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Extreme value theory for spatial random fields – with application to a Lévy-driven field

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Abstract First, we consider a stationary random field indexed by an increasing sequence of subsets of $\mathbb{Z}^d$. Under certain mixing and anti–clustering conditions combined with a very broad assumption on how the sequence of spatial index sets increases, we obtain an extremal result that relates a normalized version of the distribution of the maximum of the field over the index sets to the tail distribution of the individual variables. Furthermore, we identify the limiting distribution as an extreme value distribution.

Secondly, we consider a continuous, infinitely divisible random field indexed by $\mathbb{R}^d$ given as an integral of a kernel function with respect to a Lévy basis with convolution equivalent Lévy measure. When observing the supremum of this field over an increasing sequence of (continuous) index sets, we obtain an extreme value theorem for the distribution of this supremum. The proof relies on discretization and a conditional version of the technique applied in the first part of the paper, as we condition on the high activity and light–tailed part of the field.

Keywords Extreme value theory · Spatial models · Lévy–based modeling · Geometric probability · Intrinsic volumes · Convolution equivalence · Random fields

Mathematics Subject Classification (2010) Primary · 60G70 · 60G60 · Secondary · 60E07 · 60D05 · 52A39

1 Introduction

In classical extreme value theory the aim is to describe the asymptotic distributional properties of

$$M_n = \max\{\xi_1, \ldots, \xi_n\}$$

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as $n \to \infty$, where $\xi_1, \xi_2, \dots$ are independent and identically distributed. A main result is that if there exist sequences (a_n) and (b_n) such that $\mathbb{P}(a_n(M_n - b_n) \le x) \to G(x)$ for all x, where G is a non–degenerate distribution function, then G is in fact the distribution function of a distribution belonging to one of three specific types called *extreme value distributions*; see e.g the monographs Embrechts et al. (1997); Leadbetter et al. (1983); Resnick (2008) for thorough treatments of both the classical extreme value theory and many important extensions and applications.

A very useful result, when relating the limiting extremal distribution with the distribution of the individual ξ–variables, is the following theorem, cf. Leadbetter et al. (1983, Theorem 1.5.1).

Theorem 1 *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed real random variables and let $0 \le \tau \le \infty$. Then for a real sequence $(x_n)_{n\in\mathbb{N}}$,*

$$n\mathbb{P}(\xi_1 \le x_n) \to \tau \quad \text{as } n \to \infty$$

if and only if

$$\mathbb{P}(M_n \le x_n) \to e^{-\tau} \quad \text{as } n \to \infty.$$

A relevant extension, studied in the literature, of the independent case is assuming that $(\xi_n)_{n\in\mathbb{N}}$ is a stationary sequence of random variables that are not necessarily independent. Obtaining results on the extremal behavior of the maximum of dependent variables will now be a question of controlling this dependence. A key property in this framework is an adapted version of Theorem 1: The result of the theorem is still true under additional mixing and anti–clustering conditions ensuring that the sequence on a large scale behaves like an independent sequence; see Leadbetter et al. (1983, Chapter 3) for a detailed exposition. A generalization to stationary stochastic processes in continuous time can be found in Leadbetter et al. (1983, Chapter 13).

In the present paper we extend the index set from a one–dimensional time axis to a spatial setting with indices in $\mathbb{Z}^d$ and $\mathbb{R}^d$. The contribution of the paper will be two–fold. First we consider a stationary random field $(\xi_v)_{v\in\mathbb{Z}^d}$ and a sequence of finite, increasing index sets $(D_n)_{n\in\mathbb{N}}$ with $D_n \subseteq \mathbb{Z}^d$. A main result will be a new version of Theorem 1 in this setting, where we relate the convergence of $\mathbb{P}(\xi_v > x_n)$ to that of $\mathbb{P}(M_\xi(D_n) \le x_n)$ with the notation $M_\xi(D_n) = \max_{v\in D_n} \xi_v$, which will be used throughout. This is formulated as Theorem 5 in Section 3 below. To obtain this result, we need to impose conditions on the spatial dependence structure in the same spirit as the mixing and anti–clustering conditions needed in the one–dimensional case, cf. the conditions $\mathscr{D}(x_n; K_n)$ and $\mathscr{D}'(x_n)$ defined below.

While the index set in the one–dimensional case always has the form $D_n = \{1, \dots, m_n\}$, most often with $m_n = n$, a challenging part of the generalization to a spatial setting is to obtain the desired result under realistic and useful assumptions on the increasing behavior of the sequence of index sets (D_n). It is needed that D_n expands in a particularly nice way such that it can be approximated by a certain class of expanding cubes. The sufficient assumption, Assumption 1, on (D_n) will be given in Section 2 together with the introduction of relevant geometrical notation. The assumption is formulated in terms of the so–called intrinsic volumes of the (continuous version of) the sets.

The geometric assumption on (D_n) is formulated in a large generality, but as illustrated in Example 1 it includes the simple, but useful, situation, where a fixed set is scaled up by an increasing sequence (r_n),

$$D_n = (r_n C) \cap \mathbb{Z}^d.$$

Here C is a union of finitely many bounded, convex and full–dimensional sets.

Extreme value theory formulated in a spatial setting has until recently, to the best of the authors' knowledge, been rare in the literature. However, in Leadbetter and Rootzén (1998) a coordinate–wise spatial mixing condition is formulated. Furthermore it is obtained that under this condition the limiting distribution of a normalization of $M_\xi(D_n)$ is an extreme value distribution in the asymptotic scenario, where the index sets D_n constitutes an increasing sequence of boxes.

Very recently, a spatial setting has been studied in Jakubowski and Soja-Kukieła (2019), Ling (2019) and Soja-Kukieła (2019). Most prominently in relation to the present work is Jakubowski and Soja-Kukieła (2019, Theorem 5.2), where rather general extremal results, under conditions similar to $\mathscr{D}(x_n;K_n)$ and $\mathscr{D}'(x_n)$ below, are derived, when the index sets are increasing boxes and the random field (ξ_n) has a so–called extremal index. An extremal index of a general size θ would, in particular, mean

$$\sup_{x\in\mathbb{R}}|\mathbb{P}(M_n \leq x) - \mathbb{P}(\xi_0 \leq x)^{|D_n|}| \to 0,$$

where $|D_n|$ denotes the number of elements in D_n. The specific case of an extremal index of size 1 corresponds to the setting in the present paper.

In particular, the implication from (20) to (21) in Theorem 5 below, would follow directly from Jakubowski and Soja-Kukieła (2019, Theorem 5.2) in the case of the index sets being increasing boxes. However, still assuming the index sets being boxes, the results there are more general: Apart from dealing with the generality of extremal indices, the assumption $\mathscr{D}'(x_n)$ is relaxed to only cover sums of the following form, with notation that will be introduced later, $\sum_{|v-0|>m, v\in J_0^{n,k}}$, i.e. only over observations of a certain distance m from 0. Instead a more specific, necessary and sufficient, condition involving the extremal index is introduced.

In comparison, the contribution in this paper, as presented in Section 3 below, has a different merit – apart from giving the opposite implication between (20) to (21). While we focus solely on fields with an extremal index of 1, we allow the index sets to be rather general increasing geometric objects. Therefore, most of the work throughout the proofs of the section is devoted to handle this generality. In particular, the proofs of Lemma 4 and Theorem 4 could be reduced considerably in the case of index sets formed as boxes, and all other results rely on the geometric properties derived in Theorem 3, which in itself would be a triviality in a box–scenario.

The second part of the paper concerns a random field $(X_v)_{v\in\mathbb{R}^d}$ defined by

$$X_v = \int_{\mathbb{R}^d} f(|v-u|)\Lambda(\mathrm{d}u), \tag{1}$$

where Λ is an infinitely divisible, independently scattered random measure on $\mathbb{R}^d$ and f is an appropriate kernel function. We furthermore assume that the Lévy measure of the random measure Λ has a convolution equivalent right tail; see Cline (1986, 1987) for details about convolution equivalent distributions and Pakes (2004) for the relation between convolution equivalence and infinite divisibility. Examples of convolution equivalent distributions that are also infinitely divisible counts the important cases of the inverse Gaussian and the normal inverse Gaussian distribution; see examples 2.1 and 2.2 in Rønn-Nielsen and Jensen (2016).

Lévy–driven moving average models as defined in (1) form a very flexible framework that recently has been used for multiple modeling purposes. This includes modeling of turbulent flows (Barndorff-Nielsen and Schmiegel, 2004), growth processes (Jónsdóttir et al., 2008), Cox point processes (Hellmund et al., 2008), and brain imaging data (Jónsdóttir et al., 2013; Rønn-Nielsen et al., 2017)

For a sequence $(C_n)_{n\in\mathbb{N}}$ of index sets satisfying a continuous version of the assumption imposed on discrete sets $(D_n)_{n\in\mathbb{N}}$ above, and under mild restrictions on the kernel function f, we obtain the main result that for all $x \in \mathbb{R}$,

$$\mathbb{P}\Big(a_n^{-1}\big(\sup_{v\in C_n} X_v - b_n\big) \le x\Big) \to \exp\Big(-\mathrm{e}^{-x}\mathbb{E}\mathrm{e}^{\beta X_u}\rho\big((1,\infty)\big)\Big) \tag{2}$$

as $n \to \infty$, where a_n, b_n are norming constants chosen according to the extremal behavior of the Lévy measure ρ relative to the volume $|C_n|$ of the index set, such that $\lim_n |C_n|\rho\big((a_n x + b_n, \infty)\big) = \mathrm{e}^{-x}\rho\big((1,\infty)\big)$ for all $x \in \mathbb{R}$.

The proof of this result relies on a discretization, writing the supremum as a maximum of suprema over unit cubes. Here a result from Rønn-Nielsen and Jensen (2016) becomes rather beneficial: The tail distribution of e.g. $\sup_{v\in[0,1]^d} X_v$ is asymptotically equivalent with that of the underlying Lévy measure ρ. Obtaining the result will, however, not be a direct application of the result for stationary, discretely indexed random fields from the first part of the paper. Showing directly that the mixing and anti–clustering conditions $\mathscr{D}(x_n; K_n)$ and $\mathscr{D}'(x_n)$ are satisfied is rather challenging, while it is easier to apply an independent decomposition $X = Z + Y$, where Z is a compound Poisson random field with relatively heavy tails, and Y represents the part of X with infinite activity but lighter tails. The proof strategy will be to condition on $Y = y$ and then establish a conditional result on the extremal behavior of the field $Z + y$. The final result is obtained by applying ergodic properties of the conditioning Y–field. This proof strategy has the additional advantage that an extension of (2) follows directly with X replaced by $X + \tilde{Y}$, where $\tilde{Y}$ is a sufficiently light–tailed, stationary and ergodic random field that is independent of X, cf. Theorem 8 below.

In Fasen (2008) a stochastic process $(X_t)_{t\ge 0}$ on the form

$$X_t = \int_{\mathbb{R}_+\times\mathbb{R}} f(r, t-s)\Lambda(\mathrm{d}r, \mathrm{d}s)$$

is considered. Note that the index set here is one–dimensional. Furthermore, the kernel function f is assumed to satisfy $f(r,s) = 0$ for $s < 0$. Under very similar conditions on the random measure Λ, in particular it is assumed that ρ has a convolution equivalent right tail, an asymptotic result, similar to (2), is obtained for $\sup_{t\in[0,T]} X_t$ as $T \to \infty$.

An asymptotic result for a discretely observed field on the form (1) in a scenario with an increasing spatial index set can be found in Rønn-Nielsen and Jensen (2019). The paper proposes estimators for the mean and covariance function of the field and furthermore provides central limit theorems for these estimators. The spatial asymptotic scenario of the index sets is, however, less restrictive compared to the requirements in the present paper. There it is only needed that the discrete index set D_n increases in a way, such that the surface of D_n is asymptotically inferior to the volume of D_n.

The paper is organized as follows. In Section 2 we introduce a few geometrical concepts and formulate the assumption on the increasing sequence of index sets both in the discrete and continuous setting. In Section 3 we state and prove a spatial version of Theorem 1 under the assumption on (D_n) from Section 2 and additional mixing and anti–clustering conditions. We furthermore show that the limiting distribution of a normalized version of $M_\xi(D_n)$ is an extreme value distribution.

Section 4 is devoted to the introduction of the Lévy–based model (1) and the proofs of the main extremal results. The proofs that relates to the conditioning (Y_v)–field – in particular results on its ergodic behavior – are found in Section 5, while proofs of some further technical lemmas, used in the proof of the main theorem, are deferred to Section 6.

2 Geometrical assumption and result

The main purpose of this section will be to formulate the sufficient assumption on the sequence of index sets. It turns out that in both the discrete and the continuous setting a relevant property should be formulated through conditions on sets in $\mathbb{R}^d$. That means that for the sequence (D_n) of sets in $\mathbb{Z}^d$ used for the discrete result we will assume the existence of a sequence (C_n) of sets in $\mathbb{R}^d$ satisfying certain conditions and that $D_n = C_n \cap \mathbb{Z}^d$.

The relevant condition on the sequence of sets can be found in Assumption 1 below. However, the formulation of this assumption requires a few concepts and results from the classical theory of geometry for convex sets. This is summarized below in Theorem 2 and the properties immediately thereafter; see e.g. Schneider (1993, Chapter 4) for a detailed exposition of this topic.

Throughout the paper, the following notation will be used. For a subset A of either $\mathbb{Z}^d$ or $\mathbb{R}^d$, we let $|A|$ denote the size of the set A. Hence, $|A|$ is the number of points in $A \subseteq \mathbb{Z}^d$, and $|A|$ is the Lebesgue measure of $A \subseteq \mathbb{R}^d$. However, we will only consider the Lebesgue measure of unions of convex and full–dimensional sets in $\mathbb{R}^d$, so $|A|$ will in fact also be the Hausdorff measure of A. For a set $A \subseteq \mathbb{R}^d$ and a constant $\lambda > 0$ we define $\lambda A = \{\lambda x \mid x \in A\}$. For two sets $A, B \subseteq \mathbb{R}^d$, we define the Minkowski sum by $A \oplus B = \{a + b \mid a \in A, b \in B\}$. We let $0 \in \mathbb{R}^d$ denote the origin of $\mathbb{R}^d$, and we define $B(r)$ as the closed ball in $\mathbb{R}^d$ centered in 0 and with radius $r \geq 0$.

A compact convex set with non–empty interior will in the following be called a convex body. The following theorem, called Steiner's Theorem, constitutes a classical result from convex geometry.

Theorem 2 *Assume that C is a convex body and let $r > 0$. Then the volume $|C \oplus B(r)|$ is a polynomial in r, i.e.*

$$|C \oplus B(r)| = \sum_{j=0}^{d} \omega_{d-j} V_j(C) r^{d-j},$$

where $V_j(C)$ are coefficients that only depend on C, and ω_j is the volume of the j–dimensional unit ball.

The coefficients $V_j(C)$ are called intrinsic volumes of the convex body C and satisfy

1. $V_0(C) = 1$,
2. $V_1(C) = \frac{d\omega_d}{2\omega_{d-1}} b(C)$, where $b(C)$ is the mean width of C,
3. $V_{d-1}(C) = F(C)/2$, where $F(C)$ denotes the surface area of C, i.e. the $d-1$ dimensional Hausdorff measure of the surface ∂C,
4. $V_d(C) = |C|$.

Furthermore, the functionals $V_j : \mathscr{K} \to \mathbb{R}$, where $\mathscr{K}$ denotes the collection of all convex bodies, satisfy some important properties, among which we mention

(i) They are non–negative, i.e. $V_j(C) \geq 0$ for all $C \in \mathscr{K}$.
(ii) They are motion invariant, i.e. $V_j(gC) = V_j(C)$ for any euclidean motion g.
(iii) They are homogeneous, i.e. $V_j(\gamma C) = \gamma^j V_j(C)$ for all $\gamma > 0$.
(iv) They are monotone, i.e. $C \subseteq D$ implies $V_j(C) \leq V_j(D)$.

Corollary 1 *Let $C \subseteq \mathbb{R}^d$ be a convex body and $r > 0$. Then*

$$\sum_{j=0}^{d-1} \omega_{d-j} V_j(C) r^{d-j} \leq |\partial C \oplus B(r)| \leq 2 \sum_{j=0}^{d-1} \omega_{d-j} V_j(C) r^{d-j}.$$

Proof For the first inequality, we apply Theorem 2 to find

$$|\partial C \oplus B(r)| \geq |\big(C \oplus B(r)\big) \setminus C| = |C \oplus B(r)| - |C| = \sum_{j=0}^{d-1} \omega_{d-j} V_j(C) r^{d-j}\,.$$

For the second inequality, we define $C_{-r} = C \setminus \big(\partial C \oplus B(r)\big)$. Then C_{-r} is a convex set: Let $x, y \in C_{-r}$ and choose a point z on the line segment from x to y. If we can show $B(r) + z \subseteq C$, we will have $z \in C_{-r}$ and thus convexity is obtained. For this, choose $z' \in z + B(r)$, i.e. there is $b \in B(r)$ such that $z' = z + b$. Then z' is on the line segment from $x + b$ to $y + b$, and furthermore both $x + b$ and $y + b$ are in C. The convexity of C gives the desired result.

Arguing as above, using monotonicity of the intrinsic volumes and the fact that $C = C_{-r} \oplus B(r)$, we find

$$|C \setminus C_{-r}| = |C_{-r} \oplus B(r)| - |C_{-r}| \leq \sum_{j=0}^{d-1} \omega_{d-j} V_j(C) r^{d-j}.$$

Since $\partial C \oplus B(r) = \big((C \oplus B(r)) \setminus C\big) \cup \big(C \setminus C_{-r}\big)$, we can deduce the desired inequality.

Definition 1 A set $C \subseteq \mathbb{R}^d$ is said to be p–convex, if it is connected and has the form

$$C = \bigcup_{i=1}^{p} C_i\,,$$

where $C_1, \ldots, C_p$ are convex bodies in $\mathbb{R}^d$.

In the following we give the assumption on the index sets (C_n) used in the paper. Due to stationarity of all fields involved, we can without loss of generality in later results assume that $0 \in C_n$ for all $n \in \mathbb{N}$. Although not formulated in the assumption, this will be assumed throughout the paper.

Assumption 1 *The sequence $(C_n)_{n\in\mathbb{N}}$ consists of p–convex bodies, where*

$$C_n = \bigcup_{i=1}^{p} C_{n,i}$$

and $|C_n| \to \infty$ as $n \to \infty$. Furthermore,

$$\frac{\sum_{i=1}^{p} V_j(C_{n,i})}{|C_n|^{j/d}} \quad \textit{is bounded for each } j = 1, \ldots, d-1. \tag{3}$$

Concerning the sequence $(D_n)_{n\in\mathbb{N}}$ of discrete sets in $\mathbb{Z}^d$, we say that $(D_n)_{n\in\mathbb{N}}$ satisfies Assumption 1 if there exists a sequence $(C_n)_{n\in\mathbb{N}}$ of sets in $\mathbb{R}^d$ satisfying the assumption, such that $D_n = C_n \cap \mathbb{Z}^d$.

In proving our main results of Sections 3 and 4, we approximate the index sets D_n by a union of certain increasing boxes $J_z^{n,k,\lambda}$, $z \in \mathbb{Z}^d$. The full approximation scheme is given after this paragraph. The parameters $k \in \mathbb{N}$ and λ close to 1 determines the size of these boxes relative to the size of D_n. In particular, as it will be shown in Theorem 3(ii), we can approximate D_n by a union of approximately k such boxes, with the approximation improving for increasing k. In principle, it would suffice to control this relative size only by the use of $k \in \mathbb{N}$ and not λ, however, in the proof of Theorem 4 we control the approximation by the parameters separately.

For a sequence $(C_n)_{n\in\mathbb{N}}$ satisfying Assumption 1 we will for each $k \in \mathbb{N}$ and a fixed λ close to 1, define $t_{n,k,\lambda} = \lfloor \sqrt[d]{|C_n|/(\lambda k)} \rfloor$. Most often we will choose $\lambda = 1$, but in Theorems 3 and 4 and Lemmas 3 and 4 we will use the full generality. Subsequent to Theorem 4, we set $\lambda = 1$ and λ will be suppressed from all notation. For $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$ and n large enough relative to k we furthermore define the cube $I_z^{n,k,\lambda}$ as

$$I_z^{n,k,\lambda} = \bigtimes_{i=1}^{d} \left[z_i t_{n,k,\lambda}, (z_i+1) t_{n,k,\lambda} \right).$$

The idea of the sets $I_z^{n,k,\lambda}$ is that they can be used to approximate the C_n–sets by increasing k, and that the approximation has a "tuning" parameter λ. In fact, we do not directly approximate C_n but rather its discrete subset $C_n \cap \mathbb{Z}^d$. For this reason, let $P_{n,k,\lambda}$ be the set of indices z for which $I_z^{n,k,\lambda}$ is contained in C_n and $Q_{n,k,\lambda}$ be the set of indices z for which $I_z^{n,k,\lambda}$ is intersected by C_n, respectively. I.e.

$$P_{n,k,\lambda} = \{ z \in \mathbb{Z}^d \,:\, I_z^{n,k,\lambda} \subseteq C_n \}, \quad \text{and} \quad Q_{n,k,\lambda} = \{ z \in \mathbb{Z}^d \,:\, I_z^{n,k,\lambda} \cap C_n \neq \emptyset \}.$$

Let furthermore

$$p_{n,k,\lambda} = |P_{n,k,\lambda}| \qquad \text{and} \qquad q_{n,k,\lambda} = |Q_{n,k,\lambda}|.$$

Note that by construction, $p_{n,k,\lambda} \le \lambda k$ and $q_{n,k,\lambda} \ge \lambda k$ for values of n large enough relative to k. Let $J_z^{n,k,\lambda} = I_z^{n,k,\lambda} \cap \mathbb{Z}^d$ be the integer numbers in $I_z^{n,k,\lambda}$. Define

$$D^-_{n,k,\lambda} = \bigcup_{z \in P_{n,k,\lambda}} J_z^{n,k,\lambda} \qquad \text{and} \qquad D^+_{n,k,\lambda} = \bigcup_{z \in Q_{n,k,\lambda}} J_z^{n,k,\lambda}.$$

With $D_n = C_n \cap \mathbb{Z}^d$ we have $J_z^{n,k,\lambda} \subseteq D_n$ for all $z \in P_{n,k,\lambda}$, and that $z \in Q_{n,k,\lambda}$ for all $J_z^{n,k,\lambda}$ with $J_z^{n,k,\lambda} \cap D_n \neq \emptyset$. That gives

$$D^-_{n,k,\lambda} \subseteq D_n \subseteq D^+_{n,k,\lambda}. \tag{4}$$

Theorem 3 *Let $(C_n)_{n\in\mathbb{N}}$ satisfy Assumption 1, and let $D_n = C_n \cap \mathbb{Z}^d$. Then*

(i) *$|D_n| \sim |C_n|$ as $n \to \infty$,*

(ii) *for all λ the sequences $p_{n,k,\lambda}$ and $q_{n,k,\lambda}$, defined above, satisfy that*

$$\liminf_{n\to\infty} p_{n,k,\lambda} \sim \lambda k \qquad \text{and} \qquad \limsup_{n\to\infty} q_{n,k,\lambda} \sim \lambda k$$

as $k \to \infty$,

(iii) *for each k, λ and n with n large enough relative to k, it holds that*

$$D^+_{n,k,\lambda} \subseteq \bigcup_{z \in N_{k,\lambda}} J_z^{n,k,\lambda},$$

where $N_{k,\lambda}$ is on the form $N_{k,\lambda} = [-c_{k,\lambda}\,,\, c_{k,\lambda}]^d \cap \mathbb{Z}^d$ for some $c_{k,\lambda} < \infty$,

(iv) *there exists $c < \infty$ such that for all $n \in \mathbb{N}$*

$$D^+_{n,k,\lambda} \subseteq K_n = \left[-c \cdot |C_n|^{1/d}\,,\, c \cdot |C_n|^{1/d} \right]^d \cap \mathbb{Z}^d.$$

Remark 1 In (iii) above the important property is that $N_{k,\lambda}$ does not depend on n. That means in particular that for k and λ fixed, all D_n are included in the same finite collection of (increasing) cubes $J_z^{n,k,\lambda}$.

Proof We start by demonstrating statement (ii). Defining $\tilde{t}_{n,k,\lambda} = \left(\frac{|C_n|}{\lambda k}\right)^{1/d}$ for each $k,n \in \mathbb{N}$ and using Corollary 1, we find

$$|\partial C_n \oplus B(\tilde{t}_{n,k,\lambda})| \leq \sum_{i=1}^{p} |\partial C_{n,i} \oplus B(\tilde{t}_{n,k,\lambda})| \leq 2 \sum_{j=0}^{d-1} \omega_{d-j} \left(\sum_{i=1}^{p} V_j(C_{n,i}) \right) \tilde{t}_{n,k,\lambda}^{d-j} .$$

From straightforward calculations,

$$\frac{1}{\lambda k} \frac{1}{\tilde{t}_{n,k,\lambda}^d} |\partial C_n \oplus B(\tilde{t}_{n,k,\lambda})| \leq 2 \sum_{j=0}^{d-1} \omega_{d-j} \frac{\sum_{i=1}^{p} V_j(C_{n,i})}{|C_n|^{j/d}} \left(\frac{1}{\lambda k} \right)^{\frac{d-j}{d}} ,$$

such that (3) gives

$$\limsup_{n\to\infty} \frac{1}{\lambda k} \frac{1}{\tilde{t}_{n,k,\lambda}^d} |\partial C_n \oplus B(\tilde{t}_{n,k,\lambda})| \to 0$$

as $k \to \infty$. Using that $\tilde{t}_{n,k,\lambda/c^d} = c\tilde{t}_{n,k,\lambda}$ and replacing λ by λ/c^d in the limit above, we find that in fact

$$\limsup_{n\to\infty} \frac{1}{\lambda k} \frac{1}{\tilde{t}_{n,k,\lambda}^d} |\partial C_n \oplus B(c\tilde{t}_{n,k,\lambda})| \to 0$$

as $k \to \infty$ for all $c > 0$. Since $t_{n,k,\lambda} \leq \tilde{t}_{n,k,\lambda}$, and $t_{n,k,\lambda} \sim \tilde{t}_{n,k,\lambda}$ as $n \to \infty$, we also have

$$\limsup_{n\to\infty} \frac{1}{\lambda k} \frac{1}{t_{n,k,\lambda}^d} |\partial C_n \oplus B(ct_{n,k,\lambda})| \to 0 \tag{5}$$

as $k \to \infty$ for all $c > 0$. Recall that the side length of $I_z^{n,k,\lambda}$ is $t_{n,k,\lambda}$, so if $I_z^{n,k,\lambda} \cap \partial C_n \neq \emptyset$ then clearly $I_z^{n,k,\lambda} \subseteq \partial C_n \oplus B(2\sqrt{d}\, t_{n,k,\lambda})$. Using that $|I_z^{n,k,\lambda}| = t_{n,k,\lambda}^d$, we then find

$$\frac{q_{n,k,\lambda} - p_{n,k,\lambda}}{\lambda k} \leq \frac{1}{\lambda k} \frac{1}{t_{n,k,\lambda}^d} \Bigg| \bigcup_{I_z^{n,k,\lambda} \cap \partial C_n \neq \emptyset} I_z^{n,k,\lambda} \Bigg| \leq \frac{1}{\lambda k} \frac{1}{t_{n,k,\lambda}^d} |\partial C_n \oplus B(2\sqrt{d}\, t_{n,k,\lambda})|.$$

Together with (5) and the fact that $p_{n,k,\lambda} \leq \lambda k \leq q_{n,k,\lambda}$ for n chosen large enough relative to k, this gives the statement in (ii).

For statement (i) we now use that $|D_{n,k,\lambda}^-| = p_{n,k,\lambda} \cdot t_{n,k,\lambda}^d$ and $|D_{n,k,\lambda}^+| = q_{n,k,\lambda} \cdot t_{n,k,\lambda}^d$ together with (4) to find

$$\frac{p_{n,k,\lambda} \cdot t_{n,k,\lambda}^d}{|C_n|} \leq \frac{|D_n|}{|C_n|} \leq \frac{q_{n,k,\lambda} \cdot t_{n,k,\lambda}^d}{|C_n|}.$$

Letting $n \to \infty$ and subsequently $k \to \infty$ gives the desired result.

To see statement (iii), we have for each k and λ that the sequence $(q_{n,k,\lambda})_n$ is bounded by a constant $c_{k,\lambda}$ for n large enough. Since $0 \in C_n$, we have $0 \in Q_{n,k,\lambda}$. Furthermore, C_n is connected, so $Q_{n,k,\lambda}$ consists of at most $c_{k,\lambda}$ points that are pairwise neighbors. Therefore,

$$Q_{n,k,\lambda} \subseteq [-c_{k,\lambda}\, ,\, c_{k,\lambda}]^d \cap \mathbb{Z}^d$$

which, again, implies the desired result. Finally, (iv) follows easily from (iii).

Example 1 Let $C = \cup_{i=1}^p \overline{C}_i$ be a p–convex set and define the sequence $(C_n)_{n\in\mathbb{N}}$ by

$$C_n = r_n C = \bigcup_{i=1}^p r_n \overline{C}_i,$$

where $r_n \uparrow \infty$ as $n \to \infty$. Then $V_j(C_n) = V_j(r_n C) = r_n^j V_j(C)$ and $V_j(r_n\overline{C}_i) = r_n^j V_j(\overline{C}_i)$ for $j = 0,\dots,d$. In particular,

$$\frac{\sum_{i=1}^p V_j(r_n\overline{C}_i)}{|C_n|^{j/d}} = \frac{\sum_{i=1}^p V_j(\overline{C}_i)}{|C|^{j/d}}$$

will be constant for $j = 1,\dots,d-1$. Thus the sequence $(C_n)_{n\in\mathbb{N}}$ satisfies Assumption 1.

Before moving on to the main sections of this paper, we give a useful result concerning the integral of a decreasing function over convex sets and their complements:

Lemma 1 *Let C be a convex body and let $r \geq 0$ be fixed. For any decreasing function $g : [0,\infty) \to [0,\infty)$ satisfying $\int_0^\infty g(x)x^{d-1}\mathrm{d}x < \infty$, it holds that*

$$\int_{(C\oplus B(r))^c} \sup_{v\in C} g(|v-u|)\mathrm{d}u = \sum_{j=0}^{d-1} \mu_j V_j(C) \int_r^\infty g(x)x^{d-j-1}\mathrm{d}x \qquad \textit{and} \tag{6}$$

$$\int_C \sup_{v\in(C\oplus B(r))^c} g(|v-u|)\mathrm{d}u \leq \sum_{j=0}^{d-1} \mu_j V_j(C) \int_r^\infty g(x)x^{d-j-1}\mathrm{d}x \tag{7}$$

for constants μ_j, $j = 0,\dots,d-1$, independent of C.

Proof For notational convenience we define the convex body $\bar{C}_r = C \oplus B(r)$.

We start by illustrating (6). Since C is a convex body and using Steiner's theorem (Theorem 2), we find for any $s < t$ that

$$\begin{aligned} |C\oplus B(t)| - |C\oplus B(s)| &= \sum_{j=0}^{d-1} \omega_{d-j} V_j(C)\big(t^{d-j} - s^{d-j}\big) \\ &= \sum_{j=0}^{d-1} \mu_j V_j(C) \int_s^t x^{d-j-1}\mathrm{d}x, \end{aligned} \tag{8}$$

for appropriate finite constants μ_j. Since g is decreasing, we find for all $u \in \bar{C}_r^c$ that

$$\sup_{v\in C} g(|v-u|) = g\big(\mathrm{dist}(u,C)\big),$$

where $\mathrm{dist}(u,C) \geq r$ by construction of $\bar{C}_r$. Approximating g by a simple function and using (8), we find by a standard extension argument that

$$\int_{\bar{C}_r^c} \sup_{v\in C} g(|v-u|)\mathrm{d}u = \int_{\bar{C}_r^c} g\big(\mathrm{dist}(u,C)\big)\mathrm{d}u = \sum_{j=0}^{d-1} \mu_j V_j(C) \int_r^\infty g(x)x^{d-j-1}\mathrm{d}x$$

as claimed.

We now show (7). For this, we reuse the notation $C_{-s} = C \setminus \big(\partial C \oplus B(s)\big)$ previously defined. Using Steiner's theorem and the monotinicity of the intrinsic volumes, we obtain for any $s < t$ that

$$\begin{aligned}
|C_{-s}| - |C_{-t}| &= \Big(|C_{-t} \oplus B(t)| - |C_{-t}|\Big) - \Big(|C_{-s} \oplus B(s)| - |C_{-s}|\Big) \\
&= \sum_{j=0}^{d-1} \omega_{d-j} \Big(V_j(C_{-t}) t^{d-j} - V_j(C_{-s}) s^{d-j}\Big) \\
&\leq \sum_{j=0}^{d-1} \mu_j V_j(C) \int_s^t x^{d-j-1} \mathrm{d}x,
\end{aligned}$$

since $C_{-t} \subseteq C_{-s} \subseteq C$. Using the fact that g is decreasing,

$$\sup_{v \in \bar{C}_r^c} g(|v-u|) = g\big(\mathrm{dist}(u, \bar{C}_r^c)\big)$$

for any $u \in C$, where $r \leq \mathrm{dist}(u, \bar{C}_r^c) \leq r_0$ for some $r_0 > 0$ given by the diameter of C. By another extension argument we conclude that

$$\int_C \sup_{v \in \bar{C}_r^c} g(|v-u|) \mathrm{d}u = \int_C g\big(\mathrm{dist}(u, \bar{C}_r^c)\big) \mathrm{d}u \leq \sum_{j=0}^{d-1} \mu_j V_j(C) \int_r^{r_0} g(x) x^{d-j-1} \mathrm{d}x,$$

which shows (7).

3 Extreme value theory for a spatial stationary field on $\mathbb{Z}^d$

In this section we consider a stationary random field $\xi = (\xi_v)_{v \in \mathbb{Z}^d}$ and a sequence of index sets $(D_n)_{n \in \mathbb{N}}$ in $\mathbb{Z}^d$ satisfying Assumption 1. Below we formulate the sufficient mixing condition for ξ in terms of the behavior on the sufficient sets (K_n) associated to (D_n) by Theorem 3, however, before doing so we introduce some further notation which will be used in this and the remaining sections of the paper.

We say that two subsets A, B of $\mathbb{Z}^d$ (or $\mathbb{R}^d$) are r-separated if $\mathrm{dist}(A,B) = \inf\{|a-b| : a \in A, b \in B\} \geq r$ and there are two disjoint sets $A', B' \subseteq \mathbb{R}^d$, which are both connected, such that $A \subseteq A'$ and $B \subseteq B'$. Moreover, when talking about a d–dimensional cube with side–length $s > 0$, we mean a box with all side–lengths equal to s. If the side–lengths of a box are not identical, we list them all.

Condition ($\mathscr{D}(x_n; K_n)$) *The condition $\mathscr{D}(x_n; K_n)$ is satisfied for the stationary field $(\xi_v)_{v \in \mathbb{Z}^d}$ if there exists an increasing sequence $\gamma_n = o(\sqrt[d]{|D_n|})$ such that for all $n \in \mathbb{N}$ and all γ_n-separated sets $A, B \subseteq K_n$ where at least one is a box, it holds that*

$$\big|\mathbb{P}(M_\xi(A \cup B) \leq x_n) - \mathbb{P}(M_\xi(A) \leq x_n)\mathbb{P}(M_\xi(B) \leq x_n)\big| \leq \alpha_n, \tag{9}$$

where $\alpha_n \to 0$ as $n \to \infty$.

We need an "approximate independence" similar to (9) for more than two separated sets. The lemma below follows easily by induction (and possibly a reordering), using the fact that $\cup_{i=1}^{r-1} A_i$ and A_r are γ_n-separated if all A_i, $i = 1, \ldots, r$, are pairwise γ_n-separated for all $i \neq j$.

Lemma 2 *Let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathscr{D}(x_n;K_n)$, and let for $r\in\mathbb{N}$ the boxes $A_1,\dots,A_r\subseteq K_n$ be pairwise γ_n-separated. Then*

$$\Big|\mathbb{P}\Big(\bigcap_{i=1}^{r}\{M_\xi(A_i)\le x_n\}\Big)-\prod_{i=1}^{r}\mathbb{P}(M_\xi(A_i)\le x_n)\Big|\le(r-1)\alpha_n.$$

In Lemma 3 below we essentially approximate the distribution of $M_\xi(D_n)$ by a certain power of the distribution function of $M_\xi(J_z^{n,k,\lambda})$. The idea is to construct γ_n-separated sub-boxes of $J_z^{n,k,\lambda}$ and use Lemma 2 above.

Assuming $\mathscr{D}(x_n;K_n)$ we have $\gamma_n<t_{n,k,\lambda}$ for n sufficiently large relative to a fixed k. For $z\in N_{k,\lambda}$ we divide each $J_z=J_z^{n,k,\lambda}$ into two disjoint subsets, $H_z=H_z^{n,k,\lambda}$ and $H_z^*=H_z^{*,n,k,\lambda}$, where

$$H_z^{n,k,\lambda}=\big\{u\in\mathbb{Z}^d\,:\,z_jt_{n,k,\lambda}\le u_j\le(z_j+1)t_{n,k,\lambda}-1-\gamma_n,\text{ for all }j=1,\dots,d\big\}.$$

From now on we simply write H_z and $H_z^*=J_z\setminus H_z$ to ease notation, but bare in mind the dependence on n,k,λ. The boxes $(H_z)_z$ are constructed in such a way that they are pairwise γ_n-separated, and thus Lemma 2 can be applied to the sequence. Moreover, H_z^* is the union of the (n,k,λ-dependent) overlapping boxes $L_{z,1}^*,\dots,L_{z,d}^*$ given by

$$L_{z,j}^*=\big\{u\in J_z\,:\,(z_j+1)t_{n,k,\lambda}-\gamma_n\le u_j\le(z_j+1)t_{n,k,\lambda}-1\big\}$$

for all $j=1,\dots,d$, which satisfy $|L_{z,j}^*|\sim t_{n,k,\lambda}^{d-1}\gamma_n$ as $n\to\infty$. In the proof of Lemma 9 in the second part of the paper, we specifically use that $L_{z,j}^*$ is in the shape of a box.

Lemma 3 *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of sets satisfying Assumption 1, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathscr{D}(x_n;K_n)$. Then*

$$\begin{aligned}&\big|\mathbb{P}(M_\xi(D_{n,k,\lambda}^-)\le x_n)-\mathbb{P}^{p_{n,k,\lambda}}(M_\xi(J_0)\le x_n)\big|\\&\le 2p_{n,k,\lambda}\mathbb{P}(M_\xi(H_0)\le x_n<M_\xi(H_0^*))+(p_{n,k,\lambda}-1)\alpha_n,\end{aligned}\tag{10}$$

and similarly

$$\begin{aligned}&\big|\mathbb{P}(M_\xi(D_{n,k,\lambda}^+)\le x_n)-\mathbb{P}^{q_{n,k,\lambda}}(M_\xi(J_0)\le x_n)\big|\\&\le 2q_{n,k,\lambda}\mathbb{P}(M_\xi(H_0)\le x_n<M_\xi(H_0^*))+(q_{n,k,\lambda}-1)\alpha_n.\end{aligned}\tag{11}$$

Proof Since H_z is a subset of J_z, it is easily seen that

$$\begin{aligned}0&\le\mathbb{P}\Big(\bigcap_{z\in P_{n,k,\lambda}}\{M_\xi(H_z)\le x_n\}\Big)-\mathbb{P}(M_\xi(D_{n,k,\lambda}^-)\le x_n)\\&\le\mathbb{P}\Big(\bigcup_{z\in P_{n,k,\lambda}}\{M_\xi(H_z)\le x_n<M_\xi(H_z^*)\}\Big)\\&\le p_{n,k,\lambda}\mathbb{P}(M_\xi(H_0)\le x_n<M_\xi(H_0^*))\end{aligned}$$

by stationarity. Turning to Lemma 2 and using stationarity again show that

$$\Big|\mathbb{P}\Big(\bigcap_{z\in P_{n,k,\lambda}}\{M_\xi(H_z)\le x_n\}\Big)-\mathbb{P}^{p_{n,k,\lambda}}(M_\xi(H_0)\le x_n)\Big|\le(p_{n,k,\lambda}-1)\alpha_n,$$

and realizing that

$$\begin{aligned}0 &\le \mathbb{P}^{p_{n,k,\lambda}}(M_\xi(H_0) \le x_n) - \mathbb{P}^{p_{n,k,\lambda}}(M_\xi(J_0) \le x_n)\\ &\le p_{n,k,\lambda}\big(\mathbb{P}(M_\xi(H_0) \le x_n) - \mathbb{P}(M_\xi(J_0) \le x_n)\big)\\ &= p_{n,k,\lambda}\mathbb{P}(M_\xi(H_0) \le x_n < M_\xi(H_0^*))\end{aligned}$$

concludes (10). The claim (11) follows similarly.

Lemma 4 *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of sets satisfying Assumption 1, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathscr{D}(x_n;K_n)$. Then*

$$\big|\mathbb{P}(M_\xi(D_n) \le x_n) - \mathbb{P}^{\lambda k}(M_\xi(J_0^{n,k,\lambda}) \le x_n)\big| \le R_{n,k,\lambda}, \tag{12}$$

where $R_{n,k,\lambda}$ satisfies

$$\limsup_{n\to\infty} R_{n,k,\lambda} = o(1) \tag{13}$$

as $k\to\infty$.

Proof Let $R^p_{n,k,\lambda} \ge 0$ and $R^q_{n,k,\lambda} \ge 0$ denote the upper bounds in (10) and (11), respectively. First we show that

$$\mathbb{P}(M_\xi(H_0) \le x_n < M_\xi(H_0^*)) \to 0 \tag{14}$$

as $n\to\infty$. For any integer r it holds for all n sufficiently large that $t_{n,k,\lambda} \ge (2r+1)\gamma_n$. Now fix $r\in\mathbb{N}$ and such an n. Then, for all $j=1,\dots,d$, we can write $L^*_{0,j}$ as the union of $2(d-1)$ overlapping boxes each with side-lengths $\gamma_n,(t_{n,k,\lambda}-\gamma_n),\dots,(t_{n,k,\lambda}-\gamma_n)$, where we need one box (interval) if $d=1$. Now let E_j^* be such a box. Since $t_{n,k,\lambda} \ge (2r+1)\gamma_n$, we can construct boxes $E_1,\dots,E_r$ contained in H_0 which are simply translations of E_j^*, such that they are separated from E_j^* and each other by γ_n. Hence, by Lemma 2,

$$\begin{aligned}&\mathbb{P}(M_\xi(H_0) \le x_n < M_\xi(E_j^*))\\ &\quad\le \mathbb{P}\Big(\bigcap_{s=1}^{r}\{M_\xi(E_s) \le x_n < M_\xi(E_j^*)\}\Big)\\ &\quad= \mathbb{P}\Big(\bigcap_{s=1}^{r}\{M_\xi(E_s) \le x_n\}\Big) - \mathbb{P}\Big(\bigcap_{s=1}^{r}\{M_\xi(E_s) \le x_n\}\cap\{M_\xi(E_j^*) \le x_n\}\Big)\\ &\quad\le x^r - x^{r+1} + 2r\alpha_n,\end{aligned}$$

where $x = \mathbb{P}(M_\xi(E_s) \le x_n) = \mathbb{P}_\xi(M(E_j^*) \le x_n)$ for all s by stationarity. For $x\in[0,1]$ the mapping $x\mapsto x^r - x^{r+1}$ is bounded by $1/r$, and we conclude that

$$\mathbb{P}(M_\xi(H_0) \le x_n < M_\xi(E_j^*)) \le \frac{1}{r} + 2r\alpha_n.$$

This can be done for all the $2(d-1)$ sub-boxes of $L^*_{0,j}$ for all $j=1,\dots,d$. Since $H_0^* = \cup_{j=1}^d L^*_{1,j}$, we therefore find that

$$\mathbb{P}(M_\xi(H_0) \le x_n < M_\xi(H_0^*)) \le 2d(d-1)\Big(\frac{1}{r} + 2r\alpha_n\Big).$$

Letting $n\to\infty$ and then $r\to\infty$ show that (14) is satisfied, which implies that also

$$\lim_{n\to\infty} R^p_{n,k,\lambda} = \lim_{n\to\infty} R^q_{n,k,\lambda} = 0. \tag{15}$$

Now define $\tilde{R}^p_{n,k,\lambda} \geq 0$ and $\tilde{R}^q_{n,k,\lambda} \geq 0$ by

$$\tilde{R}^p_{n,k,\lambda} = R^p_{n,k,\lambda} + \mathbb{P}^{p_{n,k,\lambda}}(M_\xi(J_0) \leq x_n) - \mathbb{P}^{\lambda k}(M_\xi(J_0) \leq x_n), \qquad \text{and}$$
$$\tilde{R}^q_{n,k,\lambda} = R^q_{n,k,\lambda} + \mathbb{P}^{\lambda k}(M_\xi(J_0) \leq x_n) - \mathbb{P}^{q_{n,k,\lambda}}(M_\xi(J_0) \leq x_n).$$

Since $D^-_{n,k,\lambda} \subseteq D_n \subseteq D^+_{n,k,\lambda}$, it is seen that

$$\begin{aligned}
\mathbb{P}^{\lambda k}(M_\xi(J_0) \leq x_n) - \tilde{R}^q_{n,k,\lambda} &\leq \mathbb{P}(M_\xi(D^+_{n,k,\lambda}) \leq x_n) \\
&\leq \mathbb{P}(M_\xi(D_n) \leq x_n) \\
&\leq \mathbb{P}(M_\xi(D^-_{n,k,\lambda}) \leq x_n) \\
&\leq \mathbb{P}^{\lambda k}(M_\xi(J_0) \leq x_n) + \tilde{R}^p_{n,k,\lambda},
\end{aligned}$$

and defining $R_{n,k,\lambda} = \max\{\tilde{R}^p_{n,k,\lambda}, \tilde{R}^q_{n,k,\lambda}\}$ then shows (12).

It remains to show that (13) is satisfied, which follows if both $\limsup_{n\to\infty} \tilde{R}^p_{n,k,\lambda}$ and $\limsup_{n\to\infty} \tilde{R}^q_{n,k,\lambda}$ are of order $o(1)$ as $k \to \infty$. We only show this for $\tilde{R}^p_{n,k,\lambda}$ as the behavior for $\tilde{R}^q_{n,k,\lambda}$ follows similarly. First, since $p_{n,k,\lambda} \leq \lambda k$, the mapping $x \mapsto x^{p_{n,k,\lambda}} - x^{\lambda k}$ is bounded by $(1 - \frac{p_{n,k,\lambda}}{\lambda k})$ for all $x \in [0,1]$. Hence

$$\mathbb{P}^{p_{n,k,\lambda}}(M_\xi(J_0) \leq x_n) - \mathbb{P}^{\lambda k}(M_\xi(J_0) \leq x_n) \leq 1 - \frac{p_{n,k,\lambda}}{\lambda k},$$

and, using (15) and (ii) in Theorem 3, we conclude that

$$0 \leq \limsup_{n\to\infty} \tilde{R}^p_{n,k,\lambda} \leq 1 - \frac{\liminf_{n\to\infty} p_{n,k,\lambda}}{\lambda k} \to 0$$

as $k \to \infty$. Equation (13) is thus satisfied.

The next theorem shows that under Assumption 1 and $\mathscr{D}(x_n; K_n)$, and when normalized correctly, the limiting distribution of $M(D_n)$ is necessarily an extreme value distribution; see Leadbetter et al. (1983, Chapter 1) for a thorough exposition of extreme value distributions and their connection to max–stable distributions.

Theorem 4 *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of sets satisfying Assumption 1, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field. Assume that there are sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n > 0$ and $(b_n)_{n\in\mathbb{N}}$ such that $\mathscr{D}(a_n x + b_n; K_n)$ is satisfied for all $x \in \mathbb{R}$. Assume furthermore that there exists a constant $a > 0$ such that*

$$|D_n| \sim a \cdot n. \tag{16}$$

If there exists a non–degenerate distribution function G such that for all $x \in \mathbb{R}$

$$\mathbb{P}(M_\xi(D_n) \leq a_n x + b_n) \to G(x), \tag{17}$$

then G is the distribution function of an extreme value distribution.

Note that (16) is in particular satisfied if $C_n = r_n C$ for a p–convex set C and a sequence (r_n) with $r_n \sim c\sqrt[d]{n}$, cf. Example 1 and (i) in Theorem 3.

Proof According to Leadbetter et al. (1983, Theorem 1.3.1) it suffices to show that there exists a sequence of distribution functions $(F_n)_{n\in\mathbb{N}}$ such that for all $x \in \mathbb{R}$ and all $k \in \mathbb{N}$

$$F_n(a_{nk}x + b_{nk}) \to G^{1/k}(x) \tag{18}$$

as $n \to \infty$. To obtain this, we first relate the limiting distribution G to the distribution of $M(\tilde{D}_n^{k_0})$, where $(\tilde{D}_n^{k_0})_{n\in\mathbb{N}}$ is a sequence of discrete cubes defined as follows: Choose $k_0 \in \mathbb{N}$ such that $2/\sqrt[d]{\lambda k_0} \le c$ for all $\lambda \in (1-\varepsilon, 1+\varepsilon)$, with $\varepsilon > 0$ and c defined in Theorem 3, and define

$$\tilde{C}_n^{k_0,\lambda} = \left[0, \sqrt[d]{|C_n|/(\lambda k_0)}\right]^d \qquad \text{and} \qquad \tilde{D}_n^{k_0,\lambda} = \tilde{C}_n^{k_0,\lambda} \cap \mathbb{Z}^d,$$

which are easily seen to satisfy Assumption 1. Note in addition that $2\tilde{D}_n^{k_0,\lambda} \subseteq K_n$ and $|\tilde{C}_n^{k_0,\lambda}| = \frac{|C_n|}{\lambda k_0}$ for all n. Now, with a notation very similar to the notation from Section 2, we define for each $k \in \mathbb{N}$

$$\begin{aligned} \tilde{P}_{n,k}^{k_0,\lambda} &= \{z \in \mathbb{Z}^d : I_z^{n,k_0\cdot k,\lambda} \subseteq \tilde{C}_n^{k_0,\lambda}\} \qquad \text{and} \\ \tilde{Q}_{n,k}^{k_0,\lambda} &= \{z \in \mathbb{Z}^d : I_z^{n,k_0\cdot k,\lambda} \cap \tilde{C}_n^{k_0,\lambda} \neq \emptyset\}. \end{aligned}$$

With the same arguments as before and in the proof of Theorem 3 it is easily seen that

$$|\tilde{P}_{n,k}^{k_0,\lambda}| \le k \le |\tilde{Q}_{n,k}^{k_0,\lambda}|$$

and

$$\liminf_{n\to\infty} \left|\tilde{P}_{n,k}^{k_0,\lambda}\right| \sim k \qquad \text{resp.} \qquad \limsup_{n\to\infty} \left|\tilde{Q}_{n,k}^{k_0,\lambda}\right| \sim k$$

as $k \to \infty$. Using that $\tilde{D}_n^{k_0,\lambda}$ is approximated by roughly k of the cubes $J_z^{n,k_0\cdot k,\lambda}$ in exactly the same way as D_n can be approximated by roughly k of the cubes $J_z^{n,k,1}$, we find from Lemma 4 that

$$\left|\mathbb{P}\big(M_\xi(\tilde{D}_n^{k_0,\lambda}) \le a_n x + b_n\big) - \mathbb{P}^k\big(M_\xi(J_0^{n,k_0\cdot k,\lambda}) \le a_n x + b_n\big)\right| \le \tilde{R}_{n,k,\lambda},$$

where $\tilde{R}_{n,k,\lambda}$ satisfies $\limsup_{n\to\infty} \tilde{R}_{n,k,\lambda} = o(1)$ as $k \to \infty$. Using that $x \mapsto x^{\lambda k_0}$ is Lipschitz continuous when restricted to $[0,1]$, we find that also

$$\left|\mathbb{P}^{\lambda k_0}\big(M_\xi(\tilde{D}_n^{k_0,\lambda}) \le a_n x + b_n\big) - \mathbb{P}^{\lambda k_0\cdot k}\big(M_\xi(J_0^{n,k_0\cdot k,\lambda}) \le a_n x + b_n\big)\right| \le \tilde{R}'_{n,k,\lambda},$$

where $\limsup_{n\to\infty} \tilde{R}'_{n,k,\lambda} = o(1)$ as $k \to \infty$. Since we already have from Lemma 4 that

$$\left|\mathbb{P}\big(M_\xi(D_n) \le a_n x + b_n\big) - \mathbb{P}^{\lambda k_0\cdot k}\big(M_\xi(J_0^{n,k_0\cdot k,\lambda}) \le a_n x + b_n\big)\right| \le R_{n,k,\lambda},$$

we can, when combining with (17), conclude that for all $x \in \mathbb{R}$

$$\mathbb{P}^{\lambda k_0}\big(M_\xi(\tilde{D}_n^{k_0,\lambda}) \le a_n x + b_n\big) \to G(x)$$

as $n \to \infty$, and thereby also

$$\mathbb{P}^{k_0}\big(M_\xi(\tilde{D}_n^{k_0,\lambda}) \le a_n x + b_n\big) \to G^{1/\lambda}(x).$$

Now define

$$\overline{C}_n^{k_0} = \left[0, \sqrt[d]{a\cdot n/k_0}\right]^d \qquad \text{and} \qquad \overline{D}_n^{k_0} = \overline{C}_n^{k_0} \cap \mathbb{Z}^d.$$

By (16) and (i) in Theorem 3 we find for all $\lambda^- < 1 < \lambda^+$ that

$$\begin{aligned}\mathbb{P}\big(M_\xi(\tilde{D}_n^{k_0,\lambda^-}) \le a_n x + b_n\big) &\le \mathbb{P}\big(M_\xi(\overline{D}_n^{k_0}) \le a_n x + b_n\big)\\ &\le \mathbb{P}\big(M_\xi(\tilde{D}_n^{k_0,\lambda^+}) \le a_n x + b_n\big)\end{aligned}$$

for n large enough. Letting $\lambda^- \uparrow 1$ and $\lambda^+ \downarrow 1$ this gives

$$\mathbb{P}^{k_0}\big(M_\xi(\overline{D}_n^{k_0}) \le a_n x + b_n\big) \to G(x) \tag{19}$$

as $n \to \infty$. Let $k \in \mathbb{N}$ and replace k_0 by $k_0 \cdot k$ and n by $n \cdot k$ in (19). Since furthermore

$$\overline{D}_{n\cdot k}^{k_0\cdot k} = \overline{D}_n^{k_0}$$

by construction, we find that for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$ the convergence

$$\mathbb{P}^{k_0}\big(M_\xi(\overline{D}_n^{k_0}) \le a_{nk} x + b_{nk}\big) \to G^{1/k}(x)$$

as $n \to \infty$ is satisfied. This shows (18) as desired.

For the remainder of this paper we let $\lambda = 1$ and suppress it from the notation. So far, assuming $\mathscr{D}(x_n;K_n)$ has made it possible to relate the limiting distribution of $M_\xi(D_n)$ to the distribution of e.g. $M_\xi(J_0^{n,k})$. However, to be able to relate this to the distribution of the individual ξ–variables, it is necessary to make the following further assumption.

Condition ($\mathscr{D}'(x_n)$) *The condition $\mathscr{D}'(x_n)$ is satisfied for the stationary field $(\xi_v)_{v\in\mathbb{Z}^d}$ if*

$$S_{n,k} = t_{n,k}^d \sum_{0\ne v\in J_0^{n,k}} \mathbb{P}(\xi_0 > x_n, \xi_v > x_n)$$

satisfies

$$\limsup_{n\to\infty} S_{n,k} = o(k^{-1})$$

as $k \to \infty$.

Theorem 5 *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of sets satisfying Assumption 1, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathscr{D}(x_n;K_n)$ and $\mathscr{D}'(x_n)$. Let $0 \le \tau < \infty$. Then, for any $v \in \mathbb{Z}^d$,*

$$|D_n|\mathbb{P}(\xi_v > x_n) \to \tau \qquad \text{as } n \to \infty, \tag{20}$$

if and only if

$$\mathbb{P}(\max_{v\in D_n} \xi_v \le x_n) \to \mathrm{e}^{-\tau} \qquad \text{as } n \to \infty. \tag{21}$$

Proof Let F_ξ denote the common distribution function of $(\xi_v)_{v\in\mathbb{Z}^d}$, i.e.

$$F_\xi(x) = 1 - \overline{F}_\xi(x) = \mathbb{P}(\xi_v \le x), \qquad \text{for all} \quad v \in \mathbb{Z}^d.$$

Writing $J_0 = J_0^{n,k}$, it is not difficult to see that

$$\begin{aligned}\sum_{v\in J_0} \mathbb{P}(\xi_v > x_n) - \sum_{v<v'\in J_0} \mathbb{P}(\xi_v > x_n, \xi_{v'} > x_n)&\\ \le \mathbb{P}(M_\xi(J_0) > x_n)&\\ \le \sum_{v\in J_0} \mathbb{P}(\xi_v > x_n),&\end{aligned}$$

where the summation over $\{v < v' \in J_0\}$ indicates the double sum of points in $v \in J_0$ and subsequent points $v' \in J_0$ falling strictly after v under some underlying enumeration. This notation will be used in the following sections as well. By stationarity of (ξ_v) the above implies that

$$t_{n,k}^d \overline{F}_\xi(x_n) - S_{n,k} \leq \mathbb{P}(M_\xi(J_0) > x_n) \leq t_{n,k}^d \overline{F}_\xi(x_n),$$

where $S_{n,k}$ satisfies $\limsup_n S_{n,k} = o(k^{-1})$ as $k \to \infty$ according to $\mathscr{D}'(x_n)$. By Lemma 4 this implies that

$$\begin{aligned}
&(1 - t_{n,k}^d \overline{F}_\xi(x_n))^k - R_{n,k} \\
&\quad \leq \mathbb{P}(M_\xi(D_n) \leq x_n) \\
&\quad \leq (1 - t_{n,k}^d \overline{F}_\xi(x_n) + S_{n,k})^k + R_{n,k}.
\end{aligned} \tag{22}$$

Now assume that (20) is satisfied, that is, $|D_n|\overline{F}_\xi(x_n) \to \tau$ and equivalently $t_{n,k}^d \overline{F}_\xi(x_n) \to \tau/k$. Then (13) and (22) imply that

$$\begin{aligned}
\Big(1 - \frac{\tau}{k}\Big)^k + o(1) &\leq \liminf_{n\to\infty} \mathbb{P}(M_\xi(D_n) \leq x_n) \\
&\leq \limsup_{n\to\infty} \mathbb{P}(M_\xi(D_n) \leq x_n) \\
&\leq \Big(1 - \frac{\tau}{k} + o(k^{-1})\Big)^k + o(1).
\end{aligned}$$

Taking the limit $k \to \infty$ and using the equivalence $\log(1-y) \sim -y$ $(y \to 0)$ show that $\lim_{n\to\infty} \mathbb{P}(M_\xi(D_n) \leq x_n) = \mathrm{e}^{-\tau}$.

Now assume that (21) is satisfied, i.e. $\mathbb{P}(M_\xi(D_n) \leq x_n) \to \mathrm{e}^{-\tau}$. Using this assumption, (13) and (22) imply that

$$\begin{aligned}
1 - \big(\mathrm{e}^{-\tau} + o(1)\big)^{1/k} &\leq k^{-1} \liminf_{n\to\infty} |D_n|\overline{F}_\xi(x_n) \\
&\leq k^{-1} \limsup_{n\to\infty} |D_n|\overline{F}_\xi(x_n) \\
&\leq 1 - \big(\mathrm{e}^{-\tau} + o(1)\big)^{1/k} + o(k^{-1}).
\end{aligned}$$

Multiplying by k and taking the limit $k \to \infty$ show that $\lim_{n\to\infty} |D_n|\overline{F}_\xi(x_n) = \tau$.

The following corollary follows exactly as Leadbetter et al. (1983, Corollary 3.4.2).

Corollary 2 *For $\tau = \infty$, the conclusions of Theorem 5 hold if the conditions $\mathscr{D}(x_n; K_n)$ and $\mathscr{D}'(x_n)$ are replaced by the following: For all $\tau' < \infty$ there is a real sequence $(s_n)_{n\in\mathbb{N}}$ such that $|D_n|\mathbb{P}(\xi_v > s_n) \to \tau'$ and such that $\mathscr{D}(s_n; K_n)$ and $\mathscr{D}'(s_n)$ are satisfied.*

Below we provide an example to which the results can be used, namely that of a stationary Gaussian field, i.e. a field such that all finite dimensional distributions are multivariate Gaussian.

Corollary 3 *Let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary Gaussian field with correlation function r_v, with $v \in \mathbb{Z}^d$, satisfying*

$$\log(m) \sup_{|v|>m} |r_v| \to 0 \tag{23}$$

as $m \to \infty$. Furthermore, let $(D_n)_n$ be a sequence of sets satisfying Assumption 1. Then, for all $0 \leq \tau \leq \infty$,

$$|D_n|\mathbb{P}(\xi_v > x_n) \to \tau \qquad \text{if and only if} \qquad \mathbb{P}(\max_{v\in D_n} \xi_v \leq x_n) \to \mathrm{e}^{-\tau}$$

as $n \to \infty$.

Proof Without loss of generality we assume that (ξ_v) is a standard Gaussian field. First we note that

$$\sup_{v\neq 0}|r_v| < 1, \tag{24}$$

which is seen by the following considerations: From the assumption (23) we have

$$\sup_{|v|>m}|r_v| \to 0 \tag{25}$$

as $m \to \infty$. Now assume that $|r_v| = 1$ for some $0 \neq v \in \mathbb{Z}^d$. Then, by the Cauchy-Schwarz inequality, $\xi_0 = \pm\xi_{kv}$ almost surely for all $k \in \mathbb{N}$, and consequently $|r_{kv}| = 1$ for all $k \in \mathbb{N}$ contradicting (25). Hence, $|r_v| < 1$ for all $0 \neq v \in \mathbb{Z}^d$, which, by (25), implies (24). For all $x \in \mathbb{R}$ and subsets $A \subseteq K_n$, a trivial generalization of Leadbetter et al. (1983, Corollary 4.2.4) now gives

$$\left|\mathbb{P}\big(\max_{v\in A}\xi_v \le x\big) - \Phi(x)^{|A|}\right| \le K|K_n| \sum_{0\neq v\in K_n}|r_v|\exp\Big(-\frac{x^2}{1+|r_v|}\Big), \tag{26}$$

where K is an appropriate constant, and Φ denotes the standard normal distribution function, i.e. the distribution of ξ_v. If furthermore $|K_n|(1-\Phi(x_n))$ is bounded, the right-hand side of (26) tends to 0 as $n \to \infty$: Let $\delta = \sup_{v\neq 0}|r_v|$ and choose $0 < \alpha < (1-\delta)/(1+\delta)$. Splitting the sum into the two parts for which $|v| \le |K_n|^{\alpha/d}$ and $|v| > |K_n|^{\alpha/d}$, respectively, the result follows by the same arguments as in the proof of Leadbetter et al. (1983, Lemma 4.3.2), realizing that

$$\log|K_n| \sup_{|v|>|K_n|^{\alpha/d}}|r_v| = \tfrac{d}{\alpha}\log|K_n|^{\alpha/d} \sup_{|v|>|K_n|^{\alpha/d}}|r_v| \to 0$$

as $n \to \infty$ by (23). It is now not difficult to see that $\mathscr{D}(x_n;K_n)$ and $\mathscr{D}'(x_n)$ are satisfied.

Suppose that $|K_n|(1-\Phi(x_n))$ is bounded, and hence $\mathscr{D}(x_n;K_n)$ and $\mathscr{D}'(x_n)$ are satisfied. Then the claim follows for all $0 \le \tau < \infty$ from Theorem 5. Now suppose that $|K_n|(1-\Phi(x_n))$ is unbounded. Let $\tau' < \infty$ and define the sequence (s_n) such that $|K_n|(1-\Phi(s_n)) = \tau'$. Then, from the considerations above, the conditions $\mathscr{D}(s_n;K_n)$ and $\mathscr{D}'(s_n)$ are satisfied, and the claim follows for $\tau = \infty$ from Corollary 2.

Theorem 6 *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of sets satisfying Assumption 1, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field. Assume that there are sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n > 0$ and $(b_n)_{n\in\mathbb{N}}$ such that $\mathscr{D}(a_nx+b_n;K_n)$ and $\mathscr{D}'(a_nx+b_n)$ are satisfied for all $x \in \mathbb{R}$. Assume furthermore that $|D_{n+1}|/|D_n| \to 1$ as $n \to \infty$. If there exists a non–degenerate distribution function G such that for all $x \in \mathbb{R}$*

$$\mathbb{P}\big(\max_{v\in D_n}\xi_v \le a_nx+b_n\big) \to G(x),$$

then G is the distribution function of an extreme value distribution.

The assumption $|D_{n+1}|/|D_n| \to 1$ ensures that the growth of the (D_n) is not too explosive. It will in particular be satisfied under the assumption given by (16). It will also be fulfilled if e.g. $C_n = r_nC$ for a p–convex set C and a sequence (r_n), where $r_{n+1}/r_n \to 1$.

Proof Let again F_ξ denote the common distribution function of $(\xi_v)_{v\in\mathbb{Z}^d}$. By Theorem 5 we have

$$|D_n|\overline{F}_\xi(a_nx+b_n) \to -\log G(x) \tag{27}$$

for all $x \in \mathbb{R}$. Now define the two sequences $(k_m)_{m\in\mathbb{N}}$ and $(\ell_m)_{m\in\mathbb{N}}$ by

$$k_m = \max\{|D_n| : |D_n| < m\} \qquad \text{and} \qquad \ell_m = \min\{|D_n| : |D_n| \geq m\}.$$

Then $k_m < m \leq \ell_m$ and by assumption $k_m/\ell_m \to 1$ as $m \to \infty$. We find

$$\frac{k_m}{\ell_m}\ell_m \overline{F}_\xi(a_{\ell_m}x + b_{\ell_m}) < m\overline{F}_\xi(a_{\ell_m}x + b_{\ell_m}) \leq \ell_m \overline{F}_\xi(a_{\ell_m}x + b_{\ell_m}).$$

Define $a'_m = a_{\ell_m}$ and $b'_m = b_{\ell_m}$ and let $m \to \infty$. Using the limit (27) then gives

$$m\overline{F}_\xi(a'_m x + b'_m) \to -\log G(x)$$

for all $x \in \mathbb{R}$. Let $(\xi'_m)_{m\in\mathbb{N}}$ be an independent and identically distributed sequence with common distribution function F_ξ and define $M_m = \max\{\xi'_1, \ldots, \xi'_m\}$. Then, by Leadbetter et al. (1983, Theorem 1.5.1) (which is in fact Theorem 1 from the introduction), we find

$$\mathbb{P}\big(M_m \leq a'_m x + b'_m\big) \to G(x)$$

for all $x \in \mathbb{R}$, showing by the classical Extremal Types Theorem (Leadbetter et al., 1983, Theorem 1.4.2) that G is indeed an extreme value distribution.

4 Extreme value theory for stationary Lévy-driven random fields

In this section we consider a stationary random field $(X_v)_{v\in\mathbb{R}^d}$, given as an integral of a kernel function with respect to a Lévy basis, and we wish to characterize the tail behavior of $\sup_{v\in C_n} X_v$, where $(C_n)_{n\in\mathbb{N}}$ is a sequence of index sets in $\mathbb{R}^d$ satisfying Assumption 1.

We define a Lévy basis to be an infinitely divisible and independently scattered random measure. The random measure Λ on $\mathbb{R}^d$ is independently scattered if for all disjoint Borel sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^d$ the random variables $(\Lambda(A_n))_{n\in\mathbb{N}}$ are independent and furthermore satisfy $\Lambda(\cup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \Lambda(A_n)$. The random measure Λ is infinitely divisible if $\Lambda(A)$ is infinitely divisible for all Borel sets $A \subseteq \mathbb{R}^d$.

Moreover, in this paper we assume Λ to be a stationary and isotropic Lévy basis on $\mathbb{R}^d$. With $C(\lambda \dagger Y) = \log \mathbb{E}e^{i\lambda Y}$ denoting the cumulant function for a random variable Y, this means that the random variable $\Lambda(A)$ has Lévy-Khintchine representation

$$C(\lambda \dagger \Lambda(A)) = i\lambda a|A| - \frac{1}{2}\lambda^2\theta|A| + \int_{A\times\mathbb{R}} \big(e^{i\lambda x} - 1 - i\lambda x 1_{[-1,1]}(x)\big) F(\mathrm{d}u, \mathrm{d}x).$$

Here $a \in \mathbb{R}$, $\theta \geq 0$ and F is the product measure $m \otimes \rho$ of the Lebesgue measure m and a Lévy measure ρ.

We assume that the Lévy measure ρ is convolution equivalent with index $\beta > 0$, and we write $\rho \in \mathscr{S}_\beta$. Convolution equivalence is a property of the right tail of a finite measure, and thus we may equivalently define it through the restriction ρ_1 of ρ to the set $(1, \infty)$. Note that $\mathscr{S}_\beta$ usually denotes the class of convolution equivalent distributions, however, we say that ρ is in $\mathscr{S}_\beta$ if its $(1,\infty)$-restriction $\rho_1 \in \mathscr{L}_\beta$, the class of finite measures with an exponential right tail with index β, i.e.

$$\frac{\rho_1((x-y,\infty))}{\rho_1((x,\infty))} \to e^{\beta y} \qquad \text{as } x \to \infty, \tag{28}$$

for all $y \in \mathbb{R}$, and if it furthermore satisfies the convolution property

$$\frac{(\tilde{\rho}_1 * \tilde{\rho}_1)((x,\infty))}{\tilde{\rho}_1((x,\infty))} \to 2\int_{\mathbb{R}} \mathrm{e}^{\beta y}\tilde{\rho}_1(\mathrm{d}y) < \infty \qquad \text{as}\, x \to \infty,$$

where $*$ denotes convolution and $\tilde{\rho}_1$ is the normalization of ρ_1. Moreover, we note that ρ_1 (equivalently ρ) lies in the *maximum domain of attraction* of the Gumbel distribution, by which we mean that there are norming constants $\tilde{a}_n > 0$ and $\tilde{b}_n \in \mathbb{R}$ such that

$$n\tilde{\rho}_1((\tilde{a}_n x + \tilde{b}_n, \infty)) \to \mathrm{e}^{-x}, \qquad (n \to \infty),$$

for all $x \in \mathbb{R}$. This is seen by Embrechts et al. (1997, Theorem 3.3.27) and (28) choosing the function $\tilde{a}(\,\cdot\,)$ (as in the formulation of the theorem) constantly equal to $1/\beta$. The norming constants can be seen to satisfy

$$\tilde{a}_n \to \beta^{-1}, \qquad \text{and} \qquad \tilde{b}_n \to \infty$$

as $n \to \infty$, where we refer to Embrechts et al. (1997, Chapter 3) for a description on the extreme value distributions, their maximum domains of attraction, and the associated norming constants. For convenience, we collect the assumptions on Λ and ρ in the following. Assumption 2 is assumed to be satisfied in the remainder of this paper.

Assumption 2 *The Lévy basis Λ on $\mathbb{R}^d$ is stationary and isotropic with a Lévy measure $\rho \in \mathscr{S}_\beta$ where $\beta > 0$. Moreover, ρ satisfies*

$$\int_{|y|>1} |y|^k \rho(\mathrm{d}y) < \infty \qquad \forall k \in \mathbb{N}. \tag{29}$$

The integrability of ρ along its right tail is already given from the fact that $\rho \in \mathscr{S}_\beta$, and, since ρ is a Lévy measure, it also satisfies $\int_{[-1,1]} y^2 \rho(\mathrm{d}y) < \infty$. The additional integrability along the left tail of ρ is sufficient to ensure continuous sample paths of the field; see Stehr and Rønn-Nielsen (2021, Theorem 5.1). This will be used repeatedly throughout the paper. Moreover, by Sato (1999, Theorem 25.3), the integrability $\mathbb{E}|\Lambda(A)|^k < \infty$ with $|A| > 0$ is equivalent to (29).

We consider the Lévy-driven field $X = (X_v)_{v \in \mathbb{R}^d}$ defined by

$$X_v = \int_{\mathbb{R}^d} f(|v-u|)\Lambda(\mathrm{d}u), \tag{30}$$

which, by Rajput and Rosinski (1989, Theorem 2.7), is well-defined if only the Lévy measure satisfies (29) for $k = 1$, and if the integration kernel $f : [0,\infty) \to [0,\infty)$ is bounded and satisfies $\int_{\mathbb{R}^d} f(|u|)\mathrm{d}u < \infty$. However, below we make a set of stronger assumptions on f which are assumed satisfied throughout this paper.

Note that we require the kernel function to be non-negative. A kernel function attaining negative values is also possible, but this requires extra conditions on the left tail of the Lévy measure ensuring that the right tail determines the extremal behavior.

Combined with Assumption 2 on the basis, the kernel assumption below guarantees the existence of a continuous version of $(X_v)_{v \in \mathbb{R}^d}$, and furthermore they give a sufficient mixing structure of the field.

Assumption 3 *The integration kernel* $f : [0,\infty) \to [0,\infty)$ *satisfying*

$$f(0) = 1, \qquad f(x) < 1 \textit{ for } x > 0, \tag{31}$$

is bounded from above by a decreasing function g such that

$$\int_{\mathbb{R}^d} g(|u|)\mathrm{d}u < \infty. \tag{32}$$

Moreover, the kernel f is Lipschitz continuous, that is, there is a constant C_L *such that*

$$|f(x_1) - f(x_2)| \leq C_L|x_1 - x_2|$$

for all $x_1, x_2 \geq 0$.

We remark that the integrability of g and the fact that it is decreasing in particular implies that

$$\int_{\mathbb{R}^d} \sup_{v\in[0,1]^d} g(|v-u|)\mathrm{d}u < \infty.$$

This will be used when referring to the results of Rønn-Nielsen and Jensen (2016) below.

Before proceeding we present a few examples of Lévy bases and kernel functions satisfying our assumptions.

Example 2 For the Lévy basis to satisfy Assumption 2 it suffices that it is infinitely divisible with a Lévy measure ρ having a left tail with finite moments of any order, and a right tail

$$\rho((x,\infty)) \sim ax^{-\delta}\mathrm{e}^{-\beta x} \qquad (x \to \infty)$$

for some $a > 0$, $\delta > 1$ and $\beta > 0$. This is seen by Pakes (2004, Lemma 2.3) which gives an even more general representation of convolution equivalent measures. Examples of infinitely divisible bases satisfying all of the above include the inverse Gaussian basis with distribution

$$\Lambda(A) \sim \mathrm{IG}(\eta|A|, \beta), \qquad \eta, \beta > 0,$$

and the normal inverse Gaussian basis with distribution

$$\Lambda(A) \sim \mathrm{NIG}(\beta+\alpha, \alpha, \mu|A|, \eta|A|), \qquad |\alpha| < \beta+\alpha,\ \mu \in \mathbb{R},\ \eta > 0.$$

We refer to Rønn-Nielsen and Jensen (2016) for details.

Example 3 Assumption 3 provides much flexibility on the kernel function. For instance, quickly decreasing functions such as the exponential kernel $f(x) = \mathrm{e}^{-\sigma x}$, $\sigma > 0$, and the Gaussian kernel $f(x) = \mathrm{e}^{-\sigma x^2}$, $\sigma > 0$, both satisfy Assumption 3 choosing $g = f$. The assumption also allows for functions with a much slower decrease rate: If f is a Lipschitz continuous integration kernel satisfying (31), and if the decreasing upper bound g is a regularly varying function of index $-(d+\varepsilon)$, $\varepsilon > 0$, then (32) is satisfied; see Embrechts et al. (1997, Appendix A3) for the definition and examples of regularly varying functions. This includes the simple case where $g(x) = c(1+x)^{-(d+\varepsilon)}$ is a power function of order $-(d+\varepsilon)$.

We write $\Lambda = \Lambda_1 + \Lambda_2$ as the independent sum of two Lévy bases with Lévy–Khintchine representations

$$C(\lambda \dagger \Lambda_1(A)) = \int_{A\times\mathbb{R}} \big(\mathrm{e}^{i\lambda x} - 1\big) F_1(\mathrm{d}u, \mathrm{d}x)$$
$$C(\lambda \dagger \Lambda_2(A)) = i\lambda a|A| - \frac{1}{2}\lambda^2\theta|A| + \int_{A\times\mathbb{R}} \big(\mathrm{e}^{i\lambda x} - 1 - i\lambda x 1_{[-1,1]}(x)\big) F_2(\mathrm{d}u, \mathrm{d}x),$$

respectively. Here, $F_i = m \otimes \rho_i$, where ρ_1 and ρ_2 are the restrictions of ρ to $(1,\infty)$ and $(-\infty, 1]$, respectively. Similarly, we decompose the field $(X_v)_v$ into a sum of two independent random fields $X_v = Z_v + Y_v$ for all $v \in \mathbb{R}^d$, where

$$Z_v = \int_{\mathbb{R}^d} f(|v-u|)\Lambda_1(\mathrm{d}u)$$

and

$$Y_v = \int_{\mathbb{R}^d} f(|v-u|)\Lambda_2(\mathrm{d}u) \tag{33}$$

are both stationary. By Stehr and Rønn-Nielsen (2021, Theorem 5.1) all three fields have continuous versions on compact sets.

In Rønn-Nielsen and Jensen (2016, Theorem 4.2) an equivalence between the tail of the supremum of a field defined as (30) and its associated convolution equivalent Lévy measure is given. The result relies on a set of assumptions on the Lévy basis and the kernel function, which differ slightly from Assumptions 2 and 3 given here. However, by Stehr and Rønn-Nielsen (2021, Theorem 5.1) and Assumptions 2 and 3, the field $(X_v)_{v\in\mathbb{R}^d}$ is continuous on compact sets, and it can easily be seen that the results of Rønn-Nielsen and Jensen (2016) are valid. In particular, if $C \subseteq [0,1]^d$ (or a translation hereof) then

$$\frac{\mathbb{P}(\sup_{v\in C} X_v > x)}{\rho((x,\infty))} \to |C|\mathbb{E}\mathrm{e}^{\beta X_u}, \qquad (x\to\infty), \tag{34}$$

and

$$\frac{\mathbb{P}(\sup_{v\in C} Z_v > x)}{\rho((x,\infty))} \to |C|\mathbb{E}\mathrm{e}^{\beta Z_u}, \qquad (x\to\infty), \tag{35}$$

where $u \in \mathbb{R}^d$ is arbitrarily chosen.

In our arguments below we need to find a lower bound of Z_v, which (as a field) is approximately independent for large lag, and which has essentially the same limiting behavior as Z_v. Since $f \geq 0$ and $\Lambda_1 \geq 0$, the field $Z_v^{(t)}$ defined by

$$Z_v^{(t)} = \int_{\{|v-u|\leq t\}} f(|v-u|)\Lambda_1(\mathrm{d}u)$$

therefore bounds Z_v from below for all $t > 0$. Also, $Z_v^{(t)}$ and $Z_{v'}^{(t)}$ are independent for all $v, v' \in \mathbb{R}^d$ satisfying $|v - v'| > 2t$. An equivalence similar to (35) for the field $(Z_v^{(t)})$ cannot directly be concluded from the results of Rønn-Nielsen and Jensen (2016) as the kernel in their paper is required to be continuous. However, following their arguments, it can be seen that the equivalence holds if only $t \geq \sqrt{d}$, ensuring that any two points of $C \subseteq [0,1]^d$ are at most a distance t apart. Hence, for such t,

$$\frac{\mathbb{P}(\sup_{v\in C} Z_v^{(t)} > x)}{\rho((x,\infty))} \to |C|\mathbb{E}\mathrm{e}^{\beta Z_u^{(t)}}, \qquad (x\to\infty), \tag{36}$$

where $u \in \mathbb{R}^d$ is arbitrarily chosen.

For the remainder of the paper, we assume that $(C_n)_{n\in\mathbb{N}}$ satisfies Assumption 1.

As mentioned, the conditions $\mathscr{D}$ and $\mathscr{D}'$ do not show easily when C_n is properly discretized, and we therefore continue by studying the extremal behavior of the semi-deterministic field $(Z_v + y_v)_{v\in\mathbb{R}^d}$, where $(y_v)_v$ is seen as a realization of the field $(Y_v)_v$. Having characterized the extremal behavior of $(Z_v + y_v)_v$, we conclude the behavior of $(X_v)_v$ by an independence argument. To discretize, we need the notion of a unit-cube $C(z)$ in $\mathbb{R}^d$: For a point $z = (z_1,\dots,z_d) \in \mathbb{Z}^d$, we let $C(z) \subseteq \mathbb{R}^d$ denote the closed unit-cube given by

$$C(z) = \bigtimes_{j=1}^{d} [z_j, z_j + 1],$$

and we say that the *corner* $z \in \mathbb{Z}^d$ has (associated) *unit-cube* $C(z)$, and vice versa. We note that

$$\bigcup_{z\in D_{n,k}^-} C(z) \subseteq C_n \subseteq \bigcup_{z\in D_{n,k}^+} C(z)$$

Thus,

$$\max_{z\in D_{n,k}^-} \sup_{u\in C(z)} (Z_u + y_u) \le \sup_{v\in C_n} (Z_v + y_v) \le \max_{z\in D_{n,k}^+} \sup_{u\in C(z)} (Z_u + y_u). \tag{37}$$

Before proceeding, we introduce some notation which will be convenient in the formulation and proof of the results of this section. These should be read in the context described above, however, the main result, Theorem 7, is self-contained. For any discrete set $A \subseteq \mathbb{Z}^d$, we let $M_y(A)$ and $M_y^{(t)}(A)$ be the supremas over the union of (continuous) unit-cubes with corners in A,

$$M_y(A) = \max_{z\in A} \sup_{u\in C(z)} (Z_u + y_u), \qquad \text{and} \qquad M_y^{(t)}(A) = \max_{z\in A} \sup_{u\in C(z)} (Z_u^{(t)} + y_u).$$

Hence, with this notation, (37) translates to

$$M_y(D_{n,k}^-) \le \sup_{v\in C_n} (Z_v + y_v) \le M_y(D_{n,k}^+). \tag{38}$$

From now on, we let $(x_n)_{n\in\mathbb{N}}$ be a real sequence given by

$$x_n = a_n x + b_n, \qquad x \in \mathbb{R},$$

where a_n, b_n denotes the norming constants relative to $|C_n|$, i.e

$$|C_n|\tilde{\rho}_1((a_n x + b_n, \infty)) \to \mathrm{e}^{-x}$$

for all $x \in \mathbb{R}$ as $n \to \infty$. Then $x_n \to \infty$ as $n \to \infty$, and consequently we conclude from (34)–(36) that

$$|C_n|\mathbb{P}(\sup_{u\in C(v)} X_u > x_n) \to \mathrm{e}^{-x}\rho((1,\infty))\mathbb{E}\mathrm{e}^{\beta X_0}$$

$$|C_n|\mathbb{P}(\sup_{u\in C(v)} Z_u > x_n) \to \mathrm{e}^{-x}\rho((1,\infty))\mathbb{E}\mathrm{e}^{\beta Z_0}, \tag{39}$$

$$|C_n|\mathbb{P}(\sup_{u\in C(v)} Z_u^{(t)} > x_n) \to \mathrm{e}^{-x}\rho((1,\infty))\mathbb{E}\mathrm{e}^{\beta Z_0^{(t)}} \tag{40}$$

as $n \to \infty$. For each fixed $x \in \mathbb{R}$ we let for notational convenience τ and $\tau^{(t)}$ be defined by

$$\tau = \mathrm{e}^{-x}\rho((1,\infty))\mathbb{E}\mathrm{e}^{\beta X_0}, \qquad \text{and} \qquad \tau^{(t)} = \mathrm{e}^{-x}\rho((1,\infty))\mathbb{E}\mathrm{e}^{\beta(Z_0^{(t)}+Y_0)},$$

where $t > 0$. Note that $\tau^{(t)} \to \tau$ as $t \to \infty$ by monotone convergence.

For the results below, it is important that the tails of $(Z_v + y_v)_v$ and $(Z_v^{(t)} + y_v)_v$ behave essentially like those of the stationary fields $(X_v)_v$ and $(Z_v^{(t)} + Y_v)_v$, respectively. The following lemma will be shown in Section 6.

Lemma 5 *Let $(Z_v)_v$, $(Z_v^{(t)})_v$ and $(Y_v)_v$ be given as above. Then, for almost all realizations $(y_v)_v$ of $(Y_v)_v$, it holds for all $z \in N_k$ that*

$$\frac{|C_n|}{|J_z|}\sum_{v\in J_z}\mathbb{P}\big(\sup_{u\in C(v)}(Z_u + y_u) > x_n\big) \to \tau \tag{41}$$

and

$$\frac{|C_n|}{|J_z|}\sum_{v\in J_z}\mathbb{P}\big(\sup_{u\in C(v)}(Z_u^{(t)} + y_u) > x_n\big) \to \tau^{(t)}. \tag{42}$$

The result also holds true if J_z is replaced with a subset of J_z in the shape of a box, which increases in size asymptotically as J_z.

The following lemma, which will be proved in Section 6, gives that a conditional version of $\mathscr{D}(x_n;K_n)$ is satisfied for the field.

Lemma 6 *Let $(Z_v)_v$ and $(Y_v)_v$ be given as above. There is a sequence $\gamma_n = o(\sqrt[d]{|C_n|})$ such that for all γ_n-separated sets $A, B \subseteq K_n$, where at least one is a box, it holds that*

$$\big|\mathbb{P}(M_y(A\cup B) \le x_n) - \mathbb{P}(M_y(A) \le x_n)\mathbb{P}(M_y(B) \le x_n)\big| \le \alpha_{y,n},$$

where $\alpha_{y,n} \to 0$ as $n \to \infty$ for almost all realizations $(y_v)_v$ of $(Y_v)_v$.

As in the stationary case, the following generalization follows by induction.

Lemma 7 *Let $(Z_v)_v$ and $(Y_v)_v$ be given as above, and let $(y_v)_v$ be a realization of $(Y_v)_v$. Let for $r \in \mathbb{N}$ the boxes $A_1,\ldots,A_r$ be pairwise γ_n-separated. Then*

$$\Big|\mathbb{P}\Big(\bigcap_{i=1}^{r}\{M_y(A_i) \le x_n\}\Big) - \prod_{i=1}^{r}\mathbb{P}(M_y(A_i) \le x_n)\Big| \le (r-1)\alpha_{y,n}.$$

We state the following lemma without proof, as it follows by the exact arguments as Lemma 3 taking the lack of stationarity of the discretely indexed field $(\sup_{u\in C(v)}(Z_u + y_u))_{v\in\mathbb{Z}^d}$ into account.

Lemma 8 *Let $(Z_v)_v$ and $(Y_v)_v$ be given as above, and let $(y_v)_v$ be a realization of $(Y_v)_v$. Then it holds that*

$$\begin{aligned}&\Big|\mathbb{P}(M_y(D_{n,k}^-) \le x_n) - \prod_{z\in P_{n,k}}\mathbb{P}(M_y(J_z) \le x_n)\Big| \\ \le\,& 2\sum_{z\in P_{n,k}}\mathbb{P}(M_y(H_z) \le x_n < M_y(H_z^*)) + (p_{n,k}-1)\alpha_{y,n},\end{aligned} \tag{43}$$

and similarly

$$\begin{aligned}&\Big|\mathbb{P}(M_y(D_{n,k}^+) \le x_n) - \prod_{z\in Q_{n,k}}\mathbb{P}(M_y(J_z) \le x_n)\Big| \\ \le\,& 2\sum_{z\in Q_{n,k}}\mathbb{P}(M_y(H_z) \le x_n < M_y(H_z^*)) + (q_{n,k}-1)\alpha_{y,n}.\end{aligned} \tag{44}$$

For the lemma below we recall the set N_k defined in Theorem 3(iii). Note that it is independent of n. It constitutes a set of indices $z \in \mathbb{Z}^d$ for which the union of corresponding boxes $J_z = J_z^{n,k}$ contains $D_{n,k}^+$ for all n.

Lemma 9 *Let $(Z_v)_v$ and $(Y_v)_v$ be given as above. Then, for almost all realizations $(y_v)_v$ of $(Y_v)_v$, the following is satisfied*

$$\begin{aligned}\Big(\liminf_{n\to\infty}\min_{z\in N_k}\mathbb{P}(M_y(J_z)\le x_n)\Big)^{\tilde{q}_k} &\le \liminf_{n\to\infty}\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\ &\le \limsup_{n\to\infty}\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\ &\le \Big(\limsup_{n\to\infty}\max_{z\in N_k}\mathbb{P}(M_y(J_z)\le x_n)\Big)^{\tilde{p}_k},\end{aligned}\tag{45}$$

where $\tilde{p}_k = \liminf_n p_{n,k}$ and $\tilde{q}_k = \limsup_n q_{n,k}$.

Proof Let $R_{n,k}^p \ge 0$ and $R_{n,k}^q \ge 0$ denote the upper bounds in (43) and (44), respectively. For all $z \in N_k$, we have

$$\begin{aligned}\mathbb{P}(M_y(H_z)\le x_n < M_y(H_z^*)) &\le \mathbb{P}(M_y(H_z^*)>x_n)\\ &\le \sum_{j=1}^d \mathbb{P}(M_y(L_{z,j}^*)>x_n)\\ &\le \sum_{j=1}^d\sum_{v\in L_{z,j}^*}\mathbb{P}\big(\sup_{u\in C(v)}(Z_u+y_u)>x_n\big),\end{aligned}$$

where we recall that $|L_{z,j}^*| \sim t_{n,k}^{d-1}\gamma_n$ as $n\to\infty$. We then find that $|L_{z,j}^*| = o(|J_z|)$ as $n\to\infty$ for all $j=1,\dots,d$, and in particular $J_z\setminus L_{z,j}^*$ is a box, which increases in size asymptotically as J_z. Since the limit in (41) is finite and $|J_z|, |J_z\setminus L_{z,j}^*|$ and $|C_n|$ are asymptotically of the same order, we conclude by Lemma 5 that

$$\begin{aligned}\mathbb{P}(M_y(H_z)\le x_n < M_y(H_z^*)) \le &\sum_{j=1}^d\sum_{v\in J_z}\mathbb{P}\big(\sup_{u\in C(v)}(Z_u+y_u)>x_n\big)\\ &-\sum_{j=1}^d\sum_{v\in J_z\setminus L_{z,j}^*}\mathbb{P}\big(\sup_{u\in C(v)}(Z_u+y_u)>x_n\big)\\ \to\ &0\end{aligned}$$

almost surely for all $z\in N_k$. By Lemma 6 it now follows that

$$\lim_{n\to\infty}R_{n,k}^p = \lim_{n\to\infty}R_{n,k}^q = 0$$

almost surely. Turning to (38) and using Lemma 8 show that

$$\begin{aligned}\liminf_n\prod_{z\in Q_{n,k}}\mathbb{P}(M_y(J_z)\le x_n) &\le \liminf_n\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\ &\le \limsup_n\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\ &\le \limsup_n\prod_{z\in P_{n,k}}\mathbb{P}(M_y(J_z)\le x_n).\end{aligned}$$

Since the involved factors are probabilities and thus lie in the interval $[0,1]$, we easily obtain (45) as desired.

The following lemma shows that a conditional version of the anti–clustering condition $\mathscr{D}'(x_n)$ is satisfied. The proof is deferred to Section 6.

Lemma 10 *Let $(Z_v^{(t)})_v$ and $(Y_v)_v$ be given as above. Then there is a function h of order $h(k)=o(k^{-1})$ as $k\to\infty$ such that*

$$\limsup_{n\to\infty}\sum_{v<v'\in J_z}\mathbb{P}\big(\sup_{u\in C(v)}(Z_u^{(t)}+y_u)>x_n,\ \sup_{u\in C(v')}(Z_u^{(t)}+y_u)>x_n\big)\le h(k) \tag{46}$$

for almost all realizations $(y_v)_v$ of $(Y_v)_v$ and all t large enough.

The following theorem is the main result of the section and in the formulation we explicitly state the assumptions under which the limit holds.

Theorem 7 *Let $(X_v)_{v\in\mathbb{R}^d}$ be a Lévy-driven stationary field given by* (30) *where the Lévy basis Λ satisfies Assumption 2 and the kernel function f satisfies Assumption 3. Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathbb{R}^d$ satisfying Assumption 1, and let a_n,b_n be the norming constants of the Lévy measure ρ relative to $|C_n|$, i.e.* $\lim_n|C_n|\rho((a_nx+b_n,\infty))=\mathrm{e}^{-x}\rho((1,\infty))$ *for all $x\in\mathbb{R}$. Then, as $n\to\infty$,*

$$\mathbb{P}\Big(a_n^{-1}\big(\sup_{v\in C_n}X_v-b_n\big)\le x\Big)\to\exp\Big(-\mathrm{e}^{-x}\mathbb{E}\mathrm{e}^{\beta X_u}\rho((1,\infty))\Big) \tag{47}$$

for all $x\in\mathbb{R}$, where $u\in\mathbb{R}^d$ is arbitrarily chosen.

Proof We use the same notation already used throughout this section. In particular, $x_n=a_nx+b_n$, $\tau=\mathrm{e}^{-x}\mathbb{E}\mathrm{e}^{\beta X_u}\rho((1,\infty))$ and $\tau^{(t)}=\mathrm{e}^{-x}\mathbb{E}\mathrm{e}^{\beta(Z_u^{(t)}+Y_u)}\rho((1,\infty))$, where $u\in\mathbb{R}^d$ is arbitrarily chosen due to stationarity. Moreover, $\tau^{(t)}\to\tau$ as $t\to\infty$.

Similarly as in the previous section, we can find upper and lower bounds to the probability $\mathbb{P}(M_y(J_z)\le x_n)$, now taking the lack of stationarity into account. Using these in combination with Lemma 9 and the fact that $\sup_{C(v)}(Z_u^{(t)}+y_u)\le\sup_{C(v)}(Z_u+y_u)$ imply that

$$\begin{aligned}
&\Big(\liminf_n\min_{z\in N_k}\Big(1-\sum_{v\in J_z}\mathbb{P}\big(\sup_{u\in C(v)}(Z_u+y_u)>x_n\big)\Big)\Big)^{\tilde q_k}\\
&\quad\le\liminf_n\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\
&\quad\le\limsup_n\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\
&\quad\le\Big(\limsup_n\max_{z\in N_k}\Big(1-\sum_{v\in J_z}\mathbb{P}\big(\sup_{u\in C(v)}(Z_u^{(t)}+y_u)>x_n\big)+S_{n,k}(z)\Big)\Big)^{\tilde p_k},
\end{aligned} \tag{48}$$

where

$$S_{n,k}(z)=\sum_{v<v'\in J_z}\mathbb{P}\big(\sup_{u\in C(v)}(Z_u^{(t)}+y_u)>x_n,\ \sup_{u\in C(v')}(Z_u^{(t)}+y_u)>x_n\big),$$

and $\tilde q_k=\limsup q_{n,k}$ and $\tilde p_k=\liminf_n p_{n,k}$. By Lemma 10, $\limsup_n S_{n,k}(z)=o(k^{-1})$ as $k\to\infty$ uniformly in z. Since $t_{n,k}^d\sim|C_n|/k$, we find by Lemma 5 and (48) that

$$\begin{aligned}
\Big(1-\frac{\tau}{k}\Big)^{\tilde q_k}&\le\liminf_n\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\
&\le\limsup_n\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\
&\le\Big(1-\frac{\tau^{(t)}}{k}+o(k^{-1})\Big)^{\tilde p_k}
\end{aligned}$$

almost surely. First taking the limit $k \to \infty$ combined with the fact that $\tilde{p}_k \sim \tilde{q}_k \sim k$, and secondly taking the limit $t \to \infty$ show

$$\lim_{n\to\infty} \mathbb{P}\big(\sup_{v\in C_n}(Z_v + y_v) \le x_n\big) = \exp(-\tau)$$

almost surely. Let π denote the distribution of the field $(Y_v)_v$. Then, by independence and dominated convergence,

$$\mathbb{P}\big(\sup_{v\in C_n} X_v \le x_n\big) = \int \mathbb{P}\big(\sup_{v\in C_n}(Z_v + y_v) \le x_n\big)\pi(\mathrm{d}y) \to \exp(-\tau)$$

as $n \to \infty$. This is exactly (47).

Going through the arguments leading up to Theorem 7 it is seen that the Lévy based form behind the field $(Y_v)_v$ is only used to obtain that it is independent of $(Z_v)_v$ and furthermore stationary, ergodic and satisfying the integrability result of Lemma 12. Therefore, Theorem 7 is immediately extended to

Theorem 8 *Let $(X_v)_{v\in\mathbb{R}^d}$ be a Lévy-driven stationary field given by* (30) *where the Lévy basis Λ satisfies Assumption 2 and the kernel function f satisfies Assumption 3. Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathbb{R}^d$ satisfying Assumption 1, and let a_n, b_n be the norming constants of the Lévy measure ρ relative to $|C_n|$, i.e. $\lim_n |C_n|\rho((a_n x + b_n, \infty)) = \mathrm{e}^{-x}\rho((1,\infty))$ for all $x \in \mathbb{R}$. Let furthermore $(\tilde{Y}_v)_v$ be a continuous, stationary and ergodic field independent of $(X_v)_v$ which satisfies*

$$\mathbb{E}\exp\big(\gamma \sup_{v\in C(0)} \tilde{Y}_v\big) < \infty$$

for some $\gamma > \beta$, where β is introduced in Assumption 2. Then, as $n \to \infty$,

$$\mathbb{P}\Big(a_n^{-1}\big(\sup_{v\in C_n}(X_v + \tilde{Y}_v) - b_n\big) \le x\Big) \to \exp\Big(-\mathrm{e}^{-x}\mathbb{E}\mathrm{e}^{\beta(X_u+\tilde{Y}_u)}\rho((1,\infty))\Big)$$

for all $x \in \mathbb{R}$, where $u \in \mathbb{R}^d$ is arbitrarily chosen.

5 Proofs related to the (Y_v)–field

We start this section by considering the tail of a distribution in $\mathscr{L}_\beta$, and we give a bound which will be useful in the proofs below. In the formulation of the result, y^+ is defined as $y^+ = \max\{y, 0\}$ for all $y \in \mathbb{R}$.

Lemma 11 *Let $G \in \mathscr{L}_\beta$ be a distribution with exponential right tail with index $\beta \ge 0$. Let $\overline{G}$ be its tail. For all $\gamma > \beta$ there is $x_0 \in \mathbb{R}$ and $\tilde{C} \in \mathbb{R}$ such that*

$$\overline{G}(x - y) \le \overline{G}(x)\tilde{C}\exp(\gamma y^+) \tag{49}$$

for all $x \ge x_0$ and $y \in \mathbb{R}$.

Proof Since $G \in \mathscr{L}_\beta$ it follows that

$$\overline{G}(x) = a(x)\exp\Big(-\int_0^x \beta(z)\mathrm{d}z\Big),$$

where $a(x) \to a > 0$ and $\beta(x) \to \beta$ as $x \to \infty$. This is due to Karamata's representation theorem (see e.g. Embrechts et al. (1997, Theorem A3.3)) and the fact that $\overline{G} \circ \log$ is a regularly varying function of index $-\beta$. Fix $\gamma > \beta$ and find x_0 such that

$$\beta(x) < \gamma, \qquad \text{and} \qquad |a(x) - a| \le a/3 \tag{50}$$

for all $x \ge x_0$.

Now consider only $x \ge x_0$ and $y \ge 0$. If $y \le x - x_0$ and hence $x - y \ge x_0$, we find from (50) above that

$$\begin{aligned}\overline{G}(x-y) &= \overline{G}(x)\frac{a(x-y)}{a(x)}\exp\Big(\int_{x-y}^x \beta(z)\mathrm{d}z\Big)\\ &\le \overline{G}(x)2\exp(\gamma y).\end{aligned}$$

If on the other hand $y > x - x_0$, we see that

$$\begin{aligned}\overline{G}(x-y) \le \frac{\overline{G}(x)}{\overline{G}(x)} &= \overline{G}(x)\Big[a(x)\exp\Big(-\int_0^{x_0} \beta(z)\mathrm{d}z\Big)\Big]^{-1}\exp\Big(\int_{x_0}^x \beta(z)\mathrm{d}z\Big)\\ &\le \overline{G}(x)\tilde{C}_0\exp(\gamma y),\end{aligned}$$

where the constant $\tilde{C}_0$ can be chosen as

$$\tilde{C}_0 = \Big[\frac{2a}{3}\exp\Big(-\int_0^{x_0} \beta(y)\mathrm{d}y\Big)\Big]^{-1}.$$

Choosing $\tilde{C} = \max\{2, \tilde{C}_0\}$ shows (49) for $y \ge 0$.

If $y < 0$ the claim (49) reads $\overline{G}(x-y) \le \overline{G}(x)\tilde{C}$, which is clearly true since $x \mapsto \overline{G}(x)$ is decreasing.

The following result will be used repeatedly in the subsequent proofs and helps ensuring that the field $(Y_v)_v$ has minor importance when determining the extremal behaviour of $(X_v)_v$.

Lemma 12 *Let the field* $(Y_v)_{v\in\mathbb{R}^d}$ *be given by* (33). *Then*

$$\mathbb{E}\exp\big(\gamma \sup_{v\in C(0)} Y_v\big) < \infty \tag{51}$$

for all $\gamma > 0$.

Note that the result in Lemma 12 is equivalent with $\mathbb{E}\exp\big(\gamma(\sup_{C(0)} Y_v)^+\big) < \infty$ for all $\gamma > 0$, which will be used specifically in Section 6.

Proof We can write Y_v as the independent decomposition $Y_v = \overline{Y}_v + Y_v^-$, where Y_v^- is the negative compound Poisson field given by the restriction of ρ to $(-\infty, -1)$, and $\overline{Y}_v$ is the high-activity field with Lévy measure $\rho_{[-1,1]}$ (the restriction to $[-1,1]$). Since (Y_v^-) is a non-positive field, it is clear that (51) in particular follows if

$$\mathbb{E}\exp\big(\gamma\sup_{v\in B}|\overline{Y}_v|\big) < \infty \tag{52}$$

for all $\gamma > 0$. By considerations as in Rønn-Nielsen and Jensen (2016, 2017) and Stehr and Rønn-Nielsen (2021), the countable field $(\overline{Y}_v)_{v\in\mathbb{Q}^d}$ is infinitely divisible with characteristic function as in Braverman and Samorodnitsky (1995, Eq. (1.1)), where its Lévy measure ν on $\mathbb{R}^{\mathbb{Q}^d}$ is given as follows: Define $H : \mathbb{R}^d \times [-1,1] \to \mathbb{R}^{\mathbb{Q}^d}$ as

$$H(u,x) = (xf(|v-u|))_{v\in\mathbb{Q}^d}.$$

With m denoting the Lebesgue measure, we let $\nu = (m \otimes \rho_{[-1,1]}) \circ H^{-1}$ be the image-measure of $m \otimes \rho_{[-1,1]}$. Since $v \mapsto Y_v$ is continuous on compact sets, we in particular have

$$\mathbb{P}(\sup_{v\in C(0)} |\overline{Y}_v| < \infty) = \mathbb{P}(\sup_{v\in C(0)\cap\mathbb{Q}^d} |\overline{Y}_v| < \infty) = 1.$$

Moreover, $\nu(\{z \in \mathbb{R}^{\mathbb{Q}^d} : \sup_{C(0)\cap\mathbb{Q}^d} |z_v| > 1\}) = 0$, and (52) now follows from Braverman and Samorodnitsky (1995, Lemma 2.1).

In the remainder of this section we establish some useful ergodic properties of the field $(Y_v)_v$. First, we recast some notation and a result from Krengel (1985). Let $(S, \mathscr{A}, \mu)$ be a probability space, and assume that $T_i : S \to S$ is a measurable map for $i = 1,\dots,d$ such that $T_1,\dots,T_d$ commute, i.e. $T_i \circ T_j = T_j \circ T_i$ for all i, j. Furthermore, assume for all $i = 1,\dots,d$ that

$$T_i(\mu) = \mu.$$

Let $\mathbb{V} = \mathbb{Z}_+^d$, and define for each $v = (v_1,\dots,v_d) \in \mathbb{V}$ the map $T_v : S \to S$ by

$$T_v = T_1^{v_1} T_2^{v_2} \cdots T_d^{v_d},$$

where e.g. $T_1^{v_1}$ means the composition of T_1 with itself v_1 times. Note that μ is also T_v–invariant for all $v \in \mathbb{V}$.

We define a subset $I \subseteq \mathbb{V}$ to be a box if it has the form

$$I = \Big(\bigtimes_{i=1}^{d} [u_i, v_i[\Big) \cap \mathbb{Z}^d,$$

for $u = (u_1,\dots,u_d), v = (v_1,\dots,v_d) \in \mathbb{V}$. The set of all such boxes in $\mathbb{V}$ will be denoted $\mathscr{I}$.

Definition 2 A sequence $I_1, I_2, \dots \subseteq \mathbb{V}$ is said to be regular if there exists an increasing sequence $I_1' \subseteq I_2' \subseteq \dots \in \mathscr{I}$ and $c < \infty$ such that $I_i \subseteq I_i'$ and $|I_i'| \le c|I_i|$ for each i.

Now we can formulate an ergodic theorem, which can be found as Theorem 6.2.8 in Krengel (1985). The theorem is followed by a few definitions and theorems also know from the classical ergodic theory.

Theorem 9 *Assume that $(S, \mathscr{A}, \mu)$ and $(T_i)_{i=1,\dots,d}$ satisfies the above. Let furthermore $g : S \to \mathbb{R}$ be measurable and μ–integrable, and assume that $I_1, I_2, \dots \in \mathscr{I}$ is a regular sequence. Then*

$$\frac{1}{|I_n|} \sum_{v\in I_n} g \circ T_v \to \mathbb{E}[g \mid \mathbb{I}]$$

μ–almost everywhere as $n \to \infty$, where $\mathbb{I}$ is the invariant σ–algebra, i.e. the σ–algebra consisting of all sets in $\mathscr{A}$ invariant to T_i for $i = 1,\dots,d$.

If the invariant σ–algebra $\mathbb{I}$ is trivial, we say that the family $(T_v)_{v\in\mathbb{V}}$ is ergodic.

Definition 3 The family $(T_v)_{v\in\mathbb{V}}$ is mixing if

$$\mu(F\cap T^{-v_n}(G))\to\mu(F)\mu(G) \tag{53}$$

for all $F,G\in\mathscr{A}$ and $(v_n)_{n\in\mathbb{N}}\subset\mathbb{V}$ with $|v_n|\to\infty$.

Theorem 10 *If $(T_v)_{v\in\mathbb{V}}$ is mixing, then it is ergodic.*

Proof Let $F\in\mathbb{I}$ and choose any sequence $(v_n)\subset\mathbb{V}$ with $|v_n|\to\infty$. Then $T^{-v_n}(F)=F$, so

$$\mu(F)=\mu(F\cap T^{-v_n}(F))\to\mu(F)^2$$

leading to F being a trivial set.

The following theorem is obtained by a standard extension argument.

Theorem 11 *For $(T_v)_{v\in\mathbb{V}}$ being mixing, it is sufficient that* (53) *is satisfied for all F and G in an intersection stable generating system for $\mathscr{A}$.*

We will apply the ergodic theory to the random field $Y=(Y_v)_{\mathbb{R}}$ defined in (33). Thus we let $S=C(\mathbb{R}^d)$ be the set of all continuous functions on $\mathbb{R}^d$, and let $\mathscr{A}$ be the corresponding σ–algebra generated by all coordinate projections. Finally, we let $\mu=Y(\mathbb{P})$. Each map $T_i:S\to S$ is defined as

$$T_i\big((x_{t_1,\ldots,t_d})_{(t_1,\ldots,t_d)\in\mathbb{R}^d}\big)=\big((x_{t_1,\ldots,t_{i-1},t_i+1,t_{i+1},\ldots,t_d})_{(t_1,\ldots,t_d)\in\mathbb{R}^d}\big),$$

which obviously commutes with T_j, i.e. $T_i\circ T_j=T_j\circ T_i$ for all $i,j=1,\ldots,d$. Moreover, by stationarity of $(Y_v)_v$, the maps satisfy $T_i(\mu)=\mu$ for all i.

Lemma 13 *Let $u_1,\ldots,u_p,v_1,\ldots,v_q\in\mathbb{R}^d$ and $(z_n)_{n\in\mathbb{N}}\subseteq\mathbb{V}$ with $|z_n|\to\infty$ be given. Then*

$$(Y_{u_1},\ldots,Y_{u_p},Y_{v_1+z_n},\ldots,Y_{v_q+z_n})\to(Y_{u_1},\ldots,Y_{u_p})(\mathbb{P})\otimes(Y_{v_1},\ldots,Y_{v_q})(\mathbb{P})$$

in distribution.

Proof For $\lambda_1,\ldots,\lambda_p,\beta_1,\ldots,\beta_q\in\mathbb{R}$ we show that

$$\begin{aligned}&\log\mathbb{E}\big[\mathrm{e}^{i(\lambda_1Y_{u_1}+\cdots+\lambda_pY_{u_p}+\beta_1Y_{v_1+z_n}+\cdots+\beta_qY_{v_q+z_n})}\big]\\&\qquad\to\log\mathbb{E}\big[\mathrm{e}^{i(\lambda_1Y_{u_1}+\cdots+\lambda_pY_{u_p})}\big]+\log\mathbb{E}\big[(\mathrm{e}^{i(\beta_1Y_{v_1}+\cdots+\beta_qY_{v_q})}\big]\end{aligned}$$

as $n\to\infty$. Defining $g_1(w)=\sum_{i=1}^p\lambda_if(|u_i-w|)$ and $g_2(w)=\sum_{i=1}^q f(|v_i-w|)$, and utilizing that $\int_{\mathbb{R}}y^2\rho_2(\mathrm{d}y)<\infty$, we can write

$$\begin{aligned}&\log\mathbb{E}\big[\mathrm{e}^{i(\lambda_1Y_{u_1}+\cdots+\lambda_pY_{u_p}+\beta_1Y_{v_1+z_n}+\cdots+\beta_qY_{v_q+z_n})}\big]\\&\quad=ia'\int_{\mathbb{R}^d}g_1(w)+g_2(w-z_n)\mathrm{d}w-\frac{1}{2}\theta^2\int_{\mathbb{R}^d}\big(g_1(w)+g_2(w-z_n)\big)^2\mathrm{d}w\\&\qquad+\int_{\mathbb{R}^d}\int_{\mathbb{R}}\mathrm{e}^{i(g_1(w)+g_2(w-z_n))x}-1-i(g_1(w)+g_2(w-z_n))x\rho_2(\mathrm{d}x)\,\mathrm{d}w\end{aligned} \tag{54}$$

for an appropriate constant a'. The proof is complete, when it is shown that the limit of (54) is

$$\sum_{j=1}^2\Big[ia'\int_{\mathbb{R}^d}g_j(w)\mathrm{d}w-\frac{1}{2}\theta^2\int_{\mathbb{R}^d}g_j(w)^2\mathrm{d}w+\int_{\mathbb{R}^d}\int_{\mathbb{R}}\mathrm{e}^{g_j(w)x}-1-ig_j(w)x\rho_2(\mathrm{d}x)\,\mathrm{d}w\Big].$$

For the first term in (54) we have the equality

$$\int_{\mathbb{R}^d} g_1(w) + g_2(w - z_n)\mathrm{d}w = \int_{\mathbb{R}^d} g_1(w)\mathrm{d}w + \int_{\mathbb{R}^d} g_2(w)\mathrm{d}w.$$

The integral in the second term of (54) equals

$$\int_{\mathbb{R}^d} g_1(w)^2\mathrm{d}w + \int_{\mathbb{R}^d} g_2(w)^2\mathrm{d}w + 2\int_{\mathbb{R}^d} g_1(w)g_2(w - z_n)\mathrm{d}w,$$

where the last term converges 0 due to dominated convergence, since g_1 is integrable, and $g_2(w - z_n)$ is bounded and has pointwise limit 0 by assumptions on the kernel f.

Finally, for the convergence of the third term in (54) we let $\varepsilon > 0$ be given and choose $D > 0$ such that for $i = 1,2$,

$$\int_{B(D)^c} g_i(w)^2\mathrm{d}w \cdot C < \varepsilon/4, \tag{55}$$

where $C = \int_{\mathbb{R}} x^2 \rho_2(\mathrm{d}x)$. With $h(w,x)$ denoting the integrand in the third term of (54), the integral, denoted $\mathscr{I}_n$, can for large n be rewritten as

$$\begin{aligned}\mathscr{I}_n = &\int_{B(D)}\int_{\mathbb{R}} h(w,x)\rho_2(\mathrm{d}x)\,\mathrm{d}w + \int_{B(D)}\int_{\mathbb{R}} h(w+z_n,x)\rho_2(\mathrm{d}x)\,\mathrm{d}w \\ &+ \int_{(B(D)\cup(B(D)+z_n))^c}\int_{\mathbb{R}} h(w,x)\rho_2(\mathrm{d}x)\,\mathrm{d}w.\end{aligned} \tag{56}$$

Using $|e^{ix} - 1 - ix| \leq x^2$ it is seen that the third term in (56) is bounded from above by

$$\begin{aligned}&C\cdot\Big(\int_{B(D)^c} g_1(w)^2\mathrm{d}w + \int_{(B(D)+z_n)^c} g_2(w-z_n)^2\mathrm{d}w + 2\int_{\mathbb{R}^d} g_1(w)g_2(w-z_n)\mathrm{d}w\Big) \\ &\quad\leq \varepsilon/2 + 2C\int_{\mathbb{R}^d} g_1(w)g_2(w-z_n)\mathrm{d}w,\end{aligned}$$

where the integral has limit 0 with an argument similarly as above.

The limit of the sum of the first two terms in (56) is

$$\begin{aligned}&\int_{B(D)}\int_{\mathbb{R}} \mathrm{e}^{ig_1(w)x} - 1 - ig_1(w)x\,\rho_2(\mathrm{d}x)\,\mathrm{d}w \\ &\quad + \int_{B(D)}\int_{\mathbb{R}} \mathrm{e}^{ig_2(w)x} - 1 - ig_2(w)x\,\rho_2(\mathrm{d}x)\,\mathrm{d}w\end{aligned}$$

due to dominated convergence. Collecting the limit results for the three terms of (56) and referring to (55) again, we find

$$\begin{aligned}\limsup_{n\to\infty}\Big|\mathscr{I}_n &- \int_{\mathbb{R}^d}\int_{\mathbb{R}} \mathrm{e}^{ig_1(w)x} - 1 - ig_1(w)x\,\rho_2(\mathrm{d}x)\,\mathrm{d}w \\ &- \int_{\mathbb{R}^d}\int_{\mathbb{R}} \mathrm{e}^{ig_2(w)x} - 1 - ig_2(w)x\,\rho_2(\mathrm{d}x)\,\mathrm{d}w\Big| \leq \varepsilon\,,\end{aligned}$$

from which the desired conclusion is obtained, since ε was chosen arbitrarily.

Theorem 12 *Let $(Y_v)_v$ be defined as above and let $(I_n)_n$ be a regular sequence of boxes in $\mathbb{Z}_+^d$. If $g : S \to \mathbb{R}$ satisfies $\mathbb{E}|g((Y_u)_u)| < \infty$ then*

$$\frac{1}{|I_n|}\sum_{v\in I_n} g\big((Y_{u+v})_u\big) \to \mathbb{E}g\big((Y_u)_u\big)$$

almost surely as $n \to \infty$.

Proof Let A be the set of continuity points for the distribution of, say, Y_0, which due to the stationarity is the set of continuity points for all Y_v. Note that A is dense in $\mathbb{R}$. Lemma 13 implies that for all $a_1,\dots,a_p, b_1,\dots,b_q \in A$,

$$\begin{aligned}&\mathbb{P}(Y_{u_1} \le a_1,\dots,Y_{u_p} \le a_p, Y_{v_1+z_n} \le b_1,\dots,Y_{v_q+z_n} \le b_p)\\ &\qquad \to \mathbb{P}(Y_{u_1} \le a_1,\dots,Y_{u_p} \le a_p)\mathbb{P}(Y_{v_1} \le b_1,\dots,Y_{v_q} \le b_p).\end{aligned}$$

Since sets on the form

$$\{y \in S : y_{u_1} \le a_1,\dots,y_{u_p} \le a_p\},$$

where $p \in \mathbb{N}_0$, $u_1,\dots,u_p$, $a_1,\dots,a_p \in A$ constitutes an intersection stable generating system for $\mathscr{A}$, we have from Theorems 9 to 11 that for any μ–integrable map $g : S \to \mathbb{R}$,

$$1 = \mu\Big(\frac{1}{|I_n|}\sum_{v\in I_n} g\circ T_v \to \int g\,\mathrm{d}\mu\Big) = \mathbb{P}\Big(\frac{1}{|I_n|}\sum_{v\in I_n} g(T_v(Y)) \to \mathbb{E}(g(Y))\Big),$$

This concludes the proof.

The following corollary adapts Theorem 12 into the specific setting that will be useful in the further arguments.

Corollary 4 *Let the field $(Y_v)_{v\in\mathbb{R}^d}$ be given by* (33)*, and let g be a function satisfying $\mathbb{E}|g((Y_u)_{u\in C(0)})| < \infty$. For all $z \in N_k$ it then holds that*

$$\frac{1}{|J_z|}\sum_{v\in J_z} g\big((Y_{u+v})_{u\in C(0)}\big) \to \mathbb{E}g\big((Y_u)_{u\in C(0)}\big)$$

almost surely as $n \to \infty$. The result also holds true if J_z is replaced with a subset of J_z in the shape of a box, which increases in size asymptotically as J_z.

Proof In principle, we can only apply Theorem 12 to the collection $J_z^{n,k}$ of sets contained in $\mathbb{Z}_+^d$. But by symmetry, the same result can be obtained for $J_z^{n,k}$ sets in all the 2^d other placements relative to 0.

Let $z \in N_k$ be fixed and consider $J_z = J_z^{n,k}$. Furthermore, let $K_{n,k} = \bigcup_{v\in N_k} J_v^{n,k}$ be the discrete cube defined in Theorem 3 (iii) (with $\lambda = 1$). By construction, $(K_{n,k})_n$ is an increasing sequence of cubes such that $J_z^{n,k} \subseteq K_{n,k}$ and $|K_{n,k}| = |N_k|\cdot|J_z^{n,k}|$. In particular, $(J_z^{n,k})_n$ is a regular sequence of boxes in $\mathbb{Z}^d$, and the first part of the result thus follows from Theorem 12.

Now let $L_z^n \subseteq J_z^n$ be a box such that $|J_z^n|/|L_z^n| \to c \in [1,\infty)$ as $n \to \infty$. Then, for sufficiently large n, the relation $|K_{n,k}| \le 2c\,|N_k|\cdot|L_z^n|$ holds, and $(L_z^n)_n$ is thus a regular sequence of boxes in $\mathbb{Z}^d$. The proof is completed by another application of Theorem 12.

6 Remaining proofs

In what follows, $C_r(v)$ is an r-cube with corner $v \in \mathbb{R}^d$, that is, a box in $\mathbb{R}^d$ with side-length equal to $r > 0$. Moreover, as up until now, we let $C(v) = C_1(v)$ denote the unit-cube with corner v.

Proof of Lemma 5 We only show the convergence (41) as (42) and the expressions for J_z replaced by an asymptotically size-equivalent box follow identically.

Let $L \in \mathbb{N}$ be fixed. For all $v \in \mathbb{Z}^d$ define $A_L(v)$ as the set of corners in a grid with distance $1/L$ for which the associated $1/L$-cubes are contained in $C(v)$, i.e.

$$A_L(v) = \{u \in (L^{-1}\mathbb{Z})^d \,:\, C_{1/L}(u) \subseteq C(v)\}.$$

With this construction it follows that

$$C(v) = \bigcup_{u \in A_L(v)} C_{1/L}(u).$$

For $v \in \mathbb{Z}^d$, define $y^*(v) = \sup_{u \in C(v)} y_u$. Similarly, for all $u \in (L^{-1}\mathbb{Z})^d$, define $y_L^*(u) = \sup_{s \in C_{1/L}(u)} y_s$, and $\overline{y}_L(u) = \inf_{s \in C_{1/L}(u)} y_s$. Let F_L denote the distribution of $\sup_{s \in C_{1/L}(u)} Z_s$, which, by stationarity, is independent of $v \in \mathbb{Z}^d$ and $u \in A_L(v)$. Then, from (39),

$$|C_n|\overline{F}_L(x_n) \to L^{-d}\mathrm{e}^{-x}\rho((1,\infty))\mathbb{E}\mathrm{e}^{\beta Z_0} \tag{57}$$

as $n \to \infty$, where $\overline{F}_L$ is the tail of F_L. Since $\rho \in \mathscr{L}_\beta$ the equivalence (35) implies that also $\overline{F}_L \in \mathscr{L}_\beta$. From Lemma 11 we conclude for any $\gamma > \beta$ the existence of a finite constant $\tilde{C}_L$ such that

$$\overline{F}_L(x_n - y) \le \overline{F}_L(x_n)\tilde{C}_L \exp(\gamma y^+), \qquad \text{for all} \quad y \in \mathbb{R}, \tag{58}$$

for n sufficiently large.

Writing the supremum over $C(v)$ as a maximum of supremas over $C_{1/L}(u)$ for $u \in A_L(v)$, it is not difficult to see that

$$\begin{aligned}\sum_{v \in J_z} \sum_{u \in A_L(v)} & \overline{F}_L(x_n - \overline{y}_L(u)) - \sum_{v \in J_z} S_L(v) \\ & \le \sum_{v \in J_z} \mathbb{P}\big(\sup_{u \in C(v)} (Z_u + y_u) > x_n\big) \\ & \le \sum_{v \in J_z} \sum_{u \in A_L(v)} \overline{F}_L(x_n - y_L^*(u)),\end{aligned} \tag{59}$$

where

$$S_L(v) = \sum_{u < u' \in A_L(v)} \mathbb{P}\big(\sup_{s \in C_{1/L}(u)} Z_s > x_n - y^*(v), \sup_{s \in C_{1/L}(u')} Z_s > x_n - y^*(v)\big).$$

First, we consider the upper bound in (59). Since $F_L \in \mathscr{L}_\beta$, we find that the convergence $\overline{F}_L(x_n - y)/\overline{F}_L(x_n) \to \exp(\beta y)$, $n \to \infty$, is uniform for $y \le K$ for all $K \in \mathbb{N}$; see e.g Pakes

(2004, Definition 2.1). Using Corollary 4 and this uniform convergence, (57) and (58) now imply for all fixed $K \in \mathbb{N}$ that

$$\begin{aligned}
&\frac{|C_n|}{|J_z|}\sum_{v\in J_z}\sum_{u\in A_L(v)}\overline{F}_L(x_n-y_L^*(u))\\
&\quad\le |C_n|\overline{F}_L(x_n)\frac{1}{|J_z|}\sum_{v\in J_z}\sum_{u\in A_L(v)}\Big(\frac{\overline{F}_L(x_n-y_L^*(u))}{\overline{F}_L(x_n)}1_{y^*(v)\le K}\\
&\qquad\qquad\qquad\qquad\qquad\qquad +\tilde{C}_L\exp(\gamma(y_L^*(u))^+)1_{y^*(v)>K}\Big)\\
&\quad\to \mathrm{e}^{-x}\rho((1,\infty))\mathbb{E}\mathrm{e}^{\beta Z_0}\mathbb{E}\Big[\exp\big(\beta\sup_{v\in C_{1/L}(0)}Y_v\big)1_{\sup_{v\in C_1(0)}Y_v\le K}\\
&\qquad\qquad\qquad\qquad\qquad\qquad +\tilde{C}_L\exp\big(\gamma(\sup_{v\in C_{1/L}(0)}Y_v)^+\big)1_{\sup_{v\in C_1(0)}Y_v>K}\Big]
\end{aligned}$$

almost surely as $n\to\infty$, where the stationarity of $(Y_v)_v$ has also been used. Since (51) holds for all $\gamma>0$ and $v\mapsto Y_v$ is continuous, letting $K\to\infty$ and then $L\to\infty$ show by dominated convergence that

$$\begin{aligned}
\limsup_{n\to\infty}\frac{|C_n|}{|J_z|}\sum_{v\in J_z}\mathbb{P}\big(\sup_{u\in C(v)}(Z_u+y_u)>x_n\big) &\le \mathrm{e}^{-x}\rho((1,\infty))\mathbb{E}\mathrm{e}^{\beta(Z_0+Y_0)}\\
&= \mathrm{e}^{-x}\rho((1,\infty))\mathbb{E}\mathrm{e}^{\beta X_0}=\tau
\end{aligned}$$

almost surely. Concerning the lower bound of (59), we find for all fixed $u\neq u'\in(L^{-1}\mathbb{Z})^d$ that

$$\frac{1}{\overline{F}_L(x_n)}\mathbb{P}\big(\sup_{s\in C_{1/L}(u)}Z_s>x_n-y,\ \sup_{s\in C_{1/L}(u')}Z_s>x_n-y\big)\to 0 \tag{60}$$

uniformly for $y\le K$, for all $K\in\mathbb{N}$. This is easily seen from (35) and the inclusion-exclusion principle, with the convergence being uniform due to $\overline{F}_L\in\mathscr{L}_\beta$. Turning to (58) and repeating the arguments above, we conclude for all $L\in\mathbb{N}$ that

$$\frac{|C_n|}{|J_z|}\sum_{v\in J_z}S_L(v)\to 0$$

almost surely as $n\to\infty$. For the former term of the lower bound, arguing as for the upper bound shows that

$$\liminf_{n\to\infty}\frac{|C_n|}{|J_z|}\sum_{v\in J_z}\mathbb{P}\big(\sup_{u\in C(v)}(Z_u+y_u)>x_n\big)\ge\tau$$

almost surely, which concludes the proof.

The following technical lemma will be useful when determining the sequence (γ_n) needed in Lemma 6. Although the lemma is formulated for a general function g, we repeat the notation from Assumption 3, since the function from that assumption will be our only application.

Lemma 14 *Let $g : [0,\infty) \to [0,\infty)$ be a decreasing function such that $\int_0^\infty g(x)x^{d-1}\mathrm{d}x < \infty$ for fixed $d \geq 1$. There exists an increasing function $\eta : [0,\infty) \to [0,\infty)$ with $\eta(z) \to \infty$ and $\eta(z) = o(z)$ as $z \to \infty$ such that*

$$z^{d-1}\int_{\eta(z)}^{\infty} g(x)\mathrm{d}x \to 0 \tag{61}$$

as $z \to \infty$.

Proof The result is trivially true with any choice of $\eta(z) \to \infty$ with $\eta(z) = o(z)$, if either $d = 1$ or there is x_0 such that $g(x) = 0$ for all $x \geq x_0$. So assume that $d > 1$ and that g is strictly positive. Define $\zeta : [0,\infty) \to [0,\infty)$ by

$$\zeta(z) = \frac{z}{\left(\sqrt{\int_z^\infty g(x)x^{d-1}\mathrm{d}x}\right)^{1/(d-1)}}$$

and note that ζ is a continuous, strictly increasing function with $\zeta(z) \to \infty$. In fact, since the denominator is decreasing to 0, it holds that

$$\frac{\zeta(z)}{z} \to \infty \tag{62}$$

as $z \to \infty$. Furthermore,

$$\zeta(z)^{d-1}\int_z^\infty g(x)\mathrm{d}x = \frac{z^{d-1}\int_z^\infty g(x)\mathrm{d}x}{\sqrt{\int_z^\infty g(x)x^{d-1}\mathrm{d}x}} \leq \sqrt{\int_z^\infty g(x)x^{d-1}\mathrm{d}x}, \tag{63}$$

which has limit 0 as $z \to \infty$ by the integrability of $g(x)x^{d-1}$. The desired result follows by defining $\eta(z) = \zeta^{-1}(z)$ and replacing z by $\eta(z)$ in (62) and (63).

Proof of Lemma 6 Let $\gamma_n = \eta(\sqrt[d]{|C_n|})$, where η is chosen according to Lemma 14 and g is given in Assumption 3. Let A and B be γ_n–separated sets as given in the lemma and define

$$\mathscr{A} = \bigcup_{v\in A} C(v) \qquad \text{and} \qquad \mathscr{B} = \bigcup_{v\in B} C(v) \qquad \text{and} \qquad \mathscr{K}_n = \bigcup_{v\in K_n} C(v).$$

Throughout the proof, we assume that A is a box, and thus $\mathscr{A}$ is a continuous box and in particular a convex body. Note that $\mathscr{A}$ and $\mathscr{B}$ are then also γ_n-separated (strictly speaking they are only $(\gamma_n - 1)$-separated, however, this is equivalent as $n \to \infty$). Recall that $B(r)$ denotes the closed ball in $\mathbb{R}^d$ of radius $r \geq 0$ with center in $0 \in \mathbb{R}^d$, and define $\mathscr{A}_n = \mathscr{A} \oplus B(\gamma_n/2)$ and $\mathscr{B}_n = \big(\mathscr{A} \oplus B(\gamma_n)\big)^c = \big(\mathscr{A}_n \oplus B(\gamma_n/2)\big)^c$, where $\mathscr{B} \subseteq \mathscr{B}_n$ since $\mathscr{A}$ and $\mathscr{B}$ are γ_n-separated, and $\mathscr{A}_n$ and $\mathscr{B}_n$ are disjoint. For all $v \in \mathscr{A}$ let

$$Z_v^A = \int_{\mathscr{A}_n} f(|v-u|)\Lambda_1(\mathrm{d}u), \qquad \text{and} \qquad \overline{Z}_v^A = \int_{\mathscr{A}_n^c} f(|v-u|)\Lambda_1(\mathrm{d}u).$$

Similarly, for all $v \in \mathscr{B}$,

$$Z_v^B = \int_{\mathscr{A}_n^c} f(|v-u|)\Lambda_1(\mathrm{d}u), \qquad \text{and} \qquad \overline{Z}_v^B = \int_{\mathscr{A}_n} f(|v-u|)\Lambda_1(\mathrm{d}u).$$

Since Λ_1 is a positive measure all Z_v^A, $\overline{Z}_v^A$, Z_v^B and $\overline{Z}_v^B$ are non–negative, and we have

$$\sup_{v\in\mathscr{A}} \overline{Z}_v^A \leq \int_{\mathscr{A}_n^c} \sup_{v\in\mathscr{A}} f(|v-u|)\Lambda_1(\mathrm{d}u) \tag{64}$$

and

$$\sup_{v\in\mathscr{B}} \overline{Z}_v^B \leq \int_{\mathscr{A}_n} \sup_{v\in\mathscr{B}} f(|v-u|)\Lambda_1(\mathrm{d}u)\,.$$

By the definition $\gamma_n/2 = \eta(\sqrt[d]{|C_n|})$ we have from (61) that we may choose a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \downarrow 0$ and

$$\frac{1}{\varepsilon_n}|C_n|^{(d-1)/d}\int_{\gamma_n/2}^{\infty} g(x)\mathrm{d}x \to 0 \tag{65}$$

as $n\to\infty$. Recall that g is a decreasing upper bound to f. Define the events S_n^A and S_n^B by

$$S_n^A = \big(\sup_{v\in\mathscr{A}} \overline{Z}_v^A \leq \varepsilon_n\big) \qquad \text{and} \qquad S_n^B = \big(\sup_{v\in\mathscr{B}} \overline{Z}_v^B \leq \varepsilon_n\big).$$

Let Λ_1' be the so-called spot variable of the Lévy basis Λ_1, that is, a random variable equivalent in distribution to $\Lambda_1(S)$ with $S\subseteq\mathbb{R}^d$ satisfying $|S|=1$. Note that Λ_1' has finite first moment since the underlying Lévy measure ρ_1 has so. As $\mathscr{A}$ is a convex body, we find by Markov's inequality, equation (64), and Lemma 1 (equation (6)) that

$$\begin{aligned}
\mathbb{P}\big((S_n^A)^c\big) &\leq \frac{1}{\varepsilon_n}\mathbb{E}(\Lambda_1')\int_{\mathscr{A}_n^c}\sup_{v\in\mathscr{A}} f(|v-u|)\mathrm{d}u \\
&\leq \frac{1}{\varepsilon_n}\mathbb{E}(\Lambda_1')\Big[\sum_{j=0}^{d-1}\mu_j V_j(\mathscr{A})\int_{\gamma_n/2}^{\infty} g(x)x^{d-j-1}\mathrm{d}x\Big] \\
&\leq \frac{1}{\varepsilon_n}\mathbb{E}(\Lambda_1')\Big[\sum_{j=0}^{d-1}\tilde{\mu}_j |C_n|^{j/d}\int_{\gamma_n/2}^{\infty} g(x)x^{d-j-1}\mathrm{d}x\Big],
\end{aligned} \tag{66}$$

for certain $\mathscr{A}$- and n-independent constants $\tilde{\mu}_j$, where we in the last inequality used the monotinicity and homogeneity of the instrinsic volumes to write

$$\mu_j V_j(\mathscr{A}) \leq \mu_j V_j(\mathscr{K}_n) \leq \tilde{\mu}_j |C_n|^{j/d}.$$

By similar arguments using Lemma 1 (equation (7)) we also find (up to a B- and n-independent constant) that

$$\mathbb{P}\big((S_n^B)^c\big) \leq \frac{1}{\varepsilon_n}\mathbb{E}(\Lambda_1')\Big[\sum_{j=0}^{d-1}\tilde{\mu}_j |C_n|^{j/d}\int_{\gamma_n/2}^{\infty} g(x)x^{d-j-1}\mathrm{d}x\Big]. \tag{67}$$

Similarly to the notation introduced in Section 4, we define

$$\begin{aligned}
M_y^A(A) &= \max_{v\in A}\sup_{u\in C(v)}(Z_u^A + y_u) = \sup_{v\in\mathscr{A}}(Z_v^A + y_v), \qquad \text{and} \\
M_y^B(B) &= \max_{v\in B}\sup_{u\in C(v)}(Z_u^B + y_u) = \sup_{v\in\mathscr{B}}(Z_v^B + y_v).
\end{aligned}$$

Utilizing that M_y^A and M_y^B are independent since $\mathscr{A}_n$ and $\mathscr{A}_n^c$ er disjoint, it can be seen from straightforward calculations that

$$\begin{aligned}
&\big|\mathbb{P}(M_y(A\cup B)\le x_n)-\mathbb{P}(M_y(A)\le x_n)\mathbb{P}(M_y(B)\le x_n)\big|\\
&\quad\le \mathbb{P}(M_y^A(A)\le x_n,S_n^A)\mathbb{P}(M_y^B(B)\le x_n,S_n^B)\\
&\qquad -\mathbb{P}(M_y^A(A)\le x_n-\varepsilon_n,S_n^A)\mathbb{P}(M_y^B(B)\le x_n-\varepsilon_n,S_n^B)\\
&\qquad +2\Big(\mathbb{P}\big((S_n^A)^c\big)+\mathbb{P}\big((S_n^B)^c\big)\Big)\\
&\quad\le \mathbb{P}(M_y^A(A)\le x_n,S_n^A)-\mathbb{P}(M_y^A(A)\le x_n-\varepsilon_n,S_n^A)\\
&\qquad +\mathbb{P}(M_y^B(B)\le x_n,S_n^B)-\mathbb{P}(M_y^B(B)\le x_n-\varepsilon_n,S_n^B)\\
&\qquad +2\Big(\mathbb{P}\big((S_n^A)^c\big)+\mathbb{P}\big((S_n^B)^c\big)\Big).
\end{aligned}\tag{68}$$

To obtain the desired conclusion it suffices to show that all three terms of (68) have upper bounds which are independent of A and B and which tend to 0 as $n\to\infty$. Concerning the first term we see that

$$\begin{aligned}
0&\le \mathbb{P}(M_y^A(A)\le x_n,S_n^A)-\mathbb{P}(M_y^A(A)\le x_n-\varepsilon_n,S_n^A)\\
&\le \mathbb{P}(\exists v\in A\,:\,x_n-\varepsilon_n<M_y^A(\{v\})\le x_n,S_n^A)\\
&\le \mathbb{P}(\exists v\in A\,:\,x_n-\varepsilon_n<M_y(\{v\})\le x_n+\varepsilon_n,S_n^A)\\
&\le \mathbb{P}(\exists v\in K_n\,:\,x_n-\varepsilon_n<M_y(\{v\})\le x_n+\varepsilon_n)\\
&\le \sum_{v\in K_n}\mathbb{P}(M_y(\{v\})>x_n-\varepsilon_n)-\sum_{v\in K_n}\mathbb{P}(M_y(\{v\})>x_n+\varepsilon_n).
\end{aligned}$$

Since $\varepsilon_n\to 0$, the considerations that led to (41) also show that the two sums above have the same limit as $n\to\infty$ for almost all realizations $(y_v)_v$ of $(Y_v)_v$. As neither sum depend on the choice of A and B, we conclude that the first term of (68) satisfy the desired convergence. The convergence of the second term follows identically. Concerning the convergence of the third term, we see from (66) and (67) above that $\mathbb{P}\big((S_n^A)^c\big)$ and $\mathbb{P}\big((S_n^B)^c\big)$ have upper bounds independent of A and B, which, due to (65) tend to 0 as $n\to\infty$. This completes the proof.

Proof of Lemma 10 With a slight change of notation (as compared with F_L defined in the proof of Lemma 5), we now let $F_t\in\mathscr{L}_\beta$ denote the distribution of $\sup_{u\in C(v)}Z_u^{(t)}$ for $t>0$. Hence, by (40),

$$|C_n|\overline{F}_t(x_n)\to\tau^{(t)}\tag{69}$$

as $n\to\infty$, for t large enough.

For all $z\in N_k$ and all $v,v'\in J_z$ with $|v-v'|>2(t+\sqrt{d})$ we have by construction that $Z_v^{(t)}$ and $Z_{v'}^{(t)}$ are independent. Writing $y^*(v)=\sup_{u\in C(v)}y_u$ for all $v\in\mathbb{Z}^d$, turning to Lemma 11, we find for all $\gamma>\beta$ that there is a constant $\tilde{C}$ such that

$$\begin{aligned}
&\sum_{\substack{v<v'\in J_z\\|v-v'|>2(t+\sqrt{d})}}\mathbb{P}\big(\sup_{u\in C(v)}(Z_u^{(t)}+y_u)>x_n,\ \sup_{u\in C(v')}(Z_u^{(t)}+y_u)>x_n\big)\\
&\quad\le\sum_{v<v'\in J_z}\overline{F}_t(x_n-y^*(v))\overline{F}_t(x_n-y^*(v'))\\
&\quad\le\tilde{C}\Big(\sum_{v\in J_z}\overline{F}_t(x_n)\exp(\gamma(y^*(v))^+)\Big)^2
\end{aligned}$$

for sufficiently large n and all $z \in N_k$. Since the field $(Y_v)_v$ satisfies (51), and since $|C_n|/k \sim |J_z|$ as $n \to \infty$, we conclude by Corollary 4 and (69) that

$$\limsup_{n\to\infty} \tilde{C}\Big(\sum_{v\in J_z} \overline{F}_t(x_n)\exp(\gamma|y^*(v)|)\Big)^2 = \frac{1}{k^2}\tilde{C}\Big(\tau^{(t)}\mathbb{E}\exp\big(\gamma(\sup_{v\in C(0)} Y_v)^+\big)\Big)^2$$

almost surely. This is independent of z and of order $o(k^{-1})$ as $k \to \infty$. This shows (46) for the terms in the sum with indices more than $2(t+\sqrt{d})$ apart.

Now consider $v, v' \in J_z$ such that $|v - v'| \le 2(t+\sqrt{d})$. As in (60), we have for all fixed $v \ne v' \in \mathbb{Z}^d$ that

$$\frac{1}{\overline{F}_t(x_n)}\mathbb{P}\big(\sup_{u\in C(v)} Z_u^{(t)} > x_n - y, \sup_{u\in C(v')} Z_u^{(t)} > x_n - y\big) \to 0$$

uniformly for $y \le K$, for all $K \in \mathbb{N}$. Define $y^*(v,v') = \max\{y^*(v), y^*(v')\}$ and similarly for $(Y_v)_v$ and note that

$$\mathbb{E}\exp(\gamma(Y^*(v,v'))^+) \le 2\mathbb{E}\exp(\gamma(\sup_{v\in C(0)} Y_v)^+) < \infty \tag{70}$$

for all $\gamma > 0$ by the Cauchy-Schwarz inequality and (51). Arguing as in the proof of Lemma 5, the uniform convergence combined with Corollary 4 and the stationarity of $(Y_v)_v$ then yield

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{\substack{v<v'\in J_z\\ |v-v'|\le 2(t+\sqrt{d})}} \mathbb{P}\big(\sup_{u\in C(v)} (Z_u^{(t)} + y_u) > x_n, \sup_{u\in C(v')} (Z_u^{(t)} + y_u) > x_n\big)\\
&\quad\le \limsup_{n\to\infty} \sum_{\substack{v<v'\in J_z\\ |v-v'|\le 2(t+\sqrt{d})}} \mathbb{P}\big(\sup_{u\in C(v)} Z_u^{(t)} > x_n - y^*(v,v'), \sup_{u\in C(v')} Z_u^{(t)} > x_n - y^*(v,v')\big)\\
&\quad\le \frac{\tilde{C}}{k} \sum_{|v|\le 2(t+\sqrt{d})} \mathbb{E}\Big[\exp(\gamma(Y^*(0,v))^+)\, 1_{Y^*(0,v)>K}\Big]
\end{aligned}$$

for all $\gamma > 0$ and $K \in \mathbb{N}$, where the constant $\tilde{C}$ is chosen according to (49). Since this is independent of z, and due to the fact that there are only finitely many terms in the sum, we conclude (46) by letting $K \to \infty$ using a dominated convergence argument, which is justified by (70).

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Conflict of interest

The authors declare that they have no conflict of interest.

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