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Extremes of subexponential Lévy-driven random fields in the Gumbel domain of attraction

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Abstract We consider a spatial Lévy-driven moving average with an underlying Lévy measure having a subexponential right tail, which is also in the maximum domain of attraction of the Gumbel distribution. Assuming that the left tail is not heavier than the right tail, and that the integration kernel satisfies certain regularity conditions, we show that the supremum of the field over any bounded set has a right tail equivalent to that of the Lévy measure. Furthermore, for a very general class of expanding index sets, we show that the running supremum of the field, under a suitable scaling, converges to the Gumbel distribution.

Keywords Extreme value theory · Lévy-based modeling · Geometric probability · Subexponential distributions · Random fields

Mathematics Subject Classification (2010) Primary · 60G70 · 60G60 · Secondary · 60G500 · 60D05

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1 Introduction

In this paper we investigate asymptotic distributional properties of a spatial Lévy-driven moving average given by

$$X_v = \int_{\mathbb{R}^d} f(v-u)\Lambda(\mathrm{d}u), \qquad v \in \mathbb{R}^d, \tag{1}$$

where Λ is a Lévy basis, i.e. an infinitely divisible and independently scattered random measure on $\mathbb{R}^d$, and $f : \mathbb{R}^d \to \mathbb{R}$ is an appropriate integration kernel (we refer to (Rajput and Rosinski, 1989, Theorem 2.7) for conditions ensuring the existence of fields as in (1)). Such Lévy-based models constitute a broad framework, which has been used for a multitude of modeling purposes. This includes, but is not limited to, the modeling of financial assets (Barndorff-Nielsen and Shephard (2001)), turbulent flows (Barndorff-Nielsen and Schmiegel (2004)), brain imaging data (Jónsdóttir et al. (2013)) and wind power prices (Benth and Pircalabu (2018)). In Rønn-Nielsen and Jensen (2019) estimators for the mean and variogram of fields as in (1) are suggested, and corresponding central limit theorems for these estimators are given.

We assume that the Lévy measure ρ of the basis Λ has a subexponential right tail that furthermore lies in the maximum domain of attraction (MDA) of the Gumbel distribution; see for instance (Embrechts et al., 1997, Sections 3.3 & A.3.2). This class of distributions contains heavy-tailed distributions which are not too heavy, in the sense that they have lighter tails than any regularly varying distribution (Embrechts et al., 1997, Appendix A.3), but they have heavier tails than any convolution equivalent distribution of index $\beta > 0$ (Cline (1986, 1987)). In fact, the present paper constitutes the final investigation of similar asymptotic properties of fields as in (1) with the different Lévy measures just described: In Stehr and Rønn-Nielsen (2021) the Lévy measure is convolution equivalent of strictly positive index, which in particular means that it is in the MDA of the Gumbel distribution; in Rønn-Nielsen and Stehr (2021) the Lévy measure is regularly varying of strictly positive index, which is equivalent to being in the MDA of the Fréchet distribution.

In contrast to the papers Stehr and Rønn-Nielsen (2021) and Rønn-Nielsen and Stehr (2021) we here allow the integration kernel f to attain negative values. On the other hand, we then impose a restriction on the left tail of ρ ensuring that it is at most as heavy as its right tail. Compared to the framework in Stehr and Rønn-Nielsen (2021), we here also allow for the kernel function to have more than one maximal point. In fact, this was also possible within the model assumptions of Rønn-Nielsen and Stehr (2021), but handling this was somewhat simpler in the regularly varying framework than in the present. Despite these differences, many considerations from the papers Stehr and Rønn-Nielsen (2021) and Rønn-Nielsen and Stehr (2021) apply to this paper as well, and thus some arguments are shortened or even omitted here. We work under two sets of assumptions that are more or less restrictive on either the kernel function or the Lévy basis. The precise formulations are found in Assumptions 1 and 2 below. Both are further tightenings of a set of minimal requirements given in Assumption M.

The first part of the paper deals with the asymptotic distribution of the tail of $\sup_{v\in B} X_v$, where $B \subseteq \mathbb{R}^d$ is a bounded and full-dimensional index set. Using a result from Rosinski and Samorodnitsky (1993) we show that

$$\mathbb{P}\big(\sup_{v\in B} X_v > x\big) \sim \rho\big((x,\infty)\big)\, C_B \tag{2}$$

as $x \to \infty$, where $\sim$ denotes asymptotic equivalence and C_B is a constant given in terms of the set B and the index sets of maximal and minimal values of f. In fact, we show the slightly more general result that the equivalence (2) is also satisfied if we replace X with $X+Y^1$, where Y^1 is an independent field with some exponential moment.

Results similar to (2) exist in the literature for one-dimensional processes, and, as already mentioned, for spatial fields with different Lévy measures: In Fasen (2006) the stochastic process

$$X_t = \int_{\mathbb{R}} f(t-s)\Lambda(\mathrm{d}s) \tag{3}$$

is studied under the assumption that the driving Lévy process $(\Lambda(t))_{t\in\mathbb{R}}$ has subexponential increments, meaning that $\Lambda(1)$ has a subexponential distribution. Under further regularity conditions, including the assumption of a particular full-dimensional set A relating to the kernel function, they show that

$$\mathbb{P}(X_t > x) \sim |A|\mathbb{P}\big(f(U_A)\Lambda(1) > x\big),$$

where $|A|$ denotes Lebesgue measure of A, and U_A is a uniform random variable on A.

If, instead, the spatial Lévy-driven field $(X_v)_{v\in\mathbb{R}^d}$ has a convolution equivalent Lévy measure ρ of strictly positive index $\beta > 0$, it is shown in Rønn-Nielsen and Jensen (2016) that

$$\mathbb{P}\big(\sup_{v\in B} X_v > x\big) \sim \rho((x,\infty))|B|K_\beta, \tag{4}$$

where K_β is a computable constant given in terms of β–exponential moments of the field. Since subexponential distributions are in fact convolution equivalent of index 0, one might think that the claim for subexponential Lévy measures simply follows by setting $\beta = 0$ above. This is however not the case, as subexponential distributions can be more or less heavy. For instance, if the underlying Lévy measure is regularly varying of index $\alpha > 0$ (and thus in particular subexponential and very heavy), it is shown in Rønn-Nielsen and Stehr (2021) that

$$\mathbb{P}\big(\sup_{v\in B} X_v > x\big) \sim \rho((x,\infty)) \int_{\mathbb{R}^d} \big(\sup_{v\in B} f(v-u)\big)^\alpha \mathrm{d}u. \tag{5}$$

Note that this does not coincide with (4) above for $\beta = 0$. Still, if we consider subexponential and not too heavy measures as in the present paper, the equivalence (2) is actually a special case of both (4) and (5) choosing $\beta = 0$ and $\alpha = \infty$, respectively. The proofs however, do not generalize to these cases.

In the case of the much lighter Gaussian tails, arguments for results similar to the above-mentioned take a completely different route. For Gaussian random fields the distribution of the supremum can be approximated by the expected Euler characteristic of an excursion set (see Adler and Taylor (2007) and the references therein).

The second part of the paper deals with the distribution of the supremum $\sup_{v\in C_n} X_v$ as $n\to\infty$. Here, (C_n) is a very general sequence of index sets in $\mathbb{R}^d$ increasing appropriately. More specifically, we require that C_n is a union of connected convex bodies with intrinsic volumes sufficiently bounded relative to the volume of C_n; we refer to (Schneider, 1993, Chapter 4) for an exposition of convex bodies and their intrinsic volumes. For instance, if $d = 3$ and C_n is itself a convex body, we simply require that the mean width is asymptotically bounded by $|C_n|^{1/3}$ and that the surface area is asymptotically bounded by $|C_n|^{2/3}$. This includes the simple case where C_n is a box with side-lengths asymptotically of the same size. Knowing that the marginal tail essentially determines the distribution of the maximum for dependent sequences satisfying certain mixing and anti-clustering conditions, we expect by (2) that the running supremum converges in distribution to a certain power of the Gumbel distribution; see Embrechts et al. (1997); Leadbetter et al. (1983); Resnick (2008) for detailed treatments of classical extreme value theory, and see Jakubowski and Soja-Kukieła (2019); Soja-Kukieła (2019); Stehr and Rønn-Nielsen (2021) for generalizations to stationary, discretely indexed d-dimensional spatial fields. It turns out, however, that the appropriate mixing and anti-clustering conditions for discretely indexed fields do not easily show for our continuously indexed field. Instead, writing $X = Z + Y$ as the independent sum of a dominating compound Poisson field Z and a light-tailed and high-activity field Y, we first establish a conditional limit result for the field $Z + y$ conditioned on $Y = y$. Utilizing that the light-tailed field Y is ergodic, we conclude that there are norming constants (a_n) and (b_n) such that, as $n \to \infty$,

$$\mathbb{P}\big(a_n^{-1}\big(\sup_{v\in C_n} X_v - b_n\big) \le x\big) \to \exp\big(-\mathrm{e}^{-x}\rho((1,\infty))(1+K)\big),$$

where K is a positive constant that vanishes if either the right tail of ρ dominates the left tail or if the maximal negative value of f is strictly smaller than the maximal positive value.

In Fasen (2006) a similar result is obtained for the running supremum $\sup_{t\in[0,T]} X_t$ of the one-dimensional moving average (3) under the assumption that $\Lambda(1)$ is in the MDA of the Gumbel distribution. In Stehr and Rønn-Nielsen (2021) and Rønn-Nielsen and Stehr (2021) Lévy-driven fields with convolution equivalent and regularly varying Lévy measures, respectively, are considered under an identical asymptotic regime regarding the expansion of the index sets. In these cases, the running supremas $\sup_{v\in C_n} X_v$ converge to powers of the Gumbel distribution and the Fréchet distribution, respectively. As mentioned, the proof structure in Stehr and Rønn-Nielsen (2021) and Rønn-Nielsen and Stehr (2021) also apply to this paper, and in particular they both rely on conditioning on the light-tailed and heavy-activity part of the field. There are differences though, and these will be clarified when needed.

The paper is organized as follows. In Section 2 we define our Lévy-driven field and provide the assumptions used throughout the paper. Moreover, we formally present our main results. In Section 3 we first ensure boundedness of our fields, which subsequently is explicitly used in proving the tail representation of the supremum. Lastly, Section 4 is devoted to showing that the running supremum $\sup_{v\in C_n} X_v$ converges to the Gumbel distribution.

2 Main results

In this section we formally define our random field and state sufficient assumptions after which we can present the main results. Before doing so, we briefly clarify some notation used throughout the paper. We let $|\cdot|$ denote size in the following sense: $|v|$ is the Euclidean norm of a single (one- or multi-dimensional) point v, $|A|$ is Lebesgue measure of a full-dimensional set $A \subseteq \mathbb{R}^d$, and $|A|$ is the number of points in a discrete set $A \subseteq \mathbb{Z}^d$. However, we will at times also use the notation m for Lebesgue measure. Moreover, we let $\|\cdot\|_\infty$ denote the usual supremum-norm. An important notion throughout the paper is the so-called Minkowski sum: For two sets $A, B \subseteq \mathbb{R}^d$, we define their Minkowski sum by $A \oplus B = \{a+b \mid a \in A, b \in B\}$, and we let $A \ominus B$ denote the Minkowski sum of A and the reflection of B, i.e. $A \ominus B = A \oplus (-B) = \{a - b \,:\, a \in A, b \in B\}$.

We consider a stationary Lévy-driven random field $(X_v)_{v\in\mathbb{R}^d}$ given as

$$X_v = \int_{\mathbb{R}^d} f(v-u)\Lambda(\mathrm{d}u), \tag{6}$$

where f is an integration kernel and Λ is a so-called Lévy basis on $\mathbb{R}^d$, i.e. an infinitely divisible and independently scattered random measure on $\mathbb{R}^d$. Necessary and sufficient conditions for the field to be well-defined are found in (Rajput and Rosinski, 1989, Theorem 2.7), however, as described next, we make a set of sufficient assumptions on the pair (Λ, f) listed in Assumption M below. In particular, we normalize f and require that it is integrable and numerically bounded by an isotropic, decreasing and integrable function.

We can and do choose a strongly separable version (which is also known in the literature as a modification) of the random field $(X_v)_{v\in\mathbb{R}^d}$; see (Samorodnitsky and Taqqu, 1994, Definition 9.2.3 and Theorem 9.2.5). When we in Section 3 introduce the decomposition of $(X_v)_{v\in\mathbb{R}^d}$ into the independent fields $(Y_v)_{v\in\mathbb{R}^d}$ and $(Z_v)_{v\in\mathbb{R}^d}$, we also choose strongly separable versions of them. Similarly, when considering the fields on a specific bounded index set $B \subseteq \mathbb{R}^d$, we also choose strongly separable versions. For such fields with bounded index sets, we furthermore make sure that the versions have bounded sample paths; see Lemma 1 below.

In the present paper we assume that the Lévy basis is stationary and isotropic. This means that the cumulant function of the random variable $\Lambda(A)$ has Lévy-Khintchine form

$$C(\lambda \dagger \Lambda(A)) = i\lambda a|A| - \frac{1}{2}\lambda^2\theta|A| + \int_{A\times\mathbb{R}} \big(\mathrm{e}^{i\lambda x} - 1 - i\lambda x 1_{[-1,1]}(x)\big) F(\mathrm{d}u, \mathrm{d}x)$$

for all Borel sets $A \subseteq \mathbb{R}^d$, where $a \in \mathbb{R}$, $\theta \geq 0$ and $F = m \otimes \rho$ is the product measure of Lebesgue measure m and a Lévy measure ρ. The Lévy measure ρ is assumed to have a subexponential right tail, which is furthermore in the maximum domain of attraction $\mathrm{MDA}(\mathrm{Gum})$ of the Gumbel distribution, $\mathrm{Gum}(x) = \exp(-\mathrm{e}^{-x})$ for $x \in \mathbb{R}$. Although this is only an assumption on the right tail of ρ, we denote this distributional property by $\rho \in \mathscr{S} \cap \mathrm{MDA}(\mathrm{Gum})$. It will be used throughout the paper that the class $\mathscr{S} \cap \mathrm{MDA}(\mathrm{Gum})$ is closed under asymptotic tail equivalence. The right tail of ρ is subexponential if

$$\lim_{x\to\infty} \frac{\rho((x-y,\infty))}{\rho((x,\infty))} = 1$$

for $y \in \mathbb{R}$, and if

$$\lim_{x\to\infty} \frac{\tilde{\rho}_1 * \tilde{\rho}_1((x,\infty))}{\tilde{\rho}_1((x,\infty))} < \infty$$

exists, where $\tilde{\rho}_1(\cdot) = \rho(\cdot \cap (1,\infty))/\rho((1,\infty))$ denotes the normalized restriction of ρ to $(1,\infty)$ and $*$ is the convolution operator. Furthermore, $\rho \in \mathrm{MDA}(\mathrm{Gum})$ (that is, its right tail) if it is tail equivalent to a von Mises function (Embrechts et al., 1997, Section 3.3). This equivalently means that it is on the form

$$\rho((x,\infty)) = c(x)\exp\Big(-\int_0^x \frac{1}{a(t)}\mathrm{d}t\Big), \tag{7}$$

where $c(x) \to c > 0$ and the so-called auxiliary function a is absolutely continuous with density $a'(t) \to 0$. Additionally, $a(t) \to \infty$ is a necessary condition for the right tail of ρ to also be subexponential; see (Goldie and Resnick, 1988, Lemma 2.1). In Example 2 below we provide an easily checked sufficient condition on the function a ensuring that ρ is subexponential. Distributions with such tails constitute a class of rather heavy tailed distributions due to the subexponentiality, however, they have lighter tails than any regularly varying distribution: The right tail of ρ is rapidly varying, meaning in particular that

$$\lim_{x\to\infty} \frac{\rho((\gamma x,\infty))}{\rho((x,\infty))} = 0 \qquad \text{for all } \gamma > 1, \tag{8}$$

and that $\int_1^\infty y^\alpha \rho(\mathrm{d}y) < \infty$ for all $\alpha > 0$ (Embrechts et al., 1997, Corollary 3.3.32). In fact, the auxiliary function $a(t)$ is of order $o(t)$ as $t \to \infty$, and hence, for all $\varepsilon > 0$, the bound $a(t) \le \varepsilon t$ holds for all t sufficiently large. Thus, for all $\alpha > 0$ there are constants C, x_0 such that

$$\frac{\rho((\gamma x, \infty))}{\rho((x, \infty))} \le C\gamma^{-\alpha} \tag{9}$$

for all $x \ge x_0$ and $\gamma \ge 1$, which in particular implies the rapid variation (8). Concerning the left tail of ρ, we assume that it is at most as heavy as the right tail.

We collect the assumptions on the basis combined with minimal requirements on the kernel f below, where the following notation is used to distinguish between the positive and negative parts of f and its points of maximum and minimum: $f^+(u) = f(u)1_{\{f(u)\ge 0\}}$, $f^-(u) = -f(u)1_{\{f(u)<0\}}$, $M^+ = \{u \in \mathbb{R}^d : f^+(u) = \|f^+\|_\infty\}$ and $M^- = \{u \in \mathbb{R}^d : f^-(u) = \|f^-\|_\infty\}$.

Assumption M *The Lévy basis Λ on $\mathbb{R}^d$ is stationary and isotropic with a Lévy measure ρ having a subexponential right tail which is in the maximum domain of attraction of the Gumbel distribution. Moreover, the tail-balance condition*

$$\lim_{x\to\infty} \frac{\rho((-\infty, -x))}{\rho((x, \infty))} = \frac{1-p}{p} \tag{10}$$

is satisfied for some $p \in (0,1]$. The integration kernel $f : \mathbb{R}^d \to \mathbb{R}$ satisfying $\|f^-\|_\infty \le \|f^+\|_\infty = 1$ has bounded index sets M^+ and $M^- \subseteq \mathbb{R}^d$ of maximas of f^+ and f^-, respectively, and f^+ and f^- are both lower semi-continuous. Furthermore, f is numerically bounded by a decreasing càdlàg function g, in the sense that $|f(u)| \le g(|u|)$ for all $u \in \mathbb{R}^d$. The function g additionally satisfies

$$\int_{\mathbb{R}^d} g(|u|)\mathrm{d}u < \infty. \tag{11}$$

Note that the integrability in (11) is equivalent to $\int_0^\infty g(x)x^{d-1}\mathrm{d}x < \infty$. Moreover, the lower semi-continuity of the functions f^+ and f^- combined with the fact that g is càdlàg and decreasing ensures that $u \mapsto \sup_{v\in B} f(v-u)$ and $u \mapsto \sup_{v\in B} g(|v-u|)$ are also integrable for any bounded set $B \subseteq \mathbb{R}^d$.

Example 1 For any infinitely divisible distribution with a subexponential right tail there is asymptotic equivalence between the tail of the distribution and the tail of the Lévy measure; see (Embrechts et al., 1979, Theorem 1). In particular, the tail of $\Lambda(A)$ will be subexponential and in the maximum domain of attraction of the Gumbel distribution if and only if the Lévy measure ρ is so.

Example 2 If $\rho \in \mathrm{MDA(Gum)}$ with tail representation (7), a sufficient condition for $\rho \in \mathscr{S} \cap \mathrm{MDA(Gum)}$ is that the auxiliary function a is eventually non-decreasing and that there exists a $t > 1$ such that

$$\liminf_{x\to\infty} \frac{a(tx)}{a(x)} > 1,$$

see (Goldie and Resnick, 1988, Corollary 2.5). This includes distributions which are asymptotically equivalent to the Benktander-type-I, Benktander-type-II, Weibull and Lognormal distributions; (Embrechts et al., 1997, Example 3.3.35). In general, a sufficient tail representation for any ρ to be in $\mathscr{S} \cap \mathrm{MDA(Gum)}$ is the following:

$$\rho((x,\infty)) = c(x)\exp\big(-H(x)\big),$$

where $c(x) \to c > 0$ and the function H has an eventually non-increasing derivative and furthermore satisfies $H(x) = x\int_x^\infty \ell(t)t^{-1}\mathrm{d}t$ for some slowly varying function ℓ; see (Goldie and Resnick, 1988, Theorem 2.7). In particular, the following close-to-exponential tails have the desired form: $\exp(-x(\log x)^\alpha)$ for $\alpha < 0$, and $\exp(-x(\log\log x)/\log x)$.

Example 3 If the decreasing upper bound g is a regularly varying function of index $-(d+\varepsilon)$ for some $\varepsilon > 0$, then (11) is satisfied. This includes the simple case where $g(x) = c(1+x)^{-(d+\varepsilon)}$ is a power function of order $-(d+\varepsilon)$.

Example 4 If f has the form $f(u) = 1_A(u)$ for an open and bounded set A, then f satisfies the requirement from Assumption M with g defined by $g(x) = 1_{[0,R)}(x)$, where R is the radius of an open ball centered in 0 and containing A. Additionally, M^+ is equal to A.

As mentioned in the introduction we make two different tightenings of Assumption M, the first of which requires that f is Hölder continuous.

Assumption 1 *The Lévy basis Λ and integration kernel f satisfy Assumption M. Moreover, f is Hölder continuous with some index $\zeta > 0$. That is, there is a constant C such that*

$$|f(u_1) - f(u_2)| \le C|u_1 - u_2|^\zeta$$

for all $u_1, u_2 \in \mathbb{R}^d$.

Example 5 Let A be a symmetric, positive definite matrix and define f by

$$f(u) = h(u)\exp(-u^T A^{-1} u),$$

where u^T is a row vector, u is a column vector, and $h : \mathbb{R}^d \to [-1,1]$ is a Hölder continuous function with index 1 and $h(0) = 1$. Then $f(0) = 1$, $|f| \le 1$ and f is Hölder continuous with index 1. Furthermore, $|f(\cdot)| \le g(|\cdot|)$ with $g(x) = \mathrm{e}^{-x^2/\lambda}$, where λ is the largest eigenvalue of A. Note that $M^+ = \{0\}$ and that $\|f^-\|_\infty < \|f^+\|_\infty$.

The second assumption corresponds to the case where the Lévy basis is of finite variation, by which we mean that the triple (a, θ, ρ) of its Lévy-Khintchine representation satisfies that of a Lévy process of finite variation.

Assumption 2 *The Lévy basis Λ and integration kernel f satisfy Assumption M. Moreover, Λ has Lévy-Khintchine representation with $\theta = 0$ and a Lévy measure ρ satisfying*

$$\int_{|y|\le 1} |y|\rho(\mathrm{d}y) < \infty. \tag{12}$$

Example 6 Any finite Lévy measure ρ on $\mathbb{R}$ will clearly satisfy (12). Furthermore, if e.g. ρ is given by

$$\rho((x,\infty)) = p\Big(\frac{x+1}{x}\Big)^\beta \exp(-H(x))$$
$$\rho((-\infty,-x)) = (1-p)\Big(\frac{x+1}{x}\Big)^\beta \exp(-H(x))$$

for all $x \ge 0$ with $\beta < 2$, $p \in (0,1]$, and H given as in Example 2 such that it is bounded away from $-\infty$, then ρ satisfies both (10) and (12).

We can now state the first main result of the paper.

Theorem 1 *Let $(X_v)_{v\in\mathbb{R}^d}$ be a Lévy-driven field given by* (6)*, where the Lévy basis Λ and the kernel function f satisfy either Assumption 1 or Assumption 2. For any full-dimensional and bounded set $B \subseteq \mathbb{R}^d$,*

$$\mathbb{P}\big(\sup_{v\in B} X_v > x\big) \sim \big(\big|B \ominus M^+\big| + \frac{1-p}{p}\big|B \ominus M^-\big| 1_{\{\|f^-\|_\infty = 1\}}\big)\rho((x,\infty)) \tag{13}$$

as $x \to \infty$.

The following more general tail result follows easily by the former.

Theorem 2 *Let $(X_v)_{v\in\mathbb{R}^d}$ be a Lévy-driven field given by* (6)*, where the Lévy basis Λ and the kernel function f satisfy either Assumption 1 or Assumption 2. Let $(Y_v^1)_v$ be a field independent of $(X_v)_v$ satisfying*

$$\mathbb{E}\exp\big(\varepsilon \sup_{v\in B} Y_v^1\big) < \infty \tag{14}$$

for some $\varepsilon > 0$ and for any full-dimensional and bounded set $B \subseteq \mathbb{R}^d$. Then,

$$\mathbb{P}\big(\sup_{v\in B}(X_v + Y_v^1) > x\big) \sim \big(\big|B \ominus M^+\big| + \frac{1-p}{p}\big|B \ominus M^-\big| 1_{\{\|f^-\|_\infty = 1\}}\big)\rho((x,\infty))$$

as $x \to \infty$.

We now turn to the assumption on the expansion of the index sets (C_n) in $\mathbb{R}^d$ relating to the asymptotic distribution of the running supremum. As defined below, we require that C_n is a so-called p–convex set for some $p \in \mathbb{N}$ and all $n \in \mathbb{N}$. A p–convex set is given in terms of convex bodies, which are compact, convex sets with non-empty interior; see (Schneider, 1993, Chapter 4) for a greater exposition of convex bodies and their intrinsic volumes needed below.

Definition 1 A set $C \subseteq \mathbb{R}^d$ is said to be p–convex, if it is connected and has the form

$$C = \bigcup_{i=1}^{p} \overline{C}_i\,,$$

where $\overline{C}_1, \ldots, \overline{C}_p$ are convex bodies in $\mathbb{R}^d$.

Our main geometric requirement for the behavior of the sequence of index sets (C_n) is given in Assumption 3 below, and it is formulated in terms of the so-called intrinsic volumes of the convex bodies defining the p–convex set. The jth intrinsic volume $V_j(C)$ (for $j = 1, \ldots, d$) describes the geometry of the convex body C, and additionally, the collection of intrinsic volumes constitutes an essential part in the famous Steiner formula from convex geometry. Some of the intrinsic volumes have a direct geometrical interpretation. For instance, $V_0(C) = 1$, $V_1(C)$ is proportional to the mean width, $V_{d-1}(C)$ is half the surface area, and $V_d(C) = |C|$ is the volume of C. Furthermore, intrinsic volumes satisfy some important properties, among which we mention non–negativity, i.e. $V_j(C) \geq 0$, homogeneity, i.e. $V_j(\gamma C) = \gamma^j V_j(C)$ for all $\gamma > 0$, and monotonicity, i.e. $V_j(C) \leq V_j(D)$ for $C \subseteq D$.

Assumption 3 *The sequence* $(C_n)_{n\in\mathbb{N}}$ *consists of p–convex sets, where*

$$C_n = \bigcup_{i=1}^{p} C_{n,i}$$

and $|C_n| \to \infty$ *as* $n \to \infty$*. Furthermore,*

$$\frac{\sum_{i=1}^{p} V_j(C_{n,i})}{|C_n|^{j/d}} \quad \text{is bounded in } n \text{ for each } j = 1,\dots,d-1.$$

As the following example shows, a scaling of an initial p–convex set satisfies Assumption 3. This includes the case of an increasing sequence of boxes with asymptotically equivalent side-lengths.

Example 7 Let $C = \cup_{i=1}^{p} \overline{C}_i$ be a p–convex set and let $(r_{n,1}),\dots,(r_{n,d})$ be sequences of real numbers satisfying that $r_{n,i} \sim r_n$ for some $r_n \to \infty$ for all $i = 1,\dots,d$. The sequence $(C_n)_{n\in\mathbb{N}}$ defined by

$$C_n = \{(r_{n,1}x_1,\dots,r_{n,d}x_d) \in \mathbb{R}^d \,:\, (x_1,\dots,x_d) \in C\}$$

satisfies Assumption 3.

Before stating the last main results of the paper, we recall that (the right tail of) ρ is in particular in the maximum domain of attraction of the Gumbel distribution, $\mathrm{Gum}(x) = \exp(-\mathrm{e}^{-x})$. This implies that there are sequences of norming constants $(\tilde{a}_n)_{n\in\mathbb{N}}$ and $(\tilde{b}_n)_{n\in\mathbb{N}}$ such that

$$n\rho((\tilde{a}_n x + \tilde{b}_n,\infty)) \to \mathrm{e}^{-x}\rho((1,\infty))$$

as $n \to \infty$, for all $x \in \mathbb{R}$. In the remainder of the paper, we let $a_n = \tilde{a}_{|C_n|}$ and $b_n = \tilde{b}_{|C_n|}$ denote the norming constants of ρ relative to $|C_n|$, that is

$$|C_n|\rho((a_n x + b_n,\infty)) \to \mathrm{e}^{-x}\rho((1,\infty)) \tag{15}$$

as $n \to \infty$, for all $x \in \mathbb{R}$. The norming constants satisfy

$$a_n, b_n \to \infty, \qquad \text{and} \qquad a_n = o(b_n)$$

as $n \to \infty$, and they may be chosen according to

$$b_n = \inf\big\{x \in [1,\infty) \,:\, \rho((x,\infty)) \le |C_n|^{-1}\rho((1,\infty))\big\},$$
$$a_n = a(b_n),$$

where the function $a(\cdot)$ is the auxiliary function from the tail representation (7). Note that this choice of b_n is nothing more than the $(1-|C_n|^{-1})$-quantile of the probability distribution $\tilde{\rho}_1$ defined above.

Theorem 3 *Let* $(X_v)_{v\in\mathbb{R}^d}$ *be a Lévy-driven field given by* (6)*, where the Lévy basis* Λ *and the kernel function* f *satisfy either Assumption 1 or Assumption 2. Let* $(C_n)_{n\in\mathbb{N}}$ *be a sequence of sets in* $\mathbb{R}^d$ *satisfying Assumption 3, and let* a_n, b_n *be the norming constants of the Lévy measure* ρ *relative to* $|C_n|$*; see* (15)*. Then, as* $n \to \infty$*,*

$$\mathbb{P}\Big(a_n^{-1}(\sup_{v\in C_n} X_v - b_n) \le x\Big) \to \exp\Big(-\mathrm{e}^{-x}\rho((1,\infty))\big[1 + p^{-1}(1-p)1_{\{\|f^-\|_\infty = 1\}}\big]\Big)$$

for all $x \in \mathbb{R}$*.*

It will be clear from the proof of Theorem 3 that we do not directly use the Lévy-based structure of the light-tailed field $(Y_v)_v$ mentioned in the introduction and formally given in (17) below. All that is needed is some exponential moment and an ergodic behavior. Therefore we immediately have the following extension of the theorem, where $B(t) = \{u \in \mathbb{R}^d : |u| \leq t\}$ denotes the closed ball with radius t and center in the origin $0 \in \mathbb{R}^d$. This notation will be used throughout the paper.

Theorem 4 *Let $(X_v)_{v\in\mathbb{R}^d}$ be a Lévy-driven field given by* (6)*, where the Lévy basis Λ and the kernel function f satisfy either Assumption 1 or Assumption 2. Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathbb{R}^d$ satisfying Assumption 3, and let a_n, b_n be the norming constants of the Lévy measure ρ relative to $|C_n|$; see* (15)*. Furthermore, let $(Y_v^1)_v$ be a stationary and ergodic field independent of $(X_v)_v$ satisfying*

$$\mathbb{E}\exp\big(\varepsilon \sup_{v\in B(1)} Y_v^1\big) < \infty$$

for some $\varepsilon > 0$. Then, as $n \to \infty$,

$$\mathbb{P}\Big(a_n^{-1}\big(\sup_{v\in C_n}(X_v + Y_v^1) - b_n\big) \leq x\Big) \to \exp\Big(-\mathrm{e}^{-x}\rho((1,\infty))\big[1 + p^{-1}(1-p)1_{\{\|f^-\|_\infty=1\}}\big]\Big)$$

for all $x \in \mathbb{R}$.

3 Tail behavior

We use a decomposition of (X_v) into an independent sum of a dominating and positive field and a lighter tailed, high-activity field. For this we write $\Lambda = \Lambda_+ + \Lambda_- + \Lambda_2$ as the independent sum of three Lévy bases with Lévy-Khintchine representations

$$C(\lambda \dagger \Lambda_+(A)) = \int_{A\times\mathbb{R}} \big(\mathrm{e}^{i\lambda x} - 1\big) F_{(1,\infty)}(\mathrm{d}u, \mathrm{d}x)$$
$$C(\lambda \dagger \Lambda_-(A)) = \int_{A\times\mathbb{R}} \big(\mathrm{e}^{i\lambda x} - 1\big) F_{(-\infty,-1)}(\mathrm{d}u, \mathrm{d}x)$$
$$C(\lambda \dagger \Lambda_2(A)) = i\lambda a|A| - \frac{1}{2}\lambda^2\theta|A| + \int_{A\times\mathbb{R}} \big(\mathrm{e}^{i\lambda x} - 1 - i\lambda x 1_{[-1,1]}(x)\big) F_{[-1,1]}(\mathrm{d}u, \mathrm{d}x),$$

respectively. Here, $F_D = m \otimes \rho_D$, where ρ_D is the restriction of ρ to $D \subseteq \mathbb{R}$. Now we decompose the field $(X_v)_v$ into a sum of two independent random fields $X_v = Z_v + Y_v$ for all $v \in \mathbb{R}^d$, where

$$Z_v = \int_{\mathbb{R}^d} f^+(v-u)\Lambda_+(\mathrm{d}u) - \int_{\mathbb{R}^d} f^-(v-u)\Lambda_-(\mathrm{d}u) \qquad \text{and} \tag{16}$$

$$Y_v = \int_{\mathbb{R}^d} f^+(v-u)\Lambda_-(\mathrm{d}u) - \int_{\mathbb{R}^d} f^-(v-u)\Lambda_+(\mathrm{d}u) + \int_{\mathbb{R}^d} f(v-u)\Lambda_2(\mathrm{d}u). \tag{17}$$

Recall that we only consider strongly separable versions of all three fields $(X_v)_{v\in\mathbb{R}^d}$, $(Y_v)_{v\in\mathbb{R}^d}$ and $(Z_v)_{v\in\mathbb{R}^d}$. When instead the index set is a specific bounded set $B \subseteq \mathbb{R}^d$, we additionally choose a bounded version, made possible by the following lemma. The boundedness will be essential in several arguments.

Lemma 1 *Let (X_v), (Y_v) and (Z_v) be given as above and assume that either Assumption 1 or Assumption 2 is satisfied. Then all three random fields have bounded versions on any fixed bounded index set.*

Proof Let B be some fixed, bounded index set. First, under Assumption 1 we find from (Rønn-Nielsen and Stehr, 2021, Lemma 4) that both terms in (Z_v) and the first two terms of (Y_v) have continuous versions on B. Using the decomposition

$$\int_{\mathbb{R}^d} f(v-u)\Lambda_2(\mathrm{d}u) = \int_{\mathbb{R}^d} f^+(v-u)\Lambda_2(\mathrm{d}u) - \int_{\mathbb{R}^d} f^-(v-u)\Lambda_2(\mathrm{d}u)$$

of the third term in (Y_v), referring to (Rønn-Nielsen and Stehr, 2021, Lemma 3) gives continuous versions of each term separately. The continuous versions of the fields are in particular bounded. Secondly, we have immediately from (Rønn-Nielsen and Stehr, 2021, Lemma 5) that the three fields have bounded versions under Assumption 2.

The next result is a special case of (Stehr and Rønn-Nielsen, 2021, Lemma 11). Here and in the remainder of the paper, $y_+ = y1_{\{y\geq 0\}}$ for any $y \in \mathbb{R}$.

Lemma 2 *Let G be a subexponential distribution with tail $\overline{G}$. For all $\varepsilon > 0$ there are constants $C > 0$ and $x_0 > 0$ such that*

$$\overline{G}(x-y) \leq \overline{G}(x)C\exp(\varepsilon y_+)$$

for all $x \geq x_0$ and $y \in \mathbb{R}$.

Proof of Theorem 1 Fix the bounded and full-dimensional index set $B \subseteq \mathbb{R}^d$, and let $B^+ = B \ominus M^+$ and $B^- = B \ominus M^-$. Let m denote Lebesgue measure, and define the function H by

$$H(x) = m \otimes \rho\big(\{(u,z) \in \mathbb{R}^d \times \mathbb{R} : \sup_{v\in B} z f(v-u) > x\}\big)$$

for $x > 0$. For convenience, define $s_u^+ = (\sup_{v\in B} f^+(v-u))^{-1}$ and $s_u^- = (\sup_{v\in B} f^-(v-u))^{-1}$ for all $u \in \mathbb{R}^d$, and note that

$$\frac{H(x)}{\rho((x,\infty))} = \int_{\mathbb{R}^d} \Big(\frac{\rho((s_u^+ x,\infty))}{\rho((x,\infty))} + \frac{\rho((-\infty,-s_u^- x))}{\rho((x,\infty))} \Big) \mathrm{d}u.$$

Since $\|f^+\|_\infty = 1$, we have

$$s_u^+ \begin{cases} = 1 & \text{for } u \in B^+, \\ > 1 & \text{for } u \notin B^+, \end{cases} \qquad s_u^- \begin{cases} = \|f^-\|_\infty^{-1} & \text{for } u \in B^-, \\ > \|f^-\|_\infty^{-1} & \text{for } u \notin B^-, \end{cases}$$

and consequently $s_u^- > 1$ for all $u \in \mathbb{R}^d$ if $\|f^-\|_\infty < 1$. Since ρ satisfies (8) we obtain the convergence

$$\lim_{x\to\infty} \frac{\rho((s_u^+ x,\infty))}{\rho((x,\infty))} = \begin{cases} 1 & \text{for } u \in B^+, \\ 0 & \text{for } u \notin B^+, \end{cases}$$

whereas the left tail due to (8) and (10) satisfies

$$\lim_{x\to\infty} \frac{\rho((-\infty,-s_u^- x))}{\rho((x,\infty))} = \begin{cases} p^{-1}(1-p) & \text{if } \|f^-\|_\infty = 1 \text{ and } u \in B^-, \\ 0 & \text{else.} \end{cases}$$

Since g is integrable and decreasing, we in particular obtain that f, $(s_u^+)^{-1}$ and $(s_u^-)^{-1}$ are integrable. As $s_u^+, s_u^- \geq 1$ for all $u \in \mathbb{R}^d$ we find by dominated convergence, using the bound (9) and the tail-balance condition (10), that

$$\begin{aligned}\frac{H(x)}{\rho((x,\infty))} &\to \int_{\mathbb{R}^d} \lim_{x\to\infty}\Big(\frac{\rho((s_u^+ x,\infty))}{\rho((x,\infty))} + \frac{\rho((-\infty,-s_u^- x))}{\rho((x,\infty))}\Big)\mathrm{d}u \\ &= |B^+| + p^{-1}(1-p)|B^-| 1_{\{\|f^-\|_\infty = 1\}}\end{aligned}$$

as $x \to \infty$. Since $|B^+|, |B^-| < \infty$ and $\mathscr{S} \cap \mathrm{MDA}(\mathrm{Gum})$ is closed under tail equivalence, the distribution $1 - \min\{H, 1\}$ is in particular subexponential. Recall that $(X_v)_{v\in B}$ is chosen to be strongly separable and let $T \subset B$ be the corresponding countable separating dense subset. By Lemma 1, the field (X_v) is almost surely bounded on B,

$$\mathbb{P}\big(\sup_{v\in B} |X_v| < \infty\big) = \mathbb{P}\big(\sup_{v\in T} |X_v| < \infty\big) = 1,$$

and, using that $\sup_{v\in B} X_v$ and $\sup_{v\in T} X_v$ coincide due to (Samorodnitsky and Taqqu, 1994, Theorem 9.2.4), we conclude by (Rosinski and Samorodnitsky, 1993, Theorem 3.1) that

$$\mathbb{P}\big(\sup_{v\in B} X_v > x\big) \sim H(x) \sim \rho((x,\infty))\Big[|B^+| + p^{-1}(1-p)|B^-| 1_{\{\|f^-\|_\infty = 1\}}\Big]$$

as $x \to \infty$.

Proof of Theorem 2 As in the previous proof, let $B^+ = B \ominus M^+$ and $B^- = B \ominus M^-$. Let F denote the distribution of $\sup_{v\in B} X_v$, which, by Theorem 1, is subexponential. Therefore

$$\frac{\overline{F}(x-y)}{\overline{F}(x)} \to 1 \tag{18}$$

for all $y \in \mathbb{R}$ as $x \to \infty$. Let π^1 denote the distribution of $(Y_v^1)_v$, and let $y^* = \sup_{v\in B} y_v$ for a deterministic field $(y_v)_v$. Then

$$\begin{aligned}\frac{\mathbb{P}(\sup_{v\in B}(X_v + Y_v^1) > x)}{\overline{F}(x)} &\leq \frac{\mathbb{P}(\sup_{v\in B} X_v > x - \sup_{v\in B} Y_v^1)}{\overline{F}(x)} \\ &= \int \frac{\overline{F}(x-y^*)}{\overline{F}(x)} \pi^1(\mathrm{d}y),\end{aligned}$$

with the integrand tending to 1 by (18) above. Now fix $\varepsilon > 0$ such that (14) is satisfied. By Lemma 2, the integrand has an x-independent and integrable upper bound, and we thus obtain by dominated convergence and (13) that

$$\limsup_{x\to\infty} \frac{\mathbb{P}(\sup_{v\in B}(X_v + Y_v^1) > x)}{\rho((x,\infty))} \leq |B^+| + p^{-1}(1-p)|B^-| 1_{\{\|f^-\|_\infty = 1\}}.$$

Since $\exp(\varepsilon \inf_{v\in B} Y_v^1)$ is also integrable, we similarly find that

$$\liminf_{x\to\infty} \frac{\mathbb{P}(\sup_{v\in B}(X_v + Y_v^1) > x)}{\rho((x,\infty))} \geq |B^+| + p^{-1}(1-p)|B^-| 1_{\{\|f^-\|_\infty = 1\}},$$

proving the claim.

4 Extremal results

In this section we give the detailed proof of the main result Theorem 3. Some proofs and the overall structure of arguments follow that of (Rønn-Nielsen and Stehr, 2021, Sections 5 & 6), and thus some arguments will be shortened or omitted here. Throughout the section we assume that either Assumption 1 or 2 is satisfied, which in particular ensures that Assumption M is so as well. Moreover, the geometric requirements on the index sets (C_n) presented in Assumption 3 are assumed satisfied.

First, we introduce a geometric construction of sets sufficiently approximating the index sets (C_n). We give the full construction here even though it is identical to that of Rønn-Nielsen and Stehr (2021).

In defining our approximative sets, we need the notion of cubes: When referring to a discrete or continuous cube with a certain side-length $s > 0$, we mean a box where all sides have length s. Furthermore, we let $C_r(u)$ denote the closed r-cube with corner $u \in \mathbb{R}^d$, i.e.

$$C_r(u) = u + [0,r]^d$$

for $r > 0$.

A cornerstone for the following results to hold is that Assumption 3 admits approximating C_n by a union of cubes each with a volume increasing with the same rate as $|C_n|$, such that the quality of the approximation in the limit $n \to \infty$ is controlled by the number of cubes. The precise ramifications of this approximation are found in Theorem 5 below.

We define the side length $t_{n,k,L} \in \mathbb{N}$ of the approximating cubes by

$$t_{n,k,L} = L \cdot \left\lfloor \sqrt[d]{|C_n|/k} \right\rfloor$$

for each $k, L \in \mathbb{N}$. Next, we define the cubes themselves: For each $z = (z_1, \ldots, z_d) \in \mathbb{Z}^d$ and n large enough relative to k, we define the cube $I_z^{n,k,L} \subseteq \mathbb{R}^d$ as

$$I_z^{n,k,L} = z t_{n,k,L} + [0, t_{n,k,L})^d = \bigtimes_{i=1}^{d} [z_i t_{n,k,L}, (z_i + 1) t_{n,k,L}).$$

To understand the indices, we have that z determines the placement of the cube relative to the grid defined by $t_{n,k,L}$, n refers to the index of C_n, i.e. the set that is being approximated, k determines the number of cubes being used to approximate C_n (in the limit), and L determines the size of these cubes relative to the size of C_n. Clearly, using both k and L seems to be an overparametrization but having both in the notation will be helpful in a later two-step argument.

The actual approximation of C_n is obtained by collecting the indices $z \in \mathbb{Z}^d$ for which $I_z^{n,k,L}$ is contained in C_n, and the indices z for which $I_z^{n,k,L}$ intersects C_n, in $P_{n,k,L}$ and $Q_{n,k,L}$ respectively:

$$P_{n,k,L} = \{z \in \mathbb{Z}^d \,:\, I_z^{n,k,L} \subseteq C_n\} \quad \text{and} \quad Q_{n,k,L} = \{z \in \mathbb{Z}^d \,:\, I_z^{n,k,L} \cap C_n \neq \emptyset\}.$$

Furthermore, we let the number of such inner and outer approximating boxes be denoted by $p_{n,k,L} = |P_{n,k,L}|$ and $q_{n,k,L} = |Q_{n,k,L}|$, respectively. Also note that, by construction, $p_{n,k,L} \leq k/L^d$ and $q_{n,k,L} \geq k/L^d$ for values of n large enough relative to k.

When proving our results, we will make a discretization by further dividing each $I_z^{n,k,L}$ into cubes of side-length L, $C_L(v)$ for $v \in (L\mathbb{Z})^d$. For this, define the corresponding grid points of $I_z^{n,k,L}$ by $J_z^{n,k,L} = I_z^{n,k,L} \cap (L\mathbb{Z})^d$, and note that $|J_z^{n,k,L}| = (t_{n,k,L})^d / L^d \sim |C_n|/k$ as $n \to \infty$. Lastly, we define two sets of grid points

$$D^-_{n,k,L} = \bigcup_{z \in P_{n,k,L}} J_z^{n,k,L} \quad \text{and} \quad D^+_{n,k,L} = \bigcup_{z \in Q_{n,k,L}} J_z^{n,k,L},$$

which, when continuously filled with cubes $C_L(z)$, approximate C_n from the inside and outside, respectively

$$\bigcup_{z \in D^-_{n,k,L}} C_L(z) \subseteq C_n \subseteq \bigcup_{z \in D^+_{n,k,L}} C_L(z).$$

It should be noted that the set $D^-_{n,k,L}$ will not be used in this paper. Apart from contributing to understanding the approximation scheme, it does, however, appear indirectly when we refer to results and proofs in Rønn-Nielsen and Stehr (2021), where the set plays an important role.

The following result is derived in Rønn-Nielsen and Stehr (2021) and will be used throughout the remainder of the paper. In particular, it is essential that the discrete index set $N_{k,L}$ introduced in (ii) below is independent of the choice of n. Hence, for all k, L, all the approximations $D^+_{n,k,L}$ are contained in the same finite collection of (increasing) sets of grid points $J_z^{n,k,L}$.

Theorem 5 ((Rønn-Nielsen and Stehr, 2021, Theorem 6)) *Let $(C_n)_{n \in \mathbb{N}}$ satisfy Assumption 3. Then,*

(i) *for all $L \in \mathbb{N}$ the sequences $p_{n,k,L}$ and $q_{n,k,L}$, defined above, satisfy that*

$$\liminf_{n\to\infty} p_{n,k,L} \sim \frac{k}{L^d} \qquad \text{and} \qquad \limsup_{n\to\infty} q_{n,k,L} \sim \frac{k}{L^d}$$

as $k \to \infty$,

(ii) *for each k, L and n with n large enough relative to k, it holds that*

$$D^+_{n,k,L} \subseteq \bigcup_{z \in N_{k,L}} J_z^{n,k,L},$$

where $N_{k,L}$ is on the form $N_{k,L} = [-c_{k,L}\, ,\, c_{k,L}]^d \cap \mathbb{Z}^d$ for some $0 < c_{k,L} < \infty$,

(iii) *for all $L \in \mathbb{N}$ there exists $0 < c_L < \infty$ such that for all $n \in \mathbb{N}$*

$$D^+_{n,k,L} \subseteq K_{n,L} = \big[-c_L \cdot |C_n|^{1/d}\, ,\, c_L \cdot |C_n|^{1/d}\big]^d \cap (L\mathbb{Z})^d.$$

The lemma below will be used repeatedly in this section. It ensures in particular that the field $(Y_v)_v$ has exponential moments of all orders and thus is of minor importance when determining the extremal behavior of the Lévy-driven field $(X_v)_v$. It follows by arguments similar to those of (Rønn-Nielsen and Stehr, 2021, Lemma 9): Making the independent decomposition $Y_v = Y_v^- + Y_v^C$, where Y_v^- is non-positive and Y_v^C has Lévy measure restricted to $[-1,1]$, an application of (Braverman and Samorodnitsky, 1995, Lemma 2.1) gives exponential moments of (Y_v^C) and thus of the entire field (Y_v).

Lemma 3 *Let the field $(Y_v)_{v \in \mathbb{R}^d}$ be given by (17). Then*

$$\mathbb{E}\exp\big(\varepsilon \sup_{v \in C_L(0)} Y_v\big) < \infty$$

for all $\varepsilon > 0$ and all $L \in \mathbb{N}$.

The following theorem, which utilizes that the field $(Y_v)_{v\in\mathbb{R}^d}$ is ergodic, is a simple adaption of Corollary 4 of Stehr and Rønn-Nielsen (2021). In that paper, the integration kernel is assumed positive, however, it is easily seen that the claim is also true for an arbitrary kernel. For a greater exposition of spatial ergodicity, we refer to their Section 5. In the theorem and in the remainder of the paper, we write J_z for $J_z^{n,k,L}$ to avoid too heavy notation. Furthermore, we recall the set $N_{k,L}$ defined in Theorem 5(ii) above.

Theorem 6 *Let the field* $(Y_v)_{v\in\mathbb{R}^d}$ *be given by* (17)*, and let h be a function satisfying*

$$\mathbb{E}|h((Y_u)_{u\in C_L(0)})| < \infty.$$

For all $z \in N_{k,L}$ *it then holds that*

$$\frac{1}{|J_z|}\sum_{v\in J_z} h((Y_{u+v})_{u\in C_L(0)}) \to \mathbb{E}h((Y_u)_{u\in C_L(0)})$$

almost surely as $n\to\infty$*. The result also holds true if* J_z *is replaced with a subset of* J_z *in the shape of a box, which increases in size asymptotically as* J_z*.*

In proving Theorem 3 below we make use of a field $(Z_v^{(t)})_{v\in\mathbb{R}^d}$ which approximates the field $(Z_v)_{v\in\mathbb{R}^d}$ defined in (16) as $t\to\infty$. For fixed $t>0$ we define it as

$$Z_v^{(t)} = \int_{\{|v-u|<t\}} f^+(v-u)\Lambda_+(\mathrm{d}u) - \int_{\{|v-u|<t\}} f^-(v-u)\Lambda_-(\mathrm{d}u), \tag{19}$$

and note that it by construction satisfies that $Z_v^{(t)} \le Z_v$, and that $Z_v^{(t)}$ and $Z_{v'}^{(t)}$ are independent for $|v-v'|>2t$. We use this to bound the asymptotic distribution function of $\sup_{C_n} Z_v$ with that of $\sup_{C_n} Z_v^{(t)}$, which is more easily managed due to independence, but which asymptotically behaves essentially like $\sup_{C_n} Z_v$. As it turns out, the final result will not depend on the fixed finite t as long as it is chosen sufficiently large, meaning that the ball $B(t)$ of radius t contains all the maximum points of f^+ and f^-, respectively, i.e. $M^+\cup M^- \subseteq B(t)$. Thus, we only consider t to be such a fixed sufficiently large value and to take the value $t=\infty$ corresponding to the field $Z_v^{(\infty)} = Z_v$. With this construction, the following lemma follows by the same arguments as Theorem 1 realizing that $Z_v^{(t)}$ is almost surely bounded on bounded sets.

Lemma 4 *Let* $B\subseteq\mathbb{R}^d$ *be a fixed, full-dimensional and bounded set. Then, for* $t>0$ *sufficiently large,*

$$\mathbb{P}\big(\sup_{v\in B} Z_v^{(t)} > x\big) \sim \Big(\big|B\ominus M^+\big| + \frac{1-p}{p}\big|B\ominus M^-\big| 1_{\{\|f^-\|_\infty = 1\}}\Big)\rho((x,\infty))$$

as $x\to\infty$*. In particular,* $\sup_{v\in B} Z_v^{(t)} \in \mathscr{S}\cap\mathrm{MDA}(\mathrm{Gum})$.

In the remainder of the paper we fix $x\in\mathbb{R}$ and write

$$x_n = a_n x + b_n,$$

where $(a_n), (b_n)$ are the norming constants of ρ relative to $|C_n|$ satisfying (15). Since $a_n = o(b_n)$ with $b_n \to \infty$, we in particular have that $x_n \to \infty$ as $n \to \infty$ for all fixed $x \in \mathbb{R}$. Lemma 4 therefore implies that

$$\begin{aligned}\lim_{n\to\infty} &|C_n|\mathbb{P}\big(\sup_{u\in C_L(v)} Z_u^{(t)} > x_n\big)\\ &= e^{-x}\rho((1,\infty))\big[|C_L(v)\ominus M^+| + p^{-1}(1-p)|C_L(v)\ominus M^-|1_{\{\|f^-\|_\infty=1\}}\big]\\ &= e^{-x}\rho((1,\infty))\big[|C_L(0)\ominus M^+| + p^{-1}(1-p)|C_L(0)\ominus M^-|1_{\{\|f^-\|_\infty=1\}}\big]\end{aligned}$$

for all $v \in (L\mathbb{Z})^d$. Note that this limit is asymptotically equivalent to

$$\mathrm{e}^{-x}\rho((1,\infty))L^d\big[1 + p^{-1}(1-p)1_{\{\|f^-\|_\infty=1\}}\big]$$

as $L \to \infty$. This is seen by writing e.g.

$$|C_L(0)\ominus M^+| = L^d|C_1(0)\ominus(M^+/L)|,$$

which is equivalent to L^d. For notational convenience in the remainder of the paper, we let (for fixed $x \in \mathbb{R}$)

$$\tau_L = e^{-x}\rho((1,\infty))\big[|C_L(0)\ominus M^+| + p^{-1}(1-p)|C_L(0)\ominus M^-|1_{\{\|f^-\|_\infty=1\}}\big],$$

and $\tau = \lim_{L\to\infty} L^{-d}\tau_L$ be given by

$$\tau = \mathrm{e}^{-x}\rho((1,\infty))\big[1 + p^{-1}(1-p)1_{\{\|f^-\|_\infty=1\}}\big].$$

Lemma 5 *Let $(Z_v^{(t)})_v$ and $(Y_v)_v$ be given by* (19) *and* (17)*, respectively. Then, for almost all realizations $(y_v)_v$ of $(Y_v)_v$, it holds for all $z \in N_{k,L}$ and for all $t > 0$ sufficiently large that*

$$\frac{|C_n|}{|J_z|}\sum_{v\in J_z}\mathbb{P}\big(\sup_{u\in C_L(v)}(Z_u^{(t)} + y_u) > x_n\big) \to \tau_L \tag{20}$$

as $n \to \infty$. The result also holds true if J_z is replaced with a subset of J_z in the shape of a box, which increases in size asymptotically as J_z.

Before proving the claim we give a short remark on the last statement of the lemma: In this paper we do not explicitly use the result with J_z replaced by an asymptotically size-equivalent box. However, when referring to results from Rønn-Nielsen and Stehr (2021) in the proof of Lemma 7, the result is implicitly used.

Proof We only show the convergence (20) as the expression for J_z replaced by an asymptotically size-equivalent box follows identically.

Throughout this proof we fix $\varepsilon > 0$ such that

$$\mathbb{E}\exp\Big(\varepsilon\big(\sup_{v\in C_L(0)} Y_v\big)_+\Big) < \infty, \tag{21}$$

which is possible due to Lemma 3.

Let $F_{L,t}$ denote the distribution of $\sup_{u\in C_L(v)} Z_u^{(t)}$, and note that $F_{L,t} \in \mathscr{S}\cap\mathrm{MDA}(\mathrm{Gum})$ by Lemma 4. Then,

$$|C_n|\overline{F}_{L,t}(x_n) \to \tau_L$$

as $n \to \infty$, where $\overline{F}_{L,t}$ is the tail of $F_{L,t}$. Since in particular $F_{L,t} \in \mathscr{S}$, the convergence

$$\frac{\overline{F}_{L,t}(x_n - y)}{\overline{F}_{L,t}(x_n)} \to 1$$

as $n \to \infty$ is uniform for all $|y| \leq N$, for all fixed $N \in \mathbb{N}$; see e.g. (Pakes, 2004, Definition 2.1). Hence, with $y^*(v) = \sup_{C_L(v)} y_u$, we find by an application of Lemma 2 and Theorem 6, utilizing (21), that

$$\begin{aligned} &\frac{|C_n|}{|J_z|} \sum_{v \in J_z} \overline{F}_{L,t}(x_n - y^*(v)) \\ &\quad \leq \frac{|C_n|\overline{F}_{L,t}(x_n)}{|J_z|} \sum_{v \in J_z} \Big(\frac{\overline{F}_{L,t}(x_n - y^*(v))}{\overline{F}_{L,t}(x_n)} 1_{\{|y^*(v)| \leq N\}} \\ &\qquad\qquad\qquad\qquad + C \exp\big(\varepsilon y^*(v)_+\big) 1_{\{|y^*(v)| > N\}} \Big) \\ &\quad \to \tau_L \mathbb{E}\Big(1_{\{|\sup_{C_L(0)} Y_v| \leq N\}} + C \exp\big(\varepsilon (\sup_{C_L(0)} Y_v)_+\big) 1_{\{|\sup_{C_L(0)} Y_v| > N\}} \Big) \end{aligned}$$

as $n \to \infty$, almost surely. Letting $N \to \infty$ shows by monotone convergence that

$$\limsup_{n \to \infty} \frac{|C_n|}{|J_z|} \sum_{v \in J_z} \overline{F}_{L,t}(x_n - y^*(v)) \leq \tau_L$$

almost surely. By similar arguments we find that also

$$\liminf_{n \to \infty} \frac{|C_n|}{|J_z|} \sum_{v \in J_z} \overline{F}_{L,t}(x_n - \overline{y}(v)) \geq \tau_L,$$

where we used the notation $\overline{y}(v) = \inf_{C_L(v)} y_u$. As

$$\overline{F}_{L,t}(x_n - \overline{y}(v)) \leq \mathbb{P}\big(\sup_{u \in C_L(v)} (Z_u^{(t)} + y_u) > x_n \big) \leq \overline{F}_{L,t}(x_n - y^*(v)),$$

the convergence (20) follows.

We now define the last pieces of convenient notation. For a subset $A \subseteq (L\mathbb{Z})^d$ and a deterministic field $(y_v)_v$ we let

$$M_{L,y}^{(t)}(A) = \max_{z \in A} \sup_{u \in C_L(z)} (Z_u^{(t)} + y_u).$$

When considering the field (Z_v), we simply write $M_{L,y}$ for $M_{L,y}^{(\infty)}$. Furthermore, two subsets A, B of $\mathbb{Z}^d$ or $\mathbb{R}^d$ are said to be γ_n-separated if $\operatorname{dist}(A,B) = \inf\{|a-b| \,:\, a \in A, b \in B\} \geq \gamma_n$ and there are disjoint sets $A', B' \subseteq \mathbb{R}^d$, both of which are connected, such that $A \subseteq A'$ and $B \subseteq B'$.

The following lemma gives that conditioned on (Y_v) the (X_v)–field exhibits a mixing behavior in the tail. The proof follows the lines of (Stehr and Rønn-Nielsen, 2021, Lemma 6) but uses an enhanced notation due to the increased complexity of the kernel function.

Lemma 6 *There is a sequence $\gamma_n = o(\sqrt[d]{|C_n|}) \geq L$ such that for all γ_n-separated sets $A, B \subseteq K_{n,L} \subseteq (L\mathbb{Z})^d$, where at least one is a box, and where $K_{n,L}$ is defined in Theorem 5*(iii)*, it holds that*

$$\left|\mathbb{P}(M_{L,y}(A \cup B) \leq x_n) - \mathbb{P}(M_{L,y}(A) \leq x_n)\mathbb{P}(M_{L,y}(B) \leq x_n)\right| \leq \alpha_{y,n,L},$$

where $\alpha_{y,n,L} \to 0$ as $n \to \infty$ for all $L \in \mathbb{N}$ and almost all realizations $(y_v)_v$ of $(Y_v)_v$.

Proof We let $\gamma_n/2 = \eta(\sqrt[d]{|C_n|})$, where η is a function chosen according to Lemma 14 of Stehr and Rønn-Nielsen (2021) and g is an upper bound to $|f|$ given in Assumption M. This choice of γ_n in particular ensures that

$$|C_n|^{(d-1)/d} \int_{\gamma_n/2}^{\infty} g(x)\mathrm{d}x \to 0 \tag{22}$$

as $n \to \infty$. Let the sets A and B be given as in the lemma and define the continuous expansions

$$\mathscr{A} = \bigcup_{v \in A} C_L(v), \qquad \mathscr{B} = \bigcup_{v \in B} C_L(v), \qquad \mathscr{K}_{n,L} = \bigcup_{v \in K_{n,L}} C_L(v),$$

where we assume without loss of generality that A is a box. Hence, $\mathscr{A}$ is a continuous box and in particular a convex body. With $B(r)$ denoting the closed ball in $\mathbb{R}^d$ of radius $r \geq 0$ with center $0 \in \mathbb{R}^d$, we define $\mathscr{A}_n = \mathscr{A} \oplus B(\gamma_n/2)$ and $\mathscr{B}_n = \big(\mathscr{A} \oplus B(\gamma_n)\big)^c = \big(\mathscr{A}_n \oplus B(\gamma_n/2)\big)^c$. Note that $\mathscr{B} \subseteq \mathscr{B}_n$ since $\mathscr{A}$ and $\mathscr{B}$ are γ_n-separated (strictly speaking they are only $(\gamma_n - L)$-separated, however, this is equivalent as $n \to \infty$).

For all $v \in \mathscr{A}$ let

$$Z_v^A = \int_{\mathscr{A}_n} f^+(v-u)\Lambda_+(\mathrm{d}u) - \int_{\mathscr{A}_n} f^-(v-u)\Lambda_-(\mathrm{d}u), \qquad \text{and}$$
$$\overline{Z}_v^A = \int_{\mathscr{A}_n^c} f^+(v-u)\Lambda_+(\mathrm{d}u) - \int_{\mathscr{A}_n^c} f^-(v-u)\Lambda_-(\mathrm{d}u).$$

Similarly, for all $v \in \mathscr{B}$,

$$Z_v^B = \int_{\mathscr{A}_n^c} f^+(v-u)\Lambda_+(\mathrm{d}u) - \int_{\mathscr{A}_n^c} f^-(v-u)\Lambda_-(\mathrm{d}u), \qquad \text{and}$$
$$\overline{Z}_v^B = \int_{\mathscr{A}_n} f^+(v-u)\Lambda_+(\mathrm{d}u) - \int_{\mathscr{A}_n} f^-(v-u)\Lambda_-(\mathrm{d}u).$$

By construction, that is, since Λ_+ is a positive measure and Λ_- is a negative measure, we have

$$\sup_{v \in \mathscr{A}} \overline{Z}_v^A \leq \int_{\mathscr{A}_n^c} \sup_{v \in \mathscr{A}} f^+(v-u)\Lambda_+(\mathrm{d}u) - \int_{\mathscr{A}_n^c} \sup_{v \in \mathscr{A}} f^-(v-u)\Lambda_-(\mathrm{d}u)$$

and

$$\sup_{v \in \mathscr{B}} \overline{Z}_v^B \leq \int_{\mathscr{A}_n} \sup_{v \in \mathscr{B}} f^+(v-u)\Lambda_+(\mathrm{d}u) - \int_{\mathscr{A}_n} \sup_{v \in \mathscr{B}} f^-(v-u)\Lambda_-(\mathrm{d}u).$$

By (22) above, we may choose a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that $\varepsilon_n \downarrow 0$ and

$$\frac{1}{\varepsilon_n}|C_n|^{(d-1)/d} \int_{\gamma_n/2}^{\infty} g(x)\mathrm{d}x \to 0. \tag{23}$$

With this choice we define the events

$$S_n^A = \big(\sup_{v \in \mathscr{A}} \overline{Z}_v^A \leq \varepsilon_n\big) \qquad \text{and} \qquad S_n^B = \big(\sup_{v \in \mathscr{B}} \overline{Z}_v^B \leq \varepsilon_n\big).$$

Let Λ'_+ and Λ'_- be the so-called spot variables corresponding to Λ_+ and Λ_-. That is variables which in distribution are equal to $\Lambda_+(S)$ and $\Lambda_-(S)$, respectively, for a set $S \subseteq \mathbb{R}^d$ with $|S| = 1$. Utilizing that the underlying Lévy measures of Λ_+ and Λ_- have finite first moments, that g is decreasing, and that $\mathscr{A}$ is a convex body, we have by stochastic dominance and Markov's inequality that

$$\begin{aligned}\mathbb{P}\big((S_n^A)^c\big) &\le \frac{1}{\varepsilon_n}\big(\mathbb{E}(\Lambda'_+)\int_{\mathscr{A}_n^c}\sup_{v\in\mathscr{A}} f^+(v-u)\mathrm{d}u - \mathbb{E}(\Lambda'_-)\int_{\mathscr{A}_n^c}\sup_{v\in\mathscr{A}} f^-(v-u)\mathrm{d}u\big)\\ &\le \frac{1}{\varepsilon_n}(\mathbb{E}(\Lambda'_+)-\mathbb{E}(\Lambda'_-))\Big[\sum_{j=0}^{d-1}\tilde{\mu}_j|C_n|^{j/d}\int_{\gamma_n/2}^{\infty} g(x)x^{d-j-1}\mathrm{d}x\Big]\end{aligned} \tag{24}$$

for $\mathscr{A}$- and n-independent constants $\tilde{\mu}_j$. Similarly, we find

$$\mathbb{P}\big((S_n^B)^c\big) \le \frac{1}{\varepsilon_n}(\mathbb{E}(\Lambda'_+)-\mathbb{E}(\Lambda'_-))\Big[\sum_{j=0}^{d-1}\tilde{\mu}_j|C_n|^{j/d}\int_{\gamma_n/2}^{\infty} g(x)x^{d-j-1}\mathrm{d}x\Big]. \tag{25}$$

Equivalently to the definition of $M_{L,y}(A)$, we define

$$M_{L,y}^A(A) = \sup_{v\in\mathscr{A}}(Z_v^A + y_v), \qquad \text{and} \qquad M_{L,y}^B(B) = \sup_{v\in\mathscr{B}}(Z_v^B + y_v).$$

Then, using straightforward manipulations, we find

$$\begin{aligned}&\mathbb{P}(M_{L,y}^A(A) \le x_n-\varepsilon_n, S_n^A)\mathbb{P}(M_{L,y}^B(B) \le x_n-\varepsilon_n, S_n^B)\\ &\quad\le \mathbb{P}(M_{L,y}(A) \le x_n)\mathbb{P}(M_{L,y}(B) \le x_n)\\ &\quad\le \mathbb{P}(M_{L,y}^A(A) \le x_n, S_n^A)\mathbb{P}(M_{L,y}^B(B) \le x_n, S_n^B) + \mathbb{P}\big((S_n^A)^c\big) + \mathbb{P}\big((S_n^B)^c\big),\end{aligned} \tag{26}$$

and similarly, also using that $M_{L,y}^A$ and $M_{L,y}^B$ are independent since $\mathscr{A}_n$ and $\mathscr{A}_n^c$ are disjoint, we find

$$\begin{aligned}&\mathbb{P}(M_{L,y}^A(A) \le x_n-\varepsilon_n, S_n^A)\mathbb{P}(M_{L,y}^B(B) \le x_n-\varepsilon_n, S_n^B) - \mathbb{P}\big((S_n^A)^c\big) - \mathbb{P}\big((S_n^B)^c\big)\\ &\quad\le \mathbb{P}(M_{L,y}(A\cup B) \le x_n)\\ &\quad\le \mathbb{P}(M_{L,y}^A(A) \le x_n, S_n^A)\mathbb{P}(M_{L,y}^B(B) \le x_n, S_n^B) + \mathbb{P}\big((S_n^A)^c\big) + \mathbb{P}\big((S_n^B)^c\big).\end{aligned} \tag{27}$$

Combining (26) and (27) gives

$$\begin{aligned}&\big|\mathbb{P}(M_{L,y}(A\cup B) \le x_n) - \mathbb{P}(M_{L,y}(A) \le x_n)\mathbb{P}(M_{L,y}(B) \le x_n)\big|\\ &\quad\le \mathbb{P}(M_{L,y}^A(A) \le x_n, S_n^A)\mathbb{P}(M_{L,y}^B(B) \le x_n, S_n^B)\\ &\qquad - \mathbb{P}(M_{L,y}^A(A) \le x_n-\varepsilon_n, S_n^A)\mathbb{P}(M_{L,y}^B(B) \le x_n-\varepsilon_n, S_n^B)\\ &\qquad + 2\Big(\mathbb{P}\big((S_n^A)^c\big) + \mathbb{P}\big((S_n^B)^c\big)\Big)\\ &\quad\le \mathbb{P}(M_{L,y}^A(A) \le x_n, S_n^A) - \mathbb{P}(M_{L,y}^A(A) \le x_n-\varepsilon_n, S_n^A)\\ &\qquad + \mathbb{P}(M_{L,y}^B(B) \le x_n, S_n^B) - \mathbb{P}(M_{L,y}^B(B) \le x_n-\varepsilon_n, S_n^B)\\ &\qquad + 2\Big(\mathbb{P}\big((S_n^A)^c\big) + \mathbb{P}\big((S_n^B)^c\big)\Big),\end{aligned} \tag{28}$$

where the first term of (28) satisfies

$$
\begin{aligned}
0 &\le \mathbb{P}(M^A_{L,y}(A) \le x_n, S^A_n) - \mathbb{P}(M^A_{L,y}(A) \le x_n - \varepsilon_n, S^A_n) \\
&\le \sum_{v \in K_{n,L}} \mathbb{P}(M_{L,y}(\{v\}) > x_n - \varepsilon_n) - \sum_{v \in K_{n,L}} \mathbb{P}(M_{L,y}(\{v\}) > x_n + \varepsilon_n)
\end{aligned}
$$

with $K_{n,L}$ given as in Theorem 5. Since $\varepsilon_n \to 0$, we have, with an argument identical to the proof of Lemma 5, that the two sums have the same limit as $n \to \infty$ for almost all realizations (y_v) of (Y_v). Note that neither sum depends on the choice of A and B. The convergence of the second term in (28) follows similarly. That the third term has an upper bound, independent of A and B and converging to 0, follows from (24) and (25) in combination with (23). This concludes the proof.

The next result, which builds upon the mixing behavior from Lemma 6, shows that the limit distribution of $\sup_{v \in C_n}(Z_v + y_v)$ behaves approximately as if the field (Z_v) was independent on disjoint cubes C_L. The result follows by a range of lemmas from Rønn-Nielsen and Stehr (2021)

Lemma 7 *Let $(Z_v)_v$ and $(Y_v)_v$ be given by* (16) *and* (17)*, respectively. Then, for almost all realizations $(y_v)_v$ of $(Y_v)_v$, the following is satisfied*

$$
\begin{aligned}
\Big(\liminf_{n\to\infty} \min_{z \in N_{k,L}} \mathbb{P}(M_{L,y}(J_z) \le x_n)\Big)^{\tilde{q}_{k,L}} &\le \liminf_{n\to\infty} \mathbb{P}\big(\sup_{v \in C_n}(Z_v + y_v) \le x_n\big) \\
&\le \limsup_{n\to\infty} \mathbb{P}\big(\sup_{v \in C_n}(Z_v + y_v) \le x_n\big) \\
&\le \Big(\limsup_{n\to\infty} \max_{z \in N_{k,L}} \mathbb{P}(M_{L,y}(J_z) \le x_n)\Big)^{\tilde{p}_{k,L}},
\end{aligned}
$$

where $\tilde{p}_{k,L} = \liminf_n p_{n,k,L}$ and $\tilde{q}_{k,L} = \limsup_n q_{n,k,L}$, with $p_{n,k,L}$ and $q_{n,k,L}$ and $N_{k,L}$ as given in Theorem 5(i) *and* (ii)*, respectively.*

Proof The lemma follows directly by Lemmas 12–14 of Rønn-Nielsen and Stehr (2021) and the set-constructions therein.

In the remainder of the paper, we will let $\{v < v' \in J_z\}$ denote all pairs (v, v') in $J_z \times J_z$, where v' falls strictly after v according to some underlying enumeration. We note that $<$ is not related to the usual partial ordering on $J_z \times J_z$, but instead it is related to some enumeration such as e.g. the lexicographical ordering on lattices.

Lemma 8 *Let $(Z^{(t)}_v)_v$ and $(Y_v)_v$ be given by* (19) *and* (17)*, respectively. For all $t < \infty$ sufficiently large there is a sequence of functions g_L satisfying*

$$
\limsup_{k\to\infty} k\, g_L(k) = o(L^d) \tag{29}
$$

as $L \to \infty$, such that

$$
\limsup_{n\to\infty} \sum_{v < v' \in J_z} \mathbb{P}\big(\sup_{u \in C_L(v)} (Z^{(t)}_u + y_u) > x_n, \sup_{u \in C_L(v')} (Z^{(t)}_u + y_u) > x_n\big) \le g_L(k) \tag{30}
$$

for all $z \in N_{k,L}$ and almost all realizations $(y_v)_v$ of $(Y_v)_v$.

Proof Throughout this proof we fix $\varepsilon > 0$ such that

$$\mathbb{E}\exp\Big(\varepsilon\big(\sup_{v\in C_L(0)} Y_v\big)_+\Big) < \infty,$$

which is possible due to Lemma 3. Furthermore, let $F_{L,t} \in \mathscr{S} \cap \mathrm{MDA}(\mathrm{Gum})$ denote the subexponential distribution of $\sup_{u\in C_L(v)} Z_u^{(t)}$, and recall that

$$|C_n|\overline{F}_{L,t}(x_n) \to \tau_L \tag{31}$$

as $n \to \infty$.

Fix $L > 2t$. By construction, $\sup_{C_L(v)} Z_u^{(t)}$ and $\sup_{C_L(v')} Z_u^{(t)}$ are independent for all non-neighbors $v, v' \in J_z$ (recall that $J_z \subseteq (L\mathbb{Z})^d$), that is, for all $v, v' \in J_z$ such that $|v - v'| > L\sqrt{d}$. Writing $y^*(v) = \sup_{u\in C_L(v)} y_u$ and turning to Lemma 2, we find a constant $C > 0$ such that

$$\begin{aligned}
&\sum_{\substack{v<v'\in J_z\\ |v-v'|>L\sqrt{d}}} \mathbb{P}\big(\sup_{u\in C_L(v)} (Z_u^{(t)} + y_u) > x_n, \sup_{u\in C_L(v')} (Z_u^{(t)} + y_u) > x_n\big)\\
&\quad\le \sum_{v<v'\in J_z} \overline{F}_{L,t}(x_n - y^*(v))\overline{F}_{L,t}(x_n - y^*(v'))\\
&\quad\le \big(\overline{F}_{L,t}(x_n)\big)^2\Big(\sum_{v\in J_z} C\exp\big(\varepsilon y^*(v)_+\big)\Big)^2
\end{aligned}$$

for sufficiently large n and all $z \in N_{k,L}$. Since $|C_n|/k \sim |J_z|$ as $n \to \infty$, we conclude by Theorem 6 and (31) that

$$\begin{aligned}
&\limsup_{n\to\infty} \big(\overline{F}_{L,t}(x_n)\big)^2\Big(\sum_{v\in J_z} C\exp\big(\varepsilon y^*(v)_+\big)\Big)^2\\
&\quad= \frac{1}{k^2}\Big(\tau_L C\,\mathbb{E}\exp\big(\varepsilon(\sup_{v\in C_L(0)} Y_v)_+\big)\Big)^2
\end{aligned}$$

almost surely. This is independent of z and of order $o(k^{-1})$ as $k \to \infty$ for all $L > 2t$. This shows (29) and (30) for the terms in the sum with indices more than $L\sqrt{d}$ apart.

For notational convenience, define the mapping

$$B \mapsto R(B) = |B \ominus M^+| + p^{-1}(1-p)|B \ominus M^-| 1_{\{\|f^-\|_\infty = 1\}}$$

for a bounded set $B \subseteq \mathbb{R}^d$. Now consider $v, v' \in J_z$ such that $|v - v'| \le L\sqrt{d}$. Due to Lemma 4 we have for all fixed $v \ne v'$ that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\overline{F}_{L,t}(x_n)} \mathbb{P}\big(\sup_{u\in C_L(v)} Z_u^{(t)} > x_n - y, \sup_{u\in C_L(v')} Z_u^{(t)} > x_n - y\big)\\
&\quad\to 2 - \frac{R\big(C_L(v) \cup C_L(v')\big)}{R\big((C_L(0)\big)}
\end{aligned}$$

uniformly for $|y| \le N$, for all fixed $N \in \mathbb{N}$. Define $y^*(v, v') = \max\{y^*(v), y^*(v')\}$ and similarly for $(Y_v)_v$ and note that

$$\mathbb{E}\exp\big(\varepsilon Y^*(v, v')_+\big) \le 2\,\mathbb{E}\big(\varepsilon(\sup_{v\in C_L(0)} Y_v)_+\big) < \infty. \tag{32}$$

Combining the uniform convergence above with Theorem 6, Lemma 2, equation (31) and the fact that $|C_n|/k \sim |J_z|$ then yield

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{\substack{v<v'\in J_z \\ |v-v'|\le L\sqrt{d}}} \mathbb{P}\big(\sup_{u\in C_L(v)} (Z_u^{(t)}+y_u) > x_n, \sup_{u\in C_L(v')} (Z_u^{(t)}+y_u) > x_n\big) \\
&\quad\le \limsup_{n\to\infty} \sum_{\substack{v<v'\in J_z \\ |v-v'|\le L\sqrt{d}}} \mathbb{P}\big(\sup_{u\in C_L(v)} Z_u^{(t)} > x_n - y^*(v,v'), \sup_{u\in C_L(v')} Z_u^{(t)} > x_n - y^*(v,v')\big) \\
&\quad\le \frac{\tau_L}{k} \sum_{\substack{v\in(L\mathbb{Z})^d \\ 0<|v|\le L\sqrt{d}}} \mathbb{E}\big(1_{\{|Y^*(0,v)|\le N\}}\big)\Big(2-\frac{R\big(C_L(v)\cup C_L(0)\big)}{R\big((C_L(0)\big)}\Big) \\
&\qquad + \frac{\tau_L}{k} \sum_{\substack{v\in(L\mathbb{Z})^d \\ 0<|v|\le L\sqrt{d}}} C\,\mathbb{E}\Big[\exp\Big(\varepsilon\big(Y^*(0,v)_+\big)\Big)\, 1_{\{|Y^*(0,v)|>N\}}\Big]
\end{aligned}$$

for all $N\in\mathbb{N}$. Since this is independent of z, and due to the fact that there are only finitely many terms in the sum, we conclude (30) by letting $N\to\infty$ using a dominated convergence argument justified by (32):

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{\substack{v<v'\in J_z \\ |v-v'|\le L\sqrt{d}}} \mathbb{P}\big(\sup_{u\in C_L(v)} (Z_u^{(t)}+y_u) > x_n, \sup_{u\in C_L(v')} (Z_u^{(t)}+y_u) > x_n\big) \\
&\quad\le \frac{\tau_L}{k} \sum_{\substack{v\in(L\mathbb{Z})^d \\ 0<|v|\le L\sqrt{d}}} \Big(2-\frac{R\big(C_L(v)\cup C_L(0)\big)}{R\big((C_L(0)\big)}\Big).
\end{aligned}$$

The proof is finished once we show that this upper bound satisfies (29). Since τ_L is asymptotically equivalent to $\tau\cdot L^d$, this is the case if only the (k-independent) sum is of order $o(1)$ as $L\to\infty$. As the number of terms in the sum is fixed, it is a matter of showing that the summands tend to 0. It is easily seen that

$$2L^d + 2L^d p^{-1}(1-p)1_{\{\|f^-\|_\infty=1\}} \le R\big(C_L(v)\cup C_L(0)\big) \le 2R\big(C_L(0)\big)$$

for all $v\ne 0$, and by considerations as previously, $2R\big(C_L(0)\big)$ is asymptotically equivalent to $2L^d+2L^dp^{-1}(1-p)1_{\{\|f^-\|_\infty=1\}}$ as $L\to\infty$. This concludes the proof.

Proof of Theorem 3 We continue with the notation used throughout this section. In particular,

$$\tau_L = e^{-x}\rho\big((1,\infty)\big)\big[|C_L(0)\ominus M^+| + p^{-1}(1-p)|C_L(0)\ominus M^-|1_{\{\|f^-\|_\infty=1\}}\big],$$

and

$$\tau = \mathrm{e}^{-x}\rho\big((1,\infty)\big)\big[1+p^{-1}(1-p)1_{\{\|f^-\|_\infty=1\}}\big]$$

for all fixed $x\in\mathbb{R}$ with τ satisfying $\tau=\lim_{L\to\infty}\tau_L L^{-d}$. Now consider J_z for $z\in N_{k,L}$. Recalling the definition of $M_{L,y}$ and $M_{L,y}^{(t)}$ and the fact that $M_{L,y}^{(t)}\le M_{L,y}$, it can easily be seen that

$$\sum_{v\in J_z}\mathbb{P}(M_{L,y}^{(t)}(\{v\})>x_n)-\sum_{v<v'\in J_z}\mathbb{P}(M_{L,y}^{(t)}(\{v\})>x_n,M_{L,y}^{(t)}(\{v'\})>x_n)$$
$$\le \mathbb{P}(M_{L,y}(J_z)>x_n)$$
$$\le \sum_{v\in J_z}\mathbb{P}(M_{L,y}(\{v\})>x_n).$$

We now obtain from Lemma 7 that

$$\begin{aligned}
&\Big(\liminf_{n\to\infty}\min_{z\in N_{k,L}}\Big(1-\sum_{v\in J_z}\mathbb{P}\big(\sup_{u\in C_L(v)}(Z_u+y_u)>x_n\big)\Big)\Big)^{\tilde q_{k,L}}\\
&\quad\le \liminf_{n\to\infty}\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\
&\quad\le \limsup_{n\to\infty}\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\
&\quad\le \Big(\limsup_{n\to\infty}\max_{z\in N_{k,L}}\Big(1-\sum_{v\in J_z}\mathbb{P}\big(\sup_{u\in C_L(v)}(Z_u^{(t)}+y_u)>x_n\big)+S_{n,k,L}(z)\Big)\Big)^{\tilde p_{k,L}},
\end{aligned}\tag{33}$$

where

$$S_{n,k,L}(z)=\sum_{v<v'\in J_z}\mathbb{P}\big(\sup_{u\in C_L(v)}(Z_u^{(t)}+y_u)>x_n,\ \sup_{u\in C_L(v')}(Z_u^{(t)}+y_u)>x_n\big),$$

and $\tilde q_{k,L}=\limsup_n q_{n,k,L}$ and $\tilde p_{k,L}=\liminf_n p_{n,k,L}$. By Lemma 8 there is a sequence of functions $g_L(k)$ satisfying

$$\limsup_{k\to\infty} k\,g_L(k)=o(L^d)$$

as $L\to\infty$, such that $\limsup_n S_{n,k,L}(z)\le g_L(k)$ uniformly in z. Since $|C_n|/k\sim|J_z|$, we find by Lemma 5 and (33) that

$$\begin{aligned}
\Big(1-\frac{\tau_L}{k}\Big)^{\tilde q_{k,L}} &\le \liminf_{n\to\infty}\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\
&\le \limsup_{n\to\infty}\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\
&\le \Big(1-\frac{\tau_L}{k}+g_L(k)\Big)^{\tilde p_{k,L}}
\end{aligned}$$

almost surely. Since

$$\tilde p_{k,L}\sim\tilde q_{k,L}\sim\frac{k}{L^d}$$

as $k\to\infty$, we use the equivalence $\log(1-y)\sim -y$ ($y\to 0$) to obtain, as $k\to\infty$,

$$\begin{aligned}
\exp\Big(-\frac{\tau_L}{L^d}\Big) &\le \liminf_{n\to\infty}\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\
&\le \limsup_{n\to\infty}\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)\\
&\le \exp\Big(-\frac{\tau_L}{L^d}+o(1)\Big).
\end{aligned}$$

Here the remainder $o(1)$ is with respect to the limit $L\to\infty$. Secondly, taking the limit $L\to\infty$ shows that

$$\lim_{n\to\infty}\mathbb{P}\big(\sup_{v\in C_n}(Z_v+y_v)\le x_n\big)=\exp\big(-\tau\big)$$

almost surely.

Let π denote the distribution of the field $(Y_v)_v$. Then, by independence and dominated convergence,

$$\mathbb{P}\big(\sup_{v\in C_n} X_v \le x_n\big) = \int \mathbb{P}\big(\sup_{v\in C_n}(Z_v + y_v) \le x_n\big)\pi(\mathrm{d}y) \to \exp\big(-\tau\big)$$

as $n \to \infty$. This concludes the proof.

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