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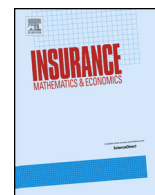
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Optimal retirement savings over the life cycle: A deterministic analysis in closed form [☆]

Marcel Fischer ^{a,b}, Bjarne Astrup Jensen ^{a,*}, Marlene Koch ^b

^a Copenhagen Business School, Department of Finance, Solbjerg Plads 3, DK-2000 Frederiksberg, Denmark

^b University of Konstanz, Department of Economics, Postbox 147, D-78457 Konstanz, Germany

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ABSTRACT

In this paper, we explore the life cycle consumption-savings problem in a stylized model with a risk-free investment opportunity, a tax-deferred retirement account, and deterministic labor income. Our closed form solutions show that liquidity constraints can be severely binding; in particular in situations with a high growth rate of labor income, in which retirement saving is optimally postponed. With a tax-deferred account, it is always optimal to save in this (illiquid) account first before saving in the (liquid) taxable account in order to satisfy the needs for consumption smoothing. The optimal retirement savings pattern is far from the widespread practice of contributing a fixed fraction of current labor income over the working life to a tax-deferred environment.

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1. Introduction

A large literature studies optimal retirement savings and related asset allocation decisions. Particular attention has been devoted to the optimal location of assets to taxable and tax-deferred retirement accounts in both theoretical work providing closed-form solutions to this question (e.g., Black, 1980; Tepper, 1981; Garlappi and Huang, 2006; Huang, 2008) and life cycle models relying on numerical methods (e.g., Shoven and Sialm, 1998, 2003; Amromin, 2003; Dammon et al., 2004; Fischer and Gallmeyer, 2017). Another strand of literature investigates the optimal draw-down of savings during retirement age (e.g., Huang et al., 2012; Huang and Milevsky, 2016). While these manuscripts have improved our understanding of the optimal location of assets to taxable and tax-deferred environments and the optimal draw-down of savings during retirement age, less attention has been paid to the optimal rate of contributions to tax-deferred retirement accounts over the life cycle. In this manuscript, we shed light on optimal contribution rates to tax-deferred retirement accounts in a setting, in which individuals have access to a taxable and a tax-deferred account, a risk-free investment opportunity and earn deterministic labor income. Our analytical solutions show that contributions to the tax-deferred retirement account are age-dependent, which contrasts with the constant contribution requirements, many countries are imposing on their citizens.

This study is inspired by some institutional features that are common in countries that rely on pension saving in an asset-backed defined contribution system (DC system) such as Denmark and the Netherlands.¹ We assume that the so-called *EET* tax regime is valid. This means that contributions to the retirement account are tax-deductible from labor income (up to certain limits) at the time the money is paid into the system (the first *E*). During the accumulation phase, the financial return is tax-exempt (the second *E*). When the pension

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^{*} Corresponding author.

E-mail addresses: marcel.fischer@uni.kn (M. Fischer), ba.fi@cbs.dk (B.A. Jensen), marlene.koch@uni.kn (M. Koch).

¹ To be precise in the terminology, we disregard the now widespread occurrence of so-called notionally defined contribution systems (NDC systems), that seemingly operate like a DC system. In an NDC system, the pension liabilities are not asset-backed, but rather rely on the contributions from the (future) working population to finance the future pension payments. For a survey of countries with such an approach, see, e.g., Holzmann and Palmer (2006).

savers reach retirement age and receive their retirement income by drawing from the accumulated reserves, the payout is taxed as income (the T). The retirement account is thus what is known as a tax-deferred account. The tax rate applicable to withdrawals may be different from the tax rate applicable at time of deduction. If so, it is typically lower.

The *EET* tax regime is commonly found in, e.g., labor market-oriented pension schemes in which individuals are required – either by labor market agreements or by law – to contribute to pension funds throughout their working life. It is also commonly found that the contributions are a constant fraction of labor income throughout the working life. A feature, which may also just be assumed as a convenient modeling device. One example is found in the influential and widely cited paper by Dammon et al. (2004):

“The treatment of the investor’s retirement account is broadly consistent with practice. Prior to retirement, the investor is assumed to contribute a constant fraction k of pre-tax non-financial income to a retirement account each year.”

(Dammon et al., 2004, p. 1013)

While the (re-)design of pension systems is an important policy topic in light of the demographic development and the longevity risk inherent in many countries’ present arrangements, whether contributing a constant fraction of labor income into a tax-deferred retirement account is optimal has received relatively little attention in the public debate. However, it is a fact of life that liquidity constraints vary over the life cycle and are most severe for younger individuals, who have recently finished their education with a possible debt to be repaid and being in the process of establishing a family with the needs for finding appropriate housing. It is also partly a result of a wage profile that is typically increasing over the working life. Classical references include Deaton (1991), Carroll (1997), Constantinides et al. (2002) and Gourinchas and Parker (2002). More recent work includes Lachance (2012), Blake et al. (2014), Schlafmann et al. (2022), Fischer et al. (2022), and Larsen and Munk (2023).

Our work challenges the idea that contributions to a retirement savings scheme should be a constant fraction of labor income; it also challenges the idea that pension saving should start early in the life cycle. If one searches Google for answers to the question: “When and how to save for retirement?”, numerous answers come up. Interestingly, they are pretty much alike. Here are two typical examples:

- The answer is simple: as soon as you can. Ideally, you’d start saving in your 20s, when you first leave school and begin earning paychecks. That’s because the sooner you begin saving, the more time your money has to grow.²
- Short answer: as soon as you begin working. Too late for that? Remember that saving will be easiest if you start now.³

These answers are fatally wrong in our view. We ask how a utility optimizing scheme looks like using a deterministic model in order to allow for analytical solutions. We provide a closed-form solution to this problem with both a taxable and a tax-deferred retirement account, a risk-free investment opportunity, and a deterministic labor income stream. Our results show that the widespread institutional feature of contributing a constant share of labor income is typically not optimal.

In the first step, the investor only has the possibility to save in an ordinary way in a taxable account, in which the current return on the investment is taxed in accordance with a mark-to-market principle; we also allow the investor to borrow against future labor income. Saving for retirement is the sole responsibility of the investor, who will behave perfectly rational in this regard.

In the next step, we impose a non-negativity constraint on financial wealth. Depending upon the profile of future labor earnings, this constraint may lead to the investor being liquidity constrained in the initial periods of the working life.

Finally, in our richest model, we allow the investor to save for retirement in a tax-deferred account. The accumulated retirement savings are locked and inaccessible for the investor until attaining retirement age. The compensation for accepting the illiquid nature of the retirement savings is the favorable tax treatment of savings in the tax-deferred account. Furthermore, the tax rate applicable to payouts from the pension system may be lower than the one applicable at the time of deduction. Both these features make it optimal for the investor to exploit the tax-advantages offered by accumulating retirement savings in the tax-deferred account instead of accumulating them in the fully liquid taxable account.

During working life, the taxable account is used for intertemporal consumption smoothing and for securing access to funds in cases, in which a stochastic environment triggers bad outcomes, such as low labor income earnings. The latter is well-known as precautionary savings/buffer stock savings motive (see, for example, Carroll, 1997; Gourinchas and Parker, 2002, and Amromin, 2003). Both motives imply that – depending on the model parameters – it may be optimal to (also) build up savings in the taxable account. The precautionary savings/buffer stock savings motive is absent in a deterministic setting, whereas intertemporal consumption smoothing via the taxable account may still be needed.

This paper proceeds as follows: In Section 2, we solve an unrestricted version of the consumption-savings problem in which individuals can borrow against future labor income. In Section 3, we extend the model introduced in Section 2 by imposing a non-negativity constraint on wealth and then derive optimal consumption-savings decisions. In Section 4, we solve a consumption-savings model that additionally offers individuals the possibility to save in a tax-deferred retirement account. In Section 5, we outline some generalizations to our model. Finally, Section 6 concludes.

2. A simple unconstrained model

The simplest possible scenario is an unconstrained variant of the classical intertemporal consumption-savings problem (see, for example, Hakansson, 1970; Merton, 1969, and Merton, 1971), where the consumption stream, C_t , is financed by labor income, and where initial wealth, W_{0-} , is defined to be 0⁴:

² Source: <https://money.cnn.com/retirement/guide>, last retrieved September 19, 2022.

³ Source: <https://investor.vanguard.com/investor-resources-education>, last retrieved September 19, 2022.

⁴ We relax the assumption of $W_{0-} = 0$ in Section 5.2 by allowing for pre-existing initial wealth.

$$\max_{\{W_t\}_{t=0}^{T-1}} \frac{1}{1-\gamma} \left[\sum_{t=0}^{T-1} \rho^t C_t^{1-\gamma} + K \rho^T C_T^{1-\gamma} \right] \tag{1}$$

subject to

$$C_t = L_t(1-\tau) + W_{t-1}(1+r(1-\tau)) - W_t, \quad t = 0, 1, \dots, T-1 \tag{2}$$

$$C_T = W_{T-1}(1+r(1-\tau)), \tag{3}$$

in which γ is the individual's risk aversion, $\rho < 1$ its time discount rate, measuring how much the individual prefers present over future consumption, r is the pre-tax interest rate that is earned on savings, τ is a common tax rate applicable to labor income and interest earnings, respectively, and W_t is the wealth invested from time t to $t + 1$ in order to finance future consumption. The utility optimizing individual is assumed to have no initial wealth, but receives a deterministic stream of labor income L_t , $t = 0, 1, \dots, T - 1$. The weighting factor K is supposed to summarize the discounted utility stream from consumption during the retirement period.⁵

The first-order conditions, also known as Euler conditions, for this utility optimization problem are:

$$\rho^t C_t^{-\gamma} = \rho^{t+1} (1+r(1-\tau)) C_{t+1}^{-\gamma}, \quad [W_t], \quad t = 0, 1, 2, \dots, T-2 \tag{4}$$

$$\rho^{T-1} C_{T-1}^{-\gamma} = (1+r(1-\tau)) K \rho^T C_T^{-\gamma}, \quad [W_{T-1}], \tag{5}$$

with solutions:

$$C_t = C_0 \varepsilon^t, \quad t = 1, 2, \dots, T-1 \tag{6}$$

$$C_T = C_0 \varepsilon^T K^{\frac{1}{\gamma}}, \quad \varepsilon \equiv (\rho(1+r(1-\tau)))^{\frac{1}{\gamma}}. \tag{7}$$

In order to derive an analytically tractable solution, we assume a wage process with a uniform gross growth factor g , i.e. $L_t = L_0 g^t$, $t = 1, \dots, T - 1$.⁶ The dynamics of the savings account for $t = 0, 1, \dots, T - 1$ then become:

$$W_t = (1+r(1-\tau))^{-(T-t)} C_0 \varepsilon^T K^{1/\gamma} + \sum_{s=t+1}^{T-1} [C_0 \varepsilon^s - L_0(1-\tau)g^s] (1+r(1-\tau))^{-(s-t)}. \tag{8}$$

The level of C_0 is found by using the initial budget constraint $W_0 = L_0(1-\tau) - C_0$, which implies:

$$0 = (1+r(1-\tau))^{-T} C_0 \varepsilon^T K^{1/\gamma} + \sum_{s=0}^{T-1} [C_0 \varepsilon^s - L_0(1-\tau)g^s] (1+r(1-\tau))^{-s}. \tag{9}$$

From here, the consumption-savings policy for $t = 0, 1, \dots, T - 1$ is given in closed form by:

$$C_t = C_0 \varepsilon^t = L_0(1-\tau) \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} \varepsilon^t \tag{10}$$

$$x = \frac{\varepsilon}{1+r(1-\tau)}, \quad y = \frac{g}{1+r(1-\tau)}.$$

The shape of the consumption pattern and the relative share of current disposable income, which is consumed in every period, is given solely by the relation between g and ε :

$$\frac{C_t}{L_t(1-\tau)} = \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} \left(\frac{\varepsilon}{g} \right)^t. \tag{11}$$

If $g < \varepsilon$ ($\Leftrightarrow y < x$), which means that labor income is relatively slowly increasing, we have an increasing profile of consumption out of disposable income and, vice versa, a decreasing profile of the savings rate. For $g < \varepsilon$, it is thus optimal to already build up savings at the beginning of the life cycle in order to smooth consumption in accordance with the preferences for intertemporal consumption.

If $g > \varepsilon$ ($\Leftrightarrow x < y$), labor income is relatively steeply increasing. In that case, we have a decreasing profile of consumption out of disposable income and, vice versa, an increasing profile of the savings rate out of disposable income. The optimal savings rate out of disposable income then has an increasing trend in line with the increasing profile of labor income over the lifetime. Savings may even be negative and allow the individual to borrow against future labor income in the beginning and first turn positive at some later point in time. Saving for the retirement period is optimally postponed as a response to the increasing labor income. To be precise, the borderline between saving and borrowing at time t is determined by whether current income after tax is sufficient to cover current expense, i.e., by the following inequality:

$$L_0(1-\tau)g^t \geq L_0(1-\tau) \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} \varepsilon^t \Leftrightarrow \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} \left(\frac{\varepsilon}{g} \right)^t \leq 1. \tag{12}$$

⁵ It is well-known that with a time-additive utility function of the HARA type, this sum has the same functional form as the individual utility functions in the sum. In Online Appendix C we give examples as to how parameter K could be chosen.

⁶ We allow for time-varying growth rates of labor income in Section 5.3.

This set of inequalities determines when and how individuals' budget constraints become binding when we add restrictions to the consumption–investment problems in the following Theorems 1 and 2 and is a key ingredient in the detailed proofs of these theorems.

In both cases ($g < \varepsilon$ and $g > \varepsilon$), a constant savings rate out of disposable income is not part of the optimal solution. Only the borderline situation with $\varepsilon = g$ leads to a constant savings rate out of disposable income.

The value of the utility function, disregarding the factor $1/(1 - \gamma)$ in front, has the analytical form:

$$\left[L_0(1 - \tau) \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} \right]^{1-\gamma} \left[\sum_{t=0}^{T-1} (\varepsilon^{1-\gamma})^t \rho^t + \varepsilon^{(1-\gamma)T} \rho^T K^{\frac{1}{\gamma}} \right] = \left[L_0(1 - \tau) \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} \right]^{1-\gamma} \left[\frac{(1 - \rho^T \varepsilon^{(1-\gamma)T})}{1 - \rho \varepsilon^{1-\gamma}} + \rho^T \varepsilon^{(1-\gamma)T} K^{\frac{1}{\gamma}} \right]. \tag{13}$$

Once the intertemporal structure of the consumption policy is found, the remaining step is to insert equation (7) into the intertemporal budget constraint and equate the present value of the labor income stream to the present value of consumption expenditures. x is then the discount factor for a geometrically growing stream of consumption, whereas y is the discount factor for a geometrically growing stream of labor income.

3. Non–negative financial wealth

We next account for the fact that, in reality, the individual is not able to finance a consumption level in excess of current financial resources; i.e., the individual is constrained to always be financially solvent and unable to borrow against future labor income. This reflects the usual institutional feature that individuals are unable to collateralize their human capital, e.g., due to moral hazard considerations. The optimization problem and the budget equations are the same as in equations (2)–(3), but the solvency constraints $W_t \geq 0$, $t = 0, 1, 2, \dots, T - 1$, are added.

Variants of this problem appear in many contexts in the economic literature. The first–order conditions now involve the complementary slackness conditions associated with the non–negativity constraints⁷:

$$\rho^t C_t^{-\gamma} \geq \rho^{t+1} (1 + r(1 - \tau)) C_{t+1}^{-\gamma} \Leftrightarrow \varepsilon C_t \leq C_{t+1} \tag{14}$$

$$[C_{t+1} - \varepsilon C_t] W_t = 0, \quad t = 0, 1, 2, \dots, T - 2$$

$$C_{T-1}^{-\gamma} = (1 + r(1 - \tau)) K \rho C_T^{-\gamma} \Leftrightarrow C_T = K^{1/\gamma} \varepsilon C_{T-1}. \tag{15}$$

Whereas the discounting mechanism is well defined and exogenously given by the after–tax interest rate in the unrestricted model outlined in Section 2, this is no longer true in the current setting with the non–negativity constraint. When the non–negativity constraint is binding, the marginal value of a future unit of wealth is less than the value given by simple discounting as the marginal utility of present consumption calls for the opportunity to convert future consumption into present consumption. The before–tax interest rate (the market rate) is only relevant when the individual is actually saving and has already built up savings in previous periods, i.e., if $W_t > 0$. If $W_t = 0$, the individual might want to borrow against future labor income, but is unable to use the financial market for this purpose.

However, for a labor income process with a constant growth rate, we are still able to derive the optimal consumption–savings profile in closed form:

Theorem 1. *When labor income has a uniform gross growth rate g , there are two possible scenarios, one of which leads to an optimal solution. Which of the two solutions applies depends on how the complementary slackness conditions (14) and (15) are fulfilled:*

1. If $\varepsilon \geq g$, we have $C_0 < L_0(1 - \tau)$, and the unconstrained solution is valid throughout.
2. If $\varepsilon < g$, we may have a sequence of initial periods in which $C_t = L_t(1 - \tau)$, followed by a saving for retirement period, during which the profile follows the unconstrained solution with the reduced remaining time to retirement. The switch point s^* , the last time index at which $C_t = L_t(1 - \tau)$, is determined by

$$\frac{\sum_{s=s^*}^{T-1} y^{(s-s^*)}}{\left[\sum_{s=s^*}^{T-1} x^{s-s^*} + K^{\frac{1}{\gamma}} x^{T-s^*} \right]} > 1 > \frac{\sum_{s=s^*+1}^{T-1} y^{s-(s^*+1)}}{\left[\sum_{s=s^*+1}^{T-1} x^{s-(s^*+1)} + K^{\frac{1}{\gamma}} x^{T-(s^*+1)} \right]}. \tag{16}$$

Proof. See Online Appendix A. \square

In case of $\varepsilon < g$, i.e., in case 2 of Theorem 1, the unrestricted solution may still be feasible. If the solvency constraint is not binding at any point in time, there is no need to search for a switch point s^* . This happens when

$$\sum_{s=0}^{T-1} y^s < \sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T. \tag{17}$$

⁷ For more details, see Online Appendix A.

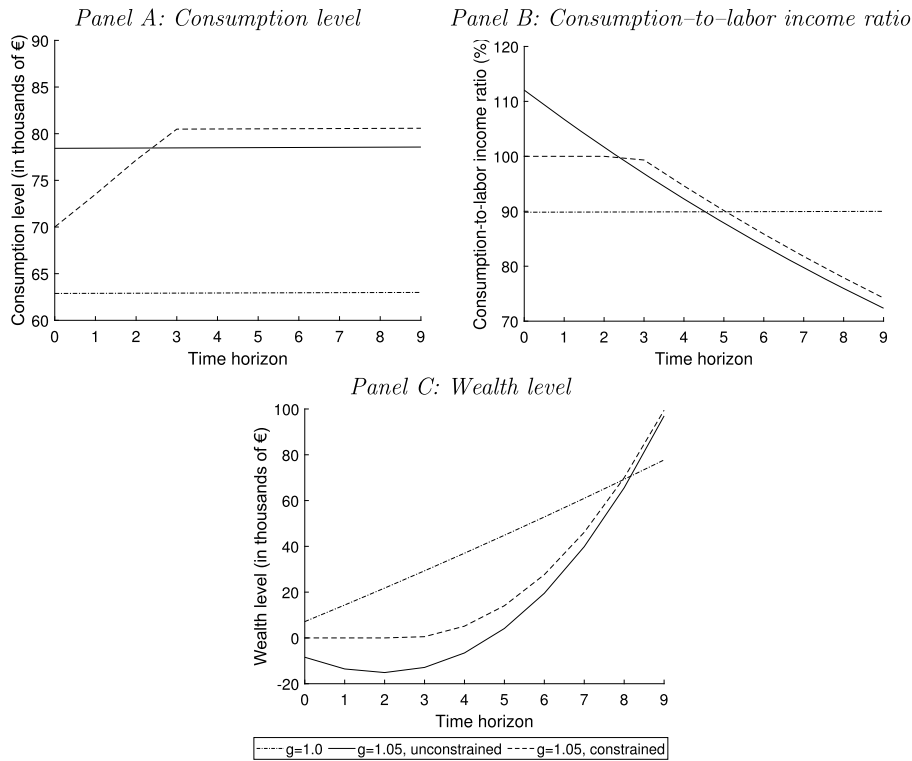


Fig. 1. Comparison of consumption-savings patterns in the constrained and unconstrained case. This figure depicts the results for consumption-savings decisions in the constrained and unconstrained case of our model, i.e., we compare the models with and without a non-negativity constraint on wealth, as presented in Section 2 and Section 3, respectively. We depict results for two different growth rates of labor income, $g = 1.05$ and $g = 1.0$. Panel A depicts the evolution of the consumption level over the time horizon $T = 10$, Panel B of the consumption-to-labor income ratio, and Panel C depicts the evolution of the wealth level over the time horizon. The solid (dashed) lines report results for the unconstrained (constrained) case with $g = 1.05$. The dash-dotted lines report the results for the case with $g = 1.0$ in which the solvency constraint is not binding. The other parameters are chosen as follows: $L_0 = 100$, $K = 2$, $\gamma = 3$, $\rho = 0.98$, $r = 3\%$, $\tau = 30\%$.

When $\varepsilon = g$ ($\Leftrightarrow x = y$), the inequality in (17) is clearly fulfilled. The left-hand side of (17) is increasing in g with no upper limit, whereas the right-hand side is independent of g . Hence, for some value of g the inequality in (17) is no longer valid. The exact condition on how large the growth in labor income, g , must be for the inequality in (17) not to hold and the non-negativity constraint to be binding in some initial period(s) depends on the parameters of the model—in particular on K .

The inequalities in (16) can be rewritten as follows:

$$\frac{\sum_{s=s^*}^{T-1} y^{(s-s^*)}}{\sum_{s=s^*}^{T-1} x^{s-s^*} + K^{1/\gamma} x^{T-s^*}} = \frac{\sum_{s=0}^{T-1-s^*} y^s}{\sum_{s=0}^{T-1-s^*} x^s + K^{1/\gamma} x^{T-s^*}} \tag{18}$$

The numerator in the right-hand side of equation (18) is now the “present” value of labor income starting from $L_0 = 1$, when the clock is reset to start at time s^* (see the standard formula for discounting a geometrically growing series). Similarly, the denominator is the “present” value of consumption starting from $C_0 = 1$, when the clock is reset to start at time s^* .

When $s^* > 0$, the individual is unable to reach the desired unrestricted level of consumption that will fulfill the Euler conditions. The marginal utility is forced to be too high in the beginning, which is captured by a strict inequality in (14). At some point in time, the labor income is high enough in order for the unrestricted solution to work for the remaining periods (Bellman’s optimality principle). So with a sufficiently high growth in labor income, saving is postponed.

We illustrate the impact of the non-negativity condition for wealth graphically in Fig. 1. We choose a time horizon of $T = 10$ periods and compare two different growth rates of labor income: (i) $g = 1.0$ and (ii) $g = 1.05$. Panel A depicts the evolution of the individual’s consumption level over the time horizon and Panel B of the consumption-to-labor income ratio. Panel C shows the evolution of total wealth over the time horizon.

The example illustrates that the financial solvency constraint significantly affects the intertemporal behavior of consumption and saving. Without the constraint and when individuals face growth in labor income, i.e., in the setting depicted by the solid line in Fig. 1, individuals borrow against human capital, i.e., the capitalized value of future labor income, to finance a consumption level well above current earnings in the early periods. Therefore, the consumption-to-labor income ratio, as depicted in Panel B of Fig. 1, is above 100% in early periods and consequently total wealth as depicted in Panel C is negative.

The case, in which individuals face labor income growth and are subject to the non-negativity constraint on financial wealth, is depicted by the dashed line in Fig. 1. In this case, the consumption profile is increasing over the considered time horizon. Panel A shows that the borrowing constraint forces a lower starting level of consumption with a marginal utility well above what it is in the unconstrained case. As a result, the optimal behavior is to wait until the solvency constraint is no longer binding and let the consumption level increase

afterwards.⁸ Panel B and Panel C of Fig. 1 show that as long as the solvency constraint is binding, individuals consume 100% of their labor income and therefore, postpone accumulating savings until the constraint is no longer binding.

In the case of no labor income growth, i.e., in the setting in which $g = 1.0$, that is depicted by the dash-dotted lines in Fig. 1, the solvency constraint is not binding and the consumption rate is almost constant over the time horizon considered. This can be explained by the fact that there is a discounting effect (in the utility function) which expresses a time preference for early consumption rather than late consumption. But there is also a need for saving for retirement, and with no growth in labor income it is necessary to start early despite the preference for early consumption.

Panel C of Fig. 1 shows that for the case with a relatively high growth rate of labor income ($g = 1.05$), the solvency constraint is binding. In order to attain the same level of utility in the constrained and in the unconstrained case, the wage profile has to be lifted by approximately 1% throughout the 10 year period. Although the liquidity constraint skews the time profile of consumption, the effect is rather small for growth rates of labor income relatively close to the risk-free interest rate r . Increasing this growth rate leads to a markedly larger effect. With a 7% growth rate in labor income, the necessary lift in the labor income profile is approximately 3%, and with a 10% growth rate in labor income it becomes 7.3%.

In all settings, it is clear that the optimal intertemporal consumption-savings decision is far from a policy with a constant contribution rate out of current disposable income.

4. A tax-deferred retirement account

Introducing the possibility to save in a tax-deferred retirement account further increases the complexity of the intertemporal consumption-savings problem. The savings decision now additionally involves an asset location decision, i.e., individuals no longer only have to decide how much to save, but also in which account—the taxable or the tax-deferred retirement account. Such a tax-deferred retirement account is a common institutional device for defined-contribution pension systems.

We assume that (i) contributions to such an account are tax-deductible from labor income, (ii) it is not possible to liquidate any of these savings before retirement, and (iii) upon liquidation the withdrawals are subject to the same tax rate, τ , as the one that was applied when the contribution was made.⁹ We also impose the solvency constraint introduced in Section 3; otherwise, there would be obvious and unlimited arbitrage opportunities by moving borrowed money from the taxable account to the tax-deferred retirement account.

We denote the contribution to the tax-deferred retirement account by Z_t . The optimization problem can then be written as:

$$\max_{\{(W_t \geq 0)_{t=0}^{T-2}, \{Z_t \geq 0\}_{t=0}^{T-1}\}} \frac{1}{1-\gamma} \left[\sum_{t=0}^{T-1} \rho^t C_t^{1-\gamma} + \rho^T K C_T^{1-\gamma} \right] \tag{19}$$

$$C_t = (L_t - Z_t)(1 - \tau) + W_{t-1}(1 + r(1 - \tau)) - W_t, \quad t = 0, 1, 2, \dots, T - 2 \tag{20}$$

$$C_{T-1} = (L_{T-1} - Z_{T-1})(1 - \tau) + W_{T-2}(1 + r(1 - \tau)) \tag{21}$$

$$C_T = (1 - \tau) \sum_{s=0}^{T-1} Z_s(1 + r)^{T-s}. \tag{22}$$

Assuming again that the wage dynamics are given by a constant gross growth rate g and setting $W_{0-} = 0$, we can rewrite the budget dynamics as:

$$C_t = (L_0 g^t - Z_t)(1 - \tau) + W_{t-1}(1 + r(1 - \tau)) - W_t, \quad t = 0, 1, 2, \dots, T - 2 \tag{23}$$

$$C_{T-1} = (L_0 g^{T-1} - Z_{T-1})(1 - \tau) + W_{T-2}(1 + r(1 - \tau)) \tag{24}$$

$$C_T = (1 - \tau) \sum_{s=0}^{T-1} Z_s(1 + r)^{T-s}. \tag{25}$$

We use the same type of additional variables as in Theorem 1, but we need to distinguish between “before tax” and “after tax” versions. We denote the before tax versions by subscript 1 and the after tax versions by subscript 2:

$$\varepsilon_1 \equiv (\rho(1+r))^{\frac{1}{\gamma}}, \quad x_1 = \frac{\varepsilon_1}{1+r}, \quad y_1 = \frac{g}{1+r}, \tag{26}$$

$$\varepsilon_2 \equiv (\rho(1+r(1-\tau)))^{\frac{1}{\gamma}}, \quad x_2 = \frac{\varepsilon_2}{1+r(1-\tau)}, \quad y_2 = \frac{g}{1+r(1-\tau)}. \tag{27}$$

Given these definitions, the first-order optimality conditions are:

$$\begin{aligned} \rho^t C_t^{-\gamma} &\geq (1+r(1-\tau))\rho^{t+1} C_{t+1}^{-\gamma}, \quad \left[\rho^t C_t^{-\gamma} - (1+r(1-\tau))\rho^{t+1} C_{t+1}^{-\gamma} \right] W_t = 0 \iff \\ \varepsilon_2 C_t &\leq C_{t+1}, \quad [\varepsilon_2 C_t - C_{t+1}] W_t = 0, \quad t = 0, 1, 2, \dots, T - 2 \end{aligned} \tag{28}$$

⁸ The consumption levels for the unconstrained cases are not constant, although they appear to be. This is due to the fact that the growth rate of consumption is determined by $\varepsilon = 1.00312$, almost equal to one.

⁹ The latter is not a restriction, since any difference between τ and the tax rate applicable to the proceeds from liquidation can be captured by the preference weighting parameter K (see Online Appendix C).

$$\begin{aligned} \rho^t C_t^{-\gamma} &\geq K \rho^T C_T^{-\gamma} (1+r)^{T-t}, \quad \left[\rho^t C_t^{-\gamma} - K \rho^T C_T^{-\gamma} (1+r)^{T-t} \right] Z_t = 0 \quad \Leftrightarrow \\ K^{1/\gamma} \varepsilon_1^{(T-t)} C_t &\leq C_T, \quad \left[K^{1/\gamma} \varepsilon_1^{(T-t)} C_t - C_T \right] Z_t = 0, \quad t = 0, 1, 2, \dots, T-1. \end{aligned} \tag{29}$$

In a model with stochastic outcomes and the possibility of unfavorable shocks, there is a clear “precautionary savings” motive to build a buffer of liquid savings. This result is found in much of the literature (see, for example, Carroll, 1997; Munk, 2000; and Amromin, 2003). However, under certainty and with a uniform growth rate of labor income, the savings motive is only driven by two incentives: (i) the fundamental desire to finance consumption in the retirement period and (ii) the desire to smooth consumption over the entire investment horizon in accordance with preferences. The vehicle for the former is the tax-deferred retirement account, and the vehicle for the latter is the taxable account. Theorem 2 shows that in our framework the former motive is the dominating one. The contributions to the tax-deferred retirement account always precede contributions to the taxable account; and for many parameter constellations the taxable account is never used, because the growth in labor income is sufficient in itself to satisfy the desire to smooth consumption over time.

Theorem 2. *When labor income has a uniform gross growth rate g , there are three possible scenarios, one of which leads to an optimal solution. These solutions differ in how the complementary slackness conditions (28) and (29) are fulfilled:*

1. *If $\varepsilon_1 \leq g$, there may be a sequence of initial periods, in which $C_t = L_t(1 - \tau)$, followed by a saving for retirement period where the profile follows the unconstrained solution with the reduced remaining time to retirement. The switch point s^* , the last time index for which $C_t = L_t(1 - \tau)$, is given by the first value of s^* , for which the following inequality is fulfilled:*

$$\frac{\sum_{s=s^*}^{T-1} y_1^{(s-s^*)}}{\left[\sum_{s=s^*}^{T-1} x_1^{s-s^*} + K^{1/\gamma} x_1^{T-s^*} \right]} > 1 > \frac{\sum_{s=s^*+1}^{T-1} y_1^{(s-s^*-1)}}{\left[\sum_{s=s^*+1}^{T-1} x_1^{s-s^*-1} + K^{1/\gamma} x_1^{(T-s^*-1)} \right]}. \tag{30}$$

The contributions to the tax-deferred retirement account develop as

$$Z_t = L_t \left(1 - \frac{\sum_{s=s^*+1}^{T-1} y^{(s-(s^*+1))}}{\sum_{s=s^*+1}^{T-1} x^{s-(s^*+1)} + K^{1/\gamma} x^{(T-(s^*+1))}} \left(\frac{\varepsilon_1}{g} \right)^{t-(s^*+1)} \right), \quad t = s^* + 1, \dots, N-1, \tag{31}$$

and the taxable account is not used.

2. *If $\varepsilon_1 > g$, saving for retirement starts immediately and stops after a switch point s^* , given by the first value of s^* , for which the following inequality is fulfilled:*

$$\frac{\sum_{s=s^*+1}^{T-1} y_2^s \varepsilon_2^{s^*+1}}{\sum_{s=s^*+1}^{T-1} x_2^s} \leq 1. \tag{32}$$

$$\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T} \varepsilon_1^{s^*+1}$$

This switch point is the last period, in which contributions are made to the tax-deferred retirement account; at the switch point there are two possibilities:

- a) *There are only contributions to the tax-deferred retirement account at time s^* . This is the relevant possibility whenever*

$$\frac{\sum_{s=s^*+1}^{T-1} y_2^s \varepsilon_2^{s^*}}{\sum_{s=s^*+1}^{T-1} x_2^s} \geq 1. \tag{33}$$

$$\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T} \varepsilon_1^{s^*}$$

Then

$$C_t = L_0(1 - \tau) \frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T} \varepsilon_1^t, \quad t = 0, 1, \dots, s^* \tag{34}$$

$$C_t = L_0(1 - \tau) \frac{\sum_{s=s^*+1}^{T-1} y_2^s}{\sum_{s=s^*+1}^{T-1} x_2^s} \varepsilon_2^t, \quad t = s^* + 1, s^* + 2, \dots, T-1 \tag{35}$$

$$Z_t = L_t \left[1 - \frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T} \left(\frac{\varepsilon_1}{g} \right)^t \right], \quad t = 0, 1, \dots, s^*. \tag{36}$$

- b) *There are contributions to both, the taxable and the tax-deferred retirement account at time s^* . This is the relevant possibility whenever*

$$\frac{\sum_{s=s^*+1}^{T-1} y_2^s \varepsilon_2^{s^*}}{\sum_{s=s^*+1}^{T-1} x_2^s} < 1. \tag{37}$$

$$\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T} \varepsilon_1^{s^*}$$

Then

$$W_{s^*} = C_{s^*} \sum_{s=1}^{T-1-s^*} x_2^s - L_{s^*} (1 - \tau) \sum_{s=1}^{T-1-s^*} y_2^s \tag{38}$$

$$Z_{s^*} = L_{s^*} \sum_{s=0}^{T-1-s^*} y_2^s - \frac{C_{s^*}}{1 - \tau} \sum_{s=0}^{T-1-s^*} x_2^s \tag{39}$$

$$C_t = L_0(1 - \tau) \frac{\sum_{s=0}^{s^*} y_1^s + y_1^{s^*} \sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=0}^{s^*} x_1^s + x_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s + K^{1/\gamma} x_1^T} \varepsilon_1^t, \quad t = 0, 1, \dots, s^* \tag{40}$$

$$C_t = L_0(1 - \tau) \frac{\sum_{s=0}^{s^*} y_1^s + y_1^{s^*} \sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=0}^{s^*} x_1^s + x_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s + K^{1/\gamma} x_1^T} \varepsilon_1^{s^*} \varepsilon_2^{t-s^*}, \quad t = s^* + 1, \dots, T - 1. \tag{41}$$

Proof. See Online Appendix B. □

The nature of s^* as the first time index satisfying the respective inequalities in Theorem 2 resembles the property of a stopping time in a stochastic model.

From Theorem 2, the optimal consumption–investment strategy at any given point in time t depends on the model parameters, which determine if and when the complementary slackness conditions are fulfilled with equality. In total, three different cases for the optimal consumption–investment strategy can be considered, corresponding to item 1, item 2a, and item 2b in Theorem 2.

In item 1, the growth rate of labor income is relatively large. More specifically, if $\varepsilon_1 \leq g$, the non–negativity condition on savings becomes binding in earlier periods. In that case, the high growth rate of labor income renders initially deferring saving to a later point in time and consuming all income in the earlier periods optimal.

In the two cases characterized by items 2a and 2b, it is optimal to start saving in the tax–deferred retirement account already in the first period and eventually stop it when the inequality in (32) is fulfilled. The two cases in item 2 are characterized by a lower growth rate of labor income ($g < \varepsilon_1$) and differ by the optimal location of contributions at the point in time s^* , when the last contribution to the tax–deferred account is made.

In case 2a, contributions are solely made to the tax–deferred retirement account, but not to the taxable account. In case 2b, there are contributions to both the taxable and the tax–deferred retirement account. In this case, the savings in the taxable account do not benefit from the tax–exemption of profits, applicable to savings in the tax–deferred account. Taxable savings may be built up if the growth rate of labor income is so low that parts of the savings are already meant for consumption prior to retirement age.

Fig. 2 illustrates the different cases from Theorem 2 graphically. For that purpose, it compares the evolution of the consumption level (Panel A) and of the consumption–to–labor income ratio (Panel B) as well as the evolution of the contributions to the tax–deferred retirement account in percent of labor income (Panel C) and the taxable account in percent of income (Panel D) for a time horizon of $T = 10$ for two different growth rates of labor income, g , and a tax rate of $\tau = 30\%$.¹⁰ More specifically, the values of g are set to $g = 1.05$ (dashed line) and $g = 0.97$ (solid line), respectively. These two choices are useful to illustrate the two different items in Theorem 2.

In the case with $g = 1.05$ (dashed line), labor income grows over the investment horizon. In that case, the individual is initially consumption–constrained as can be seen from the upward–sloping consumption–level in Panel A and the consumption–to–labor income ratio being at 100% from time $t = 0$ to $t = 2$. That is, the individual’s situation is characterized by item 1 of Theorem 2. With a growing level of labor income, the consumption–to–labor income ratio required for consumption smoothing is monotonically decreasing after time $t = 2$. Instead, starting from time $t = 3$, the individual monotonically increases its contributions to the tax–deferred retirement account to build up sufficient savings for retirement age.

In the case with $g = 0.97$ (solid line), labor income drops over the investment horizon. In this case, the individual is able to smooth consumption over the life cycle (Panel A) as the consumption constraint does not bind (Panel B). That is, for $g = 0.97$, the individual is not ending up in the case characterized by item 1 of Theorem 2. Instead, the case characterized by item 2 of Theorem 2 becomes relevant.

At the beginning of the investment horizon, in period $t = 0$ to $t = 4$, the individual solely contributes to the tax–deferred retirement account. Contributions to the retirement account in percent of labor income are monotonically decreasing, reflecting that due to the decrease in labor income, an increasing share of it is required to uphold the consumption level and smooth consumption over the life cycle. From time $t = 6$, no further contributions to the tax–deferred retirement account are made, i.e., $s^* = 5$. From this point in time, the accumulated savings (plus the past and future interest earned) are already sufficient to cover consumption expenses during retirement age. From time $t = 5$ to $t = 7$, the individual contributes to the taxable account (Panel D) to sustain its consumption level prior to retirement age in periods $t = 8$ to $t = 9$, despite declining labor income. In particular, at time $s^* = 5$, the individual contributes to both the taxable and the tax–deferred retirement account. That is, the individual is in the case characterized by item 2b in Theorem 2.¹¹

From Panel D, contributions to the taxable account increase monotonically from time $t = 5$ to $t = 6$, reflecting the decline in contributions to the tax–deferred retirement account. After $t = 6$, contributions to the taxable account monotonically decrease as a growing share of the declining labor income is needed for consumption smoothing. In period $t = 8$ and $t = 9$, labor income has dropped so much that the taxable savings need to be consumed to sustain the desired level of consumption and smooth consumption over the life cycle.

Overall, our illustrative example in Fig. 2 shows that a constant contribution rate to a tax–deferred retirement account can be far from optimal—both for a decent growth rate of labor income ($g = 1.05$) as well as a slightly declining one ($g = 0.97$).

¹⁰ We also explored other levels of the tax rate. Given that changes to the tax rate only affect our results quantitatively, but not qualitatively, we do not report these different cases here. These results are, however, available from the authors upon request.

¹¹ The case characterized by item 2a in Theorem 2 is the limiting case, in which it is optimal to still build up tax–deferred savings at time $t = 8$ and to consume the entire income at time $t = 9$, thus avoiding the build–up of taxable savings. It is attained for $g = 0.9815$ (not depicted graphically in Fig. 2, figure is available upon request).

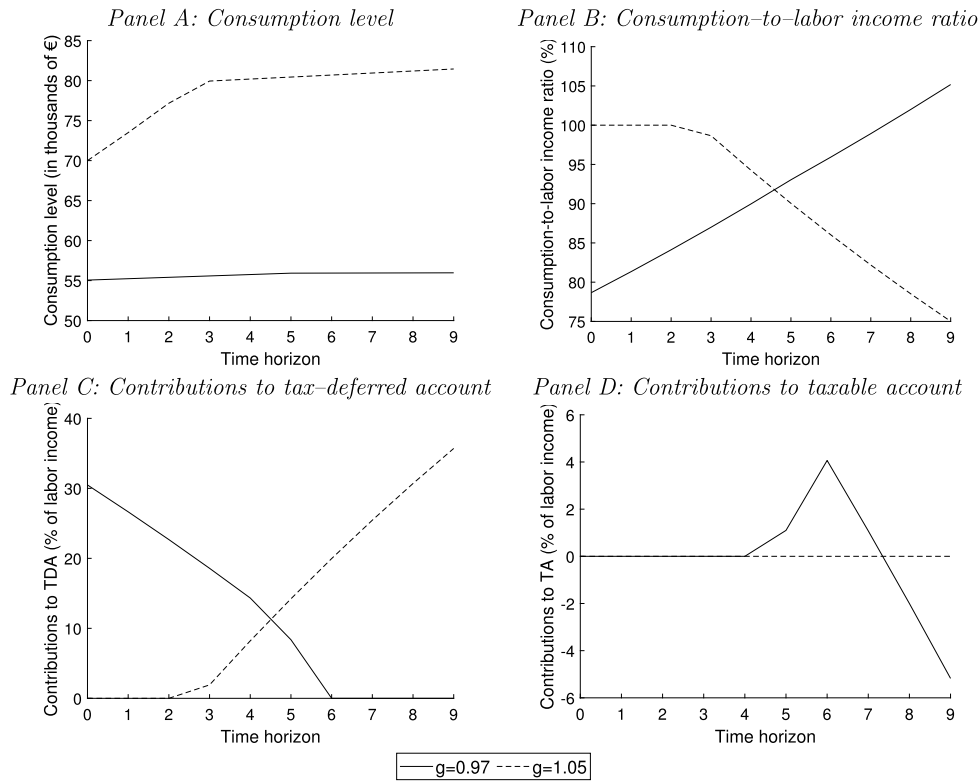


Fig. 2. Consumption–savings decisions in the model with a tax–deferred retirement account for various growth rates of labor income. This figure depicts the results for investors having access to a tax–deferred retirement account, i.e., results for the model presented in Section 4. Panel A depicts the evolution of the consumption level over the time horizon $T = 10$ and Panel B of the consumption–to–labor income ratio. Panel C depicts the evolution of contributions to the tax–deferred retirement account in percent of labor income received in the period over the time horizon $T = 10$ and Panel D the evolution of contributions to the taxable account in percent of labor income received in the period. In all panels we compare the results for two different growth rates of labor income: (i) $g = 0.97$ (solid line) and (ii) $g = 1.05$ (dashed line). The other parameters are chosen as follows: $L_0 = 100$, $K = 2$, $\gamma = 3$, $\rho = 0.98$, $r = 3\%$, $\tau = 30\%$.

5. Generalizations

Throughout we have assumed an identical tax rate τ for labor income and interest earnings. We have also assumed zero initial wealth, i.e., that the individual does not have any pre–existing savings, and a uniform growth rate of labor income. These assumptions can be relaxed without the loss of the existence of a closed–form solution as we outline in this section.

5.1. Different tax rates

In this subsection, we ask how allowing for a different tax rate, τ_l , on labor income, affects our closed form solution. A different tax rate on labor income affects the amount of labor income that is available to be consumed and invested after taxes. It does, however, not alter the growth rate of the labor income stream. Hence, by replacing the available labor income after taxes of $L_0(1 - \tau)$ in equation (2) and subsequently by its adjusted after–tax version

$$L_0(1 - \tau_l), \tag{42}$$

our closed form solution remains otherwise unaltered. That is, when allowing for a different tax rate on labor income, we can still provide a closed form solution.

5.2. Initial wealth

The case with pre–existing initial wealth, $W_{0-} > 0$ is more challenging to generalize than the case with different tax rates on labor and other income. This reflects that pre–existing initial wealth may affect whether, and if so, when, the budget constraint becomes binding. Yet, we are still able to derive a generalized analytical solution for the optimal consumption–savings profile.

For the simple unconstrained model from Section 2 we must alter the initial budget constraint $W_0 = L_0(1 - \tau) - C_0 + W_{0-}$ in equation (9) and have W_{0-} on the left hand side instead of 0. The analytical solution then becomes:

$$\begin{aligned}
 C_t &= C_0 \varepsilon^t = \frac{L_0(1-\tau) \sum_{s=0}^{T-1} y^s + W_{0-}}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} \varepsilon^t \\
 &= \left(L_0(1-\tau) + \frac{W_{0-}}{\sum_{s=0}^{T-1} y^s} \right) \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} \varepsilon^t \tag{43}
 \end{aligned}$$

$$\begin{aligned}
 &= \left(L_0 + \frac{W_{0-}}{(1-\tau) \sum_{s=0}^{T-1} y^s} \right) (1-\tau) \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} \varepsilon^t \\
 \frac{C_t}{L_t(1-\tau)} &= \left(1 + \frac{W_{0-}}{L_0(1-\tau) \sum_{s=0}^{T-1} y^s} \right) \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} \left(\frac{\varepsilon}{g} \right)^t. \tag{44}
 \end{aligned}$$

The effect of a pre-existing financial wealth is tantamount to lifting the entire profile of before-tax labor income by the factor $\left(1 + \frac{W_{0-}}{L_0(1-\tau) \sum_{s=0}^{T-1} y^s} \right)$. Hence, the optimal solution is identical to the one obtained with the corresponding higher level of labor income. However, it will decrease the savings rate, when consumption is measured against labor income.

A useful interpretation of this is that the initial wealth W_{0-} is converted to a growing annuity with the same present value that is added to the wage earnings:

$$\frac{W_{0-}}{\sum_{s=0}^{T-1} y^s} [1, g, g^2, \dots, g^{T-1}]. \tag{45}$$

In this way, the property of a constant growth rate is preserved. In this simple case, when the individual does not face any constraints, the consumption-constraint can (by assumption) not become binding, implying that a positive amount of initial wealth is economically equivalent to the payment stream from a growing annuity.

In the presence of possibly binding budget constraints, an amount of initial wealth may relax these constraints. Hence, we should expect initially pre-existing wealth to have a second effect on our closed-form solution through this channel. More technically, it turns out that the optimal solutions for the situations in Theorems 1 and 2 with a positive initial wealth follow the same reasoning as in the unrestricted model. The initial wealth W_{0-} is converted to a growing annuity with the same growth factor as labor income; i.e., instead of $L_t = L_0 g^t$ we use the optimal solution for the situation with labor income

$$L_t(1-\tau) + \frac{W_{0-} g^t}{\sum_{s=0}^{T-1} y^s} = \left(L_0(1-\tau) + \frac{W_{0-}}{\sum_{s=0}^{T-1} y^s} \right) g^t. \tag{46}$$

The initial wealth is kept in the taxable account and amortized accordingly, but it must be separated from other savings in this account as it may have an impact on whether the borrowing constraint binds or not. This means that the borrowing constraint must be reinterpreted; it is binding, when the unrestricted solution calls for a level of consumption in excess of $\left(L_0(1-\tau) + \frac{W_{0-}}{\sum_{s=0}^{T-1} y^s} \right) g^t$. The analogue of the expression in (12), i.e. the criterion for whether borrowing at time 0 takes place or not, and consequently whether the unrestricted solution is feasible also under the no borrowing constraint, is expressed in the following inequality, similar to the condition expressed in (17):

$$\left(L_0(1-\tau) + \frac{W_{0-}}{\sum_{s=0}^{T-1} y^s} \right) \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{1/\gamma} x^T} < \left(L_0(1-\tau) + \frac{W_{0-}}{\sum_{s=0}^{T-1} y^s} \right) \Leftrightarrow \tag{47}$$

$$\frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{1/\gamma} x^T} < 1. \tag{48}$$

When the inequality (48) no longer holds for a sufficiently high growth rate of labor income, we will be in a situation where $L_0(1-\tau) + \frac{W_{0-}}{\sum_{s=0}^{T-1} y^s}$ is consumed at time 0. This is analogous to the analysis in Section 3, and the proof follows the exact same line of reasoning as the proof in Online Appendix A. The same holds for the situation with a tax-deferred account in Theorem 2.

5.3. Time varying growth rates of labor income

The assumption of a constant growth rate of labor income can also be generalized to a situation with a time-varying, though still deterministic, growth rate of labor income without affecting the existence of closed-form solution. Assume that the growth rate at time v is g_v instead of just the constant g . The only change to the expression in equation (10) is found in the following expression with $g_0 = 1$ for notational convenience:

$$C_t = C_0 \varepsilon^t = L_0(1-\tau) \frac{1 + \sum_{s=1}^{T-1} (1+r(1-\tau))^{-s} \prod_{v=0}^s g_v}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} \varepsilon^t \tag{49}$$

$$\frac{C_t}{L_t(1-\tau)} = \frac{C_0}{L_0} \frac{\varepsilon^t}{\prod_{v=0}^t g_v}, \tag{50}$$

and a similar generalization of the growing annuity in equation (45). This reinterpretation of the constant parameter y to become time-varying also goes for the analogous expressions in Theorems 1 and 2.

However, for our proofs to still hold, we have to impose restrictions on the variability of the time-dependent growth rate of labor income, g_v . For example, if it has to hold that $(\varepsilon_1/g) > 1$, the variability of g_v must be restricted to situations where $\varepsilon_1^t / \prod_{v=0}^t g_v$ is increasing and larger than one. These restrictions on g_v apply whenever the monotonicity of $\varepsilon_1^t / \prod_{v=0}^t g_v$ is important for the validity of the proof.

In total, our results in this section show that our model can be generalized to cases with different tax rates on labor income and interest earnings, positive initial wealth, and time-varying growth rates of labor income.

6. Conclusion

We provide analytical solutions to variants of a classical intertemporal consumption-savings problem in a deterministic setting with a time-additive CRRA utility function. We assume a constant growth rate in labor income and solve for

- (1) the unrestricted problem, for which it is possible to borrow against future labor income,
- (2) the restricted problem with non-negative wealth constraints added, and
- (3) the restricted problem with non-negative wealth constraints and the possibility to save in a tax-deferred retirement account.

The benefit of assuming a deterministic setting is that we provide analytical solutions to all cases. Moving from (1) to (2), we show that the non-negativity constraint can be severely binding and that saving for retirement is optimally postponed when the labor income process has a sufficiently high growth rate. Moving from (2) to (3), we show that the tax advantage of saving in a tax-deferred retirement account implies that saving in this account precedes any saving in the taxable account, where the latter is based on a consumption smoothing effect. We also demonstrate that the optimal retirement savings profile is far from the widespread practice of allocating a fixed fraction of labor income to a tax-deferred retirement account during the working life.

Declaration of competing interest

No competing interest.

Data availability

No data was used for the research described in the article.

Appendix A. Supplementary material

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.insmatheco.2023.05.010>.

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