An Introduction to Linear Programming with Applications

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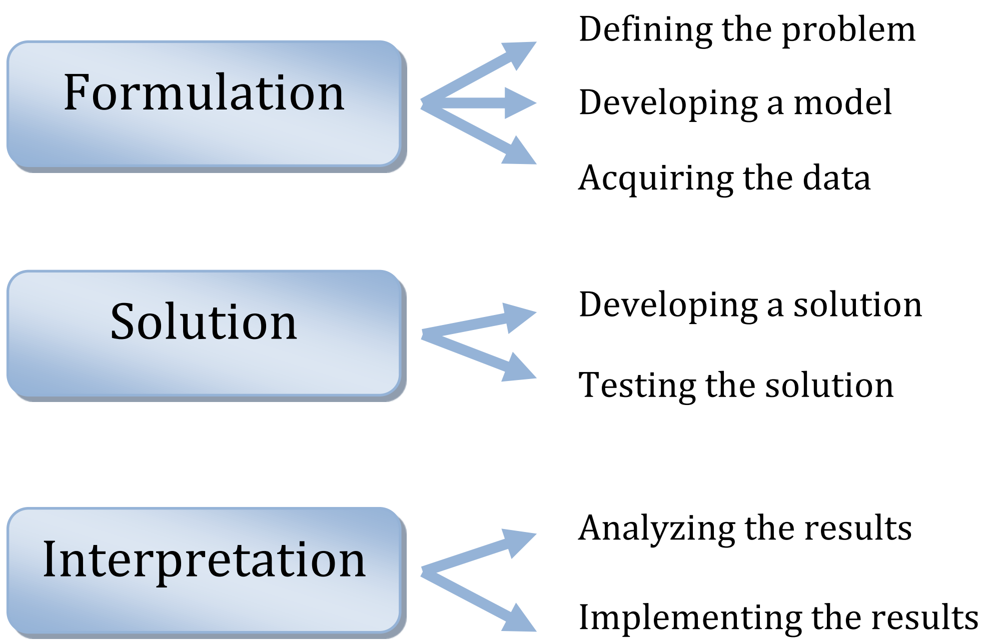
Content

1. Introduction

Decision making in a business are often restricted by the limitation of available resources and at the same time, a business manager has to meet specified goals. In this paper, quantitative analysis is importance for the managerial decision process. Nevertheless, keep in mind that quantitative analysis can never provide the entire answer for all strategic decisions. Use it with care and use it as a systematic way of working through a complex managerial decision process. Decision modeling is a scientific approach to managerial decision making where we develop a mathematical model of a real-world problem. The model should be such that the decision-making process is not affected by personal bias, emotions and guesswork.

2. The process of quantitative analysis

Quantitative analysis is a scientific method that can help in the managerial decision making process. The decision modeling process involves three distinct steps:



2.1 Formulation

In this part, each part of a managerial problem is translated and expressed in terms of a mathematical model. One very common pitfall is that the problem cannot be formulated because the problem is too complex (then one could break down the problem into smaller pieces), another that technical or managerial information are not available (often there has not been formulated an objective of the managerial problem). The aim in formulation is to ensure that the mathematical model addresses all the relevant issues to the managerial problem at hand. Formulation of the problem should further be classified into a) Defining the problem, b) Developing a model, and c) Acquiring the data.

- Defining the problem is perhaps the most important problem. When the managerial problem is difficult to quantify, it sometimes may be necessary to develop specific, measurable objectives. One objective in managerial economics could be that of maximizing the profit and another could be that of minimizing the costs. Sometimes there are multiple conflicting goals, and if this is the case, one could solve it by minimizing the distance between the goals.

- Developing the model is the step where different types of models can be specified. These models are expressed as equations or inequalities with one or more variables and parameters.

  Use the following three-step procedure to define and identify

  - Decision variables, which represent the unknown entities in a managerial problem. Decision variables could be cars of type I (X_1) and cars of type II (X_2).
  - Objective function, which states the goal of the managerial problem, can be quantified in a function like

    Max profit = (50000€ per car type I) × (number of cars type I produced) +
    (60000€ per car type II) × (number of cars type II produced)

    If we introduce

    Decision variable X_1 = number of cars type I and
    Decision variable X_2 = number of cars type II,
    we have the objective function

    Max. profit = (50000€ per car type I) × X_1 +
    (60000€ per car type II) × X_2

Other types of common managerial problems are maximizing (profit, efficiency, ...) or minimizing (cost, time, labor,...).

- Constraints can be classified into technical and economic constraints. A technical constraint could be the space or number or workers that are available. An economic constraint could be the amount of available money to invest in a project.

- Acquiring input data is the step where input data to be used in the model are obtained. This detailed as well as technical information are collected and implemented into the model. You will have to use knowledge from the basis of managerial economics. Obtaining accurate data is essential, improper data will result in misleading results. For real world problems, collecting accurate data can be one of the most difficult and challenging aspects of decision modeling.

2.2 Solution

It is the step where the mathematical expressions from your formulation process are solved to identify an optimal solution of the managerial economics problem. Because we can use software packages to find a solution, our focus has shifted away from detailed algorithm (the simplex algorithm) and towards the best of these packages. We distinguish between a) developing a solution and b) testing the solution.

- Developing a solution is the step that involves the use of an algorithm that consists of a series of procedures, and the accuracy of the solution depends on the accuracy of the input data and the model. Do not waste time with specialized programming skills. Instead, it is much easier to use a standard software program that is able to handle and solve the problem. For small problems with only two decision variables, you can do a graphical analysis, and for larger problems, you will have to use the analytical solution.

- Testing the solution means that one has to be sure that new data from another source behaves in a similar way as the original data. One has to check data as well as the model itself to make sure that they really represent the managerial economic problem.

2.3 Interpretation

After a solution has been found you, have to take a closer look at solution and all decision variables. The most important job for any managerial economists would be to ask the following question:

"*what if*".

The "*what if*" question has to be asked so that you can have an idea of how sensitive the solution is. We distinguish between a) analyzing the results and b) implementing the results.

Analyzing the results begins with determining the implications of the solution. It is important to ask the following question: "*What is the interpretation of the* results?".

Next question to ask is "*What if we change one or more of the input data?*". This sensitivity analysis is important because it shows how sensitive the solution is to changes.

Implementation is the step that most often is neglected in real world. Most people forget this part and very often a well-specified and prepared plan are missing the last and final step: implementation in the organization.

3. Practical applications

A number of practical applications have documented the use of the quantitative process as well as the use of a more complex mathematical formulation. In the following, we will only be looking at applications that use a specific mathematical model called linear programming. The linear programming model is a simple and important mathematical optimization model, and it can sometimes be applied to a number of economic problems.

A couple of old applications are:

- The linear programming model has been used for bank asset management and it has been implemented to include comprehensive risk constraints, various policy considerations, economic and institutional realities of the marketplace, and a variety of different dynamic effects, which must be considered in order to make optimal asset management decisions. Conceptual problems concerning the formulation of the bank's goals were considered as well.[1]
- A linear programming formulation of Shell's distribution network between four sources of product and a large number of transshipment points and terminals has been implemented, with emphasis on cost logistics and ranking of various alternatives.[2]
- Critical to an airline's operation is the effective use of its reservations inventory. American Airlines began research in the early 1960s in managing revenue from this inventory. Because of the problem's size and difficulty, American Airlines Decision Technologies has developed a series of OR models that effectively reduce the large problem to three much smaller and far more manageable sub problems: overbooking, discount allocation, and traffic management. The results of the sub problem solutions are combined to determine the final inventory levels. American Airlines estimates the quantifiable benefit at $1.4 billion over the last three years and expects an annual revenue contribution of over $500 million to continue into the future.[3]

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[1] Cohen and Hammer (1967): "Linear Programming and Optimal Bank Asset Management Decisions". *Journal of Finance*, 147-165.

[2] Zierer, Mitchell, and White (1976): "Practical applications of linear programming to Shell's distribution problems". *Interfaces*, 13-26.

[3] Smith., Leimkuhler, and Darrow (1992): "Yield Management at American Airlines". *Interfaces*, 8-31.

- A linear programming model with several objectives was developed in choosing media plans.[4]

And a couple of recent applications are:

- Jan de Wit Company implemented a decision-support system based on linear programming as a production planning and trade tool for the management of its lily flower business. The LP maximizes the farm's total contribution margin, subject to such constraints as market defined sales limits, market requirements, characteristics of the production cycle duration, technical requirements, bulb inventory, and greenhouse limitations. The main decision variable to be calculated is the number of flowerbeds in a specific greenhouse, from a specific bulb batch, of a specific variety, for a specific purpose, taking into consideration planting and expected harvesting weeks. Between 1999 and 2000, company revenue grew 26 percent, sales increased 14.8 percent for pots of lilies and 29.3 percent for bunches of lilies, costs fell from 87.9 to 84.7 percent of sales, income from operations increased 60 percent, return on owner's equity went from 15.1 to 22.5 percent, and best quality cut lilies jumped from 11 to 61 percent of the quantities sold.[5]

- U.S. Coast Guard used linear programming in an extensive preventative maintenance program for its Sikorsky HH60J helicopters based on helicopter flight time. The model must consider different maintenance types, maintenance capacity and various operational requirements.[6]

- Recent reform of European agricultural policy has resulted in substantial changes to the criteria by which premia payments are made. Beef farmers, who have been particularly dependent on premia payments to maintain margins, must re-evaluate their systems to identify optimal systems in these new circumstances. A linear programming model has been used to

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[4] Charnes, Cooper, Devoe, Learner, and Reinecke (1968): "A Goal Programming Model for Media Planning". *Management Science,* 14, B423-B430.

[5] Filho, José, Neto, and Maarten (2002): "Optimization of the Production Planning and Trade of Lily Flowers at Jan de Wit Company". *Interfaces,* 35-46.

[6] Hahn and Newmanb (2008): "Scheduling United States Coast Guard helicopter deployment and maintenance at Clearwater Air Station, Florida". *Computers & Operations Research* 35 (2008) 1829—1843.

identify optimal beef production systems in Ireland. The objective function maximizes farm gross margin and the model is primarily constrained by animal nutritional requirements.[7]

- Scheduling the Italian Major Football League (the so-called "Serie A") consists in finding for that league a double round robin tournament schedule that takes into account both typical requirements such as conditions on home-away matches and specific requests of the Italian Football Association such as twin-schedules for teams belonging to the same hometown. The model takes into account specific cable television companies requirements and satisfying various other operational constraints while minimizing the total number of violations on the home-away matches conditions.[8]

- A linear programming model was implemented using Activity Based Costing for calculating unit product cost, and dynamic Activity Based Management for assessing the feasibility of prospective production plans. The model is used to optimize the business plan at a steel manufacturer.[9]

- Minnesota's Nutrition Coordination Center used linear programming to estimate content of commercial food products.[10]

- Linear Programming was used to decide the daily routes of logging trucks in forestry. Aspects such as pickup and delivery with split pickups, multiple products, time windows, several time periods, multiple depots, driver changes and a heterogeneous truck fleet.[11]

- Airline seat is the most perishable commodity in the world. Each time an airliner takes off with an empty seat, a revenue opportunity is lost. Delta Air Lines used a LP model with more

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[7] Crosson, O'Kiely, O'Mara and Wallace (2006): "The development of a mathematical model to investigate Irish beef production systems". *Agricultural Systems*, 349-370.

[8] Croce and Oliveri (2006): "Scheduling the Italian Football League: an ILP-based approach". *Computers & Operations Research*, 1963-1974.

[9] Singer and Donoso (2006): "Strategic decision-making at a steel manufacturer assisted by linear programming". *Journal of Business Research*, 387-390.

[10] Westrich, Altmann, and Potthoff (1988): "Minnesota's Nutrition Coordinating Center Uses Mathematical Optimization to Estimate Food Nutrient Values". *Interfaces*, 86-99.

[11] Flisberg, P., B. Lidén, M. Rönnqvist (2009): "A hybrid method based on linear programming and tabu search for routing of logging trucks". *Computers & Operations Research*, 1122-1144.

than 40,000 constraints and 60,000 variables to solve this empty seat problem. They saved more than $220,000 per day.[12]

- Allocation of train capacity among multiple travel segments on an Indian Railways train route with several stops. Due to historical and social reasons, Indian Railways splits its train capacity based on user and type of travel. The determination of the optimal split of such capacity is nontrivial. Their LP model was applied 17 Indian Railways trains, and they increased revenue from 2.6 to 29.3 percent in revenue, 6.7 to 30.8 percent in load factors, and 8.4 to 29 percent in passengers carried.[13]

- In the shipping and transportation industry, there are several types of standard containers with different dimensions and different associated costs. Investigation of the multiple container loading cost minimization problem, where the objective is to load products of various types into containers of various sizes to minimize the total cost.[14]

- In the forest industry, a linear programming model was used based on ecological capabilities classification to determine the land area of different species for plantation. The appropriate species based on ecological capabilities were ash, elm, maple, oak and bald cypress. Results showed that maple and bald cypress were appropriate for plantation at the site and their plantation areas should be 151.3 and 355.3 ha, respectively.[15]

- Carbon mitigation strategies are an urgent and overdue tourism industry imperative. The tourism response to climate action has been to engage businesses in technology adoption, and to encourage more sustainable visitor behaviour. These strategies however are insufficient to mitigate the soaring carbon footprint of tourism. Building upon the concepts of optimization and eco-efficiency, we put forward a novel carbon mitigation approach, which seeks to pro-actively determine, foster, and develop a long-term tourist market portfolio. This can be achieved through intervening and

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[12] Subramanian, R. (1994): "Coldstart: Fleet Assignment at Delta Air Lines". *Interfaces*, 104-120.

[13] Gopalakrishnan and Narayan (2010): "Capacity Management on Long-Distance Passenger Trains of Indian Railways". *Interfaces*, 291-302.

[14] Chan Hou Che a, Weili Huang a, Andrew Lim a, Wenbin Zhu (2011): "The multiple container loading cost minimization problem". *European Journal of Operational Research*, 501-511.

[15] Mohammadi, Z., Mohammadi, S., & Shahraji, T.R. (2017). "Linear programming approach for optimal forest plantation". *Journal of Forestry Research*, 229-307.

> reconfiguring the demand mix with the fundamental aim of promoting low carbon travel markets. The concept and the analytical framework that quantitatively inform optimization of the desired market mix are presented. Combining the "de-growth" and "optimization" strategies, it is demonstrated that in the case study of Taiwan, great potential exists to reduce emissions and sustain economic yields.[16]

These practical applications belong to a class of business problems classified as allocation problems and under certain conditions, the linear programming model can achieve a solution.

[16] Sun, Ya-Yen, Pei-Chun Lin, and James Higham (2020): "Managing tourism emissions through optimizing the tourism demand mix: Concept and analysis", Tourism Management, 81, 1-11.

4. Linear programming defined[17]

We define a linear programming problem as an allocation problem wherein the values of decision variables must be determined to meet a goal, under a set of limitations based on available resources.

> **Linear programming models**
>
> A linear programming problem may be defined as the problem of maximizing (or minimizing) a linear objective function subject to a set of linear constraints.
>
> The constraints may be equalities or inequalities.

A Linear Programming model is based on the following properties:

1. Proportionality	This means that the contribution of each decision variable to the value of the objective function and the left hand side of constraints are directly proportional to the level of the decision variable. In other words, we are talking about constant returns to scale.
2. Nonnegativity	Decision variables are not allowed to be negative. This means that solutions variables of say costs cannot be negative, or the number of workers allocated to a job cannot be negative.
3. Additivity	The objective function or the function of the left hand side of a functional constraint is the sum of the individual contributions of each variable.
4. Divisibility	Decision variables are allowed to fractional values equal to or above zero. Fractional values for the decision variables, such as 1.25 are allowed.
5. Certainty	This assumption asserts that the objective and constraints coefficient the LP model are deterministic. This means that they are known fixed constants.

[17] We use the Dantzig (1963) specification. Dantzig, G.B. (1963): Linear Programming and Extensions. *Princeton University Press.*

4.1 The managerial perspective

It is important to realize that linear programming is not a panacea. Instead, linear programming is a mathematical tool that sometimes approximates a managerial problem quite well.

If we take a closer look at the assumptions underlying the linear programming model, we can very easily specify applications where the assumptions are not meet.

- Instead of having constant returns to scale, we could have increasing or decreasing returns to scale.[18]
- If we were unsure of the exact number of resources that are needed in a manufacturing problem, then the certainty assumption would be violated.
- Data that are used in a linear programming model are uncertain because they cannot be measured precisely and because they can fluctuate in different unpredictable ways. Just think of machine breakdowns, absence of workers or power failures, etc. The certainty assumption - a rare occurrence in real life, where data are more likely to be presented by probabilistic distributions. If the standard deviations of these distributions are sufficiently small, then the approximation is acceptable.[19]
- The profit or the cost approximates an uncertain amount. The actual value depends on current price of raw materials, defects during manufacturing or changing in inventory costs, etc.

To sum up one should have in mind that uncertainty in the data is one reason why models are just approximate.

[18] Decreasing returns to scale can be treated by introducing a new decision variable that covers the decreasing returns to scale. In this case we would have two decision variables one that take care of constant returns to scale and another to take of decreasing returns to scale. Decreasing and increasing returns to scale can also be solved by nonlinear programming.

[19] Large standard deviations can be accounted by applying sensitivity analysis to the optimal solution.

5. A simple production problem 1

5.1 Problem description

The IT Communication Company assembles and then tests two models of communication accessories and must decide how many of each model to assemble and then test. There are no accessories in inventory from previous month, and because these models are going to be changed after this month, the company does not want to hold any inventory after this month. We look at two types of accessories - the XL and the YL.

The company believes the most it can sell this month are 600 XL and 1200 YL. Each XL sells for €300 and each YL sells for €450. The cost component parts for a XL is €150, the cost component parts for a YL is €225.

Labor is required for assembly and testing. There are at most 10000 assembly hours and 3000 testing hours available. Each labor hour for assembling costs €11 and each labor hour for testing costs €15. Each XL requires five hours for assembling and one hour for testing, and each XL requires six hours for assembling and two hours for testing.

The IT Communication Company wants to know how many of each model it should produce to maximize its net profit, but it cannot use more labor hours than are available, and it does not want to produce more than it can sell.

Decision variables

We introduce two decision variables. Let our decision variables be

XL = number of XL accessories to assemble and test

YL = number of YL accessories to assemble and test

Objective function

The profit this month is equal to Z.

Per unit	XL	YL
Labor hours for assembly	5	6
Labor hours for testing	1	2
Assembling cost per labor hour	€11	€11
Testing cost per labor hour	€15	€15
Selling price	€300	€450
Cost of component parts	€150	€225
Total cost of producing	€220 (=150+5(11)+1(15))	€321 (=225+6(11)+2(15))
Profit this month	€80	€129

The profit per unit is XL = €80 as the profit from production of XL, and YL = €129 as the profit from production of YL.

The IT Communication Company has as their objective to maximize the total profit from assembling and testing, which means that our objective function is equal to:

Max Z = 80 XL + 129 YL

One first and feasible solution is not to produce anything that is XL = 0 and YL = 0 which yields a total profit of zero. However, any positive solutions of XL and YL give more production and greater profit.

If XL = 2 and YL = 5 then the profit is equal to Z = 80(2) + 129(5) = €400. Unfortunately, the maximization of the objective function has to meet a couple of constraints, and we have to consider these constraints in order to find an optimal solution.

Negative values of X_1 and X_2 makes no managerial sense.

Constraints

From the problem, we have four restrictions where we distinguish between technical and economic constraints.

Technical restriction 1

Assembly hours

- The number of hours to assemble one unit of XL is 5 hours and to assemble one unit of YL is 6 hours. Total hours required to assemble XL and YL is 5XL + 6YL.
  With 10000 hours available to assemble XL and YL we formulate the assemble constraint as:
  $5XL+6YL \leq 10000$

Technical restriction 2

Testing hours

- The number of hours to test one unit of XL is 1 hour and to test one unit of YL is 2 hours. Total hours to test XL and YL is: 1XL + 2YL

  With 2800 hours available to test XL and YL, we have the testing constraint as:

  $1XL+2YL \leq 2800$

Economic restriction 1

Maximum sale of XL

- The most that can be sold of XL is 600.

  The sale constraint on XL is:

  $1XL \leq 600$

Economic restriction 2

Maximum sale of YL

- The most that can be sold of XY is 1200.

  The sale constraint on YL is:

  $1YL \leq 1200$

5.2 The model

We summarize the problem as follows. We introduce X_1 units of normal accessories and X_2 units of luxury accessories such that the total profit Z is maximized[20] with respect to available resources:

Maximize Z =	80 XL	+	129 YL		
Subject to constraints					
Assemble:	5 XL	+	6 YL	≤	10000
Testing:	1 XL	+	2 YL	≤	2800
Max XL:	1 XL			≤	600
Max YL:			1 YL	≤	1200
and					
XL , YL ≥ 0					

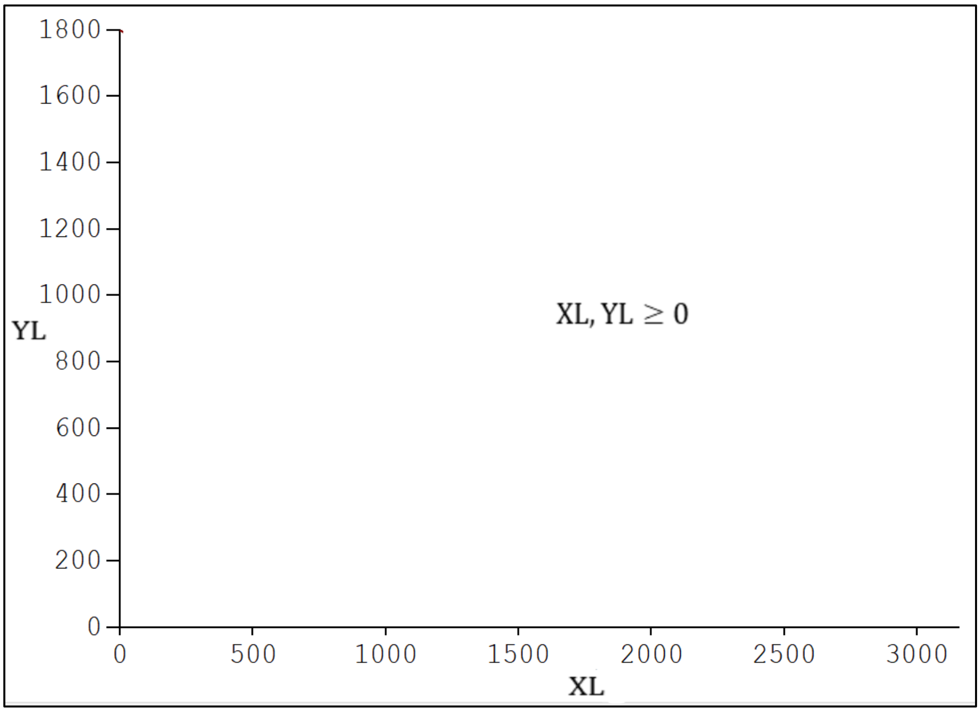
[20] The objective function as specified maximizes the profit. Remember that profit is equal to income minus expenditure. If our objective function was specified as minimization of costs, we could do that easily just by multiplying our objective function with minus one. Here we use the fact that cost is equal to expenditure minus income.

5.3 Performing the analysis

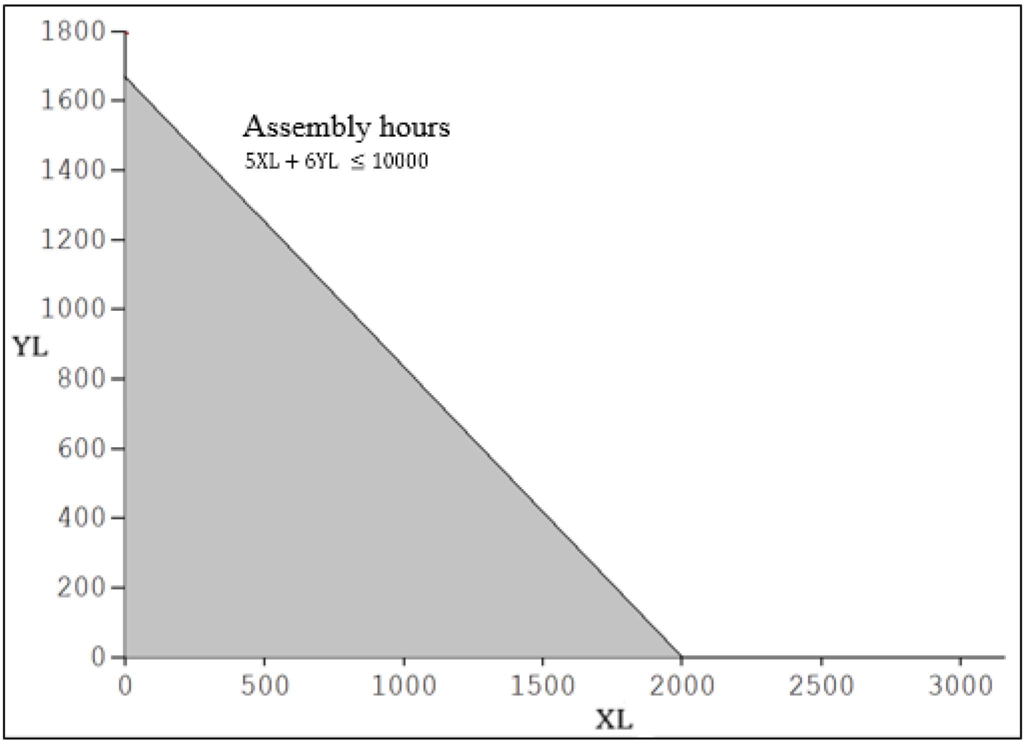
5.3.1 Graphical Solution

A graphical solution to LP-problems can only be shown with two variables.

We begin with a map that expresses only nonnegative constraints.

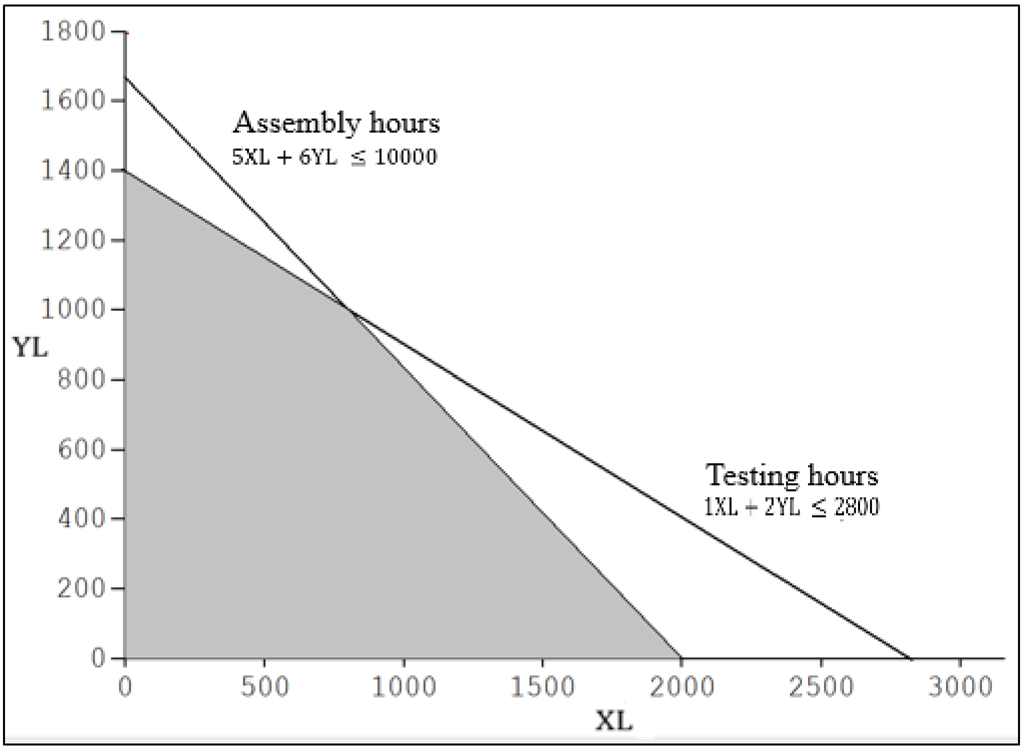


Then we add the assembly and testing constraints, and end with the two sales constraints.

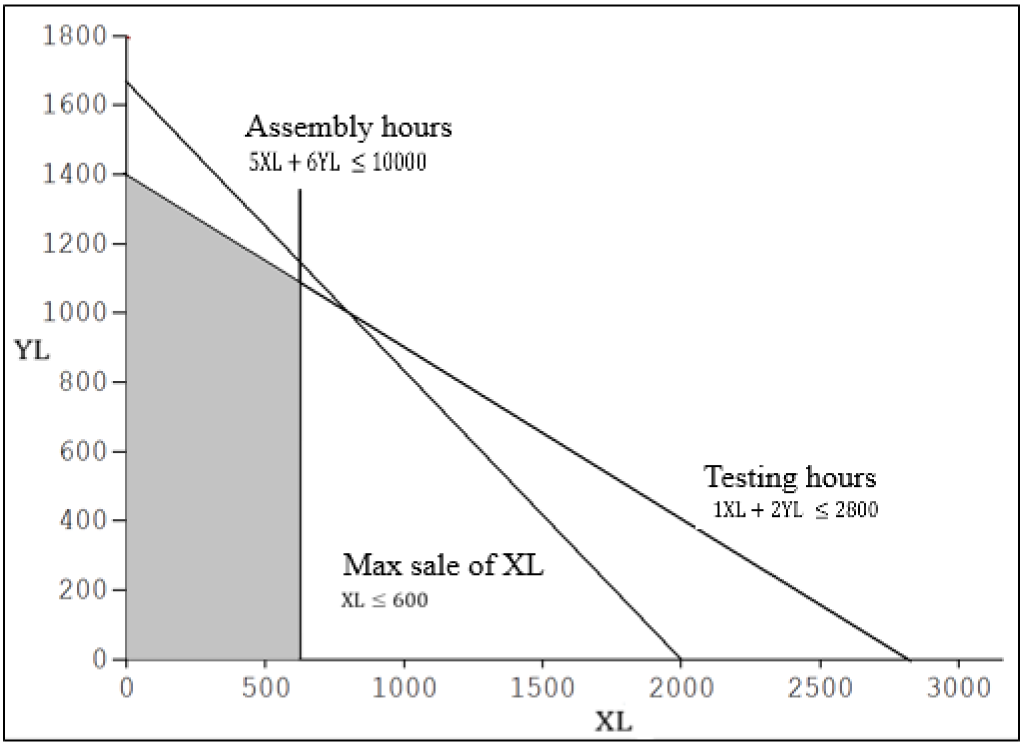


5.3.2 The feasible region

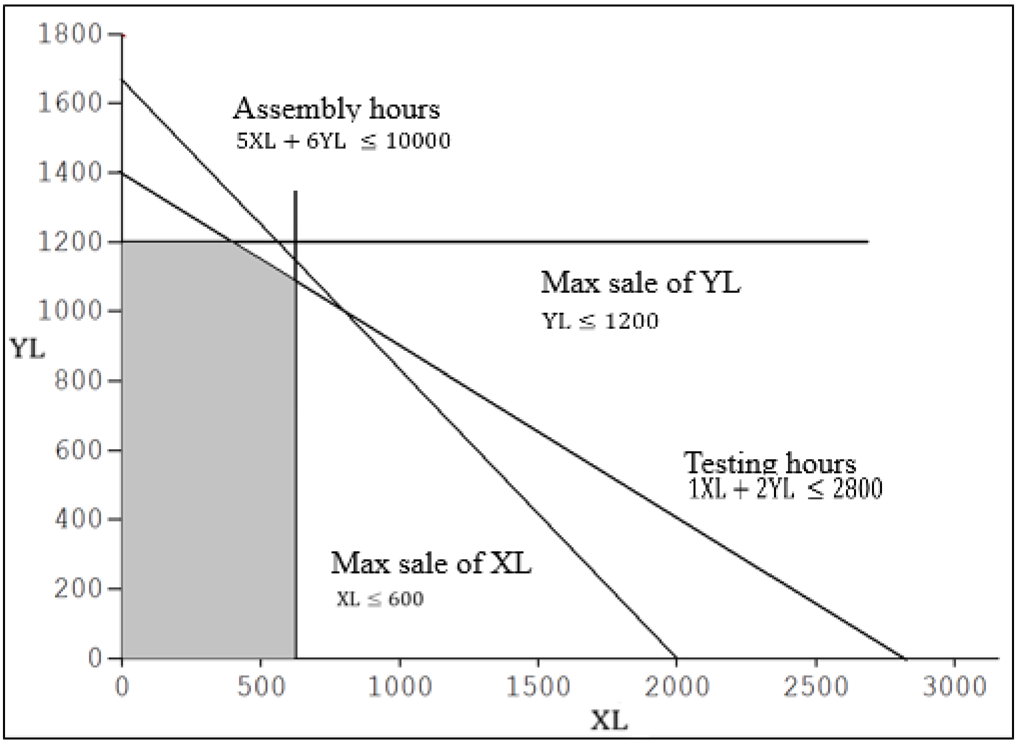
The set of all feasible solutions is called the feasible region. In the figure the feasible region is the shaded area. The feasible region is the set of all points that satisfy all the constraints.



For instance, the pair (XL=500; YL = 800) lie within the shaded region.

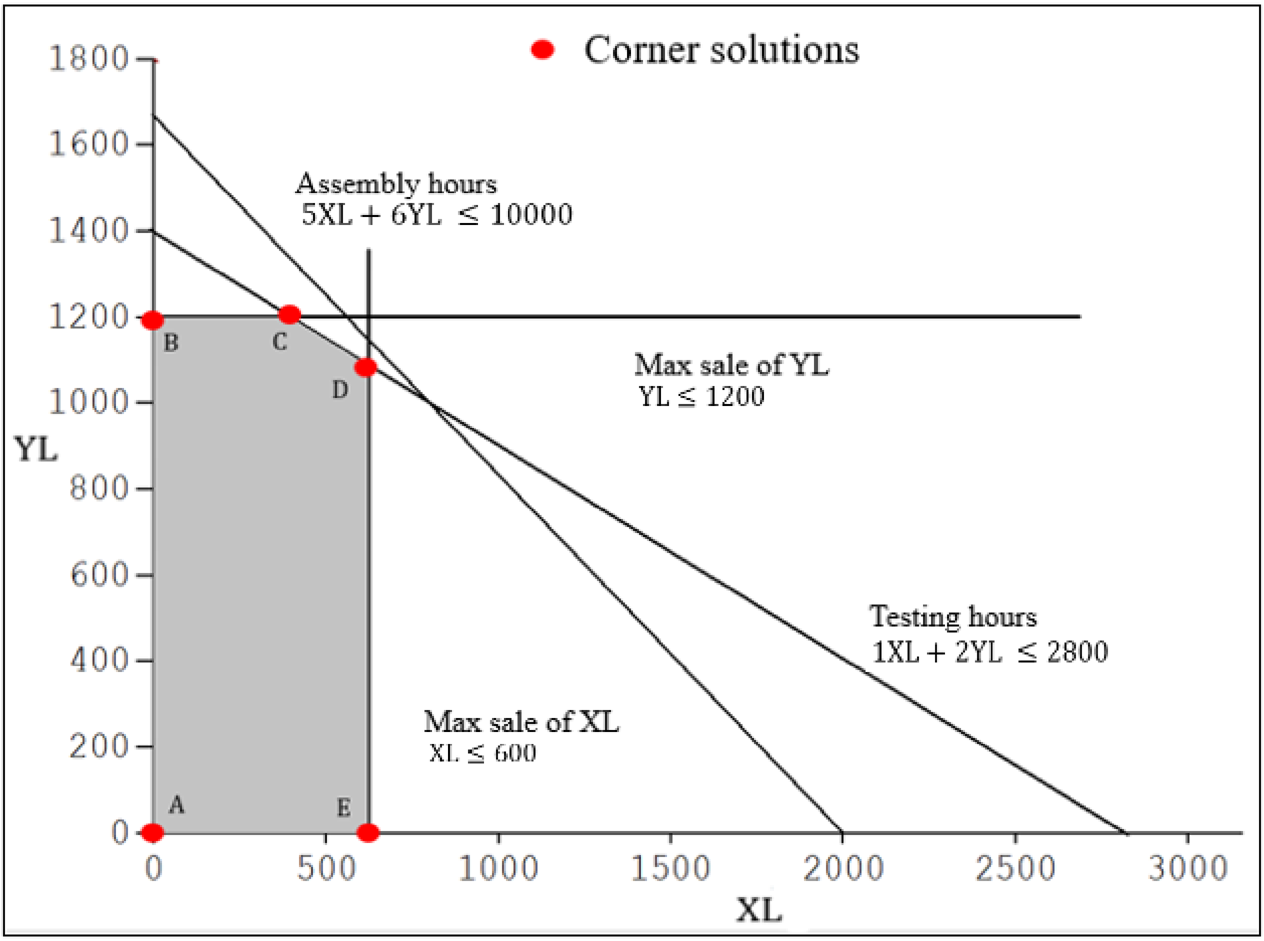


The final graphical solution has a feasible region bounded by the four constraints and the assumptions of nonnegativity. Points where two or more constraints intersect are called corner points (or corner solution).



5.3.3 Corner solutions

If we solve graphically for an optimal solution, we can evaluate the points where constraints interests as part of the feasible. These points are called corner solutions. In the figure below there are corner solutions marked with a red bullet at point {A, B, C, D and E}. Each of these points can be evaluated with the objective function.



Evaluation of corner points using the objective function Z = 80XL + 129YL

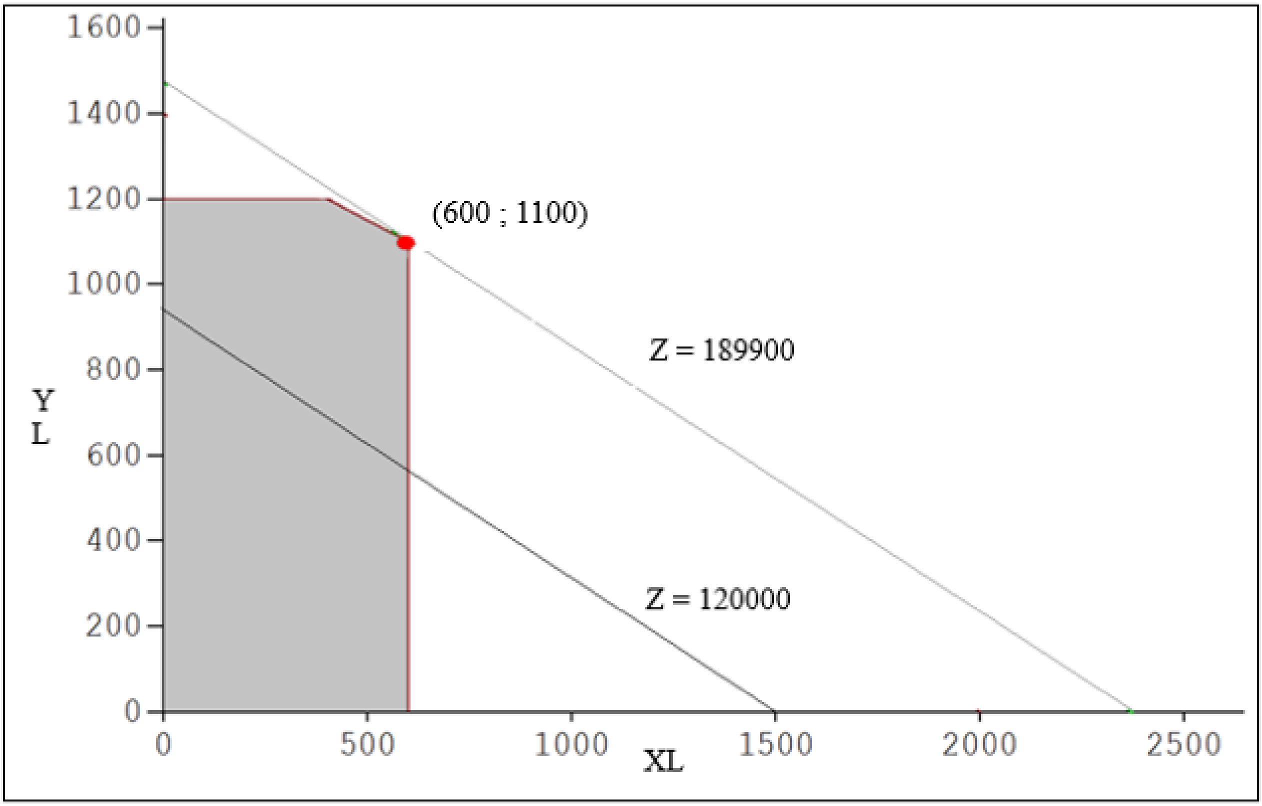
Corner solution	XL	YL	Value of objective function Z
A	0	0	0
B	0	1200	154800
C	400	1200	186800
D	600	1100	189900
E	600	0	48000

We have a maximum at corner solution D with $Z = 80(600) + 129(1100) = 189900$

At this point the value of Z is maximized and the other four corner solutions have a Z value less than the optimal corner solution D.

5.3.4 Profit lines

Until now the figure provides no information about the objective function which is to maximize the profit. If we introduce the profit line as a line which profit is constant and equal to the objective function, we can evaluate the profit line with different values of XL and YL.



In the figure we see two profit lines. We use the objective function Z = 80 XL + 129 YL to evaluate. The profit line with Z = 120000 contains the point (1500;0) and (0;930). The profit line with Z = 189900 contains the point (2373.75;0) and (1472.09).

We see that as the profit line moves further away from origo, the value of Z gets higher. At the point XL = 600 and YL = 1100 the value of Z has its maximum value equal to

Z = 80(600) + 129(1,100) = 189900.

5.4 Utilization of resources

At the optimal corner solution just found we had XL=600 and YL=1100. At that corner solution, we can see that not all available resources are used.

5.4.1 Slack variable

We will introduce an extra variable so that an inequality can be view as an equality. If do we that this extra variable can be viewed as an unused resource.

<u>Technical restriction 1</u>

Assembly hours

- 5XL+6YL ≤ 10000

  Use: 5(600) + 6(1,100) = 9600

  If we add an extra variable, S_1, to the constraint we can write the constraint as an equality as

  5XL + 6YL + 1S_1 = 10000

  The difference between the resources actually used and those available equals

  S_1 = 10000 – 9600 = 400

  This means that not all assembling hours are used. We name S_1 as slack variable number 1. Because the value of slack variable 1 is equal to 400 we say that the constraint is not binding and the number of unused hours are 400.

<u>Technical restriction 2</u>

Testing hours

- 1XL+2YL ≤ 2800

  Use: 1(600) + 2(1100) = 2800

If we add an extra variable, slack variable number S_2, to the constraint we can write the constraint as an equality $1XL + 2YL + 1S_2 = 2800$

The difference between resources used and those available equals slack variable number 2

$S_2 = 2800 - 2800 = 0$.

This means that all testing hours are used. We say this constraint is binding and the slack are equal to 0, meaning that all available testing hours are used.

Economic restriction 1

- Sales constraint $1XL \leq 600$

  Use: + 1(600) + 0(1100) = 600

  If we add an extra variable, slack variable number S_3, to the constraint we can write the constraint as an equality $1XL + 1S_3 = 600$

  The difference between resources used and those available equals $S_3 = 600 – 600 = 0$.

  This means that the sales constraint is binding and the slack is equal to zero.

Economic restriction 2

- Sales constraint $1YL \leq 1200$

  Use: + 0(600) + 1(1,100) = 1100

  If we add an extra variable, slack variable number S_4, to the constraint we can write the constraint as an equality $1YL + 1S_4 = 1200$

  The difference between resources used and those available equals $S_4 = 1200 – 1100 = 100$.

  This means that the second sales constraint is not binding and the slack is equal to 100.

We have introduced an important part in linear programming and that is the introduction of slack variables. As we have seen, a slack variable is equal to unused amount of a resource.

> **Slack Variable**
>
> A slack variable contains the difference between
>
> the resources actually used and those available.

If we once more look at the corner solution points the objective function Z = 80XL + 129YL

and include slack variables, we have the following:

Corner solution	XL	YL	S_1	S_2	S_3	S_4	Value of objective function Z
A	0	0	10000	2800	600	1200	0
B	0	1200	2800	400	600	0	154800
C	400	1200	800	0	200	0	186800
D	600	1100	400	0	0	100	189900
E	600	0	7000	2200	0	1200	48000

From this, we see that at each corner solution slack variables have a value that correspond to the resources not being used at the specific corner solution. At the optimal solution, corner solution D, not all assembly hours are used. There are further S_1=400 hours available. And we see that all testing hours are used, S_2=0. With S_3=0 we see that the sale restriction on XL is binding and the restriction is meet. Finally, S_4=100, we see, that the restriction on sale of YL is not binding, and further 100 units of Y could be sold.

5.5 The Managerial Perspective

From the above solution and conclusion with respect to used or unused resources further should be mentioned. When all resources are used one could ask what if we could have more resources allocated, and if we could choose freely between the binding constraints which of these should be the first to have more resources allocated. From an economic perspective the best way resources are allocated are with an eye to what the alternatives are. If the only alternatives are between two binding constraints, then the resources should be allocated to that constraint that increase the value of the objective function most.

5.5.1 Shadow price

If one extra resource is allocated to testing, we now have 2,801 hours available. Our slack variable was equal to $S_1=0$ and with one extra hour we have 2801 hours available. We end up with a new optimal solution where XL = 600 and YL = 1100.5 and a new objective function Z = 189964.5. The change of the objective function from 189900 to 189964.5 is equal to 64.5

The one extra resource allocated to testing changed the value of the objective function with 64.5, and so the change can be traced directly back to the extra resource. The economic value of this extra resource is thus equal to 64.5. Every constraints has an underlying economic value, and we call this value the constraints "Shadow price". A Shadow price indicates the change in the value of objective function when a resource changes by one unit.

At corner solution D not all resources were used. We found that constraint 1 (assembly) had further 400 hours available ($S_1=400$). With unused resources further resources would not add value to the objective function. This means that the shadow price of assembly must equal to zero.

With constraint 2 (testing) all resources were used ($S_2=0$). With all resources being used further resources would add value to the objective function. We found that one extra resource allocated to testing had an economic value of 64.5.

Constraint 3 (max sale of XL) we found that this constraint is binding and this constraint has a slack value equal to zero ($S_3=0$). One extra unit of XL sold would add extra value to the objective function. If we change the right hand side from 600 to 601, the objective function would change from 189900 to 189915.5. This means that the shadow price of constraint 3 is equal to 15.5.

The last constraint 4 (max. value of YL) we found that the constraint is not binding. If we added further to the maximum sale of YL that would not add to the value of the objective function. With slack value equal to S_4=100 further 100 unit of YL could be sold. The shadow price of this constraint is equal to zero. We see that the shadow price must be equal to zero because a new restriction on the sale of YL would not add new value to the objective function.

We have demonstrated the connection between slack variables and shadow prices and we summarize the results at corner solution D:

Corner solution D	XL = 600 and YL = 1,100	Z = 189900
Constraint	Value of slack variable	Value of shadow price
Assembly	400	0
Testing	0	64.5
Max XL	0	15.5
Max YL	100	0

We introduce a new variable "unit worth of a resource" which we will call a Shadow Price.

Shadow Price
A shadow price contains the amount the optimal objective function value changes per unit increase in the right hand side value of the constraint.

If a resource constraint is binding in the optimal solution, the shadow price shows how much one is willing to pay up to some amount, to obtain more of the resource. But, there is a decreasing marginal effect. As one adds more and more of a resource, the shadow price tends to decrease.

A constraint with "less than or equal to" (≤) will always have a positive sign. A constraint with "greater than or equal sign" (≥) will always have a negative sign. A constraint with an equality sign (=) may have a positive, negative, or zero shadow price.

Consider a constraint with a less than or equal to sign (≤) and a shadow price equal to 5. Adding points to this constraint can only improve the optimal object function. Having one additional resource will add 5 to the object function.

Consider a constraint with a greater than or equal to sign (≥) and a shadow price different from zero. An increase of this constraint eliminates points from the feasible region. Thus the optimal value of the object function must decrease. This means that the shadow price can only be negative.

5.6 Summary.

The example above is an optimization problem where we maximize a linear function of the decision variables subject to a set of constraints. Each constraint must be a linear equation or a linear inequality. We have formulated the example as an allocation problem in which limited resources are allocated to a number of activities. We have categorized the constraints into technical constraints (assembly and testing) and economic constraints (maximum sales).

In general terms a LP model can be written as follows:

Maximize Z =	$c_1X_1 + c_2X_2 + ... + c_nX_n$
Subject to	
	$a_{11}X_1 + a_{12}X_2 + ... + a_{1n}X_n \leq b_1$
	$a_{21}X_1 + a_{22}X_2 + ... + a_{2n}X_n \leq b_2$
	...
	$a_{m1}X_n + a_{m2}X_2 + ... + a_{mn}X_n \leq b_m$
	$X_1, X_2, \ldots, X_n \geq 0$

In the general LP model, we have m activities whose levels are represented by n decision variables, $X_1, X_2, ..., X_n$.

Each unit of activity j use an amount a_{ij} of resource i, and the m resources are given by $b_1, b_2, ..., b_m$.

The first constraint has a slack variable S_1 and a shadow price λ_1, the second constraint has a slack variable S_2 and a shadow price λ_2, and so forth.

6. A simple production problem 2

6.1 Problem description

Efficient allocation of resources is important in many fields. Engineers design systems in ways that maximize quality and minimize cost. Managers organize activities that maximize profit. Economists allocate resources in an efficient way. As we can see resource allocation take many forms.

Consider a company that assemble three different models of a recreational vehicle: Standard, Fancy and Luxury. The production problem has five departments A, B, C, D and E. The table contains relevant data. The capacities in the table are expressed in hours per week.

Capacities, manufacturing and profit of each vehicle that is made

		Manufactoring times		
Department	Capacity	Standard	Fancy	Luxury
A	120	3	2	1
B	80	1	2	3
C	96	2	0	0
D	102	0	3	0
E	40	0	0	2
Profit		€840	€1120	€1200

The company seeks a product mix that maximizes the profit earned per week. How many vehicles of each type should be produced each week.

Decision variables

The decision variables are the number of vehicles of each type to manufacture each week. Let

S = number of Standard model vehicles made per week

F = number of Fancy model vehicles made per week

L = number of Luxury model vehicles made per week

Objective function

The numbers of vehicles that maximizes profit each week

$$\max_{S,F,L\geq 0} Z = 840S + 1120F + 1200L$$

Constraints

From the table we have five constraints.

$$\begin{aligned}
A:&\quad 3S + 2F + 1L \leq 120\\
B:&\quad 1S + 2F + 3L \leq 80\\
C:&\quad 2S \leq 96\\
D:&\quad 3F \leq 102\\
E:&\quad 2L \leq 40
\end{aligned}$$

and

A, B, C, D, E ≥ 0

6. Using the computer

Linear Programming can be solved with software like Excel where you need to do the correct mathematical setup. Alternatively, you can use a predesigned software where the mathematical setup has been done in advance.

We continue using the example where the LP-model was:

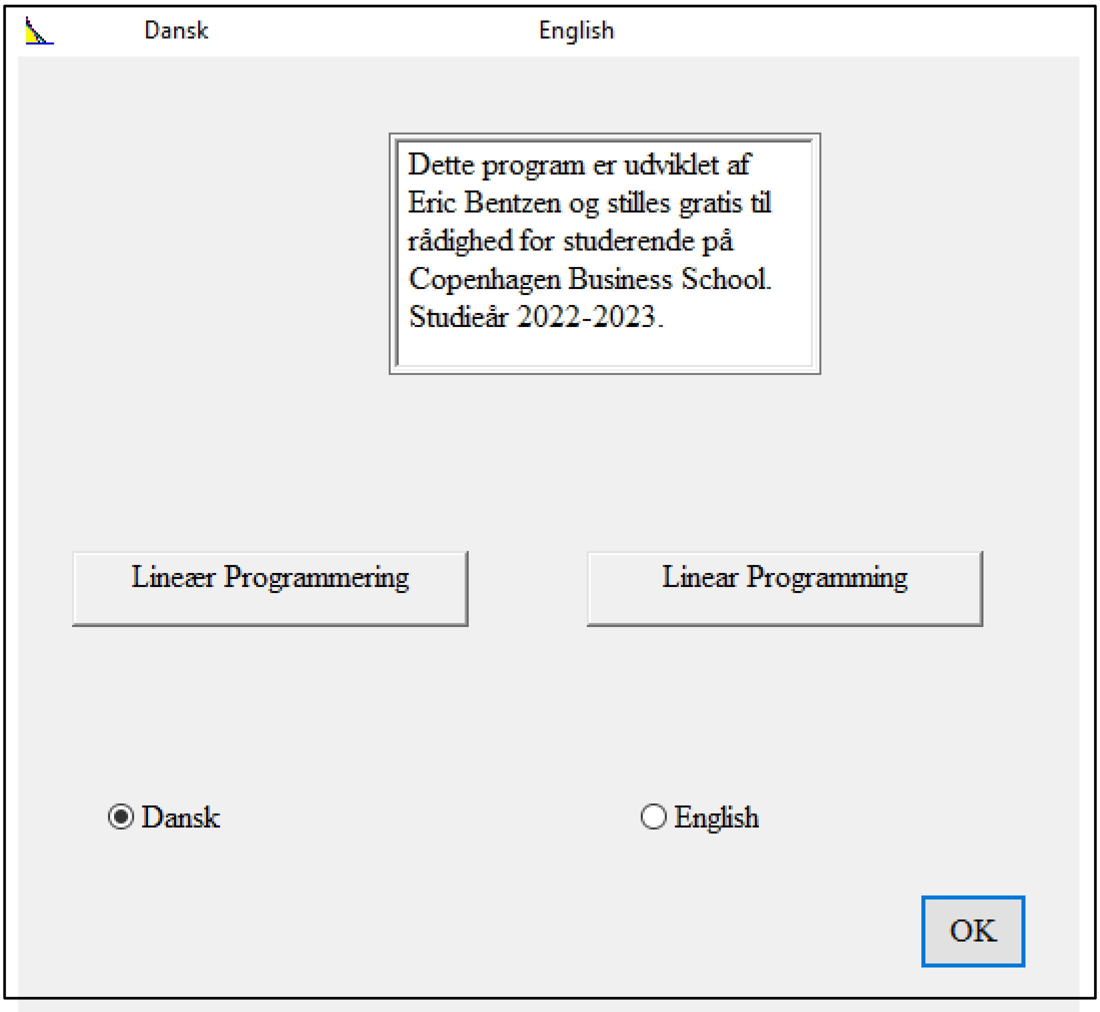
Maximize Z =	80 XL	+	129 YL		
Subject to constraints					
Assembly	5 XL	+	6 YL	≤	10000
Testing	1 XL	+	2 YL	≤	2800
Max sale XL	1 XL			≤	600
Max sale YL			1 YL	≤	1200
and	XL, YL ≥ 0				

6.1 Linear Programming using a predesigned software program

The predesigned software program “LP” can be found on Learn and is available for free for both platforms, Windows and Mac.

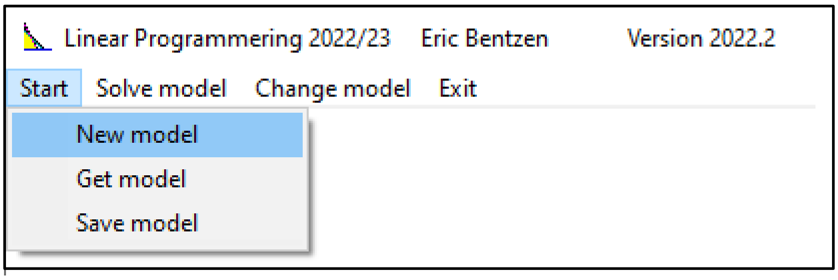
After installation of the program to your computer, you proceed as follows.

Start LP on your computer and you see the following:

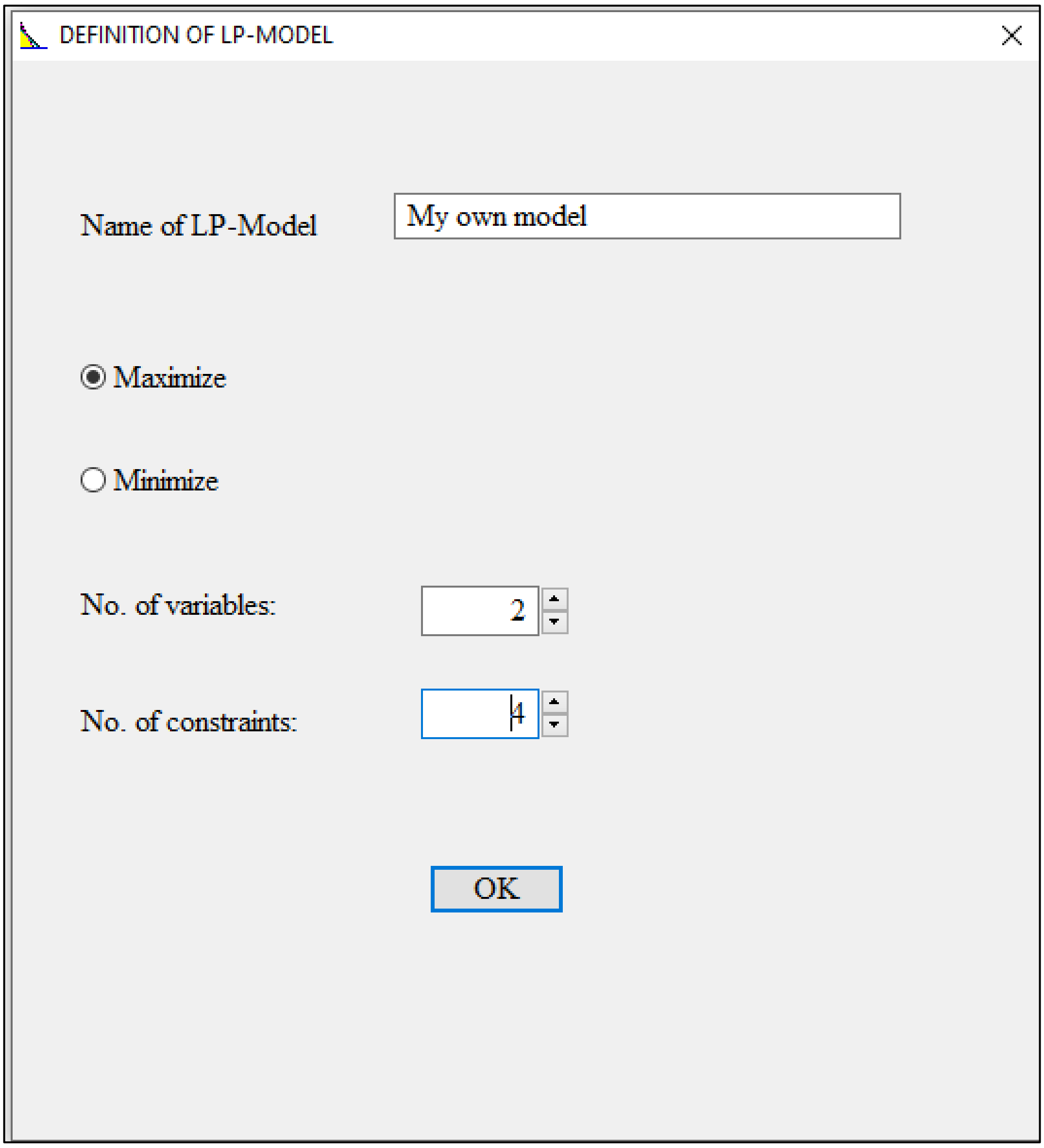


Continue with an OK and you are ready to go.

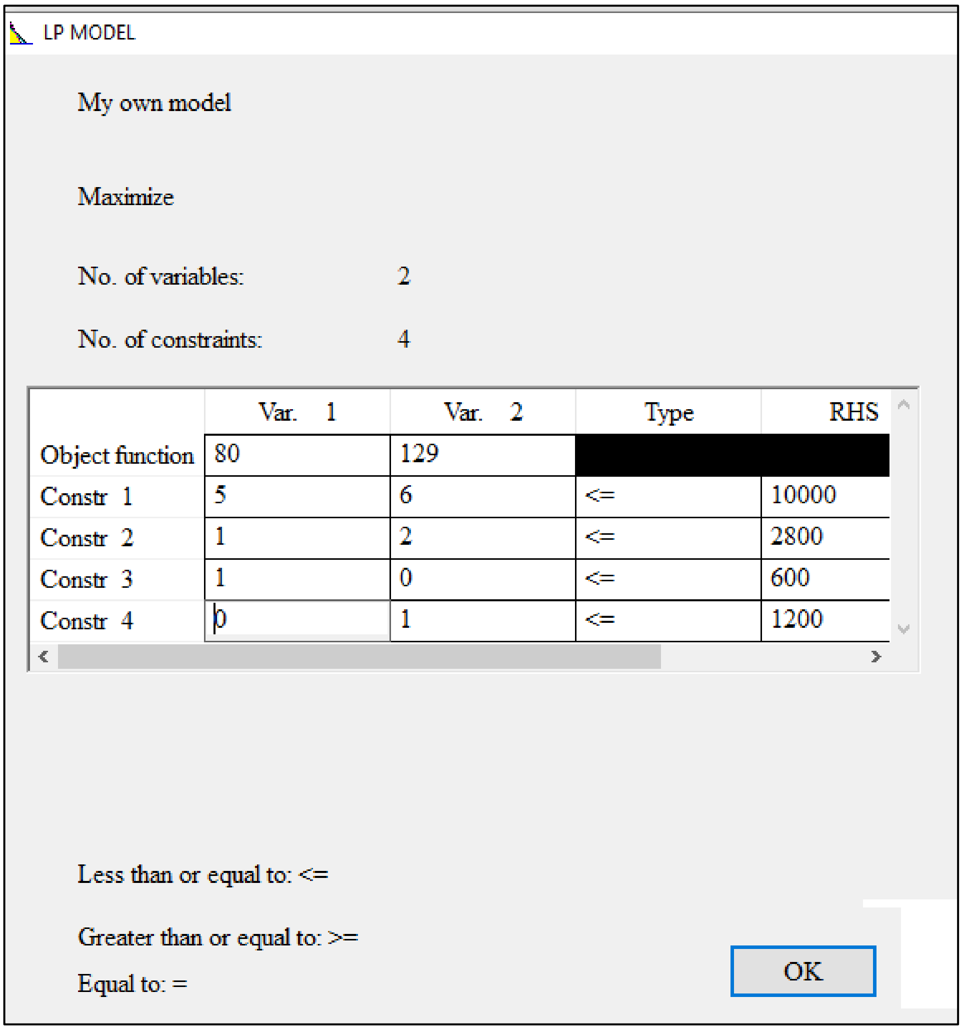
Start a New model.



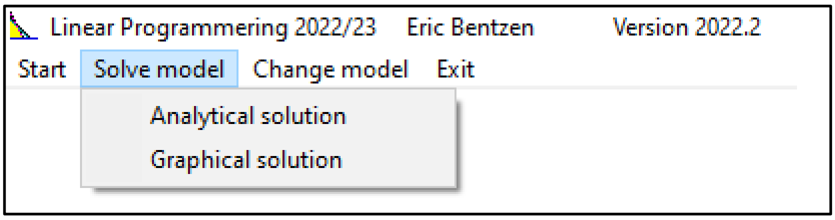
We start by maximizing the objective function and we have 2 variables and 4 constraints.



Type in the coefficients for each variable in the Object function and the 4 constraints.

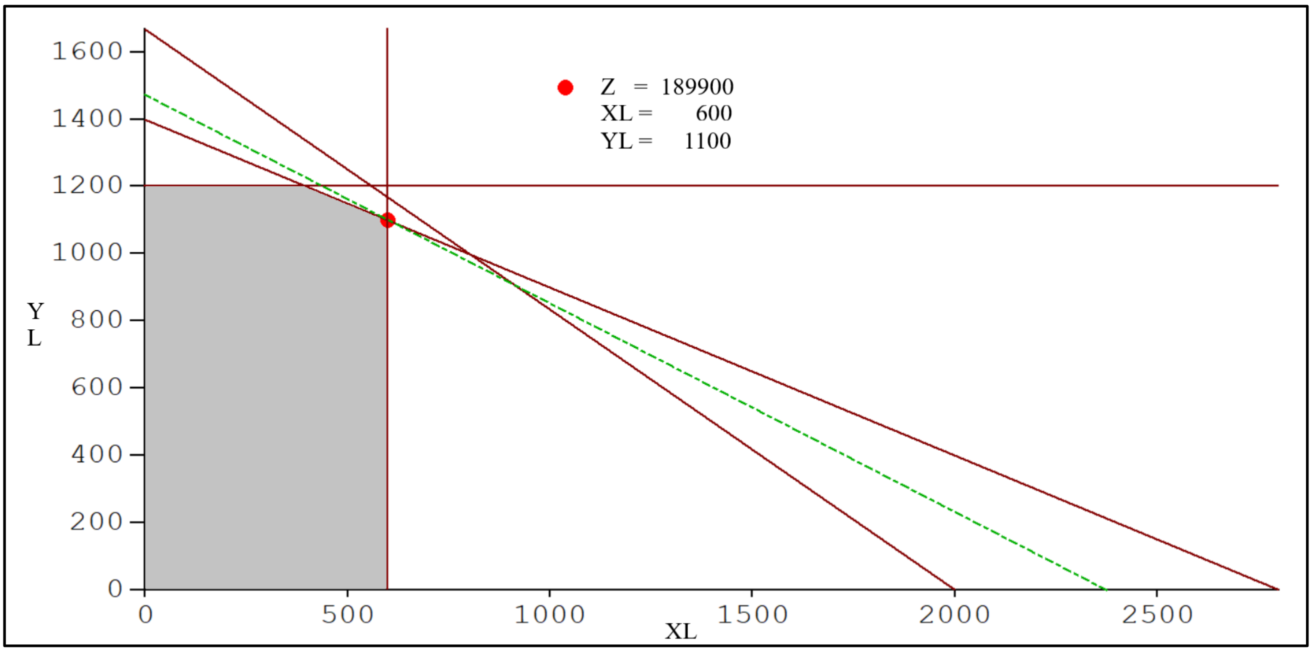


Solve the problem.



Here you can choose between a Graphical solution or an Analytical solution.

The Graphical solution can be solved only with 2 variables.



From the output we see the four constraints and the dotted line is the object function. An optimal solution is found where XL = 600 and YL = 1,100. This gives an optimal value of the objective function Z = 189,900.

The Analytical solution has the following output.

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 189900.0000

PARTIAL SENSITIVITY ANALYSIS

VARIABLE	SOLUTION VALUE	REDUCED COST	OPPORTU. COST	OBJECTIVE FUNCTION RANGES LOWER	GIVEN	UPPER
XL	600.0000	0.0000	80.0000	64.5000	80.0000	+INFINITY
YL	1100.0000	0.0000	129.0000	0.0000	129.0000	160.0000

CONSTRAINT	TYPE	SHADOW PRICE	SLACK	RIGHT HAND SIDE RANGES LOWER	GIVEN	UPPER
Assembly	≤	0.0000	400.0000	9600.0000	10000.0000	+INFINITY
Testing	≤	64.5000	0.0000	600.0000	2800.0000	2933.3333
Max sale XL	≤	15.5000	0.0000	400.0000	600.0000	800.0000
Max sale YL	≤	0.0000	100.0000	1100.0000	1200.0000	+INFINITY

6.2 Interpretation of results

From the output, we see the following results:

- The optimal solution Z equals 189900 (This is equal to corner solution D)
- The two decision variables have final values of XL = 600 and YL = 1100.
- The objective function for each decision variable also computes ranges. The coefficient value of XL has a "lower" range = 64.5 and an "upper" range = +∞. These ranges tell us that the solution of the decision variable XL = 600 will remain the same within the "lower" and "upper" ranges. If we change the given coefficient from 80 to 79 the solution will remain the same XL = 600, but we have a new optimal objective function that is decreased with 600 to a new value equal to 189300.
- The Right hand side of the four constraints has their own ranges.

  The Assembly constraint has a "Shadow price" = 0. This shadow price will remain the same in the interval of the Right hand side from (9,600 ; +∞). If we change the Right hand side from 10000 to 10001 the value of the objective function will not change.

  If we change the Testing constraint from 2,800 to 2801 we see with a Shadow price = 64.5 that the objective function will change from 189900 to 189964.5. The value of the shadow price from the testing constraint remains the same (64.5) in the interval from (600;2933.33)

- The Reduced cost[21] column shows the value of solution variables when these variables have solutions equal to zero.

Reduced cost
The reduced cost of an unused activity is the amount by which profits will decrease if one unit of this activity is forced into the solution. Obviously, a variable that already appears in the optimal solution will have a zero reduced cost.

In a *maximization* problem, the reduced cost of each variable equals its marginal profit. That is its contribution less opportunity cost of the resources that are needed to make one extra unit. In a *minimization* problem, the reduced cost of each variable equals its marginal cost. That is its direct cost plus the opportunity cost of the resources that must be freed up to make one extra unit.

- The Opportunity Cost column is equal to the opportunity cost of the resources that would need to be diverted to make one unit that it represents. The opportunity cost of a commodity is the sacrifice of producing one additional unit of that commodity, measured in terms of alternative production opportunities that must be forgone.

Opportunity Cost
The contribution of each participating activity equals the opportunity cost of the resources that are needed to engage in one unit of it.

[21] Reduced cost = [(change in optimal objective function value)/(unit increase of variable=0)]

6.3 The Managerial Perspective

Linear programming is a mathematical tool, which sometimes fit or approximates a managerial situation. Just like a hammer. A hammer can be used to hammer nails, but it can also be used to hammer screws, holes, and bolts. It is obvious that the above tasks are more efficiently done by a screwdriver, a drill, but sometimes there is no appropriate tool except from the hammer. In this case, the hammer is the best available tool, and that is the same with linear programming. Linear programming is the best mathematical tool to describe many managerial situations. A manager should not go into all the technical details of how linear programming models are solved. Instead, a manager can help with the formulation of economic coherence and later translate the results in a managerial content and take a decision.

6.4 Sensitivity of decision variables

From the output we see that changing the values of the coefficient of decision variables must have an effect on the value of the objective function.

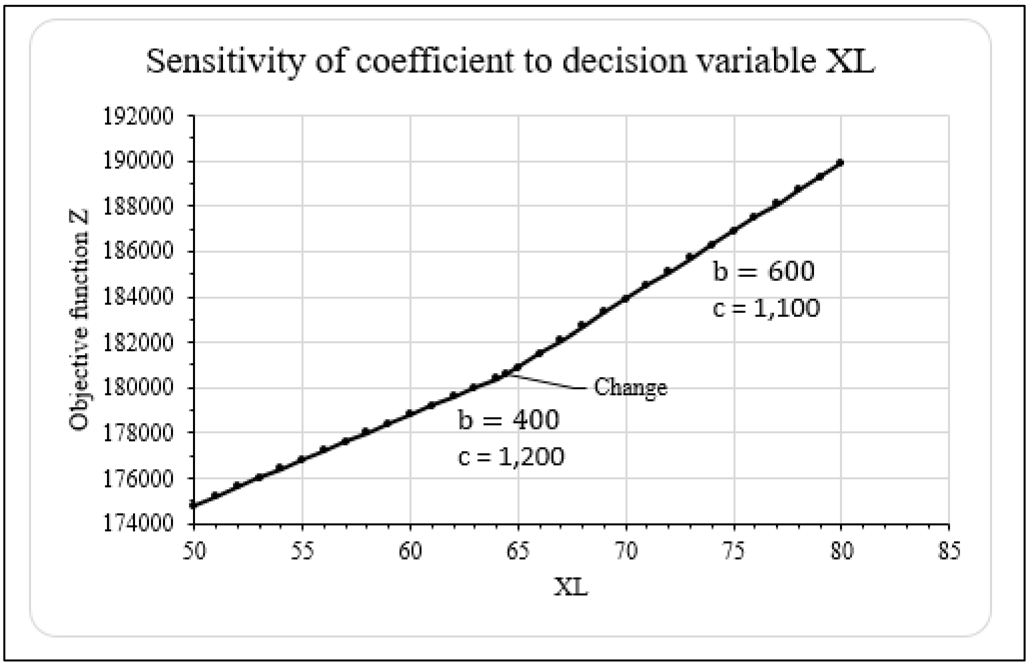
Looking at decision variable XL we see that it has a range from $[64.5 + \infty]$. In that interval the objective function will continue to grow with a constant value of 600. If the coefficient (that is the profit) of the decision variable change from 80 to 81, the objective function will change from 189900 to 190500. We see that the value 600 remains the same to infinity. If the coefficient of the decision variable drop from 80 to 79 there will be a change in the objective function equal to 600 from 189900 to 189300.

If we take a closer look and change the value of the coefficient to XL below 64.5 the solution value will no longer be 600. Instead there will be a change in the value of the objective function. The change from 65 to 64 will change the solution value from 600 to 400. The sensitivity between XL and the objective function can be described with $\mathrm{Z} = \mathrm{bXL} + \mathrm{cYL}$.

In the interval XL = [65;80] we have b = 600 and c = 1100.

In the interval XL = [50;64] we have b = 400 and c = 1200.

In the graph below this link has been indicated.



6.5 Economic interpretation of the LP concept

The sensitivity of decision variables is important because it shows that the LP-model is a stylized representation of reality. With the sensitivity analysis we get insight of the linear program that is being modeled.

The presentation of sensitivity analysis has a direct connection to the interpretation of the economics.

We have used the following connection to economics:

A variable cost is a cost that depends on the action that is being contemplated, and a fixed cost is a cost that is independent of the action. Contribution is equal to revenue less variable cost. This means when resources are allocated, a profit-maximizing decision maker should maximize the contribution.

The opportunity cst of doing something is equal to the reduction in contribution yjat occurs if you set aside rhe resources that are needed to do that thing.

The marginal profit for doing something is equal to its contribution less its opportunity cost.

The optimal solution in LP model include a set of shadow prices, one for each constraint. Each constraint's shadow price is the breakeven price at which a profit-maximizing decision maker is indifferent to buing or selling one incremental unit of the resource whose consumption that constraint measures.

In a profit-maximizing LP model, the shadow prices let us compute the opportunity cost of the resources needed to make one unit of each variable. To compute a variable's opportunity cost, multiply the coefficient of the variable in each constraint by that constraint's shadow price, and sum.

In a LP model, the shadow prices satisfy the following two rules

Each slack constraint has zero as its shadow price

Each variable that is positive has zero as its reduced cost

LP models seem to require linearity, but they readily accommodate decreasing marginal return in the contribution of each decision variable.

LP models fail to accommodate increasing marginal return in the contribution of any decision variable

The optimal solotion to LP models exhibits decreasing marginal returnin each right-hand-side.

7. Using Excel and Solver

As an alternative to the predesigned LP-program you can use Excel and Solver to find the optimal solution. It is important that you follow the following steps. We use the same model as before.

LP-model:

Maximize Z =	80 XL	+	129 YL			
Subject to constraints						
Assembly	5 XL	+	6 YL	≤	10000	
Testing	1 XL	+	2 YL	≤	2800	
Max sale XL	1 XL			≤	600	
Max sale YL			1 YL	≤	1200	
And	XL, YL ≥ 0					

We start Excel and do the following setup.

Excel setup for a linear program

	A	B	C	D	E	F	G	H	I	J	K	L	M
1													
2	Decision variables		XL	YL									
3	Solution values												
4					Total								
5	Maximize Z		80	129					E5:=SUMPRODUCT($C$3:$D$3;C5:D5)				
6													
7	Subject to constraints				LHS	Sign	RHS						
8	Assembly		5	6		≤	10000		E8:=SUMPRODUCT($C$3:$D$3;C8:D8)				
9	Testing		1	2		≤	2800		E9:=SUMPRODUCT($C$3:$D$3;C9:D9)				
10	Max sale of XL		1	0		≤	600		E10:=SUMPRODUCT($C$3:$D$3;C10:D10)				
11	Max sale of YL		0	1		≤	1200		E11:=SUMPRODUCT($C$3:$D$3;C11:D11)				

The total value of Maximizing the profit in cell E5 can be computed multiplying cell by cell

E5:=C3*C5 + D3*D5

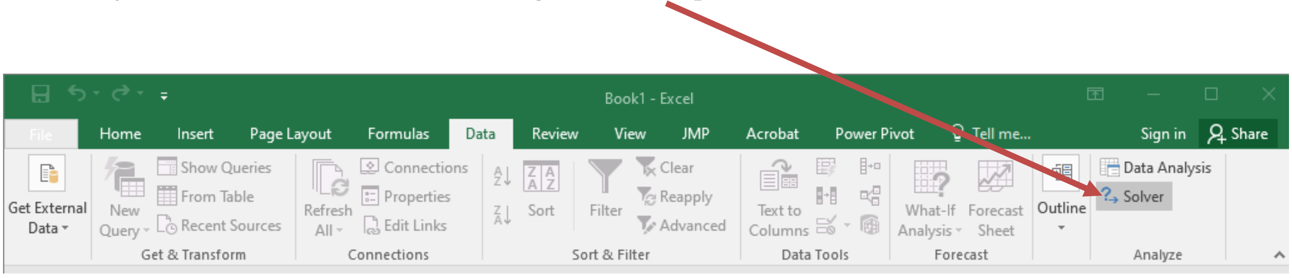
or

using SUMPRODUCT and the result in cell E5 is equal to
E5:=SUMPRODUCT($C$3:$D$3;C5:D5)

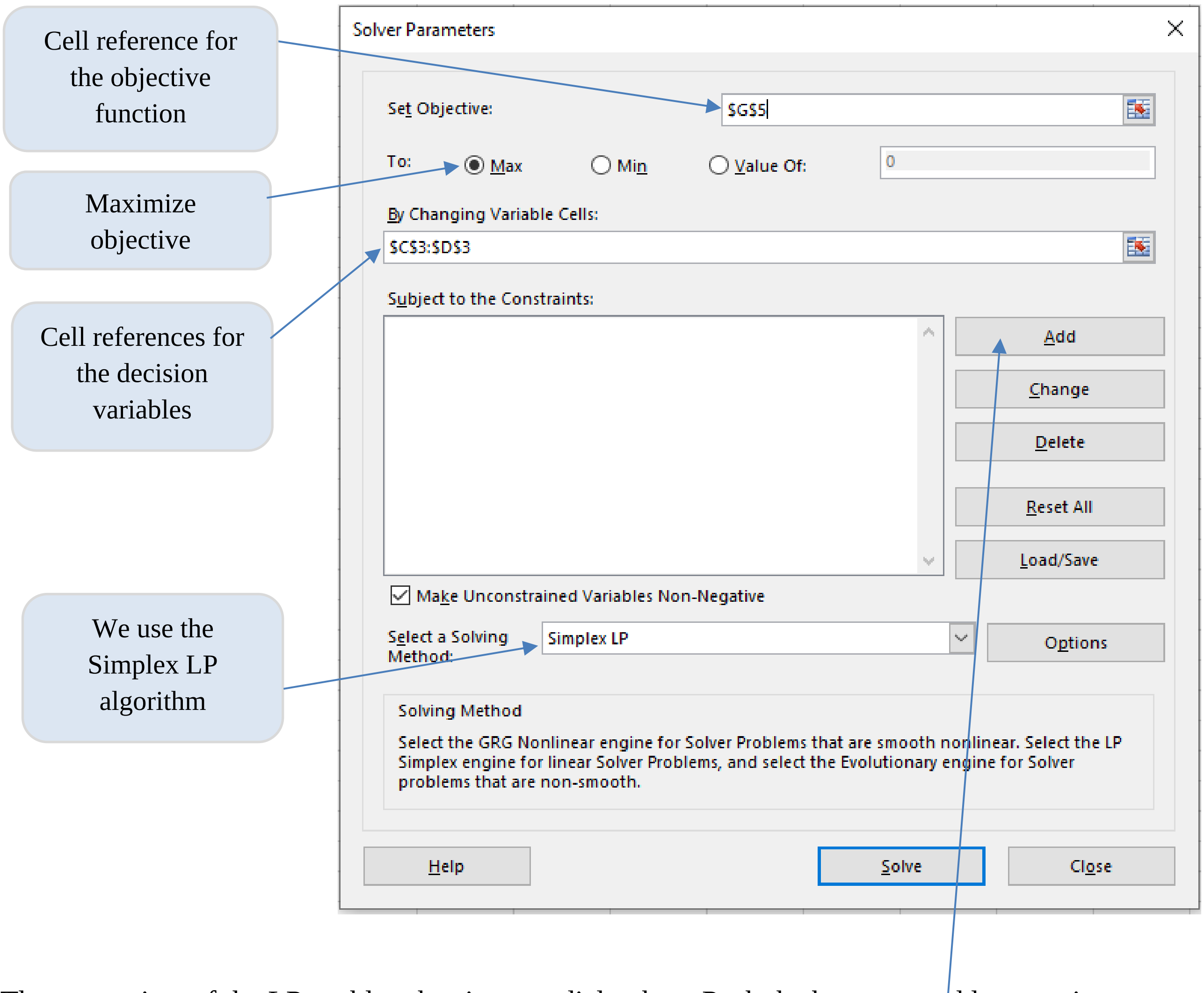
Column with left hand side (LHS), cells E8:E11, can also be computed using SUMPRODUCT and the result are equal to

E8:=SUMPRODUCT($C$3:$D$3;C8:D8)
E9:=SUMPRODUCT($C$3:$D$3;C9:D9)
E10:=SUMPRODUCT($C$3:$D$3;C10:D10)
E11:=SUMPRODUCT($C$3:$D$3;C11:D11)

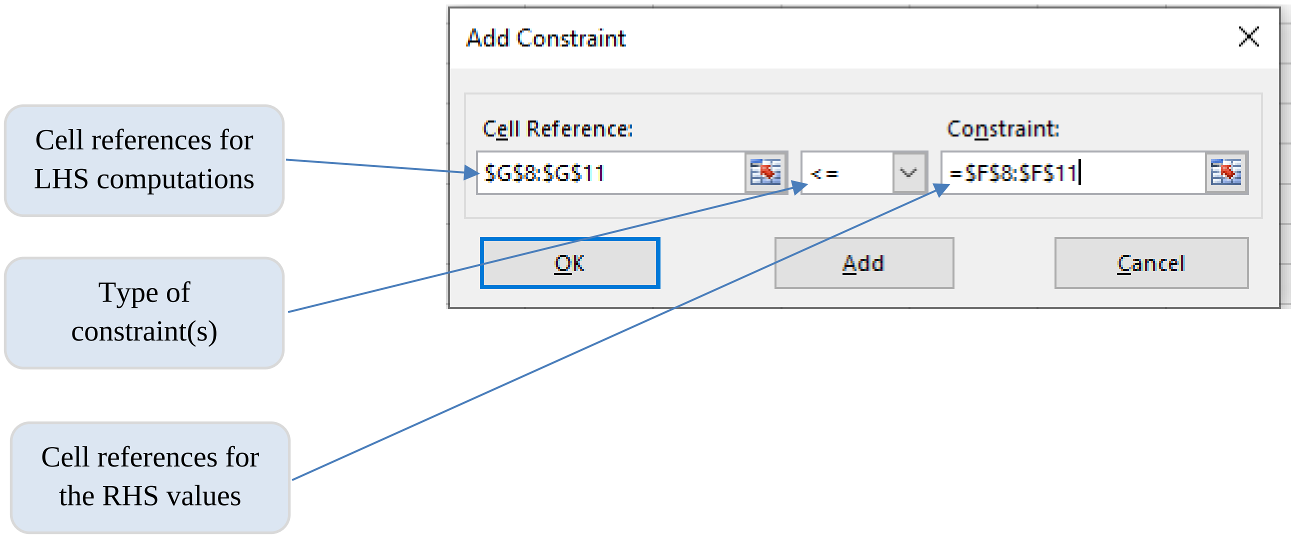
The analytical solution can be found using the Solver procedure in Excel.



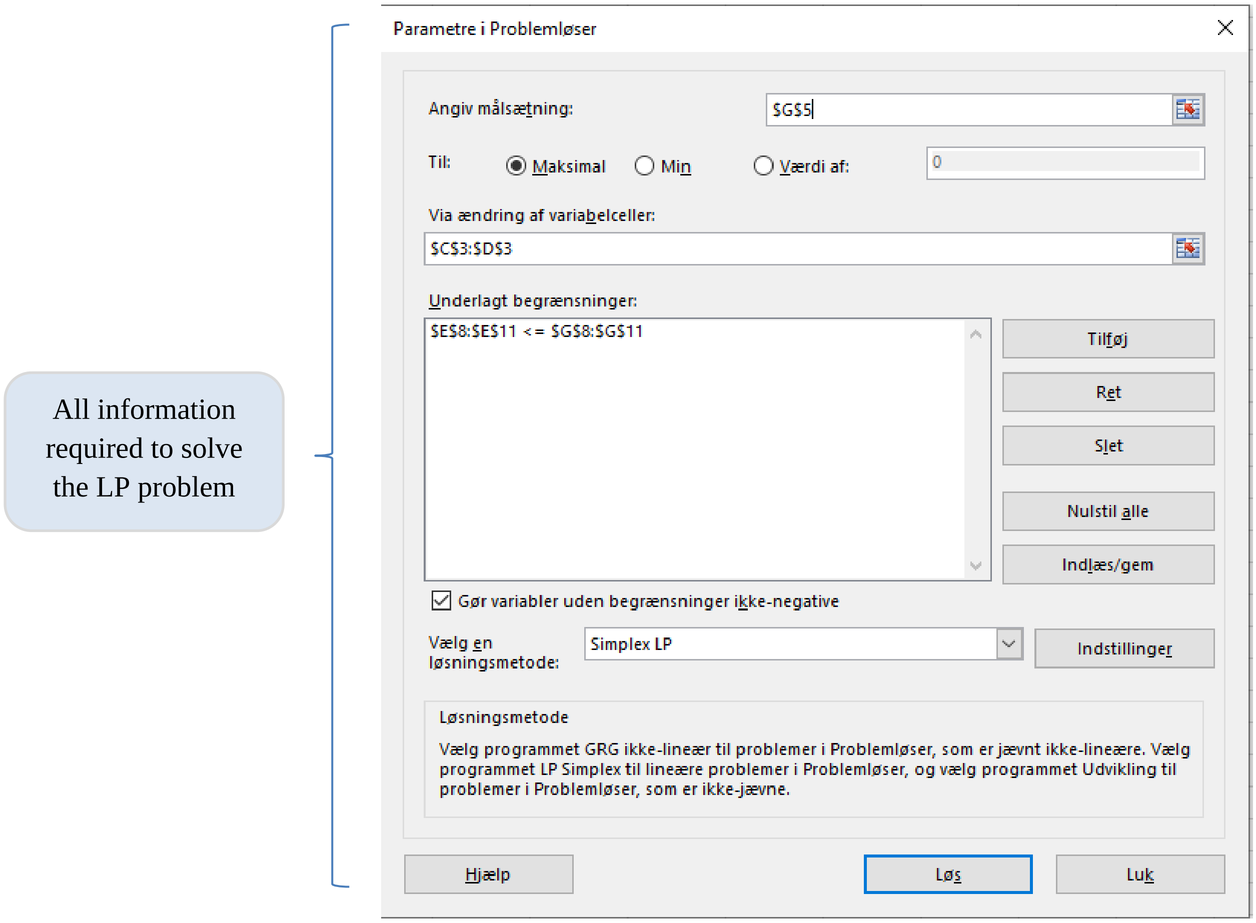
The Solver add-in procedure can be used to find the solution to a LP problem. Solver uses a dialog box that needs to be filled with the necessary information.



The constraints of the LP problem has its own dialog box. Push the bottom to add constraints.



The final dialog box has the content required to Solve the problem.



7.1 Solver solution in Excel

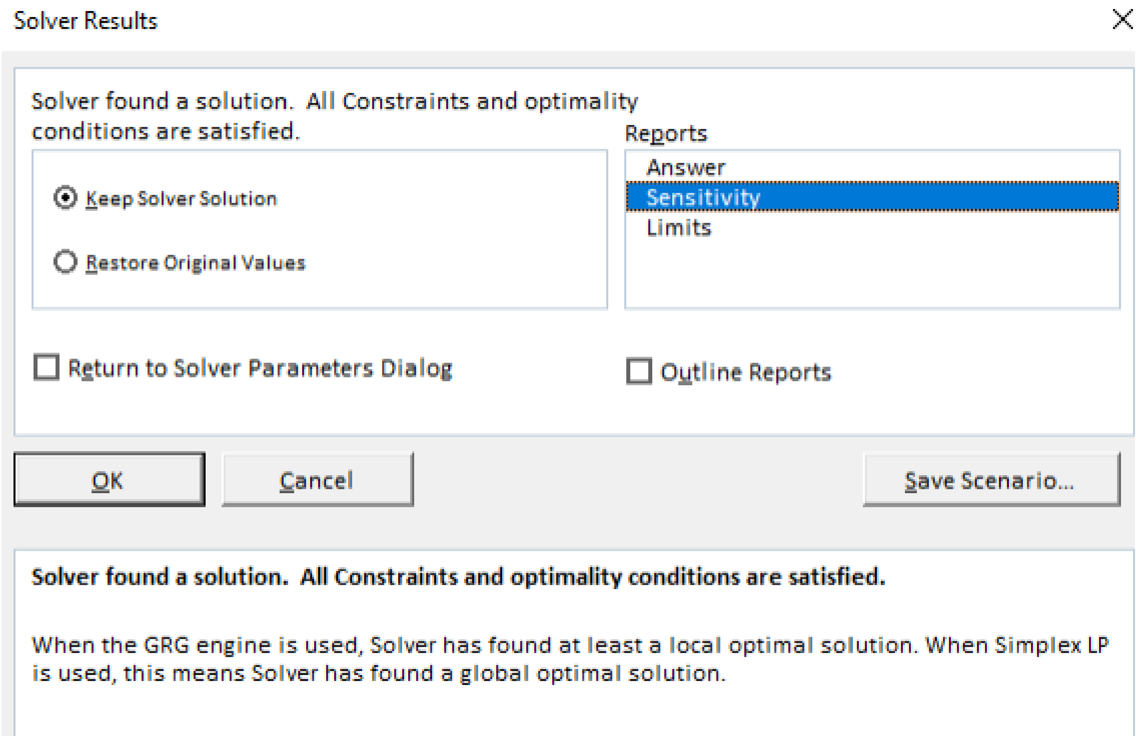
Solver comes up with the solution as shown below.

Solver solution

	A	B	C	D	E	F	G	H	I
1									
2	Decision variables		XL	YL					
3	Solution values		600	1100					
4					Total				
5	Maximize Z		80	129	189900				E5:=SUMPRODUCT($C$3:$D$3;C5:D5)
6									
7	Subject to constraints				LHS	Sign	RHS		
8	Assembly		5	6	9600	≤	10000		E8:=SUMPRODUCT($C$3:$D$3;C8:D8)
9	Testing		1	2	2800	≤	2800		E9:=SUMPRODUCT($C$3:$D$3;C9:D9)
10	Max sale of XL		1	0	600	≤	600		E10:=SUMPRODUCT($C$3:$D$3;C10:D10)
11	Max sale of YL		0	1	1100	≤	1200		E11:=SUMPRODUCT($C$3:$D$3;C11:D11)

The value of Z is maximized to 189900, and the values of the two decision variables are XL = 600 and YL = 1100.

Solver has an extra facility to give more results. From the dialog we find the Sensitivity report.



Push the OK button and Solver comes with the following sheet.

The next output from Solver is the Sensitivity report.

Microsoft Excel 16.0 Sensitivity Report
Worksheet: [Book1.xlsx]Problem 1
Report Created: 11-08-2022 09:35:21

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$C$3	Solution values XL	600	0	80	1E+30	15,5
$D$3	Solution values YL	1100	0	129	31	129

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$G$8	≤ LHS	9600	0	10000	1E+30	400
$G$9	≤ LHS	2800	64,5	2800	133,3333333	2200
$G$10	≤ LHS	600	15,5	600	200	200
$G$11	≤ LHS	1100	0	1200	1E+30	100

From the Sensitivity report we find the ranges of the objective function

Decision variable	Lower	Given	Upper
XL	80 – 15.5 = 64.5	80	∞
YL	129 – 129 = 0	129	129 + 31 = 160

Ranges of right hand sides

Constraint	Lower	Given	Upper	Shadow price
Assembly	10000 – 400 = 9600	10000	∞	0
Testing	2800 – 2200 = 600	2800	2800 + 133.33 = 2933.33	64.5
Max sale X	600 – 200 = 400	600	600 + 200 = 800	15.5
Max sale YL	1200 – 100 = 1100	1200	∞	0

8. Linear Programming application I: Media selection

Consider a company wishing to set up an advertising campaign in preparation for the introduction of a new product. Several types of audiences have been identified as target audiences for the new product. In addition, there is a selection of media available to reach the various targets. However, there is no medium that will reach all audiences. Consequently, several media need to be selected at the same time in order to cover all targets. The company wants to investigate various strategic advertising choices. The goal is not to stay within an a priori fixed budget, but to minimize the total cost of selecting media for each of the strategic choices.

The company advertises in a variety of 30-second television ads, and these ads can be placed in a number of television shows. The cost of the ads varies by the different shows, some are more expensive than others are, and by the type of viewers, they are likely to reach. With 6 mutually exclusive categories: males age 18 to 35, 36 to 55, and over 55; females age 18 to 35, 36 to 55, and over 55.

A marketing rating company has supplied data on the expected numbers of viewers in each of these categories who will watch a 30-second ad on any particular television show. Such a viewer is called an exposure. The company has determined the required number of exposures it wants to obtain for each group. It wants to know how many ads to place on each of several television shows to obtain these required exposures at minimum cost.

The data on costs per ad, numbers of exposures per ad, and minimal required exposures are as follows:

Viewer group / TV Show	Show 1	Show 2	Show 3	Show 4	Show 5	Show 6	Show 7	Show 8	Minimal required exposures
Men 18-35	5	6	5	0.5	0.7	0.1	0.1	3	60
Men 36-55	3	5	2	0.5	0.2	0.1	0.2	5	60
Men above 55	1	3	0	0.3	0	0	0.3	4	28
Women 18-35	6	1	4	0.1	0.9	0.6	0.1	3	60

Women 36-55	4	1	2	0.1	0.1	1.3	0.2	5	60
Women above 55	2	1	0	0	0	0.4	0.3	4	28
Cost per Ad	140	100	80	9	13	15	8	140	

In the table, numbers of exposures are expressed in millions, and costs are in thousands of Euro.

Decision variables

X_1 = number of ads placed in show 1

X_2 = number of ads placed in show 2

X_3 = number of ads placed in show 3

X_4 = number of ads placed in show 4

X_5 = number of ads placed in show 5

X_6 = number of ads placed in show 6

X_7 = number of ads placed in show 7

X_8 = number of ads placed in show 8

Objective

Minimize the cost advertising strategy that meets minimum exposure constraints,

Minimize $Z = 140 X_1 + 100 X_2 + 80 X_3 + 9 X_4 + 13 X_5 + 15 X_6 + 8 X_7 + 140 X_8$

Constraints

1. Men 18-35: $5 X_1 + 6 X_2 + 5 X_3 + 0.5 X_4 + 0.7 X_5 + 0.1 X_6 + 0.1 X_7 + 3 X_8 \geq 60$

2. Men 36-55: $3 X_1 + 5 X_2 + 2 X_3 + 0.5 X_4 + 0.2 X_5 + 0.1 X_6 + 0.2 X_7 + 5 X_8 \geq 60$

3. Men above 55: $1 X_1 + 3 X_2 + 0 X_3 + 0.3 X_4 + 0 X_5 + 0 X_6 + 0.3 X_7 + 4 X_8 \geq 28$

4. Women 18-35: $6 X_1 + 1 X_2 + 4 X_3 + 0.1 X_4 + 0.9 X_5 + 0.6 X_6 + 0.1 X_7 + 3 X_8 \geq 60$

5. Women 36-55: $4 X_1 + 1 X_2 + 2 X_3 + 0.1 X_4 + 0.1 X_5 + 1.3 X_6 + 0.2 X_7 + 5 X_8 \geq 60$

6. Women above 55: $2 X_1 + 1 X_2 + 0 X_3 + 0 X_4 + 0 X_5 + 0.4 X_6 + 0.3 X_7 + 4 X_8 \geq 28$

and

$X_1 , X_2 , X_3 , X_4 , X_5 , X_6 , X_7 , X_8 \geq 0$

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 1870.0000

PARTIAL SENSITIVITY ANALYSIS

	SOLUTION	REDUCED	OPPORTU.	OBJECTIVE FUNCTION RANGES		
VARIABLE	VALUE	COST	COST	LOWER	GIVEN	UPPER
X1	0.0000	-10.0000	-130.0000	130.0000	140.0000	+INFINITY
X2	0.0000	-7.5000	-92.5000	92.5000	100.0000	+INFINITY
X3	8.7187	0.0000	-80.0000	50.9091	80.0000	81.7439
X4	20.6250	0.0000	-9.0000	8.5493	9.0000	9.7619
X5	0.0000	-0.5000	-12.5000	12.5000	13.0000	+INFINITY
X6	6.8750	0.0000	-15.0000	13.8966	15.0000	17.2857
X7	0.0000	-2.2500	-5.7500	5.7500	8.0000	+INFINITY
X8	6.3125	0.0000	-140.0000	133.0435	140.0000	151.0345

		SHADOW	SLACK	RIGHT HAND SIDE RANGES		
CONSTRAINT	TYPE	PRICE		LOWER	GIVEN	UPPER
Men 18-35	≥	0.0000	-13.5312	-INFINITY	60.0000	73.5312
Men 36-55	≥	15.0000	0.0000	54.8837	60.0000	104.0000
Men >55	≥	0.0000	-3.4375	-INFINITY	28.0000	31.4375
Women 18-35	≥	10.0000	0.0000	45.0690	60.0000	71.0000
Women 36-55	≥	5.0000	0.0000	55.1111	60.0000	104.8889
Women >55	≥	2.5000	0.0000	20.4138	28.0000	34.2857

The total cost is minimized to 1,870 Euro and selected ads on shows are

$X_1 = 8.7187$; $X_4 = 20.625$; $X_6 = 6.875$; $X_8 = 6.3125$.

The constraint on exposures to men at the age of 36-55 has a shadow price equal to $\lambda_2 = 15$. If the company change, the constraint from 60 to 59 it would save 15000 Euro.

The advertising model has a dual representation. The company has in fact two competing objectives. The first is to obtain as many exposures as possible at the lowest cost, and the second is to maximize the total number of excess exposures and put a budget constraint on total cost. Here excess exposures are those above the minimum required level.

9. The Dual Problem

Linear programming models have brought to economists the meaning of duality[22]. We know that economists are interested in production and cost, prices and quantities, but duality possesses a meaning that transcends linear programming and its economic interpretation. Most problems have a twofold representation and we call this the duality. Associated with every linear programming problem is a symmetric dual linear programming problem.

For every maximization problem, there exists a symmetrical minimization problem. For every minimization problem, there exists a symmetrical maximization problem.

Pairs of related maximization and minimization problems are known as primal and dual linear programming problems. The concept of duality demonstrates the symmetry between the value of a firm's products and the value of resources used in production. Because of the symmetry between primal and dual problem specifications, either one can be constructed from the other and the solution to either problem can be used to solve both. This is helpful because it is sometimes easier to obtain the solution to the dual problem than to the original or primal problem.

The duality concept also allows one to evaluate the solution to a constrained decision problem in terms of the activity required for optimization and in terms of the economic impact of constraint conditions. Analysis of the constraint conditions and slack variable solution frequently provides important information for long-range planning.

The primal solution is often described as a tool for short-run operating decisions. The dual solution is often seen as a tool for long-range planning. The duality concept shows how operating decisions and long-range planning are related.

[22] The word duality can be found in the works by Euler and Legendre in 1750 where they were interested in methods for solving differential equations. Later Boole (1859) wrote: "*There exists in partial differential equations a remarkable duality, in virtue of which each equation stands connected with some other equation of the same order by relations of a perfectly reciprocal character*". Boole, G. B. A Treatise On Differential Equations. New York, Chelsea Pub. Co., 5th, 1859.

The solution of linear programming problem contains information that can be useful in making marginal resource-allocation decisions. And the relevant marginal information is contained in the dual variables of the linear programming problem.

The key to duality is that relevant costs are not the acquisition cost of inputs but, rather, the economic costs of using them. For resources that is available in a fixed amount, this cost is not acquisition cost but opportunity cost. Because the economic value of constrained resources is determined by their value in use rather than by historical acquisition costs, such amounts are called implicit values or shadow prices. The term shadow price is used because it represents the price that a manager would be willing to pay for additional units of a constrained resource. Comparing the shadow price of a resource with its acquisition price indicates whether the firm has an incentive to increase or decrease the amount acquired during future production periods. If shadow prices exceed acquisition prices, the resource's marginal value exceeds marginal cost and the firm has an incentive to expand employment. If acquisition cost exceeds the shadow price, there is an incentive to reduce employment.

In the production example, the company had a number of available processing hours, but we did not use all the hours. We found that 60 processing hours was not used. The question is what the economic value associated with the last processing hour is worth? The less we use of available processing hours the better. Any free processing hour can be used for another purpose.

10. Linear Programming application II: Production planning I

Consider a company that produce furniture, Chair1 and Chair2, which they sell for €2500 and €2000. Both types of furniture are produced with limited resources as 500 labor-hours and 700 machine-hours. The company has 150 cubic meter of wood that they can use.

There are technical related information regarding the production of chairs. It takes 2 labor hours and 1 machine hour to produce one Chair1; it takes 1 hour and 2 machine hours to produce one Chair2. The two types of chairs are produced using 0,3 cubic meter of wood each. The technical information gives the following resource constraints:

Labor hours:	2 Chair1 + 1 Chair2 ≤ 500
Machine hours:	1 Chair1 + 2 Chair2 ≤ 700
Wood:	0.3 Chair1 + 0.3 Chair 2 ≤ 150

The problem is to find the revenue-maximizing level of Chair1 and Chair2 given these resource constraints.

Let the decision variables be

Chair1 = number of Chair1 produced

Chair2 = number of Chair2 produced

The linear programming problem is written as

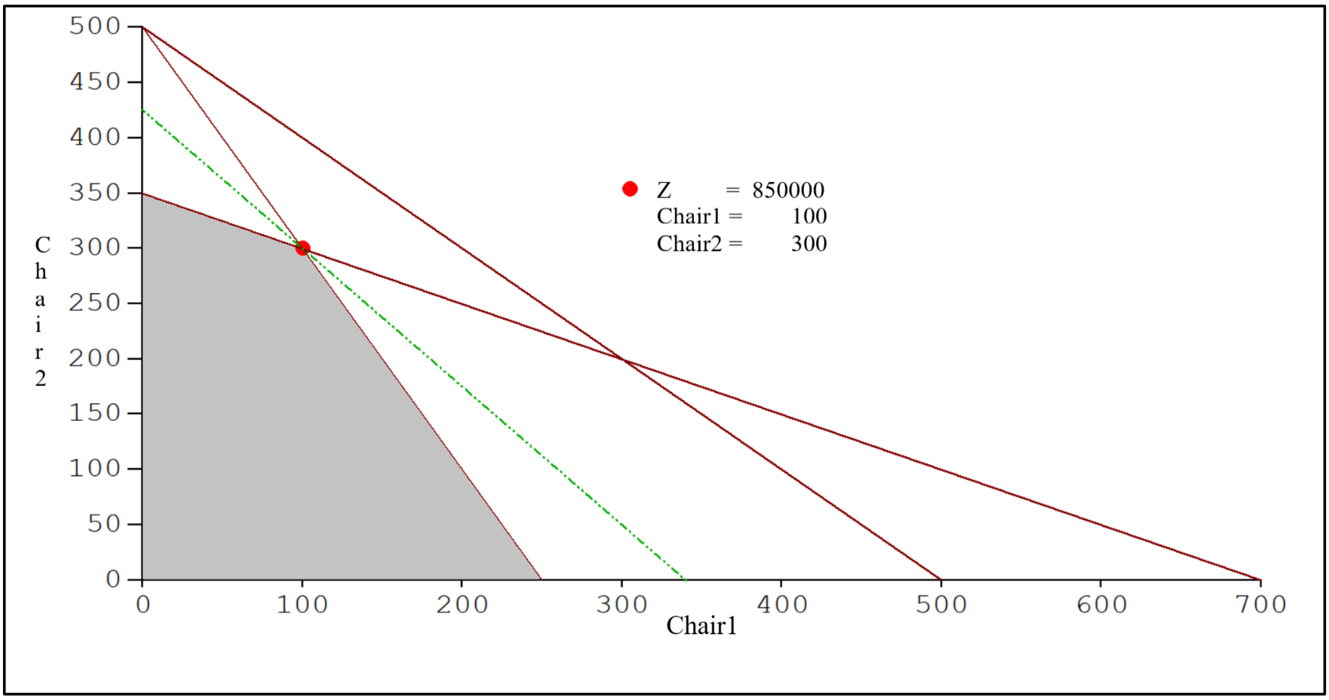
Maximize TR = €2500 Chair1 + €2000 Chair2

Subject to

Labor hours:	2 Chair1 + 1 Chair2 ≤ 500
Machine hours:	1 Chair1 + 2 Chair2 ≤ 700
Wood:	0.3 Chair1 + 0.3 Chair2 ≤ 150

and Chair1, Chair2 ≥ 0

If we solve the LP-problem we have the following optimal graphical solution



Optimal analytical solution:

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 850000.0000

PARTIAL SENSITIVITY ANALYSIS

VARIABLE	SOLUTION VALUE	REDUCED COST	OPPORTU. COST	OBJECTIVE FUNCTION RANGES LOWER	GIVEN	UPPER
Chair1	100.0000	0.0000	2500.0000	1000.0000	2500.0000	4000.0000
Chair2	300.0000	0.0000	2000.0000	1250.0000	2000.0000	5000.0000

CONSTRAINT	TYPE	SHADOW PRICE	SLACK	RIGHT HAND SIDE RANGES LOWER	GIVEN	UPPER
Labor	≤	1000.0000	0.0000	350.0000	500.0000	799.9999
Machine	≤	500.0000	0.0000	250.0000	700.0000	999.9999
Wood	≤	0.0000	30.0000	120.0000	150.0000	+INFINITY

In the optimal solution we find the value of the two decision variables Chair1 = 100 and Chair2 = 300. The value of the objective function is equal to 2500(100) + 2000(300) = €850000. The value of the three shadow variables are $\lambda_1 = 1000$, $\lambda_2 = 500$, $\lambda_3 = 0$.

At the optimal solution, the shadow price for Wood is zero, $\lambda_3 = 0$. Because shadow price measures the marginal value of an input, a zero shadow price implies that this resource has a zero marginal value to the firm. Adding more wood adds nothing to the firm's maximum obtainable profit. A zero shadow price for wood is consistent with the primal solution that wood is not a binding constraint. Excess capacity exists in wood, so additional wood would not increase production of either Chair1 or Chair2. At the optimal solution there are 30 cubic meter as indicated with slack variable $S_3 = 30$. The shadow price for input Labor of 1,000 implies that this fixed resource imposes a binding constraint. If an additional unit of Labor is added, the firm can increase total profit by 1000. And at the optimal solution all Labor hours has been used as indicated with slack variable $S_1 = 0$.

Associated with every linear programming problem is a symmetrical dual linear programming problem. If the objective in the original problem is the maximization of an objective function, the objective of the dual is the minimization of a related function. The solution of a linear programming problem contains information that can be useful in making marginal resource-allocation decisions. The relevant information is contained in the dual variables of the linear programming problem. Thus the concept of duality is the symmetry between the value of outputs and the value of resources used. The key to duality is that relevant costs are not the acquisition cost of inputs but, rather, the economic costs of using them. For resources that are fixed, the economic cost is the opportunity cost. The term shadow price is used to describe implicit values because it represents the price that a manager would be willing to pay for additional units of a limited resource. Comparing the shadow prices of resources with their acquisition prices indicates whether an incentive exists to increase or decrease the amount used in future production periods.

To model the dual problem, it helps to ask the following three questions in order:

1) what are the variables
2) what is our objective in terms of these variables
3) what are the constraints

The decision variables in the dual problem are the shadow prices of the resources.

Decision variables

We introduce three dual decision variables.

Let

λ_1=shadow price of labor hours
λ_2=shadow price of machine hours
λ_3=shadow price of wood

Objective function

Our objective is to minimize the shadow prices of the resources. We specify our linear objective function by

Minimize TR' = $500\lambda_1+700\lambda_2+150\lambda_3$

Constraints

In our original problem (the primal problem), we maximized the total revenue. In the dual problem, we minimize the available hours with respect to having the revenue from the two types of chairs as large as possible.

We want to have Chair1 as large as possible (the larger the better):

- Chair1: $2\lambda_1 + 1\lambda_2 + 0.3\lambda_3 \geq 2500$

We want to have Chair2 as large as possible (the larger the better):

- Chair2: $1\lambda_1 + 2\lambda_2 + 0.3\lambda_3 \geq 2000$

The Dual Model

The dual objective and constraints are as follows:

Minimize Dual TR = $500\ \lambda_1 + 700\ \lambda_2 + 150\ \lambda_3$

Subject to

Chair1: $2\ \lambda_1 + 1\ \lambda_2 + 0.3\ \lambda_3 \geq 2500$

Chair2: $1\ \lambda_1 + 2\ \lambda_2 + 0.3\ \lambda_3 \geq 2000$

and λ_1 , $\lambda_2 \geq 0$

Using the computer, we have

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 850000.0000

PARTIAL SENSITIVITY ANALYSIS

VARIABLE	SOLUTION VALUE	REDUCED COST	OPPORTU. COST	OBJECTIVE FUNCTION RANGES LOWER	GIVEN	UPPER
Shadow 1	1000.0000	0.0000	-500.0000	350.0000	500.0000	799.9999
Shadow 2	500.0000	0.0000	-700.0000	250.0000	700.0000	999.9999
Shadow 3	0.0000	-30.0000	-120.0000	120.0000	150.0000	+INFINITY

CONSTRAINT	TYPE	SHADOW PRICE	SLACK	RIGHT HAND SIDE RANGES LOWER	GIVEN	UPPER
Chair1	≥	100.0000	0.0000	1000.0000	2500.0000	4000.0000
Chair2	≥	300.0000	0.0000	1250.0000	2000.0000	5000.0000

First we see that the optimal solution of TR' = 500(1000) + 700(500) + 150(0) = 850000 which is identical to the optimal solution of the original primal problem where TR=850000. As can be seen the values of the three decision variables are λ_1= 1000 λ_2= 500, λ_3= 0. Remember from our presentation above in the primal problem that the three shadow prices where: {1000; 500; 0}. This is an important observation.

We have the correspondence between the original problems - the primal problem - to the dual problem, that shadow prices in the primal problem are equal to solution values in the dual problem.

The shadow prices in the dual problem are equal to solution values in the primal problem. Shadow prices in the dual problem have solution values equal to {100 ; 300}. These values are equal to the solution values from the primal problem {100 ; 300}.

We also have a reduced cost that is not equal to zero. We see in this dual solution that the reduced cost of Shadow 3 is equal to -30 and this is exactly equal to the value of the slack value (with opposite sign) in the primal problem.

Dual price

The dual price of a constraint measures the rate at which the solution value improves as the right-hand side is increased.

There is a direct link between a primal and a dual problem. Duality is nothing else than the specification of one problem from two different points of view. Once a problem is specified, we refer to it as the primal problem. A second, associated problem very often exists and it is called the dual specification. In managerial economics we are often more interested in the dual problem because it contains mostly economic information. Sometimes a manager is less concerned about the profit from a solution than the use of available resources. This is because a manager has more control over the use of available resources compared with the profits. The dual solution gives a manager the information about the value of resources, and this in turn is important when he has to decide if more resources should be allocated and how much to pay for these extra resources.

To fully understand the duality relations in linear programming, it is important to know how to set up dual pairs of problems properly and how to interpret all their components in economic terms.

We have seen that primal problem constraints state that the total amount of each input used to produce X_1 and X_2 must be equal to or less than the available quantity of input. In the dual, the constraints state that the total value of inputs used to produce 1 unit of X_1 or 1 unit of X_2 must not be less than the profit contribution provided by a unit of these products.

11. Linear Programming application III: Product mix

The features of a product mix problem are that a collection of products compete for a finite set of resources. Consider a plant that can manufacture five different products in any combination. Each product requires time on each of three machines as follows:

(numbers in minutes/unit)

	Machine		
Product	1	2	3
A	12	8	5
B	7	9	10
C	8	4	7
D	10	0	3
E	7	11	2

Each machine is available 128 hours per week.

The three products are competitive, and any amounts made may be sold at respective prices €5, €4, €5. The first 20 units of D and E produced per week can be sold at €4 each, but all made in excess of 20 can only be sold at €3 each. Variable costs are €4 per hour for machines 1 and 2 and €3 per hour for machine 3. Material costs are €2 for products A and C and €1 for products B, D, and E. The firm wish to maximize profit.

Decision variables

The decision variables specify the number of units produced per week

Let A = number of units of A produced per week

B = number of units of B produced per week

C = number of units of C produced per week

D = number of units of $D_1 + D_2$ produced per week

D_1 = number of units of D not in excess of 20 produced/week

D_2 = number of units of D produced in excess of 20 per week

E = number of units of $E_1 + E_2$ produced per week

E_1 = number of units of E not in excess of 20 produced/week

E_2 = number of units of E produced in excess of 20 per week

M_1 = hours of machine 1 used per week

M_2 = hours of machine 2 used per week

M_3 = hours of machine 3 used per week

Objective function

Maximize profits = revenue minus costs

Decision variable	Price €	Variable cost €	Material cost €
A	5		2
B	4		1
C	5		2
D_1	4		1
D_2	3		1
E_1	4		1
E_2	3		1
M_1		4	
M_2		4	
M_3		3	

Objective function:

$$\text{Max } Z = 3A + 3B + 3C + 3D_1 + 2D_2 + 3E_1 + 2E_2 - 4M_1 - 4M_2 - 3M_3$$

Constraints

Product limits.

- $D_1 \leq 20$
- $E_1 \leq 20$

Machine availability.

- $M_1 \leq 128$
- $M_2 \leq 128$
- $M_3 \leq 128$

Minutes used equals minutes run on each machine.

The first three constraints have the units of "minutes" and specify the hours of machine time as a function of the number of units produced. We specify the machine time available for machine 1, as

$$\frac{12\mathrm{A} + 7\mathrm{B} + 8\mathrm{C} + 10\mathrm{D}_1 + 10\mathrm{D}_2 + 7\mathrm{E}_1 + 7\mathrm{E}_2}{60} = \mathrm{M}_1$$

This is equal to

$$12\mathrm{A} + 7\mathrm{B} + 8\mathrm{C} + 10\mathrm{D}_1 + 10\mathrm{D}_2 + 7\mathrm{E}_1 + 7\mathrm{E}_2 = 60\mathrm{M}_1$$

equal to

$$12\mathrm{A} + 7\mathrm{B} + 8\mathrm{C} + 10\mathrm{D}_1 + 10\mathrm{D}_2 + 7\mathrm{E}_1 + 7\mathrm{E}_2 - 60\mathrm{M}_1 = 0$$

The three constraints then becomes

- $12\mathrm{A} + 7\mathrm{B} + 8\mathrm{C} + 10\mathrm{D}_1 + 10\mathrm{D}_2 + 7\mathrm{E}_1 + 7\mathrm{E}_2 - 60\mathrm{M}_1 = 0$
- $8\mathrm{A} + 9\mathrm{B} + 4\mathrm{C} + 11\mathrm{E}_1 + 11\mathrm{E}_2 - 60\mathrm{M}_2 = 0$
- $5\mathrm{A} + 10\mathrm{B} + 7\mathrm{C} + 3\mathrm{D}_1 + 3\mathrm{D}_2 + 2\mathrm{E}_1 + 2\mathrm{E}_2 - 60\mathrm{M}_3 = 0$

Final LP model

$$\text{Max Z} = 3A + 3B + 3C + 3D_1 + 2D_2 + 3E_1 + 2E_2 - 4M_1 - 4M_2 - 3M_3$$

Subject to

Product D_1: $D_1 \leq 20$

Product E_1: $E_1 \leq 20$

Machine time 1: $M_1 \leq 128$

Machine time 2: $M_2 \leq 128$

Machine time 3: $M_3 \leq 128$

Machine 1: $12A + 7B + 8C + 10D_1 + 10D_2 + 7E_1 + 7E_2 - 60M_1 = 0$

Machine 2: $8A + 9B + 4C + 11E_1 + 11E_2 - 60M_2 = 0$

Machine 3: $5A + 10B + 7C + 3D_1 + 3D_2 + 2E_1 + 2E_2 - 60M_3 = 0$

$A, B, C, D_1, D_2, E_1, E_2, M_1, M_2, M_3 \geq 0$

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 1777.6250

PARTIAL SENSITIVITY ANALYSIS

VARIABLE	SOLUTION VALUE	REDUCED COST	OPPORTU. COST	OBJECTIVE FUNCTION RANGES LOWER	GIVEN	UPPER
A	0.0000	-1.3583	4.3583	-INFINITY	3.0000	4.3583
B	0.0000	-0.1854	3.1854	-INFINITY	3.0000	3.1854
C	942.5000	0.0000	3.0000	2.8967	3.0000	3.0929
D1	0.0000	-0.1292	3.1292	-INFINITY	3.0000	3.1292
D2	0.0000	-1.1292	3.1292	-INFINITY	2.0000	3.1292
E1	20.0000	0.0000	3.0000	2.9187	3.0000	+INFINITY
E2	0.0000	-0.9188	2.9188	-INFINITY	2.0000	2.9187
M1	128.0000	0.0000	-4.0000	-17.8750	-4.0000	+INFINITY
M2	66.5000	0.0000	-4.0000	-4.6500	-4.0000	-1.9773
M3	110.6250	0.0000	-3.0000	-4.3478	-3.0000	-1.8182

CONSTRAINT	TYPE	SHADOW PRICE	SLACK	RIGHT HAND SIDE RANGES LOWER	GIVEN	UPPER
Limit D1	<=	0.0000	20.0000	0.0000	20.0000	+INFINITY
Limit E1	<=	0.0813	0.0000	0.0000	20.0000	512.0000
Machine 1	<=	13.8750	0.0000	2.3333	128.0000	147.8571
Machine 2	<=	0.0000	61.5000	66.5000	128.0000	+INFINITY
Machine 3	<=	0.0000	17.3750	110.6250	128.0000	+INFINITY
Time 1	=	0.2979	0.0000	-7540.0000	0.0000	1191.4286
Time 2	=	0.0667	0.0000	-3690.0000	0.0000	3990.0000
Time 3	=	0.0500	0.0000	-1042.5000	0.0000	6637.5000

The optimal solution gives a value of the objective function equal to 1777.6250

The optimal values of decision variables.

Variable	Solution value
A	0
B	0
C	942.5
D_1	0
D_2	0
E_1	20
E_2	0
M_1	128
M_2	66.5
M_3	110.625

12. Linear programming application IV: Investment

We have introduced the general structure of the linear programming problem and characteristics of the solution. Now let us look at an investment problem.

A company has 12 mill. € for investment within the company. There are five possible projects under consideration.

Purchasing	Expected return	Maximum investment (mill. of €)
Improved materials-handling equipment	15%	3
Automating packaging operations	10%	5
Purchasing raw materials in anticipation of price increase	18%	6
Paying up outstanding notes	8%	4
Additional promotion for a new product line	20%	1

There is no minimum investment required for any project. Project 1 and 2 are classified as capital expenditure projects, projects 3 and 5 are speculative investment projects, and project 4 is a financial project.

The company has the following requirements:

- Investment in capital expenditures must at least be 40% of the total
- The investment in speculative projects must be no more than 50% of the amount used to pay notes.

Decision variables

The decision variables must specify the amount of money invested in each of the alternatives. We introduce 6 decision variables that specify the amount invested in each of the alternatives.

Let X_1= investment in Improved materials-handling equipment

X_2 = investment in Automating packaging operations

X_3 = investment in Purchasing raw materials in anticipation of price increase

X_4 = investment in Paying up outstanding notes

X_5 = investment in Additional promotion for a new product line

Objective function

Determine how much to invest in each project to maximize annual return. We have

Max Z=annual return

$Z = 0.15X_1+0.10X_2 + 0.18X_3 +0.08X_4+0.20X_5$

Constraints

From the problem specification, we have a number of constraints

- Available cash (mill. €) $1X_1+1X_2+1X_3+1X_4+1X_5 \leq 12$
- Max project 1 $1X_1 \leq 3$
- Max project 2 $1X_2 \leq 5$
- Max project 3 $1X_3 \leq 6$
- Max project 4 $1X_4 \leq 4$
- Max project 5 $1X_5 \leq 1$
- Spec. 50% notes $1X_3+1X_5 \leq 0.5x_4$
- Capital expenditure at least 40% $(1X_1+1X_2)/(1X_1+1X_2+1X_3+1X_4+1X_5) \geq 0.4$

We summarize the LP portfolio selection problem as follows.

We introduce decision variables X_1 ,..., X_5

Max $Z = 0.15X_1+0.10X_2 + 0.18X_3 +0.08X_4+0.20X_5$	
Subject to constraints	
Available cash (mill. €)	$1X_1+1X_2+1X_3+1X_4+1X_5 \leq 12$
Max project 1	$1X_1 \leq 3$
Max project 2	$1X_2 \leq 5$
Max project 3	$1X_3 \leq 6$
Max project 4	$1X_4 \leq 4$
Max project 5	$1X_5 \leq 1$
Spec. 50% notes	$1X_3 – 1X_4 + 1X_5 \leq 0$
Capital exp. at least 40%	$0.6X_1+0.6X_2-0.4X_3-0.4X_4-0.4X_5 \geq 0$
and	
$X_1,X_2,X_3,X_4,X_5,X_6 \geq 0$	

The solution to the portfolio problem gives the following optimal solution:

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 1.5860

				PARTIAL SENSITIVITY ANALYSIS		
	SOLUTION	REDUCED	OPPORTU.	OBJECTIVE FUNCTION RANGES		
VARIABLE	VALUE	COST	COST	LOWER	GIVEN	UPPER
Project 1	3.0000	0.0000	0.1500	0.1000	0.1500	+INFINITY
Project 2	1.8000	0.0000	0.1000	-0.1950	0.1000	0.1300
Project 3	2.6000	0.0000	0.1800	0.1200	0.1800	0.2000
Project 4	3.6000	0.0000	0.0800	0.0200	0.0800	0.1800
Project 5	1.0000	0.0000	0.2000	0.1800	0.2000	+INFINITY

		SHADOW	SLACK	RIGHT HAND SIDE RANGES		
CONSTRAINT	TYPE	PRICE		LOWER	GIVEN	UPPER
Cash	≤	0.1180	0.0000	7.5000	12.0000	13.3333
Max proj 1	≤	0.0500	0.0000	0.0000	3.0000	4.8000
Max proj 2	≤	0.0000	3.2000	1.8000	5.0000	+INFINITY
Max proj 3	≤	0.0000	3.4000	2.6000	6.0000	+INFINITY
Max proj 4	≤	0.0000	0.4000	3.6000	4.0000	+INFINITY
Max proj 5	≤	0.0200	0.0000	0.0000	1.0000	3.6000
Notes	≤	0.0500	0.0000	-0.8000	0.0000	6.8000
Cap. Exp.	≥	0.0300	0.0000	-0.8000	0.0000	3.2000

The optimal solution gives a value of the objective function equal to 1.586.

This means an annual return = 1.586 mill € on a 12 mill € investment.

From the output, we can find the values of the decision variables.

Variable	Investment (in mill. €)
Project 1	3
Project 2	1.8
Project 3	2.6
Project 4	3.6
Project 5	1.0
Total	12

From the output we see that the shadow price on cash is equal to 0.118, this means a 11.8% return on additional cash; The shadow price on max project 1 is equal to 0.05, this means 5% additional return for additional investment permitted in project 1; project 5 has shadow price equal to 0.02, this means 2% for additional project 4 funds; notes has shadow price equal to 0.05, this means 5% additional return for each percentage point less than 50% permitted.

13. Linear programming application V: Supply Chain I

Consider a pharmaceutical company that has three production plants factories, at which it produces its product. Products are distributed through three warehouses, which are located at distances from the factories. Each month the warehouses supply the central office with their expected demands for the product and management must determine from which factories to supply each warehouse. Warehouses may be supplied from more than one factory.

The three factories have the following capacity each month:

Factory	A	B	C	Total
Capacity in units each month	50	55	70	175

For the coming month, the warehouses have the following expected demand:

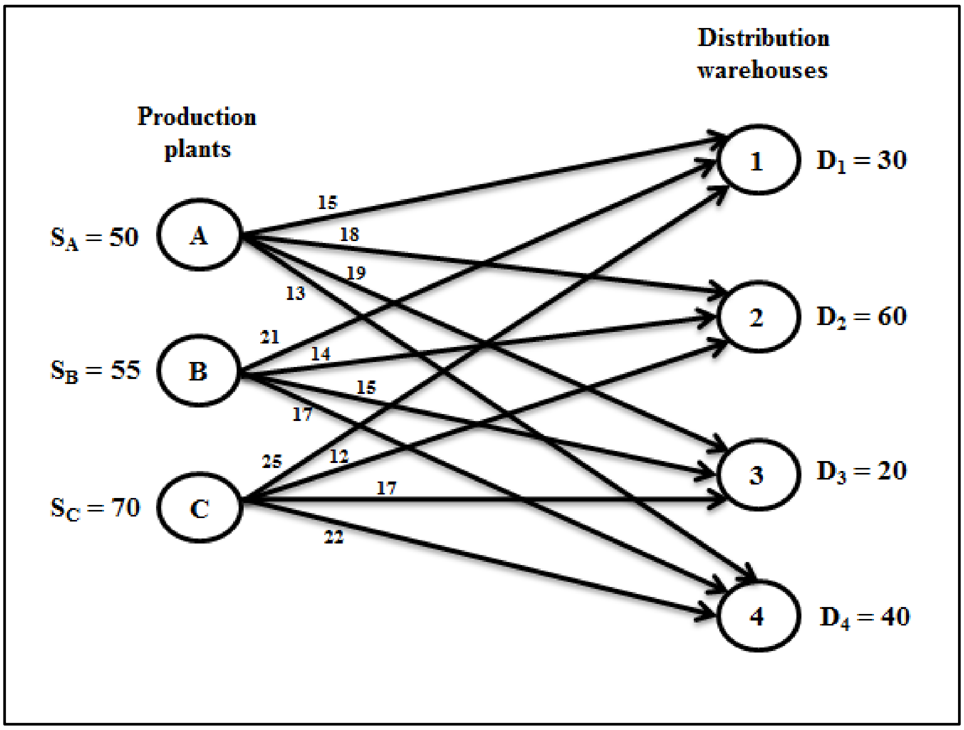
Warehouse	1	2	3	4	Total
Demand	30	60	20	40	150

The per-unit cost of shipping a unit from each factory to each warehouse is:

		To Warehouse 1	2	3	4
From Factory	A	15	18	19	13
	B	21	14	15	17
	C	25	12	17	22

The company wishes to find the best way to supply each warehouse's demand without exceeding the capacity of product at each factory.

From the tables we can represent the transportation network in the following figure:



Transportation network

The objective of the decision maker is to determine how to supply each warehouse's demand to minimize the total cost.

It is necessary to define a variable representing the number of units shipped from each production plant to each warehouse. Since there are three plants are four warehouses, there will be a total of (i=A,B,C) and (j=1,2,3,4) = 3×4=12 variables. Let

X_{ij} = number of units shipped from plant i and sent to warehouse j

	Number of units shipped	
Variable	From Plant	To Warehouse
X_{A1}	A	1
X_{A2}	A	2
X_{A3}	A	3
X_{A4}	A	4
X_{B1}	B	1
X_{B2}	B	2
X_{B3}	B	3
X_{B4}	B	4
X_{C1}	C	1
X_{C2}	C	2
X_{C3}	C	3
X_{C4}	C	4

In terms of these variables, we will have two types of constraints. The first type will state that, for each plant, the sum of the shipments from that plant to all warehouses may not exceed the available units. The second type state that, for each warehouse, the sum of shipments to that warehouse must be at least equal to its demand.

Plant constraints:

Plant A: $X_{A1} + X_{A2} + X_{A3} + X_{A4} \leq 50$

Plant B: $X_{B1} + X_{B2} + X_{B3} + X_{AB} \leq 55$

Plant C: $X_{C1} + X_{C2} + X_{C3} + X_{C4} \leq 70$

Warehouse constraints:

Warehouse 1: $X_{A1} + X_{B1} + X_{C1} \geq 30$

Warehouse 2: $X_{A2} + X_{B2} + X_{C2} \geq 60$

Warehouse 3: $X_{A3} + X_{B3} + X_{C3} \geq 20$

Warehouse 4: $X_{A4} + X_{B4} + X_{C4} \geq 40$

<u>The objective function</u> must state the total cost of the shipments. Thus

Cost of shipping from plant A: $15X_{A1} + 18X_{A2} + 19X_{A3} + 13X_{A4}$

Cost of shipping from plant B: $21X_{B1} + 14X_{B2} + 15X_{B3} + 17X_{B4}$

Cost of shipping from plant C: $25X_{C1} + 12X_{C2} + 17X_{C3} + 22X_{C4}$

Because all the X_{ij}'s must be nonnegative, we add the sign restrictions $X_{ij} \geq 0 \quad (i = A, B, C; j = 1,2,3,4)$.

Combining the objective function, supply constraints, demand constraints, and sign restrictions yields the following linear programming formulation:

$$\begin{aligned} \text{Min } Z = \; & 15X_{A1} + 18X_{A2} + 19X_{A3} + 13X_{A4} + \\ & 21X_{B1} + 14X_{B2} + 15X_{B3} + 17X_{B4} + \\ & 25X_{C1} + 12X_{C2} + 17X_{C3} + 22X_{C4} \end{aligned}$$

Subject to

$$\left.\begin{aligned} & X_{A1} + X_{A2} + X_{A3} + X_{A4} \leq 50 \\ & X_{B1} + X_{B2} + X_{B3} + X_{AB} \leq 55 \\ & X_{C1} + X_{C2} + X_{C3} + X_{C4} \leq 70 \end{aligned}\right\} \text{(Supply constraints)}$$

$$\left.\begin{aligned} & X_{A1} + X_{B1} + X_{C1} \geq 30 \\ & X_{A2} + X_{B2} + X_{C2} \geq 60 \\ & X_{A3} + X_{B3} + X_{C3} \geq 20 \\ & X_{A4} + X_{B4} + X_{C4} \geq 40 \end{aligned}\right\} \text{(Demand constraints)}$$

$X_{ij} \geq 0 \quad (i = A, B, C; j = 1,2,3,4)$.

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 2070.0000

PARTIAL SENSITIVITY ANALYSIS

VARIABLE	SOLUTION VALUE	REDUCED COST	OPPORTU. COST	OBJECTIVE FUNCTION RANGES LOWER	GIVEN	UPPER
XA1	30.0000	0.0000	-15.0000	-4.0000	15.0000	17.0000
XA2	0.0000	-10.0000	-8.0000	8.0000	18.0000	+INFINITY
XA3	0.0000	-8.0000	-11.0000	11.0000	19.0000	+INFINITY
XA4	20.0000	0.0000	-13.0000	11.0000	13.0000	17.0000
XB1	0.0000	-2.0000	-19.0000	19.0000	21.0000	+INFINITY
XB2	0.0000	-2.0000	-12.0000	12.0000	14.0000	+INFINITY
XB3	20.0000	0.0000	-15.0000	0.0000	15.0000	17.0000
XB4	20.0000	0.0000	-17.0000	13.0000	17.0000	19.0000
XC1	0.0000	-6.0000	-19.0000	19.0000	25.0000	+INFINITY
XC2	60.0000	0.0000	-12.0000	0.0000	12.0000	14.0000
XC3	0.0000	-2.0000	-15.0000	15.0000	17.0000	+INFINITY
XC4	0.0000	-5.0000	-17.0000	17.0000	22.0000	+INFINITY

CONSTRAINT	TYPE	SHADOW PRICE	SLACK	RIGHT HAND SIDE RANGES LOWER	GIVEN	UPPER
Plant A	≤	4.0000	0.0000	35.0000	50.0000	70.0000
Plant B	≤	0.0000	15.0000	40.0000	55.0000	+INFINITY
Plant C	≤	0.0000	10.0000	60.0000	70.0000	+INFINITY
Warehouse 1	≥	19.0000	0.0000	10.0000	30.0000	45.0000
Warehouse 2	≥	12.0000	0.0000	0.0000	60.0000	70.0000
Warehouse 3	≥	15.0000	0.0000	0.0000	20.0000	35.0000
Warehouse 4	≥	17.0000	0.0000	20.0000	40.0000	55.0000

As we can see from the output, the minimum shipping cost of 2,070 is obtained by the following shipping pattern:

Variable	From	To	Units	Costs	Total	Reduced cost
XA1	Plant A	Warehouse 1	30	15	450	0
XA2	Plant A	Warehouse 2	0	18	0	-10
XA3	Plant A	Warehouse 3	0	19	0	-8
XA4	Plant A	Warehouse 4	20	13	260	0
XB1	Plant B	Warehouse 1	0	21	0	-2
XB2	Plant B	Warehouse 2	0	14	0	-2
XB3	Plant B	Warehouse 3	20	15	300	0
XB4	Plant B	Warehouse 4	20	17	340	0
XC1	Plant C	Warehouse 1	0	25	0	-6
XC2	Plant C	Warehouse 2	60	12	720	0
XC3	Plant C	Warehouse 3	0	17	0	-2
XC4	Plant C	Warehouse 4	0	22	0	-5
Total			150		2070	

Interpreting the Reduced costs: Because this is a minimization problem the reduced cost for shipment between Plant A and Warehouse 2 (XA2) is equal to -10.
This means that

1) the cost for shipments on this route must be reduced by at least 10 (to at most Opportunity cost=8) before it becomes economically feasible to utilize this route.
2) if this route is used under the current cost structure, then for each item that is shipped along this route, the total cost will increase by 10.

Capacity	Available	Used	Excess	Shadow price
Plant A	50	50	0	4
Plant B	55	40	15	0
Plant C	70	60	10	0

Demand	Expected	Used	Excess	Shadow price
Warehouse 1	30	30	0	19
Warehouse 2	60	60	0	12
Warehouse 3	20	20	0	15
Warehouse 4	40	40	0	17

The shadow prices for the plants convey the cost savings realized for each extra case of pharmaceutical available at the plant, the shadow prices are the cost savings resulting from having an extra case demanded at the warehouse. Remember, that extra demand at a warehouse means that total demand exceeds total supply. Thus, while there may be cost savings in the cases shipped, they come at the expense of some warehouse not obtaining its quantity of pharmaceutical products.

14. Linear programming application VI: Production/Inventory I

A company that produces footballs must decide how many footballs to produce for the next couple of month. The company has decided to use a 6-month planning horizon, and they have forecasted demands for the next 6 months are 10000, 15000, 30000, 35000, 25000, and 10000. The company wants to meet these demands on time, knowing that it currently has 5,000 footballs in inventory and that it can use a given month's production to help meet the demand for that month. We assume that production occurs during the month, and demand occurs at the end of a month. During each month, there is enough production capacity to produce up to 30000 footballs, and there is enough storage capacity to store up to 10000 footballs at the end of the month, after demand has occurred. The forecasted production costs per football for the next 6 months are €12.50, €12.55, €12.70, €12.80, €12.85, and €12.95, respectively. The holding cost per football held in inventory at the end of any month is figured at 5% of the production cost for that month. This cost includes the cost of storage and the cost of money tied up in inventory. The selling price for footballs is not considered relevant to the production decision because the company plans to satisfy all customer demand exactly when they occurs – at whatever the selling price is. The company wants to determine the production schedule that minimizes the total production and holding costs.

Decision variables

The company wishes to schedule production of footballs over the next 6 month. Thus, it wants to find the values for the following decision variables:

Month	Production	Inventory
1	P1	I1
2	P2	I2
3	P3	I3
4	P4	I4
5	P5	I5
6	P6	I6

Objective

Use a linear programming model to find the production schedule that meets the demand and minimizes total production costs and inventory holding costs.

Tables introducing Unit costs

We first have 6 production variables, one for each month:

Variable	Production cost
P1	12.50
P2	12.55
P3	12.70
P4	12.80
P5	12.85
P6	12.95

Then we have 6 inventory variables, one for each month:

Variable	Inventory cost
I1	0.625
I2	0.6275
I3	0.635
I4	0.64
I5	0.6425
i6	0.6475

The objective function can now be written as follows:

$$\text{Min } Z = 12.50P1 + 12.55P2 + 12.70P3 + 12.80P4 + 12.85P5 + 12.95P6 + 0.625I1 + 0.6275I2 + 0.635I3 + 0.64I4 + 0.6425I5 + 0.6475I6$$

Constraints

Production capacity has 6 constraints, one for each month:

$P1 \leq 30000$ (Capacity month1)

$P2 \leq 30000$ (Capacity month2)

$P3 \leq 30000$ (Capacity month3)

$P4 \leq 30000$ (Capacity month4)

$P5 \leq 30000$ (Capacity month5)

$P6 \leq 30000$ (Capacity month6)

Inventory capacity has 6 constraints, one for each month:

$I1 \leq 10000$ (Inventory month1)

$I2 \leq 10000$ (Inventory month2)

$I3 \leq 10000$ (Inventory month3)

$I4 \leq 10000$ (Inventory month4)

$I5 \leq 10000$ (Inventory month5)

$I6 \leq 10000$ (Inventory month6)

Each month the total production plus beginning of the month inventory must at least be as great as the scheduled demand. Any extra production is the amount stored for the next month.

Thus, for each month:

(Beginning inventory) + (Monthly production) = (Demand) + (Amount stored)

This relationship yields the following production constraints:

$5000 + P1 = 10000 + I1$ (Month 1)

$I1 + P2 = 15000 + I2$ (Month 2)

$I2 + P3 = 30000 + I3$ (Month 3)

$I3 + P4 = 35000 + I4$ (Month 4)

$I4 + P5 = 25000 + I5$ (Month 5)

$I5 + P6 = 10000 + I6$ (Month 6)

The final LP model

Rearranging the terms of monthly production constraints and adding the nonnegative constraints, we obtain the following model:

	P1	P2	P3	P4	P5	P6	I1	I2	I3	I4	I5	I6		RHS
Z	12.5	12.55	12.7	12.8	12.85	12.95	.625	.6275	.635	.64	.6425	.6475		
Cap1	1												≤	30000
Cap2		1											≤	30000
Cap3			1										≤	30000
Cap4				1									≤	30000
Cap5					1								≤	30000
Cap6						1							≤	30000
Inv1							1						≤	10000
Inv2								1					≤	10000
Inv3									1				≤	10000
Inv4										1			≤	10000
Inv5											1		≤	10000
Inv6												1	≤	10000
Mon1	1						-1						=	5000
Mon2		1					1	-1					=	15000
Mon3			1					1	-1				=	30000
Mon4				1					1	-1			=	35000
Mon5					1					1	-1		=	25000
Mon6						1					1	-1	=	10000

All variables ≥ 0

Optimal solution

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 1535562.5000

PARTIAL SENSITIVITY ANALYSIS

VARIABLE	SOLUTION VALUE	REDUCED COST	OPPORTU. COST	OBJECTIVE FUNCTION RANGES LOWER	GIVEN	UPPER
P1	5000.0000	0.0000	12.5000	11.9250	12.5000	+INFINITY
P2	20000.0000	0.0000	12.5500	12.0725	12.5500	13.1250
P3	30000.0000	0.0000	12.7000	-INFINITY	12.7000	13.1775
P4	30000.0000	0.0000	12.8000	-INFINITY	12.8000	13.8125
P5	25000.0000	0.0000	12.8500	12.3075	12.8500	14.4525
P6	10000.0000	0.0000	12.9500	-0.6475	12.9500	13.4925
I1	0.0000	0.5750	0.0500	0.0500	0.6250	+INFINITY
I2	5000.0000	0.0000	0.6275	0.1500	0.6275	+INFINITY
I3	5000.0000	0.0000	0.6350	-0.3775	0.6350	+INFINITY
I4	0.0000	1.6025	-0.9625	-0.9625	0.6400	+INFINITY
I5	0.0000	0.5425	0.1000	0.1000	0.6425	+INFINITY
I6	0.0000	13.5975	-12.9500	-12.9500	0.6475	+INFINITY

CONSTRAINT	TYPE	SHADOW PRICE	SLACK	RIGHT HAND SIDE RANGES LOWER	GIVEN	UPPER
Pro1	≤	0.0000	25000.0000	5000.0000	30000.0000	+INFINITY
Pro2	≤	0.0000	10000.0000	20000.0000	30000.0000	+INFINITY
Pro3	≤	-0.4775	0.0000	25000.0000	30000.0000	35000.0000
Pro4	≤	-1.0125	0.0000	25000.0000	30000.0000	35000.0000
Pro5	≤	0.0000	5000.0000	25000.0000	30000.0000	+INFINITY
Pro6	≤	0.0000	20000.0000	10000.0000	30000.0000	+INFINITY
Inv1	≤	0.0000	10000.0000	0.0000	10000.0000	+INFINITY
Inv2	≤	0.0000	5000.0000	5000.0000	10000.0000	+INFINITY
Inv3	≤	0.0000	5000.0000	5000.0000	10000.0000	+INFINITY
Inv4	≤	0.0000	10000.0000	0.0000	10000.0000	+INFINITY
Inv5	≤	0.0000	10000.0000	0.0000	10000.0000	+INFINITY
Inv6	≤	0.0000	10000.0000	0.0000	10000.0000	+INFINITY
Mon1	=	12.5000	0.0000	0.0000	5000.0000	30000.0000
Mon2	=	12.5625	0.0000	-5000.0000	15000.0000	25000.0000
Mon3	=	13.1875	0.0000	25000.0000	30000.0000	35000.0000
Mon4	=	13.8125	0.0000	30000.0000	35000.0000	40000.0000
Mon5	=	12.8750	0.0000	0.0000	25000.0000	30000.0000
Mon6	=	12.9375	0.0000	0.0000	10000.0000	30000.0000

We now evaluate the results from the LP output in the following table.

Month	1	2	3	4	5	6
Production costs	12.50	12.55	12.70	12.80	12.85	12.95
Production	5000	20000	30000	30000	25000	10000
Inventory	0	5000	5000	0	0	0
Demand	10000	15000	30000	35000	25000	10000

Month	1	2	3	4	5	6	Total
Production costs	62500	251000	381000	384000	321250	129500	1529250
Holding costs	0	3137.5	3175	0	0	0	6312.5
Total costs	62500	254137,5	384175	384000	321250	129500	1535563

From the table we see that the total cost has been minimized to a value equal to 1535563.

We also see that the inventory costs are low. This could be traced back to the inventory costs of 5%.

15. Linear programming application VII: Production/Inventory II

A company produces equipment used in two types of caravans. Equipment produced for normal caravans are made from less expensive materials compared with equipment produced for exclusive caravans. The company has recently distributed to its individual manufacturing partners the production quotas required for the upcoming summer quarter. The scheduled production requirements for one partner (Partner A) are the following:

Production requirements – Partner A

	April	May	June
Normal caravans	250	250	150
Exclusive caravans	100	300	400

Equipment for a normal caravan requires 3 working hours to produce, and equipment for an exclusive caravan requires 5 working hours to produce. The average wage is €18 per hour, but in May and June where many part-time workers are being used the average wage is €14 and €16 per hour, respectively. Partner A has available 2100 working hours in April, 1.500 working hours in May and 1200 working hours in June. During any given month Partner A can schedule up to 50% additional working hours, using overtime at 1½ times the standard rate. Material costs for the normal caravan are €146 and for the exclusive caravan material costs are €210.

Partner A expects to have 25 normal caravans and 20 exclusive caravans assembled in storehouse at the beginning of April, and at the beginning of July they expect to have at least 10 normal caravans and 25 exclusive caravans to cover extraordinary demand.

Partner A has storage facilities to capable of holding up to 300 caravans in any one month. The costs for storing caravans from one month to the next month are estimated to be €60 for 1 normal caravan and €90 for 1 exclusive caravan.

Partner A would like to setup a production plan to determine the number of normal and exclusive caravans to produce in each of the next three months, and how many should be produced during normal working hours, and how many should be produced during overtime hours. Partner A would like to minimize the total costs over the three months.

Decision variables

Decision variables covers production during normal working hours and during overtime hours. Furthermore we also need to separate between the month where the production will take place, and how much will be store for the next month.

We use the following decision variables:

	April			May			June		
	Normal	Overtime	Storage	Normal	Overtime	Storage	Normal	Overtime	Storage
Normal caravan	NAN	NAO	NAS	NMN	NMO	NMS	NJN	NJO	NJS
Exclusive caravan	EAN	EAO	EAS	EMN	EMO	EMS	EJN	EJO	EJS

where NAN = Normal caravan, April, Normal hours;

…

EJS = Exclusive caravan, end of June, Storage

Objective

The objective is to minimize the total costs over the quarter.

All number in €

Decision variable	Material costs	Labor costs	Total unit costs
NAN	146	3(14)=42	188
NAO	146	3(21)=63	209
NMN	146	3(16)=48	194
NMO	146	3(24)=72	218
NJN	146	3(18)=54	200
NJO	146	3(27)=81	227
EAN	210	5(14)=70	280
EAO	210	5(21)=105	315
EMN	210	5(16)=80	290
EMO	210	5(24)=120	330
EJN	210	5(18)=90	300
EJO	210	5(27)=135	345
NAS			60
NMS			60
NJS			60
EAS			90
EMS			90
EJS			90

Final objective function

$$\text{Min } Z = 188NAN + 209NAO + 194NMN + 218NMO + 200NJN + 227NJO + 280EAN + 315EAO + 290EMN + 330EMO + 300EJN + 345EJO + 60NAS + 60NMS + 60NJS + 90EAS + 90EMS + 90EJS$$

Constraints

Monthly production

Each month, the total production plus beginning of the month inventory must be at least as great as the shipping quotas scheduled. Any extra production is the amount stored for the next month. Thus, for each month:

(Beginning inventory) + (Monthly production) = (Shipping quotas) + (Amount stored)

Normal caravan, April: $25 + (NAN + NAO) = 250 + NAS$

Normal caravan, May: $NAS + (NMN + NMO) = 250 + NMS$

Normal caravan, June: $NMS + (NJN + NJO) = 150 + NJS$

Exclusive caravan, April: $20 + (EAN + EAO) = 100 + EAS$

Exclusive caravan, May: $EAS + (EMN + EMO) = 300 + EMS$

Exclusive caravan, June: $EMS + (EJN + EJO) = 400 + EJS$

July in-stock requirements

Normal caravan: $NJS \geq 10$

Exclusive caravan: $EJS \geq 25$

Production hours

The limits on the number of regular working hours and overtime hours available can be expressed by the following relationships:

$(\text{Normal working hours used}) \leq (\text{Normal working hours available})$

$(\text{Overtime working hours used}) \leq (\text{Overtime working hours available})$

April, regular time: $3\text{NAN} + 5\text{EAN} \leq 2100$

April, overtime: $3\text{NAO} + 5\text{EAO} \leq 1050$

May, regular time: $3\text{NMN} + 5\text{EMN} \leq 1500$

May, overtime: $3\text{NMO} + 5\text{EMO} \leq 750$

June, regular time: $3\text{NJN} + 5\text{EJN} \leq 1200$

June, overtime: $3\text{NJO} + 5\text{EJO} \leq 600$

Maximum storage limits

Partner A is restricted to a maximum storage of 500 finished caravans in any one month. Thus, for each month:

$$(\text{Normal caravan stored}) + (\text{Exclusive caravan stored}) \leq 300$$

April limit: $\text{NAS} + \text{EAS} \leq 300$

May limit: $\text{NMS} + \text{EMS} \leq 300$

June limit: $\text{NJS} + \text{EJS} \leq 300$

and

NAN,NAO,NAS,NMN,NMO,NMS,NJN,NJO,NJS,EAN,EAO,EAS,EMN,EMO,EMS,EJN,EJO,EJS ≥ 0

Optimal solution

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 367969.0000

PARTIAL SENSITIVITY ANALYSIS

VARIABLE	SOLUTION VALUE	REDUCED COST	OPPORTU. COST	OBJECTIVE FUNCTION RANGES LOWER	GIVEN	UPPER
XJR	225.0000	0.0000	188.0000	185.0000	188.0000	+INFINITY
XJO	0.0000	0.0000	209.0000	209.0000	209.0000	+INFINITY
XAR	250.0000	0.0000	194.0000	193.4000	194.0000	197.0000
XAO	0.0000	0.0000	218.0000	218.0000	218.0000	+INFINITY
XSR	160.0000	0.0000	200.0000	-29.4000	200.0000	200.6000
XSO	0.0000	3.6000	223.4000	223.4000	227.0000	+INFINITY
YJR	285.0000	0.0000	280.0000	-UENDELIG	280.0000	285.0000
YJO	95.0000	0.0000	315.0000	280.0000	315.0000	321.0000
YAR	150.0000	0.0000	290.0000	285.0000	290.0000	291.0000
YAO	131.0000	0.0000	330.0000	324.0000	330.0000	336.0000
YSR	144.0000	0.0000	300.0000	299.0000	300.0000	339.0000
YSO	0.0000	6.0000	339.0000	339.0000	345.0000	+INFINITY
SXJ	0.0000	3.0000	3.0000	3.0000	6.0000	+INFINITY
SXA	0.0000	0.6000	5.4000	5.4000	6.0000	+INFINITY
SXS	10.0000	0.0000	6.0000	-223.4000	6.0000	+INFINITY
SYJ	300.0000	0.0000	9.0000	-UENDELIG	9.0000	12.0000
SYA	281.0000	0.0000	9.0000	-30.0000	9.0000	10.0000
SYS	25.0000	0.0000	9.0000	-339.0000	9.0000	+INFINITY

CONSTRAINT	TYPE	SHADOW PRICE	SLACK	RIGHT HAND SIDE RANGES LOWER	GIVEN	UPPER
Constraint 9	≤	-7.0000	0.0000	1525.0000	2100.0000	2575.0000
Constraint 10	≤	0.0000	575.0000	475.0000	1050.0000	+INFINITY
Constraint 11	≤	-8.0000	0.0000	1405.0000	1500.0000	2155.0000
Constraint 12	≤	0.0000	95.0000	655.0000	750.0000	+INFINITY
Constraint 13	≤	-7.8000	0.0000	1105.0000	1200.0000	1855.0000
Constraint 14	≤	0.0000	600.0000	0.0000	600.0000	+INFINITY
Constraint 15	≤	-6.0000	0.0000	281.0000	300.0000	415.0000
Constraint 16	≤	0.0000	19.0000	281.0000	300.0000	+INFINITY
Constraint 17	≤	0.0000	265.0000	35.0000	300.0000	+INFINITY
Constraint 7	≥	-229.4000	0.0000	0.0000	10.0000	41.6667
Constraint 8	≥	-348.0000	0.0000	0.0000	25.0000	44.0000
Constraint 1	=	209.0000	0.0000	66.6667	225.0000	416.6667
Constraint 2	=	218.0000	0.0000	31.6667	250.0000	281.6667
Constraint 3	=	223.3750	0.0000	-10.0000	150.0000	181.6667
Constraint 4	=	315.0000	0.0000	-15.0000	80.0000	195.0000
Constraint 5	=	330.0000	0.0000	169.0000	300.0000	319.0000
Constraint 6	=	339.0000	0.0000	269.0000	400.0000	419.0000

Based on the optimal solution we find the following information.

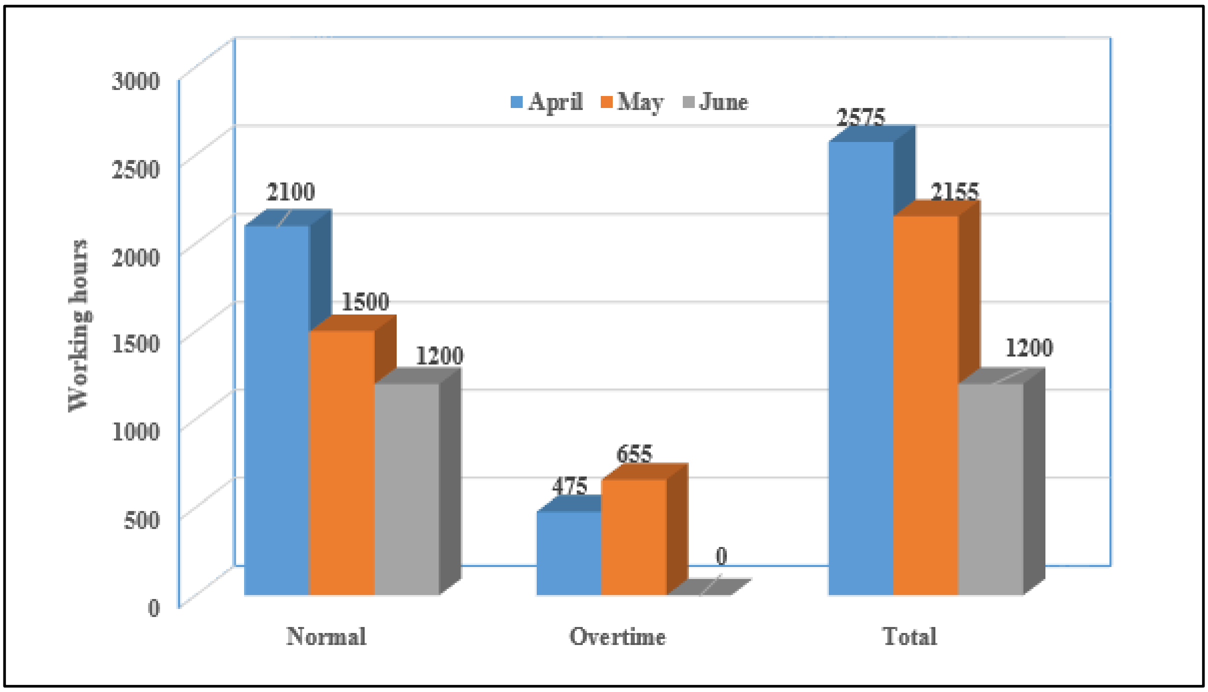
	Beginning inventory	Production		Monthly shipments	Quantity stored
		Normal	Overtime		
April					
Normal caravan	25	225	0	250	0
Exclusive caravan	20	285	95	100	300
May					
Normal caravan	0	250	0	250	0
Exclusive caravan	300	150	131	300	281
June					
Normal caravan	0	160	0	150	10
Exclusive caravan	281	144	0	400	25

Labor utilization

The production schedule utilizes all the regular time available each month and requires the use of 475 overtime in April and 655 overtime hours in May; each of these is below the maximum amount of overtime that could be scheduled for the month. No overtime is required in June.

Additional hires to increase the amount of normal working time will be economically beneficial to Partner A only if workers can be found less than €7 per hour in April, €8 in May, and €7.80 in June.

Since overtime hours cost Partner A less in April than in May, it may seem that more overtime should be scheduled during May. However, this would result in production that would exceed the storage limitation of 300 caravans for the month.



Additional storage

If additional storage could be rented, the ability to store 115 additional caravans would reduce the total quarterly costs by €690.

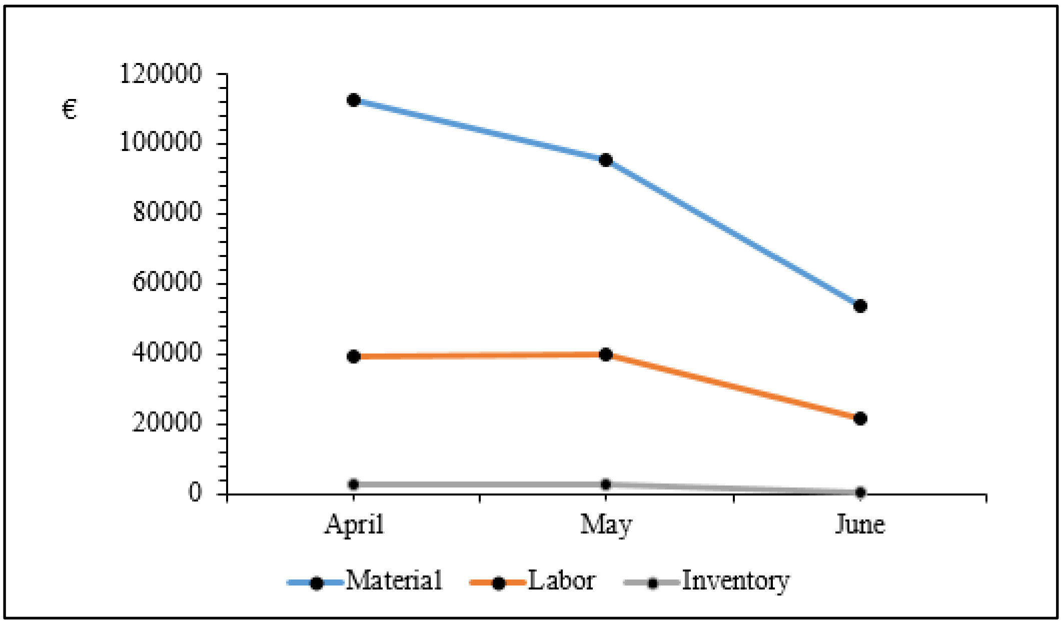
Budget costs

The quarterly cost information for budgeting purposes shows the following.

	Production		Labor		Material cost	Inventory cost	Total cost
	Normal	Overtime	Normal	Overtime			
April							
Normal	225	0	9450	0	32850	0	42300
Exclusive	285	95	19950	9975	79800	2700	112425
Total	510	95	29400	9975	112650	2700	154725
May							
Normal	250	0	12000	0	32500	0	48500
Exclusive	150	131	12000	15720	59010	2529	89259
Total	400	131	2400	15720	95010	2529	137759
June							
Normal	160	0	8640	0	23360	60	32060
Exclusive	144	0	12960	0	30240	225	43425
Total	304	0	21600	0	53600	285	75485
Quarterly totals	1214	226	75000	25695	261760	5514	367969

According to this schedule, the need for capital to purchase materials sharply declines during each month over the quarter.

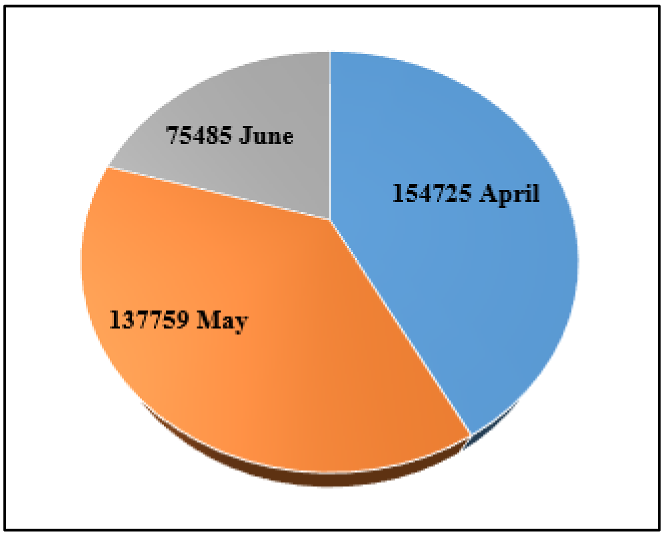
A figure shows this.



Overall recommendation

Changes in the material, labor, or storage costs or availabilities could affect the recommendation.

Monthly cost requirements.



16. Linear Programming application VIII: Solid Waste[23]

Consider a local community with the objective to minimize the total cost of disposing of solid waste in the local area.
It is expected that there is a daily solid waste of 700 tons. Some of the waste has a net cost, and other have a net revenue.

We have the following decision variables.
X1 = tons of solid waste annually sorted at private sites (home, commercial, charitable) for resale to secondary markets
X2 = tons of solid waste annually hauled from urban area to the central facility either for transfer to the landfill or for processing
X3 = tons of solid waste annually sorted at the center to recover materials for resale on secondary markets
X4 = tons of solid waste annually shredded and air classified at the processing plant
X5 = tons of solid waste annually removed by separation for resale in secondary markets
X6 = tons of solid waste annually hauled from the processing center to the landfill
X7 = tons of solid waste annually recovered for fuel production

The local community has collected data and estimated varios constraints (processing, material, bounding), and we can represent the total complex model as follows.
Let net cost be the difference between the cost of the activity and the value of the associated recoverable materials, then we have the following minimization of total net cost of solid waste LP-model:

$$\text{Min Cost} = -0.1\mathrm{X}_1 + 3.33\mathrm{X}_2 - 3\mathrm{X}_3 + 1.81\mathrm{X}_4 - 9.2\mathrm{X}_5 + 4.07\mathrm{X}_6 - 1.59\mathrm{X}_7$$

subject to

1) $X_1 \leq 25{,}550$
2) $X_1 + X_3 \leq 30{,}660$
3) $0.1X_1 + 0.5X_3 + X_5 \leq 15{,}330$
4) $0.7X_1 + 0.25X_3 + X_7 \leq 166{,}075$
5) $X_1 + X_2 = 255{,}500$
6) $X_2 - X_3 - X_4 = 0$
7) $X_2 - X_3 - X_5 - X_6 - X_7 = 0$
8) $-0.7X_4 + X_7 \leq 0$
9) $-0.4X_4 + X_5 \leq 0$
10) $X_1, X_2, \dots, X_7 \geq 0$

[23] See J. Friedman: "A comparative cost analysis of the production of energy from solid waste in Oregon". Western Agricultural Economics Research Counsil, Report 5, 1974, 15-31.

If we solve this LP-problem we get the following optimal solution:

```
                     OPTIMAL SOLUTION

 OBJECTIVE FUNCTION     1112547.1339

                                                                            PARTIAL SENSITIVITY ANALYSIS

                       SOLUTION      REDUCED        OPPORTU.                  OBJECTIVE FUNCTION
RANGES
VARIABLE                 VALUE        COST            COST          LOWER        GIVEN        UPPER
 ------------------------------------------------------------------------------------------------
 X1              22483.9996      -0.2000          0.1000        -2.2170      -0.1000       3.0910
 X2             233016.0004       6.6600         -3.3300         0.1390       3.3300       5.4470
 X3               8176.0004      -6.0000          3.0000        -6.1910      -3.0000      -0.8830
 X4             224840.0000       3.6200         -1.8100        -2.6897       1.8100    +INFINITY
 X5               8993.5998     -18.4000          9.2000      -121.6913      -9.2000      -1.2225
 X6              67554.1997       8.1400         -4.0700        -0.6344       4.0700      67.8900
 X7             148292.2005      -3.1800          1.5900        -7.4527      -1.5900       3.1144

                                SHADOW          SLACK           RIGHT HAND SIDE RANGES
 CONSTRAINT              TYPE   PRICE                           LOWER        GIVEN        UPPER
 ------------------------------------------------------------------------------------------------
 1                        <=      0.0000     3066.0004    22483.9996   25550.0000    +INFINITY
 2                        <=      4.4996        0.0000    11108.6961   30660.0000   33326.0873
 3                        <=      5.2925        0.0000    14103.5998   15330.0000   23415.1530
 4                        <=      5.6600        0.0000    17782.7995  166075.0000  175170.7969
 8                        <=      0.0000     9095.7969    -9095.7969       0.0000    +INFINITY
 9                        <=      7.9775        0.0000    -8085.1530       0.0000    1226.4002
 6                        =       1.4909        0.0000   -30660.0048       0.0000   12209.1235
 7                        =       4.0700        0.0000     -INFINITY       0.0000   67554.1997
 5                        =      -8.8909        0.0000   243290.8765  255500.0000  286160.0048
```

Minimize cost = $1112547

Solution	Units of tons
Private	22484
Hauling	233016
Sorting	8176
Shredding	224840
Sorting	8994
Landfill	67554
Fuel	1482292

Constraint	Shadow price	Slack
1	0	3066
2	5	0
3	5.3	0
4	5,66	0
5	-8,9	0
6	1,5	0
7	4,1	0
8	0	9096

17. Linear Programming application IX: Supply Chain II

A company manufactures two styles of boxes (G50 and H90) that various companies supply to their customers when a specific service is established in the EU. Different companies require different models. The company has four production facilities located in City 1 (the original plant), City 2, City 3 and City 4. The manufactured items are shipped from the plants to regional distribution centers located in City 5, City 6, and City 7. From these locations they are distributed around the EU.

Because of a decrease in demand for the services and technological changes in the industry, the demand for the products is currently less than the total capacities at its four plants. As a result, management is contemplating closing one or more of its facilities.

Each plant has a fixed operating cost, and, because of the unique conditions at each facility, the production costs, production time per unit, and total monthly production time available vary from plant to plant, as summarized in the following table:

Production costs, times, availability

Plant	Fixed cost/month (€1,000)	Production cost (per unit)		Production Time (Hr/unit)		Available hours per month
		G50	H90	G50	H90	
City 1	40	10	14	0.06	0.06	640
City 2	35	12	12	0.07	0.08	960
City 3	20	8	10	0.09	0.07	480
City 4	30	13	15	0.05	0.09	640

The boxes are sold at the same price within the EU: €22 for the G50, and €28 for the H90.

Current monthly demand projections at each distribution center for both products are in the following table:

Monthly demand projections

Product	Demand		
	City 5	City 6	City 7
G50	2000	3000	5000
H90	5000	6000	7000

To remain viable in each market, the company must meet at least 70% of the demand for each product at each distribution center. The transportation costs between each plant and each distribution center are the same for either product, are as shown in the following table:

Transportation costs per units

in €	To		
From	City 5	City 6	City 7
City 1	200	300	500
City 2	100	100	400
City 3	200	200	300
City 4	300	100	100

The company wants to develop an optimal distribution policy utilizing all four of its operational plants. It also wants to determine whether closing any of the production facilities will result in higher company profits.

LP formulation when all plants are operational

Determine the number of G50 and H90 boxes to be produced at each plant.

Determine a shipping pattern from the plants to the distribution centers.

Maximize monthly profit.

Not exceed the production capacities at any plant.

Ensure that each distribution center received between 70% and 100% of its demand

Let the four plants be given by i=1,2,3,4

Let the three cities be given by j=1,2,3

We have the following representation of plants and cities:

Plant	Distribution center
i = 1 → City 1	j = 1 → City 5
i = 2 → City 2	j = 2 → City 6
i = 3 → City 3	j = 3 → City 7
i = 4 → City 4	

Decision variables

Management must decide how many G50 and H90 boxes it will produce at each plant (i):

G_i = hundreds of G50 boxes produced at plant i monthly

H_i = hundreds of H90 boxes produced at plant i monthly

Given these production schedules, management must then decide how many units of each product will be shipped from each plant i to each production center j:

G_{ij} = hundreds of G50 boxes shipped from plant i to distribution center j each month

H_{ij} = hundreds of H90 boxes shipped from plant i to distribution center j each month

We then have the following decision variables.

Shipment of G50 boxes

Plant	City5	City6	City7	Total produced
City1	G11	G12	G13	G1
City2	G21	G22	G23	G2
City3	G31	G32	G33	G3
City4	G41	G42	G43	G4
Total	G5	G6	G7	G

Shipment of H90 boxes

Plant	City5	City6	City7	Total produced
City1	H11	H12	H13	H1
City2	H21	H22	H23	H2
City3	H31	H32	H33	H3
City4	H41	H42	H43	H4
Total	H5	H6	H7	H

Objective function

$\text{Profit} = (\text{Revenue per unit}) - (\text{Production cost per unit})$

The profit is given by

€22 (G50 produced) + €28 (H90 produced) – (Production cost) – (Transportation cost)

Thus the final objective function is:

$$\begin{aligned}\text{Max Profit} = {} & 22G + 28H - 10G1 - 12G2 - 8G3 - 13G4 - 14H1 - 12H2 - 10H3 - 15H4 \\ & - 2G11 - 3G12 - 5G13 - 1G21 - 1G22 - 4G23 - 2G31 - 2G32 - 3G33 \\ & - 3G41 - 1G42 - 1G43 - 2H11 - 3H12 - 5H13 - 1H21 - 1H22 - 4H23 \\ & - 2H31 - 2H32 - 3H33 - 3H41 - 1H42 - 1H43\end{aligned}$$

Constraints

1. The LP model must ensure that the total amount shipped from a plant equals the amount produced in the plant.

   G50 boxes:
   City1: $G11 + G12 + G13 = G1$
   City2: $G21 + G22 + G23 = G2$
   City3: $G31 + G32 + G33 = G3$
   City4: $G41 + G42 + G43 = G4$
   Total: $G1 + G2 + G3 + G4 = G$

   H90 boxes:
   City1: $H11 + H12 + H13 = H1$
   City2: $H21 + H22 + H23 = H2$
   City3: $H31 + H32 + H33 = H3$
   City4: $H41 + G42 + H43 = H4$
   Total: $H1 + H2 + H3 + H4 = H$

2. Constraints for total shipments

   G50 boxes:
   City5: $G11 + G21 + G31 + G41 = G5$
   City6: $G12 + G22 + G32 + G42 = G6$
   City5: $G13 + G23 + G33 + G43 = G7$

   H90 boxes:
   City5: $H11 + H21 + H31 + H41 = H5$
   City6: $H12 + H22 + H32 + H42 = H6$
   City5: $H13 + H23 + H33 + H43 = H7$

3. Production time used at each plant cannot exceed the time available:
   City1: $0.06G1 + 0.066H1 \leq 640$
   City2: $0.077G2 + 0.08H2 \leq 960$
   City3: $0.099G3 + 0.07H3 \leq 480$
   City4: $0.05G4 + 0.09H4 \leq 640$

4. The amount of each product received by a distribution center cannot exceed its total demand or be less than 70% of its demand:

Minimum (≥70%):

City5: $G5 \geq 1400$

City5: $H5 \geq 3500$

City6: $G6 \geq 2100$

City6: $H6 \geq 4200$

City7: $G7 \geq 3500$

City7: $H7 \geq 4900$

Maximum (≤70%):

City5: $G5 \leq 2000$

City5: $H5 \leq 5000$

City6: $G6 \leq 3000$

City6: $H6 \leq 6000$

City7: $G7 \leq 5000$

City7: $H7 \leq 7000$

All $G, H, Gi, Hi, Gij, Hij \geq 0$

Solving this LP model with 32 decision variables and 24 constraints, are:

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 356571.4285

PARTIAL SENSITIVITY ANALYSIS

VARIABLE	SOLUTION VALUE	REDUCED COST	OPPORTU. COST	OBJECTIVE FUNCTION RANGES LOWER	GIVEN	UPPER
G	10000.0000	0.0000	22.0000	14.0000	22.0000	+INFINITY
H	18000.0000	0.0000	28.0000	16.0000	28.0000	+INFINITY
G1	3857.1426	0.0000	-10.0000	-10.0000	-10.0000	-9.0000
G2	1142.8574	0.0000	-12.0000	-13.2000	-12.0000	-12.0000
G3	0.0000	-0.8571	-7.1429	-INFINITY	-8.0000	-7.1429
G4	5000.0000	0.0000	-13.0000	-14.0000	-13.0000	-12.0000
H1	0.0000	-3.0000	-11.0000	-INFINITY	-14.0000	-11.0000
H2	11000.0000	0.0000	-12.0000	-12.0000	-12.0000	-10.9286
H3	6857.1428	0.0000	-10.0000	-10.6667	-10.0000	-6.0000
H4	142.8572	0.0000	-15.0000	-16.6667	-15.0000	-15.0000
G11	2000.0000	0.0000	-2.0000	-3.0000	-2.0000	+INFINITY
G12	1857.1426	0.0000	-3.0000	-3.0000	-3.0000	-2.0000
G13	0.0000	-1.0000	-4.0000	-INFINITY	-5.0000	-4.0000
G21	0.0000	-1.0000	0.0000	-INFINITY	-1.0000	0.0000
G22	1142.8574	0.0000	-1.0000	-2.0000	-1.0000	-1.0000
G23	0.0000	-2.0000	-2.0000	-INFINITY	-4.0000	-2.0000
G31	0.0000	-1.0000	-1.0000	-INFINITY	-2.0000	-1.0000
G32	0.0000	0.0000	-2.0000	-3.0000	-2.0000	-1.1429
G33	0.0000	0.0000	-3.0000	-INFINITY	-3.0000	-3.0000
G41	0.0000	-4.0000	1.0000	-INFINITY	-3.0000	1.0000
G42	0.0000	-1.0000	0.0000	-INFINITY	-1.0000	0.0000
G43	5000.0000	0.0000	-1.0000	-2.0000	-1.0000	+INFINITY
H11	0.0000	0.0000	-2.0000	-3.0000	-2.0000	-2.0000
H12	0.0000	-1.0000	-2.0000	-INFINITY	-3.0000	-2.0000
H13	0.0000	0.0000	-5.0000	-INFINITY	-5.0000	-5.0000
H21	5000.0000	0.0000	-1.0000	-1.0000	-1.0000	0.0000
H22	6000.0000	0.0000	-1.0000	-2.0000	-1.0000	+INFINITY
H23	0.0000	0.0000	-4.0000	-INFINITY	-4.0000	-4.0000
H31	0.0000	-2.0000	0.0000	-INFINITY	-2.0000	0.0000
H32	0.0000	-2.0000	0.0000	-INFINITY	-2.0000	0.0000
H33	6857.1428	0.0000	-3.0000	-3.6667	-3.0000	-1.3333
H41	0.0000	-5.0000	2.0000	-INFINITY	-3.0000	2.0000
H42	0.0000	-3.0000	2.0000	-INFINITY	-1.0000	2.0000
H43	142.8572	0.0000	-1.0000	-2.6667	-1.0000	-1.0000
G5	2000.0000	0.0000	0.0000	-10.0000	0.0000	+INFINITY
G6	3000.0000	0.0000	0.0000	0.0000	0.0000	+INFINITY
G7	5000.0000	0.0000	0.0000	-8.0000	0.0000	+INFINITY
H5	5000.0000	0.0000	0.0000	-15.0000	0.0000	+INFINITY
H6	6000.0000	0.0000	0.0000	-15.0000	0.0000	+INFINITY
H7	7000.0000	0.0000	0.0000	-12.0000	0.0000	+INFINITY

The LP solution has an optimal value equal to €356.571. Subtracting the €125.000 in fixed costs results in a monthly profit of €231.571.

CONSTRAINT	TYPE	SHADOW PRICE	SLACK	RIGHT HAND SIDE RANGES LOWER	GIVEN	UPPER
1	<=	0.0000	408.5715	231.4285	640.0000	+INFINITY
2	<=	0.0000	0.0000	880.0000	960.0000	1090.0000
3	<=	42.8571	0.0000	186.6667	480.0000	490.0000
4	<=	0.0000	377.1429	262.8571	640.0000	+INFINITY
5	<=	10.0000	0.0000	1400.0000	2000.0000	8809.5243
6	<=	15.0000	0.0000	3500.0000	5000.0000	6000.0003
7	<=	9.0000	0.0000	2100.0000	3000.0000	9809.5243
8	<=	15.0000	0.0000	4375.0002	6000.0000	7000.0003
9	<=	8.0000	0.0000	3500.0000	5000.0000	12542.8569
10	<=	12.0000	0.0000	6857.1428	7000.0000	11190.4759
11	>=	0.0000	600.0000	-INFINITY	1400.0000	2000.0000
12	>=	0.0000	1500.0000	-INFINITY	3500.0000	5000.0000
13	>=	0.0000	900.0000	-INFINITY	2100.0000	3000.0000
14	>=	0.0000	1800.0000	-INFINITY	4200.0000	6000.0000
15	>=	0.0000	1500.0000	-INFINITY	3500.0000	5000.0000
16	>=	0.0000	2100.0000	-INFINITY	4900.0000	7000.0000
17	=	-12.0000	0.0000	-6809.5243	0.0000	3857.1426
18	=	-10.0000	0.0000	-1142.8574	0.0000	1857.1426
19	=	-11.0000	0.0000	0.0000	0.0000	1857.1426
20	=	-9.0000	0.0000	-7542.8569	0.0000	5000.0000
21	=	-22.0000	0.0000	-INFINITY	0.0000	10000.0000
22	=	-17.0000	0.0000	0.0000	0.0000	1624.9998
23	=	-16.0000	0.0000	-1000.0003	0.0000	1624.9998
24	=	-15.0000	0.0000	-4190.4759	0.0000	142.8572
25	=	-13.0000	0.0000	-4190.4759	0.0000	142.8572
26	=	-28.0000	0.0000	-INFINITY	0.0000	18000.0000
27	=	10.0000	0.0000	-2000.0000	0.0000	6809.5243
28	=	9.0000	0.0000	-1857.1426	0.0000	6809.5243
29	=	8.0000	0.0000	-5000.0000	0.0000	7542.8569
30	=	15.0000	0.0000	-1624.9998	0.0000	1000.0003
31	=	15.0000	0.0000	-1624.9998	0.0000	1000.0003
32	=	12.0000	0.0000	-142.8572	0.0000	4190.4759

Given the current situation at the four plants in operation, we have the following production schedule that should maximize the monthly profit for the company:

Production/transportation schedule

Plant	Product	Amount	X5	X6	X7
City1:					
G1	G50	3.857	2.000	1.857	
H1	H90	0			
City2:					
G2	G50	1.143			1.143
H2	H90	11.000	5.000	6.000	
City3:					
G3	G50	0			
H3	H90	6.857			6.857
City4:					
G4	G50	5.000			5.000
H4	H90	143			143
Total:					
G	G50	10.000	2.000	3.000	5.000
H	H90	18.000	5.000	6.000	7.000

Production time (hours) all plants operational

Plant	G50	H90	Total	Total capacity	Excess capacity
City1	231	0	231	640	409
City2	80	880	960	960	0
City3	0	480	480	480	0
City4	250	13	263	640	377
Total	561	1.371	1.934	2.720	786

Monthly revenues and costs all plants operational

Plant	Revenue	Costs			Total costs	Profit
		Production	Transportation	Operations		
City1	84.854	38.570	9.571	40.000	88.141	-3.287
City2	333.146	145.716	12.143	35.000	192.859	140.287
City3	191.996	68.570	20.571	20.000	109.141	82.855
City4	114.004	67.145	5.143	30.000	102.288	11.716
Total	724.000	320.000	47.428	125.000	492.429	231.571

From the tables we see, that this production plan meets the full demand at the distribution centers. No G50 models are produced in City3, no H90 models are produced in City1, and very few H90 are produced in City4. As a result, there is considerable excess capacity at both the City1 and City4 plants.

Under this plan, the company will be utilizing only 1934 production hours, or 71% of available production capacity. Also, observe that plant City1 will have negative profit, and City4 plant will have a marginal profit.

LP formulation when closing plants

With the assumption that all plants are operational, we need to investigate, because of the large fixed cost components at each plant, if this is best solution. The company should consider which plants it wishes to keep operational.

We introduce four decision variables:

Y1 = number of City1 plants

Y2 = number of City2 plants

Y3 = number of City3 plants

Y4 = number of City4 plants

The fixed operating costs can be accounted for in the objective function by subtracting from the previos objective function the expression

40000Y1 + 35000Y2 + 20000Y3 + 30000Y4

The production constraints are modified as:

$$0.06\text{G1} + 0.06\text{H1} \leq 640\text{Y1}$$

$$0.07\text{G2} + 0.08\text{H2} \leq 960Y2$$

$$0.09\text{G3} + 0.07\text{H3} \leq 480Y3$$

$$0.05\text{G4} + 0.09\text{H4} \leq 640Y4$$

If plant i is closed ($Y_i = 0$), then these constraints will force the corresponding production (G_i and H_i) to be 0, which, in turn, implies that all shipments from plant i $\text{G}_{\text{ij}}$and H_{ij}) will also be 0.

Changing the original LP problem by subtracting the fixed operating costs in the objective function and modifying any constraint that is dependent on a facility being operational, gives the following profit function for the facility location problem faced by the company:

$$\begin{aligned}\text{Max Profit} = 22\text{G} + 28\text{H} - 10\text{G1} - 12\text{G2} - 8\text{G3} - 13\text{G4} - 14\text{H1} - 12\text{H2} - 10\text{H3} - 15\text{H4}\\ - 2\text{G11} - 3\text{G12} - 5\text{G13} - 1\text{G21} - 1\text{G22} - 4\text{G23} - 2\text{G31} - 2\text{G32} - 3\text{G33}\\ - 3\text{G41} - 1\text{G42} - 1\text{G43} - 2\text{H11} - 3\text{H12} - 5\text{H13} - 1\text{H21} - 1\text{H22} - 4\text{H23}\\ - 2\text{H31} - 2\text{H32} - 3\text{H33} - 3\text{H41} - 1\text{H42} - 1\text{H43} - 40000\text{Y1} - 35000\text{Y2}\\ - 20000\text{Y3} - 30000\text{Y4}\end{aligned}$$

1. The LP model must ensure that the total amount shipped from a plant equals the amount produced in the plant.

   G50 boxes:
   City1: $G11 + G12 + G13 = G1$
   City2: $G21 + G22 + G23 = G2$
   City3: $G31 + G32 + G33 = G3$
   City4: $G41 + G42 + G43 = G4$
   Total: $G1 + G2 + G3 + G4 = G$

   H90 boxes:
   City1: $H11 + H12 + H13 = H1$
   City2: $H21 + H22 + H23 = H2$
   City3: $H31 + H32 + H33 = H3$
   City4: $H41 + G42 + H43 = H4$
   Total: $H1 + H2 + H3 + H4 = H$

2. Constraints for total shipments

   G50 boxes:
   City5: $G11 + G21 + G31 + G41 = G5$
   City6: $G12 + G22 + G32 + G42 = G6$
   City5: $G13 + G23 + G33 + G43 = G7$

   H90 boxes:
   City5: $H11 + H21 + H31 + H41 = H5$
   City6: $H12 + H22 + H32 + H42 = H6$
   City5: $H13 + H23 + H33 + H43 = H7$

3. Production time used at each plant cannot exceed the time available:
   $0.06G1 + 0.06H1 - 640Y1 \leq 0$

   $0.07G2 + 0.08H2 - 960Y2 \leq 0$

   $0.09G3 + 0.07H3 - 480Y3 \leq 0$

   $0.05G4 + 0.09H4 - 640Y4 \leq 0$

4. The amount of each product received by a distribution center cannot exceed its total demand or be less than 70% of its demand:

Minimum (≥70%):

$\text{City5: G5} \geq 1400$

$\text{City5: H5} \geq 3500$

$\text{City6: G6} \geq 2100$

$\text{City6: H6} \geq 4200$

$\text{City7: G7} \geq 3500$

$\text{City7: H7} \geq 4900$

Maximum (≤70%):

$\text{City5: G5} \leq 2000$

$\text{City5: H5} \leq 5000$

$\text{City6: G6} \leq 3000$

$\text{City6: H6} \leq 6000$

$\text{City7: G7} \leq 5000$

$\text{City7: H7} \leq 7000$

$\text{All Xi, Zi, Xij, Zij} \geq 0$

$\text{All Yi} = 0 \text{ or } 1$

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 265071.4289

VARIABLE	SOLUTION VALUE	REDUCED COST	OPPORTU. COST	PARTIAL SENSITIVITY ANALYSIS OBJECTIVE FUNCTION RANGES LOWER	GIVEN	UPPER
G	10000.0000	0.0000	22.0000	14.7500	22.0000	+INFINITY
H	18000.0000	0.0000	28.0000	16.0000	28.0000	+INFINITY
G2	2000.0000	0.0000	-12.0000	-12.2723	-12.0000	-11.2500
G3	0.0000	-0.2723	-7.7277	-INFINITY	-8.0000	-7.7277
G4	8000.0000	0.0000	-13.0000	-13.7500	-13.0000	-11.7500
H2	10250.0002	0.0000	-12.0000	-12.8571	-12.0000	-11.6888
H3	6857.1428	0.0000	-10.0000	-10.1627	-10.0000	4.3360
H4	892.8570	0.0000	-15.0000	-16.4286	-15.0000	-14.6592
G21	2000.0000	0.0000	-1.0000	-1.2723	-1.0000	-0.2500
G22	0.0000	-0.7500	-0.2500	-INFINITY	-1.0000	-0.2500
G23	0.0000	-3.7500	-0.2500	-INFINITY	-4.0000	-0.2500
G31	0.0000	0.0000	-2.0000	-2.7500	-2.0000	-1.7277
G32	0.0000	-0.7500	-1.2500	-INFINITY	-2.0000	-1.2500
G33	0.0000	-1.7500	-1.2500	-INFINITY	-3.0000	-1.2500
G41	0.0000	-1.2500	-1.7500	-INFINITY	-3.0000	-1.7500
G42	3000.0000	0.0000	-1.0000	-1.7500	-1.0000	+INFINITY
G43	5000.0000	0.0000	-1.0000	-2.7500	-1.0000	+INFINITY
H21	4250.0002	0.0000	-1.0000	-1.8571	-1.0000	-1.0000
H22	6000.0000	0.0000	-1.0000	-1.0000	-1.0000	+INFINITY
H23	0.0000	-2.0000	-2.0000	-INFINITY	-4.0000	-2.0000
H31	749.9998	0.0000	-2.0000	-2.0000	-2.0000	-1.1429
H32	0.0000	0.0000	-2.0000	-INFINITY	-2.0000	-2.0000
H33	6107.1430	0.0000	-3.0000	-3.3408	-3.0000	-2.0000
H41	0.0000	-3.0000	0.0000	-INFINITY	-3.0000	0.0000
H42	0.0000	-1.0000	0.0000	-INFINITY	-1.0000	0.0000
H43	892.8570	0.0000	-1.0000	-2.0000	-1.0000	-0.6592
G5	2000.0000	0.0000	0.0000	-7.2500	0.0000	+INFINITY
G6	3000.0000	0.0000	0.0000	-8.0000	0.0000	+INFINITY
G7	5000.0000	0.0000	0.0000	-8.0000	0.0000	+INFINITY
H5	5000.0000	0.0000	0.0000	-13.0000	0.0000	+INFINITY
H6	6000.0000	0.0000	0.0000	-13.0000	0.0000	+INFINITY
H7	7000.0000	0.0000	0.0000	-12.0000	0.0000	+INFINITY

The optimal LP solution to this fixed charge problem is to close the City1 plant and to schedule monthly production according to the quantities shown in the output. This gives a monthly profit of €265071, which is €34544 per month greater than the optimal profit of €231571 obtained when all four plants are operational. This is an annual increase of profit equal to €402.000.

CONSTRAINT	TYPE	SHADOW PRICE	SLACK	RIGHT HAND SIDE RANGES LOWER	GIVEN	UPPER
1	<=	45.8333	0.0000	0.0000	0.0000	51.4286
2	<=	25.0000	0.0000	-141.9048	0.0000	60.0000
3	<=	42.8571	0.0000	-124.1667	0.0000	62.5000
4	<=	0.0000	159.6429	-159.6429	0.0000	+INFINITY
5	<=	7.2500	0.0000	1400.0000	2000.0000	4027.2109
6	<=	13.0000	0.0000	4250.0002	5000.0000	6773.8095
7	<=	8.0000	0.0000	2100.0000	3000.0000	6192.8573
8	<=	13.0000	0.0000	5250.0002	6000.0000	7773.8095
9	<=	8.0000	0.0000	3500.0000	5000.0000	8192.8573
10	<=	12.0000	0.0000	6107.1430	7000.0000	8773.8095
11	>=	0.0000	600.0000	-INFINITY	1400.0000	2000.0000
12	>=	0.0000	1500.0000	-INFINITY	3500.0000	5000.0000
13	>=	0.0000	900.0000	-INFINITY	2100.0000	3000.0000
14	>=	0.0000	1800.0000	-INFINITY	4200.0000	6000.0000
15	>=	0.0000	1500.0000	-INFINITY	3500.0000	5000.0000
16	>=	0.0000	2100.0000	-INFINITY	4900.0000	7000.0000
17	=	-9.2500	0.0000	0.0000	0.0000	857.1426
18	=	-8.2500	0.0000	-2027.2109	0.0000	857.1426
19	=	-9.2500	0.0000	0.0000	0.0000	857.1426
20	=	-9.0000	0.0000	-3192.8573	0.0000	8000.0000
21	=	-22.0000	0.0000	-INFINITY	0.0000	10000.0000
22	=	-15.0000	0.0000	0.0000	0.0000	749.9998
23	=	-14.0000	0.0000	-1773.8095	0.0000	749.9998
24	=	-15.0000	0.0000	-1773.8095	0.0000	892.8570
25	=	-13.0000	0.0000	-1773.8095	0.0000	892.8570
26	=	-28.0000	0.0000	-INFINITY	0.0000	18000.0000
27	=	7.2500	0.0000	-857.1426	0.0000	2027.2109
28	=	8.0000	0.0000	-3000.0000	0.0000	3192.8573
29	=	8.0000	0.0000	-5000.0000	0.0000	3192.8573
30	=	13.0000	0.0000	-749.9998	0.0000	1773.8095
31	=	13.0000	0.0000	-749.9998	0.0000	1773.8095
32	=	12.0000	0.0000	-892.8570	0.0000	1773.8095

The following tables gives the production and distribution schedule that would result from closing the City1 plant.

Production/transportation schedule City1 plant closed

Production/transportation schedule

Plant	Product	Amount	X5	X6	X7
City2:					
G2	G50	2.000	2.000		
H2	H90	10.250	4.250	6.000	
City3:					
G3	G50	0			
H3	H90	6.857	750		6.107
City4:					
G4	G50	8.000		3.000	5.000
H4	H90	893			893
Total:					
G	G50	10.000	2.000	3.000	5.000
H	H90	18.000	5.000	6.000	7.000

Production time (hours) three plants operational

Plant	G50	H90	Total	Total capacity	Excess capacity
City2	80	880	960	960	0
City3	252	228	480	480	0
City4	303	337	640	640	0
Total	635	1.445	2.080	2.080	0

Monthly revenues and costs three plants operational

Plant	Revenue	Costs			Total costs	Profit
		Production	Transportation	Operations		
City2	331.000	147.000	12.250	35.000	194.250	136.750
City3	191.996	68.570	19.821	20.000	108.391	83.605
City4	201.004	117.395	8.893	30.000	156.288	44.716
Total	724.000	332.965	40.964	85.000	458.929	265.071

Based on the analysis, the City1 plant should be closed. The remaining three plants will then be fully utilized, and all projected demand will be met.

18. Linear Programming application X: Data Envelopment Analysis[24]

The measure of efficiency of a group of firms operating in the same industry, or of departments within the firm or of the firm itself over time can be investigated using Data Envelopment Analysis (DEA). A procedure developed by Charnes, Cooper, and Rhodes (1978) has been developed and presented using the linear programming model.

In the following we take a closer look at the efficiency among three hospitals.[25]

We will consider three hospitals. We assume that each hospital has two inputs and three outputs.

The two inputs used by each hospital are

X_1 = capital measured by the number of hospital beds

X_2 = labor measured in thousands of labor hours used during a month

The outputs produced by each hospital are

Y_1 = hundreds of patient-days during month for patients under age 14

Y_2 = hundreds of patient-days during month for patients between 14 and 65

Y_3 = hundreds of patient-days during month for patients over 65

The table provides the relevant data for the year in question:

	Inputs		Outputs		
Hospital	1	2	1	2	3
1	5	14	9	4	15
2	8	15	5	7	10
3	7	12	4	9	13

To determine whether a hospital is efficient, we rank each hospital by an efficiency ratio, namely, by the ratio of the total value of its outputs to the total value of its inputs.

Using the data we find the efficiency of each hospital to be as follows:

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[24] Charnes, A., Cooper, W.W., Rhodes, E. (1978): “Measuring the Efficiency of Decision Making Units”, European Journal of Operational Research, 2(6), 429-444.

[25] Callen, J.L. (1991): “Data Envelopment Analysis: Partial Survey and Applications for Management Accounting”. *Journal of Management Accounting Research*, Fall, 35-56.

$$\text{Efficiency ratio of hospital 1} = \frac{9Y_1 + 4Y_2 + 16Y_3}{5X_1 + 14X_2}$$

$$\text{Efficiency ratio of hospital 2} = \frac{5Y_1 + 7Y_2 + 10Y_3}{8X_1 + 15X_2}$$

$$\text{Efficiency ratio of hospital 3} = \frac{4Y_1 + 9Y_2 + 13Y_3}{7X_1 + 12X_2}$$

The DEA approach then use the following to determine if a hospital is efficient.

- No hospital can be more than 100% efficient. Thus, the efficiency of each hospital must be less than or equal to 1. For Hospital 1 we find

  Hospital 1: $\frac{9Y_1+4Y_2+16Y_3}{5X_1+14X_2} \leq 1$

  We convert this to the following LP constraint

  Hospital 1: $9Y_1 + 4Y_2 + 16Y_3 \leq 5X_1 + 14X_2$

  Hospital 1: $5X_1 + 14X_2 - 9Y_1 - 4Y_2 - 16Y_3 \geq 0$

- If the efficiency of a hospital is equal to 1, then it is efficient.

- To simplify computations, we scale the output so that the input of hospital1's inputs equal1 1.
  For hospital 2 we add the constraint that $8X_1 + 15X_2 = 1$

- We must ensure that each input and output is strickly positive. We add the following constraint to each variables:

  $X_1 \geq 0.0001$
  $X_2 \geq 0.0001$
  $X_3 \geq 0.0001$
  $Y_1 \geq 0.0001$
  $Y_2 \geq 0.0001$

We now have the following 3 models for testing the efficiency of each hospital.

<u>Hospital 1</u>

Max Z =	$9X_1$	$+ 4 X_2$	$+ 16X_3$			
Subject to	$-9X_1$	$- 4X_2$	$-16X_3$	$+ 5Y_1$	$14Y_2$	≥ 0
1	$-5X_1$	$- 7X_2$	$- 10X_3$	$+ 8Y_1$	$+ 15Y_2$	≥ 0
2	$-4X_1$	$- 9X_2$	$- 13X_3$	$+ 7Y_1$	$+ 12Y_2$	≥ 0
3				$+ 5Y_1$	$+ 14Y_2$	$= 1$
4	X_1					≥ 0.0001
5		X_2				≥ 0.0001
6			X_3			≥ 0.0001
7				Y_1		≥ 0.0001
8					Y_2	≥ 0.0001

Hospital 2

Max Z =	$5X_1$	$+ 7X_2$	$+ 10X_3$			
Subject to	$-9X_1$	$- 4X_2$	$-16X_3$	$+ 5Y_1$	$14Y_2$	≥ 0
1	$-5X_1$	$- 7X_2$	$- 10X_3$	$+ 8Y_1$	$+ 15Y_2$	≥ 0
2	$-4X_1$	$- 9X_2$	$- 13X_3$	$+ 7Y_1$	$+ 12Y_2$	≥ 0
3				$+ 8Y_1$	$+ 15Y_2$	$= 1$
4	X_1					≥ 0.0001
5		X_2				≥ 0.0001
6			X_3			≥ 0.0001
7				Y_1		≥ 0.0001
8					Y_2	≥ 0.0001

Hospital 3

Max Z =	$4X_1$	$+ 9X_2$	$+ 13X_3$			
Subject to	$-9X_1$	$- 4X_2$	$-16X_3$	$+ 5Y_1$	$14Y_2$	≥ 0
1	$-5X_1$	$- 7X_2$	$- 10X_3$	$+ 8Y_1$	$+ 15Y_2$	≥ 0
2	$-4X_1$	$- 9X_2$	$- 13X_3$	$+ 7Y_1$	$+ 12Y_2$	≥ 0
3				$+ 7Y_1$	$+ 12Y_2$	$= 1$
4	X_1					≥ 0.0001
5		X_2				≥ 0.0001
6			X_3			≥ 0.0001
7				Y_1		≥ 0.0001
8					Y_2	≥ 0.0001

The output from each of the three models are.

Hospital 1

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 1.0000

PARTIAL SENSITIVITY ANALYSIS

VARIABLE	SOLUTION VALUE	REDUCED COST	OPPORTU. COST	OBJECTIVE FUNCTION RANGES LOWER	GIVEN	UPPER
X1	0.0856	0.0000	9.0000	9.0000	9.0000	9.0000
X2	0.0571	0.0000	4.0000	-2.5116	4.0000	4.0000
X3	0.0001	0.0000	16.0000	16.0000	16.0000	318.1530
Y1	0.0001	0.0000	0.0000	-INFINITY	0.0000	0.0000
Y2	0.0714	0.0000	0.0000	0.0000	0.0000	+INFINITY

CONSTRAINT	TYPE	SHADOW PRICE	SLACK	RIGHT HAND SIDE RANGES LOWER	GIVEN	UPPER
Constr 1	≥	-1.0000	0.0000	-0.9262	0.0000	0.6172
Constr 2	≥	0.0000	0.2432	-INFINITY	0.0000	0.2432
Constr 3	≥	0.0000	0.0000	-0.3677	0.0000	0.4117
Constr 5	≥	0.0000	0.0855	-INFINITY	0.0001	0.0856
Constr 6	≥	0.0000	0.0570	-INFINITY	0.0001	0.0571
Constr 7	≥	0.0000	0.0000	0.0000	0.0001	0.0605
Constr 8	≥	0.0000	0.0000	0.0000	0.0001	0.1997
Constr 9	≥	0.0000	0.0713	-INFINITY	0.0001	0.0714
Constr 4	=	-1.0000	0.0000	0.0030	1.0000	+INFINITY

Hospital 2

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 0.7730

PARTIAL SENSITIVITY ANALYSIS

VARIABLE	SOLUTION VALUE	REDUCED COST	OPPORTU. COST	OBJECTIVE FUNCTION RANGES LOWER	GIVEN	UPPER
X1	0.0798	0.0000	5.0000	4.3442	5.0000	15.7500
X2	0.0533	0.0000	7.0000	3.5849	7.0000	8.0568
X3	0.0001	0.0000	10.0000	-INFINITY	10.0000	12.7846
Y1	0.0001	0.0000	0.0000	-INFINITY	0.0000	0.2482
Y2	0.0666	0.0000	0.0000	-0.4654	0.0000	+INFINITY

CONSTRAINT	TYPE	SHADOW PRICE	SLACK	RIGHT HAND SIDE RANGES LOWER	GIVEN	UPPER
Constr 1	≥	-0.2615	0.0000	-0.8641	0.0000	0.5758
Constr 2	≥	0.0000	0.2270	-INFINITY	0.0000	0.2270
Constr 3	≥	-0.6615	0.0000	-0.3431	0.0000	0.3840
Constr 5	≥	0.0000	0.0797	-INFINITY	0.0001	0.0798
Constr 6	≥	0.0000	0.0532	-INFINITY	0.0001	0.0533
Constr 7	≥	-2.7846	0.0000	0.0000	0.0001	0.0564
Constr 8	≥	-0.2482	0.0000	0.0000	0.0001	0.1248
Constr 9	≥	0.0000	0.0665	-INFINITY	0.0001	0.0666
Constr 4	=	-0.7733	0.0000	0.0035	1.0000	+INFINITY

Hospital 3

OPTIMAL SOLUTION

OBJECTIVE FUNCTION 1.0000

PARTIAL SENSITIVITY ANALYSIS

VARIABLE	SOLUTION VALUE	REDUCED COST	OPPORTU. COST	OBJECTIVE FUNCTION RANGES LOWER	GIVEN	UPPER
X1	0.0998	0.0000	4.0000	4.0000	4.0000	4.0000
X2	0.0666	0.0000	9.0000	9.0000	9.0000	+INFINITY
X3	0.0001	0.0000	13.0000	-109.3529	13.0000	13.0000
Y1	0.0001	0.0000	0.0000	-999999.0000	0.0000	+INFINITY
Y2	0.0833	0.0000	0.0000	-7.0748	0.0000	91.9337

CONSTRAINT	TYPE	SHADOW PRICE	SLACK	RIGHT HAND SIDE RANGES LOWER	GIVEN	UPPER
Constr 1	≥	0.0000	0.0000	-1.0807	0.0000	0.7202
Constr 2	≥	0.0000	0.2836	-INFINITY	0.0000	0.2836
Constr 3	≥	-1.0000	0.0000	-0.4287	0.0000	0.4803
Constr 5	≥	0.0000	0.0997	-INFINITY	0.0001	0.0998
Constr 6	≥	0.0000	0.0665	-INFINITY	0.0001	0.0666
Constr 7	≥	0.0000	0.0000	0.0000	0.0001	0.0706
Constr 8	≥	0.0000	0.0000	0.0000	0.0001	0.1427
Constr 9	≥	0.0000	0.0832	-INFINITY	0.0001	0.0833
Constr 4	=	-1.0000	0.0000	0.0029	1.0000	+INFINITY

The output from the three models give the three objective function values:

Hospital 1 efficiency = 1

Hospital 2 efficiency = 0.773

Hospital 3 efficiency = 1

Thus we find that hospital 2 is in-efficient and hospital 1 and hospital 3 are efficient.

The dual from Hospital 2 gives extra insight to the efficiency of that hospital.

If we average the output vectors and input vectors for these hospitals we obtain the following.

Averaged output vector

$$0.261538\begin{bmatrix}9\\4\\16\end{bmatrix}+0.661538\begin{bmatrix}4\\9\\13\end{bmatrix}=\begin{bmatrix}5\\7\\12.785\end{bmatrix}$$

Averaged input vector

$$0.261538\begin{bmatrix}5\\14\end{bmatrix}+0.661538\begin{bmatrix}7\\12\end{bmatrix}=\begin{bmatrix}5.938\\11.6\end{bmatrix}$$

Now, suppose that we create a composite hospital by combining 0.261538 of hospital 1 with 0.661538 of hospital 3. The averaged output tells us that the composite hospital produces the same amount of outputs 1 and 2 as hospital 2, but the composite hospital produces 12.785 – 10 = 2.785 more of out 3 (patient days for more than 65). From the averaged input for the composite hospital, we find that the composite hospital uses less of each input than does hospital 2. We now see exactly where hospital 2 is inefficient.

19. Linear Programming XI: Marketing

A marketing company is considering placing ads in five media: late night TV (LNTV), prime time TV (PTTV), billboards (BLB), newspapers (NEW), and radio (RAD). The ads are intended to reach seven different demographic groups.

In the following table the number of exposures, (the degree to which a company's target market is exposed to the company's communications about its product/services) obtained in each of seven markets per Euro of advertising in each of the five media. The second last row lists the minimum required number of exposures in each of the seven markets. The strategy is to reach the minimum number of customers, regardless of its cost. The last row is the saturation level for each market. The strategy is that exposure beyond this level is of no value. Exposures between these two limits will be termed as useful exposures.

	Exposure in 1000s per €1000 spent market group						
	1	2	3	4	5	6	7
LNTV		10	4	50	5		2
PTTV		10	30	5	12		
BLB	20					5	3
NEW	8					6	10
RAD		6	5	10	11	4	
Min number of exposures needed in 1000s	25	40	60	120	40	11	15
Saturation level in 1000s of exposures	60	70	120	140	80	25	55

How much should be spent on advertising in each medium if the company has a budget limit on €11.000 ?

Model

The cost should be as low as possible, and the exposure should be as high as possible.

Decision variables

Let LNTV, PTTV, BLB, NEW, RAD = Euro in 1000s on advertising in the 5 media

UX1,…,UX7 is the number of exposures obtained in each of the 7 markets beyond the minimum (min = saturation level – actual exposure

Cost = total amount spnt on advertising

USEFULX = total useful exposures

Constraints

There will be two main sets of constraints:

1) $\text{exposures in a market} \geq \text{minimum required} + \text{useful excess exposure beyond minimum}$

2) $\text{useful excess exposures in a market} \leq \text{saturation level} - \text{minimum required}$

The LP model is:

Max USEFULLX

Subject to

LIMCOST) COST ≤ 11 (a limt in €1000)

LIMEXP) USEFULX ≥ 0 (required exposures)

DEFCOST) $LNTV + PTTV + BLB + NEW + RAD = COST$

DEFEXP) $UX1 + UX2 + UX3 + UX4 + UX5 + UX6 + UX7 = USEFULX$

MRKT1) $20BLB + 8NEW - UX1 \geq 25$

MRKT2) $10LNTV + 10PTTV + 6RAD - UX2 \geq 40$

MRKT3) $4LNTV + 30PTTV + 5RAD - UX3 \geq 60$

MRKT4) $50LNTV + 5PTTV + 10RAD - UX4 \geq 120$

MRKT5) $5LNTV + 12PTTV + 11RAD - UX5 \geq 40$

MRKT6) $5BLB + 6NEW + 4RAD - UX6 \geq 11$

MRKT7) $2LNTV + 3BLB + 10NEW - UX7 \geq 15$

RANGE1) $UX1 \leq 35$

RANGE2) $UX2 \leq 30$

RANGE3) $UX3 \leq 60$

RANGE4) $UX4 \leq 20$

RANGE5) $UX5 \leq 40$

RANGE6) $UX6 \leq 14$

RANGE7) $UX7 \leq 40$

OPTIMAL SOLUTION

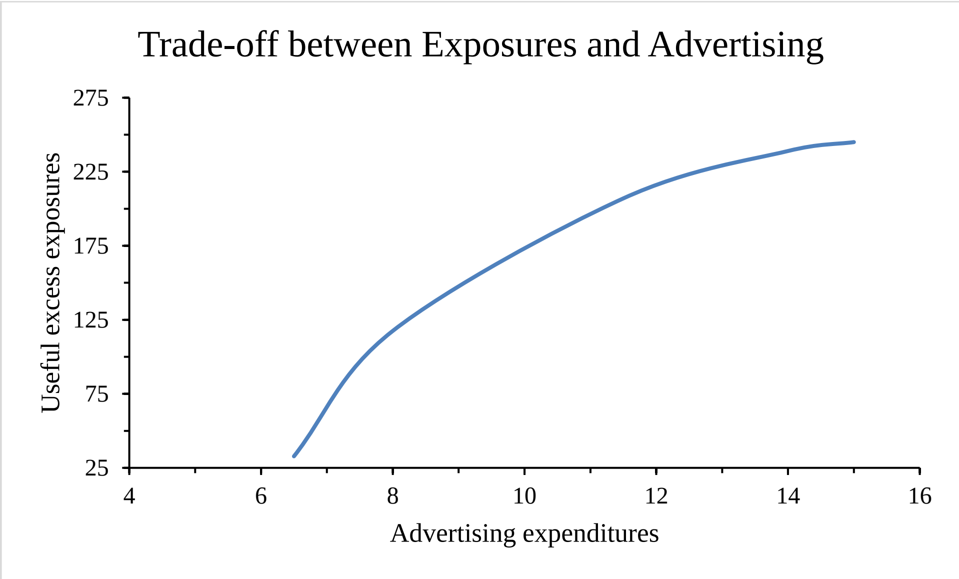
OBJECTIVE FUNCTION 196.7626

				PARTIAL SENSITIVITY ANALYSIS		
	SOLUTION	REDUCED	OPPORTU.	OBJECTIVE FUNCTION RANGES		
VARIABLE	VALUE	COST	COST	LOWER	GIVEN	UPPER
usefulx	196.7626	0.0000	1.0000	0.0000	1.0000	+INFINITY
cost	11.0000	0.0000	0.0000	-21.4388	0.0000	+INFINITY
lntv	1.9976	0.0000	0.0000	-4.2273	0.0000	0.2857
pttv	3.7074	0.0000	0.0000	-0.5110	0.0000	0.0000
blp	2.9089	0.0000	0.0000	-0.1667	0.0000	11.5361
new	0.2278	0.0000	0.0000	-23.1000	0.0000	0.0667
rad	2.1583	0.0000	0.0000	-0.0444	0.0000	0.4627
ux1	35.0000	0.0000	0.0000	-0.3291	0.0000	+INFINITY
ux2	30.0000	0.0000	0.0000	-0.1115	0.0000	+INFINITY
ux3	60.0000	0.0000	0.0000	-1.0000	0.0000	+INFINITY
ux4	20.0000	0.0000	0.0000	-0.8892	0.0000	+INFINITY
ux5	38.2182	0.0000	0.0000	-0.0091	0.0000	0.7130
ux6	13.5444	0.0000	0.0000	-0.0333	0.0000	0.1129
ux7	0.0000	-0.0072	0.0072	-INFINITY	0.0000	0.0072

		SHADOW	SLACK	RIGHT HAND SIDE RANGES		
CONSTRAINT	TYPE	PRICE		LOWER	GIVEN	UPPER
limcost	<=	21.4388	0.0000	10.0909	11.0000	11.0463
range1	<=	0.3291	0.0000	32.8223	35.0000	46.1765
range2	<=	0.1115	0.0000	29.5388	30.0000	38.9552
range3	<=	1.0000	0.0000	0.0000	60.0000	70.0048
range4	<=	0.8892	0.0000	5.1992	20.0000	35.5788
range5	<=	0.0000	1.7818	38.2182	40.0000	+INFINITY
ramge6	<=	0.0000	0.4556	13.5444	14.0000	+INFINITY
range7	<=	0.0000	40.0000	0.0000	40.0000	+INFINITY
limexp	>=	0.0000	196.7626	-INFINITY	0.0000	196.7626
mkt1	>=	-0.6709	0.0000	22.8223	25.0000	36.1765
mkt2	>=	-0.8885	0.0000	39.5388	40.0000	48.9552
mkt3	>=	0.0000	10.0048	-INFINITY	60.0000	70.0048
mkt4	>=	-0.1108	0.0000	105.1992	120.0000	135.5788
mkt5	>=	-1.0000	0.0000	38.2182	40.0000	78.2182
mkt6	>=	-1.0000	0.0000	10.5444	11.0000	24.5444
mkt7	>=	-1.0072	0.0000	12.8889	15.0000	28.3333
defcost	=	21.4388	0.0000	-0.9091	0.0000	0.0463
defexp	=	-1.0000	0.0000	-INFINITY	0.0000	196.7626

From the output we see that we advertise up to the saturation level in markets 1 to 4. In market 7 we advertise just enough to achieve the minimum required.

If we change the cost limit between values from 6 to 15, and plot the maximum possible number of useful exposures, we find the efficient frontier curve.



20. Finale note

In this note, you have been introduced to the idea of building managerial models in a scientific way. Most of the assumptions are very often requirements that are never meet in practice, but as an approximation of an uncertain world, you have a tool that points towards one feasible managerial direction. With the use of sensitivity analysis, several other directions can be included in the managerial proposal. Moreover, with the dual you have the information of the value of resources.