Essays on Private Equity

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Document Version
Final published version

Publication date:
2022

COPENHAGEN BUSINESS SCHOOL
SOLBJERG PLADS 3
DK-2000 FREDERIKSBERG
DANMARK

WWW.CBS.DK

ISSN 0906-6934

Print ISBN: 978-87-7568-059-7
Online ISBN: 978-87-7568-060-3

ESSAYS ON PRIVATE EQUITY

PhD Series 03.2022

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HANDELSHØJSKOLEN

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Copenhagen Business School

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1st edition 2022
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Online ISBN: 978-87-7568-060-3

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Acknowledgments

This thesis could not be possible without the support, advice, and encouragement of those who helped me throughout the process. I owe them a great deal, probably more than I can ever repay, and I would like to take this opportunity to say thank you.

First, I would like to thank Morten Sorensen, whose support as advisor and co-author has been invaluable. Morten taught me most of what I know about research, and working with him has been a privilege and a real pleasure. I will always be thankful for the effort and time he has put towards my development as an economist.

I owe great thanks also to Steffen Andersen, who first introduced me to research work. I was fortunate to assist Steffen in his projects during my Master's studies, and that experience convinced me to start a PhD in the first place. Steffen has supported and helped me ever since, and I owe him a great deal.

I am grateful to Copenhagen Business School and the Department of Finance for supporting my research and for providing a great working environment. I met many smart and talented colleagues at the Department of Finance. I would like to thank especially my co-author and friend, Rasmus Jørgensen, who shared with me much of the pain and pleasure of writing a PhD thesis. Thanks also to Lena Jaroszek for the many refreshing conversations about life and research.

Last but not least, I would like to thank my family for their unconditional love and support throughout the years.

Abstract

This thesis concerns risk and performance of private equity funds. Private equity funds are illiquid investments with mostly unobservable returns and with special institutional rules distinguishing them from other assets typically considered in finance. This thesis studies (i) how to risk-adjust the performance of private equity funds using data on cash flows instead of returns, and (ii) how to optimally allocate capital between private equity and publicly traded assets. It consists of three chapters which can be read independently.

The first chapter concerns risk adjustment of private equity cash flows. Recent literature has developed methods to risk-adjust private equity cash flows using stochastic discount factors (SDFs). In this chapter, we find that those methods result in unrealistic time discounting, which can generate implausible performance estimates. We propose and evaluate a modified method which estimates a set of SDF parameters so that the subjective term structure of interest rates is determined by market data. Our method is based on a decomposition of private equity performance in a risk-neutral part and a risk adjustment, and it keeps the risk-neutral part constant as we add or remove risk factors from the SDF. We show that (i) our approach allows for economically meaningful measurement and comparison of risk across models, (ii) existing methods estimate implausible performance when time discounting is particularly degenerate, and (iii) our approach results in lower variation of performance across funds.

The second and third chapters study optimal portfolio allocation with private equity funds and publicly traded assets. In the second chapter, we study the portfolio problem of an investor (or limited partner, LP) that invests in stocks, bonds, and private equity funds. Stocks and bonds are liquid assets, while private equity is illiquid. The LP repeatedly commits capital to private equity funds. This capital is only gradually contributed and eventually distributed back to the LP, requiring the LP to hold a liquidity buffer for its uncalled commitments. We solve the problem numerically for LPs with different

risk aversion, and we find that optimal private equity allocation is not monotonically declining in risk aversion, despite private equity being riskier than stocks. We investigate the optimal dynamic investment strategy of two LPs at opposite ends of the risk aversion spectrum, and we find two qualitatively different strategies with intuitive heuristics. Further, we introduce a secondary market for private equity partnership interests to study optimal trading in this market and implications for the LP's optimal investments.

The third chapter considers a portfolio problem with private equity and several liquid assets. This chapter focuses on average portfolio allocation over time, as opposed to dynamic strategies generating that allocation, and derives an approximate closed-form solution despite complex private equity dynamics. In this chapter, optimal portfolio allocation is well approximated by static mean-variance optimization with margin requirements. Margin requirements are self-imposed by the investor, and because private equity needs capital commitment, the investor assigns greater margin requirement to private equity than liquid assets. Due to that greater margin requirement, the risky portfolio of constrained investors can optimally underweight private equity relative to the tangency portfolio, even when private equity has positive alpha and moderately high beta with respect to liquid assets.

Abstract in Danish

Denne afhandling omhandler kapitalfondes risiko og afkast. Kapitalfonde foretager illikvide investeringer hvis afkast er vanskeligt at observere før fondene udløber og som er underlagt særlige institutionelle regler hvilket adskiller dem fra de investeringer man typisk undersøger i finansiel økonomi. Denne afhandling undersøger (i) hvordan man risikojusterer kapitalfondes afkast ved hjælp af pengestrømsdata, og (ii) hvordan man allokerer kapital optimalt mellem kapitalfonde og andre aktiver som noterede aktier og obligationer. Afhandlingen består at tre kapitler der kan læses uafhængigt af hinanden.

Det første kapitel omhandler risikojustering af kapitalfondes pengestrømme. Tidligere litteratur har udviklet metoder til at risikojustere kapitalfondes pengestrømme ved hjælp af stokastiske diskonteringsfaktorer (SDF). I dette kapitel finder vi, at disse metoder resulterer i en urealistisk diskontering af fremtidige pengestrømme, hvilket kan generere urealistiske estimater. Vi fremsætter og evaluerer en modificeret metode, der estimerer et sæt af SDF-parametre således at den subjektive rentekurve er bestemt af markedsdata. Vores metode er baseret på en dekomponering af kapitalfondes performance i en risikoneutral del og en risikojustering, og metoden holder den risiko-neutrale del konstant, hvis vi tilføjer eller fjerner risikofaktorer fra den stokastiske diskonteringsfaktor. Vi viser at (i) vores metode foranlediger en økonomisk meningsfyldt måling og sammenligning af risiko på tværs af modeller, (ii) vores metode er at foretrække når diskontering af fremtidige pengestrømme er særlig problematisk og (iii) vores metode resulterer i mindre variation i afkast på tværs af fonde.

Det andet og tredje kapitel undersøger optimal porteføljeallokering mellem kapitalfonde og noterede aktiver. I det andet kapital undersøger vi porteføljeproblemet for en investor (eller limited partner, LP) der investerer i aktier, obligationer og kapitalfonde. Aktier

og obligationer er likvide aktiver, mens kapitalfonde er illikvide. LP'en giver løbende investeringstilsagn til kapitalfonde. Kapital trækkes gradvist og returneres til sidst, hvilket kræver at LP'en holder en likviditetsreserve. Vi løser modellen numerisk for LP'er med varierende grade af risikoaversion og finder, at den optimal allokering til kapitalfonde ikke er monotont aftagende i risikoaversion på trods af, at kapitalfonde er mere risikofyldte end aktier. Vi undersøger den optimale dynamiske investeringsstrategi for to LP'er, i modsatte ender af risikoaversions spekteret, og finder to kvalitativt forskellige strategier med intuitive heuristikker. Derudover introducerer vi et sekundært marked for partnerskabsandel i kapitalfonde med henblik på at undersøge optimal handel i dette marked samt implikationer for LP'ens optimale investeringer.

Det tredje kapitel undersøger et porteføljeproblem med kapitalfonde og adskillige likvide aktiver. Dette kapitel fokuserer på porteføljeallokering over tid, i modsætning til de dynamiske strategier der genererer allokeringen, og udleder en approksimativ analytisk løsning på lukket form på trods af de komplekse kapitalfonds dynamikker. I dette kapitel er den optimal kapitalfonds allokering tilnærmelsesvist givet ved statisk middelværdi-varians optimering med marginkrav, og marginkravet er opstår endogent som et resultat af investorens optimalitetsbetingelser. Fordi kapitalfonde kræver investeringstilsagn, pålægger investoren kapitalfonde større marginkrav end likvide aktiver. Som følge af det større marginkrav er kapitalfonde undervægtet relativt til tangensporteføljen, selv hvis kapitalfonde udviser positiv risikojusteret afkast og et moderat beta med hensyn til likvide aktiver.

Contents

Chapter 1

Risk Adjustment of Private Equity Cash Flows

Nicola Giommetti Rasmus Jørgensen*

Abstract

Existing stochastic discount factor methods for the valuation of private equity funds result in unrealistic time discounting. We propose and evaluate a modified method. Valuation has a risk-neutral component plus a risk adjustment, and we fix the risk-neutral part by constraining the subjective term structure of interest rates with market data. We show that (i) our approach allows for economically meaningful measurement and comparison of risk across models, (ii) existing methods estimate implausible performance when time discounting is particularly degenerate, and (iii) our approach results in lower cross-sectional variation of performance.

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Asset allocation to buyout, venture capital, and other private equity (PE) funds has increased consistently over the past decade.[1] It remains challenging, however, to estimate risk and performance of these funds, especially due to their illiquid secondary market and the consequential absence of reliable return data. PE returns can be extrapolated from cash flow data as in Ang, Chen, Goetzmann, and Phalippou (2018), but that requires restrictive assumptions on the return-generating process. To avoid those assumptions, Korteweg and Nagel (2016, KN) develops a stochastic discount factor (SDF) valuation framework that uses cash flows instead of returns, and that benchmarks PE against publicly traded assets.

Central to the SDF framework is a requirement for proper benchmarking: the SDF must price benchmark assets during the sample period. To satisfy this requirement, KN use a heuristic implementation. They build artificial funds invested in benchmark assets, and they estimate SDF parameters pricing the artificial funds.

In this paper, we propose an alternative implementation which estimates a set of SDF parameters so that the subjective term structure of interest rates is determined by market data. Theoretically, our approach is based on a decomposition of PE performance in a risk-neutral part and a risk adjustment. By construction, the risk-neutral part does not vary as we add or remove risk factors from the SDF, so we can meaningfully measure the economic cost of PE risk and compare it across models. Empirically, we evaluate our approach against the original KN implementation, and we find that KN implies unrealistic time discounting which can generate implausible performance estimates. For example, a zero-coupon bond paying \$1 at 3 years maturity can have discounted value up to \$9, and a zero-coupon bond paying \$1 at 10 years maturity can have discounted value up to \$7. Our approach avoids this problem. As a result, we estimate more stable performance across models and obtain lower variation of performance across funds.

We use our method to measure risk adjustment for two types of investor: a CAPM investor, and a long-term investor who distinguishes between permanent and transitory wealth shocks. We discount net-of-fees cash flows of 1866 PE funds started in the US between 1978 and 2009, and divided in three categories: buyout, venture capital, and

[1] Bain & Company (2021) quantify and discuss trends in PE allocation.

generalist.[2] As benchmark assets, we use the S&P 500 total return index and the 3-month T-bill. For the CAPM investor, buyout has generated 30 cents of NPV per dollar of commitment, as opposed to 7 cents of venture capital and 21 cents of generalist. Unsurprisingly, venture capital has the highest (absolute value of) risk adjustment, equal to 65 cents per dollar of commitment and about twice as large compared to 31 cents of buyout and 35 cents of generalist. Our long-term investor assigns similar NPVs; risk adjustment is only marginally smaller, about 5 cents lower than CAPM across the three categories.

Comparing our method with the KN implementation, we find the largest differences in the buyout category. With CAPM, the risk-adjusted performance of buyout is similar across the two methods, but performance components differ substantially. The KN implementation results in larger risk-neutral value which is then compensated by higher risk adjustment. Further, the standard deviation of performance across funds is 142 cents using KN, while it is 98 cents in our implementation. For the long-term investor, the KN method shows very high buyout performance, up to 80 cents of NPV per dollar of commitment, in contrast to 35 cents with our implementation. That very high NPV, however, is not driven by lower risk adjustment; instead, it is driven by a large increase in the risk-neutral value, which goes from 80 cents with CAPM up to 250 cents with the long-term model. The standard deviation of performance in the long-term model goes up to 925 cents with the KN implementation, while it remains stable at 105 cents with our method.

We find smaller differences between the two implementations for the venture capital and generalist categories. We consistently find, however, that our implementation implies a more plausible subjective term structure of interest rates, resulting in lower cross-sectional variation of performance and more stable NPV estimates across models.

With our implementation, we further decompose risk adjustment based on the timing of cash flows during a fund's life. For all three fund categories, cash flows have marginally negative risk exposure in the first three years of operations, indicating weak pro-cyclicality of contributions, and risk exposure becomes positive from the fourth year onwards. We

[2]PE data is maintained by Burgiss, and it is one of the most comprehensive PE dataset available to date.

find differences in the timing of risk across the three categories. For venture capital, the largest risk adjustment is due to cash flows from year 4 to 7 and in contrast with year 9 to 11 for buyout. For generalist funds, risk adjustment is spread more homogeneously between year 4 and 10.

An important weakness remaining in our approach is that our performance decomposition does not provide clear guidance on how to estimate risk prices for proper benchmarking. In practice, we restrict the SDF to price S&P 500 returns in the sample period at a 10-year horizon. This condition is heuristic, however, based on the typical horizon of PE funds. To address this weakness, we study the robustness of our results by changing the price of risk exogenously. We find only weak effects on risk adjustment and NPVs of buyout and generalist funds. Their NPVs remain positive over a wide range of risk prices. Venture capital, on the other hand, has higher risk exposure, and its valuation is more sensitive to the price of risk.

This paper fits into a large literature studying the risk and return of PE investments. Korteweg (2019) surveys that literature, and we build on a series of studies benchmarking PE cash flows against publicly traded assets. In this context, a popular measure of risk-adjusted performance is Kaplan and Schoar (2005)'s Public Market Equivalent (PME). The PME discounts cash flows using the realized return on a portfolio of benchmark assets. Sorensen and Jagannathan (2015) show that the PME fits into the SDF framework as a special case of Rubinstein (1976)'s log-utility model. But the log-utility model does not necessarily price benchmark assets, and in that case the PME applies the wrong risk adjustment. To fix that issue, Korteweg and Nagel (2016) propose a generalized PME, and we build on their work.[3]

Starting with Ljungqvist and Richardson (2003), several authors study the performance of PE funds adjusting for different risk factors. Franzoni, Nowak, and Phalippou (2012) along with Ang, Chen, Goetzmann, and Phalippou (2018) estimate some of the most inclusive models considering Fama-French three factors, the liquidity factor of Pástor and Stambaugh (2003), and in some cases also profitability and investment factors. With our long-term investor, we introduce a new risk factor representing shocks to investment

[3]Parallel effort by Gupta and Van Nieuwerburgh (2021) takes a different approach to benchmark PE. They try to replicate funds' cash flows with a large portfolio of synthetic dividend strips which is then priced with standard asset pricing techniques.

opportunities, or discount rate news, as in the intertemporal CAPM of Campbell (1993). Closest to the spirit of our long-term investor is the work of Gredil, Sorensen, and Waller (2020), who study PE performance using SDFs of leading consumption-based asset pricing models.

1 Risk Adjustment of Private Equity Cash Flows

We measure the risk-adjusted performance of PE funds using the Generalized Public Market Equivalent (GPME). In its most general form, the GPME of fund i is the sum of fund's cash flows, $C_{i,t}$, discounted with realized SDF:

$$\text{GPME}_i \equiv \sum_{h=0}^{H} M_{t,t+h} C_{i,t+h} \tag{1}$$

The term $M_{t,t+h}$ denotes a multi-period SDF discounting cash flows from $t+h$ to the start of the fund. Time t is the date of the first cash flow of the fund, and it depends on i despite the simplified notation. The letter H indicates the number of periods (quarters in our case) from the first to last cash flow of the fund. As a convention, we let H be the same across funds, and funds that are active for a lower number of periods have a series of zero cash flows in the last part of their life.

Functional forms of the SDF are discussed in Section 2. They typically include at least one risk factor and depend on a vector of parameters. Those parameters should be estimated such that the SDF reflects realized returns on benchmark assets during the sample period. This intuitive condition is necessary for proper benchmarking, but it is unclear how it should be translated into formal statements. Korteweg and Nagel (2016) propose a heuristic approach based on the construction of artificial funds that invest in the benchmark assets. They then estimate parameters such that the NPV of those funds is zero. In the rest of this section, we propose an alternative approach based on the GPME decomposition which we are about to describe.[4]

Investing in a random fund gives $E[\text{GPME}_i]$ as NPV, and it is useful to decompose this

[4] A related but different decomposition is discussed by Boyer, Nadauld, Vorkink, and Weisbach (2021).

quantity in a typical asset pricing way:

$$E[\text{GPME}_i] = \underbrace{\sum_{h=0}^{H} E[M_{t,t+h}]E[C_{i,t+h}]}_{\text{risk-neutral value}} + \underbrace{\sum_{h=0}^{H} \text{cov}(M_{t,t+h}, C_{i,t+h})}_{\text{risk adjustment}} \tag{2}$$

As illustrated on the right-hand side of this expression, NPV is the sum of a risk-neutral value and a risk adjustment. We make a simple consideration: by definition, the risk-neutral value should be determined by cash flows and risk-free rates, and it should not change as we add or remove risk factors from the SDF.

Further, a main objective of a benchmarking exercise like ours is to assess risk exposure of PE to different risk factors. In general, the GPME does not allow direct measurement of risk quantities, and we are left with indirect evidence based on the behavior of risk adjustment (Jeffers, Lyu, and Posenau, 2021). As we add or remove risk factors from the SDF, it is tempting to attribute differences in GPME to changes in risk adjustment, but that interpretation is robust only when the risk-neutral value is fixed.

We fix the risk-neutral value using standard asset pricing conditions on risk-free assets. We consider a \$1 investment at time t in a risk-free asset paying $R^{\text{f}}_{t,t+h}$ at time $t+h$. This investment is priced by the SDF if $E_t[M_{t,t+h}] = 1/R^{\text{f}}_{t,t+h}$. Take unconditional expectations on both sides, we get the following condition:

$$E[M_{t,t+h}] = E\left[\frac{1}{R^{\text{f}}_{t,t+h}}\right] \tag{3}$$

Imposing this restriction for all horizons h from 1 to H, the risk-neutral value can be rewritten without the SDF. Thus, risk-neutral value is determined by cash flows and risk-free rates, and remains constant as we consider different SDFs, so our initial consideration is satisfied.

Empirically, we wish to impose condition (3) to the SDF. However, the practical meaning of the expectation operator inside that condition can be elusive. How does the population condition translate into a sample condition?

To address this question, it is useful to consider the sample version of $E[\text{GPME}_i]$. We call it simply GPME, and compute it as the mean of GPME_i across N funds in a sample:

$$\text{GPME} \equiv \frac{1}{N}\sum_{i=1}^{N}\sum_{h=0}^{H} M_{t,t+h}C_{i,t+h} \tag{4}$$

It is possible to decompose this quantity similarly to its population counterpart. For each horizon h, we define $\overline{M}_h = \frac{1}{N}\sum_i M_{t,t+h}$ as the average SDF and $\overline{C}_h = \frac{1}{N}\sum_i C_{i,t+h}$ as the average cash flow across funds. With these definitions, we can write

$$\text{GPME} = \sum_{h=0}^{H}\overline{M}_h\overline{C}_h + \sum_{h=0}^{H}\overline{M}_h A_h \tag{5}$$

where $A_h = \frac{1}{N}\sum_i (M_{t,t+h}/\overline{M}_h - 1)(C_{i,t+h} - \overline{C}_h)$ is the covariance between normalized SDF and cash flows. In this decomposition, the risk-neutral value is $\sum_{h=0}^{H}\overline{M}_h\overline{C}_h$ and the risk adjustment is $\sum_{h=0}^{H}\overline{M}_h A_h$. Fixing the risk-neutral value requires restrictions on $\overline{M}_h$, and the expectation operator inside condition (3) must be implemented as a cross-sectional mean. As a result, we impose the following sample condition on the SDF:

$$\frac{1}{N}\sum_{i=1}^{N} M_{t,t+h} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{R^{\text{f}}_{t,t+h}} \tag{6}$$

This expression must hold for horizons 1 to H, and it represents H moment conditions. For large h, however, we do not observe all returns on benchmark assets.[5] In that case, we rescale N down to account for the missing observations.

In our GPME decomposition, risk adjustment is determined by risk prices inside the SDF, and the decomposition does not provide clear guidance on how to identify the appropriate risk prices. In this case, we use heuristic rules. For each benchmark asset, b, with risky return $R^{\text{b}}_{t,t+h}$, we impose the following condition:

$$\frac{1}{N}\sum_{i=1}^{N} M_{t,t+h}R^{\text{b}}_{t,t+h} = 1 \tag{7}$$

This expression is the risky counterpart of the risk-free rate condition above. In our

[5]Some funds in our data operate longer than 15 years, so that $H > 60$ quarters. However, we cannot observe returns on benchmark assets at horizon $h = 60$ for funds started in 2009, for example, because that would require knowing returns realized in 2024.

empirical applications, we impose it only for horizon $h = 40$ quarters, or 10 years, which represents the standard horizon of a PE fund. It is possible to impose this condition for every h between 1 and H, and we verify in unreported analysis that our empirical results are robust to that choice.

In summary, we restrict the SDF with conditions (6)-(7) in order to price benchmark assets. With the restricted SDF, we use expression (4) to estimate the NPV of investing in a random PE fund, and decomposition (5) to measure the two sources of value, risk-neutral vs. risk adjustment. This procedure fits into the GMM framework with the complication that sample size varies across moments. Different sample sizes do not affect point estimates, but they complicate the derivation of standard errors on moments and parameters.[6]

For statistical inference, an additional problem is performance correlation between PE funds of close vintages. This correlation can originate from exposure to the same factor shocks, and some of it could remain also after controlling for public factors.[7] To address this problem, Korteweg and Nagel (2016) integrate methods from spatial econometrics in their GMM framework. Below, we illustrate our inference, which is closely related to their method.

To compute standard error of GPME, we ignore uncertainty about SDF parameters, but we allow for correlation between overlapping PE funds. As a start, we measure the economic distance between funds i and k by their degree of overlap. Defining $T(i)$ and $T(k)$ as the last non-zero cash flow dates of fund i and j, we compute their economic distance as follows:

$$d(i,k) \equiv 1 - \frac{\min\{T(i),T(k)\} - \max\{t(i),t(k)\}}{\max\{T(i),T(k)\} - \min\{t(i),t(k)\}} \tag{8}$$

The distance is zero if the overlap is exact, and it is 1 or greater if there is no overlap. This distance is used to construct weights that account for cross-sectional correlation in the sample estimate of the asymptotic variance. Specifically, we estimate the variance of

[6] It is rare to find asymptotic GMM theories allowing for moments constructed with samples of unequal length. An exception is provided by Lynch and Wachter (2013).

[7] Ang, Chen, Goetzmann, and Phalippou (2018), for example, find a PE specific factor which is not spanned by publicly traded factors.

$\sqrt{N}$ GPME as

$$v \equiv \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{N} \max\{1 - d(i,k)/\bar{d}, 0\} \; u_i u_k \tag{9}$$

where $u_i \equiv \text{GPME}_i - \text{GPME}$. In the sum, each product $u_i u_k$ is assigned a weight between 0 and 1, and weights decrease with the distance between two funds. In our empirical work, we set $\bar{d} = 2$, and some non-overlapping pairs of funds still get positive weight. The standard error of GPME is estimated as $\sqrt{v/N}$.

The resulting standard error ignores parameters uncertainty and can be interpreted conservatively as a lower bound. Our primary objective remains obtaining point estimates of GPME that are as economically robust as possible.

2 Stochastic Discount Factor

We focus on applications with exponentially affine SDFs. To illustrate, consider the case with a generic single factor f. The SDF can be written as follows:

$$M_{t,t+h} = \exp\left(a_h - \gamma_h f_{t,t+h}\right) \tag{10}$$

In this expression, a_h and γ_h indicate a pair of parameters per horizon, and γ_h can be interpreted as the risk price of f at horizon h. In absence of other restrictions, this SDF has a total of $2H$ parameters. Korteweg and Nagel (2016) restrict $a_h = ah$ and $\gamma_h = \gamma$, so they work with only 2 parameters. We do not impose any functional form on a_h. This additional flexibility is necessary to satisfy moment conditions (6) and fix the subjective term structure of interest rates with market data. We maintain the restriction on risk prices, $\gamma_h = \gamma$, for two reasons. First, our main argument is about fixing the risk-neutral value of GPME, which is not determined by risk prices, so we maintain this part of the model as simple as possible. Second, we exploit this simplicity to study the robustness of our empirical results with respect to risk prices.

The single-factor form of the SDF can easily be extended with additional factors and corresponding risk prices. Below, we describe the form used in our empirical work.

2.1 CAPM and Long-Term Investors

We consider two risk factors. One factor is the log-return on the market, $r^{\mathrm{m}}_{t,t+h} = \ln(R^{\mathrm{m}}_{t,t+h})$. The other factor is news about future expected returns on the market, often called discount rate (DR) news in the literature. DR news arriving between t and $t+h$ is denoted $N^{\mathrm{DR}}_{t,t+h}$, and is defined as follows:

$$N^{\mathrm{DR}}_{t,t+h} \equiv (E_{t+h} - E_t) \sum_{j=1}^{\infty} \rho^j r^{\mathrm{m}}_{t+h+j} \tag{11}$$

In this expression, ρ is an approximation constant just below 1, and the right-hand side measures cumulative news between t and $t+h$ about market returns from $t+h$ onwards. In a simple model with homoscedastic returns, this factor summarizes variation in investment opportunities, and positive news corresponds to better opportunities (Campbell, 1993).[8]

With the two risk factors, we construct the following SDF:

$$M_{t,t+h} = \exp\left(a_h - \omega\gamma\, r^{\mathrm{m}}_{t,t+h} - \omega(\gamma-1) N^{\mathrm{DR}}_{t,t+h}\right) \tag{12}$$

This is a two-factor version of (10) with risk price $\omega\gamma$ for market return and $\omega(\gamma-1)$ for DR news. Appendix A connects this SDF with theory, and shows that the parameter γ can be interpreted as the investor's relative risk aversion, while ω is the portfolio weight in the market, with $1-\omega$ being invested in the risk-free asset. Throughout our main analysis and unless otherwise specified, we assume $\omega = 1$ representing an investor fully allocated to the market.

This SDF recognizes that the same realized market return implies different marginal utility depending on expected returns. If expected returns are constant, $N^{\mathrm{DR}}_{t,t+h}$ is zero, and the SDF simplifies to a single-factor CAPM model. If expected returns vary over time, $N^{\mathrm{DR}}_{t,t+h}$ appears as an additional risk factor with positive risk price for investors with $\gamma > 1$. These investors are particularly averse to portfolio losses arriving jointly with

[8] Formally, $\rho \equiv 1 - \exp(x)$ where x is the mean of the investor's log consumption-wealth ratio. In our empirical applications, one period corresponds to one quarter, and we set $\rho = 0.95^{1/4}$ corresponding to a mean consumption-wealth ratio of approximately 5% per year.

negative news about expected returns. These losses are permanent in the sense that they are not compensated by higher expected returns, and a risk-averse, long-term investor fears them in particular (Campbell and Vuolteenaho, 2004).

We compare GPME estimates obtained with different restrictions on (12). In one case, we impose $a_h = 0$ and $\gamma = 1$. These restrictions correspond to a log-utility investor and result in the same SDF of Kaplan and Schoar (2005)'s PME. In addition to the log-utility investor, we consider two other types, CAPM and long-term investors, differentiated only by $N^{\text{DR}}_{t,t+h}$, which is zero with CAPM and estimated below for long-term investors. The SDFs of these two investors require estimation of a_h and γ, and we compare our method with that of Korteweg and Nagel (2016). Since Korteweg and Nagel (2016) impose $a_h = ah$, we refer to their method as 'single intercept' and we refer to ours as 'multiple intercepts'.

2.2 A Model of Discount Rate News

To estimate DR news, we follow a large literature starting with Campbell (1991) that models expected market returns using vector autoregression (VAR).[9] We assume that the data are generated by a first-order VAR:

$$x_{t+1} = \mu + \Theta x_t + \varepsilon_{t+1} \tag{13}$$

In this expression, μ is a $K \times 1$ vector and Θ is a $K \times K$ matrix of parameters. Furthermore, x_{t+1} is $K \times 1$ vector of state variables with $r^{\text{m}}_{t+1} - r^{\text{f}}_{t+1}$ as first element, and $\boldsymbol{\varepsilon}_{t+1}$ is a i.i.d $K \times 1$ vector of shocks with variance Σ_{ε}.

In this model, DR news is a linear function of the shocks:

$$N^{\text{DR}}_{t,t+1} = \lambda \varepsilon_{t+1} \tag{14}$$

The vector of coefficients for DR news is defined as $\lambda = \rho e' \Theta (I - \rho\Theta)^{-1}$, where I is the identity matrix and $\boldsymbol{e}' = (1, 0, 0, \ldots, 0)$. Those coefficients measure the long-run

[9] See for example Campbell (1991, 1993, 1996), Campbell and Vuolteenaho (2004), Lustig and Van Nieuwerburgh (2006), Cochrane (2011).

sensitivity of expected returns to each element of x_t.

Combining the VAR model with the definition of multi-period DR news from (11), we obtain the following result:

$$N^{\mathrm{DR}}_{t,t+h} = \lambda\varepsilon_{t+h} + (\lambda - e'\rho\Theta)\,\varepsilon_{t+h-1} + \left(\lambda - e'\rho\Theta - e'\rho^2\Theta^2\right)\varepsilon_{t+h-2} + \dots$$
$$\dots + (\lambda - e'\textstyle\sum_{i=1}^{h-1}\rho^i\Theta^i)\varepsilon_{t+1} \tag{15}$$

This equation expresses multi-period DR news in terms of observables, and it constitutes the empirical specification of the risk factor. In section 3, we obtain two versions of this factor by estimating two VAR models that differ in the choice of state variables.

3 Expected Returns and Discount Rate News

3.1 Public Market Data

The VAR vector x_t contains data about publicly traded assets at a quarterly frequency from 1950 to 2018. The first element of x_t is the difference between the log-return on the value-weighted S&P 500 and the log-return on quarterly T-bills. For this element, data is taken from the Center of Research in Security Prices (CRSP). The remaining elements of x_t are candidate predictors of expected returns and DR news. We consider (1) the log dividend-price ratio, (2) the term premium, (3) a credit spread of corporate bond yields and (4) the value spread. The log dividend-price ratio, term premium, and credit spread are constructed using data from Amit Goyal's website. The log dividend-price ratio is defined as the sum of the last 12 months dividends divided by the current price of the S&P 500. Term premium is the difference between the annualized yield on 10-year constant maturity Treasuries and the annualized quarterly T-bill yield. Credit spread is the difference between the annualized yield on BAA-rated corporate bonds and AAA-rated corporate bonds. For the value spread, we rely on data from Kenneth French's data library. We construct the value spread as the difference in log book-to-market ratio of small-value and small-growth stock portfolios. These portfolios are generated from a double sort on market capitalization and book-to-market ratio.

Table 1 reports summary statistics of public market data in our sample period. From

Panel A, the quarterly log equity premium is 1.6%, corresponding to 6.4% annualy, with a quarterly standard deviation of 8%. Further, our candidate return predictors are highly persistent, especially the log dividend-price ratio with autocorrelation coefficient of 0.982. Panel B reports correlations between contemporaneous and lagged state variables. The first column reports univariate correlations between one-period ahead excess market return ($r_t^{\mathrm{m}} - r_t^{\mathrm{f}}$) and lagged predictors. Market return is positively correlated with lagged dividend-price ratio, credit spread and term premium, and negatively correlated with lagged value spread.

3.2 VAR Estimation

We estimate the VAR model using OLS at a quarterly frequency in the post-war period from 1950 to 2018. We consider two different specifications: (1) a parsimonious specification including only the log dividend-price ratio as predictor and (2) a specification including the full set of predictors.

Table 2 reports the two VAR estimations. Panel A reports the parsimonious DP specification including only the dividend-price ratio, and Panel B reports the full specification. Each row corresponds to an equation in the VAR. The first row of each panel corresponds to the market return prediction equation. Standard errors are reported in brackets and the last two columns report R^2 and F-statistic for each forecasting equation. Panel A shows that dividend-price ratio significantly predicts excess market returns with a coefficient of 0.025. The R^2 is 2.8 percent and the F-statistic is statistically different from zero, consistent with the dividend-price ratio and lagged market return jointly predicting excess market returns. Panel B also includes the value spread, credit spread, and term premium in the VAR. The first row shows that the lagged market return, dividend-price ratio and term premium positively predict excess returns. The coefficients on the dividend-price ratio and term premium are statistically significant at the five percent level. The value spread and credit spread negatively predict market return, although the corresponding coefficients are statistically insignificant.

Table 3 shows properties of one-period DR news, $N_{t,t+1}^{\mathrm{DR}}$, implied by the two VAR estimations. Panel A reports the vector of coefficients, λ, measuring the sensitivity of DR news to each element of ε_t. The "DP only" column shows that shocks to the dividend-price ratio are a significant determinant of DR news. The “Full VAR” column shows that shocks to the dividend-price ratio and term premium are significant determinants of DR news in the full VAR specification. These coefficients, however, do not represent a complete picture of how much each variable affects DR news; they do not account for the fact that elements of ε_t have different variances. We therefore decompose the unconditional variance of DR news to compare the importance of shocks to different variables.

Panel B of Table 3 decomposes the variance of DR news, $\lambda\Sigma_{\varepsilon}\lambda'$, into variance contributions from each variable's shock. The column “DP only” reports the decomposition for the parsimonious VAR. In this specifications, 105% of DR news variance originates from shocks to the dividend-price ratio and negative 5% stems from the lagged market return. Shocks to DR news are almost exclusively determined by shocks to the dividend-price ratio. The “Full VAR” column shows that the dividend-price ratio is the largest contributor to DR news variance also in the full specification. Even in this specification, the dividend-price ratio contributes nearly 100% percent of the variance. The value spread and term premium contribute only 3% and 4%, respectively, while the credit spread's contribution is essentially zero. These results suggest that both specifications rely almost exclusively on shocks to the dividend-price ratio to determine DR news.

4 Private Equity Performance

4.1 Funds Data

We analyze PE data maintained by Burgiss. Our final sample contains net-of-fees cash flows of 1866 PE funds started in the US between 1978 and 2009, and divided in three mutually exclusive categories: buyout, venture capital, and generalist funds.

Burgiss provides at least two levels of classification for each fund. At the most general level (Tier 1), funds are primarily classified as ‘equity’, ‘debt’, or ‘real assets’. We focus exclusively on equity. At a more detailed level (Tier 2), we distinguish between equity funds classified as buyout, venture capital, and generalist. We define buyout and venture capital funds using the homonymous Tier 2 classes. In our generalist category, we include funds with Tier 2 classification of ‘generalist’, ‘expansion capital’, ‘unknown’, and ‘not elsewhere classified’.

To obtain our final sample, we exclude funds with less than 5 million USD of commitment from the raw data. We also exclude funds whose majority of investments are not liquidated by 2019. For that, we impose two conditions. First, we only include funds of vintage year 2009 or earlier. Second, among those funds, we exclude those with a ratio of residual NAV over cumulative distributions larger than 50%. Finally, we normalize cash flows and residual NAV by each fund's commitment.

Figure 1 plots the aggregate sum of normalized contributions, distributions, and net cash flows for the three fund categories over time. For all the categories, our sample constitutes mostly of cash flows observed between 1995 and 2019. For venture capital, we see uniquely large distributions in year 2000, corresponding to the dot-com bubble; those distributions are almost 10 times larger than distributions and contributions observed at any other time.

Table 4 summarizes our PE data. Panel A reports descriptive statistics and shows that the sample consists of 652 buyout funds, 971 venture capital funds and 243 generalist funds. The median (average) fund size is $421 ($1099) million for buyout, $126 ($222) million for venture capital and $225 ($558) million for generalist funds. The average number of years between the first and last buyout fund cash flow is 14.16 years, 15.52 years for venture capital, and 14.42 years for generalist funds. The average number of cash flows per fund is approximately 36 for buyout, 28 for venture capital and 33 for generalists. The sample includes 311 unresolved buyout funds with an average NAV-to-Distributions ratio of 0.10, 353 unresolved venture capital funds with NAV-to-Distributions ratio of 0.14, and 88 unresolved generalist funds with a NAV-to-Distributions ratio of 0.12.

Panel B of Table 4 reports Total Value to Paid-In ratios (TVPI) across vintage years.[10] Across the three categories, TVPIs fluctuates over time and are typically higher for earlier vintages. Furthermore, venture capital shows peculiarly high TVPIs between 1993 and 1996. These large multiples can be ascribed, at least in part, to funds of these vintages deploying capital in the period leading up to the 2001 dot-com bubble and exiting investments before 2001.

[10] If there are less than five funds per vintage year, figures are omitted due to confidentiality.

4.2 Buyout

Table 5 reports GPME estimation for buyout funds. The first estimation corresponds to "Log-Utility" and uses the inverse return on the market as SDF. The resulting GPME is a reformulation of Kaplan and Schoar (2005)'s PME defined as the sum of discounted cash flows rather than the ratio of discounted distributions over contributions. The log-utility GPME is 0.20 and significantly different from zero at the one percent level; buyout funds provide log-utility investors with 20 cents of abnormal profits per dollar of committed capital.

Other than the log-utility model, Table 5 reports GPME estimation for CAPM and long-term (LT) investors assuming they are fully invested in the market ($\omega = 100\%$). The "Single Intercept" columns estimate only one intercept parameter a, with $a_h = ah$, and the estimations use the moment conditions of Korteweg and Nagel (2016). The "Multiple Intercepts" columns use our method as described in Section 1, which does not impose a functional restriction on a_h and estimates multiple intercepts linking the subjective term-structure of interest rates to market data.

4.2.1 Buyout Value for CAPM Investors

Table 5 shows a GPME of 0.28 for the CAPM SDF using a single intercept. The estimate is statistically different from zero at the ten percent level. With multiple intercepts, The GPME is 0.30 and statistically significant at the one percent level. These numbers are close, and they imply that buyout funds provide CAPM investors with 28-30 cents of NPV per dollar of committed capital. CAPM investors thus derive 8-10 cents more value than log-utility investors from a marginal allocation to buyout funds.

Even though GPME estimates for CAPM investor are similar across the two methods, we show below that the different restrictions placed on the single and multiple intercepts specifications imply markedly different SDF properties. To further explore differences between the two methods, Table 5 also reports the two GPME components from the decomposition of Section 1. The first component is $\sum_h \overline{M}_h \overline{C}_h$ and corresponds to the risk-neutral part of GPME. For each horizon, we take the average cash flow across funds and discount it with the average SDF across funds. We then sum over horizons. The second component is $\sum_h \overline{M}_h A_h$ and corresponds to the total risk adjustment inside GPME.

For the CAPM investor, the risk-neutral component is 0.61 with multiple intercepts and 0.79 with a single intercept. Risk adjustment, instead, is -0.31 with multiple intercepts and -0.51 with a single intercept. In this case, the single intercept specification achieves similar GPME estimate by assigning higher risk-neutral value, but also more negative risk adjustment, relative to the multiple intercepts case. Differences between the two methods become more evident considering long-term investors below.

4.2.2 Buyout Value for Long-Term Investors

In Table 5, the "LT" columns report GPME estimations for long-term investors. In particular, the "LT (DP)" columns use the VAR specification with only the dividend-price ratio as return predictor to measure DR news. With a single intercept, the LT (DP) investor assigns a GPME of 0.80 to buyout. This point estimate is considerably higher relative to the CAPM investor, but it is not statistically significant due to the large standard error. Further, a NPV of 80 cents per dollar of commitment is large enough to appear economically implausible, and our decomposition shows that this GPME results from summing a risk-neutral value of 2.50 with a risk adjustment of -1.69. Compared to the CAPM, this large value is not generated by a change in risk adjustment. Instead, it comes from a large increase of the risk-neutral component. With multiple intercepts, we estimate a GPME of 0.35 for the LT (DP) investor. This point estimate is statistically significant, and it is only marginally higher compared to CAPM. By construction, the difference relative to CAPM is entirely due to risk adjustment.

The "LT (Full)" columns of Table 5 use DR news computed with the full VAR specification which includes the dividend-price ratio, term premium, credit spread, and value spread as return predictors. With single intercept, the resulting GPME is 0.41, although not statistically significant, and substantially lower than 0.80 obtained in the LT (DP) case. With multiple intercepts, GPME is 0.34, statistically significant at the one percent level, and close to the 0.35 estimated in the LT (DP) case. While the single intercept methodology suggests that the two VAR specifications result in DR news which generate large GPME differences, the multiple intercepts method suggests similar GPME implications of the two VAR specifications. In Section 3, we show that both VAR estimations generate DR news that vary almost exclusively from shocks to the dividend-price ratio, suggesting that the full VAR might have similar dynamics to the parsimonious VAR.

Consistent with this interpretation, the multiple intercepts method estimates virtually identical GPMEs for LT (DP) and LT (Full) investors.

4.2.3 Time Discounting of Buyout Funds

Figure 2 plots the average SDF across funds as a function of horizon. At each horizon h, the average SDF measures the present value of one dollar paid for certain at that horizon by all funds in the sample. The figure compares single intercept and multiple intercepts specifications from Table 5. By construction, multiple intercepts estimations using the same sample imply the same average SDF as a function of horizon. Single intercept estimations impose less structure to the SDF, and the average SDF at each horizon varies depending on the risk factors considered.

The figure shows unrealistic time discounting for the single intercept estimation with a peculiar pattern of negative time discounting in the first 3 years, positive discounting from year 4 to 8, and negative discounting again from year 8 to 11. This pattern is qualitatively consistent across investors and it is quantitatively strongest for the LT (DP) model, suggesting that the very large GPME estimate obtained with this model might be due to this implausible time discounting pattern resulting from the single intercept method.

With multiple intercepts, Figure 2 shows that time discounting is consistently positive and stable across horizons, and this result corresponds to more stable GPME estimates across models, as shown in Table 5. It also corresponds to lower variation of GPME_i across funds, as we show below.

4.2.4 Cross-Sectional Variation of Performance

Table 6 summarizes the cross-sectional distribution of GPME_i resulting from the different estimations. The table contains results for all three fund categories. We focus primarily on buyout, and a similar discussion applies for venture capital and generalist funds studied below. For each estimation, we report the mean of GPME_i, which corresponds to the GPME estimates of Table 5. Below the mean, we report the standard deviation and selected percentiles of the GPME_i distribution.

Differences in the distribution of GPME_i are interesting because the multiple intercepts estimations, just like the single intercept ones, restrict the SDF using exclusively public market data, and ignoring any information about PE cash flows. Thus, differences in the GPME_i distribution are a result which is not imposed by construction. We find that the multiple intercepts estimations imply consistently lower variation of GPME_i across funds, relative to single intercept.

For buyout, the log-utility model generates the lowest standard deviation of of GPME_i, equal to 0.64. The single intercept CAPM model implies a standard deviation of 1.42, while the multiple intercepts CAPM model implies a standard deviation of 0.98. Thus, the multiple intercepts model generates substantially lower standard deviation with CAPM, even though the two models have similar mean (0.28 vs. 0.30). Further, the lower standard deviation of the multiple intercepts CAPM model comes with less extreme tail observations as indicated by the reported percentiles. For long-term investors, we see qualitatively similar differences between the single intercept and multiple intercepts methods, with more extreme magnitudes. Especially for the LT (DP) investor, the GPME_i standard deviation of the single intercept model is extremely high, 9.25, relative to 1.05 obtained with multiple intercepts model.

4.2.5 Components of Buyout Performance across Horizons

With multiple intercepts, we take the GPME decomposition one step further. Not only do we decompose GPME in a risk-neutral part and a risk adjustment, but we also decompose the risk-neutral part and the risk adjustment based on the contribution of each horizon. To illustrate, we decompose the risk-neutral part as follows:

$$\sum_{h=0}^{H} \overline{M}_h \overline{C}_h = \overline{M}_0 \overline{C}_0 + \sum_{h=1}^{4} \overline{M}_h \overline{C}_h + \sum_{h=5}^{8} \overline{M}_h \overline{C}_h + \cdots + \sum_{h=53}^{56} \overline{M}_h \overline{C}_h + \sum_{h=57}^{H} \overline{M}_h \overline{C}_h \tag{16}$$

These components correspond to values coming from year 0, year 1, year 2, ..., year 14, and year 15 or higher. A similar decomposition is done for risk adjustment.

Figure 3 plots the resulting GPME components against horizon for selected models with multiple intercepts. The figure focuses on CAPM and LT (DP) models. By construction, the risk-neutral component from each horizon (grey bars) is identical across models, and differences originate exclusively from components of risk adjustment plotted as black bars for CAPM and white bars for LT (DP).

Decomposing the risk-neutral part, the grey bars in Figure 3 show the "J-curve" typical of PE cash flows.[11] Investors contribute capital primarily in the first 4 years, corresponding to negative average cash flows at short horizons. Average cash flows turn positive from year 5, as funds distribute capital.

Decomposing risk adjustment, Figure 3 shows that CAPM and LT (DP) models are similar not only on the overall risk adjustment but also on its components across horizons. Surprisingly perhaps, risk adjustment is moderately positive in the first years of fund operations and turn negative only after the third year. At short horizons, net cash flows are dominated by contributions, and a positive risk adjustment suggest that buyout funds tend to call less capital in bad times with high SDF realizations. This tendency decreases risk and has small but positive effect on GPME. Further, this result is consistent with Robinson and Sensoy (2016), who also find pro-cyclicality in contributions.

The components of risk adjustment turn negative at longer horizons, after year 3, and they are most negative between year 9 and 11. Interestingly, risk adjustment is small for years 6 to 8, even though average cash flows are high during those years. This result appears consistent with Gupta and Van Nieuwerburgh (2021) finding that buyout funds generate cash flows that appear to be risk-free in part of their harvesting period.

4.3 Venture Capital

Table 7 reports GPME estimations for venture capital funds. As a starting point, we estimate a log-utility GPME of 0.14 and statistically indistinguishable from zero. For comparison, Korteweg and Nagel (2016) find a marginally positive log-utility GPME of 0.05 for venture capital funds. The higher GPME in our sample might come from a larger number of funds in pre-1998 vintages. Historically, those vintages have high risk-adjusted performance for venture capital.

Considering the GPME decomposition for log-utility, venture capital has risk-neutral

[11] Grey bars represent the risk-neutral present value of average cash flows at each horizon. Instead, the J-curve is typically plotted as the average cash flows at each horizon without discounting. Nonetheless, the two quantities are close, especially at horizons shorter than 10 years.

value of 0.48 and risk adjustment of -0.35. Compared to buyout, this risk adjustment is considerably larger (-0.35 vs. -0.09). The log-utility model has constant risk price of $\gamma = 1$ across samples, and differences in risk adjustment are entirely due to different covariance between cash flows and market returns. Thus, higher risk adjustment for venture capital suggests higher market exposure of venture capital's cash flows relative to buyout. Larger risk adjustment for venture capital is consistent with Driessen, Lin, and Phalippou (2012), who estimate a market beta of 2.4 for venture capital and 1.3 for buyout, and with Ang et al. (2018), who estimate a market beta 1.8 for venture capital and 1.2 for buyout.

4.3.1 Venture Capital for CAPM and Long-Term Investors

In Table 7, the GPME estimate for the CAPM investor is -0.15 with single intercept and 0.07 with multiple intercepts. Both methods indicate that CAPM implies lower GPME relative to log-utility, and this qualitative difference is consistent with the findings of Korteweg and Nagel (2016) in a smaller sample. The single intercept and multiple intercepts methods disagree on the GPME sign and magnitude, however, and multiple intercepts result in marginally positive GPME for venture capital.

Differences in GPME between single intercept and multiple intercepts can arise from differences in the SDF intercepts, a_h, but also from differences in the estimated risk price, γ. Table 7 shows that CAPM's risk price is 2.93 with single intercept and 2.03 with multiple intercepts. We show in Section 5.1 that differences in γ do not entirely explain this GPME difference between the two methods, as the GPME of the CAPM investor with multiple intercepts remain higher relative to single intercept even assuming the same risk price of 2.93 for both models.

For the long-term investor, we observe qualitatively similar differences between single and multiple intercepts. With single intercept, we estimate negative GPMEs of -0.19 for LT (DP) and -0.08 for LT (Full). With multiple intercepts, we estimate positive GPMEs of 0.13 for LT (DP) and 0.15 for LT (Full). In Section 5.1, we also show that this difference is not entirely explained by lower risk prices with multiple intercepts.

Comparing GPMEs between CAPM and long-term investors, we find that the long-term investor assigns higher value to venture capital, relative to the CAPM investor, with multiple intercepts. GPME estimates with multiple intercepts under LT (DP) and LT (Full) are almost double the GPME estimate under CAPM. Considering the single intercept method, we find more stable performance across investors for venture capital relative to buyout. To investigate this result, we plot the implied discounting of the different venture capital estimations.

For venture capital, Figure 4 plots the cross-sectional average SDF as a function of horizon. As we do for buyout, the figure distinguishes between multiple intercepts and the three models with single intercept. This figure confirms that multiple intercepts imply a more stable time discounting across horizons. The single intercept method implies qualitatively similar time discounting between buyout and venture capital estimations. With venture capital, however, time discounting of the single intercept method does not vary as widely across the three models.

4.3.2 Components of Venture Capital Performance across Horizons

Following the discussion of buyout results, we also decompose GPMEs by horizon for venture capital. Figure 5 plots the decomposition for CAPM and LT (DP) models with multiple intercepts. We focus on the decomposition of risk adjustment.

In the figure, risk adjustment varies similarly with horizon across the two models. For both models, risk adjustment is marginally positive from year 0 to 2. This result suggests that contributions tend to be slightly pro-cyclical, and it is similar to buyout funds although quantitatively smaller. Starting from year 3, risk adjustment turns negative, and it is most important between year 4 and 7. This result contrasts with buyout, whose risk adjustment tends to be small especially in year 6 and 7. Distributions of venture capital funds show substantially different risk across horizons, relative to buyout.

4.4 Generalist

Our analysis of generalist funds is similar to that of buyout and venture capital. Here we provide an overview of the results.

Table 8 shows the results of our GPME estimations for generalist funds. With log-utility, we estimate a statistically significant GPME of 0.16, which is lower relative to buyout and marginally higher relative to venture capital. For CAPM and long-term investors, we estimate consistently positive GPME with single intercept and multiple intercepts methods. GPME estimates are higher and more stable across estimations with multiple intercepts relative to single intercept.

Figure 6 shows differences in time discounting across methods plotting the average SDF by horizons implied by the different estimations. Qualitatively, the figure shows results similar to buyout and venture capital. With multiple intercepts, time discounting is positive, constant across investors, and stable across horizons. With single intercept, we observe time discounting being negative in the first 3 years, positive in the next 5 years, and negative again in the next 3 years.

Figure 7 plots the risk-neutral value and risk adjustment components by horizon. From year 0 to 2, the figure shows risk adjustment similar to the other fund categories and consistent with contributions hedging some risk for PE investors. From year 3 onwards, risk adjustment turns negative. Compared to the other categories, risk adjustment of generalist fund can be attributed more homogeneously to cash flows received from year 4 to 10.

5 Robustness

In this section, we focus exclusively on the method with multiple intercepts, and we explore the robustness of our results with respect to two parameters. First, we study the sensitivity of GPME as we exogenously change risk aversion, γ. Second, we estimate GPMEs for investors whose portfolio weight in the market is either $\omega = 50\%$ or 200%, as opposed to 100% in Section 4.

5.1 Risk Aversion

Our GPME decomposition does not provide clear guidance on how to estimate risk prices for proper benchmarking of PE cash flows. As described in Section 1, our method identifies risk prices by constraining the SDF to price risky benchmark returns at a 10 year horizon. This is a heuristic approach based on the typical horizon of PE funds, and we study the sensitivity to this heuristic by changing risk prices exogenously. Specifically, we change risk aversion, γ, since risk prices are primarily determined by this parameter for our investors.

As we change risk aversion exogenously, we do not need to run new estimations. Instead, the resulting GPME with multiple intercepts can be computed as follows:

$$\text{GPME}(\gamma) = \sum_{h=1}^{H} \left(\frac{1}{N} \sum_{i=1}^{N} \frac{1}{R^{\text{f}}_{t,t+h}} \right) \left(\overline{C}_h + A_h(\gamma) \right) \tag{17}$$

To obtain this expression, we rewrite the GPME decomposition (5) using time discounting restrictions: $\frac{1}{N}\sum_{i=1}^{N} M_{t,t+h} = \frac{1}{N}\sum_{i=1}^{N} 1/R^{\text{f}}_{t,t+h}$. We use this notation to highlight that A_h is the only term of GPME affected by risk prices. Further, A_h is the only term affected by the SDF, but it does not depend on intercept parameters.[12]

Using expression (17), we compute GPMEs with risk aversion between 1 and 12 for each type of investor in each fund category. Figure 8 plots the resulting GPMEs as a function of γ. For each category, the three lines correspond to different investors. The solid line represents CAPM, the dotted line represents LT (DP), and the dash-dotted line represents LT (Full). Further, there are two circles over each line. The black circle represents the combination of GPME and γ estimated for that investor with multiple intercepts in Table 5, Table 7, or Table 8, depending on the category. For comparison, the white circle corresponds to γ estimated with single intercept and GPME computed with expression (17).

The top-left panel of Figure 8 shows results for buyout. For the CAPM investor, GPME ranges from 0.5 to 0.1, it is monotonically decreasing in risk aversion, and it remains positive even at risk aversion of 12. For long-term investors, the LT (DP) and LT (Full) models imply approximately the same GPME across all levels of risk aversion. For these investors, GPME ranges from 0.5 to 0.3, and it is non-monotonic in risk aversion. Overall, we find robustly positive performance of buyout funds, and only moderate sensitivity to risk prices.

The top-right panel of Figure 8 shows results for venture capital. As opposed to buyout,

[12]To see why A_h does not depend on intercept parameters, recall that $A_h = \frac{1}{N}\sum_i (M_{t,t+h}/\overline{M}_h - 1)(C_{i,t+h} - \overline{C}_h)$ with $\overline{M}_h = \frac{1}{N}\sum_i M_{t,t+h}$. The SDF enters A_h only through its normalized form, $M_{t,t+h}/\overline{M}_h$, and intercepts cancel out because of the normalization.

we find high sensitivity of venture capital's GPME to risk prices. This sensitivity is highest for the CAPM investor, whose GPME estimate goes from 0.35 to -0.25 as risk aversion increases. For long-term investors, GPME shows marginally lower sensitivity, ranging from 0.35 to -0.15. Across all investors, we find that venture capital's GPME is most sensitive to risk aversion in the range of risk aversion between 1 and 3, which contains the three point estimates of γ with multiple intercepts from Table 7.

In the bottom panel of Figure 8, we report sensitivity results for generalist funds. Similarly to buyout, the GPME of generalist funds display only moderate sensitivity to risk prices, and it remains positive for all investors at all levels of risk aversion between 1 and 12. GPME ranges from 0.35 to 0.05 for the CAPM investor, and from 0.35 to 0.15 for long-term investors.

Across investors and fund categories, we find tendency for GPME to decrease in risk aversion, especially for risk aversion between 1 and 5, which is typically the most relevant range. For buyout and generalist funds, we find quantitatively modest GPME sensitivity to risk prices, and GPME remains positive across a wide range of risk aversion. For venture capital, instead, we find high sensitivity of GPME to risk prices, with positive GPME for risk aversion below 2 and negative GPME for risk aversion above 3. Because of this high sensitivity, venture capital seems the most problematic category to evaluate.

5.2 Investor Leverage

An additional way to compare CAPM and long-term investors is by looking at the effect of investor's leverage on GPME. A natural measure of leverage in our model is the portfolio weight in the market, ω, and while we assume $\omega = 100\%$ in most of the paper, here we consider two different values representing a conservative investor with low leverage ($\omega = 50\%$) and an aggressive investor with high leverage ($\omega = 200\%$).

After further inspection, SDF expression (12) presented in Section 2 suggests two considerations about leverage. First, CAPM investors with different leverage assign the same GPMEs. For those investors, ω enters the SDF only in the product $\omega\gamma$ that determines market risk price, and it is a redundant parameter. For CAPM investors with higher leverage, our GPME estimation will mechanically result in proportionally lower risk aversion.

Second, long-term investors with different leverage can assign different GPMEs. For long-term investors, ω affects the importance of market risk price, $\omega\gamma$, relative DR news risk price, $\omega(\gamma - 1)$. Since $\gamma > \gamma - 1$, risk from DR news is less important for aggressive investors with large ω, and if DR news matters when evaluating PE, leverage can affect GPMEs.

In Table 9, we report GPME estimations similar to the multiple intercepts part of Table 5, Table 7, and Table 8, except that we do not assume $\omega = 100\%$. Instead, we assume $\omega = 50\%$ in first part of the table and $\omega = 200\%$ in the second part. For CAPM investors, the table confirms that leverage has no effect on GPME, and higher leverage is mechanically offset by lower risk aversion. For long-term investors, we find some differences in GPME across leverage. The conservative long-term investor assigns GPME of 0.35 to buyout, 0.25 to generalist, and in the 0.15-0.20 range to venture capital. The aggressive long-term investor, instead, assigns GPME of 0.33 to buyout, 0.22 to generalist, and 0.07 to venture capital. Thus, the conservative investor assigns higher value to PE across all three fund categories, and by construction, these differences are entirely due to different risk adjustments. Quantitatively, however, these GPME differences across leverage seem largely negligible at least for the case of buyout and generalist funds.

Overall, we estimate that all three fund categories provide positive values to both CAPM investors and long-term investors across a wide range of leverage levels.

6 Conclusion

PE funds are illiquid investments whose true fundamental return is typically unobservable. Since investment returns are unobservable, risk and performance cannot be estimated with standard approaches, and the literature has developed methods to evaluate these investments by discounting funds' cash flows with SDFs. In this paper, we show that existing SDF methods for the valuation of PE funds result in unrealistic time discounting, which can generate implausible performance estimates. We propose a modified method and compare it to existing ones.

Theoretically, our approach is based on a standard asset pricing decomposition of PE performance in a risk-neutral part and a risk adjustment. We fix the risk-neutral part by constraining the SDF such that the subjective term structure of interest rates is determined by market data. By construction, the risk-neutral part does not vary as we add or remove risk factors from the SDF, so we can meaningfully measure the economic cost of PE risk and compare it across models. Empirically, we evaluate our approach against existing methods, and find that our approach results in more stable PE performance across models and lower variation of performance across funds.

We use our method to measure PE performance and risk adjustment for two types of investors: a CAPM investor, and a long-term investor who distinguishes between permanent and transitory wealth shocks. We discount net-of-fees cash flows of 1866 PE funds started in the US between 1978 and 2009, and divided in three categories: buyout, venture capital, and generalist. We find largely negligible differences between the two investors, especially for buyout and generalist funds. Overall, we find positive performance of buyout, generalist, and venture capital funds. For venture capital, however, high risk exposure makes performance estimates particularly sensitive to estimated risk prices.

Our performance decomposition does not provide clear guidance on how to estimate risk prices for proper benchmarking of PE cash flows. Because of this, we rely on heuristic SDF restrictions, and we study the sensitivity of performance estimates with respect to risk prices. The open issue on the estimation of risk prices, among others, is left to future research.

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Figure 1
Aggregate Cash Flows

This figure plots aggregate (normalized) contributions, distributions, and net cash flows for the three fund categories. Buyout corresponds to the top-left, venture capital is in the top-right, and generalist at the bottom. The blue area represents distributions, the red area represents contributions, and the solid line represents net cash flows. The grey shaded areas correspond to NBER recessions.

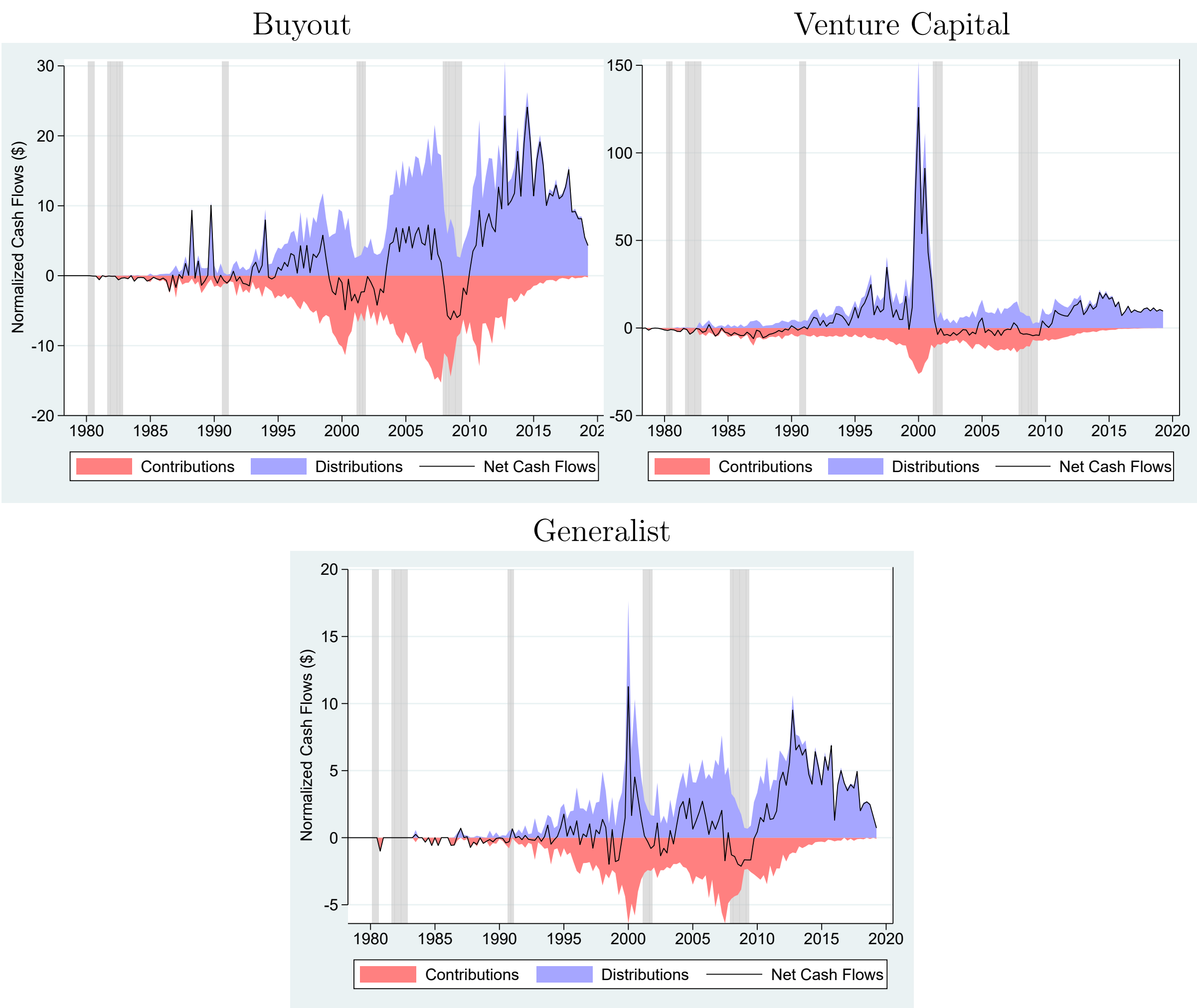

Figure 2

Time Discounting for Buyout

This figure plots average multi-period SDF, $\overline{M}_h = \frac{1}{N}\sum_i M_{t,t+h}$, across buyout funds every quarter. We consider different SDFs resulting from the estimations of Table 5.

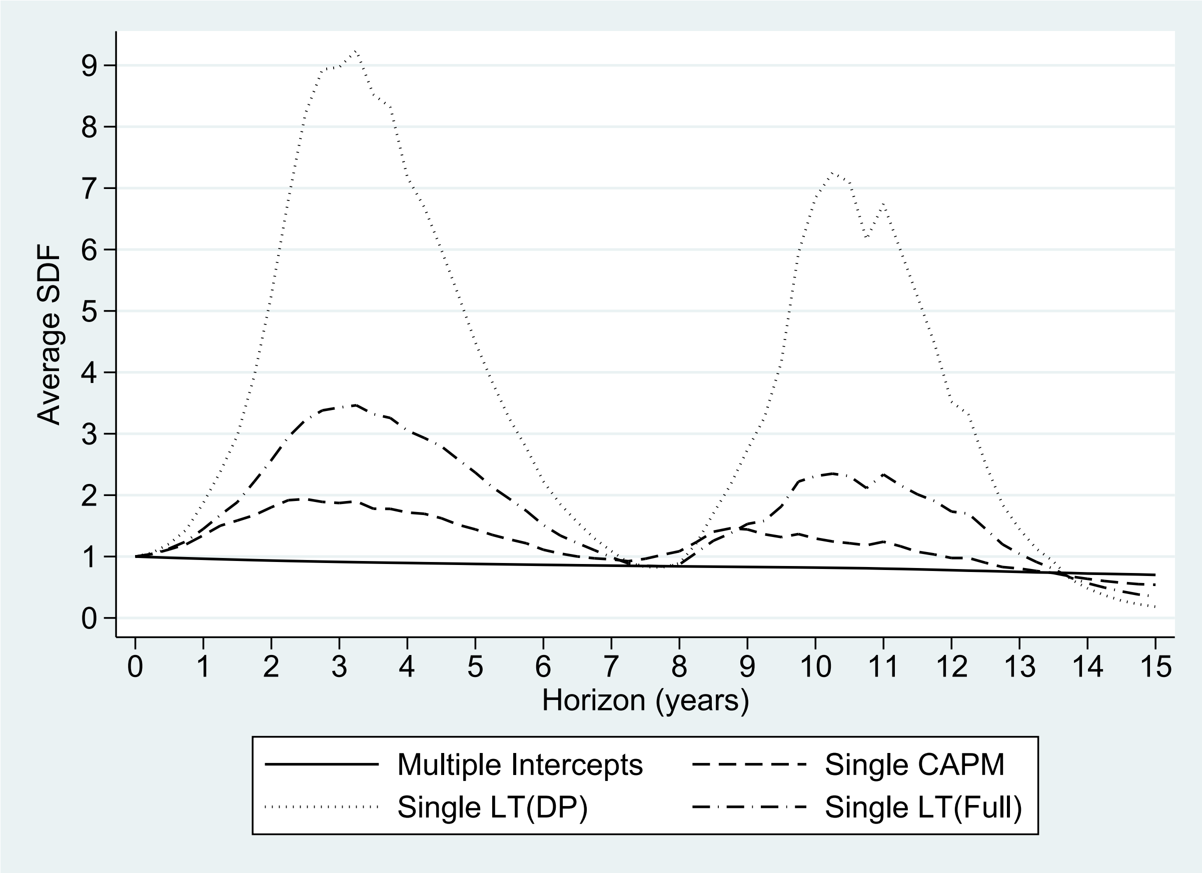

Figure 3

GPME Decomposition for Buyout

In this figure, we decompose GPMEs estimated in Table 5 for the CAPM and LT(DP) models with multiple intercepts. Discounted Value of Avg. Cash Flow is $\overline{C}_0$ in year 0, $\sum_{h=1}^{4}\overline{M}_h\overline{C}_h$ in year 1, $\sum_{h=4y-3}^{4y}\overline{M}_h\overline{C}_h$ in year y between 2 and 14, and $\sum_{h=57}^{H}\overline{M}_h\overline{C}_h$ in year 15. By construction, Discounted Value of Avg. Cash Flow is the same across the two models. Discounted Value of Risk is 0 in year 0, $\sum_{h=1}^{4}\overline{M}_h A_h$ in year 1, $\sum_{h=4y-3}^{4y}\overline{M}_h A_h$ in year y between 2 and 14, and $\sum_{h=57}^{H}\overline{M}_h A_h$ in year 15.

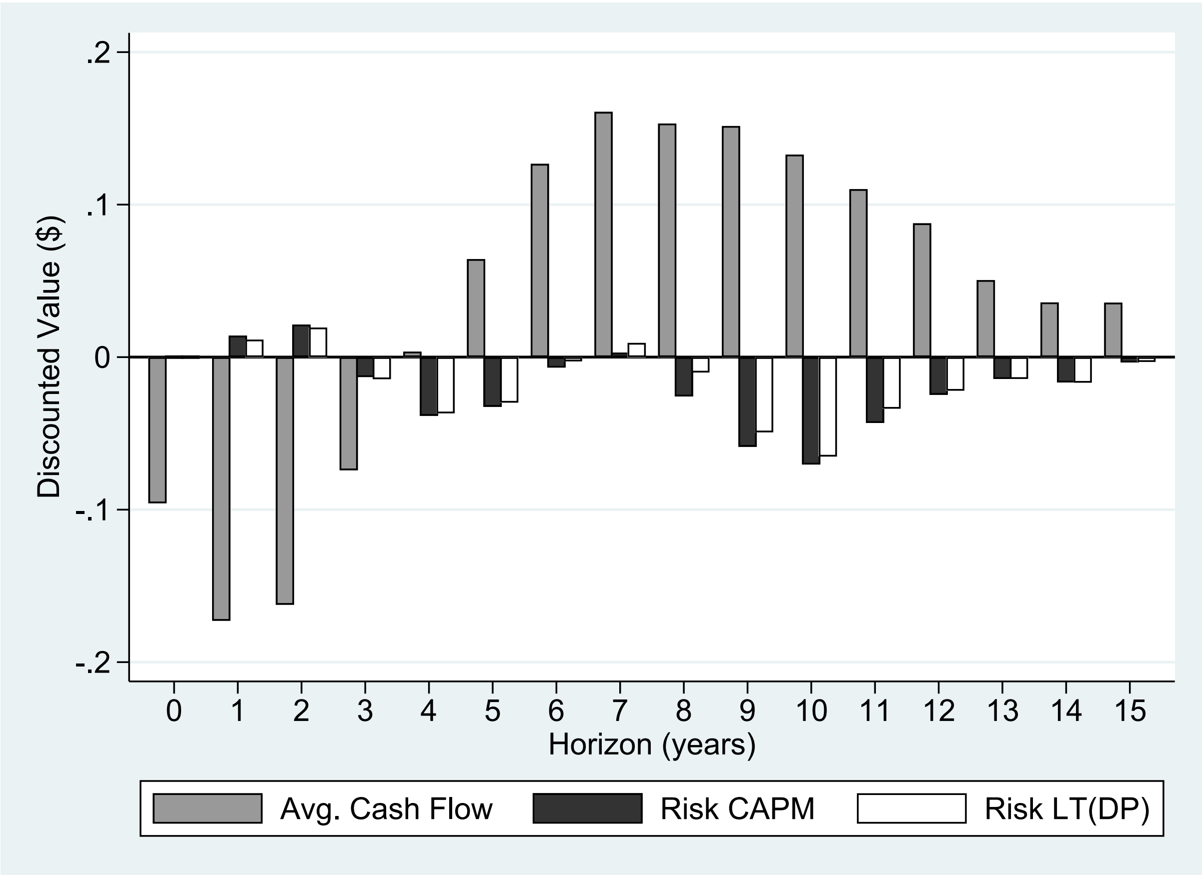

Figure 4

Time Discounting for Venture Capital

This figure plots average multi-period SDF, $\overline{M}_h = \frac{1}{N}\sum_i M_{t,t+h}$, across venture capital funds every quarter. We consider different SDFs resulting from the estimations of Table 7.

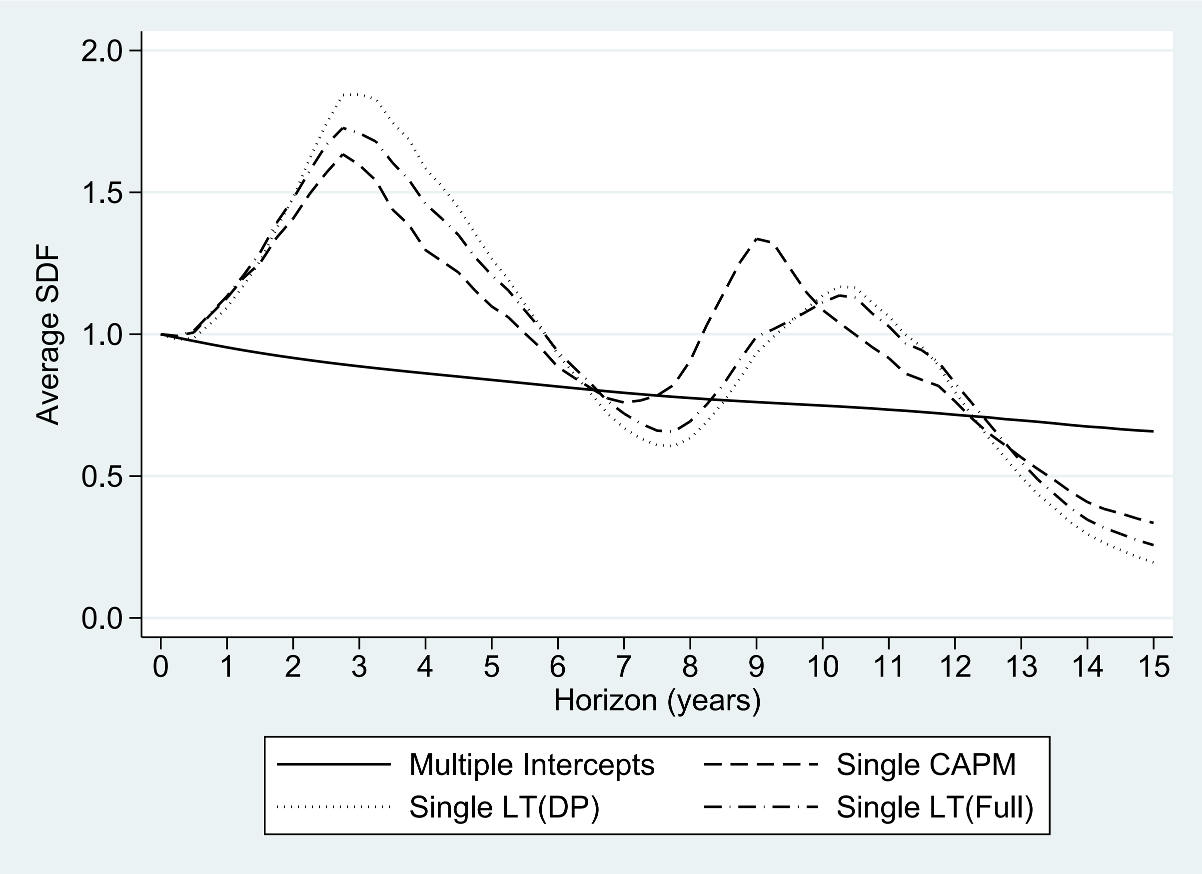

Figure 5

GPME Decomposition for Venture Capital

In this figure, we decompose GPMEs estimated in Table 7 for the Multiple Intercepts CAPM and LT(DP) models. The plot is constructed as in Figure 3.

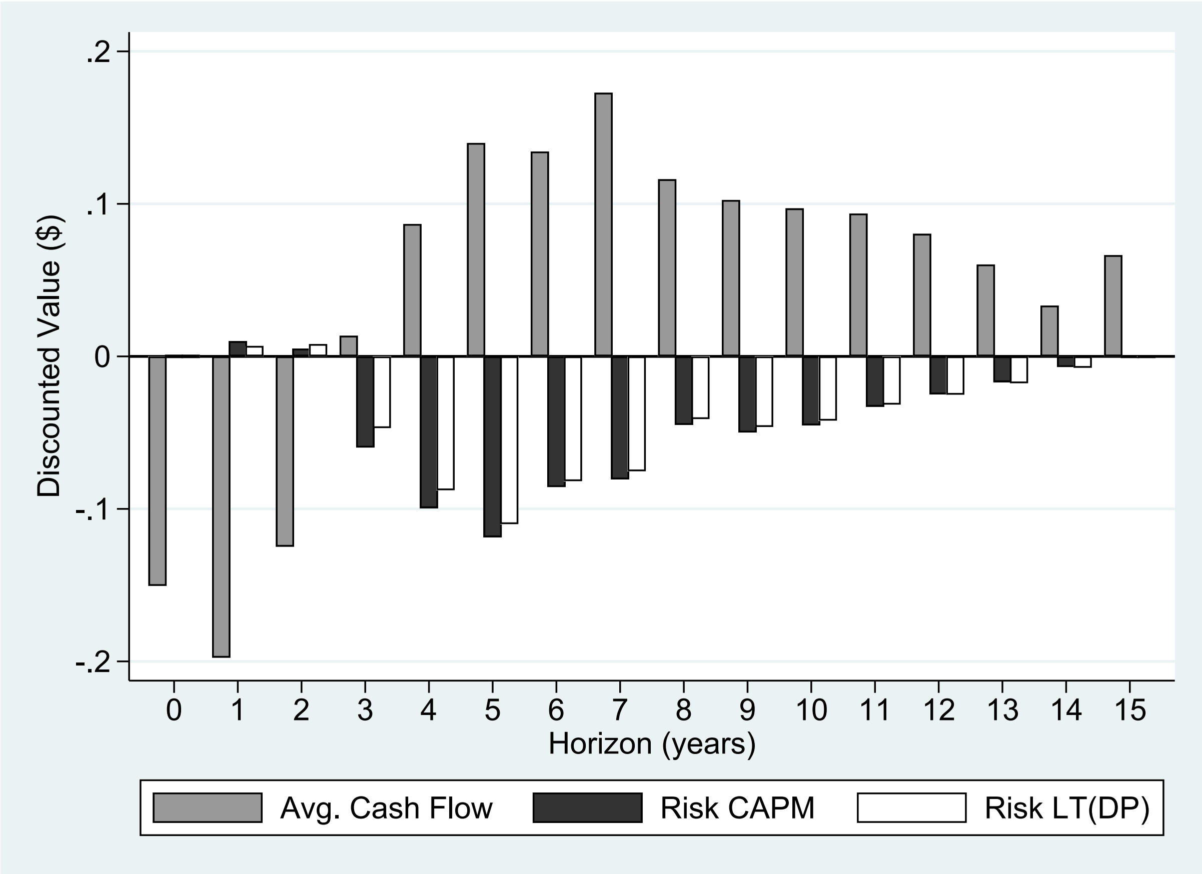

Figure 6

Time Discounting for Generalist

This figure plots average multi-period SDF, $\overline{M}_h = \frac{1}{N}\sum_i M_{t,t+h}$, across generalist funds every quarter. We consider different SDFs resulting from the estimations of Table 8.

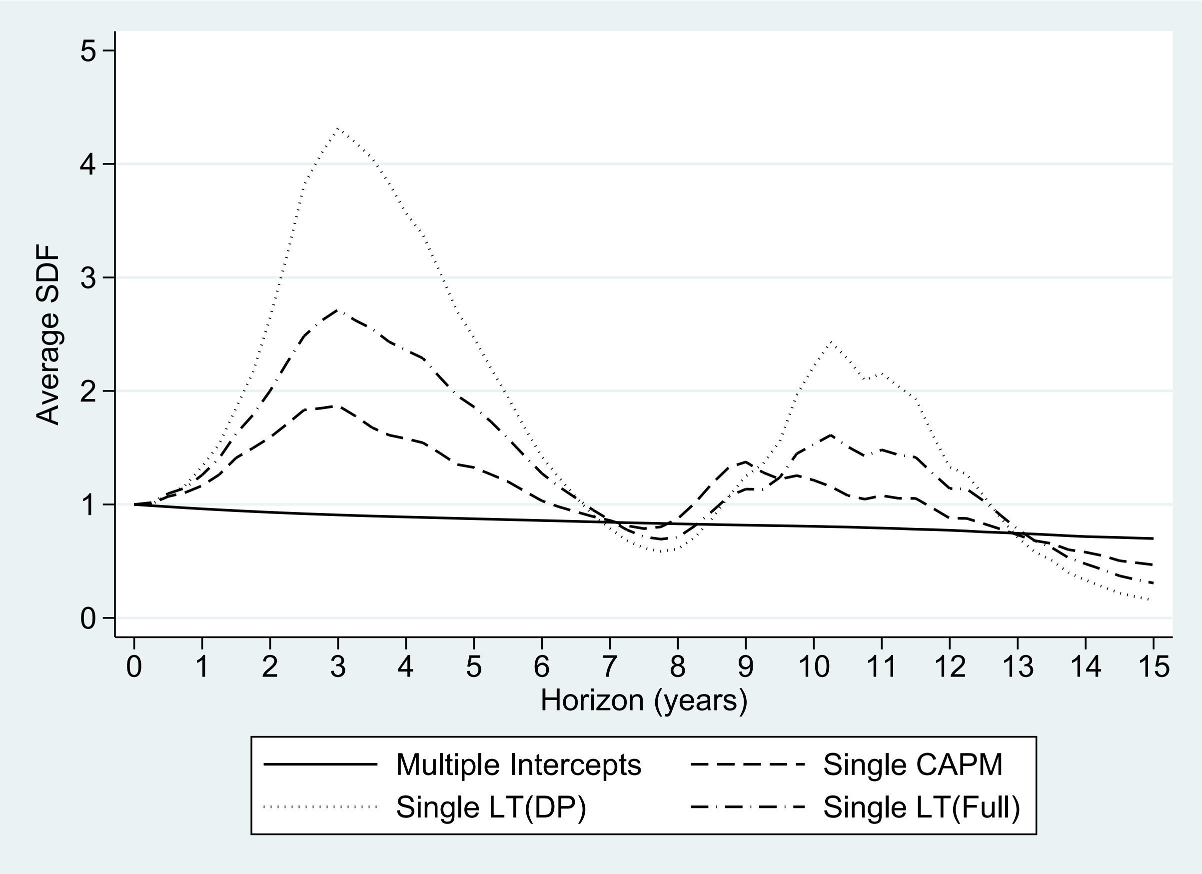

Figure 7
GPME Decomposition for Generalist

In this figure, we decompose GPMEs estimated in Table 8 for the Multiple Intercepts CAPM and LT(DP) models. The plot is constructed as in Figure 3.

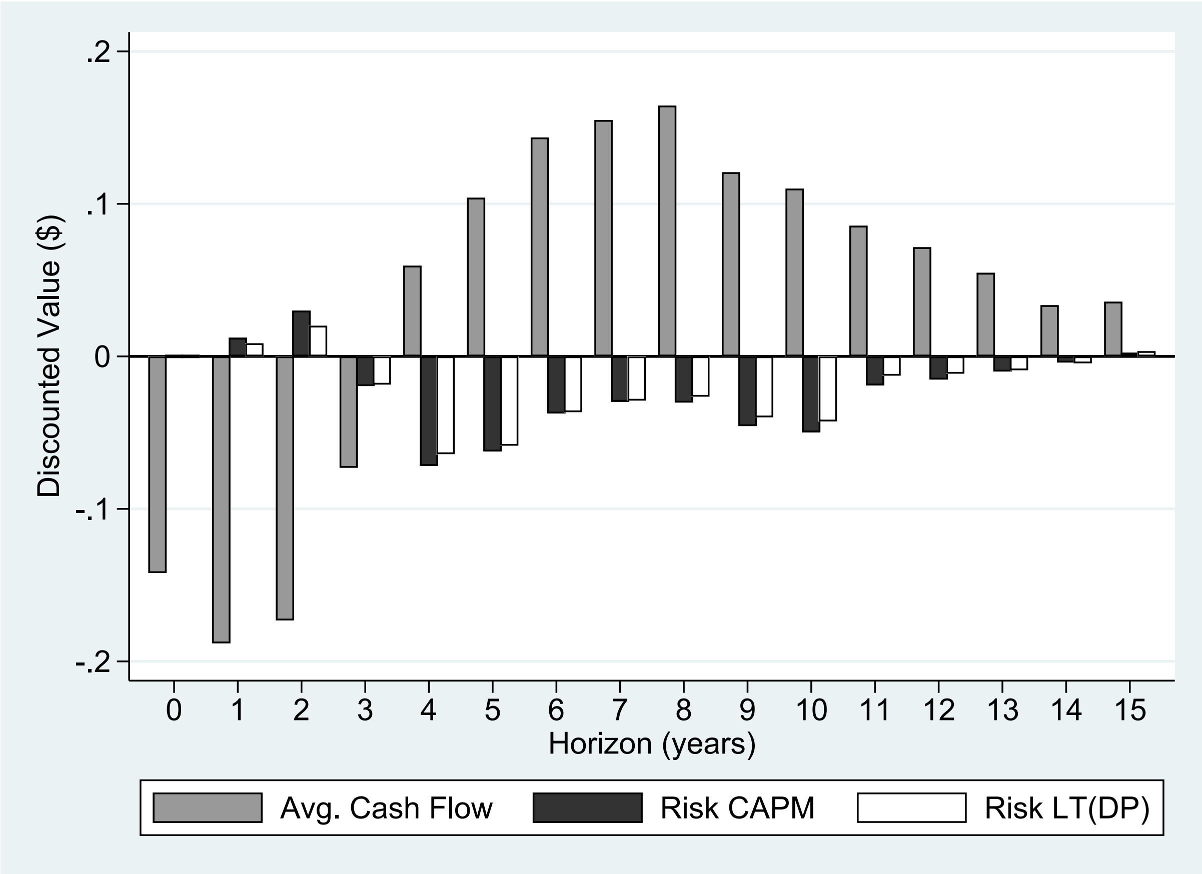

Figure 8

GPME Sensitivity to Risk Aversion Estimates

In this figure, we plot GPME for models with multiple intercepts using exogenous values of γ between 1 and 12. For each category and for each model, we compute GPME as $\sum_h (\frac{1}{N}\sum_i 1/R^{\mathrm{f}}_{t,t+h})\ (\overline{C}_h + A_h)$, where $A_h = \frac{1}{N}\sum_i (M_{t,t+h}/\overline{M}_h - 1)(C_{i,t+h} - \overline{C}_h)$. The term $M_{t,t+h}/\overline{M}_h$ is computed as $\exp(-\gamma r^{\mathrm{m}}_{t,t+h})/\frac{1}{N}\sum_i \exp(-\gamma r^{\mathrm{m}}_{t,t+h})$ for CAPM and similarly with an additional risk factor for LT(DP) and LT(Full). On each line, the black circle indicates the GPME estimate using γ from the corresponding model in Table 5 for buyout, Table 7 for venture capital, and Table 8 for generalist. The white circle indicates the GPME estimate using γ from the single intercept version of the model.

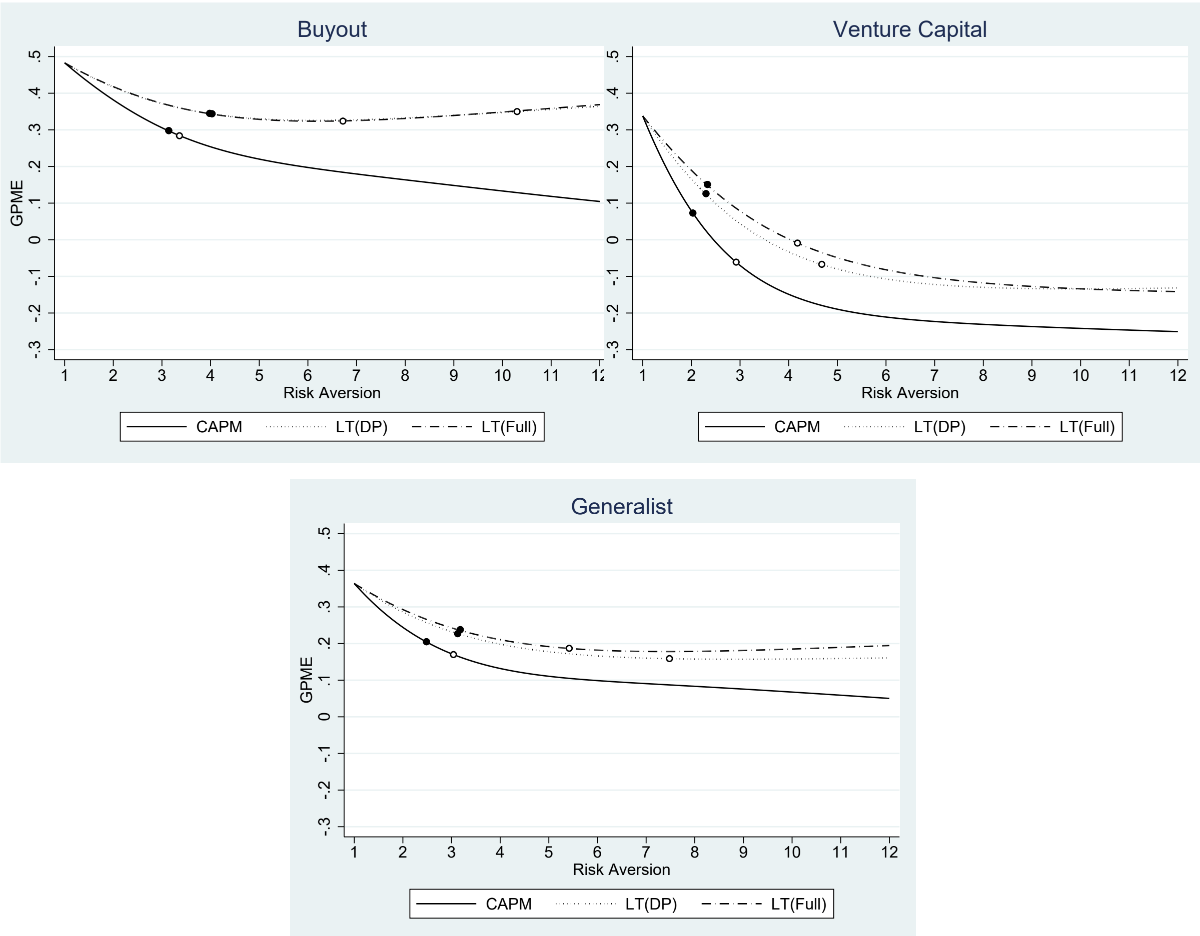

Table 1
Summary Statistics of VAR Variables

This table reports summary statistics of variables entering the full VAR model. Variables are computed quarterly from 1950 to 2018 for a total of 276 observations. The expression $r^{\mathrm{m}} - r^{\mathrm{f}}$ indicates the excess log-return on the S&P 500, DP is the logarithm of dividend yield on the S&P 500, VS is the difference in the log book-to-market ratio of small-value and small-growth stocks, CS is the yield difference between BAA and AAA rated corporate bonds, and $TERM$ is the yield difference between treasuries with 10-year and 3-month maturity. Panel A reports descriptive statistics. Panel B reports correlations between contemporaneous and lagged variables.

Panel A: Descriptive Statistics

	Mean	Std	Min	Max	Autocorr.
$r_t^{\mathrm{m}} - r_t^{\mathrm{f}}$	0.016	0.078	−0.311	0.192	0.099
DP_t	−3.536	0.424	−4.497	−2.624	0.982
VS_t	1.575	0.157	1.280	2.111	0.890
CS_t	0.010	0.004	0.003	0.034	0.878
$TERM_t$	0.017	0.014	−0.035	0.045	0.841

Panel B: Correlations

	$r_t^{\mathrm{m}} - r_t^{\mathrm{f}}$	DP_t	VS_t	CS_t	$TERM_t$
$r_{t-1}^{\mathrm{m}} - r_{t-1}^{\mathrm{f}}$	0.099	−0.069	−0.017	−0.220	0.060
DP_{t-1}	0.130	0.982	−0.439	0.159	−0.221
VS_{t-1}	−0.099	−0.429	0.890	0.030	0.326
CS_{t-1}	0.022	0.144	0.060	0.878	0.321
$TERM_{t-1}$	0.103	−0.273	0.283	0.171	0.841

Table 2
VAR Estimation

This table reports the results of two VAR estimations. In Panel A, the VAR includes only the excess log-return and the logarithm of dividend yield on the S&P 500. In Panel B, the VAR includes also the value spread, credit spread, and term premium as defined in Table 1 and in the main text. Variables are computed quarterly from 1950 to 2018 for a total of 276 observations. OLS Standard errors are reported in parenthesis, and the symbols ***, **, and * indicate significance at 1%, 5%, and 10%.

Panel A: DP only							
	Constant	$r^{\mathrm{m}}_t - r^{\mathrm{f}}_t$	DP_t	VS_t	CS_t	$TERM_t$	R^2
$r^{\mathrm{m}}_{t+1} - r^{\mathrm{f}}_{t+1}$	0.102***	0.107*	0.025**				0.028
	(0.039)	(0.060)	(0.011)				
DP_{t+1}	−0.084**	−0.100	0.977***				0.965
	(0.040)	(0.062)	(0.011)				

Panel B: Full VAR							
	Constant	$r^{\mathrm{m}}_t - r^{\mathrm{f}}_t$	DP_t	VS_t	CS_t	$TERM_t$	R^2
$r^{\mathrm{m}}_{t+1} - r^{\mathrm{f}}_{t+1}$	0.162***	0.082	0.027**	−0.040	−0.622	0.929**	0.051
	(0.055)	(0.060)	(0.012)	(0.034)	(1.118)	(0.373)	
DP_{t+1}	−0.101*	−0.085	0.975***	0.019	−0.705	−0.780**	0.966
	(0.056)	(0.062)	(0.013)	(0.035)	(1.153)	(0.385)	
VS_{t+1}	0.101**	0.064	−0.027**	0.868***	1.840*	−0.240	0.798
	(0.051)	(0.056)	(0.012)	(0.031)	(1.039)	(0.347)	
CS_{t+1}	0.001	−0.009***	0.000	0.000	0.875***	−0.011	0.798
	(0.001)	(0.002)	(0.000)	(0.001)	(0.029)	(0.010)	
$TERM_{t+1}$	−0.011**	−0.004	0.000	0.006*	0.371***	0.789***	0.725
	(0.005)	(0.006)	(0.001)	(0.003)	(0.106)	(0.036)	

Table 3
Discount Rate News

This table decomposes point estimates and variance of N_t^{DR} from the two VAR estimations of Table 2. In Panel A, we report estimates of the vector λ such that $N_t^{\mathrm{DR}} = \lambda\varepsilon_t$ where ε_t is the VAR error term. The vector is $\lambda = \rho e'\Theta(I - \rho\Theta)^{-1}$, where Θ is the matrix of VAR coefficients, I is the identity matrix, e is a column vector with 1 as first element and 0 elsewhere, and $\rho = 0.95^{1/4}$. In parentheses, we report standard errors calculated with delta method. Statistical significance is computed using the normal distribution, and the symbols ***, **, and * indicate significance at 1%, 5%, and 10%. In Panel B, we decompose the variance of N_t^{DR} expressed as $\lambda\Sigma_\varepsilon\lambda'$. We report the vector of variance components $\lambda \circ \lambda\Sigma_\varepsilon$, where $\circ$ is the element-wise product. We also report components as percentages of the total variance.

Panel A: Long-Run Coefficients

	DP only	Full VAR
	λ	λ
$r^{\mathrm{m}} - r^{\mathrm{f}}$	0.039	0.046
	(0.027)	(0.041)
DP	0.713***	0.782***
	(0.112)	(0.139)
VS		−0.110
		(0.097)
CS		−4.869
		(3.843)
$TERM$		1.977***
		(0.747)

Panel B: Variance Decomposition

	DP only		Full VAR	
	$\lambda \circ \lambda\Sigma_\varepsilon$	%	$\lambda \circ \lambda\Sigma_\varepsilon$	%
$r^{\mathrm{m}} - r^{\mathrm{f}}$ shock	−0.00016	−5.4	−0.00019	−5.4
DP shock	0.00303	105.4	0.00343	99.2
VS shock			0.00011	3.0
CS shock			−0.00002	−0.4
$TERM$ shock			0.00013	3.6
$\mathrm{Var}(N_t^{\mathrm{DR}})$	0.00287	100.0	0.00345	100.0

Table 4
Summary Statistics of Funds Data

This table reports summary statistics of funds data from Burgiss. In Panel A, *#Funds* is the sample size, and for each fund, Fund Size is total commitment, Effective Years counts the years between the first and last available cash flow, #Cash flows/Fund is the number of cash flows, TVPI is the ratio of distributions over contributions. Furthermore, #Unresolved Funds counts funds that are not fully liquidated in sample, and NAV/Distributions is the ratio of their residual NAV over total distributions. Panel B reports mean and median TVPI for different vintage years. For confidentiality, we hide figures computed with 4 funds or less.

Panel A: Descriptive Statistics

	Buyout			Venture Capital			Generalist		
	Mean	Median	St.dev.	Mean	Median	St.dev.	Mean	Median	St.dev.
#Funds	652			971			243		
Fund Size ($M)	1099.10	421.00	2118.19	222.10	126.00	282.75	557.77	225.00	1290.66
Effective Years	14.16	13.75	3.13	15.52	15.25	3.37	14.42	13.50	3.37
#Cash flows/Fund	35.83	36.00	10.45	27.60	27.00	9.62	32.99	33.00	10.53
TVPI	1.83	1.68	1.15	2.08	1.39	3.07	1.80	1.64	1.02
#Unresolved Funds	311			353			88		
NAV/Distributions	0.10	0.06	0.11	0.14	0.10	0.13	0.12	0.07	0.13

Panel B: TVPI by Vintage Year

	Buyout			Venture Capital			Generalist		
Vintage	# Funds	Mean	Median	# Funds	Mean	Median	# Funds	Mean	Median
1978-91	57	3.09	2.22	238	2.12	1.77	15	2.76	2.55
1992	8	1.97	1.64	17	3.19	1.76	5	3.12	2.62
1993	7	1.68	1.72	20	5.35	3.15	7	2.30	1.90
1994	18	1.73	1.49	16	6.15	4.50	8	2.60	2.18
1995	26	1.63	1.53	27	5.69	2.72	6	3.08	2.37
1996	17	1.64	1.70	18	6.68	3.31	8	2.00	1.43
1997	26	1.24	1.23	47	3.48	1.94	17	1.49	1.28
1998	40	1.45	1.45	53	1.97	1.18	17	1.48	1.39
1999	34	1.45	1.55	94	0.86	0.72	20	1.29	1.09
2000	50	1.79	1.68	119	0.96	0.85	26	1.43	1.41
2001	31	1.90	1.94	60	1.26	1.12	6	2.04	2.20
2002	21	1.89	1.85	21	1.08	1.11	7	1.73	1.63
2003	22	2.07	1.82	21	1.38	1.10	4	* * *	* * *
2004	38	1.77	1.63	34	1.49	0.91	10	1.66	1.74
2005	57	1.64	1.52	52	1.63	1.31	18	1.84	1.53
2006	61	1.67	1.64	54	1.56	1.47	27	1.53	1.39
2007	66	1.77	1.70	45	2.21	1.90	22	1.58	1.68
2008	55	1.72	1.69	28	2.06	1.71	13	1.96	1.81
2009	18	2.06	2.05	7	1.92	2.01	7	1.71	1.59

Table 5
Buyout Performance

For $N = 652$ buyout funds in our sample, we estimate expected GPME by summing discounted cash flows of each fund and averaging the result across funds. Cash flows are discounted with the following SDF:

$$M_{t,t+h} = \exp\left(a_h h - \omega\gamma\, r^{\mathrm{m}}_{t,t+h} - \omega(\gamma - 1)N^{\mathrm{DR}}_{t,t+h}\right)$$

In this table, the investor has stock allocation $\omega = 100\%$, and each column corresponds to different restrictions on the SDF. With Log-Utility, $a_h = 0$ and $\gamma = 1$ as in the PME of Kaplan and Schoar (2005). With Single Intercept, $a_h = a$ for all h, and the two parameters (a and γ) are estimated following Korteweg and Nagel (2016) so that the SDF prices artificial funds invested in the S&P 500 and in T-bills. With Multiple Intercepts, we estimate γ and one a_h for every h such that the SDF prices T-Bills investments at every horizon and stock investments at horizon $h = 40$ quarters. CAPM corresponds to $N^{\mathrm{DR}}_{t,t+h} = 0$, while LT(DP) and LT(Full) use $N^{\mathrm{DR}}_{t,t+h}$ from the VAR estimations of Table 2. For each specification, we report point estimates of GPME and γ. In parentheses, we report GPME standard errors that account for error dependence between overlapping funds and ignore parameters uncertainty. In brackets, we report p-values for a J-test of GPME $= 0$. In the last two rows of the table, we divide the GPME in two components, and we let $\overline{M}_h = \frac{1}{N}\sum_i M_{t,t+h}$ be the average SDF at each horizon, $\overline{C}_h = \frac{1}{N}\sum_i C_{i,t+h}$ be the average cash flow, and $A_h = \frac{1}{N}\sum_i (M_{t,t+h}/\overline{M}_h - 1)(C_{i,t+h} - \overline{C}_h)$ be a risk adjustment.

		Single Intercept			Multiple Intercepts		
	Log-Utility	CAPM	LT (DP)	LT (Full)	CAPM	LT (DP)	LT (Full)
GPME	0.203	0.279	0.803	0.405	0.298	0.350	0.346
	(0.027)	(0.159)	(0.706)	(0.285)	(0.113)	(0.118)	(0.114)
	[0.000]	[0.079]	[0.255]	[0.155]	[0.008]	[0.003]	[0.002]
γ	1.00	3.37	10.30	6.72	3.16	3.98	4.03
Components of GPME							
$\sum_h \overline{M}_h \overline{C}_h$	0.293	0.792	2.502	1.139	0.608	0.608	0.608
$\sum_h \overline{M}_h A_h$	−0.090	−0.513	−1.699	−0.735	−0.310	−0.258	−0.261

Table 6
GPME Distributions

The table reports mean, standard deviation, and selected percentiles of the GPME distribution for buyout, venture capital, and generalist funds across different models. The models are estimated in Table 5, 7, and 8.

		Single Intercept			Multiple Intercepts		
	Log-Utility	CAPM	LT (DP)	LT (Full)	CAPM	LT (DP)	LT (Full)
Buyout ($N = 652$)							
Mean	0.20	0.28	0.80	0.41	0.30	0.35	0.35
St.Dev.	0.64	1.42	9.25	2.99	0.98	1.05	1.04
Min	−1.25	−4.15	−15.39	−5.86	−2.21	−1.60	−1.66
p10	−0.36	−0.83	−4.30	−1.65	−0.47	−0.48	−0.47
p25	−0.12	−0.38	−0.85	−0.56	−0.20	−0.23	−0.20
p50	0.13	−0.04	−0.17	−0.14	0.02	0.03	0.04
p75	0.44	0.59	0.61	0.69	0.51	0.61	0.58
p90	0.82	1.72	4.06	2.76	1.35	1.56	1.51
Max	10.89	13.98	165.94	42.62	8.78	8.32	8.66
Venture Capital ($N = 971$)							
Mean	0.14	−0.15	−0.19	−0.08	0.07	0.13	0.15
St.Dev.	1.40	1.18	1.43	1.68	1.07	1.23	1.32
Min	−1.14	−2.57	−3.21	−2.84	−1.50	−1.36	−1.40
p10	−0.67	−0.95	−1.12	−1.02	−0.66	−0.65	−0.65
p25	−0.44	−0.63	−0.68	−0.63	−0.42	−0.41	−0.41
p50	−0.19	−0.33	−0.37	−0.34	−0.16	−0.16	−0.15
p75	0.23	0.04	−0.04	0.03	0.26	0.30	0.31
p90	0.88	0.72	0.73	0.90	0.87	0.91	0.98
Max	15.13	16.47	20.42	24.55	15.51	17.31	17.46
Generalist ($N = 243$)							
Mean	0.16	0.13	0.05	0.15	0.21	0.24	0.25
St.Dev.	0.52	1.01	2.28	1.58	0.75	0.79	0.79
Min	−1.15	−2.72	−6.43	−3.75	−1.50	−1.47	−1.51
p10	−0.39	−0.72	−1.95	−1.08	−0.41	−0.46	−0.42
p25	−0.17	−0.44	−0.69	−0.51	−0.24	−0.24	−0.23
p50	0.06	−0.12	−0.24	−0.17	0.00	0.03	0.06
p75	0.44	0.42	0.34	0.40	0.46	0.51	0.53
p90	0.79	1.19	2.38	2.07	1.01	1.11	1.10
Max	2.63	6.11	11.52	8.94	4.15	4.01	4.20

Table 7
Venture Capital Performance

We estimate expected GPME for $N = 971$ venture capital funds in our sample. The construction of this table follows the description of Table 5 using the sample of venture capital funds.

		Single Intercept			Multiple Intercepts		
	Log-Utility	CAPM	LT (DP)	LT (Full)	CAPM	LT (DP)	LT (Full)
GPME	0.135	−0.150	−0.188	−0.077	0.073	0.126	0.151
	(0.084)	(0.067)	(0.081)	(0.101)	(0.065)	(0.078)	(0.084)
	[0.109]	[0.026]	[0.020]	[0.447]	[0.261]	[0.108]	[0.072]
γ	1.00	2.93	4.68	4.19	2.03	2.37	2.40
Components of GPME							
$\sum_h \overline{M}_h \overline{C}_h$	0.484	0.910	0.903	0.892	0.725	0.725	0.725
$\sum_h \overline{M}_h A_h$	−0.349	−1.060	−1.091	−0.969	−0.652	−0.599	−0.574

Table 8
Generalist Performance

We estimate expected GPME for $N = 243$ generalist funds in our sample. The construction of this table follows the description of Table 5 using the sample of generalist funds.

		Single Intercept			Multiple Intercepts		
	Log-Utility	CAPM	LT (DP)	LT (Full)	CAPM	LT (DP)	LT (Full)
GPME	0.156	0.127	0.046	0.153	0.213	0.241	0.249
	(0.023)	(0.094)	(0.173)	(0.129)	(0.069)	(0.074)	(0.071)
	[0.000]	[0.177]	[0.791]	[0.234]	[0.002]	[0.001]	[0.000]
γ	1.00	3.05	7.49	5.43	2.53	3.14	3.19
Components of GPME							
$\sum_h \overline{M}_h \overline{C}_h$	0.318	0.703	1.043	0.832	0.563	0.563	0.563
$\sum_h \overline{M}_h A_h$	−0.162	−0.576	−0.998	−0.679	−0.350	−0.321	−0.313

Table 9
PE Performance and Investor's Leverage

We estimate expected GPME for of a conservative investor with 50% of wealth in stocks ($\omega = 50\%$) and an aggressive investor with 200% of wealth in stocks ($\omega = 200\%$). This table reports the results separately for buyout, venture capital, and generalist funds. The estimation follows the description of Table 5 limited to the case with Multiple Intercepts.

	Buyout			Venture Capital			Generalist		
	CAPM	LT (DP)	LT (Full)	CAPM	LT (DP)	LT (Full)	CAPM	LT (DP)	LT (Full)
Conservative Investor ($\omega = 50\%$)									
GPME	0.298	0.358	0.354	0.073	0.155	0.198	0.212	0.250	0.262
	(0.113)	(0.119)	(0.114)	(0.065)	(0.086)	(0.096)	(0.069)	(0.076)	(0.071)
	[0.008]	[0.003]	[0.002]	[0.262]	[0.071]	[0.040]	[0.002]	[0.001]	[0.000]
γ	6.32	8.29	8.41	4.08	5.07	5.17	5.07	6.66	6.78
Components of GPME									
$\sum_h \overline{M}_h \overline{C}_h$	0.608	0.608	0.608	0.725	0.725	0.725	0.563	0.563	0.563
$\sum_h \overline{M}_h A_h$	−0.310	−0.250	−0.254	−0.652	−0.570	−0.527	−0.350	−0.313	−0.301
Aggressive Investor ($\omega = 200\%$)									
GPME	0.298	0.329	0.327	0.073	0.075	0.075	0.212	0.223	0.225
	(0.113)	(0.116)	(0.114)	(0.065)	(0.065)	(0.066)	(0.069)	(0.071)	(0.069)
	[0.008]	[0.004]	[0.004]	[0.262]	[0.253]	[0.250]	[0.002]	[0.002]	[0.001]
γ	1.58	1.81	1.83	1.02	1.02	1.03	1.27	1.38	1.39
Components of GPME									
$\sum_h \overline{M}_h \overline{C}_h$	0.608	0.608	0.608	0.725	0.725	0.725	0.563	0.563	0.563
$\sum_h \overline{M}_h A_h$	−0.310	−0.279	−0.281	−0.652	−0.650	−0.649	−0.350	−0.339	−0.337

A Theoretical Stochastic Discount Factor

In this section, we connect the long-term SDF from the main text to theory by deriving its theoretical version. We use the setup of Campbell (1993) extending his results to price payoffs received several periods in the future.

The investor has infinite-horizon Epstein-Zin preferences over consumption, and these preferences correspond to the following generic form of SDF (Epstein and Zin, 1989):

$$M_{t,t+1}^{\text{theory}} = \exp\left(\theta \log \delta - \frac{\theta}{\psi}(c_{t+1} - c_t) - (1-\theta)\, r_{t+1}^{\text{W}}\right) \tag{A.1}$$

In this expression, c_t is the natural logarithm of consumption, and r_{t+1}^{W} is the log-return on wealth. Greek letters indicate parameters with δ being the subjective discount factor, ψ being the elasticity of intertemporal substitution, and $\theta = \frac{1-\gamma}{1-1/\psi}$, which depends on ψ and relative risk aversion γ.

The budget constraint of the investor can be written as follows:

$$W_{t+1} = (W_t - C_t) R_{t+1}^{\text{W}} \tag{A.2}$$

Wealth is W_t, while $C_t = \exp(c_t)$ is consumption, and $R_{t+1}^{\text{W}} = \exp(r_{t+1}^{\text{W}})$ is the return on wealth. Following Campbell (1993), we represent the investor's budget constraint with the following log-linear approximation:

$$w_{t+1} - w_t = r_{t+1}^{\text{W}} + k + \left(1 - \frac{1}{\rho}\right)(c_t - w_t) \tag{A.3}$$

In this expression, w_t is the logarithm of wealth, k and ρ are approximation constants.

A.1 Consumption

Assuming that second and higher moments of r_{t+1}^{W} are constant over time, consumption can be expressed as a function of returns. For that, we use two equations derived in Campbell (1993) under the same set of assumptions. The first equation connects expected consumption growth to expected return on wealth:

$$E_t\left[c_{t+1}-c_t\right]=\psi E_t\left[r_{t+1}^{\mathrm{W}}\right]+\underbrace{\psi\log(\delta)+\frac{1}{2}\frac{1-\gamma}{1-\psi}\mathrm{Var}_t\left(c_{t+1}-c_t-\psi r_{t+1}^{\mathrm{W}}\right)}_{\text{constant}} \tag{A.4}$$

The second equation connects the unexpected component of consumption growth to the unexpected component of r_{t+1}^{W} and to discount rate (DR) news about investor's wealth:

$$c_{t+1}-c_t-E_t\left[c_{t+1}-c_t\right]=r_{t+1}^{\mathrm{W}}-E_t\left[r_{t+1}^{\mathrm{W}}\right]+(1-\psi)\underbrace{\left(E_{t+1}-E_t\right)\sum_{j=1}^{\infty}\rho^j r_{t+1}^{\mathrm{W}}}_{N_{t+1}^{\mathrm{DR,W}}} \tag{A.5}$$

Combining (A.4) with (A.5), we get the following expression for the logarithm of consumption growth:

$$c_{t+1}-c_t=r_{t+1}^{\mathrm{W}}+(1-\psi)N_{t+1}^{\mathrm{DR,W}}-(1-\psi)E_t\left[r_{t+1}^{\mathrm{W}}\right]+\text{constant} \tag{A.6}$$

A.2 Return on Wealth

DR news on wealth, $N_{t+1}^{\mathrm{DR,W}}$, can be expressed in terms of DR news on underlying assets. For that, we specify a simple investment strategy for the investor. His portfolio is a constant combination of two assets. One asset is risk-free, while the other is a market index of public equities. The return on the risk-free asset is constant and its logarithm is denoted r^{f}. The return on the market is risky and its logarithm is denoted r_t^{m}. Expected log-return on the market can vary over time, while its variance and higher moments remain constant.

Following Campbell and Viceira (1999), log-returns on individual assets determine log-return on wealth through the following approximate relation:

$$r_{t+1}^{\mathrm{W}}=\omega\left(r_{t+1}^{\mathrm{m}}-r^{\mathrm{f}}\right)+r^{\mathrm{f}}+\frac{1}{2}\omega(1-\omega)\mathrm{var}\left(r_{t+1}^{\mathrm{m}}\right) \tag{A.7}$$

In this expression, ω is the constant portfolio weight in the market. Under our assumptions, this approximation implies:

$$N_{t+1}^{\mathrm{DR,W}} = \omega N_{t+1}^{\mathrm{DR}} \tag{A.8}$$

where N_{t+1}^{DR} is one-period DR news on the market as defined in the main text.

A.3 Theoretical Long-Term SDF

To obtain the theoretical form of the long-term SDF, we substitute (A.6) and (A.7) inside the SDF expression (A.1). As a result, we obtain the one-period theoretical version of the long-term SDF:

$$M_{t+1}^{\mathrm{theory}} = \exp\left(a_t - \omega\gamma\, r_{t+1}^{\mathrm{m}} - \omega(\gamma-1)\, N_{t+1}^{\mathrm{DR}}\right) \tag{A.9}$$

with $a_t = \omega(\gamma-1)E_t\left[r_{\mathrm{m},t+1}\right] + \text{constant}$.

A.3.1 Two Periods

The product $M_{t+1}^{\mathrm{theory}} \times M_{t+2}^{\mathrm{theory}}$ can be written as follows:

$$M_{t,t+2}^{\mathrm{theory}} = \exp\left(a_{2,t} - \omega\gamma\, r_{t,t+2}^{\mathrm{m}} - \omega(\gamma-1)N_{t,t+2}^{\mathrm{DR}}\right) \tag{A.10}$$

This expression contains the following objects:

$$a_{2,t} = \omega(\gamma-1)\left(E_t[r_{t+1}^{\mathrm{m}}] + E_t[r_{t+2}^{\mathrm{m}}]\right) + 2\cdot\text{constant} \tag{A.11}$$

$$\begin{aligned} N_{t,t+2}^{\mathrm{DR}} &= (E_{t+2}-E_t)\sum_{j=1}^{\infty}\rho^j r_{t+2+j}^{\mathrm{m}} - (1-\rho)(E_{t+1}-E_t)r_{t+2}^{\mathrm{m}} \\ &\approx (E_{t+2}-E_t)\sum_{j=1}^{\infty}\rho^j r_{t+2+j}^{\mathrm{m}} \end{aligned} \tag{A.12}$$

and the approximation of $N_{t,t+2}^{\mathrm{DR}}$ is accurate for ρ close 1.

A.3.2 Many Periods

The product $M_{t+1}^{\text{theory}} \times M_{t+2}^{\text{theory}} \times \cdots \times M_{t+h}^{\text{theory}}$ can be written as follows:

$$M_{t,t+h}^{\text{theory}} = \exp\left(a_{h,t} - \omega\gamma\, r_{t,t+h}^{\text{m}} - \omega(\gamma - 1)N_{t,t+h}^{\text{DR}}\right) \tag{A.13}$$

where

$$a_{h,t} = \omega(\gamma - 1)\sum_{s=1}^{h} E_t[r_{t+s}^{\text{m}}] + h \cdot \text{constant} \tag{A.14}$$

Furthermore, multi-period DR news is written as follows:

$$\begin{aligned}
N_{t,t+h}^{\text{DR}} &= \sum_{s=1}^{h} N_{t+s}^{\text{DR}} - \sum_{s=1}^{h-1}\left[(E_{t+s} - E_{t+s-1})\sum_{j=1}^{h-s} r_{t+s+j}^{\text{m}}\right] &\text{(A.15)}\\
&= \sum_{s=1}^{h}\left[(E_{t+s} - E_{t+s-1})\sum_{j=1}^{\infty}\rho^j r_{t+s+j}^{\text{m}}\right] - \sum_{s=1}^{h-1}\left[(E_{t+s} - E_{t+s-1})\sum_{j=1}^{h-s} r_{t+s+j}^{\text{m}}\right]\\
&= \sum_{s=1}^{h}\left[(E_{t+s} - E_{t+s-1})\sum_{j=h-s+1}^{\infty}\rho^j r_{t+s+j}^{\text{m}}\right] - \sum_{s=1}^{h-1}\left[(E_{t+s} - E_{t+s-1})\sum_{j=1}^{h-s}(1-\rho^j) r_{t+s+j}^{\text{m}}\right]\\
&\approx \sum_{s=1}^{h}\left[(E_{t+s} - E_{t+s-1})\sum_{j=h-s+1}^{\infty}\rho^j r_{t+s+j}^{\text{m}}\right]\\
&= \sum_{s=1}^{h}\left[(E_{t+s} - E_{t+s-1})\sum_{j=1}^{\infty}\rho^j r_{t+h+j}^{\text{m}}\right]\\
&= (E_{t+h} - E_t)\sum_{j=1}^{\infty}\rho^j r_{t+h+j}^{\text{m}} &\text{(A.16)}
\end{aligned}$$

The approximation is accurate for ρ close to 1.

Chapter 2

Optimal Allocation to Private Equity

Nicola Giommetti Morten Sorensen*

Abstract

We study the portfolio problem of an investor (LP) that invests in stocks, bonds, and private equity (PE) funds. The LP repeatedly commits capital to PE funds. This capital is only gradually contributed and eventually distributed back to the LP, requiring the LP to hold a liquidity buffer for its uncalled commitments. Despite being riskier, PE investments are not monotonically declining in risk aversion. Instead, there are two qualitatively different investment strategies with intuitive heuristics. We introduce a secondary market for PE partnership interests to study optimal trading in this market and implications for the LP's optimal investments.

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The number of US publicly traded companies is declining and an increasing amount of capital is invested in private equity (PE) and other non-public assets.[1] Typical PE investors, so-called limited partners (LPs), are pension funds, sovereign wealth funds, university endowments, and other institutional investors. Despite their increasing allocations to PE, it remains unclear how LPs should optimally manage their PE investments as part of their broader portfolio allocations. We study this problem.

We introduce PE investments into a standard portfolio model. PE investments are risky, illiquid, and long-term. They are risky because PE investments earn a risky return and generate risky distributions to the LP. They are illiquid because the LP must hold them to maturity, and the LP cannot liquidate (or collateralize) its PE investments to convert them into current consumption, although this is relaxed in the extension with a secondary market. They are long-term because the LP's commitments are only gradually called and invested into underlying private assets by PE funds, and the eventual distributions back to the LP only arrive after a substantial time. Thus, the LP can only adjust its PE exposure slowly, and it may need to hold a liquidity reserve to meet future capital calls arising from its current uncalled commitments. A main contribution of our analysis is to present a tractable model that captures these aspects of PE investments, particularly the LP's repeated capital commitments to an arbitrary number of PE funds, and to solve for the LP's optimal investment policies.

The particular nature of PE investments has interesting implications for the LP's optimal allocations and investment policies. Unsurprisingly, the illiquid and long-term nature of PE investments can lead the LP to reduce its PE allocation relative to the first-best allocation. Perhaps more surprisingly, the optimal PE allocation is not monotonically decreasing in the LP's risk aversion. With reasonable parameter values, the LP's PE allocation is largely constant, and a more risk averse LP may, in fact, choose a higher PE allocation than a less risk averse LP. In contrast, risk aversion significantly affects the LP's allocations to traded stocks and bonds.

Depending on risk aversion, there are two distinct investment strategies. A conservative LP, with higher risk aversion (in our specification, a relative risk aversion of $\gamma = 3$),

[1] Doidge, Karolyi, and Stulz (2017) discuss the declining number of listed companies, and Bain & Company (2019) show the increasing allocations to alternative investments.

holds relatively more liquid reserves of stocks and bonds than illiquid PE investments. This LP tends to be unconstrained and to remain close to an interior optimum, leaving it largely unaffected by the illiquid and long-term nature of PE investments. In response to a positive shock to the net asset value (NAV) of its PE investments, the conservative LP reduces new PE commitments to gradually rebalance its portfolio back towards the interior optimum, basically treating PE investments as another traded stock. In contrast, an aggressive LP with lower risk aversion ($\gamma = 1$) faces a binding liquidity constraint. It would prefer greater PE exposure, but larger PE commitments would require a greater liquidity reserve, which is costly. In response to a positive shock to the NAV of its PE investments, the aggressive LP does not rebalance its portfolio, and instead it enjoys a temporary increase in its PE exposure without a corresponding increase in its liquidity reserve.

We extend the model with a secondary market where LPs can trade PE partnership interests to investigate two effects: First, an aggressive LP assigns higher valuations to mature PE funds than a conservative LP. Mature funds provide PE exposure with relatively less liquidity requirements, which is valuable for a constrained LP, and there are gains from trade when conservative LPs can sell partnership interests in mature PE funds to aggressive LPs. Second, the presence of a secondary market where an LP can liquidate its PE investments relaxes its liquidity constraint. In times of stress an LP may choose to liquidate a fraction of its PE investments, even at a discount. And knowing that this liquidity is available, ex-post, allows the LP to hold a more aggressive allocation, ex-ante. We quantify these effects and find that the gains from a single trade, while positive, are modest. However, a secondary market that provides liquidity in times of stress substantially affects the aggressive LP's optimal investments even though the LP only rarely actually transacts in this market.

A large literature investigates the portfolio and asset pricing implications of illiquidity in general, and we do not attempt to summarize this literature. Our analysis is in the spirit of Longstaff (2001) who uses numerical methods to study a portfolio problem with illiquid assets that can only be traded in limited amounts.

In the context of PE specifically, Ang, Papanikolaou, and Westerfield (2014) analyze an investor's optimal investments with an illiquid asset that can only be rebalanced infrequently, at Poisson arrival times, to capture a situation where the investor must search for a counterparty. They find that this form of illiquidity increases the investor's effective risk aversion. Our analysis does not capture the arrival of trading opportunities. Instead, we focus on the ongoing commitments and long-term dynamics of PE investments, which have distinct implications. For example, in our model, LPs with higher risk aversion are unaffected by the illiquidity of PE investments, and it does not increase their effective risk aversion. Sorensen, Wang, and Yang (2014) model an LP that rebalances stocks and bonds while holding an illiquid asset to maturity in order to analyze the costs of the unspanned risks of the illiquid asset, taking the investment in the illiquid asset as given. Bollen and Sensoy (2020) extend this analysis with a secondary market for partnership interests. Dimmock, Wang, and Yang (2019) evaluate the "endowment model" used by some institutions that invest in alternative assets with lock-up periods. Phalippou and Westerfield (2014) study commitment risk for an investor in one, two, or an infinite number of PE funds using a Markov switching model to capture the state of the economy. Finally, there is an extensive literature about the risks and returns of PE investments, and Korteweg (2019) surveys this literature.

1 Model

We extend a standard portfolio model with illiquid PE investments that work as follows. Each period (in our specification, each year) a limited partner (LP) decides how much capital to commit to new PE funds. This capital is not immediately invested, however. Instead, it is gradually contributed to (or drawn down or called by) the PE funds over a period of several years. When capital is contributed (or drawn or called), it is paid by the LP to the PE funds, which invest the capital in underlying private assets (typically, equity stakes in portfolio companies). After holding these private assets for an extended period, typically several years, the private assets are sold (exited) by the PE funds. The proceeds from these sales, net of the PE funds' fees (management fees and carried interest), are paid out (distributed) to the LP.

The LP thus maintains a stock of uncalled commitments (drawdown obligations) representing the capital that the LP has already committed but that has not yet been contributed to the PE funds. These uncalled commitments are gradually called by the

PE funds and then invested in underlying private assets. The LP's PE exposure arises from its claim to the future distributions from the currently held private assets, and the NAV is the value of this claim, which constitutes the LP's illiquid wealth.[2] Each period the LP can decide to commit additional capital to new PE funds and increase its stock of uncalled commitments. Once committed, however, the LP cannot decide to reduce its uncalled commitments. Uncalled commitments only decline gradually, as capital is called and invested by the PE funds.

PE funds are only open for new commitments at inception and typically have ten-year lives, although funds are routinely extended. New PE funds are continuously raised, and LPs diversify their PE investments both across different contemporaneous funds and over time by committing capital to new funds on an ongoing basis, resulting in a diversified portfolio of staggered PE fund commitments. In this case, distributions received from earlier fund commitments, when available, can pay contributions to later commitments, and our model is consistent with this practice.

1.1 Preferences and Timing

The analysis is based on a standard, discrete-time, infinite-horizon, partial-equilibrium, portfolio model with i.i.d. returns and an LP with power utility (CRRA) preferences:

$$\mathrm{E}_t\left[\sum_{s=t}^{\infty}\delta^{s-t}\frac{C_s^{1-\gamma}}{1-\gamma}\right] \tag{1}$$

The LP's consumption is C_t, its time preference is δ, and γ is its relative risk aversion.

Like Ang et al. (2014) we assume a relative risk aversion greater than or equal to one, $\gamma \geq 1$, implying an elasticity of intertemporal substitution less than or equal to one. The LP has negative infinite utility in states with zero consumption. The LP is "bankrupt" when it is unable to consume, and the LP will avoid such states at all costs.[3] Below, we emphasize the results for an LP with $\gamma = 1$, which we call an aggressive LP, and for an LP with $\gamma = 3$, which we call a conservative LP.

[2] The NAV, denoted P_t below, is the price that PE investments would trade at, if they were traded. In the liquid model introduced below, we allow for such trades.

[3] Maurin, Robinson, and Stromberg (2020) and Banal-Estañol, Ippolito, and Vicente (2017) formally model LPs' liquidity constraints, distortions when LPs can default, and the penalties for defaulting.

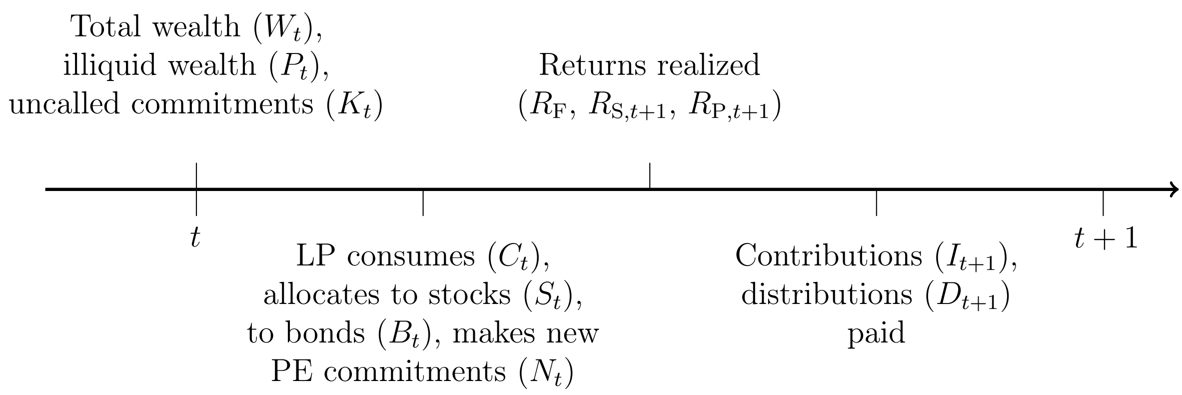


Figure 1: Timing of events during the period from time t to time $t+1$.

The timing of the model is illustrated in Figure 1. At time t, the LP has total wealth W_t, illiquid wealth P_t, and uncalled commitments K_t. After observing these three state variables, the LP consumes and invests. The LP consumes C_t out of its liquid wealth, $W_t - P_t$. Its remaining liquid wealth, $W_t - P_t - C_t$, is allocated to traded stocks, S_t, and bonds, B_t:

$$B_t = W_t - P_t - C_t - S_t \tag{2}$$

The LP decides the amount of capital to commit to new PE funds, N_t. New PE commitments increase the LP's total stock of uncalled commitments, $N_t + K_t$, but they do not directly affect the LP's wealth or budget constraint in the period when they are made.

After the LP has consumed and made its portfolio decisions, returns are realized and contributions and distributions are paid. In practice, returns, contributions, and distributions are earned and paid throughout the period. For simplicity, we assume that returns are realized first and then contributions and distributions are paid. This assumption emphasizes the long-term nature of PE investments, because new PE commitments cannot be contributed and provide PE exposure to the LP in the period when they are made. Rather, new PE commitments are contributed and invested in private assets with at least a one-period lag.

Conventionally, returns earned from time t to $t+1$ have subscript $t+1$, and the returns to stocks and PE investments during this period are denoted $R_{\mathrm{S},t+1}$ and $R_{\mathrm{P},t+1}$, respectively. Bonds earn the risk-free rate, R_{F}. All returns are gross returns. Correspondingly, distributions and contributions paid from time t to $t+1$ are denoted D_{t+1} and I_{t+1}, respectively. The LP pays contributions, I_{t+1}, out of its liquid wealth to the PE funds who invest them in private assets. Distributions, D_{t+1}, are paid from the PE funds to the LP as liquid wealth.

The three state variables update as follows. The LP's uncalled commitments, K_t, increase by the amount of new PE commitments, N_t, and decrease by the amount contributed, I_{t+1}:

$$K_{t+1} = K_t + N_t - I_{t+1} \tag{3}$$

The NAV of the PE investments, P_t, earns the risky return, $R_{\mathrm{P},t+1}$, and then it increases by the amount contributed to the PE funds, I_{t+1}, and it decreases by the amount distributed from the PE funds, D_{t+1}:

$$P_{t+1} = R_{\mathrm{P},t+1} P_t + I_{t+1} - D_{t+1} \tag{4}$$

The LP's end-of-period wealth, W_{t+1}, is the combined value of its PE investments, stocks, and bonds. Contributions and distributions made during the period do not affect the end-of-period wealth since they only reallocate wealth between liquid and illiquid investments:

$$W_{t+1} = R_{\mathrm{P},t+1} P_t + R_{\mathrm{S},t+1} S_t + R_{\mathrm{F}}(W_t - P_t - C_t - S_t) \tag{5}$$

1.2 Linear Fund Dynamics

In practice, an LP invests in many PE funds, each with its own amounts of uncalled commitments, contributions, distributions, and NAVs. Modeling each fund separately, however, is difficult due to the resulting high-dimensional state space and the curse of dimensionality in dynamic programming. Motivated by Takahashi and Alexander (2002), we assume linear fund dynamics where each fund's contributions and distributions are linear in the fund's uncalled commitments and NAV.[4] An implication of linear fund dynamics is that the LP's aggregate contributions and distributions across all its PE funds are linear in the aggregate uncalled commitments and aggregate NAV. Aggregate

[4]Our model simplifies Takahashi and Alexander (2002) in two ways. Their contribution rate varies over time, and our rate is fixed by λ_{N} and λ_{K}. Moreover, our distribution rate is fixed by λ_{D}, and their distribution rate approaches 100% as the fund winds down.

uncalled commitments and aggregate NAV are therefore sufficient statistics for the LP's portfolio of PE investments, in an arbitrary number of PE funds, which reduces the state space to three dimensions (two, in the normalized problem below) and avoids the numerical difficulties that arise when specifying each fund separately.

Formally, let PE funds be indexed by u, and let the set of PE funds that the LP has a partnership interest in at time t, including the new PE funds that the LP commits capital to at time t, be denoted U_t.[5] In practice, multiple LPs invest in a PE fund, and the uncalled commitments, contributions, distributions, and NAVs defined below represent the LP's share of each fund.

In its first year, contributions to a new fund u are the fraction λ_{N} of the fund's newly committed capital, $N_{u,t}$:

$$I_{u,t+1} = \lambda_{\mathrm{N}} N_{u,t} \tag{6}$$

In subsequent years, contributions are the fraction, λ_{K}, of the fund's remaining uncalled commitments, $K_{u,t}$:

$$I_{u,t+1} = \lambda_{\mathrm{K}} K_{u,t} \tag{7}$$

Since $N_t = \sum_{u \in U_t} N_{u,t}$ is the LP's aggregate new PE commitments, and $K_t = \sum_{u \in U_t} K_{u,t}$ is the aggregate existing commitments, the LP's aggregate contributions, I_{t+1}, are:

$$I_{t+1} = \sum_{u \in U_t} I_{u,t+1} = \lambda_{\mathrm{N}} N_t + \lambda_{\mathrm{K}} K_t \tag{8}$$

Hence, aggregate contributions, I_t, are linear in aggregate new commitments, N_t, and aggregate existing commitments, K_t. Our model allows for different intensities of contributions for new and existing commitments, but in our baseline specification $\lambda_{\mathrm{N}} = \lambda_{\mathrm{K}} = 30\%$ for simplicity.[6]

Distributions are modeled similarly. Fund u's distributions, $D_{u,t+1}$, are the fraction, λ_{D},

[5]When fund u has a ten-year life, the LP has a partnership interest in this fund starting the year of the LP's initial commitment and extending through the following nine years until the fund terminates. Note also that a new PE fund, u, that the LP commits capital to at time t starts out with zero NAV, so $P_{u,t} = 0$.

[6]For comparison, Takahashi and Alexander (2002) consider a contribution rate of 25% when the fund is one year old, 33.3% when it is two years old, and 50% after that.

of fund u's NAV, which is $R_{u,t+1}P_{u,t}$, where $R_{u,t+1}$ is fund u's gross return, so:

$$D_{u,t+1} = \lambda_{\mathrm{D}} R_{u,t+1} P_{u,t} \tag{9}$$

The LP's aggregate distributions across all its PE funds is denoted D_{t+1} and equals:

$$D_{t+1} = \sum_{u \in U_t} D_{u,t+1} = \lambda_{\mathrm{D}} R_{\mathrm{P},t+1} P_t \tag{10}$$

Here, $P_t = \sum_{u \in U_t} P_{u,t}$ is the LP's aggregate NAV, and $R_{\mathrm{P},t+1} = \sum_{u \in U_t} R_{u,t+1} \frac{P_{u,t}}{P_t}$ is the value-weighted average return of the LP's PE funds. In our specification, the intensity of distributions is $\lambda_{\mathrm{D}} = 40\%$ of the fund's remaining NAV. Even though distributions are a deterministic fraction of the NAV, the fund's NAV is stochastic, because private assets earn risky returns, and the LP receives a risky flow of distributions from its PE investments.

The above derivations are important for the generality of our analysis. In the general problem, the LP invests in an arbitrary number of contemporaneous and staggered PE funds, but specifying each fund individually is intractable, as mentioned. The above discussions shows, however, that linear fund dynamics imply that the aggregate NAV and aggregate uncalled commitments are sufficient statistics for the LP's portfolio of PE investments. Hence, the solution to the LP's general problem can be found by solving a reduced problem with a more tractable state space. Without loss of generality, we therefore focus on the LP's reduced problem below.[7]

A limitation of linear fund dynamics is that PE funds never completely end, as also discussed by Takahashi and Alexander (2002). In practice, PE funds have ten-year lives, although these lives are routinely extended.[8] In our specification the remaining economic

[7] An additional detail of the specification is how the idiosyncratic risk of the value-weighted PE return, R_{P}, depends on the LP's PE investments, P. Here, we assume that the LP has partnership interests in a diversified number of PE funds, and that changes in the amount of the LP's PE investments arise from proportional changes in the stakes in these funds, leaving the idiosyncratic risk constant. Alternatively, changes in the LP's PE investments can be associated with changes in the number of PE funds, in which case a lower amount of PE investments would be associated with an increase in the idiosyncratic risk of R_{P} due to less diversification. In principle, our model can accommodate a dependency between the idiosyncratic risk of R_{P} and P, although we have not pursued this extension. Robinson and Sensoy (2016) find that most of the volatility in PE investments can be diversified.

[8] In their analysis of long-term PE performance, Gupta and Van Nieuwerburgh (2020) report that it is common for PE funds to have activity fifteen years after inception.

value in the PE funds' later years is minimal, mitigating this limitation of the linear fund dynamics. Specifically, 83.2% of a fund's committed capital is contributed during its first five years, and 97.2% is contributed after ten years. Moreover, in steady state, the LP holds a balanced portfolio of younger and older PE funds, and specification errors in the dynamics will largely average out across the funds when aggregating their contributions and distributions.[9]

Management fees are implicit in this setup. In practice, PE funds charge annual management fees of 0.5%–2% of the total committed capital, which means that this capital is contributed by the LP to the PE funds, but this capital is not used to acquire underlying private assets. Below, we extend our model to explicit management fees, but in most of the analysis management fees are implicit, and the contributed capital, I_t, is the combined amount invested in underlying private assets and paid in management fees. In this case, the NAV, P_t, includes the value of management fees, and the PE return, $R_{\mathrm{P},t}$, is net of both management fees and carried interest.[10] An advantage of implicit management fees is that the analysis of illiquidity is more transparent, because the optimal allocations can be compared to the first-best benchmark when PE is liquid and freely traded. With liquid PE there is no notion of committed capital, however, and the liquid model does not accommodate explicit management fees. Another advantage of implicit management fees is that the return to PE investments, $R_{\mathrm{P},t}$, is the return net of both management fees and carried interest, which is the return that is typically reported in empirical studies. In Section 4, we model management fees explicitly, which requires a reinterpretation of the contributed capital, I_t, the NAV, P_t, and the PE return, $R_{\mathrm{P},t}$. The main finding is that the optimal policies are largely similar with implicit and explicit management fees and that this modeling choice is not critical for our results.

[9] To illustrate this averaging, assume that PE funds live for three periods, and let the dynamics of contributions be specified by λ_0, λ_1, and λ_2, which are the relative amounts of a fund's total committed capital that is contributed during each period. All committed capital is eventually contributed, so $\lambda_0 + \lambda_1 + \lambda_2 = 1$. A diversified LP with an equal amount, κ, committed to funds at each of the three ages has aggregate contributions of $\lambda_0\kappa + \lambda_1\kappa + \lambda_2\kappa = \kappa$. It follows that regardless of the specification of the dynamics, i.e., the particular choice of λ, the resulting amount of aggregate contributions is unaffected.

[10] Actual accounting for management fees can be complicated. PE funds with deal-by-deal carry, sometimes known as the US Market Standard, typically attribute management fees to individual deals and gross up the valuations of these deals with the attributed management fees. Another PE fund compensation structure, sometimes known as the UK Market Standard, has the general partner borrowing the amounts that correspond to the fees and repaying these amounts out of a priority share of the fund's profits, which is consistent with our model of implicit management fees.

The implied dynamics of capital calls and distributions are shown in Figures 2 to 5, which show impulse responses of an initial \$100 PE commitment in year 0, followed by no further commitments. Figure 2 shows annual capital calls. In year one, \$30 is called ($\lambda_{\mathrm{N}} = 30\%$ times $N_0 = \$100$). In year two, \$21 is called ($\lambda_{\mathrm{K}} = 30\%$ times $K_1 = \$70$). In the following years, corresponding declining amounts of capital are called.

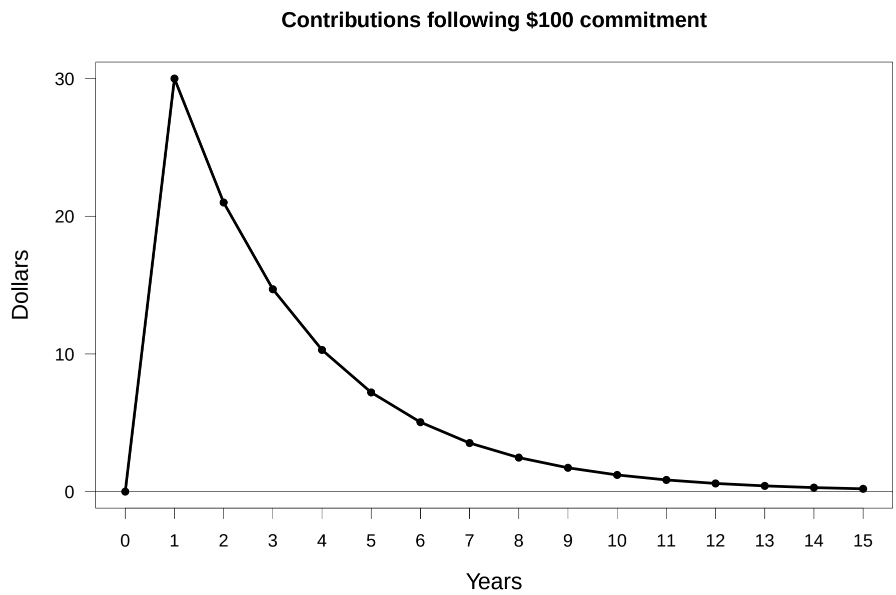


Figure 2: Impulse response function of contributions following a \$100 commitment in year 0.

Figure 3 shows the dynamics of distributions. Distributions depend on NAVs, which are risky, and the figure shows average distributions along with the 5th and 95th percentiles (distributional assumptions are provided below). The largest annual distributions arrive three to five years after the initial commitment, reflecting the long-term nature of PE investments.

Figure 4 shows the cumulative net cash flow, i.e., cumulative distributions minus cumulative contributions. The cumulative cash flow exhibits the well-known "J-curve" dynamics, where it is initially negative, meaning that the LP initially contributes more capital than it receives. With average cash flows the LP breaks even around year six, and then the cumulative cash flow becomes positive and converges to its final value as the fund winds down. The average cumulative net cash flow from an initial \$100 investment is around \$50, implying an average multiple (TVPI or MOIC) around 1.5.

Figure 5 shows the dynamics of the fund's NAV. Interestingly, \$100 of initial commitments

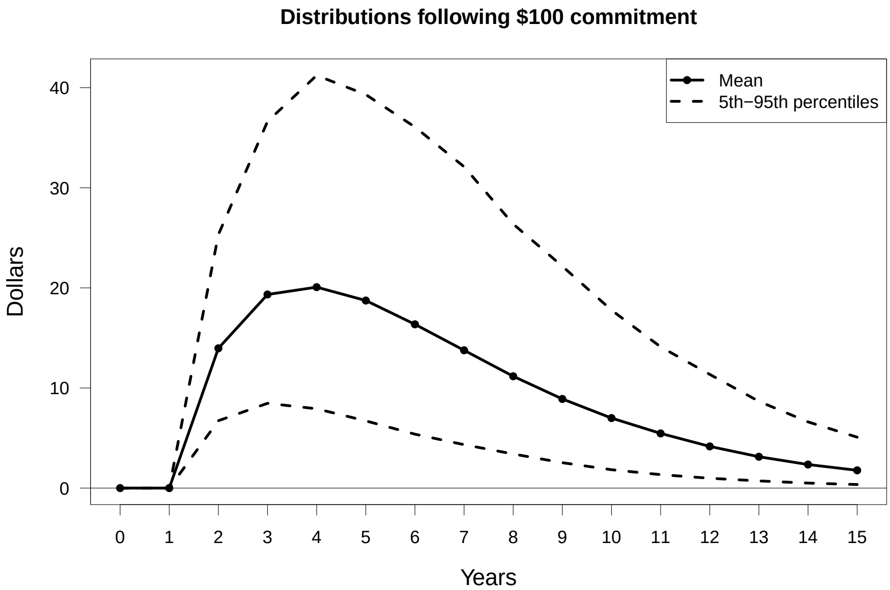


Figure 3: Impulse response function of distributions following a \$100 commitment in year 0. The solid line shows average annual distributions. The dashed lines show the 95^{th} and 5^{th} percentiles of the distribution of annual distributions.

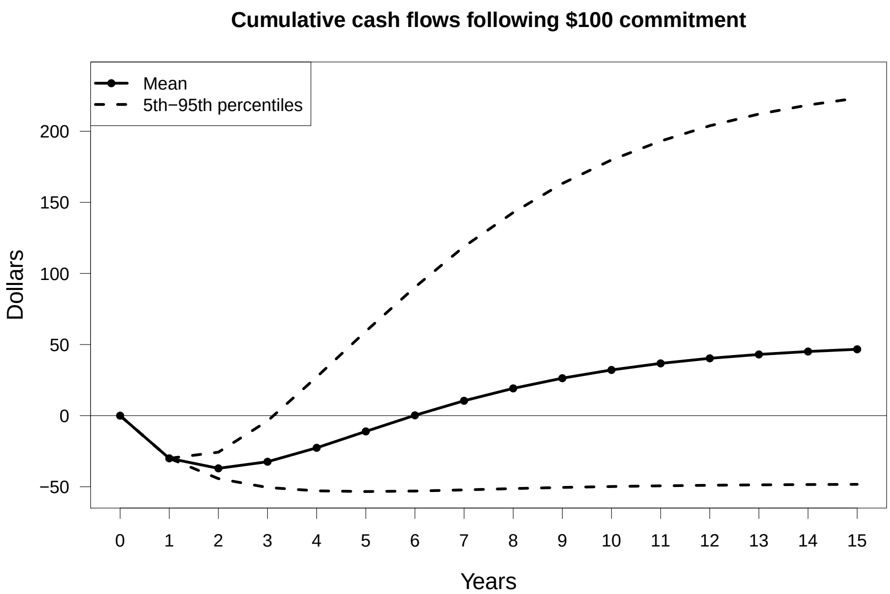


Figure 4: Impulse response function of cumulative net cash flows, the sum of past and current net cash flows, following a \$100 commitment in year 0. The solid line shows average cumulative cash flows. The dashed lines show the 95^{th} and 5^{th} percentiles.

only result in a maximal PE exposure around \$45, on average, and this maximal exposure only arises about three years after the initial commitment. A challenge when managing PE investments is thus that actual PE exposure adjusts slowly, only reaches its maximum level several years after a new commitment is made, and that this maximal level of exposure is only a fraction of the amount of committed capital.

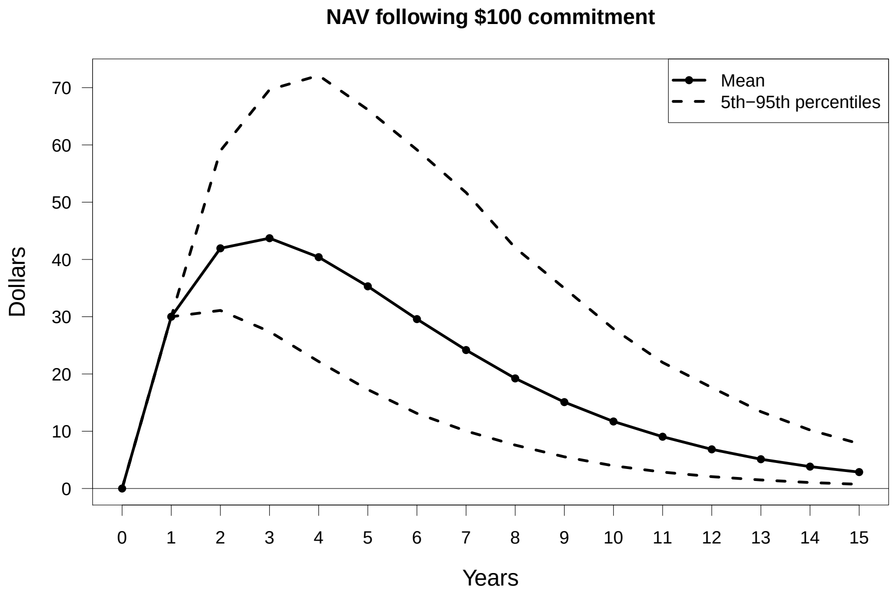


Figure 5: Impulse response function of NAV (or illiquid wealth) following a \$100 commitment in year 0. The solid line shows the average NAV. The dashed lines show the 95^{th} and 5^{th} percentiles.

1.3 Distributional Assumptions and Parameters

The returns to PE and stocks are denoted $R_{\mathrm{P},t}$ and $R_{\mathrm{S},t}$ respectively. These returns are the two sources of uncertainty in the model, and they follow standard i.i.d log-normal distributions. Let $r_{\mathrm{P},t} = \ln R_{\mathrm{P},t}$ and $r_{\mathrm{S},t} = \ln R_{\mathrm{S},t}$. Then, $r_{\mathrm{P},t}$ and $r_{\mathrm{S},t}$ are normal distributed with means μ_{P} and μ_{S}, variances σ_{P}^2 and σ_{S}^2, and covariance $\rho\sigma_{\mathrm{P}}\sigma_{\mathrm{S}}$, where ρ is the correlation between the log-returns:

$$(r_{\mathrm{P},t}, r_{\mathrm{S},t}) \sim \mathrm{N}(\mu, \Sigma)$$

Table 1 shows parameter values for our baseline specification. The log risk-free rate is $r_{\mathrm{F}} = 2\%$. The log-return on stocks has expectation $\mu_{\mathrm{S}} = 6\%$ and volatility $\sigma_{\mathrm{S}} = 20\%$. The parameters for PE investments, ρ, μ_{P} and σ_{P}, imply that PE has $\alpha = 3\%$ and

$\beta = 1.6$, as defined by the log-linear CAPM:

$$\ln \mathrm{E}[R_{\mathrm{P}}] - r_{\mathrm{F}} = \alpha + \beta \left(\ln \mathrm{E}[R_{\mathrm{S}}] - r_{\mathrm{F}}\right) \tag{11}$$

with $\beta = \rho\sigma_{\mathrm{P}}/\sigma_{\mathrm{S}}$.

Our parameters differ slightly from those in Ang et al. (2014), who assume a risk-free rate of 4%, expected log-return on stocks of 12%, and volatility of 15%. To isolate the effect of illiquidity, Ang et al. (2014) choose ρ, μ_{P}, and σ_{P} so PE investments have the same risk and return as the stock market. In our specification, PE investments are positively correlated with stocks, are more volatile than stocks, and have a positive alpha. For robustness we also solve the model for other specifications of alpha, beta, and the idiosyncratic risk, as reported in Appendix C.

Table 1: This table reports parameter values used as inputs of the model. It also includes important statistics that are implied by those parameter values.

Parameter/Statistic	Expression	Value
Draw-downs of new commitments	λ_{N}	30%
Draw-downs of older commitments	λ_{K}	30%
Distribution intensity	λ_{D}	40%
Log of risk-free rate	r_{F}	2%
Expected log-return to stocks	μ_{S}	6%
Expected log-return to PE	μ_{P}	6.6%
Volatility of log-return to stocks	σ_{S}	20%
Volatility of log-return to PE	σ_{P}	40%
Alpha of PE	α	3%
Beta of PE	β	1.6
Subjective discount factor	δ	95%
Implied:		
Log of expected return to stocks	$\ln \mathrm{E}[R_{\mathrm{S}}]$	8%
Log of expected return to PE	$\ln \mathrm{E}[R_{\mathrm{P}}]$	14.6%
Sharpe ratio of stocks	$\left(\ln \mathrm{E}[R_{\mathrm{S}}] - r_{\mathrm{F}}\right)/\sigma_{\mathrm{S}}$	0.3
Sharpe ratio of PE	$\left(\ln \mathrm{E}[R_{\mathrm{P}}] - r_{\mathrm{F}}\right)/\sigma_{\mathrm{P}}$	0.315
Correlation of log-returns to stocks and PE	ρ	0.8

1.4 Reduced Portfolio Problem

The LP's preferences, liquidity constraint, and the dynamics of contributions and distributions define the LP's reduced problem:

$$
\begin{aligned}
&V(W,P,K) = \max_{(C,N,S)} \left\{C^{1-\gamma} + \delta \mathrm{E}\left[V(W',P',K')^{1-\gamma}\right]\right\}^{\frac{1}{1-\gamma}} \\
&\text{subject to} \\
&W' = R_{\mathrm{P}}P + R_{\mathrm{S}}S + R_{\mathrm{F}}\left(W - P - C - S\right) \\
&P' = \left(1-\lambda_{\mathrm{D}}\right) R_{\mathrm{P}}P + \lambda_{\mathrm{N}}N + \lambda_{\mathrm{K}}K \\
&K' = (1-\lambda_{\mathrm{K}})K + (1-\lambda_{\mathrm{N}})N \\
&0 \le C \le W - P \\
&N \ge 0
\end{aligned}
\tag{12}
$$

The problem has three state variables: W is total wealth, P is aggregate illiquid wealth (NAV), and K is aggregate stock of uncalled commitments. Each period illiquid wealth (NAV), P, earns the return R_{P}, decreases by the distributed amount, $\lambda_{\mathrm{D}}R_{\mathrm{P}}P$, and increases by the contributed capital, $\lambda_{\mathrm{N}}N + \lambda_{\mathrm{K}}K$. The end-of-period stock of uncalled commitments, K, increases by new commitments, N, and decreases by the amount of contributed capital, $\lambda_{\mathrm{N}}N + \lambda_{\mathrm{K}}K$.

The LP maximizes the value function over three controls: First, it chooses new commitments, $N \ge 0$. Second, the LP chooses consumption, $C \ge 0$, and since the LP can only consume out of liquid wealth, $C \le W - P$. If it exhausts its liquid wealth and is unable to consume, it has negative infinite utility. Third, the LP chooses its stock allocation, S, and the remaining wealth, $W - C - P - S$, is invested in bonds.

The value function is homogeneous in wealth, and we guess the functional form:

$$V(W,P,K) = v(p,k)W \tag{13}$$

where $k = K/W$ and $p = P/W$ are the LP's uncalled commitments and illiquid wealth normalized by total wealth. Substituting into the Bellman equation (12), we verify the guess for $v(k, p)$ solving the normalized problem:

$$
\begin{aligned}
& v(p,k) = \max_{(c,n,\omega_\mathrm{S})} \left\{ c^{1-\gamma} + \delta \mathrm{E}\left[\left[(1-c) R_\mathrm{W} v(p',k') \right]^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}} \\
& \text{subject to} \\
& R_\mathrm{W} = \omega_\mathrm{P}(R_\mathrm{P} - R_\mathrm{F}) + \omega_\mathrm{S}(R_\mathrm{S} - R_\mathrm{F}) + R_\mathrm{F} \\
& p' = \left[(1-\lambda_\mathrm{D}) R_\mathrm{P} p + \lambda_\mathrm{N} n + \lambda_\mathrm{K} k \right] / \left[(1-c) R_\mathrm{W} \right] \\
& k' = \left[k + n - \lambda_\mathrm{N} n - \lambda_\mathrm{K} k \right] / \left[(1-c) R_\mathrm{W} \right] \\
& 0 \leq c \leq 1 - p \\
& n \geq 0
\end{aligned}
\tag{14}
$$

The normalized control variables are $c = C/W$, $s = S/W$, and $n = N/W$. Furthermore, it is convenient to define the portfolio weight $\omega_\mathrm{S} = S/(W-C)$, $\omega_\mathrm{P} = P/(W-C)$, and $\omega_\mathrm{B} = 1 - \omega_\mathrm{S} - \omega_\mathrm{P}$. These portfolio weights are normalized by the LP's post-consumption wealth in order to sum to one.[11]

1.5 Liquidity Constraint

The LP ensures that there is zero probability that it exhausts its liquid wealth, or, equivalently, that its illiquid wealth, P, exceeds its total wealth, W. Since $p = P/W$, the optimal policy satisfies the liquidity constraint:

$$
\mathrm{Prob}(p > 1) = 0 \tag{15}
$$

This liquidity constraint is the main friction in our analysis. One immediate implication of the constraint and the unbounded support of the log-returns to stocks and PE investments is that the LP wants to hold a liquidity reserve of safe assets, in our case risk-free bonds, at least as large as its total stock of uncalled commitments. Another immediate implication of the liquidity constraint is that the LP will not leverage its allocations to stocks and PE by holding a negative amount of bonds, because a large negative shock then leaves the LP with insufficient liquid wealth. Moreover, the LP will not short stocks, because a large positive shock then exhausts its liquid wealth. As in Longstaff (2001), the LP optimally

[11] Below, the optimal consumption ratio, c, is 4%–5% with minimal variation. Hence, the LP's PE allocation is approximately $\omega_\mathrm{P} \approx 1/0.95 \times p$.

acts as if subject to no-leverage and no-shorting constraints although these constraints are not explicitly imposed.

This liquidity constraint, however, is not straightforward to evaluate empirically. In practice LPs hold reserves in a variety of assets (the bonds in our model can be interpreted as a general reserve asset). Moreover, LPs hold other illiquid assets—such as direct investments in private companies, real estate, and natural resources—which have their own liquidity requirements and implications for the LP's liquidity reserve. Hence, a simple comparison of an LP's bond holdings to its stock of uncalled commitments may not be an adequate test of the liquidity constraint. One case where this comparison is meaningful is for the Return-Enhanced Investment Grade Notes (REIGNs) issued by KKR in 2019. These long-term notes have payouts backed by investments in PE funds as their sole illiquid asset, and they have a liquidity reserve expressly to cover the uncalled commitments of these PE funds. Consistent with our liquidity constraint, the policy for REIGNs states that "... the Liquidity Reserve is required to equal the sum of all scheduled fixed coupon payments on the Notes and all future drawdown obligations." (Kroll Bond Rating Agency, 2019).

Proposition 1 (proof in Internet Appendix) shows the dynamic implications of the liquidity constraint. In the dynamic model the LP may need to hold a reserve of bonds, depending on its new and existing commitments, to ensure that it can make current and future contributions, and the proposition captures this effect.

Proposition 1. *Any optimal strategy satisfies the following inequality:*

$$B_t \quad \geq \quad a_{\mathrm{K}} K_t + a_{\mathrm{N}} N_t \tag{16}$$

The coefficients a_{K} *and* a_{N} *are given by:*

$$a_{\mathrm{K}} = \frac{\lambda_{\mathrm{K}}}{\lambda_{\mathrm{K}} + R_{\mathrm{F}} - 1} \qquad a_{\mathrm{N}} = \frac{\frac{\lambda_{\mathrm{K}}}{R_{\mathrm{F}}} + \lambda_{\mathrm{N}}\left(1 - \frac{1}{R_{\mathrm{F}}}\right)}{\lambda_{\mathrm{K}} + R_{\mathrm{F}} - 1}$$

Proposition 1 gives the required liquidity reserve that ensures that the LP can be certain to meet future contributions. The reserve is linear in the LP's new and existing commitments, with the coefficients, a_{K} and a_{N}, both between zero and one. Economically, the proposition captures the appreciation of the liquidity reserve due to the bonds earning the risk-free rate, which relaxes the liquidity constraint. Both a_{K} and a_{N} are decreasing in R_{F}, and they attain their maximal values, $a_{\mathrm{K}} = a_{\mathrm{N}} = 1$, when $R_{\mathrm{F}} = 1$. In this case, the risk free rate is zero, and the required liquidity reserve is simply $B_t \geq K_t + N_t$. Faster capital calls increase the size of the required liquidity reserve, because the reserve has less time to appreciate before it is needed to fund contributions. Hence, a_{K} and a_{N} increase in λ_{K}, and a_{N} increases in λ_{N}. Similarly, $a_{\mathrm{N}} - a_{\mathrm{K}}$ has the same sign as $\lambda_{\mathrm{N}} - \lambda_{\mathrm{K}}$, because when new commitments are called faster than existing commitments, the new commitments have less time to appreciate and therefore require a larger reserve. Finally, the liquidity reserve is independent of the rate of distributions, λ_{D}, because distributions are risky, and the LP cannot rely on them to pay future contributions.

2 Optimal Allocation to Private Equity

We use a standard value-function iteration algorithm (Ljungqvist and Sargent, 2018) to solve the Bellman equation of the normalized problem in equation (14) for different levels of risk aversion. We simulate the model 1000 times for initial convergence ("burn-in") followed by 10,000 simulations to recover the joint distribution of returns, state variables, and choice variables. Appendix A describes the details of the numerical procedure. Panel A of Table 2 shows the resulting distributions of the state variables, choice variables, and portfolio returns. For comparison, Panels B and C also show the solutions to the liquid model and the model with a secondary market, discussed below.

The average allocations are shown in Figure 6. Unsurprisingly, LPs with higher risk aversion choose smaller stock allocations and larger bond allocations. Perhaps more surprisingly, the optimal PE allocation is not monotonically decreasing with risk aversion. It follows an inverse-U shape and largely remains in the range of 15%–25%. To illustrate, the conservative LP, with $\gamma = 3$, has an average PE allocation of $\omega_{\mathrm{P}} = 16.2\%$, and its new capital commitments are $n = 5.3\%$, on average, bringing the total amount of uncalled commitments of $n + k = 17.5\%$ each period. The aggressive LP, with $\gamma = 1$, has a PE allocation of $\omega_{\mathrm{P}} = 23.6\%$, new commitments of $n = 7.6\%$, and total uncalled commitments of $n + k = 25.9\%$, on average, each period. Their liquid holdings differ

Table 2: Summary statistics of state and choice variables implied by the LP's optimal policies. This table compares the distribution of state variables, choice variables, and portfolio returns resulting from the optimal policies of two types of LP, aggressive and conservative, across three different models. The reduced model is given in Section 1, the liquid model is in Appendix B, and the secondary market is discussed in Section 3. All figures in the table are percentages.

Panel A: Reduced Model

	Aggressive ($\gamma = 1$)					Conservative ($\gamma = 3$)				
Variable	Mean	S.D.	p5	p50	p95	Mean	S.D.	p5	p50	p95
p	22.4	2.9	18.0	22.1	27.6	15.5	2.6	12.0	15.2	20.4
k	18.3	3.4	12.9	18.1	24.0	12.3	2.9	7.0	12.3	17.0
n	7.6	3.3	1.8	7.7	12.9	5.3	3.2	0.4	5.0	10.8
$k+n$	25.9	0.3	25.5	25.9	26.2	17.5	3.7	10.0	17.8	23.3
ω_{P}	23.6	3.1	19.0	23.3	29.1	16.2	2.7	12.5	15.8	21.3
ω_{S}	50.9	3.0	45.8	50.9	55.1	24.4	4.1	16.7	25.0	30.1
ω_{B}	25.6	0.3	25.1	25.6	25.9	59.4	1.4	57.4	59.2	62.0
$R_{\mathrm{W}} - R_{\mathrm{F}}$	6.4	21.4	-22.7	3.3	45.5	3.7	12.6	-13.2	1.8	27.0
c	5.1	0.0	5.0	5.1	5.1	4.1	0.0	4.1	4.1	4.1

Panel B: Liquid Model (first-best)

	Aggressive ($\gamma = 1$)					Conservative ($\gamma = 3$)				
Variable	Mean	S.D.	p5	p50	p95	Mean	S.D.	p5	p50	p95
p	55.6	0.0	55.6	55.6	55.6	15.2	0.0	15.2	15.2	15.2
ω_{P}	58.5	0.0	58.5	58.5	58.5	15.9	0.0	15.9	15.9	15.9
ω_{S}	41.5	0.0	41.5	41.5	41.5	24.8	0.0	24.8	24.8	24.8
ω_{B}	0.0	0.0	0.0	0.0	0.0	59.3	0.0	59.3	59.3	59.3
$R_{\mathrm{W}} - R_{\mathrm{F}}$	10.5	35.9	-36.6	4.4	77.1	3.6	12.4	-13.3	1.9	26.8
c	5.0	0.0	5.0	5.0	5.0	4.1	0.0	4.1	4.1	4.1

Panel C: Secondary Market

	Aggressive ($\gamma = 1$)					Conservative ($\gamma = 3$)				
Variable	Mean	S.D.	p5	p50	p95	Mean	S.D.	p5	p50	p95
p	53.9	4.4	48.3	53.2	61.6	15.5	2.6	12.1	15.1	20.2
k	46.1	13.6	26.8	44.7	70.6	12.2	3.1	6.6	12.4	17.1
n	18.5	12.5	0.0	18.4	39.0	5.3	3.4	0.7	4.8	11.6
$k+n$	64.6	4.9	59.3	64.0	71.5	17.5	4.0	9.7	17.7	23.4
ω_{P}	56.7	4.6	50.1	56.0	64.8	16.2	2.7	12.6	15.8	21.0
ω_{S}	36.9	4.9	28.8	37.7	42.5	24.4	4.1	17.0	25.0	29.9
ω_{B}	6.4	0.9	5.8	6.3	7.0	59.4	1.4	57.5	59.2	62.0
$R_{\mathrm{W}} - R_{\mathrm{F}}$	10.1	34.0	-34.0	4.3	73.8	3.7	12.6	-13.1	1.8	26.8
c	5.0	0.0	5.0	5.0	5.1	4.1	0.0	4.1	4.1	4.1

more substantially. The conservative LP, on average, holds $w_{\mathrm{S}} = 24.4\%$ in stocks and $\omega_{\mathrm{B}} = 59.4\%$ in bonds. The aggressive LP holds $\omega_{\mathrm{S}} = 50.9\%$ in stocks, almost twice as much as the conservative LP, and it holds only $\omega_{\mathrm{B}} = 25.6\%$ in bonds. Despite PE investments being substantially more risky than both stocks and bonds, the PE allocation is much less sensitive to the LP's risk aversion.

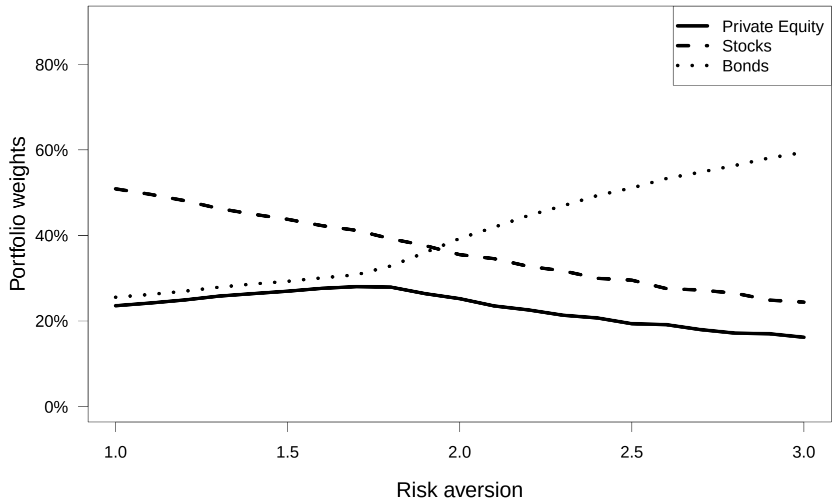


Figure 6: Average portfolio weights resulting from the model. This figure plots the average allocation of LPs solving our investment problem where PE is illiquid and requires commitment. Average allocations are computed in three steps. First, we solve numerically the Bellman equation (14). At any point in the state space, that solution provides the corresponding portfolio weights in stocks $\omega_{\mathrm{S}}^* = s^*/(1-c^*)$ and PE $\omega_{\mathrm{P}}^* = p/(1-c^*)$, while $1 - \omega_{\mathrm{S}}^* - \omega_{\mathrm{P}}^*$ is the risk-free allocation. Second, we simulate the evolution of those variables under optimal strategies. Third, we compute their averages. This procedure is repeated for relative risk aversion $\gamma \in \{1, 1.1, 1.2, \ldots, 2.9, 3\}$.

These allocations can be compared to the first-best allocations, which are the optimal allocations if PE were liquid and could be freely traded at a price equal to the NAV, effectively making PE investments another traded stock. This liquid model is standard, it is described in Appendix B, and the optimal allocations are given in Panel B of Table 2 and shown in Figure 7. The LP's PE allocation now declines in risk aversion, as expected. Moreover, for more risk averse LP's, the first-best allocations from the liquid model largely coincide with the allocations with illiquid PE investments. To illustrate, for the conservative LP the average PE allocation is $\omega_{\mathrm{P}} = 16.2\%$ which effectively equals its first-best allocation of $\omega_{\mathrm{P}} = 15.9\%$. Intuitively, the liquidity constraint tends to be slack for more risk averse LPs, and they can effectively ignore the illiquidity of PE investments in their portfolio allocations.

In contrast, for less risk averse LPs the liquidity constraint tends to bind, and these LPs allocate substantially less capital to PE and substantially more to bonds than their first-best allocations. For the aggressive LP, the average PE allocation is $\omega_\mathrm{P} = 23.6\%$ and its first-best allocation is $\omega_\mathrm{P} = 58.5\%$.

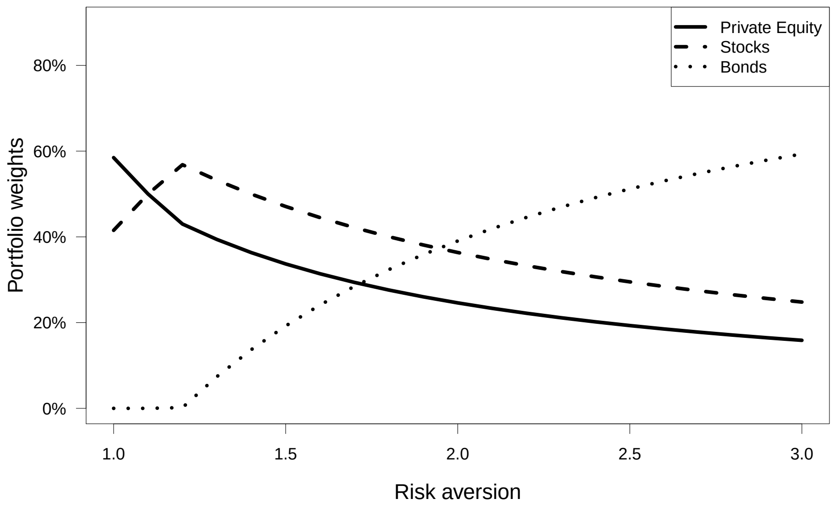


Figure 7: Optimal portfolio weights with liquid PE. This figure plots the optimal portfolio weights of investors operating in perfectly liquid markets. Optimal portfolio weights in stocks and PE are respectively ω_S and ω_P solving Bellman equation (B.2). The residual weight goes to risk-free bonds. Markets are perfectly liquid because PE can be traded freely like stocks and bonds. We consider investors with relative risk aversion coefficient $\gamma \in \{1, 1.1, 1.2, \ldots, 2.9, 3\}$.

Because the conservative LP's portfolio is largely unaffected by the illiquidity of PE investments, its returns are also largely unaffected, and it earns an average return of 3.7% both with illiquid and liquid PE investments. In contrast, the aggressive LP earns an average return of just 6.4% with illiquid PE investments, which is substantially below the 10.6% average return this LP would earn from its first-best allocation.

These allocations and returns are calculated for the baseline specification where PE has a beta of 1.6 and an alpha of 3%. We solve the model for other specifications, and Appendix C shows the optimal allocations and policies. A consistent finding across these specifications is that PE allocations are less sensitive to the LP's risk aversion than the allocations to stocks and bonds, despite PE being substantially more risky.

2.1 Value Function

Figure 8 shows the LP's normalized value function, $v(p, k)$, as a function of the two normalized state variables: illiquid wealth as a fraction of total wealth, $p = P/W$, and uncalled commitments as a fraction of total wealth, $k = K/W$. Returns are risky and transitions in the state space are stochastic. In Figure 8 the state space is divided into cells with sides of 2%, and the shading indicates how frequently each cell is visited during the simulations. Cells that are never visited are blank. Cells that are visited at least once (and less than 10%) are increasingly darker. A single cell is visited in more than 10% of the simulations, which is when then conservative LP holds $14\% < p < 16\%$ and $12\% < k < 14\%$, and it is shaded black.

For both the aggressive ($\gamma = 1$) and the conservative LP ($\gamma = 3$) the value function is decreasing in uncalled commitments, k, when uncalled commitments increases above the shaded area of the state space. This follows naturally from the first-order condition for k, because an LP can freely commit additional capital to PE, and the LP will continue to commit new capital as long as the value function is increasing.

Importantly, in Figure 8 the value function increases in illiquid wealth, p, for the aggressive LP ($\gamma = 1$), but it decreases in p for the conservative LP ($\gamma = 3$), above the shaded part of the state space. An aggressive LP would prefer to increase its PE exposure, but the binding liquidity constraint makes it costly to commit additional capital because it also requires an increase in the LP's liquidity reserve. In contrast, the conservative LP's liquidity constraint tends to be slack, and its portfolio moves around an interior optimum. Its first-best bond allocation exceeds the amount needed for its liquidity reserve, and it can freely commit additional capital to PE until it reaches its first-best allocation, given by the liquid model.

2.2 Optimal New Commitments

The aggressive and conservative LPs have different optimal policies for new PE commitments, as illustrated in Figure 9. A conservative LP ($\gamma = 3$) tends to be close to an interior optimum, and in response to a positive shock to the NAV of its PE investments this LP reduces its new PE commitments to rebalance its PE exposure back towards the

a. Aggressive ($\gamma = 1$) b. Conservative ($\gamma = 3$)

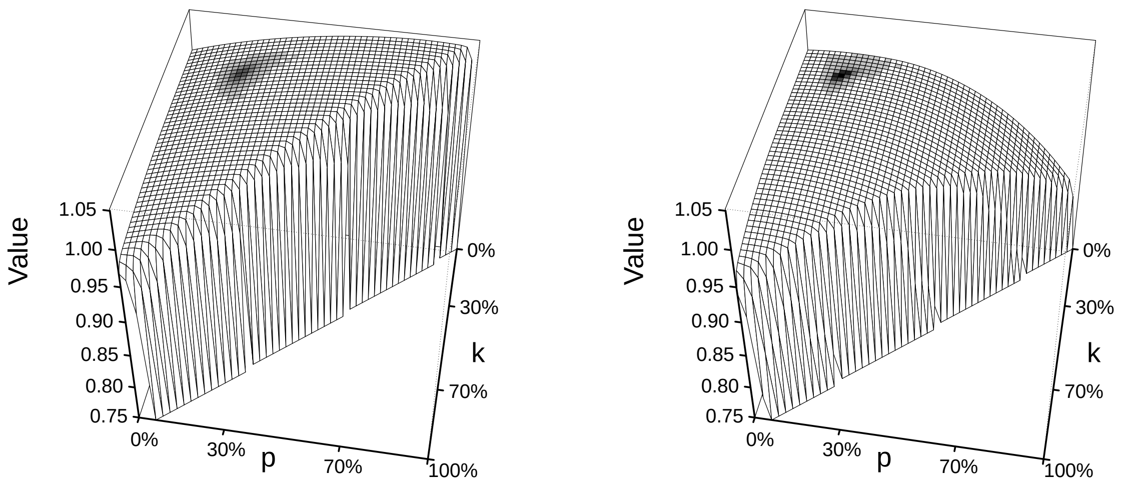


Figure 8: Value function per unit of wealth. This figure compares the value function $v(p, k)$ of the normalized Bellman equation (14) for investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). In the figure, each function is rescaled and expressed in units of value at $(p, k) = (0, 0)$. That is, we plot $v(p, k)/v(0, 0)$ for the two types of investors. We only display functional values above a certain threshold, which is 0.75 for the low risk-aversion case and 0.7 for the high risk-aversion case. The length of each axis is kept constant across the two cases, so the shapes of the functions are directly comparable. The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally.

interior optimum. In contrast, an aggressive LP prefers a greater PE exposure but it cannot commit additional capital due to the binding liquidity constraint. The aggressive LP therefore does not reduce its new PE commitments in response to a positive shock to the NAV of its existing PE investments. Instead, it enjoys a temporary increase in its PE exposure, and it continues to make new commitments at the same maximally sustainable rate, as seen in Figure 9.

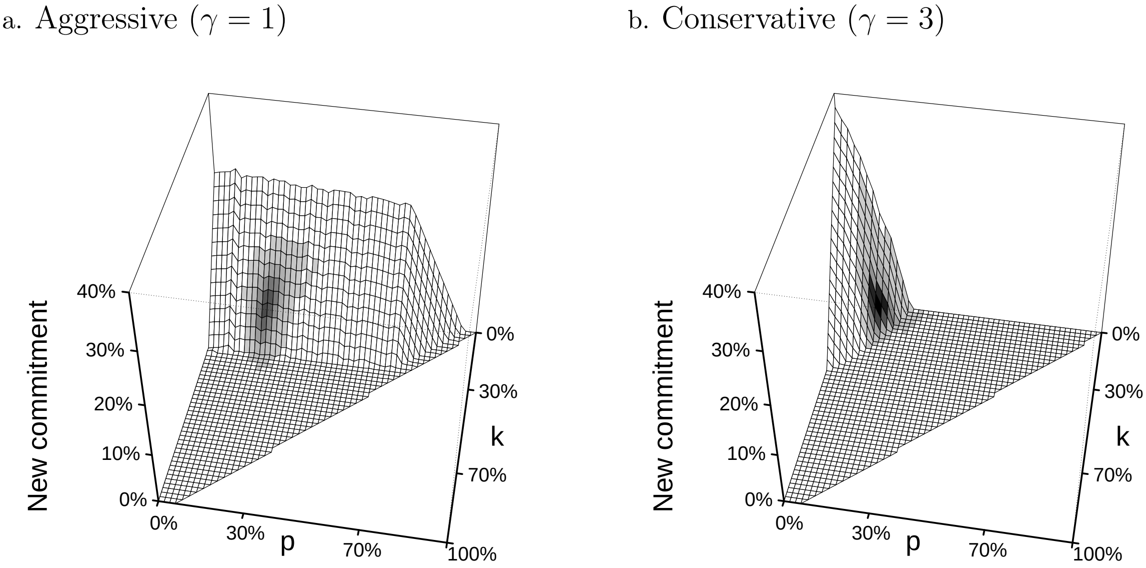


Figure 9: Optimal new commitment. This figure compares optimal commitment strategy for investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). For each type, the plots show optimal new commitment as a share of wealth $n^*(p, k)$ obtained solving the Bellman equation (14). The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally.

Figure 9 shows that the optimal policies for new PE commitments consist of multiple segments, each of which appears to be close to linear. The shading shows, however, that the state mostly moves on a single such segment, although it occasionally drops to the bottom segment with zero new commitments. In Table 2 the aggressive LP's average (standard deviation) of new commitments, n, is 7.6% (3.3%) (and n equals zero for 0.5% of the simulations). For the conservative LP, n is 5.3% (3.2%) (and n equals zero for 1.5% of the simulations).

Since the state remains mostly on a single linear segment, we can locally approximate the optimal policy by regressing the optimal new commitments, $n^*(p, k)$, on the two normalized state variables. Table 3 shows that regressions with and without an interaction term have similar coefficients, and that the R^2 only increases marginally when the interaction term is included, confirming that the segment is close to linear and is well approximated by the linear regression. Hence, we focus on specifications (1) and (3), without interaction terms, because their coefficients are easier to interpret. Figures C.5 and C.6 in the appendix show the optimal policies for other parameter choices, and Table C.7 contains the corresponding regression coefficients. In Table 3 the coefficient for PE exposure (or

Table 3: This table reports coefficients β_0, β_1, β_2, and β_3 from OLS estimations of the following model:

$$n^*(p,k) = \beta_0 + \beta_1 p + \beta_2 k + \beta_3 (p \times k) + \epsilon$$

In this equation, ϵ is an error term, while $n^*(p,k)$ is the optimal commitment function obtained solving problem (14), and plotted in figure 9 for two different levels of risk-aversion. The model is estimated twice for the case of low risk-aversion ($\gamma = 1$) and twice for the case of high risk-aversion ($\gamma = 3$). For each level of risk-aversion, the first estimation imposes $\beta_3 = 0$, while the second estimation ignores that restriction. The data for each estimation is taken from a Monte Carlo simulation where the evolution of the state variables is simulated under optimal policies. We run the simulation for 11000 periods, discarding the first 1000 periods and eventual observations for which $n^* = 0$.

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.02	−0.07	−1.45	−1.53
Commitments (k)	−0.99	−1.04	−0.98	−1.09
Interaction ($p \times k$)		0.27		0.71
Constant	0.26	0.27	0.40	0.41
R^2	1.00	1.00	0.98	0.98

NAV), p, is consistently close to zero for the aggressive LP, and it is negative for the conservative LP (see Table C.7 for the variation across other specifications). The coefficient for uncalled commitments, k, is close to -1.0 for both LPs. Hence, the optimal policy for new PE commitments is approximately:

$$n^*(p,k) + k \approx \text{constant} - \pi_{\text{P}} p \tag{17}$$

where π_{P} is the negative of the regression coefficient on p. For example, in specification (3) in Table 3, it equals $\pi_{\text{P}} = 1.45$. The LPs optimally determine their new commitments, n, to target a level certain of total uncalled commitments, $n + k$. For the aggressive LP, the optimal policy for new commitments is simple. The coefficient on p is close to zero, so π_{P} is also close to zero, and the aggressive LP targets a fixed level of uncalled commitments, $n+$

$k \approx$ constant, regardless of the performance of its other investments. For the conservative LP, the coefficient π_{P} is positive, and this LP optimally reduces the target level of uncalled commitments in response to a positive shock its PE investments.

2.3 Optimal Stock and Bond Allocations

a. Aggressive ($\gamma = 1$) b. Conservative ($\gamma = 3$)

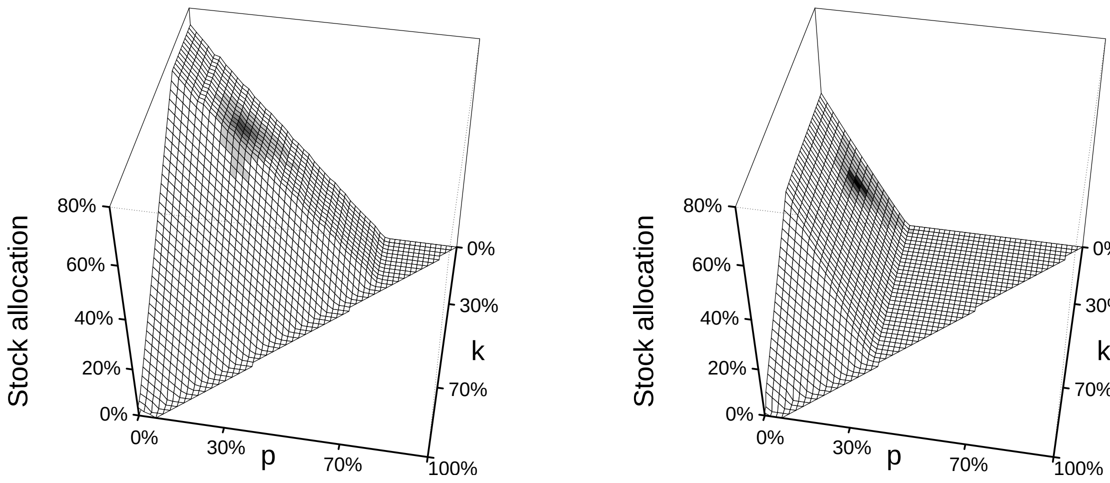


Figure 10: Optimal stock allocation. This figure compares optimal stock allocation for investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). For each type, the plots show optimal portfolio weight in stocks $\omega^*_{\mathrm{S}}(p, k)$ obtained solving the Bellman equation (14). The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally.

Figure 10 shows optimal stock allocations as functions of the state variables: illiquid wealth, p, and uncalled commitments, k. Table 2 shows that the aggressive LP's average (standard deviation) stock allocation is $\omega_{\mathrm{S}} = 50.9\%$ (3.0%), its average bond allocation is $\omega_{\mathrm{B}} = 25.6\%$ (0.3%), and its average PE allocation is $\omega_{\mathrm{P}} = 23.6\%$ (3.1%). For the conservative LP, the average stock allocation is 24.4% (4.1%), average bond allocation is 59.4% (1.4%), and average PE allocation is 16.2% (2.7%). Table 4 shows the coefficients from a regression of the optimal stock allocation on the state variables in the shaded part of the state space. As above, the optimal allocation is well approximated by a linear regression, and we focus on specifications (1) and (3) without interaction terms.

In Table 4, the coefficients on uncalled commitments, k, are close to zero, and the LPs' optimal stock allocations are largely unaffected by their aggregate uncalled PE commitments. This is natural. The LPs commit new capital, n, each period to bring their total uncalled commitments, $k+n$, to a target level. Small deviations in the LPs' uncalled commitments, k, are therefore immaterial, because the LPs can undo these deviations by adjusting their new commitments, n, to reach the same target level. Only large positive shocks to uncalled commitments can affect the LPs' optimal allocations, because new commitments cannot be reduced below zero in response to such shocks. Hence, the LPs' optimal stock allocation is approximately:

$$\omega_{\mathrm{S}}^{*}(p,k) \approx \text{constant} - \phi_{\mathrm{P}}\omega_{\mathrm{P}} \tag{18}$$

In this expression, the PE portfolio weight, ω_{P}, equals $p/(1-c)$, the coefficient ϕ_{P} is the negative of the regression coefficient on p multiplied by $1-c$, where c is the LP's average consumption-to-wealth ratio. The approximation then follows from the low volatility of the optimal consumption-to-wealth ratio seen in Table 2. For example, in specification (3) of Table 4, $\phi_{\mathrm{P}} = 1.59 \times (1-0.041) = 1.53$.

The conservative LP hedges a larger PE exposure by reducing its stock allocation. Across the specifications in Table 4, the conservative LP's coefficient for PE allocation, ϕ_{P}, is close to the β of PE. Intuitively, the conservative LP is unconstrained, and when the beta of PE equals 1.6, as in our baseline specification, the conservative LP optimally responds to a 1% increase in its PE allocation, ω_{P}, by reducing its stock allocation by 1.6% to maintain a constant risk exposure across its total portfolio.

In contrast, the aggressive LP has a ϕ_{P} close to one. This LP's liquidity constraint binds, and it maintains a maximally sustainable level of uncalled commitments along with a liquidity reserve of bonds that is consistent with this level. Since $\omega_{\mathrm{B}}^{*}(p,k) = 1 - \omega_{\mathrm{S}}^{*}(p,k) - \omega_{\mathrm{P}}$, rearranging the expression shows that the aggressive LP's optimal bond allocation, ω_{B}, is approximately constant:

$$\omega_{\mathrm{S}}^{*}(p,k) + \omega_{\mathrm{P}} \approx \text{constant} \Leftrightarrow \omega_{\mathrm{B}}^{*}(p,k) \approx 1 - \text{constant} \tag{19}$$

Table 4: This table reports coefficients β_0, β_1, β_2, and β_3 from OLS estimations of the following model:

$$\omega^*_{\mathrm{S}}(p,k) = \beta_0 + \beta_1 p + \beta_2 k + \beta_3(p \times k) + \epsilon$$

In this equation, ϵ is an error term, while $\omega^*_{\mathrm{S}}(p,k)$ is the optimal stock allocation obtained solving problem (14), and plotted in figure 10 for two different levels of risk-aversion. The model is estimated twice for the case of low risk-aversion ($\gamma = 1$) and twice for the case of high risk-aversion ($\gamma = 3$). For each level of risk-aversion, the first estimation imposes $\beta_3 = 0$, while the second estimation ignores that restriction. The data for each estimation is taken from a Monte Carlo simulation where the evolution of the state variables is simulated under optimal policies. We run the simulation for 11000 periods, discarding the first 1000 periods and eventual observations for which $\omega^*_{\mathrm{S}} = 0$.

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−1.04	−1.01	−1.59	−1.56
Commitments (k)	−0.03	0.00	0.00	0.05
Interaction ($p \times k$)		−0.13		−0.31
Constant	0.75	0.74	0.49	0.49
R^2	0.99	0.99	1.00	1.00

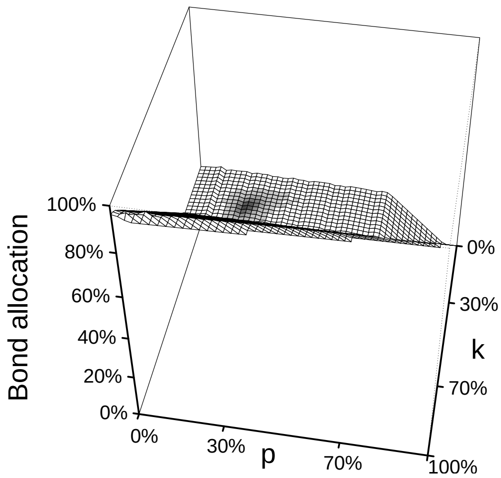
Table 5: This table reports coefficients β_0, β_1, β_2, and β_3 from OLS estimations of the following model:

$$\omega^*_{\mathrm{B}}(p,k) = \beta_0 + \beta_1 p + \beta_2 k + \beta_3(p \times k) + \epsilon$$

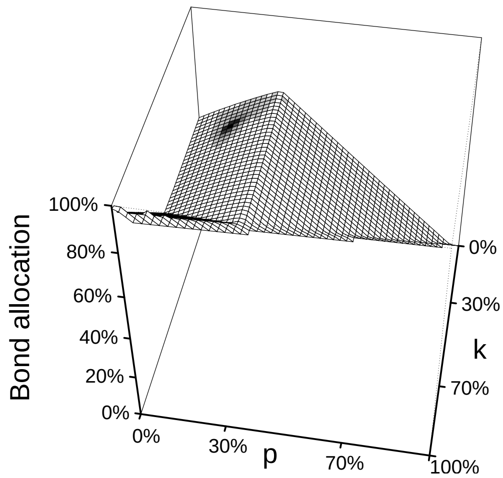
In this equation, ϵ is an error term, while $\omega^*_{\mathrm{B}}(p,k) = 1 - \omega^*_{\mathrm{S}}(p,k) - \omega^*_{\mathrm{P}}(p,k)$ is the optimal bond allocation obtained solving problem (14), and plotted in figure 11 for two different levels of risk-aversion. The model is estimated twice for the case of low risk-aversion ($\gamma = 1$) and twice for the case of high risk-aversion ($\gamma = 3$). For each level of risk-aversion, the first estimation imposes $\beta_3 = 0$, while the second estimation ignores that restriction. The data for each estimation is taken from a Monte Carlo simulation where the evolution of the state variables is simulated under optimal policies. We run the simulation for 11000 periods, discarding the first 1000 periods and eventual observations for which $\omega^*_{\mathrm{S}} = 0$.

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.02	−0.04	0.55	0.52
Commitments (k)	0.03	0.00	0.00	−0.05
Interaction ($p \times k$)		0.13		0.31
Constant	0.25	0.26	0.51	0.51
R^2	0.15	0.16	1.00	1.00

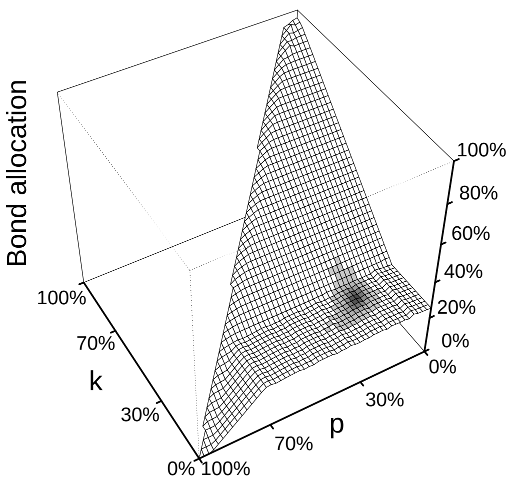
a. Aggressive ($\gamma = 1$)



b. Conservative ($\gamma = 3$)



c. Aggressive ($\gamma = 1$, rotated)



d. Conservative ($\gamma = 3$, rotated)

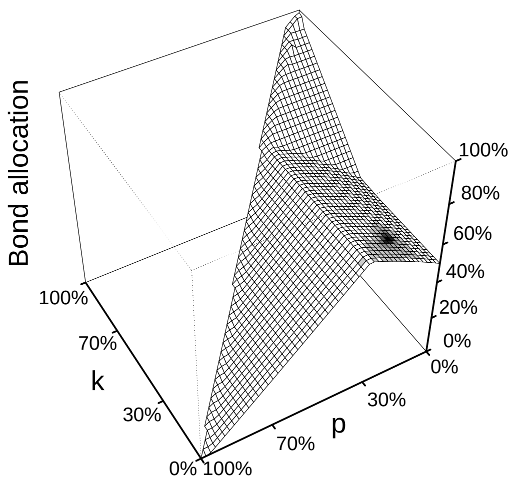


Figure 11: Optimal bond allocation. This figure compares optimal bond allocation for investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). For each type, the plots show optimal portfolio weight in bonds $\omega^*_{\mathrm{B}}(p,k) = 1 - \omega^*_{\mathrm{S}}(p,k) - \omega^*_{\mathrm{P}}(p,k)$ obtained solving the Bellman equation (14). The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally. The two plots at the bottom of the figure are rotated versions of those at the top.

2.4 Optimal Consumption

Figure 12 shows optimal consumption for the conservative and aggressive LPs. A standard result for the liquid model with i.i.d. shocks and CRRA preferences is that optimal consumption is a constant fraction of total wealth. Figure 12 confirms that with illiquid PE investments the optimal consumption rate remains close to constant in the shaded part of the state space. The aggressive LP's average (standard deviation) consumption ratio, c, is 5.1% (0.020%), and the conservative LP's is 4.1% (0.001%). Brown, Dimmock, Kang, and Weisbenner (2014) study the payout policies of university endowments and report typical payout rates of 4%–6%, which is consistent with the optimal consumption rate in our model.

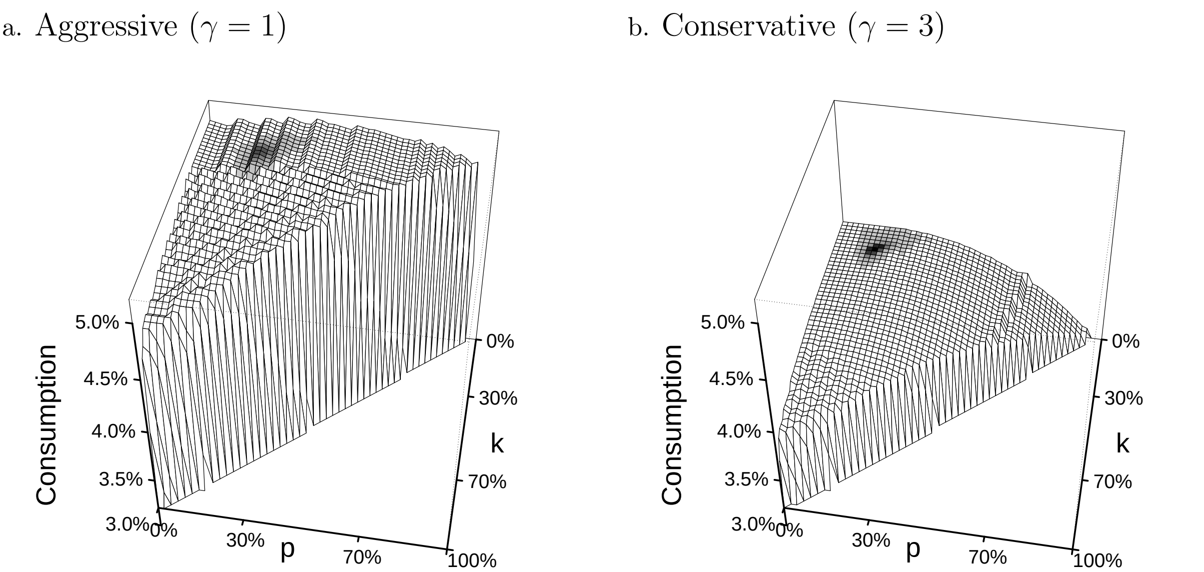


Figure 12: Optimal consumption as a fraction of wealth. This figure compares optimal consumption strategy for investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). For each type, the plots show optimal consumption to wealth ratio $c^*(p, k)$ obtained solving the Bellman equation (14). The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally.

3 Secondary Market

We now introduce a secondary market for PE investments where LPs can trade partnership interests in PE fund.[12] This creates two effects: First, there are gains from trade

[12]The secondary market for partnership interests is distinct from the market where PE funds trade underlying private assets, typically by buying and selling equity stakes in portfolio companies, which is

when unconstrained LPs can sell partnership interests in mature PE funds to constrained LPs. Mature PE funds are partly invested, with positive NAVs and fewer uncalled commitments, allowing a constrained LP to increase its PE exposure with a relatively smaller increase in its liquidity needs. Second, following a negative shock to the value of its liquid investments, a stressed LP may decide to liquidate a fraction of its PE investments, even at a discount, to raise liquidity. Moreover, access to liquidity, ex-post, relaxes the LP's liquidity constraint, ex-ante, allowing it to hold more risky allocations.

3.1 Gains from Trade

An aggressive constrained LP assigns a higher valuation to a mature PE fund than a conservative unconstrained LP. The aggressive LP prefers greater PE exposure, but it faces a binding liquidity constraint, and a mature fund provides PE exposure with a relatively smaller amount of uncalled commitments. In contrast, a conservative LP assigns no special value to mature PE funds compared to simply committing additional capital to new PE funds, even though these new commitments imply a higher level of uncalled commitments.

To quantify the gains from trade, Table 6 shows the effects of a transfer of PE investments from a conservative to an aggressive LP. In this calculation, the two LPs have the same amounts of wealth, so the transfer represents the same percentage change in their PE investments. The transfer is compensated by a payment of liquid wealth that leaves both LPs' total wealth unchanged. The two LPs start with the average levels of NAV, p, and uncalled commitments, k, that are implied by their respective optimal investment policies. For the conservative LP, these levels are $p_{\mathrm{C}} = 15.5\%$ and $k_{\mathrm{C}} = 12.3\%$ (see Table 2). For the aggressive LP, they are $p_{\mathrm{A}} = 22.4\%$ and $k_{\mathrm{A}} = 18.3\%$. The transaction transfers an amount of NAV, denoted Δp, and an amount of uncalled commitments, denoted Δk, from the conservative to the aggressive LP. The aggressive LP pays Δp of liquid wealth to the conservative LP, leaving their total wealths unchanged. After the transaction, the aggressive LP's NAV is $p_{\mathrm{A}} + \Delta p$, and its uncalled commitments are $k_{\mathrm{A}} + \Delta k$. The conservative LP has $p_{\mathrm{C}} - \Delta p$ and $k_{\mathrm{C}} - \Delta k$.

also called a "secondary market." Bollen and Sensoy (2020) model valuation with a secondary market for partnership interests. Hege and Nuti (2011), Albuquerque, Cassel, Phalippou, and Schroth (2018), and Nadauld, Sensoy, Vorkink, and Weisbach (2018) report empirical evidence of pricing and discounts for secondary transactions in the secondary market for partnership interests.

Table 6 reports normalized value functions, $v(p,k)$, for different levels of Δp and Δk. The first column of Table 6 shows the initial values, $v(p,k)$, before the transaction, i.e., with $\Delta p = \Delta k = 0\%$. The following columns show the effects of transacting PE funds with decreasing maturities, corresponding to increasing amounts of uncalled commitments, Δk, and decreasing amounts of NAV, Δp. Table 6 shows the two value functions and their sum. An increase in the sum implies that the transfer increases the two LPs' aggregate utility.

Even when the value function for the conservative LP declines, as long as the sum increases, there exists an additional transfer of liquid wealth that leaves both LPs strictly better off.

Table 6 shows that the conservative LP is largely unaffected by the transaction. This LP is unconstrained, with an excess liquidity reserve, and it compensates for the reduction in p by increasing its new commitments to rebuild its PE exposure. In the short term, it also increases its stock allocation to rebalance its risk exposure. The transfer of PE funds therefore leaves the conservative LP with only a minor and temporary deviation from its optimal risk exposures, and its value function remains effectively unchanged.

In contrast, the transaction benefits the aggressive LP by temporarily increasing its PE exposure. The benefit is larger for a more mature fund, with greater NAV and less uncalled commitments. The benefit declines for younger funds, and a transaction of just uncalled commitments, $\Delta p = 0\%$, does not benefit the aggressive LP.

In Table 6 the greatest increase in total valuations is 0.402%, and the gains from a single transaction are therefore modest. The gains are larger when the LPs transact repeatedly. Anticipating such repeated transactions, however, an unconstrained LP would increase its PE commitments to have extra PE funds to sell to a constrained LP, as these PE funds mature. In turn, a constrained LP would reduce its own commitments to new funds and instead manage its PE exposure by acquiring mature PE funds from an unconstrained LP, further relaxing its liquidity constraint. Formally, analyzing such strategic interactions among LPs is interesting, but it is well beyond the scope of this paper.

Table 6: This table quantifies the net welfare gains from a single transfer of PE partnership interests of $(\Delta p, \Delta k)$ from a conservative to an aggressive LP. The two LPs start with the same amounts of wealth and the average levels of p and k that are implied by their optimal investment policies.

Δp Δk	No trade	10% 0%	8% 2%	5% 5%	2% 8%	0% 10%
$v_{\mathrm{A}}(p_{\mathrm{A}} + \Delta p, k_{\mathrm{A}} + \Delta k)$	.0624	.0629	.0628	.0627	.0625	.0624
$v_{\mathrm{C}}(p_{\mathrm{C}} - \Delta p, k_{\mathrm{C}} - \Delta k)$	.0372	.0371	.0372	.0372	.0372	.0372
$v_{\mathrm{A}} + v_{\mathrm{C}}$	.0996	.1000	.0999	.0998	.0997	.0996
Gains from Trade	0.000%	0.402%	0.331%	0.221%	0.090%	0.000%

3.2 Insuring Liquidity Shocks

A secondary market for partnership interests can also insure LPs against liquidity shocks. To quantify this effect, we extend the model to allow the LP to sell, but not buy, a fraction of its PE investments each period. The extension is summarized below, and more details are in Appendix D. In practice, an LP must sell its currently held PE funds, with their actual mix of NAV and uncalled commitments. We capture this mix by restricting the LP to sell NAV and uncalled commitments in the same proportions as in the LP's current PE investments. Each period, t, the LP can liquidate a fraction, $0 \leq f_t \leq 1$, of its PE investments. When $f_t = 0$ the LP does not transact in the secondary market. When $f_t > 0$ the sale of PE investments reduces the LP's NAV by $f_t P_t$ and reduces the uncalled commitments by $f_t K_t$. The fraction f_t is an additional choice variable in the LP's problem. It is convenient to model the transaction using the non-normalized NAV, P_t, and non-normalized uncalled commitments, K_t (rather than the normalized p_t and k_t). Due to the discounts, described below, the LP's total wealth decreases when it transacts in the secondary market, complicating the resulting changes in the normalized state variables.

Liquidating PE investments at times of stress is costly, and they are liquidated at a discount, which represents a transfer of wealth from the selling to the buying LPs.[13] We take this discount as given. LPs that provide liquidity must be relatively unconstrained. Albuquerque et al. (2018) report that buying LPs are mostly pension funds, endowments, and banks, which are arguably relatively conservative and unconstrained investors. As

[13]Because the discount is a transfer between the transacting LPs, it is not relevant for the analysis of gains from trade in the previous section.

shown above, unconstrained LPs have lower valuations of mature funds than constrained LPs, which would naturally creates a discount for stressed sales of PE investments in the secondary market.

The discount has two parts. The LP sells $f_t P_t$ of NAV in return for $(1-\psi_{\mathrm{P}})f_t P_t$ of liquid wealth, where $\psi_{\mathrm{P}} \geq 0$ is the discount to NAV. The sale also reduces the LP's uncalled commitments by $f_t K_t$, which costs the LP $\psi_{\mathrm{K}} f_t K_t$, where $\psi_{\mathrm{K}} \geq 0$ is the cost of disposing of uncalled commitments. In practice, reported discounts for secondary transactions are typically (one minus) the transaction price divided by the NAV, which confounds these two parts of the discount. Empirically disentangling the two parts would require comparing transactions of, otherwise similar, funds with relatively more and less uncalled commitments.

In terms of timing, we assume that the LP transacts in the secondary market after contributions and distributions are paid and before new commitments are made. Uncalled commitments are now the sum of the LP's remaining existing commitments, $(1-f_t)K_t$, and its new commitments, N_t. The LP's illiquid wealth is the remaining NAV, $(1-f_t)P_t$. End-of-period wealth is now also net of the discount for transacting in the secondary market, $f_t\left(\psi_{\mathrm{P}} P_t + \psi_{\mathrm{K}} K_t\right)$.

In our baseline specification, $\psi_{\mathrm{K}} = 5\%$ and $\psi_{\mathrm{P}} = 20\%$. Nadauld et al. (2018) report average discounts to NAV from 9.0% to 13.8% (although, they do not separate the part of the discount due to uncalled commitments). Since our transactions occur at times of liquidity stress, we assume higher than average discounts. Higher discounts are also a conservative choice, giving a lower bound on the effects of a secondary market. Below, we find that it allows the aggressive LP to effectively return to its first-best allocations, and smaller discounts would only strengthen this finding.

The Bellman equation of the extended model is given in Appendix D. We solve the Bellman equation using the numerical methods from the original model (see Appendix A for details), and Proposition 2 generalizes the liquidity constraint with a secondary market.

Proposition 2. *Suppose that the LP can sell PE interests in the secondary market with* *liquidity costs ψ_{P} and ψ_{K} both between 0 and 1. Suppose also that $\lambda_{\text{N}} = \lambda_{\text{K}}$. Then, any optimal strategy satisfies the following inequality at all times:*

$$B_t \;\geq\; a_{\text{SM}}\Big((1 - f_t)K_t + N_t\Big)$$

In this expression, B_t is savings in bonds, and the coefficient a_{SM} is defined as follows:

$$a_{\text{SM}} \;=\; \frac{\psi_{\text{K}}(1 - \lambda_{\text{K}}) + \psi_{\text{P}}\lambda_{\text{K}}}{R_{\text{F}}}$$

Proof in Internet Appendix.

In Proposition 2, with the secondary market, B_t equals $W_t - C_t - S_t - P_t + f_t(1 - \psi_{\text{P}})P_t - f_t\psi_{\text{K}}K_t$. When ψ_{K} and ψ_{P} are less than one, which is natural, the coefficient a_{SM} is below the coefficient a_{K} from proposition 1. Intuitively, a_{SM} is increasing in ψ_{K} and ψ_{P}, and the effect of λ_{K} on a_{SM} depends on the relative magnitudes of ψ_{K} and ψ_{P}.

Figure 13 shows the optimal allocations with a secondary market, and we compare these allocations to the optimal allocations without a secondary market, in Figure 6, and to the first-best allocations in Figure 7. The conservative LP has similar allocations in all three figures, because this LP already effectively holds its first-best allocation even when PE is illiquid.

In contrast, the secondary market allows the aggressive LP to hold allocations substantially closer to the first-best. Its average NAV increases from $\omega_{\text{P}} = 23.6\%$ without a secondary market to $\omega_{\text{P}} = 56.7\%$ with a secondary market (first-best $\omega_{\text{P}} = 58.5\%$). Its stock allocation changes from $\omega_{\text{S}} = 50.9\%$ without a secondary market to $\omega_{\text{S}} = 36.9\%$ (first-best $\omega_{\text{S}} = 41.5\%$). And its bond allocation changes from $\omega_{\text{B}} = 25.6\%$ without a secondary market to $\omega_{\text{B}} = 6.4\%$ (first-best $\omega_{\text{B}} = 0.0\%$).

The positive allocation of 6.4% to bonds may be surprising, given that the aggressive LP's first-best bond allocation is zero. After a decline in the values of stocks and PE investments, the aggressive LP's PE investments consist of relatively more uncalled commitments, K_t, than NAV, P_t. Therefore, the discount to uncalled commitments becomes relatively more important, making it more costly for the aggressive LP to raise liquid wealth in the secondary market, net of this discount. In this situation, the LP can be unable to fund its contributions forcing it to sell its PE investments, at a discount, but with a low NAV this sale effectively requires the LP to pay a buyer to accept the LP's unfunded liabilities, and the bond reserve allows the LP make this payment and remain solvent.

The value functions with a secondary market are shown in Figure 14. The aggressive LP is now also close to the interior optimum, and its liquidity constraint tends to be slack. The secondary market, through its insurance function, increases the aggressive LP's utility by 11%.

Figure 15 shows optimal policies for trading in the secondary market, $f^*(p, k)$. As above, the state space is divided into cells, and the shading indicates how frequently a cell is visited during the simulations.[14] Cells that are not visited during the simulations are blank. Figure 15 shows that the conservative LP does not transact in the secondary market. The aggressive LP sometimes, albeit rarely, liquidates a positive fraction of its PE investments in response to a negative shock to its liquid wealth, which corresponds to a positive shock to the normalized state variables, p and k.

The Internet Appendix reports optimal allocations and policies for additional specifications. The results are robust. In general, the conservative LP is unaffected by the introduction of a secondary market. The aggressive LP benefits from the secondary market's insurance function, which allows the aggressive LP to approach its first-best allocation, even when the secondary market is characterized by relatively large discounts.

4 Explicit Management Fees

Up to now the management fees that are charged by PE funds have been implicit in the analysis. Annual fees are typically set to 0.5%–2% of a PE fund's total committed capital, and these fees often decline as the fund matures, either due to an explicit reduction in the percentage fee or due to a change in basis from committed to invested capital (see

[14]With a secondary market, the LPs visit a larger part of the state space, and we increase the size of the cells to 5% times 5% to reduce the computational workload.

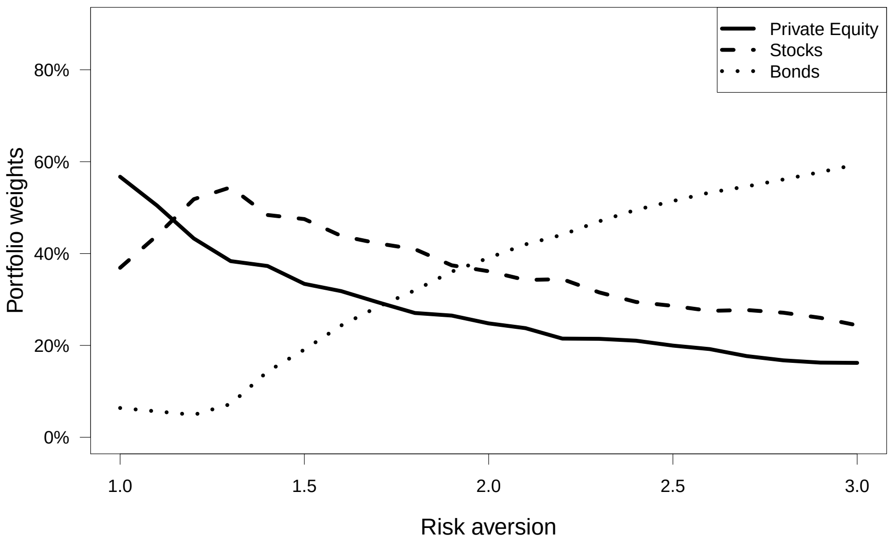
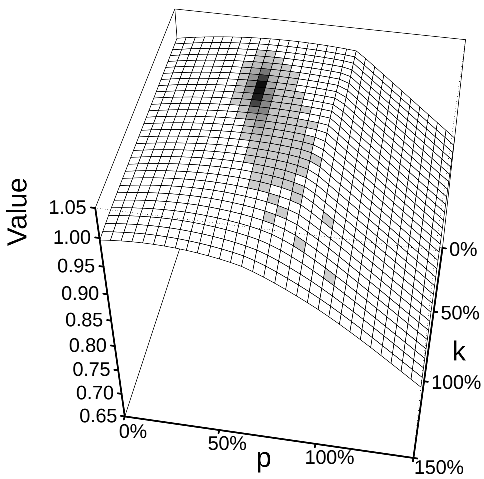


Figure 13: Average portfolio weights with secondary market. This figure plots the average allocation of LPs solving the extended model with secondary market. Average allocations are computed in three steps. First, we solve numerically the Bellman equation of the extended model (see appendix D). At any point in the state space, that solution provides the corresponding portfolio weights in stocks, private equity, and risk-free bonds. Second, we simulate the evolution of those portfolio weights under optimal strategies. Third, we compute their averages. This procedure is repeated for risk-aversion $\gamma \in \{1, 1.1, 1.2, \ldots, 2.9, 3\}$.

a. Aggressive ($\gamma = 1$)

b. Conservative ($\gamma = 3$)



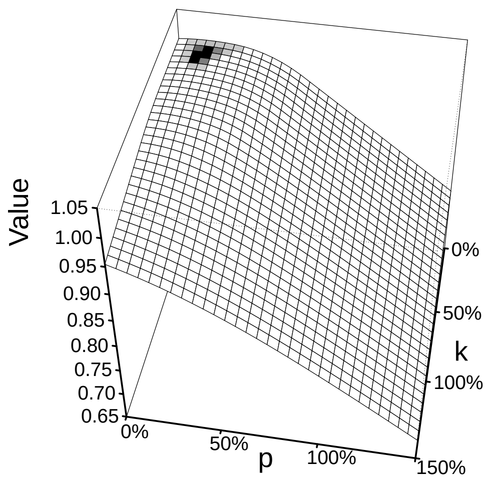


Figure 14: Value function with secondary market. This figure compares the value function of investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$) from the model with secondary market. In the figure, each function is rescaled and expressed in units of value at $(p, k) = (0, 0)$. The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally.

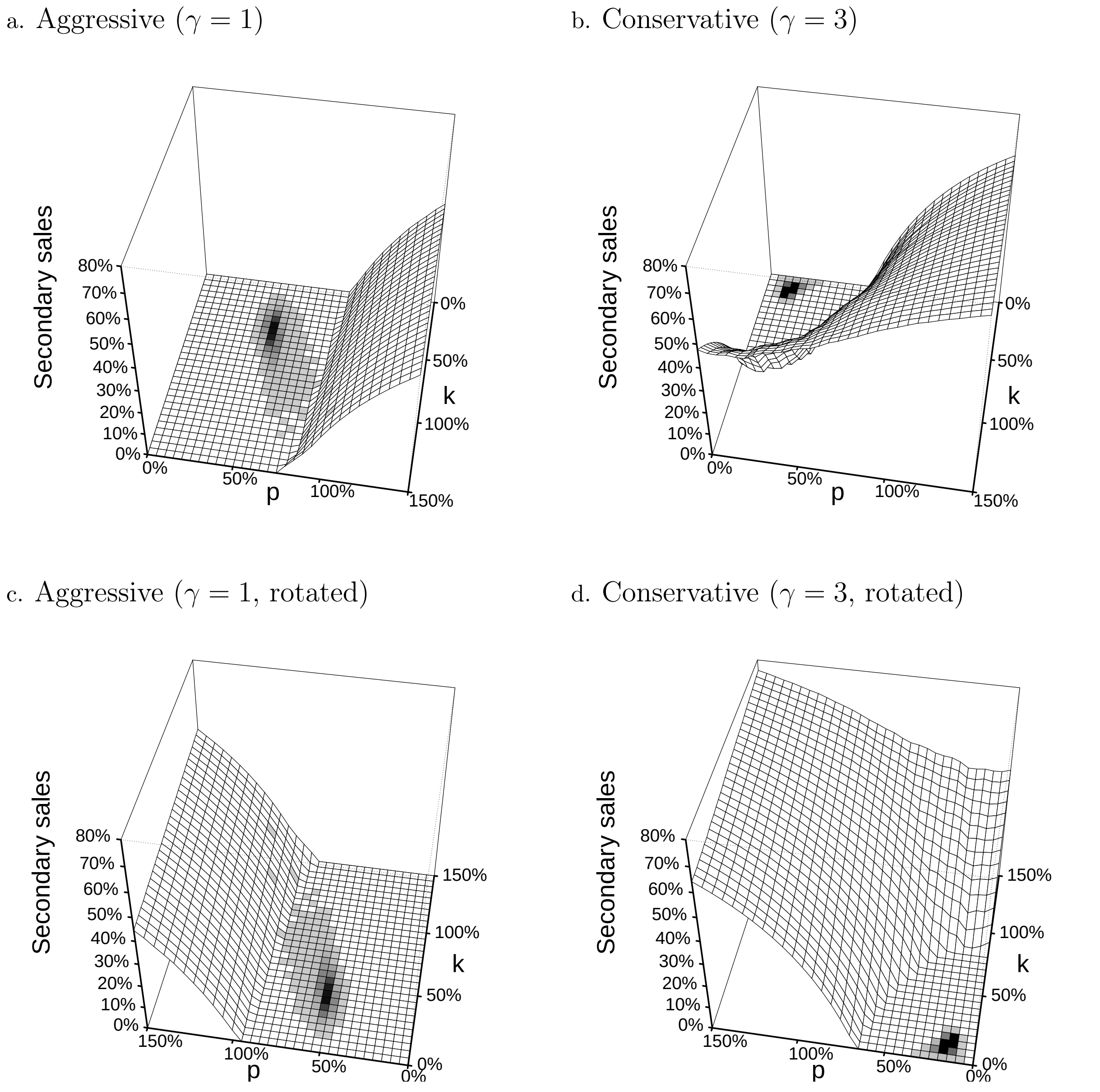


Figure 15: Optimal sales in the secondary market. This figure compares the optimal share f of PE investments sold in the secondary market at different points in the state space by investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally. The two plots at the bottom of the figure are rotated versions of those at the top.

Metrick and Yasuda, 2010). Implicit management fees are convenient, because the liquid model does not have a notion of committed capital, and it cannot accommodate explicit management fees. So a benefit of implicit fees is that the optimal allocations can be compared directly to the first-best allocations from the liquid model. Below, we extend the model to explicit management fees. The main finding is that, with the appropriate reinterpretation of the state variables and the returns, the optimal solution is largely unaffected and the specification of these fees is not critical for the analysis.

In practice, management fees are specified as fraction of the fund's committed capital, and these fees are paid to the PE firm, acting as a general partner in the fund, to cover the PE firm's ongoing expenses and to compensate it for managing the fund. This capital is contributed by the LP to the PE fund, but it is not used to acquire private assets. As a simple illustration, a PE fund with \$100 of total committed capital, a ten-year life, and annual management fees of 1.5% of its total committed capital charges \$1.50 annually in management fees. Over ten years, these fees add up to \$15, leaving \$85 to invest in private assets. Every time the LP contributes \$1 to the PE fund, the fund charges \$0.15 in fees. In the model the aggregate management fee can thus be specified as a proportion, $\theta = 15\%$, of the contributed capital. When the LP contributes I to the PE funds, then $(1-\theta)I$ is invested in private assets and the PE funds receive θI in fees.

Considering a single PE fund, this specification implies that the management fees are not constant over the fund's life. The fee is instead proportional to the contributed capital each period, as illustrated in Figure 2. We prefer this specification for three reasons. First, it is tractable and simple to implement in the model. Second, in practice, management fees tend to decline as a fund matures, which is consistent with this specification (Metrick and Yasuda, 2010). Third, only aggregate management fees are relevant for the LP's problem. Even if actual fees for younger funds are slightly lower than specified (and actual fees for mature funds are slightly higher) these differences average out in steady state when the LP's portfolio contains a balanced mix of younger and older PE funds (see footnote 9 at page for a similar argument).

Explicit fees require a reinterpretation of the state variables and the return to PE investments. Aggregate uncalled commitments, K_t, and N_t, still include the amount reserved for management fees, but the aggregate NAV, P_t, is now net of management fees. The return to PE investments, $R_{\mathrm{P},t}$, is now gross of management fees, and total wealth, W_t, is now net of the management fees paid at time t. The transition equations for P_{t+1} and W_{t+1}, corresponding to equations (4) and (5), become:

$$P_{t+1} = R_{\mathrm{P},t+1}P_t + (1-\theta)I_{t+1} - D_{t+1} \tag{20}$$

and

$$W_{t+1} = R_{\mathrm{P},t+1}P_t + R_{\mathrm{S},t+1}S_t + R_{\mathrm{F}}(W_t - P_t - C_t - S_t) - \theta I_{t+1} \tag{21}$$

Proposition 1 applies unchanged, and the normalized Bellman equation for the reduced problem with explicit management fees is:

$$\begin{aligned}
&v(p,k) = \max_{(c,n,\omega_{\mathrm{S}})} \left\{ c^{1-\gamma} + \delta \mathrm{E}\left[\left[\Big((1-c)R_{\mathrm{W}} - \theta(\lambda_{\mathrm{N}} n + \lambda_{\mathrm{K}} k) \Big) v(p',k') \right]^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}} \\
&\text{subject to} \\
&R_{\mathrm{W}} = \omega_{\mathrm{P}} R_{\mathrm{P}} + \omega_{\mathrm{S}} R_{\mathrm{S}} + (1-\omega_{\mathrm{P}}-\omega_{\mathrm{S}})\, R_{\mathrm{F}} \\
&p' = \big[\, (1-\lambda_{\mathrm{D}})\, R_{\mathrm{P}} p + (1-\theta)(\lambda_{\mathrm{N}} n + \lambda_{\mathrm{K}} k) \big] \Big/ \big[(1-c)R_{\mathrm{W}} - \theta(\lambda_{\mathrm{N}} n + \lambda_{\mathrm{K}} k) \big] \\
&k' = \big[(1-\lambda_{\mathrm{K}})k + (1-\lambda_{\mathrm{N}})n \big] \Big/ \big[(1-c)R_{\mathrm{W}} - \theta(\lambda_{\mathrm{N}} n + \lambda_{\mathrm{K}} k) \big] \\
&0 \le c \le 1-p \\
&n \ge 0
\end{aligned} \tag{22}$$

We solve the model using the same numerical methods as before (see Appendix A for details). As discussed above, we use $\theta = 15\%$ as the rate of management fees. For the other parameters, except for α and μ_{P}, we use the values from the baseline specification. The PE return is now gross of management fees, and it needs to be increased accordingly. We set $\alpha = 9\%$ and $\mu_P = 12.6\%$ to approximate the same allocations for the conservative LP.

4.1 Optimal Allocation with Explicit Management Fees

The optimal allocations with explicit management fees are in Figure 16. These allocations are almost identical to those with implicit fees, in Figure 6. As a function of risk aversion, the PE allocation starts to decline and the bond allocation starts to increase slightly earlier with explicit fees than with implicit fees, but the differences are minimal.

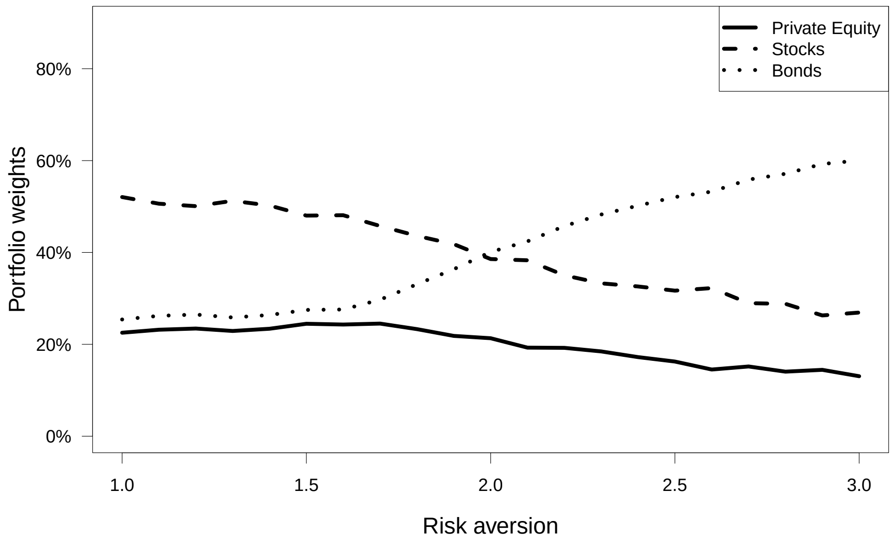


Figure 16: Average portfolio weights with explicit management fees. This figure plots the average allocation of LPs solving the model with explicit management fees. Average allocations are computed in three steps. First, we solve numerically the Bellman equation of the model with management fees (see expression (22)). At any point in the state space, that solution provides the corresponding portfolio weights in stocks, private equity, and risk-free bonds. Second, we simulate the evolution of those portfolio weights under optimal strategies. Third, we compute their averages. This procedure is repeated for risk-aversion $\gamma \in \{1, 1.1, 1.2, \dots, 2.9, 3\}$.

Figure 17 shows the value functions, $v(p, k)$, with explicit management fees. These value functions, however, are no longer comparable to the value functions with implicit fees, from Figure 8. In Figure 17, the conservative LP's value function is upward sloping in p in the shaded part of the state space, but this slope does not mean that the conservative LP is constrained, because the state variable p is now net of management fees and the value function, $v(p, k)$, is normalized by the LP's wealth. An increase in p, holding everything else equal, therefore reflects an increase in the LP's PE exposure without paying the management fees associated with this increase, which would obviously benefit the LP, but which is not feasible. These complications when interpreting the value function with explicit management fees are avoided when management fees are implicit.

Figure 18 shows the optimal policies for new PE commitments with explicit management fees. These policies are indistinguishable from the policies with implicit fees, in Figure 9. Table 7 confirms that the coefficients in the linear approximation of the optimal policy are also similar to coefficients with implicit fees, in Table 3. Overall, the optimal policies

a. Aggressive ($\gamma = 1$) b. Conservative ($\gamma = 3$)

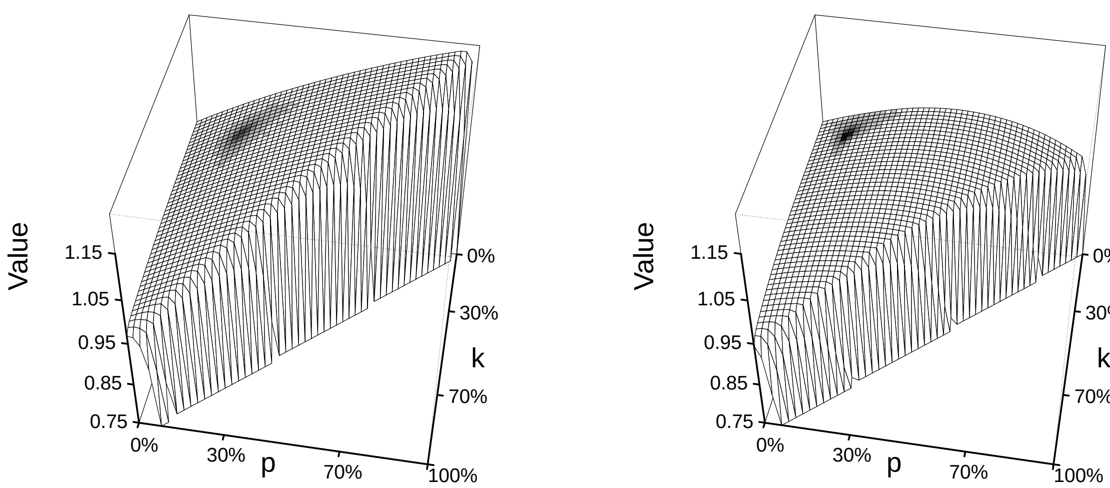


Figure 17: Value function with explicit management fees. This figure compares the value function of investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$) from the model with explicit management fees. In the figure, each function is rescaled and expressed in units of value at $(p, k) = (0, 0)$. The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally.

and allocations with explicit and implicit management fees are almost identical, and the choice between these two specifications of management fees does not appear to matter for our results.

a. Aggressive ($\gamma = 1$) b. Conservative ($\gamma = 3$)

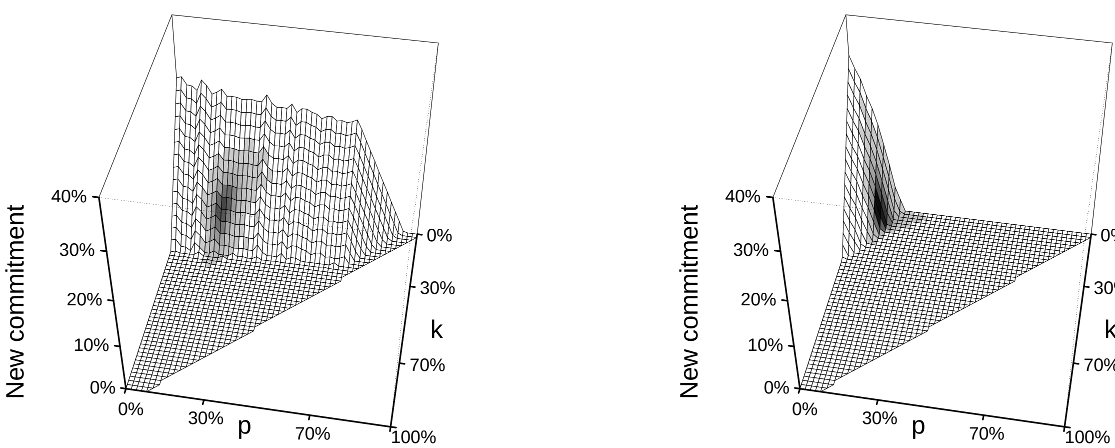


Figure 18: Optimal commitment with explicit management fees.

Table 7: Optimal commitment with explicit management fees.

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid Wealth	−0.10	0.09	−1.84	−1.99
Commitment	−0.98	−0.73	−0.90	−1.11
Interaction		−1.15		1.77
Constant	0.27	0.23	0.36	0.38
R^2	0.99	0.99	0.98	0.98

Figure 18 and Table 4 confirm that the optimal stock allocation with explicit fees is similar to the allocation with implicit fees, and that the coefficients in the regressions are similar, especially for the specifications without interaction terms.

a. Aggressive ($\gamma = 1$)

b. Conservative ($\gamma = 3$)

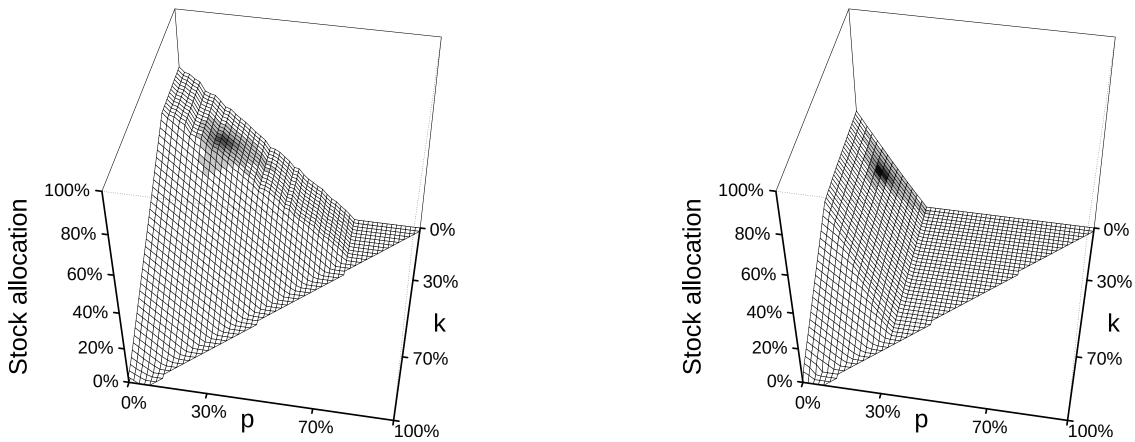


Figure 19: Optimal stock allocation with explicit management fees.

Table 8: Optimal stock allocation with explicit management fees.

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid Wealth	−0.97	−1.17	−1.79	−1.76
Commitment	−0.03	−0.28	0.00	0.05
Interaction		1.21		−0.38
Constant	0.73	0.78	0.49	0.49
R^2	0.99	0.99	1.00	1.00

5 Conclusion

We present the first formal analysis of an LP's investment problem with ongoing commitments to an arbitrary number of private equity (PE) funds. Our model captures three aspects of PE investments: they are risky, illiquid, and long-term investments. PE investments are risky because they earn an uncertain return. They are illiquid because after committing capital to PE funds, the LP must hold this investment to maturity, and the LP cannot liquidate (or collateralize) its PE investments to convert them into current consumption, although this is relaxed somewhat with a secondary market. PE investments are long-term because the LP's committed capital is not immediately invested into private assets. Rather, the LP maintains a stock of uncalled commitments, which are gradually contributed to the PE funds, and the LP's need to pay future contributions creates the main liquidity friction in our analysis. We show that linear fund dynamics substantially simplify the problem, because the LP's aggregate uncalled commitments and aggregate uncalled NAV become summary statistics for the LP's entire portfolio of PE investment in the LP's problem.

Depending on the LP's risk aversion, we find two distinct investment strategies. A conservative LP with a high risk aversion (here, a relative risk aversion of $\gamma = 3$) is unconstrained. It's first best portfolio allocation places a sufficient amount of capital in safe assets, and the LP's liquidity constraint is not binding. The LP is close to its first-best interior optimum, and it effectively treats PE investing as another traded stock. In response to a positive shock to the value of its PE investments it reduces its allocation to traded stocks, and it reduces commitments to new PE funds to rebalance and return to the optimal portfolio.

In contrast, an aggressive LP with a lower risk aversion ($\gamma = 1$) faces a binding liquidity constraint. The aggressive LP does not rebalance its portfolio to maintain constant risk exposures. It has a substantially larger allocation to the risk free asset and a lower allocation to stocks than its first-best allocation. Each period it determines the amount of new commitments to target a given level of total uncalled commitments.

We extend the analysis with a secondary market for PE partnership interests and analyze two effects. There are gains from trade when mature PE funds are traded from unconstrained to constrained LPs, because mature funds provide greater PE exposure relative to the required liquidity reserve. Moreover, a secondary market can insure a constrained LP against negative shocks to the value of its liquid investments by providing liquidity, ex-post, to the stressed LP. In turn, anticipating that it will be able to liquidate its PE investments, an aggressive LP will hold a greater PE allocation, ex-ante. In our specification, the gains from a single trade are economically small. In contrast, insuring the aggressive LP from liquidity shocks has a large effect, and it effectively allows the aggressive LP to hold its first-best portfolio.

Our model also allows for different specifications of management fees, which slightly affects the interpretation of the state variables and the return to PE investments (either net or gross of management fees). Most of our analysis assumes implicit management fees, because the resulting portfolio allocations can be compared directly to the first-best allocations from a liquid model. Another benefit of implicit fees is that the PE return in the model is net of fees, which is easier to calibrate because this is the return that is usually reported in industry studies and empirical research. We also analyze the model with explicit management fees, but after the appropriate adjustments, the solution and optimal policies are very similar, and this modeling choice appears unimportant for our analysis.

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A Numerical Methods

In this section, we describe the numerical procedure used to solve the portfolio problems in the main text. We take the reduced model as a working example.

We use a standard value function iteration algorithm to solve the following problem:

$$
\begin{aligned}
& v(p,k) = \max_{(c,n,\omega_{\mathrm{S}})} \left\{ c^{1-\gamma} + \delta \mathrm{E}\left[\left[(1-c) R_{\mathrm{W}} v(k',p') \right]^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}} \\
& \text{subject to} \\
& R_{\mathrm{W}} = \omega_{\mathrm{P}} R_{\mathrm{P}} + \omega_{\mathrm{S}} R_{\mathrm{S}} + (1-\omega_{\mathrm{P}}-\omega_{\mathrm{S}}) R_{\mathrm{F}} \\
& p' = \left[(1-\lambda_{\mathrm{D}}) R_{\mathrm{P}} p + \lambda_{\mathrm{N}} n + \lambda_{\mathrm{K}} k \right] / \left[(1-c) R_{\mathrm{W}} \right] \\
& k' = \left[k + n - \lambda_{\mathrm{N}} n - \lambda_{\mathrm{K}} k \right] / \left[(1-c) R_{\mathrm{W}} \right] \\
& 0 \le c \le \max\{1-p, 0\} \\
& n \ge 0
\end{aligned}
\tag{A.1}
$$

The value function of this problem is not smooth, so we use robust numerical methods, such as linear interpolation and discretization of choice variables, which are described below. We also exploit the inequality in Proposition 1, normalized by post-consumption wealth:

$$1 - \omega_{\mathrm{S}} - \omega_{\mathrm{P}} \ge a_{\mathrm{K}} k/(1-c) + a_{\mathrm{N}} n/(1-c)$$

The coefficients a_{K} and a_{N} are given in Proposition 1. Numerically, at each point in the state space, we ignore any combination of choice variables that does not satisfy this constraint.

In our value function iteration algorithm, we represent the state space with a grid of (p,k) values with 51×51 points, which are evenly distributed over the unit square $[0,1]^2$. In each dimension, the distance between one point and the next is 2%.

We start the algorithm with a guess for the value function that is constant across all grid points, and which equals the constant value v_{L} of the case with perfect market liquidity (see section B of this appendix).

In each iteration, we solve the constrained maximization problem to obtain the optimal policies and the resulting value at every point of the grid. This generates an updated guess for the value function that replaces the initial one. With the new guess, a new iteration begins, and the procedure is repeated until convergence. Specifically, we stop the algorithm when the absolute difference between the current and the new value function, at each point of the grid, becomes lower than $v_{\mathrm{L}} \times 10^{-4}$.

To calculate expectations, we need to integrate with respect to the risky returns, $r_{\mathrm{P}} = \ln(R_{\mathrm{P}})$ and $r_{\mathrm{S}} = \ln(R_{\mathrm{S}})$, and interpolate the value function $v(k', p')$. We use linear interpolation to evaluate this function with arguments outside the (k, p) grid. We use Gauss-Hermite quadrature to integrate with respect to the risky returns. We employ 3 quadrature points for each risky asset, resulting into a total of 9 quadrature points.

From iteration 1 to 9, we represent the choice set with a grid made of $8 \times 201 \times 101$ points in the $(c, \omega_{\mathrm{S}}, n)$ space.[15] At each (k, p) in the discretized state space, we select from that grid the point that maximizes the objective function while satisfying proposition 1. Then, at the beginning of iteration 10, we build new grids for the choice set that allow us to refine the solution. Specifically, for each (k, p) in the discretized state space, we build a new grid in the $(c, \omega_{\mathrm{S}}, n)$ space, which covers a smaller area while being finer than the starting grid. Importantly, the center of these new grids are the optimal choices obtained from iteration 9. We maintain these new grids until iteration 20, when we refine them further using the same idea. After this second refinement, we let the algorithm run unchanged for 10 more simulations. Finally, starting from iteration 30, we apply a standard policy iteration procedure in order to speed up convergence.

The algorithm is implemented in R. It takes about 5 minutes to solve the model on a standard computer running Linux.

A.1 Monte Carlo Simulation

Every time we solve the model, the solution is then used in a Monte Carlo simulation. In the simulation, we consider a LP starting without any PE interests ($k_1 = 0$ and $p_1 = 0$). The LP employs optimal strategies as it transitions randomly in the state space depending on risky returns which are drawn from their lognormal distribution. The simulation lasts for 11000 periods, and we discard the first 1000 periods to ensure that our assumption

[15] In particular, we use the vector $(0.0001, 0.01, 0.02, \ldots, 0.07)$ for c, $(0, 0.005, 0.01, 0.015, \ldots, 1)$ for ω_{S}, and $(0, 0.005, 0.01, 0.015, \ldots, 0.5)$ for n.

about initial PE interests, k_1 and p_1, does not affect the results. For the remaining 10000 periods, we save the realization of state variables, choice variables, and returns. These data are then used to compute properties of the marginal and joint distribution of those variables. We use those data, for example, to obtain average portfolio allocations, and also to compute how likely is the investor to visit a certain area of the state space.

B Liquid Model

We consider a simple liquid model where the LP can buy and sell private assets directly, instead of having to make initial commitments that are gradually called and eventually distributed. In this liquid model, PE is effectively another liquid stock. The model with two stocks is standard, and it has been studied since, at least, Samuelson (1969). Nevertheless, this liquid model provides a useful baseline.

Total wealth, W, is the only state variable of this model, and the Bellman equation is as follows:

$$\begin{aligned} & V_{\mathrm{L}}(W) = \max_{(c,\omega_{\mathrm{P}},\omega_{\mathrm{S}})} \left\{ (cW)^{1-\gamma} + \delta \mathrm{E}\left[V_{\mathrm{L}}(W')^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}} \\ & \text{subject to} \\ & R_{\mathrm{W}} = \omega_{\mathrm{P}} R_{\mathrm{P}} + \omega_{\mathrm{S}} R_{\mathrm{S}} + \left(1 - \omega_{\mathrm{P}} - \omega_{\mathrm{S}}\right) R_{\mathrm{F}}, \\ & W' = W(1-c) R_{\mathrm{W}} \\ & 0 \leq c \leq 1 \end{aligned} \tag{B.1}$$

In this expression, c is the consumption-to-wealth ratio, while ω_{P} and ω_{S} are portfolio weights in PE and stocks, respectively. It is convenient to state explicitly the return to LP's total wealth, R_{W}, even though this return is i.i.d and not a state variable.

Since the LP cannot consume more resources than what it owns, it is unable to consume in states with zero wealth. Those states, however, are infinitely painful for LPs with $\gamma \geq 1$, who will then make sure to never run out of wealth. To that end, the LP maintains a positive allocation to risk-free bonds, and does not short sell any risky asset. The resulting constraints (i.e. $\omega_{\mathrm{P}} + \omega_{\mathrm{S}} \leq 1$, $\omega_{\mathrm{P}} \geq 0$, and $\omega_{\mathrm{S}} \geq 0$) are introduced explicitly as we normalize and solve the liquid model.

The liquid model is homogeneous in wealth, and the value function is $V_{\mathrm{L}}(W) = W v_{\mathrm{L}}$

where v_{L} is a positive constant that solves the normalized problem:

$$\begin{aligned}
& v_{\mathrm{L}} = \max_{(c,\omega_{\mathrm{P}},\omega_{\mathrm{S}})} \left\{ c^{1-\gamma} + \delta \mathrm{E}\Big[[v_{\mathrm{L}}(1-c)R_{\mathrm{W}}]^{1-\gamma} \Big] \right\}^{\frac{1}{1-\gamma}} \\
& \text{subject to} \\
& R_{\mathrm{W}} = \omega_{\mathrm{P}} R_{\mathrm{P}} + \omega_{\mathrm{S}} R_{\mathrm{S}} + (1 - \omega_{\mathrm{P}} - \omega_{\mathrm{S}})\, R_{\mathrm{F}} \\
& 0 \leq c \leq 1 \\
& \omega_{\mathrm{P}} + \omega_{\mathrm{S}} \leq 1
\end{aligned} \tag{B.2}$$

We solve this normalized problem numerically for 21 evenly distributed values of $\gamma \in [1, 3]$. Figure 7 in the main text shows the resulting optimal portfolio allocations for different values of the LP's relative risk aversion, γ. Naturally, when the LP's relative risk aversion increases, the optimal allocations to PE and stocks decline, and the bond allocation increases. Conversely, as relative risk aversion decreases, the LP first reduces its bond allocation to zero. Due to the no-leverage constraint in the model, the LP cannot reduce its bond allocation below zero, and to keep increasing its PE allocation, the LP reduces its stock allocation. The LP's choice between PE and stocks depends on their relative risks and returns. In the current parametrization, PE is more attractive than stocks, and a less risk averse LP prefers to increase its PE allocation at the cost of a lower stock allocation.

C Other Specifications of PE Performance

The baseline specification of the reduced model is reported in Table 1 of the main text. In that specification, PE performance is characterized by $\alpha = 3\%, \beta = 1.6$, $\rho = 0.8$, and $\sigma_{\mathrm{P}} = 40\%$. In this section, we compare the solution of the reduced model under different assumptions about PE performance. We consider the following parameter values:

$$\alpha \in (1\%, 3\%, 5\%)$$

$$\beta \in (1.2, 1.6, 1.8)$$

$$\rho = 0.8 \quad \text{or} \quad \sigma_{\mathrm{P}} = 40\%$$

All other parameters remain constant at their baseline level. We obtain 18 ($= 3 \times 3 \times 2$) specifications of PE performance, which are reduced to 15 after eliminating duplicates. In particular, there are only 5 unique combinations of β, ρ, and σ_{P}:

β	ρ	σ_{P}
1.2	0.8	30%
1.2	0.6	40%
1.6	0.8	40%
1.8	0.8	45%
1.8	0.9	40%

These 5 combinations and the 3 possible values of α result into 15 unique cases.

We report output in the following order:

1. Average portfolio weights resulting from optimal policies (plots, p.)
2. Optimal portfolio weights in the Liquid Model (plots, p.)
3. Value function (plots, p. -)
4. Optimal new commitment (plots at p. - and tables at p.)
5. Optimal stock allocation (plots at p. - and tables at p.)
6. Optimal consumption-to-wealth ratio (plots, p. -)

In the figures of this section, the middle column of Panel A is identical to that of Panel B, and the center of each panel corresponds to the baseline specification in Table 1 of the main text. In the tables of Figure C.7 (p.) and Figure C.10 (p.), empty columns are shown when average PE uncalled commitments are less than 3% of wealth. In Figure C.10, empty columns are shown also when average portfolio weight in stocks is 3% or less.

Figure C.1: Average portfolio weights resulting from optimal policies

Panel A: $\sigma_{\mathrm{P}} = 40\%$

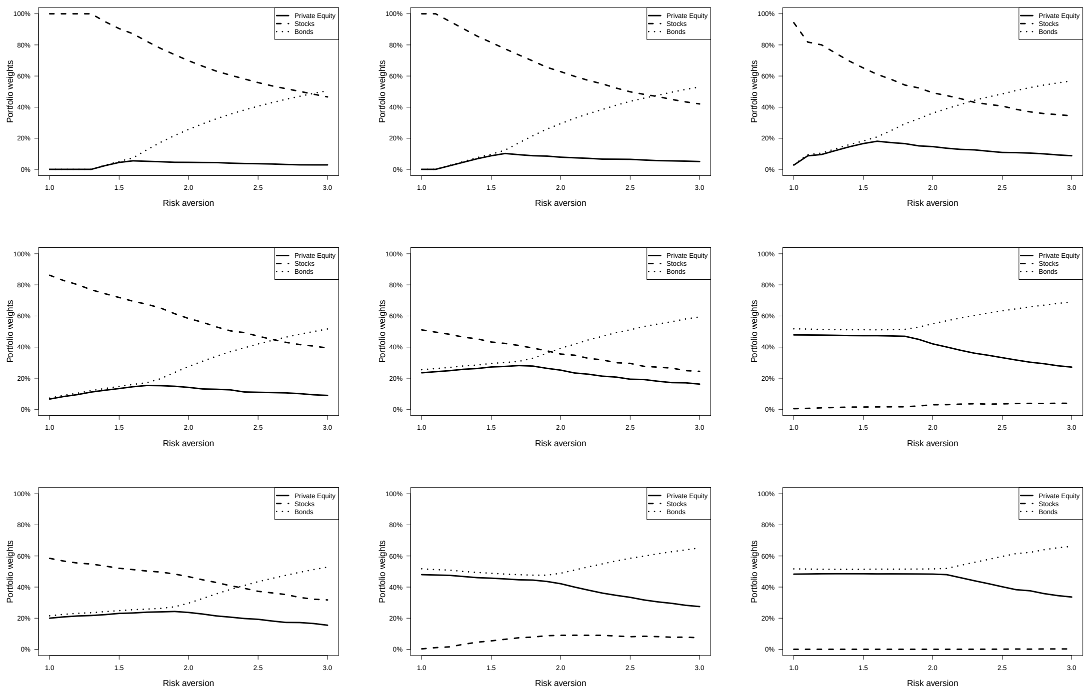
β 1.2 1.6 1.8

α

1%

3%

5%



Panel B: $\rho = 0.8$

β 1.2 1.6 1.8

α

1%

3%

5%

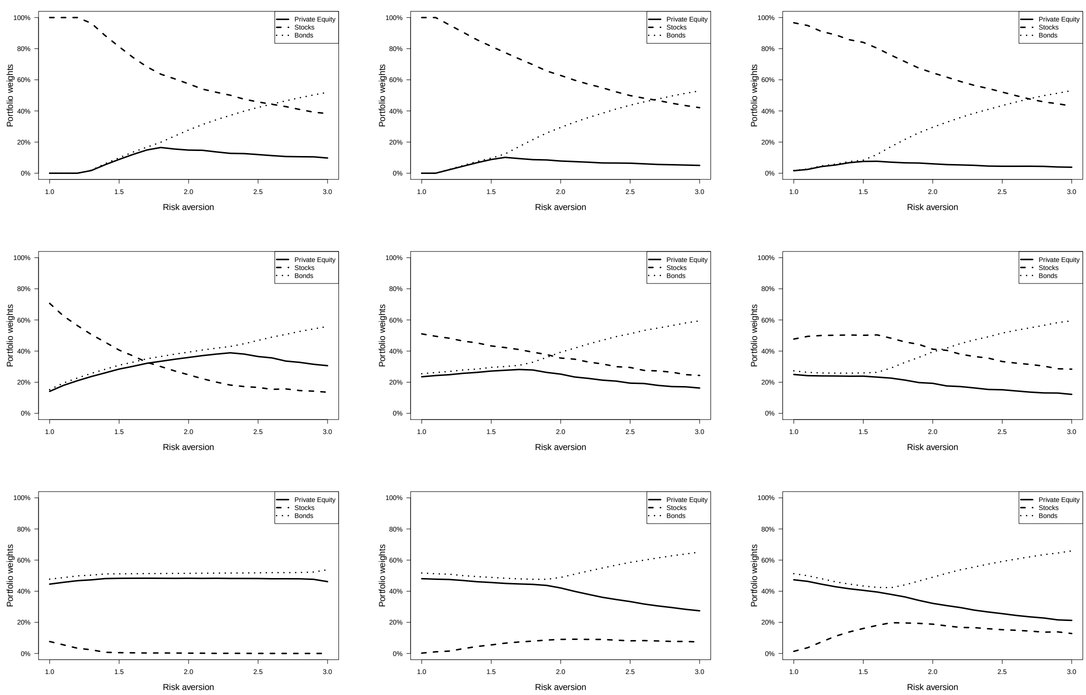
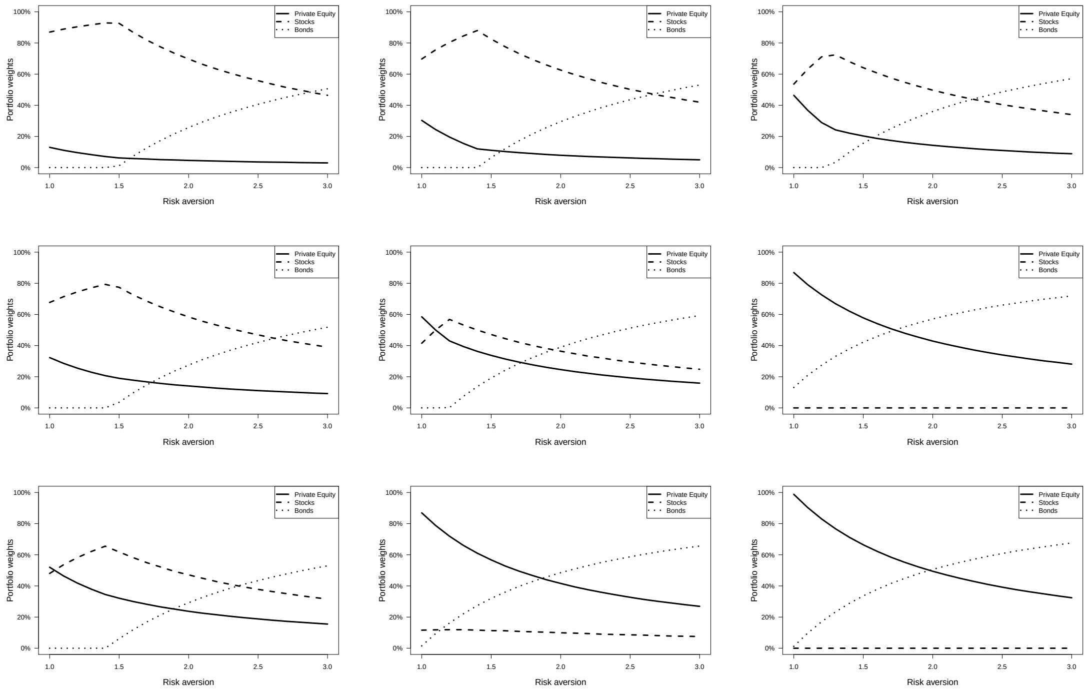


Figure C.2: Optimal portfolio weights in the Liquid Model

Panel A: $\sigma_{\mathrm{P}} = 40\%$

α \ β	1.2	1.6	1.8
1%			
3%			
5%			



Panel B: $\rho = 0.8$

α \ β	1.2	1.6	1.8
1%			
3%			
5%			

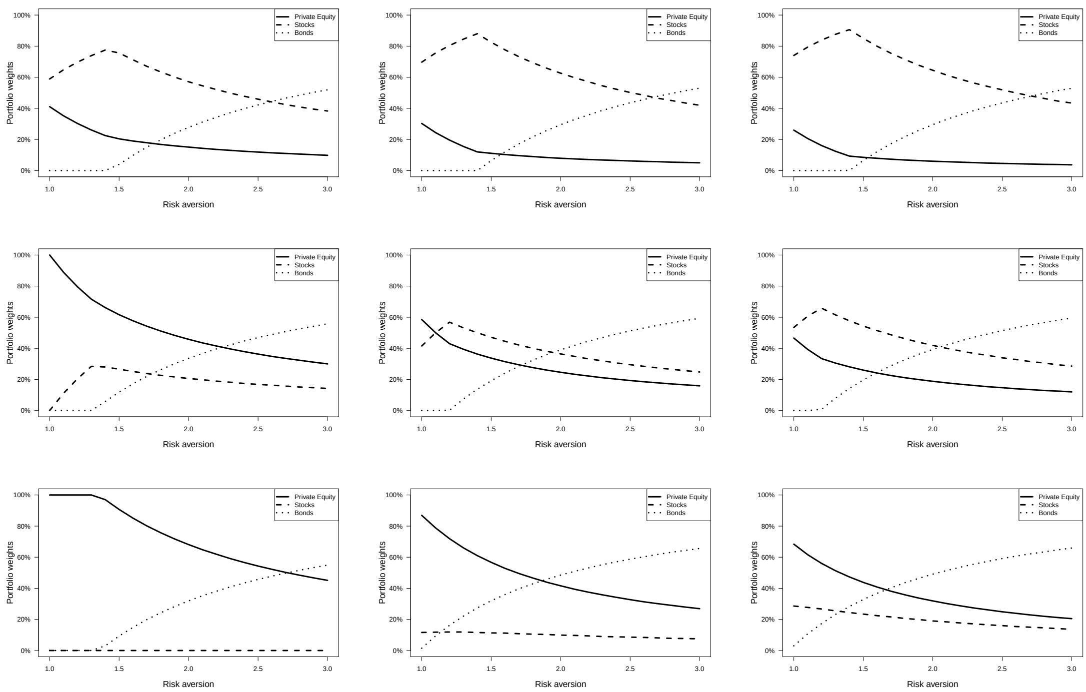
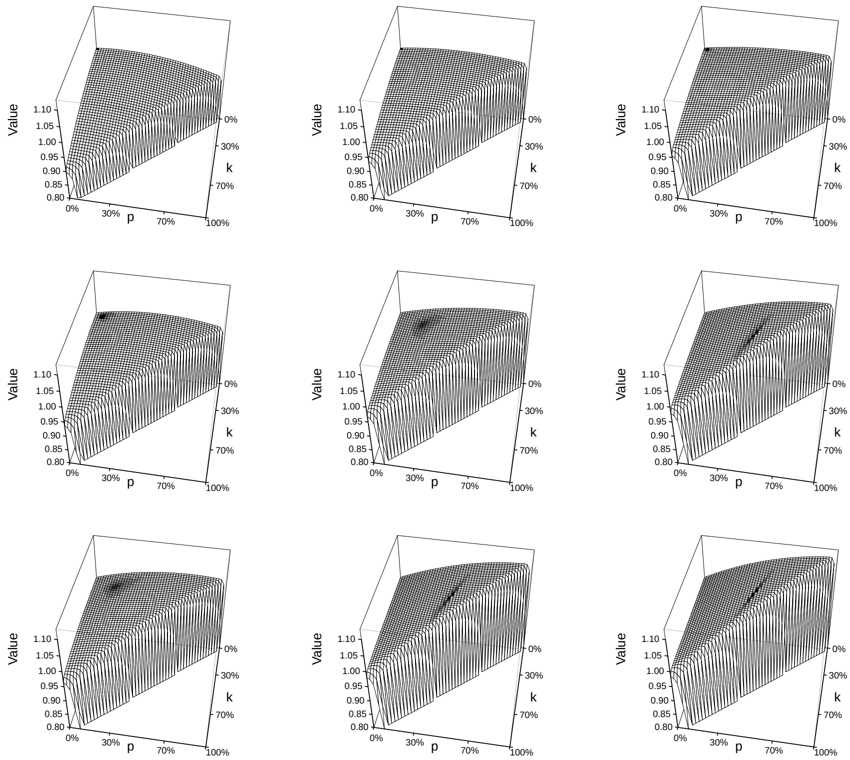


Figure C.3: Value function of aggressive LP ($\gamma = 1$)

Panel A: $\sigma_{\mathrm{P}} = 40\%$

α \ β: 1.2, 1.6, 1.8

α: 1%, 3%, 5%



Panel B: $\rho = 0.8$

α \ β: 1.2, 1.6, 1.8

α: 1%, 3%, 5%

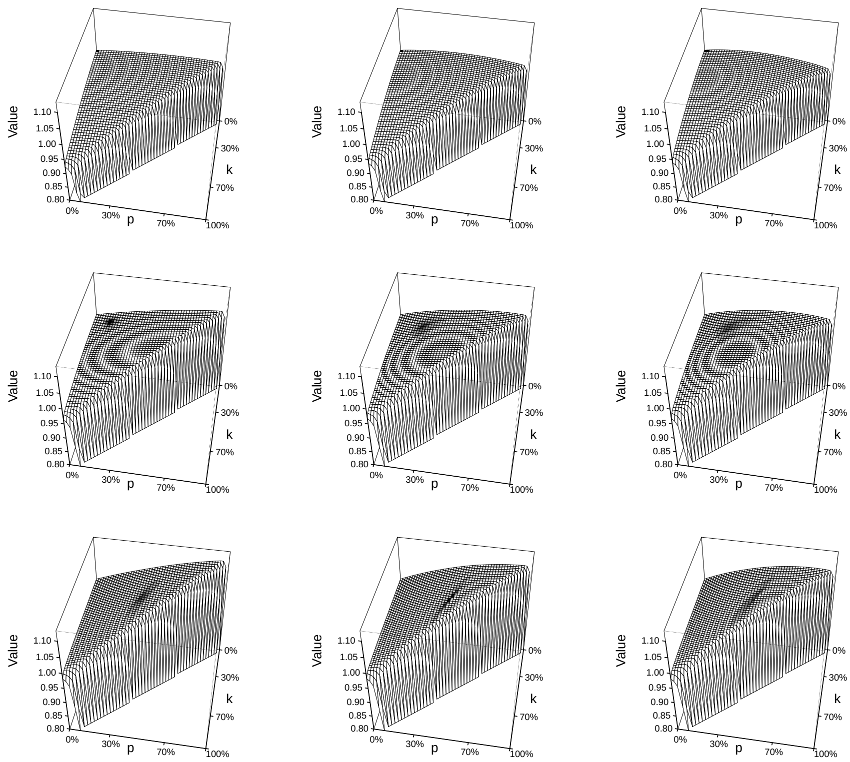
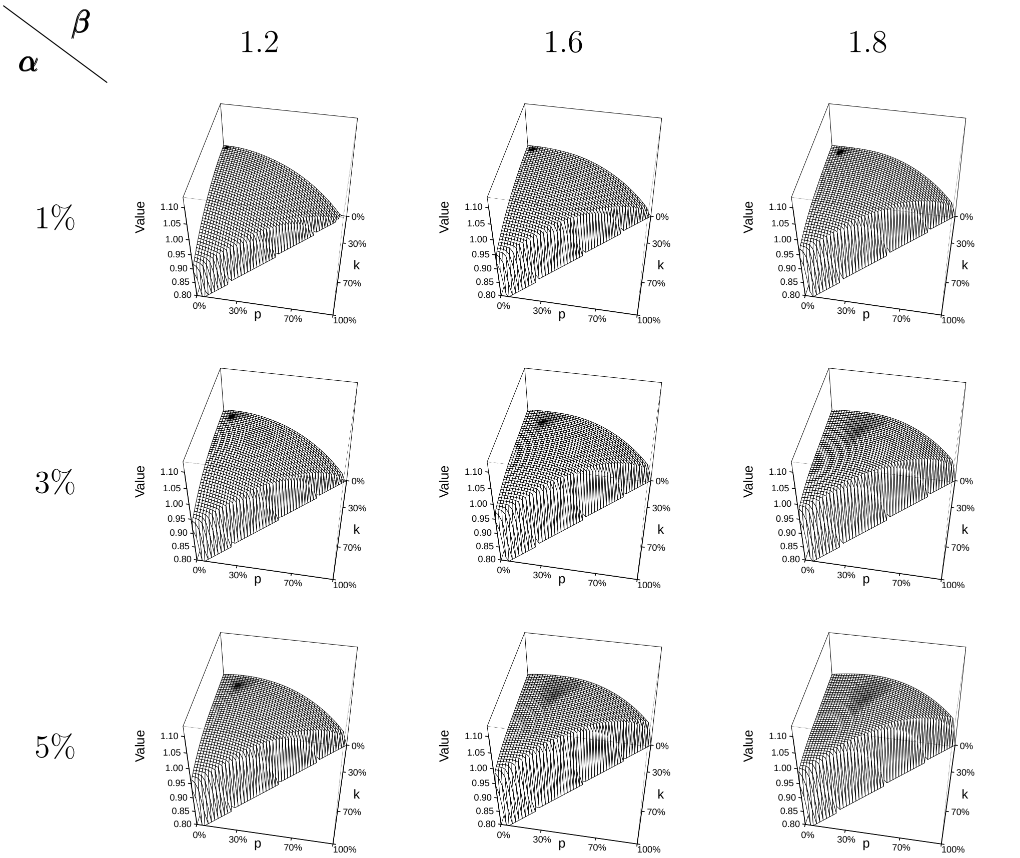


Figure C.4: Value function of conservative LP ($\gamma = 3$)

Panel A: $\sigma_{\mathrm{P}} = 40\%$



Panel B: $\rho = 0.8$

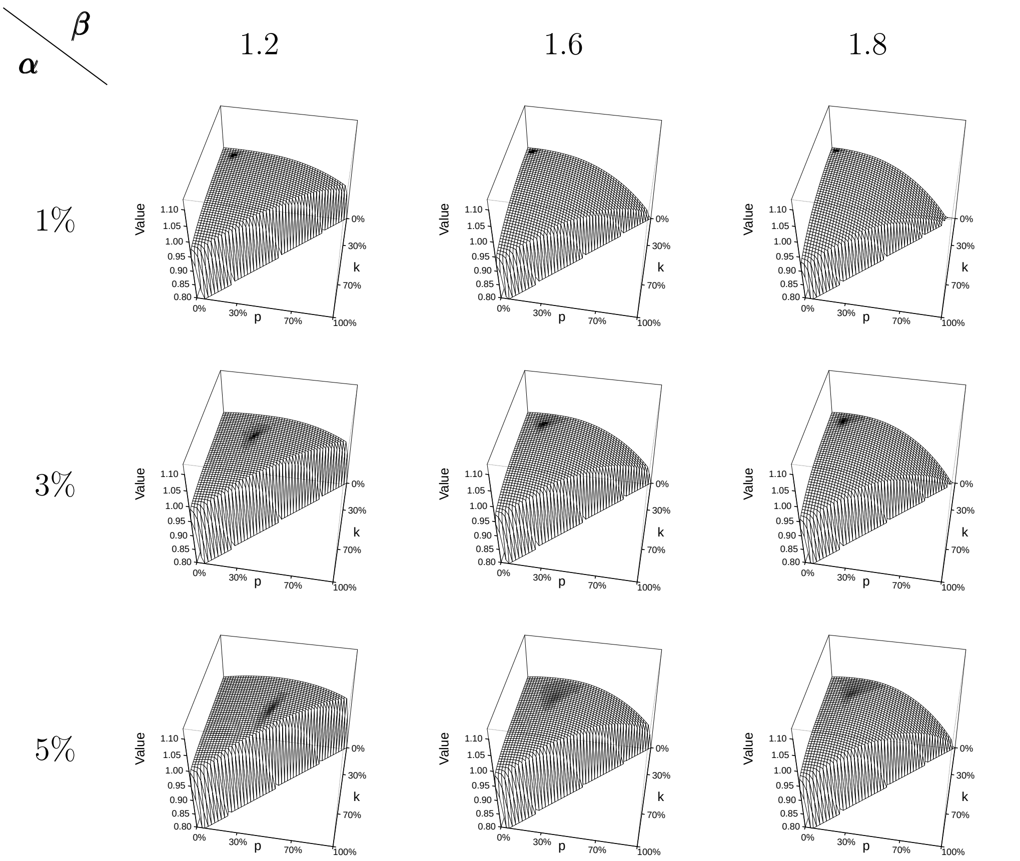


Figure C.5: New commitment of aggressive LP ($\gamma = 1$)

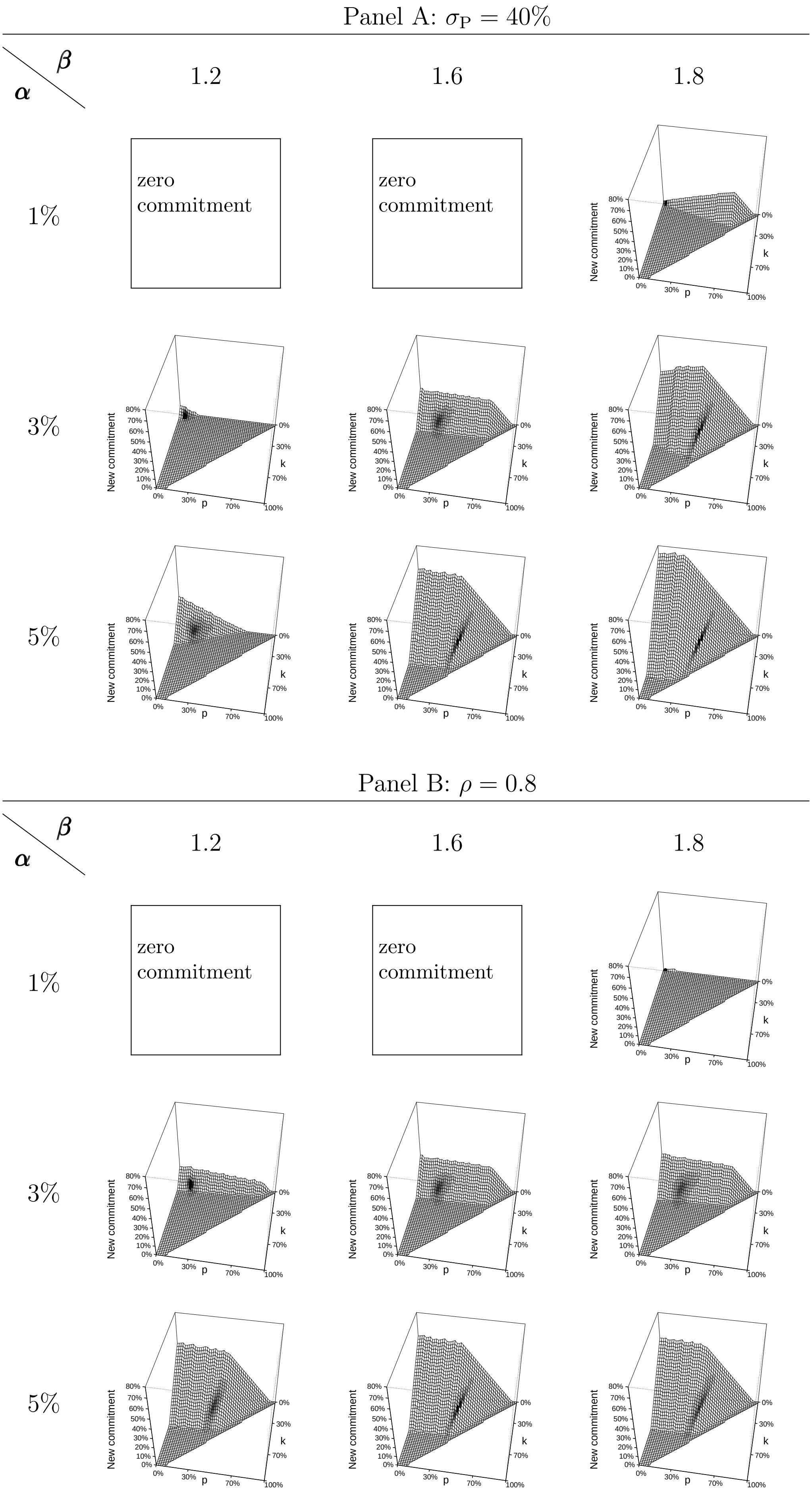
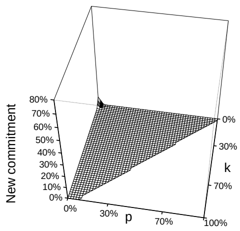
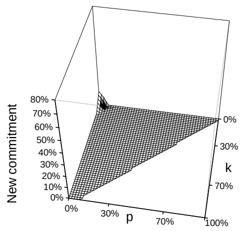
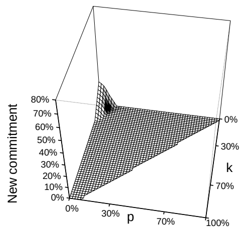
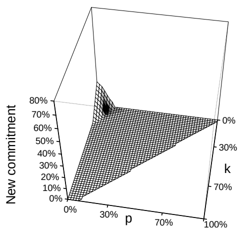
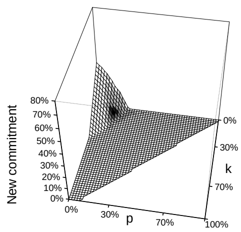
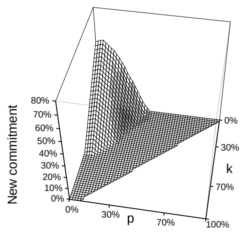
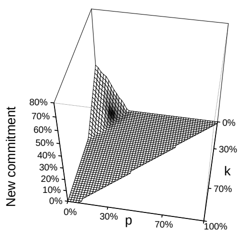
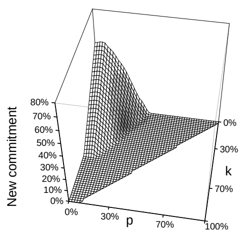
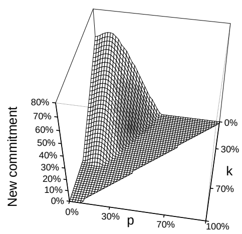


Figure C.6: New commitment of conservative LP ($\gamma = 3$)

Panel A: $\sigma_{\mathrm{P}} = 40\%$

$\boldsymbol{\alpha}$ \ $\boldsymbol{\beta}$	1.2	1.6	1.8
1%			
3%			
5%			

Panel B: $\rho = 0.8$

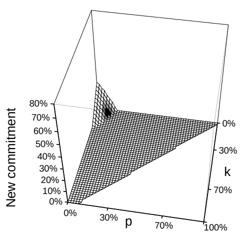
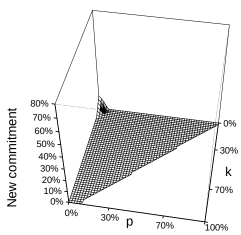
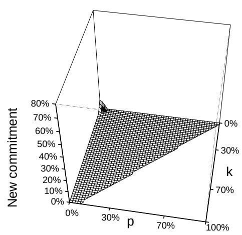
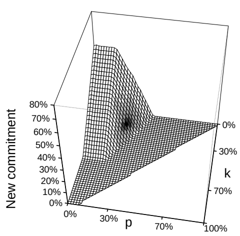
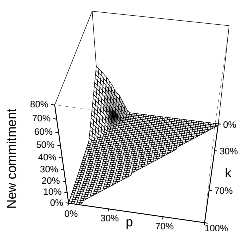
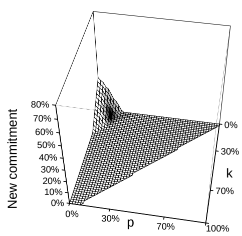
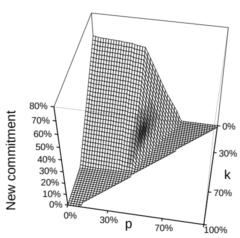
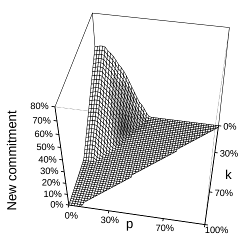
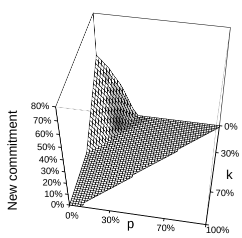
$\boldsymbol{\alpha}$ \ $\boldsymbol{\beta}$	1.2	1.6	1.8
1%			
3%			
5%			

Figure C.7: New commitment of aggressive vs. conservative LP

Panel A: $\sigma_P = 40\%$

β / α — 1.2 | 1.6 | 1.8

1%

1.2

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)				
Commitments (k)				
Interaction ($p \times k$)				
Constant				
R^2				

1.6

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.22	−1.63
Commitments (k)			−0.75	−1.32
Interaction ($p \times k$)				12.05
Constant			0.10	0.12
R^2			0.94	0.95

1.8

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	0.39	0.57	−1.53	−1.40
Commitments (k)	−0.63	−0.41	−0.91	−0.73
Interaction ($p \times k$)		−8.46		−2.19
Constant	0.01	0.01	0.22	0.21
R^2	0.95	0.95	0.96	0.96

3%

1.2

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.63	−1.07	−1.43	−1.43
Commitments (k)	−0.76	−1.39	−0.89	−0.89
Interaction ($p \times k$)		9.67		0.06
Constant	0.10	0.13	0.21	0.21
R^2	0.89	0.90	0.96	0.96

1.6

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.08	0.16	−1.46	−1.55
Commitments (k)	−0.98	−0.66	−0.98	−1.10
Interaction ($p \times k$)		−1.43		0.77
Constant	0.27	0.22	0.40	0.41
R^2	0.98	0.99	0.98	0.98

1.8

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.64	−1.46	−1.85	−1.86
Commitments (k)	−0.92	−1.92	−0.98	−1.00
Interaction ($p \times k$)		2.19		0.05
Constant	0.79	1.16	0.77	0.77
R^2	0.98	0.99	0.99	0.99

5%

1.2

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.40	−0.33	−1.55	−1.45
Commitments (k)	−0.99	−0.90	−0.96	−0.82
Interaction ($p \times k$)		−0.46		−0.97
Constant	0.29	0.28	0.39	0.38
R^2	0.97	0.98	0.98	0.98

1.6

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.73	−1.40	−1.93	−1.99
Commitments (k)	−0.93	−1.75	−0.96	−1.04
Interaction ($p \times k$)		1.79		0.33
Constant	0.83	1.14	0.79	0.80
R^2	0.98	0.99	0.99	0.99

1.8

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−1.02	−0.99	−1.93	−2.09
Commitments (k)	−0.98	−0.98	−0.98	−1.20
Interaction ($p \times k$)		0.03		0.71
Constant	0.98	0.97	0.97	1.02
R^2	1.00	1.00	1.00	1.00

Panel B: $\rho = 0.8$

β / α — 1.2 | 1.6 | 1.8

1%

1.2

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.36	−1.26
Commitments (k)			−0.97	−0.84
Interaction ($p \times k$)				−1.37
Constant			0.24	0.23
R^2			0.99	0.99

1.6

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.23	−1.66
Commitments (k)			−0.77	−1.35
Interaction ($p \times k$)				12.59
Constant			0.11	0.13
R^2			0.94	0.96

1.8

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−0.94	−1.06
Commitments (k)			−0.78	−0.94
Interaction ($p \times k$)				4.52
Constant			0.07	0.07
R^2			0.96	0.96

3%

1.2

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.60	−0.56	−1.71	−2.29
Commitments (k)	−0.92	−0.87	−0.99	−1.76
Interaction ($p \times k$)		−0.38		2.66
Constant	0.23	0.22	0.84	1.01
R^2	0.94	0.94	0.98	0.99

1.6

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.08	0.16	−1.47	−1.57
Commitments (k)	−0.98	−0.66	−0.98	−1.12
Interaction ($p \times k$)		−1.41		0.92
Constant	0.27	0.22	0.40	0.42
R^2	0.98	0.98	0.98	0.98

1.8

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.07	−0.21	−1.55	−1.37
Commitments (k)	−0.96	−1.15	−0.93	−0.67
Interaction ($p \times k$)		0.81		−2.33
Constant	0.28	0.32	0.30	0.28
R^2	0.98	0.98	0.96	0.96

5%

1.2

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.04	0.40	−1.97	−2.44
Commitments (k)	−0.99	−0.43	−0.99	−1.59
Interaction ($p \times k$)		−1.33		1.37
Constant	0.49	0.31	1.37	1.58
R^2	0.99	0.99	0.98	0.99

1.6

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.77	−1.40	−1.93	−2.01
Commitments (k)	−0.94	−1.73	−0.96	−1.06
Interaction ($p \times k$)		1.72		0.41
Constant	0.85	1.14	0.79	0.81
R^2	0.98	0.99	0.99	0.99

1.8

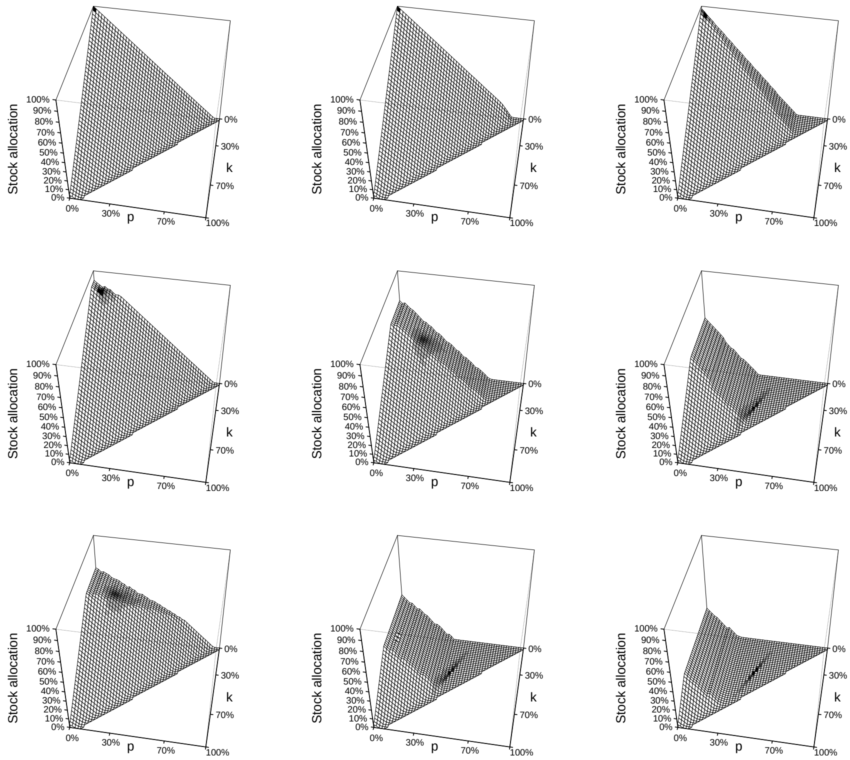
	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.51	−1.19	−1.73	−1.98
Commitments (k)	−0.94	−1.77	−0.94	−1.30
Interaction ($p \times k$)		1.82		1.88
Constant	0.73	1.04	0.56	0.61
R^2	0.99	1.00	0.99	0.99

Figure C.8: Stock allocation of aggressive LP ($\gamma = 1$)

Panel A: $\sigma_{\mathrm{P}} = 40\%$

α \ β: 1.2 | 1.6 | 1.8

Rows (α): 1% | 3% | 5%



Panel B: $\rho = 0.8$

α \ β: 1.2 | 1.6 | 1.8

Rows (α): 1% | 3% | 5%

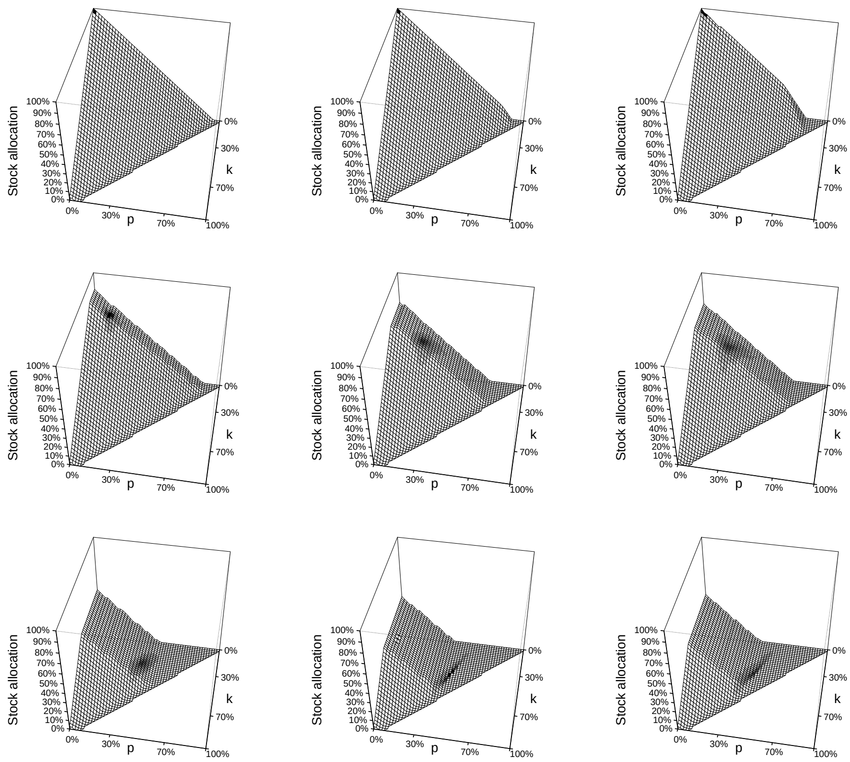
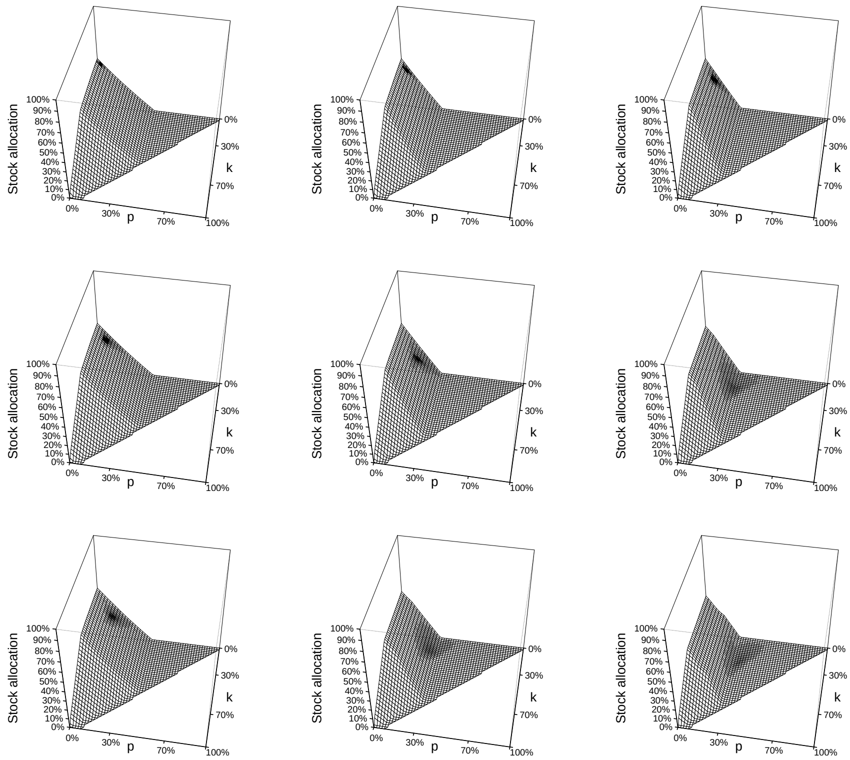


Figure C.9: Stock allocation of conservative LP ($\gamma = 3$)

Panel A: $\sigma_{\mathrm{P}} = 40\%$

$\boldsymbol{\alpha}$ \ $\boldsymbol{\beta}$	1.2	1.6	1.8
1%			
3%			
5%			



Panel B: $\rho = 0.8$

$\boldsymbol{\alpha}$ \ $\boldsymbol{\beta}$	1.2	1.6	1.8
1%			
3%			
5%			

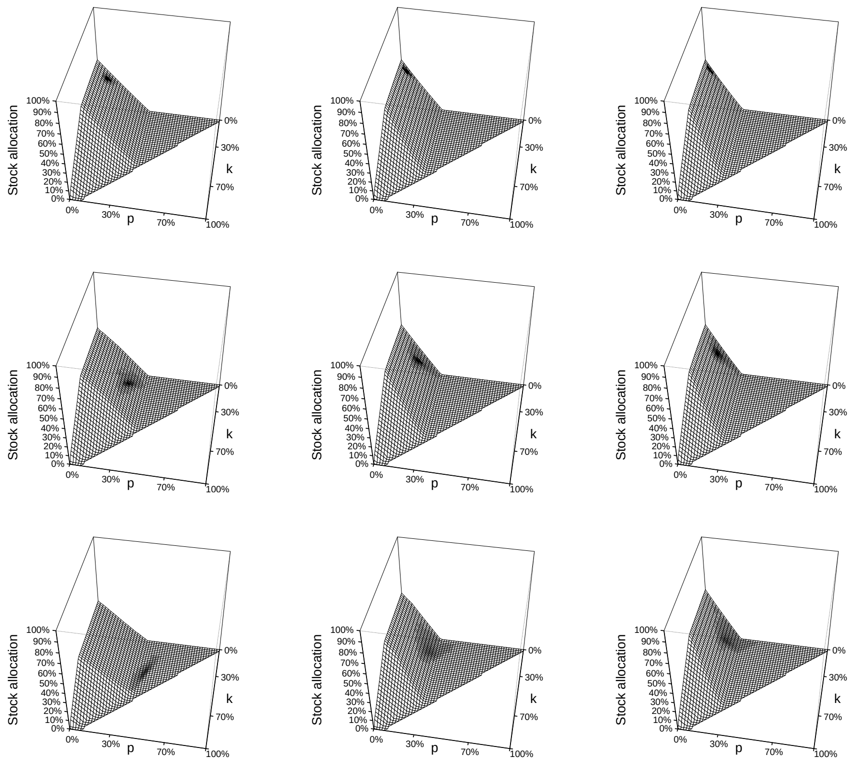


Figure C.10: Stock allocation of aggressive vs. conservative LP

Panel A: $\sigma_{\mathrm{P}} = 40\%$

$\boldsymbol{\beta}$ / $\boldsymbol{\alpha}$

1.2 | 1.6 | 1.8

1%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.24	−1.22
Commitments (k)			−0.00	0.04
Interaction ($p \times k$)				−1.31
Constant			0.50	0.50
R^2			1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.65	−1.64
Commitments (k)			0.00	0.01
Interaction ($p \times k$)				−0.21
Constant			0.50	0.50
R^2			1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−1.44	−1.60	−1.85	−1.84
Commitments (k)	−0.37	−0.57	0.00	0.01
Interaction ($p \times k$)		7.46		−0.12
Constant	0.99	0.99	0.50	0.50
R^2	0.99	0.99	1.00	1.00

3%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.46	0.04	−1.21	−1.19
Commitments (k)	−0.25	0.45	−0.00	0.02
Interaction ($p \times k$)		−10.69		−0.32
Constant	0.90	0.87	0.50	0.50
R^2	0.72	0.77	1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.98	−1.24	−1.59	−1.57
Commitments (k)	−0.03	−0.36	−0.00	0.02
Interaction ($p \times k$)		1.51		−0.16
Constant	0.73	0.79	0.49	0.49
R^2	0.98	0.98	1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.68	−1.58
Commitments (k)			−0.01	0.11
Interaction ($p \times k$)				−0.51
Constant			0.46	0.44
R^2			0.99	0.99

5%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.67	−0.72	−1.19	−1.16
Commitments (k)	−0.03	−0.10	−0.00	0.04
Interaction ($p \times k$)		0.40		−0.27
Constant	0.72	0.73	0.49	0.49
R^2	0.96	0.96	1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.52	−1.40
Commitments (k)			−0.01	0.15
Interaction ($p \times k$)				−0.61
Constant			0.47	0.44
R^2			1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)				
Commitments (k)				
Interaction ($p \times k$)				
Constant				
R^2				

Panel B: $\rho = 0.8$

$\boldsymbol{\beta}$ / $\boldsymbol{\alpha}$

1.2 | 1.6 | 1.8

1%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.23	−1.20
Commitments (k)			−0.00	0.04
Interaction ($p \times k$)				−0.47
Constant			0.50	0.50
R^2			1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.65	−1.64
Commitments (k)			0.00	0.01
Interaction ($p \times k$)				−0.18
Constant			0.50	0.50
R^2			1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−1.19	−1.24	−1.82	−1.78
Commitments (k)	−0.34	−0.42	0.00	0.06
Interaction ($p \times k$)		5.00		−1.53
Constant	0.99	0.99	0.50	0.50
R^2	0.99	0.99	1.00	1.00

3%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.48	−0.41	−1.22	−1.20
Commitments (k)	−0.09	0.01	0.00	0.02
Interaction ($p \times k$)		−0.69		−0.08
Constant	0.78	0.77	0.49	0.49
R^2	0.64	0.64	1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−0.98	−1.24	−1.59	−1.57
Commitments (k)	−0.03	−0.36	−0.00	0.03
Interaction ($p \times k$)		1.47		−0.18
Constant	0.74	0.79	0.49	0.49
R^2	0.97	0.98	1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−1.00	−0.91	−1.78	−1.74
Commitments (k)	−0.06	0.06	−0.00	0.06
Interaction ($p \times k$)		−0.52		−0.53
Constant	0.73	0.70	0.49	0.49
R^2	0.97	0.97	1.00	1.00

5%

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)	−1.02	−1.45		
Commitments (k)	−0.01	−0.56		
Interaction ($p \times k$)		1.30		
Constant	0.51	0.69		
R^2	0.96	0.97		

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.52	−1.40
Commitments (k)			−0.01	0.15
Interaction ($p \times k$)				−0.64
Constant			0.47	0.44
R^2			1.00	1.00

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid wealth (p)			−1.74	−1.71
Commitments (k)			−0.00	0.03
Interaction ($p \times k$)				−0.18
Constant			0.48	0.48
R^2			1.00	1.00

Figure C.11: Consumption of aggressive LP ($\gamma = 1$)

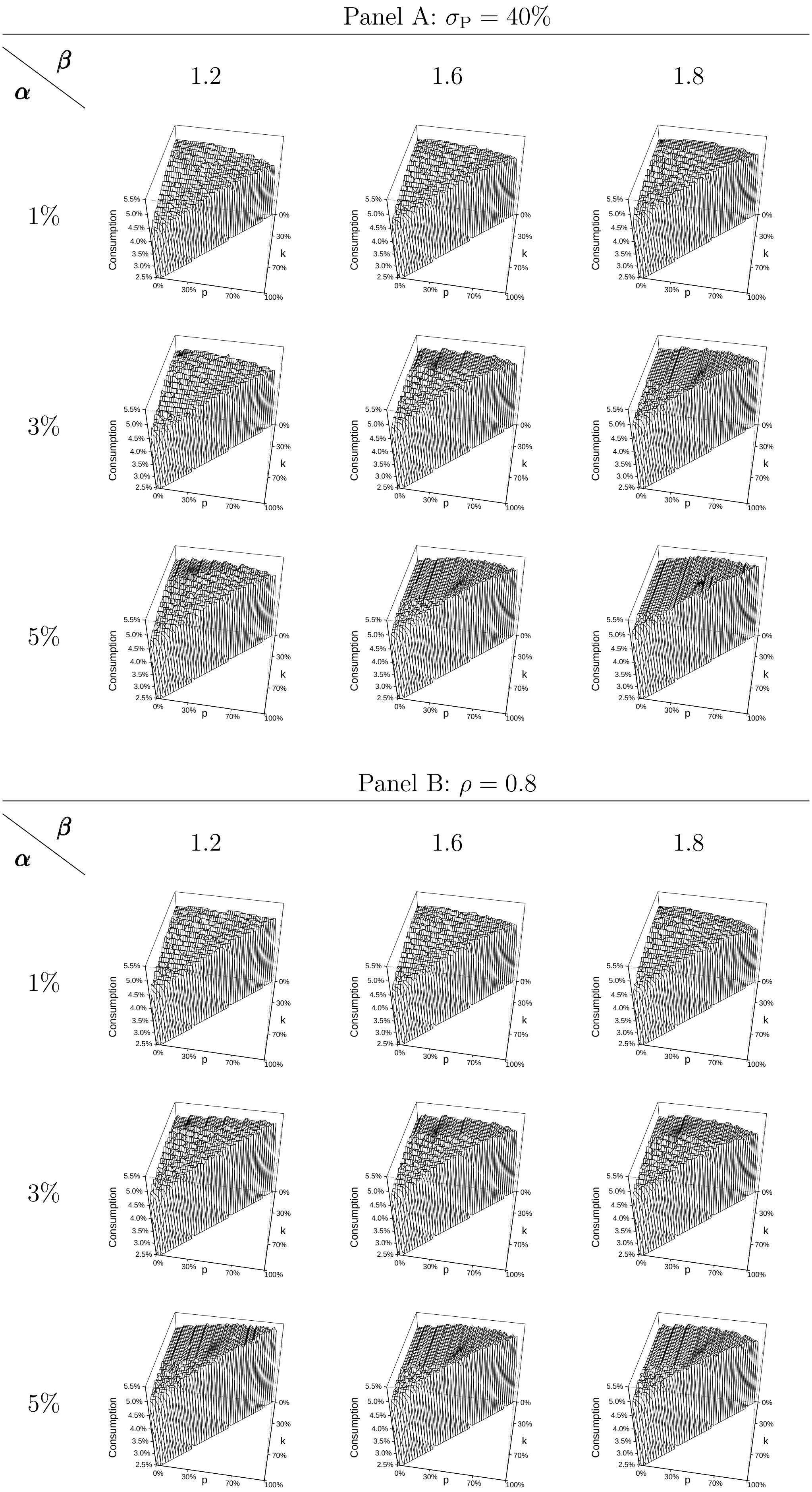
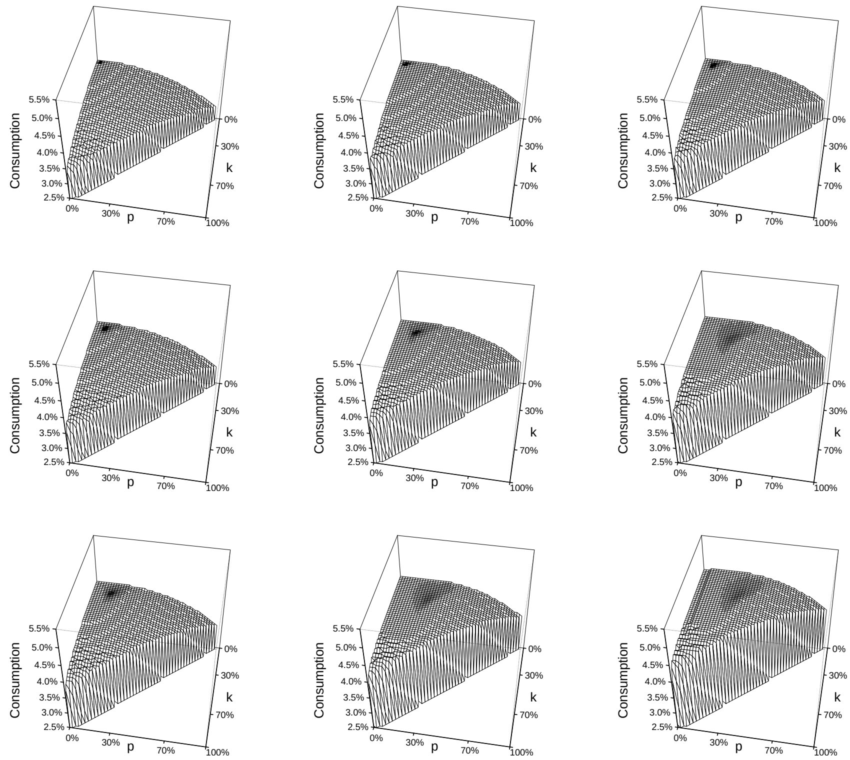


Figure C.12: CONSUMPTION OF CONSERVATIVE LP ($\gamma = 3$)

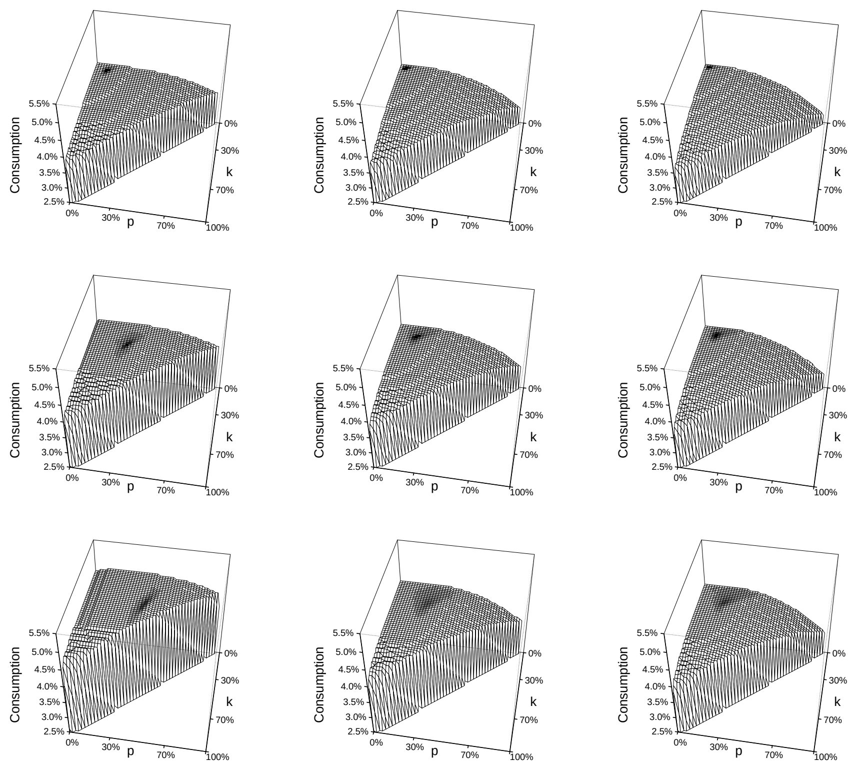
Panel A: $\sigma_{\mathrm{P}} = 40\%$

β / α	1.2	1.6	1.8
1%			
3%			
5%			



Panel B: $\rho = 0.8$

β / α	1.2	1.6	1.8
1%			
3%			
5%			



D Model with Secondary Market

The Bellman equation with a secondary market is given by the following expression:

$$
\begin{aligned}
& V(W,P,K) = \max_{(C,N,S,f)} \left\{ C^{1-\gamma} + \delta E\left[V(W',P',K')^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}} \\
& \text{subject to} \\
& W' = R_{\mathrm{P}} P(1-f) + R_{\mathrm{S}} S + R_{\mathrm{F}}\left(W - C - P(1-f) - S - f(\psi_{\mathrm{P}} P + \psi_{\mathrm{K}} K) \right) \\
& P' = (1-\lambda_{\mathrm{D}})\, R_{\mathrm{P}}(1-f)P + \lambda_{\mathrm{N}} N + \lambda_{\mathrm{K}}(1-f)K \\
& K' = (1-\lambda_{\mathrm{K}})(1-f)K + (1-\lambda_{\mathrm{N}})N \\
& 0 \le C \le W - P(1-f) - f(\psi_{\mathrm{P}} P + \psi_{\mathrm{K}} K) \\
& N \ge 0
\end{aligned}
\tag{D.1}
$$

Variables and parameters in this expression are defined in the main text. Normalized by wealth, the Bellman equation with a secondary market becomes:

$$
\begin{aligned}
& v(p,k) = \max_{(c,n,\omega_{\mathrm{S}},f)} \left\{ c^{1-\gamma} + \delta E\left[\left[\Big(1 - c - f(\psi_{\mathrm{P}} p + \psi_{\mathrm{K}} k) \Big) R_{\mathrm{W}} v(p',k') \right]^{1-\gamma} \right] \right\}^{\frac{1}{1-\gamma}} \\
& \text{subject to} \\
& R_{\mathrm{W}} = \omega_{\mathrm{P}} R_{\mathrm{P}} + \omega_{\mathrm{S}} R_{\mathrm{S}} + (1-\omega_{\mathrm{P}}-\omega_{\mathrm{S}})\, R_{\mathrm{F}} \\
& p' = \left[(1-\lambda_{\mathrm{D}})\, R_{\mathrm{P}}(1-f)p + \lambda_{\mathrm{N}} n + \lambda_{\mathrm{K}}(1-f)k \right] \Big/ \left[(1 - c - f(\psi_{\mathrm{P}} p + \psi_{\mathrm{K}} k)) R_{\mathrm{W}} \right] \\
& k' = \left[(1-\lambda_{\mathrm{K}})(1-f)k + (1-\lambda_{\mathrm{N}})n \right] \Big/ \left[(1 - c - f(\psi_{\mathrm{P}} p + \psi_{\mathrm{K}} k)) R_{\mathrm{W}} \right] \\
& 0 \le c \le 1 - p(1-f) - f(\psi_{\mathrm{P}} p + \psi_{\mathrm{K}} k) \\
& n \ge 0
\end{aligned}
\tag{D.2}
$$

In this expression, the portfolio weights in PE and stocks are defined respectively as $\omega_{\mathrm{P}} = P(1-f)\big/[W - C - f(\psi_{\mathrm{P}} P + \psi_{\mathrm{K}} K)]$ and $\omega_{\mathrm{S}} = S\big/[W - C - f(\psi_{\mathrm{P}} P + \psi_{\mathrm{K}} K)]$. The remaining variables are defined in the main text.

We solve the normalized problem numerically using an algorithm similar to the one described in Appendix A, and the rest of this section reports output from that solution. We plot optimal policies for new commitment, stock allocation, and consumption. We also include regression tables measuring the local sensitivity of optimal policies to the two state variables, k and p. Figures and tables are constructed in the same way as their counterparts from the solution of the reduced model.

D.1 Optimal Commitment with Secondary Market

a. Aggressive ($\gamma = 1$) b. Conservative ($\gamma = 3$)

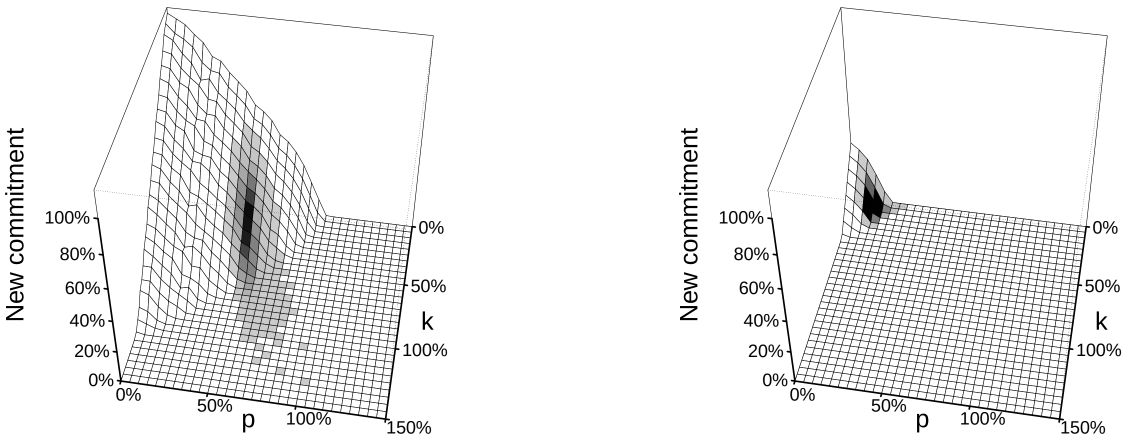


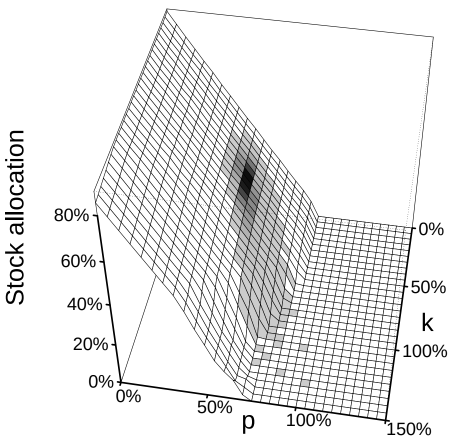
Figure D.13: Optimal commitment strategy with secondary market.

Table A1: Optimal Commitment with secondary market.

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid Wealth	−0.86	−1.35	−1.49	−1.55
Commitment	−0.97	−1.57	−0.84	−0.94
Interaction		1.12		0.66
Constant	1.08	1.34	0.39	0.40
R^2	0.99	0.99	0.96	0.96

D.2 Optimal Stock Allocation with Secondary Market

a. Aggressive ($\gamma = 1$)



b. Conservative ($\gamma = 3$)

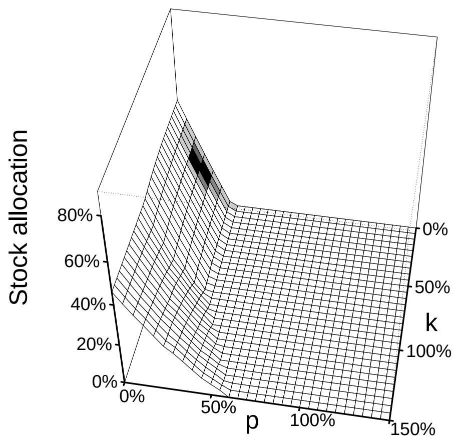


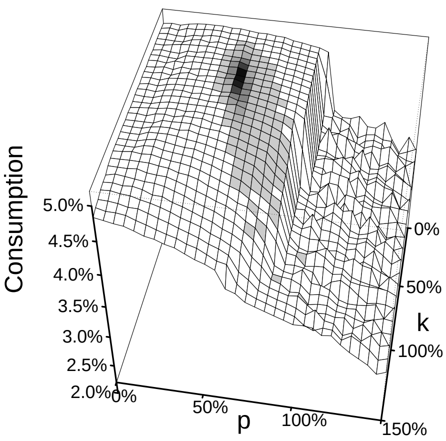
Figure D.14: Optimal stock allocation with secondary market.

Table A2: Optimal stock allocation with secondary market.

	Aggressive ($\gamma = 1$)		Conservative ($\gamma = 3$)	
	(1)	(2)	(3)	(4)
Illiquid Wealth	−1.04	−0.60	−1.58	−1.54
Commitment	−0.02	0.42	0.00	0.06
Interaction		−0.80		−0.40
Constant	0.94	0.70	0.49	0.48
R^2	0.97	0.99	1.00	1.00

D.3 Optimal Consumption with Secondary Market

a. Aggressive ($\gamma = 1$)



b. Conservative ($\gamma = 3$)

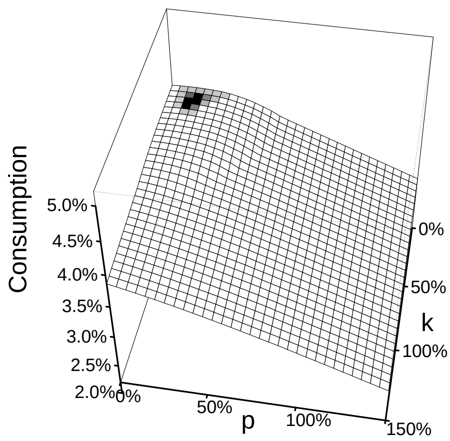


Figure D.15: Optimal consumption-to-wealth ratio with secondary market.

Chapter 3

Private Equity with Leverage Aversion

Nicola Giommetti*

Abstract

I study optimal portfolio allocation with private equity (PE) and several liquid assets. PE requires capital commitment which is gradually contributed and eventually distributed back to the investor. Optimal portfolio allocation is well approximated by static mean-variance optimization with self-imposed margin requirements. Due to capital commitment, the investor assigns greater margin requirement to PE than liquid assets, and the risky portfolio of a constrained investor can optimally underweight PE relative to the tangency portfolio, even when PE has positive alpha and moderately high beta with respect to liquid assets.

*I am grateful to Morten Sorensen for helpful suggestions and support throughout this project, and to Peter Feldhütter, Rasmus Jørgensen, Andrey Malenko, Kasper Meisner Nielsen, and seminar participants at Copenhagen Business School and LBS Private Equity Symposium for helpful discussion and comments. The author can be reached at Copenhagen Business School (ngi.fi@cbs.dk).

Limited partnerships in private equity (PE) funds are risky long-term investments with special institutional rules distinguishing them from other assets typically considered in finance. PE investors become limited partners (LPs) by committing capital at the start of the partnership. This capital is not immediately contributed to the fund, however, and fund managers are given large discretion on the timing of contributions. LPs cannot easily divest their initial commitment. They must typically ensure to cover contributions as the fund requests them, and later wait for capital distributions as the fund liquidates its underlying investments.

By the end of 2020, almost \$10 trillion were allocated to PE and other private market funds under those institutional rules.[1] Little is known, however, about the effect of those rules on optimal portfolio allocation of LPs. To address that question, this paper studies the normative implications of a portfolio problem with explicit modeling of the PE commitment mechanism.

I introduce PE in a mean-variance portfolio model with several liquid assets. As in Giommetti and Sorensen (2021), PE funds require capital commitment from the investor, and they gradually call and invest that commitment in underlying private assets. These assets appreciate at a risky rate until they are gradually liquidated, and the resulting value is distributed to the investor. PE investments cannot be sold or collateralized, and the investor must hold them to maturity. A main contribution of my analysis is to derive a closed-form solution consistent with the result in Giommetti and Sorensen (2021) that optimal PE allocation is not monotonically declining in risk aversion, despite the high risk of PE.

I compare two versions of the model. In the main version, the investor has static mean-variance objective over the return to his average portfolio allocation over time. The investor ignores time variation of portfolio weights, provided that there is always some liquid wealth available. Despite the dynamic PE mechanism, optimal average portfolio allocation is the solution to a static mean-variance problem with margin requirements assigned endogenously by the investor. Liquid risky assets have a margin of 1, while PE has greater margin, typically above 2. This greater margin is due primarily to commitments, since the investor maintains a liquidity reserve in risk-free bonds which is proportional to

[1] See Preqin dataset and Gourier, Phalippou, and Westerfield (2021) about this figure.

commitments. With plausible parameter values, greater PE margin tend to discourage constrained investors; despite the high risk and return of PE, constrained investors underweight PE and overweight liquid risky assets relative to the tangency portfolio.

In a second version of the model, the investor has dynamic mean-variance objective over the return to his current and future portfolio allocations. This version is arguably more realistic, but it cannot be reduced to a static problem. Instead, I solve it numerically for five different specifications of parameter values and for several levels of risk aversion, then I compare the resulting average portfolio allocations with those from the main model. I find similar results across the two models. Portfolio allocations are virtually identical for unconstrained investors, while they differ marginally for constrained investors. The main model tend to overestimate the optimal PE allocation of constrained investors relative to the more realistic version. The difference remains small, however, with limited and usystematic variation across specifications.

A large literature studies portfolio and asset pricing implications of trading and financing frictions. Within that broad literature, my results are related to the work of Frazzini and Pedersen (2014), who study portfolio theory and asset pricing implications of exogenous margin constraints. Assuming margins to be constant across assets, they find that constrained investors overweight assets with high market beta relative to unconstrained investors, and equilibrium alpha is positive for assets with low beta. Among PE, buyout funds are known to undertake levered investments in low-beta firms, an activity that generates valuable high-beta assets for constrained investors in the model of Frazzini and Pedersen (2014). In my model, however, margins are endogenous and investors assign high margin requirement to PE funds because of commitment. The high margin requirement can discourage constrained investors from investing heavily in PE, despite its positive alpha and moderately high beta.

Several authors study investment implications specific to PE illiquidity. Giommetti and Sorensen (2021) solve numerically a dynamic portfolio model with stocks, bonds, and illiquid PE. They find that stocks and bonds allocation are sensitive to risk aversion, while PE allocation is approximately flat across investors. I obtain the same result in a closed-form solution highlighting the main mechanism. Sorensen, Wang, and Yang (2014) develop a subjective valuation model assigning a long-lasting illiquid asset to an investor who can freely trade stocks and bonds. In their model, the investor demands a liquidity premium for PE only if its return is not fully spanned by stocks. Similarly, my results rely on PE having some idiosyncratic risk. In my model, idiosyncratic risk has the main role of generating situations in which the investor could remain without liquid wealth. Bollen and Sensoy (2020) extend the subjective valution approach with a secondary market for partnership interests. Gourier, Phalippou, and Westerfield (2021) study commitment risk with a single PE fund and show that commitment risk reduces PE allocation significantly. I find capital commitment to be essential also for my results; if the investor could buy (but not sell) PE directly, PE would have the same margin of liquid assets. Finally, Ang, Papanikolaou, and Westerfield (2014) and Dimmock, Wang, and Yang (2019) model illiquidity of PE and alternative assets more broadly.

1 Model

I consider a discrete time model with one risk-free bond and $A + 1$ risky assets $a \in \{0, 1, \ldots, A\}$. The risk-free bond appreciates at constant rate $1 + r_F$, while risky assets grow at random rates $1 + r_{a,t}$. These random rates are collected in the vector $r_t = \{r_{0,t}, r_{1,t}, \ldots, r_{N,t}\}$ and $1 + r_t$ is i.i.d log-normal with expected value $1 + \mu$ and covariance matrix Σ. All assets are freely tradeable except for $a = 0$, which denotes PE.

1.1 Private Equity

PE represents an investment in a diversified portfolio of PE funds. It is illiquid as there is no market where to transact PE directly, but it is possible to invest in PE by becoming a LP in PE funds. As a LP, the investor must be willing to commit capital to PE funds at the start of their life. Committed capital remains in the hands of the LP until it is gradually called (or drawn down) by the funds. The LP loses control of capital once it is called, and it regains control only when funds decide to liquidate some of their underlying investments and pay the realized value, net of fees, to the LP. I formalize this process using the setup of Giommetti and Sorensen (2021).

Let K_t represent the amount of capital which is committed to PE funds before period t, and which has yet to be called. I refer to this variable as uncalled commitments. At the end of each period, uncalled commitments can be incremented taking new commitments

N_t, resulting in total commitments equal to $K_t + N_t$. Then, a constant share λ_C of those total commitments is called and invested by PE funds in the next period. The updated amount of uncalled commitments is given by:

$$K_{t+1} = (K_t + N_t) - \lambda_C(K_t + N_t) \tag{1}$$

The quantity $\lambda_C(K_t + N_t)$ is a capital call, and it represents a cash flow from the LP to PE funds. This expression for capital call at $t + 1$ implies that new commitments affect capital calls with a 1-year delay. Capital calls, however, are immediately used for underlying PE investments, so capital calls in a given period become part of illiquid PE wealth, or net asset value (NAV), at the end of that period.

To illustrate the evolution of NAV, let P_t be the dollar amount of NAV at the end of period t. This variable grows with the PE return $r_{0,t}$, increases with capital calls, and decreases because PE funds liquidate a share λ_D of underlying investments each period. These three effects correspond to three components in the evolution of NAV:

$$P_{t+1} = P_t(1 + r_{0,t+1}) + \lambda_C(K_t + N_t) - \lambda_D P_t(1 + r_{0,t+1}) \tag{2}$$

Random fluctuations in value move NAV from P_t at the end of one period to $P_t(1+r_{0,t+1})$ at the beginning of the next. At that point, two things happen contemporaneously: there is a capital call equal to $\lambda_C(K_t + N_t)$ and a distribution equal to $\lambda_D P_t(1 + r_{0,t+1})$. The distribution is a cash flow from PE funds to the LP generated by the liquidation of underlying investments. This cash flow represents the last stage of the investment process.

To summarize, each time the LP makes a new commitment, it agrees to endure two waiting periods. First, the LP must wait for commitment to be invested. Commitment is invested gradually with capital calls that compel the LP to allocate capital to PE funds. After capital is called, its value starts growing at the PE return, so the LP obtains PE exposure. However, before it can dispose of that value, the LP must wait for PE funds to sell the underlying investments and distribute the corresponding value. Distributions, like capital calls, are made gradually over time.

1.2 Investor

I consider an investor whose preferences have two components: (i) a mean-variance objective function which ignores liquidity concerns, and (ii) a constraint which ensures that the investor does not exhaust liquid wealth. I describe these two components in that order, then I introduce the budget constraint.

For the objective function, a key variable is the average portfolio allocation. Specifically, let $\omega_t = \{\omega_{0,t}, \omega_{1,t}, \ldots, \omega_{A,t}\}$ be the vector of portfolio weights in risky assets at time t. The average portfolio allocation is then defined as the vector of mean portfolio weights, $\bar{\omega} = E[\omega_t]$. With this definition, the objective function can be written as

$$\bar{\omega}' (\mu - r_F) - \frac{\gamma}{2} \bar{\omega}' \Sigma \bar{\omega} \tag{3}$$

where $\mu - r_F$ is the vector of expected excess returns, Σ is the covariance matrix of returns, and γ is risk aversion. This objective function is determined by the mean-variance characteristics of the average portfolio allocation, so it is a static objective and it is unaffected by variation of portfolio weights over time. I use this assumption for tractability, and I consider a dynamic objective function in section 4. Furthermore, Appendix A shows that a sufficient condition for this objective function to rank average allocations similarly to a dynamic objective is that most variation in PE return be spanned by liquid assets.

The second component of preferences is a constraint on liquid wealth. Specifically, let W_t be the investor's total wealth, so that $W_t - P_t$ quantifies his liquid wealth. The investor imposes the following constraint:

$$\text{Prob}\big(W_t - P_t \leq 0\big) = 0 \tag{4}$$

Liquid wealth must always remain positive. This constraint is introduced exogenously based on Giommetti and Sorensen (2021), who show that the same constraint arises in a model with similar investment opportunities and power utility over consumption. Although outside of my model, the economic intuition behind the constraint is that the investor needs to maintain a positive amount of liquid wealth because the NAV of PE investments is not immediately disposable, and the investor has recurring expenses that might be costly or even impossible to delay.

I do not model any of the expenses that the investor might have in reality. In particular, I assume that the investor does not consume anything, and while this is a simplification, results would be virtually unchanged if I assumed a constant consumption-to-wealth ratio. Without consumption, wealth grows according to the following budget constraint:

$$W_{t+1} = W_t\big[1 + r_F + \omega_t'(r_{t+1} - r_F)\big] \tag{5}$$

In this expression, $r_{t+1} - r_F$ is the vector of excess returns on risky assets, and the term inside square brackets is the gross return on wealth at time $t + 1$. Accordingly, the net return on wealth is defined as follows:

$$r_{W,t+1} = r_F + \omega_t'(r_{t+1} - r_F) \tag{6}$$

1.3 Optimization Problem

The investor's preferences, the budget constrant, and the dynamics of commitments and NAV define the optimization problem. The investor has a static objective function, and it might be intuitive that he should optimize over the set of average allocations. Formally, however, the liquidity constraint (4) forces the investor to optimize over dynamic strategies to determine the set of feasible average allocations.

The investor chooses (i) a dynamic allocation strategy $\omega_{a,t} = \omega_a(\omega_{0,t}, K_t)$ in liquid risky assets $a \neq 0$ and (ii) a dynamic commitment strategy $N_t = N(\omega_{0,t}, K_t)$ that maximize

$$\bar{\omega}'\left(\mu - r_F\right) - \frac{\gamma}{2}\bar{\omega}'\Sigma\bar{\omega}$$

subject to

$$\begin{aligned}
&\text{Prob}\big(W_t - P_t \leq 0\big) = 0 \\
&P_{t+1} = P_t(1-\lambda_D)(1+r_{0,t+1}) + \lambda_C(K_t + N_t) \\
&K_{t+1} = (1-\lambda_C)(K_t + N_t) \\
&W_{t+1} = W_t(1 + r_{W,t+1}) \\
&r_{W,t+1} = r_F + \omega_t'(r_{t+1} - r_F) \\
&N_t \geq 0
\end{aligned} \tag{7}$$

The investor can condition his strategy on three state variables: W_t is total wealth, P_t is the NAV of his illiquid wealth, and K_t is uncalled commitments. Each period, the evolution of these state variables is determined as follows. Wealth grows with portfolio return, $r_{W,t+1}$. NAV grows with PE return, $r_{0,t+1}$, decreases with distributions, $\lambda_D P_t(1+r_{0,t+1})$, and increases with capital calls, $\lambda_C(K_t + N_t)$. Uncalled commitments increase at the end of the period with new commitments, N_t, and decrease next period with capital calls $\lambda_C(K_t + N_t)$.

The last constraint of the problem explicitly prevents the investor to take negative new commitments. Relaxing this constraint would allow the investor to default on uncalled commitments, so to reduce future capital calls. In practice, the investor can potentially default on commitments, but that is very costly both in terms of reputation with PE funds and in terms of contractual punishments (Banal-Estañol, Ippolito, and Vicente, 2017).

A simplification can be obtained normalizing the constraints by wealth so that the level of wealth disappears entirely from the problem. To write the normalized constraints, it is useful to define $n_t = N_t/W_t$ and $k_t = K_t/W_t$ as the fraction of new and uncalled commitments over wealth, and $\omega_{0,t}$ as the portfolio weight in PE. With those definitions, the normalized problem can be stated as follows. The investor chooses (i) a dynamic allocation strategy $\omega_{a,t} = \omega_a(\omega_{0,t}, k_t)$ in liquid risky assets $a \neq 0$ and (ii) a dynamic commitment strategy $n_t = n(\omega_{0,t}, k_t)$ that maximize

$$\bar{\omega}'\left(\mu - r_F\right) - \frac{\gamma}{2}\bar{\omega}'\Sigma\bar{\omega}$$

subject to

$$\begin{aligned}
&\text{Prob}\big(1 - \omega_{0,t} \leq 0\big) = 0 \\
&\omega_{0,t+1} = [\omega_{0,t}R_{P,t+1} + \lambda_C(k_t + n_t) - \lambda_D\omega_{0,t}R_{P,t+1}]/(1 + r_{W,t+1}) \\
&k_{t+1} = [(k_t + n_t) - \lambda_C(k_t + n_t)]/(1 + r_{W,t+1}) \\
&r_{W,t+1} = r_F + \omega_t'(r_{t+1} - r_F) \\
&n_t \geq 0
\end{aligned} \tag{8}$$

Notice that this is not a standard dynamic programming problem, and the optimal dynamic strategy is not guaranteed to be unique. Specifically, there is an infinite number of dynamic strategies that achieve the same average allocation, and the investor is indifferent between any two strategies that achieve the same average allocation while satisfying the liquidity constraint. Instead of solving for the set of optimal dynamic strategies, I show how to simplify the constraints so that the problem can be reformulated as static mean-variance optimization with leverage aversion, and the choice variable becomes $\bar{\omega}$. Using that reformulation, I solve for the optimal average allocation, which is unique.

2 Private Equity with Leverage Aversion

To derive a fully static version of the problem, I start by considering the liquidity constraint:

$$\text{Prob}\big(1 - \omega_{0,t} \leq 0\big) = 0 \tag{9}$$

With log-normally distributed returns, Giommetti and Sorensen (2021) show that this constraint can be replaced with two types of dynamic inequalities. First, the investor does not short-sell risky assets because large positive returns could otherwise exhaust his liquid wealth. For simplicity, I assume that short-selling constraints do not bind.[2] Second, the investor maintains a liquidity reserve in risk-free bonds that is sufficiently large to meet

[2] Short-selling constraints on average portfolio allocations do not bind in any of the examples I discuss below.

future capital calls even if all risky assets were to lose their entire value. I focus on this second condition, which corresponds to the following dynamic constraint:

$$1 - \omega_t' \mathbf{1} \geq \frac{\lambda_C}{\lambda_C + r_F}(k_t + n_t) \tag{10}$$

In this expression, $\mathbf{1}$ indicates a vector of ones, and the left-hand side represents the current portfolio weight in bonds. The right-hand side quantifies the minimum reserve of bonds required by the investor. This reserve is proportional to total commitments, and the constant of proportionality can be written as

$$\frac{\lambda_C}{\lambda_C + r_F} = \sum_{s=1}^{\infty} \frac{I_s}{(1 + r_F)^s} \tag{11}$$

where the terms $I_s = (1 - \lambda_C)^{s-1}\lambda_C$ represent the future stream of capital calls generated by 1 unit of commitment. For a positive risk-free rate r_F, the constant of proportionality is lower than 1, and the minimum reserve of bonds is lower than total commitments because commitments are called gradually with intensity λ_C, and the reserve has time to appreciate before being possibly used up.

From a dynamic perspective, constraint (10) imposes a trade-off between present exposure to risky assets (including PE) and future exposure to PE. This trade-off is a consequence of the liquidity constraint and the commitment mechanism of PE. The investor dislikes having no liquid wealth, and commitments represent a future obligation to invest in illiquid PE. The investor reconciliates this obligation with his liquidity preferences by making sure to always have enough liquid wealth to satisfy the obligation. This objective is achieved by maintaining a safe reserve of bonds.

From a static perspective, taking expectations on both sides of (10), the constraint imposes a trade-off between average exposure to risky assets, including PE, and average commitments. At the same time, average commitments are strictly related to the average portfolio weight in PE, and after accounting for this relationship, the static version of (10) becomes exclusively about portfolio weights. To show that, I first formalize the relationship between average commitments and average portfolio weight in PE.

Intuitively, if the investor increases average commitments, average PE allocation increases too. Formally, that can be seen from problem (8) by taking expectations on both sides of the constraint about $\omega_{0,t+1}$. Appendix B follows this approach to derive the following proposition.

Proposition 1. *Average PE allocation can be expressed approximately as a linear function of average commitments:*

$$\bar{k} + \bar{n} = L\, \bar{\omega}_0 \tag{12}$$

In this expression, $\bar{k} = E[k_t]$ *and* $\bar{n} = E[n_t]$ *are average uncalled and new commitment, and*

$$L = \frac{\lambda_D(1+\mu_0) + \mu_W - \mu_0}{\lambda_C} \tag{13}$$

where $\mu_W = E[r_{W,t}]$ *is the expected return on wealth, and* μ_0 *is the expected return to PE.*

Proposition 1 quantifies the average commitments needed to maintain a given average allocation to PE. The relationship between average commitments and average PE allocation is determined by the coefficient L, which depends on the liquidity of PE (λ_C and λ_D) and on expected returns (μ_W and μ_0).

PE liquidity has contrasting effects on L. The coefficient decreases with respect to λ_C, when commitments are called faster, and it increases with respect to λ_D, when investments are liquidated faster. I explain these two effects in order. First, if funds call less commitments each period, it takes more commitments to maintain capital calls and PE allocation at the same average level. Second, if more investments are liquidated each period, it takes larger capital calls to maintain the same average PE allocation. To increase average capital calls, the investor must increase average commitments.

Asset returns affect the coefficient L in two intuitive ways. First, L decreases with the expected return to PE, μ_0. If the value of PE investments grows faster, it takes smaller capital calls to maintain a constant average PE allocation, and smaller capital calls correspond to less commitments on average. Second, L increases with the expected return on wealth, μ_W. To understand this effect, recall that the portfolio weight in PE is the ratio of NAV over wealth. If wealth grows faster, NAV must also grow faster to maintain the same PE allocation. For NAV to grow faster, larger capital calls are needed, and they are obtained increasing average commitments.

The two conditions (10) and (12) can be combined to characterize the set of feasible average allocations implied by the constraints of problem (8). The result is shown in the following proposition.

Proposition 2. *The set of feasible average allocations in problem (8) can be expressed as margin requirements on risky assets:*

$$m_0\bar{\omega}_0 + \sum_{a=1}^{A} \bar{\omega}_a \leq 1 \tag{14}$$

The margin requirement for PE is given by:

$$m_0 = 1 + \frac{\lambda_D(1+\mu_0) + (\mu_W - \mu_0)}{\lambda_C + r_F} \tag{15}$$

This proposition determines the endogenous margin requirements consistent with liquidity constraint (9). An important characteristic of those margin requirements is that PE has higher margin than liquid assets. Liquid assets require margin of 1, while PE requires margin $m_0 > 1$, and the difference is entirely due to the fact that PE allocation can be obtained only through commitments. Commitments and capital calls are not directly affected by asset returns, and periods of low returns deplete wealth, pushing the investor towards illiquid states. To avoid excessive illiquidity, the investor saves $m_0 - 1$ dollars in bonds for every dollar of PE allocation.

Notice that the margin m_0 is not fully exogenous with respect to the other terms of constraint (14). In particular, m_0 depends on the expected return on wealth, which is affected by the average portfolio weights, $\bar{\omega}$. With realistic parameter values, however, portfolio weights have negligible effect on m_0, and I return to this point below with a numerical example.

Using Proposition 2, it is possible to find the optimal average allocation from problem (8) by solving a simpler static problem. In the static version of the problem, the investor chooses average allocation in risky assets, $\bar{\omega}$, to maximize

$$\bar{\omega}\left(\mu - r_F\right) - \frac{\gamma}{2}\bar{\omega}'\Sigma\bar{\omega}$$

$$\text{subject to} \tag{16}$$

$$m_0\bar{\omega}_0 + \sum_{a=1}^{A}\bar{\omega}_a \leq 1$$

where m_0 is given in Proposition 2.

This is a familiar problem of mean-variance optimization with margin constraints. Compared to problem (8), this version is simpler in at least two ways. Not only it is static, but it also works directly with PE allocation, bypassing commitment. In the context of public equities, a similar problem was first studied by Black (1972) and more recently extended by Frazzini and Pedersen (2014). Importantly, I do not impose exogenous margins, and the margin requirement of PE is endogenously higher than that of liquid assets.

3 Optimal Allocation

To find the optimal average allocation, it suffices to take the first-order condition of problem (16). I define $m = (m_0\ 1\ 1\ \cdots\ 1)'$ as the column vector of margin requirements, and the optimal allocation can be expressed as follows:

$$\bar{\omega}^* = \frac{1}{\gamma}\Sigma^{-1}(\mu - r_F - \psi m) \tag{17}$$

The Lagrange multiplier $\psi \geq 0$ measures the impact of margin requirements.

Risk aversion is the only parameter differentiating constrained and unconstrained investors. Specifically, there exists risk aversion $\hat{\gamma}$ such that investors with $\gamma > \hat{\gamma}$ are unconstrained, while investors with $\gamma < \hat{\gamma}$ are constrained. Unconstrained investors have $\psi = 0$, and they achieve their first-best allocation despite the margin requirements. They hold a combination of bonds and the tangency portfolio, which is the portfolio of risky assets with highest Sharpe ratio. Constrained investors have $\psi > 0$ and their first-best allocation violates margin requirements. As a result, constrained investors 'reach for yield', meaning that they overweight assets with high expected returns and low margin requirements. This strategy allows constrained investors to increase the expected return of their portfolio while also satisfying the margin constraint. A main drawback of this strategy, compared to the first-best, is that the resulting portfolio of risky assets achieves a lower Sharpe ratio.

3.1 Reaching for Yield

An interesting question is whether constrained investors overweight or underweight PE in their portfolio of risky assets relative to the tangency portfolio. Intuitively, is private equity a good or bad asset to reach for yield? To answer this question, it is useful to define two portfolios.

A relevant portfolio is the combination of liquid risky assets that minimize the variance of return, or the liquid minimum variance portfolio for short. To define this portfolio, I partition the covariance matrix of retuns as follows:

$$\Sigma = \begin{pmatrix} \sigma_0^2 & \sigma_{0A} \\ \sigma_{0A}' & \Sigma_A \end{pmatrix} \tag{18}$$

With a slight abuse of notation, Σ_A indicates the covariance matrix of liquid risky assets, σ_0^2 indicates the variance of PE, and σ_{0A} represents the row vector of covariances between PE and liquid risky assets. The liquid minimum variance portfolio is defined by the following vector of $A+1$ weights:

$$\omega_{\text{LMV}} = \begin{pmatrix} 0 & \frac{\mathbf{1}'\Sigma_A^{-1}}{\mathbf{1}'\Sigma_A^{-1}\mathbf{1}} \end{pmatrix} \tag{19}$$

where $\mathbf{1}$ is a column vector of ones.

A second relevant portfolio is the hedging portfolio, meaning the portfolio of liquid risky assets that is most correlated with PE. To define this portfolio, let $B = \Sigma_A^{-1}\sigma_{0A}'$ be the vector of coefficients from a multivariate regression of PE return on the returns to liquid risky assets. Ignoring a constant, the regression corresponds to

$$r_{0,t} = B' r_{A,t} + \varepsilon_t \tag{20}$$

where $r_{A,t}$ is the vector of returns on liquid risky assets and ε_t is the idiosyncratic error term. The hedging portfolio has weights that are proportional to B:

$$\omega_{\text{HDG}} = \begin{pmatrix} 0 & \frac{B'}{B'\mathbf{1}} \end{pmatrix} \tag{21}$$

Furthermore, notice from (20) that the beta of PE with respect to this portfolio is $B'\mathbf{1}$, the sum of all elements of B.

Using the liquid minimum variance and hedging portfolios, Proposition 3 (proof in Appendix B) provides a necessary and sufficient condition for PE allocation to increase disproportionately more than other risky assets as risk aversion decreases and margin constraints become more binding.

Proposition 3. *For $\gamma < \hat{\gamma}$, the derivative of $\bar{\omega}_0^*/(\bar{\omega}_0^* + \bar{\omega}_1^* + \ldots + \bar{\omega}_A^*)$ with respect to γ is negative if and only if the following condition is satisfied:*

$$\mu_0 - r_F > B'\mathbf{1}(\mu_{\text{HDG}} - r_F) + (m_0 - B'\mathbf{1})(\mu_{\text{LMV}} - r_F) \tag{22}$$

The terms μ_{HDG} and μ_{LMV} are expected returns on the hedging and liquid minimum variance portfolios.

At the margin, constrained investors compare PE with an alternative investment opportunity that requires the same amount of capital. Each unit of PE requires m_0 units of capital, and the alternative constitutes of purchasing $B'\mathbf{1}$ units of the hedging portfolio and $m_0 - B'\mathbf{1}$ units of the liquid minimum variance portfolio. The expected excess return of this alternative is:

$$B'\mathbf{1}(\mu_{\text{HDG}} - r_F) + (m_0 - B'\mathbf{1})(\mu_{\text{LMV}} - r_F) \tag{23}$$

As risk aversion decreases, margin constraints become more binding, and the investor substitutes the investment with lower expected return in favor of the other. PE is a preferred asset to reach for yield only if its expected return is higher than (23). The condition is satisfied if PE has a sufficiently high alpha and beta with respect to the hedging portfolio, and if m_0 is sufficiently low. The expected return of the liquid minimum variance portfolio has ambiguous effect.

A special case is when there is only one liquid risky asset. In that case, the liquid risky asset constitutes the entirety of the hedging and liquid minimum variance portfolios. Condition (22) simplifies as follows:

$$\mu_0 - r_F > m_0(\mu_1 - r_F) \tag{24}$$

If expected excess return to PE is higher than m_0 times the expected excess return on the liquid risky asset, the risky portfolio of constrained investors overweights PE relative to the tangency portfolio.

3.2 Benchmarking Private Equity

Studies such as Phalippou (2014) and Stafford (2021) argue that PE should be benchmarked against leveraged portfolios of public equities most correlated with it. A reformulation of the claim is that PE should be benchmarked against the hedging portfolio leveraged $B'\mathbf{1}$ times. This claim is correct for unconstrained investors, and to see why, expression (17) can be used to write the optimal PE allocation as follows:

$$\bar{\omega}_0^* = \frac{1}{\gamma \text{var}(\varepsilon_t)} \left[\mu_0 - r_F - B'\mathbf{1}(\mu_{\text{HDG}} - r_F) + \psi(B'\mathbf{1} - m_0)\right] \tag{25}$$

The term ε_t is the idiosyncratic component of PE return with respect to the hedging portfolio, as defined in equation (20). Unconstrained investors have $\psi = 0$, and they invest in PE only if it has positive alpha with respect to the hedging portfolio. Constrained investors, however, have $\psi > 0$ and they might want to invest in PE even if it has negative alpha with respect to the hedging portfolio, provided that $B'\mathbf{1} > m_0$.

Alternatively, several studies including Kaplan and Schoar (2005) and Harris et al. (2014) benchmark PE against the S&P500 and other stock market indices. This approach is simpler but also more difficult to justify. It is valid for unconstrained investors if liquid markets satisfy CAPM. In that case, it is possible to show that $(\mu_0 - r_F) - B'\mathbf{1}(\mu_{\text{HDG}} - r_F)$ would also be the alpha of PE with respect to the market portfolio of liquid assets. In absence of pricing restrictions, however, more assumptions are needed to justify this approach. For unconstrained investors, this approach remains valid assuming that liquid risky assets are pooled in a single portfolio which becomes the benchmark. This assumption effectively restricts the hedging portfolio to equal the stock index, and investors solve problem (16) with a risk-free bond, PE, and the index. The resulting optimal PE allocation is:

$$\bar{\omega}_0^* = \frac{1}{\gamma \text{var}(\varepsilon_t)} \left[\mu_0 - r_F - b_1(\mu_1 - r_F) + \psi(b_1 - m_0)\right] \tag{26}$$

The term μ_1 indicates the expected return of the index, and b_1 is the beta of PE with respect to the index. This expression is a special case of (25), and unconstrained investors choose positive PE allocation only if PE has positive alpha with respect to the index. For proper benchmarking, the index must be leveraged b_1 times. Constrained investors have positive Lagrange multiplier, and if b_1 is sufficiently large and m_0 sufficiently low, they invest a positive amount in PE even if it has negative alpha with respect to the index.

3.3 Numerical Example with Two Risky Assets

To illustrate the intuition of the model, it is useful to consider a numerical example in which investors can choose between a risk-free bond, PE, and a liquid risky asset such as a portfolio of stocks.

Table 1 reports the parameter values used in the numerical example. For the liquidity of PE, I use call intensity $\lambda_C = 30\%$ and distribution intensity $\lambda_D = 40\%$. These values are used also by Giommetti and Sorensen (2021), who show that this specification can generate plausible dynamics of PE cash flows. Stocks have expected return $\mu_1 = 8\%$ and volatility $\sigma_1 = 20\%$, while the risk-free rate is $r_F = 2\%$, resulting in a Sharpe ratio of 0.3. The values of μ_0, σ_0, and ρ imply that PE has $\alpha = 2.4\%$ and $\beta = 1.6$, as defined by the following CAPM:

$$\mu_0 - r_F = \alpha + \beta(\mu_1 - r_F) \tag{27}$$

with $\beta = \rho\sigma_0\sigma_1$. The specification of Table 1 is similar to that of Giommetti and Sorensen (2021) with the main difference being that PE has lower alpha compared to their 3%.

Figure 1 plots in Panel A the set of average allocations $(\bar{\omega}_0, \bar{\omega}_1)$ satisfying the margin constraint of problem (16). It compares the case of fully liquid PE, implying $m_0 = 1$, with the case in which PE is illiquid and requires commitment. In this second case, m_0 is taken from Proposition 2. The triangle delimited by the solid line contains feasible

Table 1: Parameter values. This table reports parameter values used in the numerical example with two risky assets. It also includes important statistics that are implied by those parameter values.

Parameter/Statistic	Expression	Value
Capital call intensity	λ_{C}	30%
Distribution intensity	λ_{D}	40%
Risk-free rate	r_F	2%
Expected return to stocks	μ_1	8%
Expected return to PE	μ_0	14%
Volatility of stocks	σ_1	20%
Volatility of PE	σ_0	40%
Correlation of stocks and PE	ρ	0.8
Implied:		
Sharpe ratio of stocks	$(\mu_1 - r_F)/\sigma_0$	0.3
Sharpe ratio of PE	$(\mu_0 - r_F)/\sigma_1$	0.3
Alpha of PE	α	2.4%
Beta of PE	β	1.6

allocations with liquid PE, and the shaded area contains feasible allocations when PE requires commitment. With liquid PE, the investor faces only a no-leverage constraint, and any feasible allocation has positive portfolio weights satisfying $\bar{\omega}_0 + \bar{\omega}_1 \leq 1$. The illiquidity of PE reduces substantially the set of feasible allocations. With commitment, the maximum PE allocation is lower than 50% and it would require the investor to hold the rest of wealth in bonds.

In Panel A of Figure 1, the shaded area is not an exact triangle. Its diagonal edge is not exactly a straight line, and Panel B plots the slope of that edge as a function of $\bar{\omega}_0$. This slope corresponds to the value of PE margin m_0 at the corresponding $(\bar{\omega}_0, \bar{\omega}_1)$ points, and the diagonal edge of the shaded area in Panel A is the set of points where the margin constraint is binding and m_0 is most relevant. Ranging from 2.220 to 2.235, PE margin is large and varies with $(\bar{\omega}_0, \bar{\omega}_1)$ through the expected return on wealth, μ_W, but that variation is negligible.

It remains unclear whether differences in feasible allocations between liquid PE and commitment can generate large differences in portfolio allocation. The answer depends on preferences and returns. Below, I show that commitment has large effects on the optimal PE and stocks allocation of investors with risk aversion of 1, for example.

Figure 2 illustrates the numerical example in the space of expected returns and volatility.

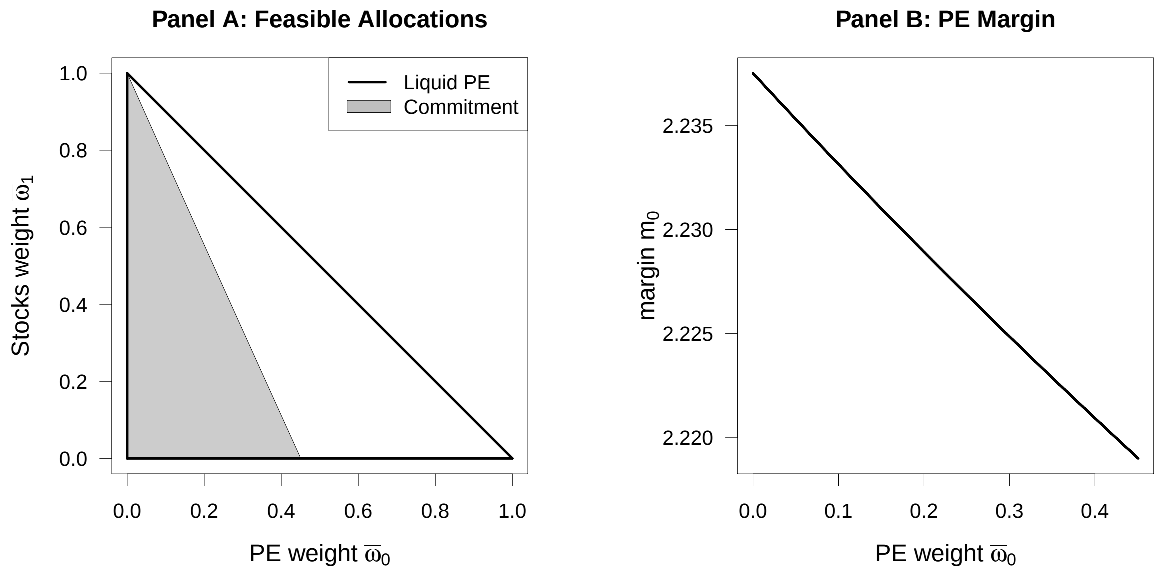


Figure 1: Feasible allocations and PE margin. This figure plots feasible allocations (Panel A) and PE margin as function of PE allocation (Panel B) with parameter values from Table 1. Panel A distinguishes the case with liquid PE from the case with commitment. Panel B plots the values of $m_0(\bar{\omega}_0, \bar{\omega}_1)$, as given by (15), along the diagonal edge of the shaded area from Panel A.

In the figure, the dashed line is the constrained efficient frontier with liquid PE, and the solid black line is the constrained efficient frontier with commitment. The solid line is always below the dashed line since the investor is more constrained (i.e. $m_0 > 1$) when PE requires commitment. In this example, the expected return to PE is not high enough to satisfy condition (24), so private equity is not a preferred asset to reach for yield, and the solid line ends with a portfolio fully invested in stocks.

I consider an investor with risk aversion $\gamma = 1$ whose indifference curves are represented with gray lines in Figure 2. If PE were liquid and $m_0 = 1$, this investor would optimally allocate 50% in PE and 50% in stocks. In that case, PE would be a preferred asset to reach for yield, and as a response to the leverage constraint, the investor would optimally tilt its risky portfolio towards PE relative to the 33% PE allocation of the tangency portfolio. With commitment, instead, the investor responds to the margin constraint by tilting his risky allocation towards stocks. His optimal portfolio invests 16% in PE, 65% stocks, and 19% in bonds.

Figure 3 plots the optimal portfolio weights of investors with risk aversion between 1 and 3, and Table 2 reports their numeric value for a subset of risk aversion. With risk aversion of 3, the margin constraint does not bind, and the investor holds 42% of wealth

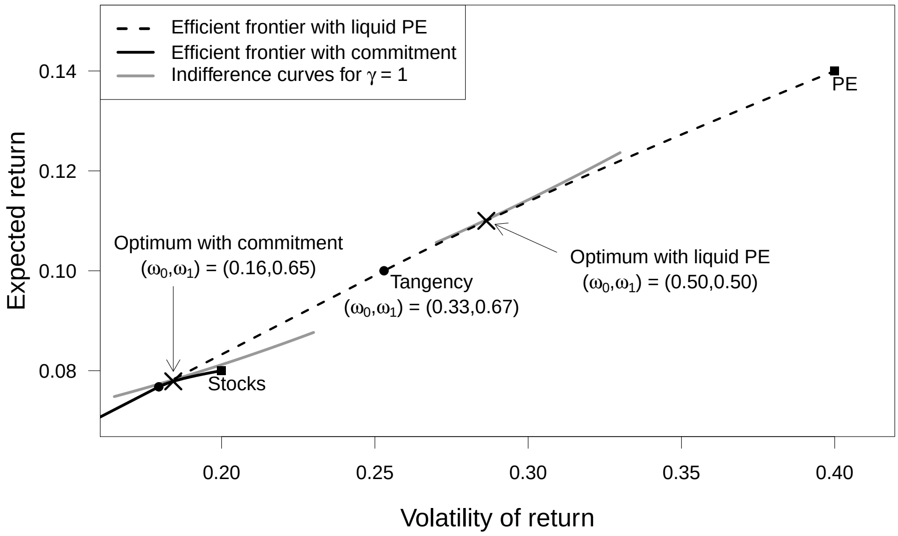


Figure 2: Efficient frontier and optimal allocation. This figure illustrates the solution to problem (16) with PE, stocks, and bonds in the space of expected returns and volatility. Using parameters values from Table 1, the figure plots efficient frontier with liquid PE and $m_0 = 1$ (dashed line), efficient frontier with commitment and m_0 given by (15) (solid line), and indifference curves for investors with risk aversion $\gamma = 1$ (gray lines).

in the tangency portfolio. The optimal allocation constitutes of 14% PE, 28% stocks, and 58% bonds. As risk aversion decreases, investors decrease their holdings of bonds and invest more in the tangency portfolio until the margin constraint starts binding. At that point, because of the large margin requirement of PE, investors prefer to increase their stocks allocation at the expense of PE. As a result, PE allocation is approximately flat across risk aversion with only a 2% difference between risk aversion of 1 and 3. Instead, investors differ mainly in their allocation to liquid assets, and with risk aversion of 1 they allocate an additional 37% of wealth in stocks compared to risk aversion of 3.

Table 2: Optimal average allocations. This table reports optimal average allocations solving static problem (16) for a subset of risk aversions between 1 and 3. Parameters are taken from Table 1 and m_0 is computed with (15). PE allocation is indicated with $\bar{\omega}_0$, stocks allocation is denoted $\bar{\omega}_1$, and $\bar{\omega}_F = 1 - \bar{\omega}_0 - \bar{\omega}_1$ represents bonds allocation. Percentages are rounded to the nearest integer.

	Risk aversion					
	1.0	1.4	1.8	2.2	2.6	3.0
$\bar{\omega}_0$	16%	21%	23%	19%	16%	14%
$\bar{\omega}_1$	65%	53%	46%	38%	32%	28%
$\bar{\omega}_F$	19%	26%	31%	43%	52%	58%

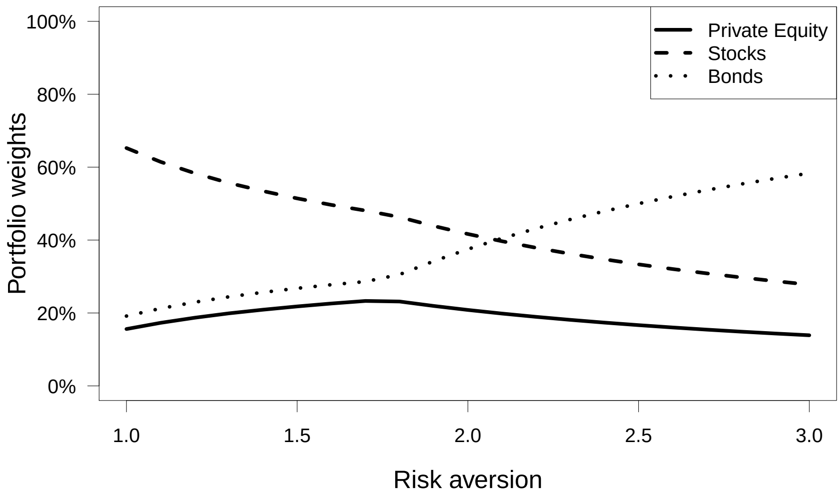


Figure 3: Optimal average allocations. This figure plots the optimal average allocation in PE, stocks, and bonds obtained solving problem (16) for different values of risk aversion. Parameters are taken from Table 1 and m_0 is computed with (15).

4 Dynamic Objective

In this section, I replace the static mean-variance objective function (3) with the following expression:

$$\sum_{t=0}^{\infty} \delta^t E_t \left[\omega_t' \left(\mu - r_F \right) - \frac{\gamma}{2} \omega_t' \Sigma \omega_t \right] \tag{28}$$

With this new objective function, the investor ranks investment strategies based on the expected mean-variance properties of portfolio allocation ω_t at current and future times. The investor discounts time with $\delta = 0.95$. Comparing the static and dynamic objective functions, Appendix A shows that an investor with dynamic objective cares not only about the mean-variance properties of the average allocation, but also about variation of portfolio weights around that average. Specifically, the investor is averse towards that variation.

I compare average allocations and margin requirements resulting from the dynamic objective function with those obtained above with static objective function. This comparison is useful for at least two reasons. First, the static objective function is a strong assumption that might not hold in reality, and it is useful to understand whether it introduces substantial differences in terms of average allocations. Second, the static margin constraint m_0 in equation (15) is based on an approximation from Proposition 1, and it is useful to assess its accuracy by comparing m_0 to the exact counterpart in a fully dynamic model.

With dynamic objective, the problem can be expressed in the form of Bellman equation. I restrict my attention to the case with only one liquid risky asset representing a portfolio of stocks. The investor chooses (i) a dynamic stocks allocation strategy $\omega_{1,t} = \omega_1(\omega_{0,t}, k_t)$ and (ii) a dynamic commitment strategy $n_t = n(\omega_{0,t}, k_t)$ solving the following problem:

$$V(\omega_{0,t}, n_t) = \max_{(n_t, \omega_{1,t})} \left\{ \omega_t' (\mu - r_F) - \frac{\gamma}{2} \omega_t' \Sigma \omega_t + \delta E_t[V(\omega_{0,t+1}, n_{t+1})] \right\}$$

subject to

$$\begin{aligned}
&\text{Prob}\big(1 - \omega_{0,t} \leq 0\big) = 0 \\
&\omega_{0,t+1} = [\omega_{0,t} R_{P,t+1} + \lambda_C(k_t + n_t) - \lambda_D \omega_{0,t} R_{P,t+1}]/(1 + r_{W,t+1}) \\
&k_{t+1} = [(k_t + n_t) - \lambda_C(k_t + n_t)]/(1 + r_{W,t+1}) \\
&r_{W,t+1} = r_F + \omega_t'(r_{t+1} - r_F) \\
&n_t \geq 0
\end{aligned} \tag{29}$$

The constraints of this Bellman equation remain unchanged from problem (8), but the new preferences do not allow a fully static reformulation of the problem as it was possible earlier with static objective. Instead, we must find optimal dynamic strategies.

I solve the Bellman equation numerically using the value-function iteration algorithm outlined in Giommetti and Sorensen (2021). I use parameter values from Table 1 as a baseline specification, and I consider investors with risk aversion between 1 and 3. For each investor, the solution is used to simulate the model and obtain the joint stationary distribution of returns, state variables, and choice variables at the optimum.

Appendix C reports the value function, optimal stocks allocation, and optimal commitment strategy for investors at opposite ends of the risk aversion spectrum, with $\gamma = 1$ and $\gamma = 3$. The optimal investment strategies of this problem are similar to those in Giommetti and Sorensen (2021) despite differences with their setup and especially different preferences compared to their assumption of power utility over consumption. As a result, average allocations are also qualitatively similar.

Average allocations resulting from the simulations are shown in Figure 4 as a function of risk aversion, and Table 3 reports the corresponding numerical values for a subset of risk aversion. These average allocations are also qualitatively similar to those with static objective in the numerical example of Section 3.3. Average PE allocation is approximately flat across risk aversion and investors vary primarily their allocation to stocks and bonds. As risk aversion decreases, investors increase their stocks allocation and decrease their bonds allocation. Quantitatively, average portfolio allocations are also close to those with static objective. For example, an investor with dynamic objective and risk aversion of 3 has average PE allocation of 14%, stocks allocation of 28%, and bonds allocation of 58%. These numbers are identical to their counterparts with static objective. A similar comparison shows some differences at low risk aversion, but numbers remain close. Compared to the case with static objective, an investor with risk aversion of 1 and dynamic objective holds more stocks (72% vs. 65%) and less PE (12% vs. 16%).

Table 3: Average allocations with dynamic objective function. This table reports the average allocations resulting from problem (29) for a subset of different risk aversions between 1 and 3. Parameters are taken from Table 1. PE allocation is indicated with $\bar{\omega}_0$, stocks allocation is denoted $\bar{\omega}_1$, and $\bar{\omega}_F = 1 - \bar{\omega}_0 - \bar{\omega}_1$ represents bonds allocation. Percentages are rounded to the nearest integer.

	Risk aversion					
	1.0	1.4	1.8	2.2	2.6	3.0
$\bar{\omega}_0$	12%	18%	23%	18%	16%	14%
$\bar{\omega}_1$	72%	60%	46%	39%	32%	28%
$\bar{\omega}_F$	16%	22%	31%	43%	52%	58%

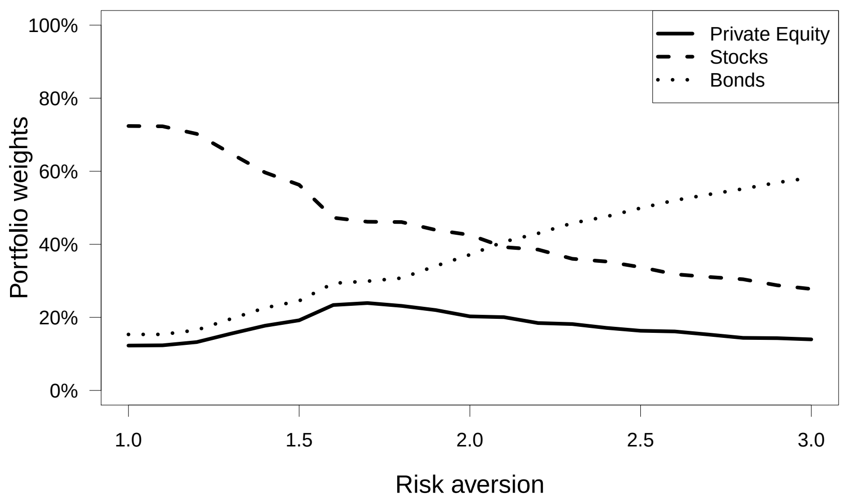


Figure 4: Average allocation with dynamic objective function. This figure plots average portfolio weights in PE, stocks, and bonds resulting from the optimal strategy of Bellman equation (29) for different levels of risk aversion. Parameters values are taken from Table 1.

Next, I use the solution of the model with dynamic objective to study the accuracy of the static margin constraint m_0 given in equation (15). To compute PE margin in the dynamic model, I use inequality (10) which determines the minimum liquidity reserve in proportion to commitment. I rewrite that inequality as follows:

$$1 - \sum_{i=1}^{A} \omega_{i,t} \geq \left(1 + \frac{\lambda_C}{\lambda_C + r_F} \frac{k_t + n_t}{\omega_{0,t}}\right) \omega_{0,t} \tag{30}$$

This condition must hold every period also with dynamic objective, and based on that, I compute PE margin averaging the following expression over time:

$$1 + \frac{\lambda_C}{\lambda_C + r_F} \frac{k_t + n_t}{\omega_{0,t}} \tag{31}$$

I plot the resulting quantity in Figure 5 as a function of risk aversion and I compare it to the margin m_0 derived in the static model. PE margin from the static model is approximately 2.2 and almost constant with respect to risk aversion. PE margin from the dynamic model is between 2.2 and 2.3, also approximately constant across risk aversion. The two margins are quantitatively close to each other, indicating that m_0 from the static model provides an accurate measure of the average margin requirement implicit in the dynamic model. Furthermore, the two margins are closest when it matters most, for investors with low risk aversion that are more likely to be constrained.

4.1 Idiosyncratic Risk

Appendix A finds that the static and dynamic objective functions should result in similar average allocations when the idiosyncratic risk of PE is small. To investigate robustness of the static model with respect to idiosyncratic risk, I compare the two models using four different specifications other than the baseline. Table 4 reports the parameter values that differ across the specifications. The table reports also the resulting variance of idiosyncratic risk, and CAPM α and β of PE.

The variance of idiosyncratic risk varies with PE volatility and with the correlation between PE and stocks:

$$\text{var}(\varepsilon_t) = (1 - \rho^2)\sigma_0^2 \tag{32}$$

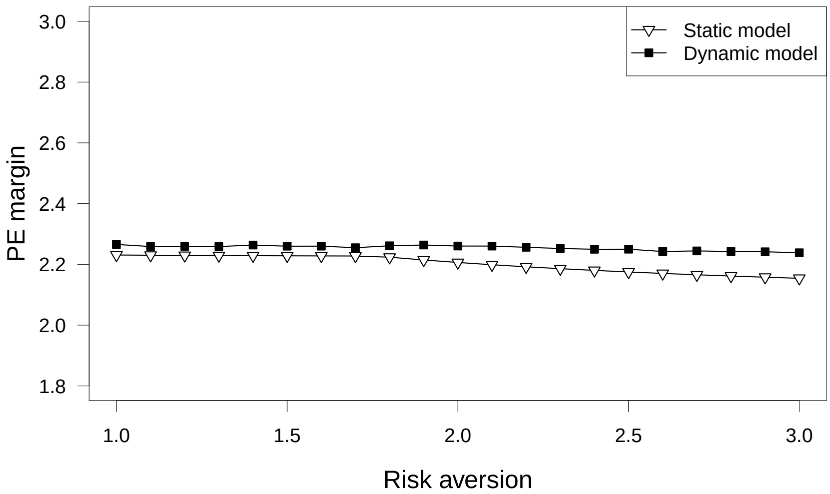


Figure 5: PE margin in the static vs dynamic model. This figure compares PE margin of the static model with the average margin implicit in the dynamic model. The line with white triangles represents m_0 from equation (15) at the average portfolio allocation solving problem (16) with risk aversion between 1 and 3. The line with black squares represents the average over time of expression (31) for investors following dynamic investment strategies that solve problem (29).

To study the effect of each variable separately, I increase $\text{var}(\varepsilon_t)$ relative to its baseline value of 0.0576 by increasing σ_0 in two specifications and decreasing ρ in two others. I consider alternative σ_0 of 45% and 50%, and alternative ρ of 0.7 and 0.6.

Other than idiosyncratic risk, a change in ρ and σ_0 affects several statistics influencing optimal PE allocation. I control for these additional effects by modifying the expected return to PE. Across specifications, I change μ_0 to fix the average PE allocation of unconstrained investors in the static model. Specifically, I set

$$\mu_0 = \gamma \bar{\omega}_0^* \,\text{var}(\varepsilon_t) + r_F + \beta(\mu_1 - r_F) \tag{33}$$

where $\gamma \bar{\omega}_0^* \approx 0.41$ is risk aversion times the optimal PE allocation of unconstrained investors in the baseline specification of the static model. Across specifications, this choice of μ_0 fixes the demand for PE in absence of frictions.

Appendix C reports the value function and optimal strategies with dynamic objective for all the specifications and for investors with risk aversion of 1 and 3. Figure 6 compares average allocations with static and dynamic objective function across specifications. For each specification, I compare the left-hand plot (static objective) with the right-hand plot (dynamic objective). For risk aversion above 2, investors are unconstrained and the two

Table 4: Alternative model specifications. This table reports alternative specifications used to compare solutions with static and dynamic objective function. The left-hand side of the table reports parameter values that differ across specifications. It ignores parameters that remain constant from Table 1. The right-hand side of the table reports implied variance of idiosyncratic PE risk, and CAPM alpha and beta of PE with respect to stocks.

	Parameters			Implications		
Specification	ρ	σ_0	μ_0	$\text{var}(\varepsilon_t)$	α	β
baseline	0.8	40%	14.00%	0.0576	2.40%	1.6
$\sigma+$	0.8	45%	15.80%	0.0729	3.00%	1.8
$\sigma++$	0.8	50%	17.75%	0.0900	3.75%	2.0
$\rho-$	0.7	40%	13.80%	0.0816	3.40%	1.4
$\rho--$	0.6	40%	13.40%	0.1024	4.20%	1.2

objective functions result in approximately the same average allocation across all specifications. Idiosyncratic risk does not seem to affect unconstrained investors differently between static and dynamic objective. For risk aversion below 2, average allocations remain qualitatively similar across objectives for all specifications. However, there is a tendency for investors with low risk aversion to invest more in PE with static than dynamic objective. Furthermore, since these investors tend to be constrained, more PE implies less stocks and more bonds. This result is consistent with the baseline specification, and I now study whether it varies quantitatively with idiosyncratic risk.

In Table 5, I focus on investors with risk aversion of 1 and report their average portfolio allocations for every specification. The table distinguishes between static and dynamic objective, and reports also average portfolio weights with static objective as a proportion of the corresponding weight with dynamic objective. For PE, that proportion varies between 1.13 and 1.30 depending on the specification. However, it does not seem systematically related to the variance of idiosyncratic risk. As σ_0 increases, the proportion goes from 1.27 in the baseline to 1.13 in specification $\sigma+$ and 1.30 in specification $\sigma++$. As ρ decreases from the baseline, the proportion remains approximately constant at 1.28 in specification $\rho-$ and decreases to 1.20 in specification $\rho--$. Overall, for investors with risk aversion of 1, the portfolio in PE with static objective is higher than the corresponding weight with dynamic objective, and the proportion between the two weights has limited and seemingly unsystematic variation across specifications.

Figure 7 compares PE margin between static and dynamic models in the four alternative specifications. The figure confirms results from the baseline. The margin m_0 from the

Figure 6: Average allocation across specifications. This figure compares optimal average allocations using the static objective from problem (16) with average allocations using the dynamic objective from problem (29). The comparison is made across four model specifications described in Table 4.

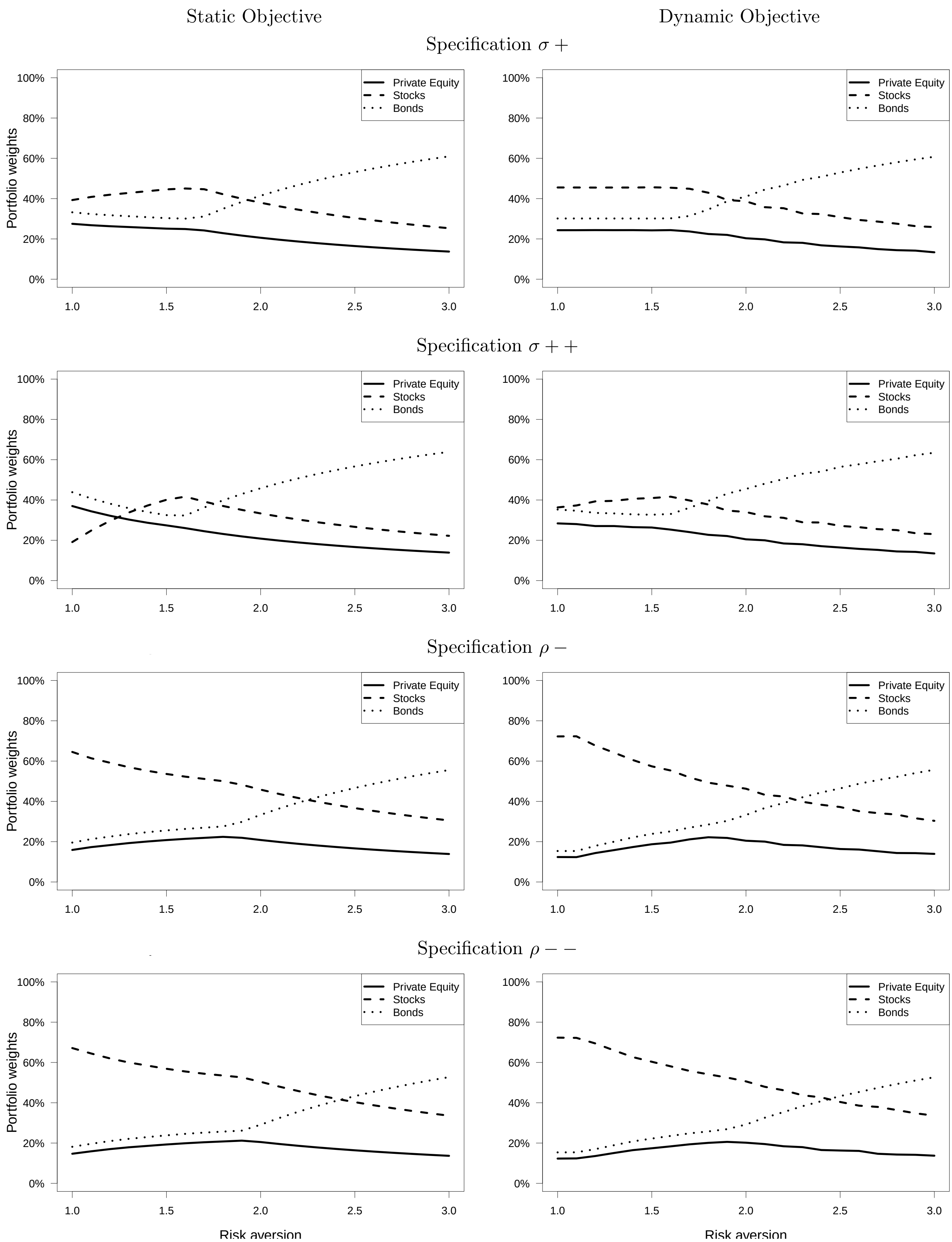


Table 5: Average allocations with low risk aversion. This table reports average allocations with risk aversion of 1 across objective function and sprecifications. Percentages are rounded to the nearest integer and they correspond to the portfolio weights plotted at risk aversion of 1 in Figure 3, 4, and 6. Ratios are rounded to two decimal digits.

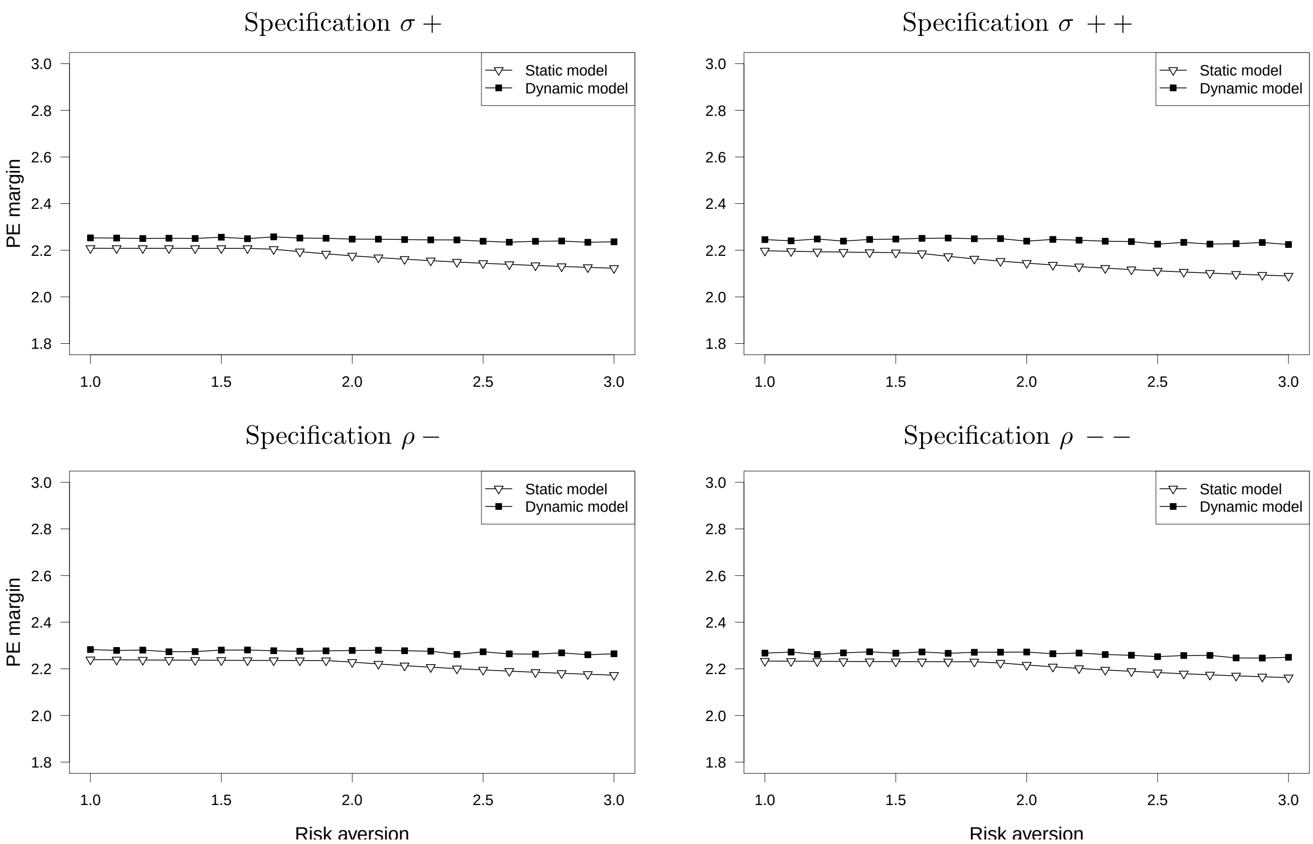
Specification:	baseline	$\sigma +$	$\sigma ++$	$\rho -$	$\rho --$
			Static Objective		
$\bar{\omega}_0$	16%	28%	37%	16%	15%
$\bar{\omega}_1$	65%	39%	19%	64%	67%
$\bar{\omega}_F$	19%	33%	44%	20%	18%
			Dynamic Objective		
$\bar{\omega}_0$	12%	24%	29%	13%	12%
$\bar{\omega}_1$	72%	46%	36%	72%	72%
$\bar{\omega}_F$	16%	30%	35%	15%	16%
			Static/Dynamic Ratio		
$\bar{\omega}_0$	1.27	1.13	1.30	1.28	1.20
$\bar{\omega}_1$	0.90	0.86	0.53	0.89	0.93
$\bar{\omega}_F$	1.25	1.10	1.24	1.27	1.18

static model is consistently lower than, but very close to, its counterpart in the dynamic model, especially for investors with low risk aversion.

5 Conclusion

I introduce PE in a mean-variance portfolio model with several liquid assets. PE requires capital commitment, which is gradually called and invested in underlying private assets. These private assets appreciate at a risky rate over time, and the resulting value is gradually liquidated and distributed to the investor, who cannot sell or collateralize PE investments, and must hold them to maturity. In the main version of the model, the investor has static mean-variance objective with respect to his average portfolio allocation over time. The investor requires always some liquid wealth available, and otherwise ignores time variation of portfolio weights. I show that the optimal portfolio allocation can be found solving a static mean-variance problem with endogenous margin requirements. The investor assigns a margin of 1 to liquid risky assets, and a greater margin to PE. With plausible parameter values, PE margin is higher than 2, and constrained investors prefer to overweight liquid risky assets and underweight PE relative to the tangency portfolio, despite the high risk and return of PE.

Figure 7: PE margin across specifications. This figure compares PE margin of the static model with the average margin implicit in the dynamic model across four specifications. The line with white triangles represents m_0 from equation (15) at the average portfolio allocation solving static problem (16) with risk aversion between 1 and 3. The line with black squares represents the average over time of expression (31) for investors solving dynamic problem (29).



I investigate the robustness of my results in a second version of the model assuming that the investor has dynamic mean-variance objective on the return to his current and future portfolio allocation. This second version cannot be reduced to a static problem. Instead, I solve it numerically for five different specifications, and I compare the resulting average portfolio allocations with those from the main model. I find similar results across the two models. Portfolio allocations are virtually identical for unconstrained investors, while they differ marginally for constrained investors. The main model tend to overestimate the optimal PE allocation of constrained investors relative to the more realistic version. The difference remains small, however, with limited and unsystematic variation across specifications.

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A Objective Function

In this appendix, I study the static objective function in relation to a more standard dynamic mean-variance objective function. The starting point is the objective function from the main text:

$$\bar{\omega}\left(\mu - r_F\right) - \frac{\gamma}{2}\bar{\omega}'\Sigma\bar{\omega} \tag{A.1}$$

Under i.i.d. returns, and for a subjective discount factor $\delta \in (0, 1)$, it is possible to show that the expression in (A.1) is proportional and economically equivalent, to the following expression:

$$\sum_{t=0}^{\infty} \delta^t \underbrace{E\left[\omega_t'\left(\mu - r_F\right) - \frac{\gamma}{2}\omega_t'\Sigma\omega_t\right]}_{\text{expected mean-variance}} + \sum_{t=0}^{\infty} \delta^t \underbrace{E\left[\frac{\gamma}{2}(\omega_t - \bar{\omega})'\Sigma(\omega_t - \bar{\omega})\right]}_{\text{scaled variance of portfolio weights}} \tag{A.2}$$

This expression shows that the objective function (A.1) can be understood as the sum of two components. The first component corresponds to the expected value of dynamic mean-variance objective with constant relative risk aversion. Similar objective functions are commonly used in the literature (e.g. Collin-Dufresne, Daniel, and Sağlam, 2020). The second component is proportional to the variance of portfolio weights scaled through the covariance matrix of returns. Formally, this component simplifies the objective function because it makes the investor risk-neutral toward variation in portfolio weights. Because of this second component, the objective function (A.1) deviates from the expected value of standard dynamic mean-variance objective.

Below, I show that portfolio weights in liquid assets can be rebalanced such that, as PE risk becomes more spanned by liquid assets, variation in portfolio weights becomes an increasingly negligible source of risk. Thus, as PE becomes more spanned, the objective function in (A.1) approximates the dynamic mean-variance objective in ranking average portfolio allocations.

The argument starts with the exact relationship between the expected value of dynamic mean-variance objective and the objective function used in the main model:

$$\sum_{t=0}^{\infty} \delta^t E\left[\omega_t'\left(\mu - r_F\right) - \frac{\gamma}{2}\omega_t'\Sigma\omega_t\right] = \sum_{t=0}^{\infty} \delta^t \left(\bar{\omega}'\left(\mu - r_F\right) - \frac{\gamma}{2}\bar{\omega}'\Sigma\bar{\omega}\right) - \frac{\gamma}{2}\sum_{t=0}^{\infty} \delta^t E\left[(\omega_t - \bar{\omega})'\Sigma(\omega_t - \bar{\omega})\right] \tag{A.3}$$

Consider an investor whose objective function is the left-hand side of this expression. The expression shows that this investor cares about two things: (i) the mean-variance characteristics of his average allocation, which is the objective function used in the main text, and (ii) variation of portfolio weights around that average allocation. This second component constitutes the difference between this hypothetical investor and the ojective function of the main model.

Proceeding with the argument, I take an exogenous sequence of PE portfolio weights $\{\omega_{0,t}\}_{t=0}^{\infty}$, and consider the following linear strategy for investing in liquid assets $a \neq 0$:

$$\omega_{a,t} = c_a - b_a\omega_{0,t} \tag{A.4}$$

The intercepts c_a remain unrestricted throughout the argument, and the investor could use them to maximize the first term on the right-hand side of (A.3). I find the vector $B = \{b_1, \ldots, b_A\}$ that minimizes the scaled variance of portfolio weights, which is the second term on the right-hand side of (A.3).

It is useful to introduce the following partitioning of the covariance matrix of returns:

$$\Sigma = \begin{pmatrix} \sigma_0^2 & \sigma_{0A} \\ \sigma_{0A}' & \Sigma_A \end{pmatrix} \tag{A.5}$$

In this expression, σ_0^2 is the variance of PE return, σ_{0A} is the covariance vector between PE and liquid assets, and Σ_A is the covariance matrix of liquid assets. Using definition (A.5) and the linear strategy (A.4), it is possible to derive the following expression:

$$E\left[(\omega_t - \bar{\omega})'\Sigma(\omega_t - \bar{\omega})\right] = \mathrm{var}(\omega_{0,t}) \begin{pmatrix} 1 \\ -B \end{pmatrix}' \begin{pmatrix} \sigma_0^2 & \sigma_{0A} \\ \sigma_{0A}' & \Sigma_A \end{pmatrix} \begin{pmatrix} 1 \\ -B \end{pmatrix} \tag{A.6}$$

This formula expresses the scaled variance of portfolio weights in terms of the variance of

PE allocation and in terms of the rebalancing coefficients, B. I minimize the right-hand side of this expression with respect to B taking the first order condition:

$$B = \Sigma_A^{-1}\sigma_{0A}' \tag{A.7}$$

Considering $\omega_{0,t}$ as exogenous, B minimizes the scaled variation of portfolio weights among linear rebalancing strategies. Notice that B is also the vector of coefficients from a multivariate regression of PE returns on the returns to liquid assets.

Substituting (A.7) inside (A.6) it is possible to find the resulting scaled variance of portfolio weights:

$$\text{var}(\omega_{0,t})\left[\sigma_0^2 - \sigma_{0A}\Sigma_A^{-1}\sigma_{0A}'\right] \tag{A.8}$$

The term in square brackets is the amount of variance that remains unexplained after regressing PE return on the returns to liquid assets. In other words, it is the idiosyncratic variance of PE, and when that is small, PE return is more spanned by liquid assets.

In summary, if PE return is more spanned by liquid assets, the investor can set the second term on the right-hand side of (A.3) closer to zero without restricting his ability to affect the average portfolio allocation; that is, without restricting his ability to maximize the first term on the right-hand side of (A.3). Therefore, as PE becomes more spanned, the second term on the right-hand side of (A.3) has increasingly negligible effect on optimal average allocation, and the objective value from the main text approximates the expected value of dynamic mean-variance objective.

I conclude with a discussion of the main assumptions and possible shortcomings of this argument. First, throughout the argument I have assumed that the variance of $\omega_{0,t}$ is exogenous to the rebalancing strategy. This assumption is violated since the rebalancing strategy determines the variance of wealth, which affects the variance of $\omega_{0,t}$. Relaxing this assumption, the argument remains valid under the weak condition that the variance of $\omega_{0,t}$ does not diverge towards infinity as idiosyncratic PE volatility approaches zero.

A second and perhaps more important shortcoming of the argument is that it focuses exclusively on the objective function of the model and ignores the constraint on liquid wealth. In particular, since the evolution of $\omega_{0,t}$ is endogenous to the rebalacing strategy, it is possible that the suggested strategy with B violates the liquidity constraint imposed by the investor:

$$\text{Prob}\big(1 - \omega_{0,t} \leq 0\big) = 0 \tag{A.9}$$

Ignoring this constraint, the argument can make use of a larger set of strategies. At the same time, however, I restrict my attention to linear rebalancing strategies, and even assuming that the proposed linear strategy fails to satisfy the liquidity constraint, there could be non linear strategies that accomplish the same goals while satisfying that constraint.

B Proofs

This appendix contains the derivations of Proposition 1 and Proposition 3. The derivation of Proposition 2 is described in the main text.

B.1 Proposition 1

This proposition uses the law of motion of $\omega_{0,t}$ from problem (8) to derive a relationship between average commitments and average PE allocation. The law of motion can be written as follows:

$$(1 + r_{W,t+1})\omega_{0,t+1} = \omega_{0,t} R_{P,t+1} + \lambda_C (k_t + n_t) - \lambda_D \omega_{0,t} R_{P,t+1} \tag{B.1}$$

In the left-hand side, it is useful to linearize $r_{W,t+1}\omega_{0,t+1}$ around the point $r_{W,t+1} = \mu_W$ and $\omega_{0,t+1} = \bar{\omega}_0$. The linearization results into the following expression:

$$r_{W,t+1}\omega_{0,t+1} \approx \omega_{0,t+1}\mu_W + \bar{\omega}_0(r_{W,t+1} - \mu_W) \tag{B.2}$$

Replacing this approximation in the left-hand side of (B.1), and taking unconditional expectations on both sides, I get the following result:

$$\bar{\omega}_0(1 + \mu_W) = \bar{\omega}_0(1 - \lambda_D)(1 + \mu_P) + (\bar{k} + \bar{n})\lambda_C \tag{B.3}$$

It is sufficient to rearrange this expression to obtain the result of the proposition.

B.2 Proposition 3

The optimal portfolio allocation of the static portfolio problem is

$$\bar{\omega}^* = \frac{1}{\gamma}\Sigma^{-1}(\mu - r_F - \psi m) \tag{B.4}$$

The elements of $\bar{\omega}^*$ are all proportional to γ, and the fraction of any two elements depends on risk aversion indirectly through ψ. Similarly, the fraction $\bar{\omega}_0^*/(\bar{\omega}_0^* + \bar{\omega}_1^* + \ldots + \bar{\omega}_A^*)$ depends on risk aversion only through ψ. Furhtermore, $\bar{\omega}_0^*/(\bar{\omega}_0^* + \bar{\omega}_1^* + \ldots + \bar{\omega}_A^*)$ increases with ψ if and only if $\bar{\omega}_0^*/(\bar{\omega}_1^* + \ldots + \bar{\omega}_A^*)$ also increases with ψ.

Using expression (B.4) and blockwise matrix inversion, the fraction $\bar{\omega}_0^*/(\bar{\omega}_1^* + \ldots + \bar{\omega}_A^*)$ can be written as follows:

$$\frac{\bar{\omega}_0^*}{\bar{\omega}_1^* + \ldots + \bar{\omega}_A^*} = \frac{v_0(\mu_0 - r_F) - \frac{\sigma_{0A}}{\sigma_0^2} v_A(\mu_A - r_F) - \psi(v_0 m_0 - \frac{\sigma_{0A}}{\sigma_0^2} v_A \mathbf{1})}{\mathbf{1}' \left[-\Sigma_A^{-1} \sigma_{0A}' v_0(\mu_0 - r_F) + v_A(\mu_A - r_F) + \psi(\Sigma_A^{-1} \sigma_{0A}' v_0 m_0 - v_A \mathbf{1}) \right]} \tag{B.5}$$

In this expression, μ_A and Σ_A indicate the vector of expected returns and covariance matrix of risky assets except PE. Similarly, μ_0 and σ_0^2 indicate the expected return and variance of PE, while σ_{0A} is a row vector of covariances between PE and the other assets. Furthermore, $\mathbf{1}$ indicates an A-dimensional column vector of ones, and the terms v_0 and v_A are defined as follows:

$$v_0 = \left(\sigma_0^2 - \sigma_{0A} \Sigma_A^{-1} \sigma_{0A}' \right)^{-1} \tag{B.6}$$

$$v_A = \left(\Sigma_A - \sigma_{0A}' \sigma_{0A} \frac{1}{\sigma_0^2} \right)^{-1} = \Sigma_A^{-1} + \Sigma_A^{-1} \sigma_{0A}' v_0 \sigma_{0A} \Sigma_A^{-1} \tag{B.7}$$

Since ψ decreases with γ, the derivative of $\bar{\omega}_0^*/(\bar{\omega}_1^* + \ldots + \bar{\omega}_A^*)$ with respect to γ has to opposite sign of the derivative $\bar{\omega}_0^*/(\bar{\omega}_1^* + \ldots + \bar{\omega}_A^*)$ with respect to ψ. With basic algebra, it is possible to show that the latter derivative is independent of ψ and has the same sign of the following expression:

$$\begin{aligned} &- v_0 m_0 \mathbf{1}' v_A(\mu_A - r_F) + \frac{\sigma_{0A}}{\sigma_0^2} v_A \mathbf{1}\mathbf{1}' \Big(v_A(\mu_A - r_F) - \Sigma_A^{-1} \sigma_{A0} v_0 (\mu_0 - r_F) \Big) \\ &+ \mathbf{1}' \Sigma_A^{-1} \sigma_{A0} v_0 m_0 \frac{\sigma_{0A}}{\sigma_0^2} v_A(\mu_A - r_F) + \mathbf{1}' v_A \mathbf{1} \Big(v_0(\mu_0 - r_F) - \frac{\sigma_{0A}}{\sigma_0^2} v_A(\mu_A - r_F) \Big) \end{aligned}$$

This expression can be simplified as follows:

$$\mathbf{1}' \Sigma_A^{-1} \mathbf{1} [(\mu_0 - r_F) - \sigma_{0A} \Sigma_A^{-1} (\mu_A - r_F)] - [m_0 - \mathbf{1}' \Sigma_A^{-1} \sigma_{0A}'] \mathbf{1}' \Sigma_A^{-1} (\mu_A - r_F) \tag{B.8}$$

Using the definition of B, and dividing this expression by $\mathbf{1}' \Sigma_A^{-1} \mathbf{1}$, which is a positive term, I find that the derivative of $\bar{\omega}_0^*/(\bar{\omega}_1^* + \ldots + \bar{\omega}_A^*)$ with respect to ψ has the same sign of the following expression:

$$(\mu_0 - r_F) - \mathbf{1}' B \left(\frac{B'}{\mathbf{1}' B} - \frac{\mathbf{1}' \Sigma_A^{-1}}{\mathbf{1}' \Sigma_A^{-1} \mathbf{1}} \right) (\mu_A - r_F) - m_0 \frac{\mathbf{1}' \Sigma_A^{-1}}{\mathbf{1}' \Sigma_A^{-1} \mathbf{1}} (\mu_A - r_F)$$

Thus, this expression has the opposite sign of the derivative of $\bar{\omega}_0^*/(\bar{\omega}_1^* + \ldots + \bar{\omega}_A^*)$ with respect to γ. This is the result in the proposition.

Finally, it is possible to generalize the result with a generic vector m_A of margin requirements for liquid risky assets. Under this assumption, the derivative of $\bar{\omega}_0^*/(\bar{\omega}_1^* + \ldots + \bar{\omega}_A^*)$ with respect to ψ has the same sign of

$$(\mu_0 - r_F) - m_A' B \left(\frac{\mathbf{1}'B}{m_A' B} \frac{B'}{\mathbf{1}'B} - \frac{\mathbf{1}'\Sigma_A^{-1}\mathbf{1}}{\mathbf{1}'\Sigma_A^{-1} m_A} \frac{\mathbf{1}'\Sigma_A^{-1}}{\mathbf{1}'\Sigma_A^{-1}\mathbf{1}} \right) (\mu_A - r_F) - m_0 \frac{\mathbf{1}'\Sigma_A^{-1}\mathbf{1}}{\mathbf{1}'\Sigma_A^{-1} m_A} \frac{\mathbf{1}'\Sigma_A^{-1}}{\mathbf{1}'\Sigma_A^{-1}\mathbf{1}} (\mu_A - r_F)$$

C Supplementary Output with Dynamic Objective

This section reports the value function and optimal strategies solving problem (29) for every specification considered in the main text. To save space, I report only the solution for two types of investor, with risk aversion of 1 and 3, which I call 'Aggressive' and 'Conservative'. For the baseline specification, I also estimate and report a linear approximation of the optimal strategies on the area of the state space that is reached more often by the investor.

C.1 Baseline Specification

(a) Aggressive ($\gamma = 1$) (b) Conservative ($\gamma = 3$)

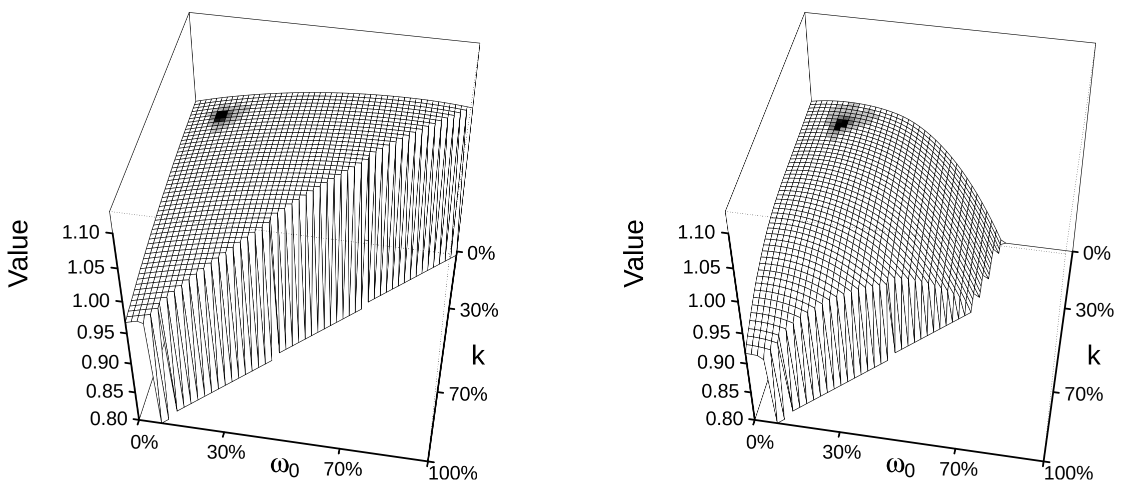


Figure C.8: Value function with dynamic objective function. This figure compares the value function $V(\omega_0, k)$ of Bellman equation (29) for investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). Each function is rescaled and expressed in units of value at $(\omega_0, k) = (0, 0)$, so I plot $v(p, k)/v(0, 0)$ for the two types of investors. I only display functional values above a certain threshold, which is 0.75 for the low risk-aversion case and 0.7 for the high risk-aversion case. The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally.

(a) Aggressive ($\gamma = 1$) (b) Conservative ($\gamma = 3$)

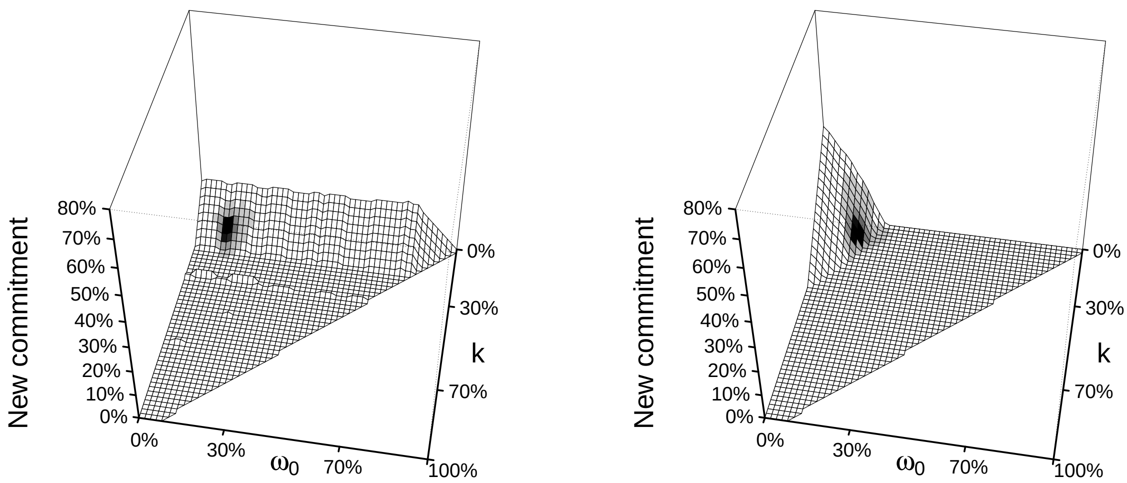


Figure C.9: Optimal commitment with dynamic objective function. This figure reports optimal commitment strategy with dynamic objective for investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). For each type, the plots show optimal new commitment $n^*(\omega_0, k)$ resulting from Bellman equation (29) with baseline specification given in Table 1. The shade on the surface of the function indicates how often investors reach a certain area of the state space. Darker areas indicate states that are reached more often. Shades are constructed using a Monte Carlo simulation where investors behave optimally.

Table A1: Local slope of the optimal commitment strategy. This table reports coefficients β_0, β_1, β_2, and β_3 from OLS estimations of the following model:

$$n^*(\omega_0, k) = \beta_0 + \beta_1\omega_0 + \beta_2 k + \beta_3(\omega_0 \times k) + \epsilon$$

In this equation, ϵ is an error term, while $n^*(\omega_0, k)$ is the optimal commitment function plotted in Figure C.9 for two different levels of risk-aversion. The model is estimated twice for the case of low risk-aversion ($\gamma = 1$) and twice for the case of high risk-aversion ($\gamma = 3$). For each level of risk-aversion, the first estimation imposes $\beta_3 = 0$, while the second estimation ignores that restriction. The data for each estimation is taken from a Monte Carlo simulation where the evolution of the state variables is simulated under optimal policies.

	$\gamma = 1$		$\gamma = 3$	
	(1)	(2)	(3)	(4)
PE allocation (ω_0)	0.17	0.40	−1.89	−2.16
Commitment (k)	−0.95	−0.69	−0.93	−1.25
Interaction ($\omega_0 \times k$)		−2.09		0.46
Constant	0.14	0.11	0.43	0.46
R^2	0.96	0.96	0.99	0.99

(a) Aggressive ($\gamma = 1$) (b) Conservative ($\gamma = 3$)

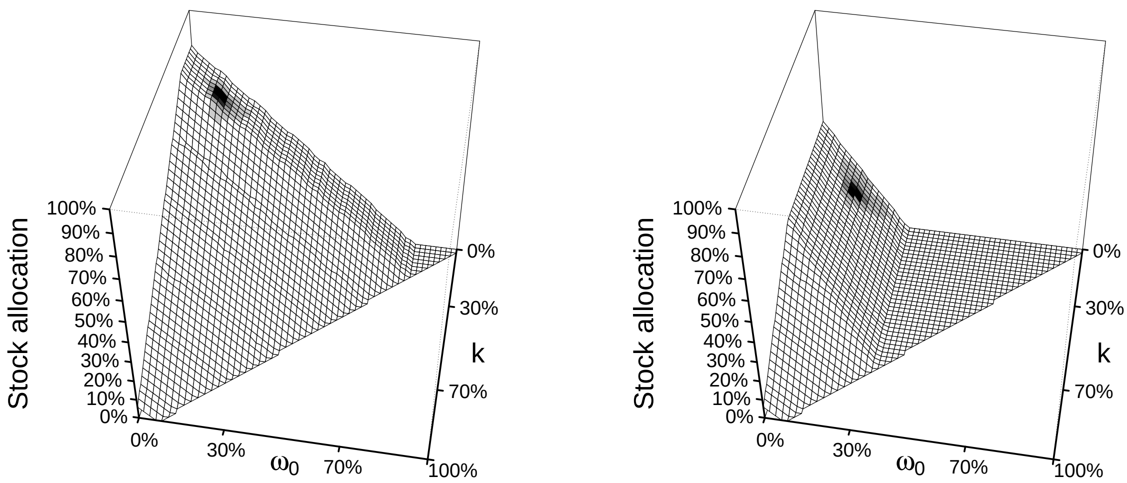


Figure C.10: Optimal stocks allocation with dynamic objective function. This figure reports optimal stocks allocation with dynamic objective for investors with low risk-aversion ($\gamma = 1$) and investors with high risk-aversion ($\gamma = 3$). For each type, the plots show optimal stocks allocation $\omega_1^*(\omega_0, k)$ resulting from Bellman equation (29) with baseline specification given in Table 1. The shade on the surface of the function is constructed as in Figure C.9.

Table A2: Local slope of optimal stocks allocation. This table reports coefficients β_0, β_1, β_2, and β_3 from OLS estimations of the following model:

$$\omega_1^*(\omega_0, k) = \beta_0 + \beta_1\omega_0 + \beta_2 k + \beta_3(\omega_0 \times k) + \epsilon$$

In this equation, ϵ is an error term, while $\omega_1^*(\omega_0, k)$ is the optimal stocks allocation plotted in Figure C.10 for two different levels of risk-aversion. The model is estimated as in Table A1.

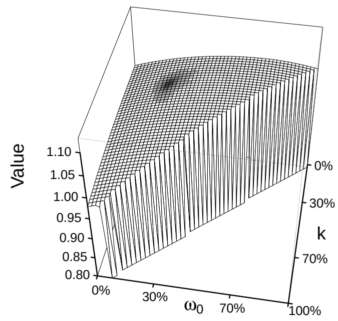
	$\gamma = 1$		$\gamma = 3$	
	(1)	(2)	(3)	(4)
PE allocation (ω_0)	−1.17	−1.48	−1.64	−1.78
Commitment (k)	−0.13	−0.47	0.02	−0.15
Interaction ($\omega_0 \times k$)		2.81		1.21
Constant	0.88	0.92	0.51	0.52
R^2	0.95	0.95	1.00	1.00

C.2 Alternative Specifications

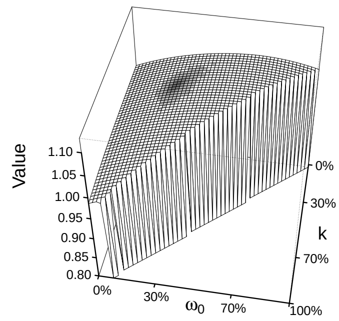
Figure C.11: Value function with dynamic objective and different parameter specifications. This figure plots the value function $V(\omega_0, k)$ of Bellman equation (29) for two types of investors (aggressive and conservative) across four different specifications described in Table 4 of the main text.

Aggressive Investor ($\gamma = 1$)

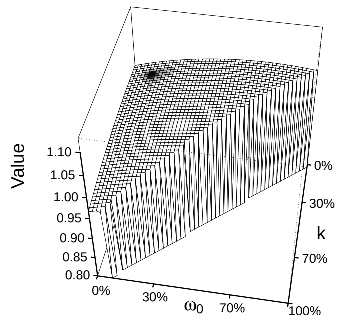
Specification $\sigma+$



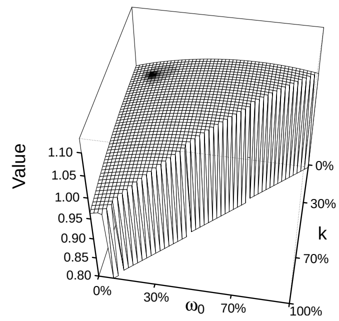
Specification $\sigma++$



Specification $\rho-$

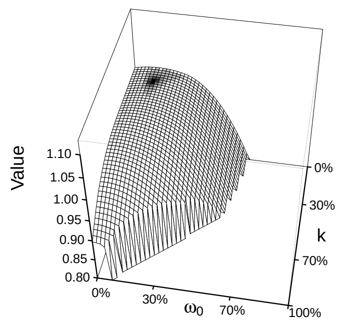


Specification $\rho--$

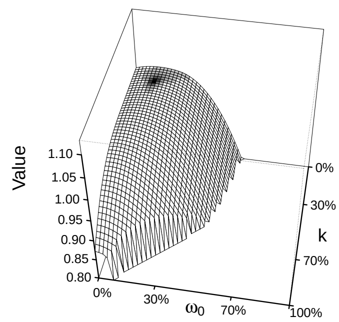


Conservative Investor ($\gamma = 3$)

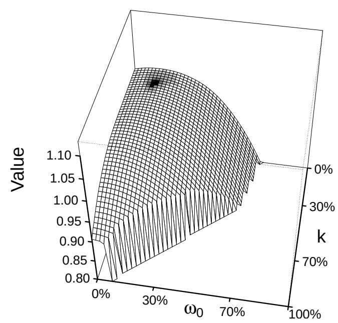
Specification $\sigma+$



Specification $\sigma++$



Specification $\rho-$



Specification $\rho--$

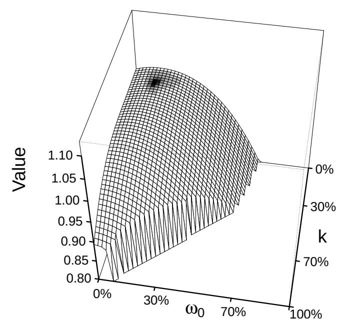
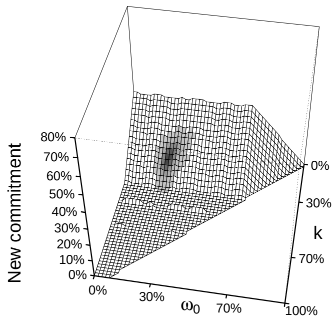


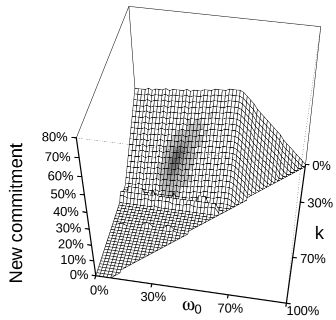
Figure C.12: Optimal commitment strategy with dynamic objective and different parameter specifications. This figure plots optimal new commitment $n^*(\omega_0, k)$ of Bellman equation (29) for two types of investors (aggressive and conservative) across four different specifications described in Table 4 of the main text.

Aggressive Investor ($\gamma = 1$)

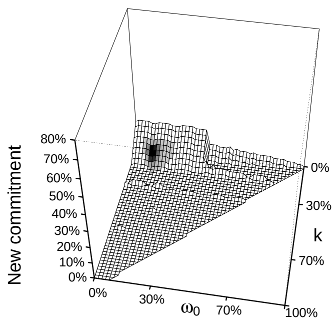
Specification $\sigma+$



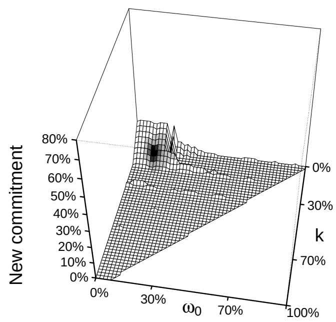
Specification $\sigma++$



Specification $\rho-$

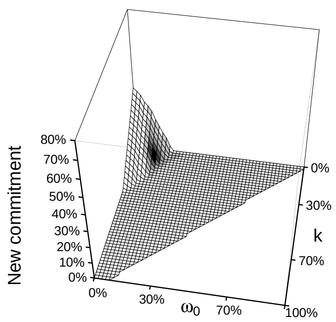


Specification $\rho--$

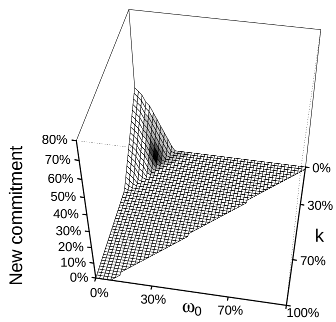


Conservative Investor ($\gamma = 3$)

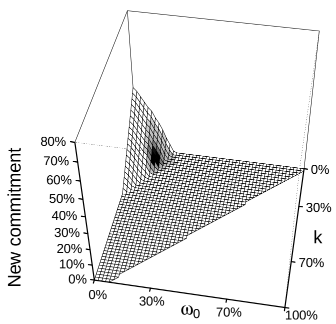
Specification $\sigma+$



Specification $\sigma++$



Specification $\rho-$



Specification $\rho--$

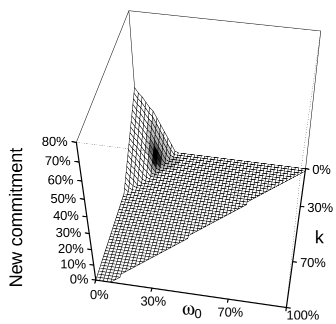
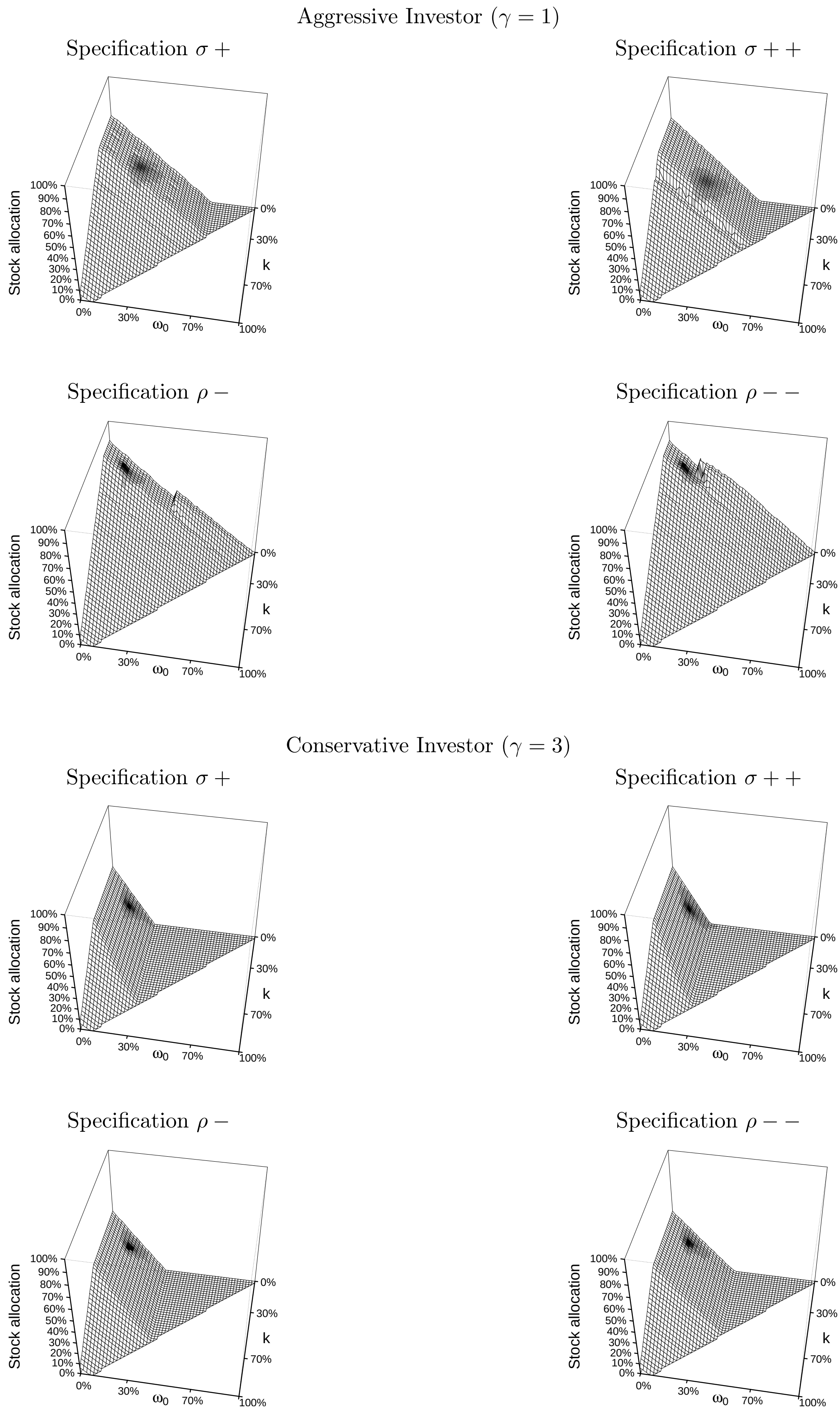


Figure C.13: Optimal stocks allocation with dynamic objective and different parameter specifications. This figure plots optimal stocks allocation $\omega_1^*(\omega_0, k)$ of Bellman equation (29) for two types of investors (aggressive and conservative) across four different specifications described in Table 4 of the main text.



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