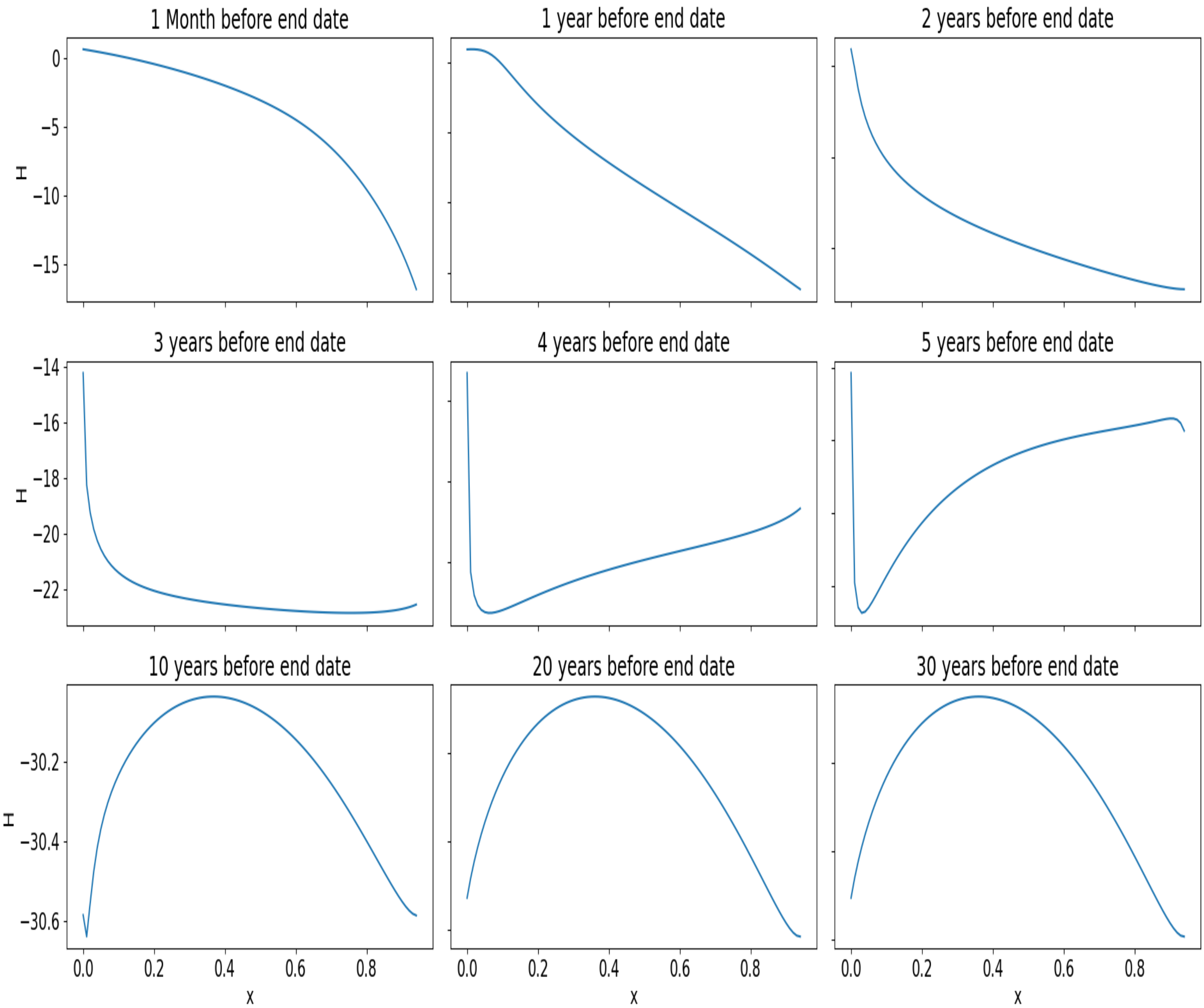
5.1 Numerical Results for Finite Time Horizon Model

In figure 5, we see a plot of the investor's value function against fraction of total wealth invested in the illiquid asset for different dates before the terminal date, when $\lambda = 1$. All other parameters are identical to the ones used in the section with infinite time horizon[8]. We see that for dates close to the terminal date, the risk-averse investor prefers not to have any amount of his total wealth invested in the illiquid asset. The reason for is that with $\lambda = 1$, the illiquid asset is only trade-able in expectation every year. So for a risk-averse investor, investing any amount of wealth in the illiquid asset, when close to the terminal date, yields very little utility, and doing so is equivalent to removing potential consumable wealth from the terminal date, as only liquid wealth can be consumed at the terminal date. In other words, the two assets are not viewed as perfect substitutes, because only the liquid asset can be used for consumption at any date. We also see that for earlier dates, that is dates which are further away from the end date (terminal date), the investor's value function converges to that of an investor with infinite time horizon. Already for an investor with a time horizon of 10 years, his optimal allocation will be roughly 36% of his total wealth in the illiquid asset, compared to an investor with infinite time horizon having an optimal allocation of roughly 37% of total wealth in the illiquid asset. We can conclude that an investor's time horizon does not necessarily have to be extremely large before he starts behaving identical to an investor with infinite time horizon. Interestingly, we see that the investor does not invest anything in the illiquid asset if his time horizon is not greater than 5 years. This comes mainly from the characterization of the investor and the degree of risk-aversion we have assumed, i.e $\gamma = 6$, and not so much from the illiquidity in it self. This naturally raises the question, "how does the relative risk aversion parameter, γ, affect the minimum time horizon the investor needs before he is willing to invest any amount in the illiquid asset?". We will later look at how this risk-aversion parameter affects the minimum time horizon for which the investor is willing to invest any amount in the illiquid asset.

[8] $\sigma = \phi = 0.15, \mu = \nu = 0.12, r = 0.04, \gamma = 6, \lambda = 1, \rho = 0$

Figure 5

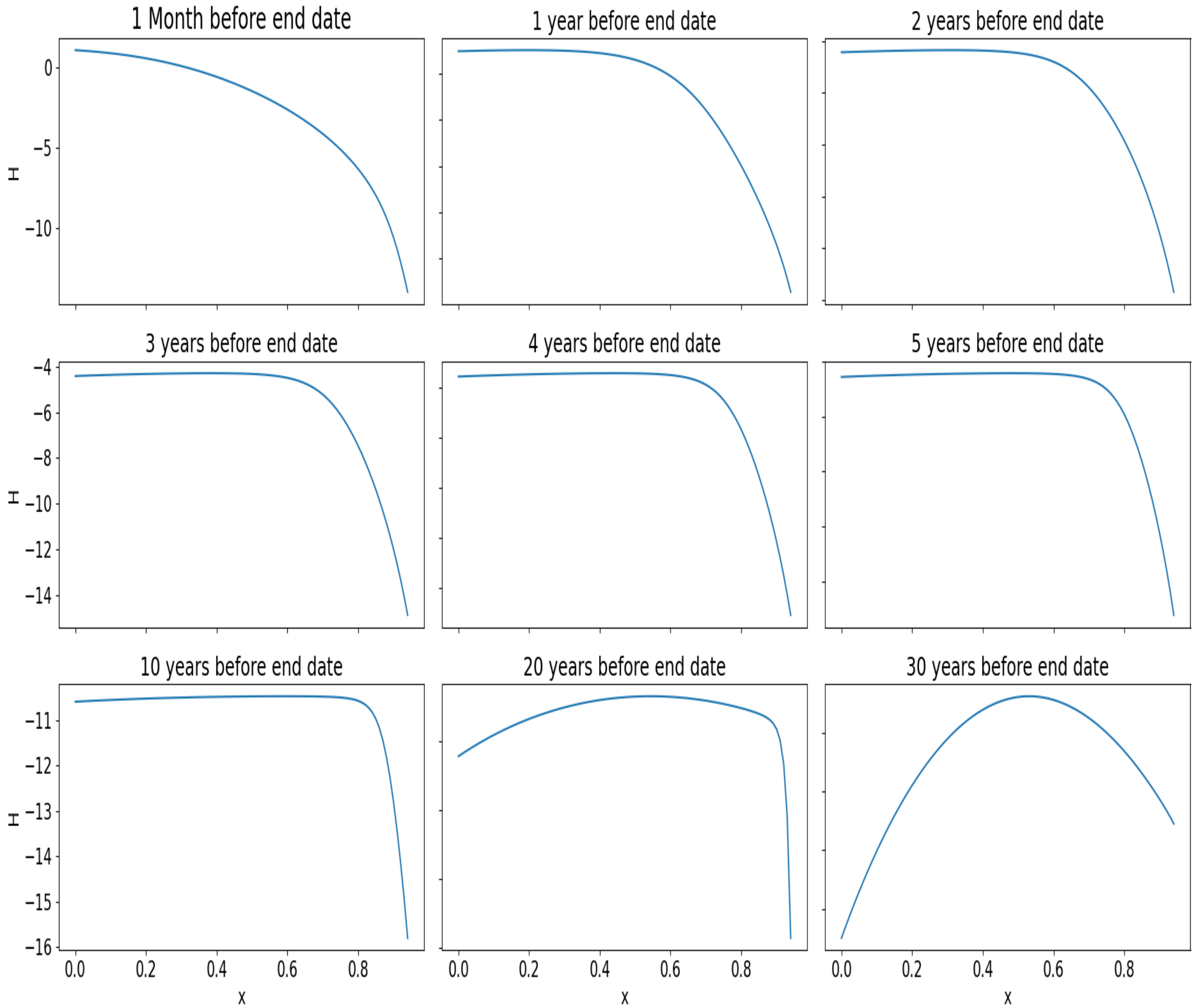
Plot of H against the fraction of total wealth invested in the illiquid asset for different time periods, when λ is equal to 1



Below in figure 6, the same plot of the investor's value function against his fraction of total wealth invested in illiquid asset can be seen, with the only difference from figure 5 being that λ has been adjusted to 10. This case is done to see what happens with the investor's preferences, when the illiquid asset becomes more liquid. With $\lambda = 10$, the investor can expect to trade the illiquid asset on average every 1/10 year. Or in terms of months, every 12/10 month. Like the situation with $\lambda = 1$, it seems that the investor will not invest in the illiquid asset at all, if his investment horizon is very small. But we also see that the investor is willing to invest with a much smaller investment horizon in the illiquid asset when $\lambda = 10$ compared to when $\lambda = 1$. A risk-averse investor with an investment horizon as small as a year is willing to buy the illiquid asset when the asset is trade-able on average every 12/10 month. This makes sense, as over a year the Poisson distribution with parameter $\lambda = 10$ will have a probability of hitting at least once very close to 1. Thus, from the investor's perspective, the risk associated with illiquidity is negligible, as he can trade the asset quasi-continuous. We also see that as the investment horizon increases, the investor's optimal allocation in the illiquid asset approaches that of an investor with infinite time horizon (60%), which is the same as a merton-investor. With a horizon of 10 year or more, the investor starts behaving like a merton investor. Again, we can conclude that the investor does not need an unreasonably large investment horizon before he starts acting like an investor with infinite time horizon.

Figure 6

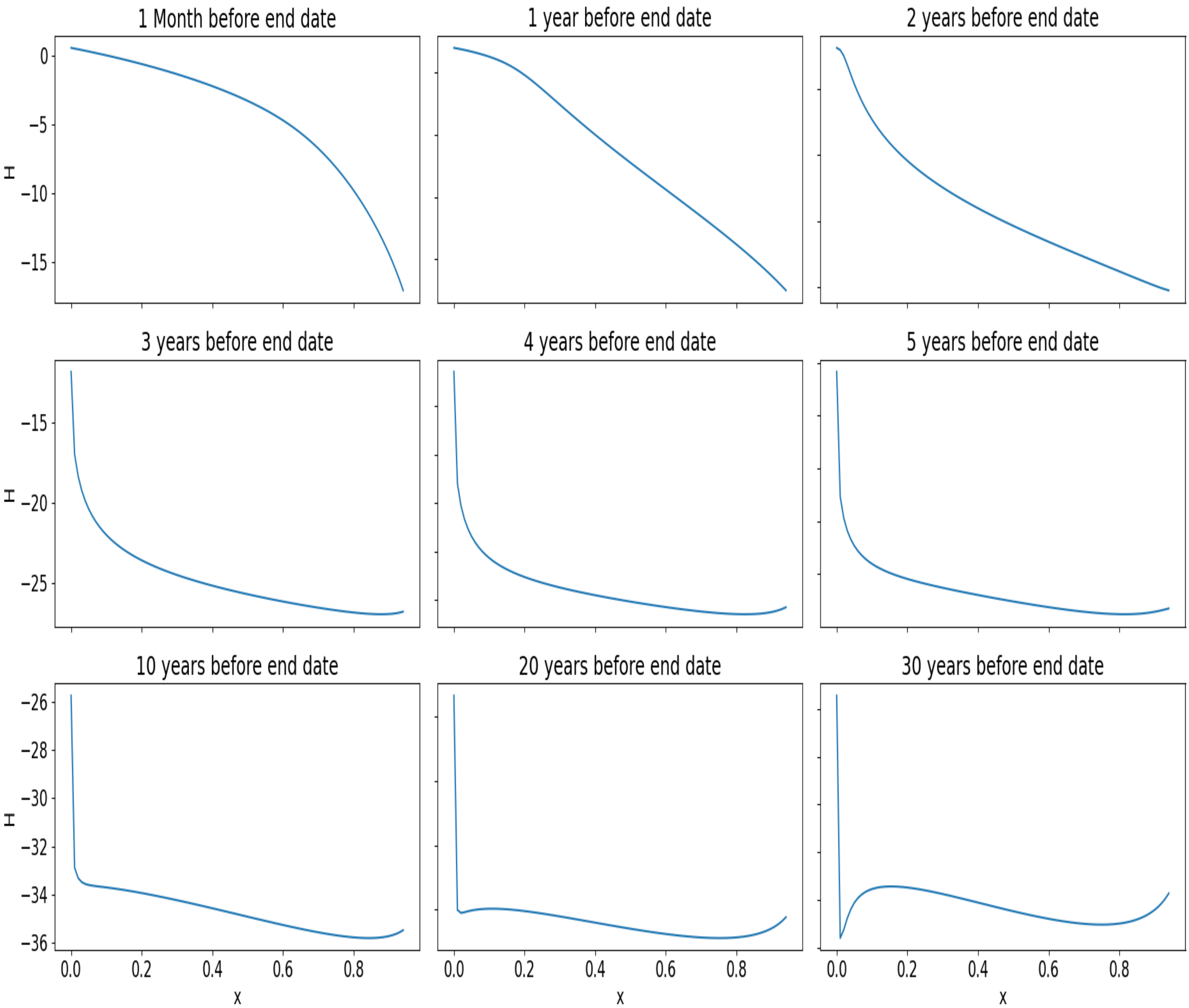
Plot of H against the fraction of total wealth invested in the illiquid asset for different time periods, when lambda is equal to 10



In figure 7, we see what happens to the investor's value function when the $\lambda \approx 0$. We see that it is not optimal for the investor, no matter the investment horizon, to hold any amount of the illiquid asset. Even for investors with very long investment horizons, the expected time between rebalancing periods is too long for investor to trade the illiquid asset. From the investors perspective, the asset is not a "feasible" asset, as he will almost surely never be able to sell it again. The investor will act as if the illiquid asset does not exist, and we will be back to a case where the investor chooses optimally between 1 risk-free asset and 1 risky asset. This is consistent with the results found earlier for an investor with infinite time horizon and the Merton 1-asset model. Surprisingly, the shape of the value function changes heavily over time, but the optimum does not. This is more of a consequence of the numerical method used, rather than any economic reasoning.

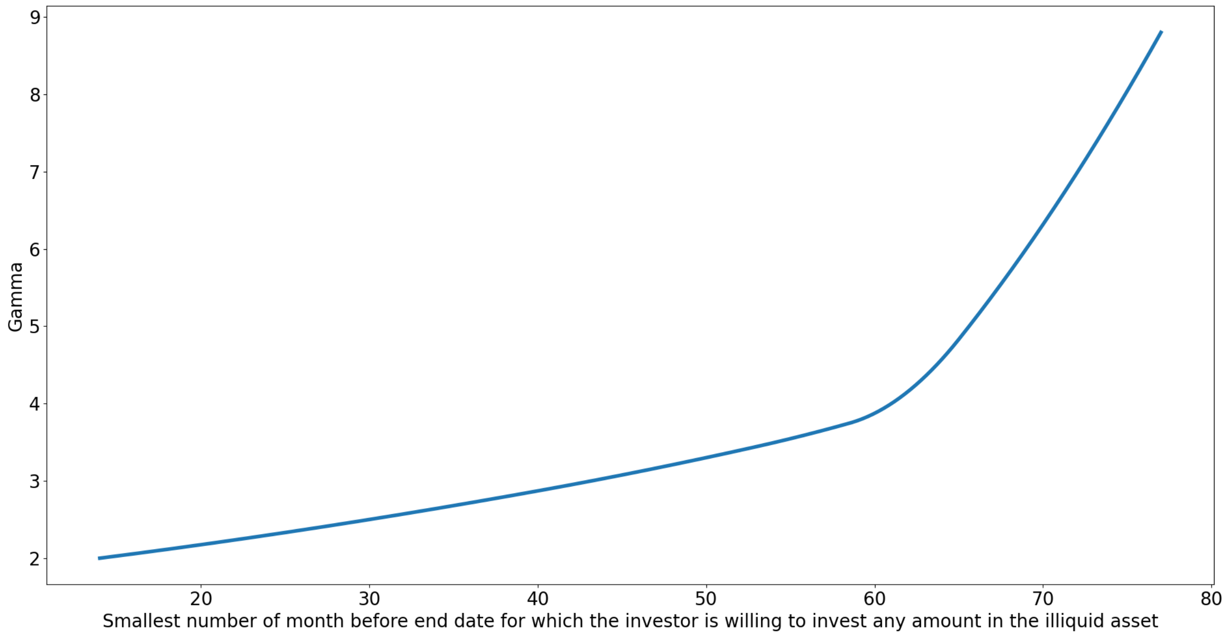
Figure 7

Plot of H against the fraction of total wealth invested in the illiquid asset for different time periods when lambda is equal to 0.00001



Risk-aversion dictates, to a great degree, at which point the investor is willing to invest any amount in the illiquid asset. In figure 8, a plot of the investor's relative risk aversion parameter γ against number of months before end date for which the investor is willing to invest any amount in the illiquid asset can be seen. We see that the curve is increasing in number of months, which means that a very risk-averse investor will need a longer time horizon before he is willing to buy any amount of the illiquid asset. For example, the latest date an investor with $\gamma = 4$ will be willing to buy any amount of the illiquid asset is the date corresponding to 64 months before the end date. We see that the curve is exploding, as the number of month increases, which indicates that even for very risk averse investors, they will invest in the illiquid risky asset within a reasonable time horizon.

Figure 8 Plot of γ against the smallest number of months before end date for which the investor is willing to buy the illiquid asset.



5.2 Conclusion

We study two models on how an investor behaves in the presence of illiquidity by extending the Merton Model to allow for infrequent and stochastic trading opportunities of the illiquid asset. We show that illiquidity distorts the optimal portfolio choice of the investor, no matter the investor's time horizon, by a significant amount. For an illiquid asset with an average time between trading dates of 1 year, a long-term investor only allocates 37% of his portfolio in that asset, compared to the Merton case, where he would have invested 60%. The presence of illiquidity also distorts the investor's willingness to take gambles in his liquid wealth, and as such illiquidity distorts both the allocation in illiquid risky asset and liquid risky asset. The main reason for this is the fact that the investor can only use liquid wealth to meet his immediate obligations, and if his allocation in illiquid wealth is large enough, the investor can not meet his obligations before the next liquidity event. Therefore, the investor acts in a more risk-averse fashion, wanting to avoid states with low liquid wealth. We also see that investors with smaller time horizon prefer not buying the illiquid asset, as they can not be sure to sell it again before the end of their investment horizon. The difference in behaviour between an investor with a finite time horizon of 10 or more years and an investor with infinite time horizon is minimal, and as such the infinite time horizon model is a good approximation for a long term investor's behaviour in the pressence of illiquidity.

5.3 Further Studies

While we have analysed the investor's portfolio choice thoroughly, we have not spent much time on consumption. For future studies, the consumption aspect of the problem would be a direction to go. In our study, and in the numerical approximations, consumption is mostly considered as an economic input, which exists in order to derive the portfolio decision of the investor. This is, however, a very bland look at consumption, and a thorough study on the effect of illiquidity on consumption is something one could look at. We would expect that consumption will also be distorted, such that the investor consumes less compared to the Merton case, because we see from our study the investor behaves in a more risk-averse fashion in the presence of illiquidity.

Another aspect one could look at for future studies would be the initial assumption. It would be interesting to see what would happen if our initial illiquidity assumption changed slightly. The initial assumption we made was that illiquid assets can only be traded at infrequent and random dates. While this is a somewhat realistic assumption, it does not quite capture reality. If an asset is very illiquid, it still holds value, and therefor investors are willing to buy it (maybe for a lower price). If an investor holds this asset, and for some reason, is in need of selling it to meet his obligations, he would not necessarily need to wait too long before he can sell it, if he is willing to sell it at a lower price. Instead of assuming that investors can only trade at specific trading dates, we could assume that investors can trade at any date, but has to pay a premium except for the liquidity event dates. The size of the premium would be dictated by the degree of the illiquidity. The conclusion would probably still be that the investor invests less in the illiquid asset, but not to such an extreme degree as we derive, since the investor's immediate obligations wouldn't be as vulnerable.

6 Appendix

Below is the code for the two algorithms.

```
from scipy.optimize import minimize
import numpy as np
import math
import matplotlib.pyplot as plt
x= np.arange(0,1,1/100)
x[0] = 0.0001
x[-1] = 0.99
h=1/100
x_optimal = 0.4
gamma = 6
beta = 0.1
mu = 0.12
v = 0.12
r = 0.04
psi = 0.15
sigma = 0.15
lmbda = 1
rho = 0



def H_N(input, x):
    c = input[0]
    theta = input[1]

    upper_p_u = h*x*(1-x)*(c+gamma*(1-x)*sigma**2*theta+v+gamma*x*psi**2) + 0.5*x**2*(1-x)**2*(sigma**2*theta**2+psi**2-2*psi*theta*sigma*rho)
    upper_p_d = -h*x*(1-x)*( -(r+(mu-r)*theta) + gamma*(2*x-1)*psi*theta*sigma*rho) + 0.5*x**2*(1-x)**2*(sigma**2*theta**2+psi**2-2*psi*theta*sigma*rho)

    C_1 = x*(1-x)*(c+gamma*(1-x)*sigma**2*theta**2+v+gamma*x*psi**2+r+(mu-r)*theta - gamma*(2*x-1)*psi*theta*sigma*rho)
    C_2 = lmbda + beta -(1-gamma)*(r+(1-x)*((mu-r)*theta-c)+x*(v-r)-0.5*gamma*(sigma**2*theta**2*(1-x)**2+psi**2*x**2+2*x*(1-x)*psi*theta*sigma*rho))
    C_3 = x**2*(1-x)**2*(sigma**2*theta**2+psi**2-2*psi*theta*sigma*rho)
    Delta_T = h**2/(h**2*C_2+h*C_1+C_3)
    p_u = Delta_T*upper_p_u/h**2
    p_d = Delta_T*upper_p_d/h**2
    if x==0:
        M = -lmbda-beta+(1-gamma)*(r+(1-x)*((mu-r)*theta -c)+x*(v-r)-0.5*gamma*(sigma**2*theta**2*(1-x)**2+psi**2*x**2+2*x*(1-x)*psi*theta*rho*sigma))
        N = c+gamma*(1-x)*sigma**2*theta**2+v+gamma*x*psi**2-r-(mu-r)*theta+gamma*(2*x-1)*psi*theta*sigma*rho

        K = 1/(-h**2*M+h*x*(1-x)*N)
        H_nn = H[int(x*100+1)]
        H_n = K*(h**2/(1-gamma)*c**(1-gamma)*(1-x)**(1-gamma)+h**2*lmbda*G*(1+x_optimal)**(1-gamma)+H_nn*h*x*(1-x))
        return max(min(H_n,H[3]),-1000000)
        #return H[int(x*100)-1]
        #H_0 = H[int(x*100)-1]
        #H_0 = 0
    else:
```

```
        H_0 = H[int(x*100-1)]

    if int(x*100) == 99:
        #return H[99]
        H_nn = H[int(x*100)]

    else:
        H_nn = H[int(x*100+1)]

    H_n = p_d*H_0 + H_nn*p_u + Delta_T*(1/(1-gamma)*c**(1-gamma)*(1-x)**(1-gamma) + lmbda*G*(1+x_optimal)**(1-gamma) )

    if x<=0.03:
        return max(min(H_n,H[3]),-1000000)
    else:
        return H_n



def perc_diff(x,y):
    x = np.array(x)
    y = np.array(y)
    x = x[!np.isnan(x)]
    x = x[x<1E308]
    x = x[-1E308<x]
    y = y[!np.isnan(y)]
    y = y[y<1E308]
    y = y[y>-1E308]
    y = y[:min(len(x),len(y))]
    x = x[:min(len(x),len(y))]
    return np.nansum(np.abs(x-y)/np.abs(x),)/len(x)

#total_fun = H
a = 0
H= np.full(100,1)
total_fun = [np.log(i) for i in np.arange(0,1,1/100)]
#total_fun = H
C_OPT = []
while True:
    print("diff = ",perc_diff(H,total_fun))
    H = total_fun
    print("opt=",H.index(np.nanmax(H[:])))
    total_fun = []

    total_C = []
    G=H[int(100*x_optimal)-1]/(1+x_optimal)**(1-gamma)
    for t in np.arange(0,1,1/100):
        print(t)
        fun_v =[]
        C = []
        for i in np.arange(0.01,1.001,1/100):
            for j in np.arange(0.01,1.001,1/100):
                x0 = np.array([i, j])
                fun_v.append(H_N(x0,t))
                C.append((x0,t))
                #if math.isinf(H_N(x0,t)):
                #    print(i,j,t)
```

```python
            total_fun.append(np.nanmax(fun_v))
            if math.isnan(np.nanmax(fun_v)):
                total_C.append(math.nan)
            else:
                total_C.append(C[fun_v.index(np.nanmax(fun_v))])
        #for i in [0,1,2]:
        #    total_fun[i] = total_fun[3]

        C_OPT = total_C
        plt.plot(np.arange(0,1,1/100)[:],[-np.log(-i) for i in total_fun[:]])
        #plt.ylim(-2260639.860577891*0.05)
        #plt.xlim(0,1)
        plt.show()
        print(total_fun)
        a = a + 1
        if a ==50:
            break




#Below is code for model with finite time horizon


#terminal utility:
terminal_utility = np.array([(1-x)**(1-gamma)*1/(1-gamma) for x in np.arange(0,1,1/
    100)])
#terminal_utility = -np.log(-terminal_utility)
#plt.plot(np.arange(0,1,1/100),terminal_utility)
#plt.show()
#terminal_utility = total_all_fun[0]
x= np.arange(0,1,1/100)
theta= 0
lmbda = 1
dt = 1/12
gamma = 6
v = 0.12
#x_optimal = 0.2

def tridiag(a, b, c, k1=-1, k2=0, k3=1):
    return np.diag(a, k1) + np.diag(b, k2) + np.diag(c, k3)

total_all_fun = []
time_t_fun = []
time_t_opt = []
time_t_index = []
for t in np.arange(1,400,1):
    if total_all_fun != []:
        terminal_utility = total_all_fun[total_optimal.index(max(total_optimal))]
    total_optimal = []
    total_all_fun = []
    total_index =[]
    for x_optimal in np.arange(0,1,1/100):
        c=0.99-0.66*x_optimal
        opt_H = -1000000000000000000000000000000
        opt_x = np.argmax(terminal_utility)
        while True:
```

```
            temp_opt_H = opt_H
            temp_opt_x = opt_x
            theta = 0
            while True:
                #print(theta)
                theta = theta + h
                #if theta>=1:
                #    break
                T_1 =   -1-dt*beta+ dt*(1-gamma)*(r+(1-x)*((mu-r)*theta-c)+x*(v-r)-0.5*gamma*(sigma**2*theta**2*(1-x)**2+psi*x**2+2*x*(1-x)*psi*theta*sigma*rho)) - (dt/h)*x*(1-x)*(c+gamma*(1-x)*sigma**2*theta**2+v+gamma*x*psi**2) - dt*lmbda + (dt/h)*x*(1-x)*(-r-1*(mu-r)*theta+gamma*(2*x-1)*psi*theta*sigma*rho)-2*dt*0.5*(1/h**2)*x**2*(1-x)**2*(sigma**2*theta**2+psi**2-2*psi*theta*sigma*rho)
                T_1 = -T_1
                T_2 = (dt/h)*x*(1-x)*(c+gamma*(1-x)*sigma**2*theta**2+v+gamma*x*psi**2+ (0.5*1/h)*x*(1-x)*(sigma**2*theta**2+psi**2-2*psi*theta*sigma*rho))
                T_2 = -T_2
                T_3 = dt*0.5*(1/h**2)*x**2*(1-x)**2*(sigma**2*theta**2+psi**2-2*psi*theta*sigma*rho) - dt*(1/h)*x*(1-x)*(-r-(mu-r)*theta +gamma*(2*x-1)*psi*theta*sigma*rho)
                T_3 = -T_3
                G=terminal_utility[int(100*x_optimal)-1]/(1+x_optimal)**(1-gamma)
                H_before = terminal_utility+ dt*(1/(1-gamma))*c**(1-gamma)*(1-x)**(1-gamma)+dt*lmbda*G*(1+x_optimal)**(1-gamma)
                A = tridiag(T_3[1:], T_1, T_2[:-1])
                H = np.linalg.solve(A,H_before)
                H = H
                if theta >=1:
                    temp_opt_H = np.nanmax(H)
                    temp_opt_x = np.nanargmax(H)
                    temp_all_H = H
                    break
                if np.nanmax(H)<temp_opt_H:
                    break
                else:
                    temp_opt_H = np.nanmax(H)
                    temp_opt_x = np.nanargmax(H)
                    temp_all_H = H

            #if theta >=1:
            #    break

            if opt_H == temp_opt_H:
                break

            opt_H = temp_opt_H
            print(opt_H)
            opt_x = temp_opt_x
            opt_all_H = temp_all_H
            print("c=",c)
            c = c-h
            if c<=0:
                break

        total_optimal.append(opt_H)
        total_all_fun.append(opt_all_H)
        total_index.append(opt_x)
```

```
    time_t_opt.append(total_optimal[total_optimal.index(max(total_optimal))])

    time_t_fun.append(total_all_fun[total_optimal.index(max(total_optimal))])

    time_t_index.append(total_index[total_optimal.index(max(total_optimal))])
```

7 Bibliography

Andrew Ang, Dimitris Papanikolaou, Mark M. Westerfield (2014). Portfolio Choice with Illiquid Assets. *Management Science* 60(11):2737-2761. http://dx.doi.org/10.1287/mnsc.2014.1986

Munk, Claus. Dynamic Asset Allocation notes (2017)

Munk, Claus. Fixed Income Modelling. *Oxford University Press*, 2011

Munk, Claus. Financial Asset Pricing Theory. *Oxford University Press*, 2013

Björk, Tomas. Arbitrage Theory in Continuous Time, *Oxford University Press*, 2020

Kushner, H. J., and P. Dupuis. Numerical Methods for Stochastic Control Prob- lems in Continuous Time, *Springer-Verlag*, Berlin, 1992.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* 3(4):373–413. https://doi.org/10.1016/0022-0531(71)90038-X

Hansen, K. A. (1998). Seasonality of moves and duration of residence. Current Population Report P70-66, Household Economic Studies, *Bureau of the Census*, U.S. Department of Commerce, Washington, DC.

Kaplan, Greg, & Violante, G. L. (2010). "How Much Consumption Insurance beyond Self-Insurance?" *American Economic Journal: Macroeconomics*, 2 (4): 53-87. https://doi.org/10.1257/mac.2.4.53

Grossman, Sanford, & Laroque, Guy, (1990). Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Goods, *Econometrica*, 58, issue 1, p. 25-51, https://EconPapers.repec.org/RePEc:ecm:emetrp:v:58:y:1990:i:1:p:25-51.

Vayanos, Dimitri. “Transaction Costs and Asset Prices: A Dynamic Equilibrium Model.”

The Review of Financial Studies, vol. 11, no. 1, 1998, pp. 1–58. JSTOR, www.jstor.org/stable/2646036. Accessed 11 May 2021.

Kahl, Matthias, Liu, Jun and Longstaff, Francis, (2003), Paper millionaires: how valuable is stock to a stockholder who is restricted from selling it?, *Journal of Financial Economics*, 67, issue 3, p. 385-410, https://EconPapers.repec.org/RePEc:eee:jfinec:v:67:y:2003:i:3:p:385-410.

Øksendahl, Bernt. Stochastic Differential Equations. *Springer*, 2003.