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Extremal clustering and cluster counting for spatial random fields

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We consider a stationary random field indexed by an increasing sequence of subsets of $\mathbb{Z}^d$ obeying a very broad geometrical assumption on how the sequence expands. Under certain mixing and local conditions, we show how the tail distribution of the individual variables relates to the tail behavior of the maximum of the field over the index sets in the limit as the index sets expand.

In a framework where we let the increasing index sets be scalar multiplications of a fixed set C, potentially with different scalars in different directions, we use two cluster definitions to define associated cluster counting point processes on the rescaled index set C; one cluster definition divides the index set into more and more boxes and counts a box as a cluster if it contains an extremal observation. The other cluster definition that is more intuitive considers extremal points to be in the same cluster, if they are close in distance. We show that both cluster point processes converge to a Poisson point process on C. Additionally, we find a limit of the mean cluster size. Finally, we pay special attention to the case without clusters.

Keywords: Extreme value theory; spatial models; random fields; intrinsic volumes; extremal index; cluster counting process; limit theorems

1. Introduction

This paper will provide a multitude of results within the realm of spatial extreme value theory, some of which will be generalizations of one-dimensional results or results from a simpler spatial framework. More precisely, we consider a stationary field $(\xi_v)_{v\in\mathbb{Z}^d}$ where $d \in \mathbb{N}$, for which we consider the extremal behavior of $(\xi_v)_{v\in D_n}$ under a very rich asymptotic regime of increasing index sets $D_n \subseteq \mathbb{Z}^d$. For detailed treatments of classical extreme value theory and its generalizations to stationary sequences in the one-dimensional case, we refer to e.g. [2] and [8].

In the literature, results for spatial objects indexed by a lattice, comparable to some of the present results, are to the best of the authors' knowledge only formulated under the assumption of $(D_n)_{n\in\mathbb{N}}$ being a sequence of increasing boxes; see for instance [5,9,11,16] that provides results like the ones found in Section 3 below under this additional geometrical assumption. In contrast, we allow the index sets D_n to expand in a much more general way; we refer to the authors' papers [13,17,18] for similar however slightly less general assumptions on the sequence of index sets, but with other and less general results. Our requirement will simply be that each D_n is the lattice points of a sufficiently nice full-dimensional subset of $\mathbb{R}^d$. To be precise, we assume that D_n is given as $D_n = C_n \cap \mathbb{Z}^d$, where C_n is a finite union of convex bodies contained in an expanding box. This essentially means that all of the convex bodies in the union, when scaled coordinate-wise with the side lengths of the surrounding box, has bounded intrinsic volumes as $n \to \infty$; see [15, Chapter 4] for an introduction to convex geometry and in particular intrinsic volumes of convex bodies, and see Assumption 2.2 below for the formal description. If $d = 3$ and C_n itself is a convex body for all $n \in \mathbb{N}$, the assumption contains — but is indeed not limited to — the case where C_n expands at a similar pace in all directions, meaning equivalently that the mean width of C_n is asymptotically bounded by the cubic root of its volume, and that the surface area is asymptotically bounded by the cubic root of the squared volume.

A related but somewhat more general geometrical framework can be found in [12], where the index set is allowed to be any set of isolated points in $\mathbb{R}^2$. Here the geometrical constraints are instead implicitly given in the subsequent mixing and clustering conditions under which the extremal results, comparable to those in Section 3 below, are derived.

The first part of the paper, see Sections 3 and 4 below, concerns the asymptotic distribution function of $\max_{v\in D_n}\xi_v$ along some sequence x_n. Here we derive results already known in the literature, but under the more general asymptotic scenario for the sequence of index sets. Having these results established in the present geometrical setting plays an important role in the subsequent sections. More precisely, we first formulate a mixing condition ensuring appropriate asymptotic independence of the field, which implies the following representation as $n\to\infty$:

$$\mathbb{P}\big(\max_{v\in D_n}\xi_v\le x_n\big)=\exp\Big(-|D_n|\mathbb{P}\big(\max_{v\in A_0^n}\xi_v\le x_n<\xi_0\big)\Big)+o(1). \tag{1}$$

Here the set A_0^n is defined by

$$A_0^n=\Big\{v\in\Big(\bigtimes_{\ell=1}^{d}[-t_{n,\ell},t_{n,\ell}]\Big)\cap\mathbb{Z}^d\ :\ 0\prec v\Big\},$$

where $(t_{n,\ell})_{n\in\mathbb{N}}$ is an increasing sequence of integers for all $\ell=1,\dots,d$, $\preceq$ is a translation invariant total order on $\mathbb{Z}^d$ (as for instance the lexicographical order) and $z\prec v$ means $z\preceq v$ and $z\neq v$. A one-dimensional (i.e. $d=1$) counterpart of this result can be found in [10, Theorem 2.1], and in [16, Theorem 3.1] a result is obtained in the special case where (D_n) is an increasing sequence of boxes in $\mathbb{Z}^d$. Secondly, we formulate a local condition describing the clustering of exceedances of $(\xi_v)_{v\in\mathbb{Z}^d}$ over the threshold (x_n) in terms of an index $\theta\in(0,1]$; the mixing and local conditions are named $\mathcal{D}(x_n;K_n;\boldsymbol{c_n})$ and $\mathcal{D}^{\ell}(x_n;\boldsymbol{k_n};\theta)$, respectively (with K_n and $\boldsymbol{k_n}$ specified in the relevant section), and will be referred to as this in the present introduction. If e.g. $(\xi_v)_{v\in\mathbb{Z}^d}$ is m-dependent, meaning in particular that any two variables a distance more than m apart are independent, the local condition $\mathcal{D}^{\ell}(x_n;\boldsymbol{k_n};\theta)$ simply reads

$$\lim_{n\to\infty}\mathbb{P}\big(\max_{v\in A_0^{(m)}}\xi_v\le x_n\mid\xi_0>x_n\big)=\theta,$$

where the set $A_0^{(m)}=\{v\in[-m,m]^d\cap\mathbb{Z}^d\ :\ 0\prec v\}$ is a fixed subset of A_0^n from the representation (1). A simple example given in the paper is the stationary field defined by

$$\xi_v=\max_{z\in v+B}Y_z,\qquad v\in\mathbb{Z}^d,$$

where B is a finite subset of $\mathbb{Z}^d$ and the field $(Y_z)_{z\in\mathbb{Z}^d}$ consists of i.i.d. variables. In fact, the field is m-dependent for sufficiently large m and it satisfies the local condition with $\theta=1/|B|$, where $|B|$ denotes the number of points in B. Using the same cardinality notation as in the example, we show under the mixing and local conditions that

$$|D_n|\mathbb{P}(\xi_0>x_n)\to\tau\quad\text{if and only if}\quad\mathbb{P}\big(\max_{v\in D_n}\xi_v\le x_n\big)\to\exp(-\theta\tau) \tag{2}$$

as $n\to\infty$ for $\tau\in[0,\infty)$. This result constitutes a substantial generalization of [17, Theorem 5] that only considers the special case of $\theta=1$ — meaning that no clustering of exceedances occur — but under a similar, but less general, asymptotic expansion of the index sets. An immediate consequence of the considerations leading to the equivalence above, is the fact that (ξ_v) has extremal index $\theta \in (0,1]$ if and only if the local condition $\mathcal{D}^\ell(x_n;\boldsymbol{k_n};\theta)$ is satisfied (for $x_n = x_n(\tau)$ for all $\tau > 0$); we refer to Section 4 for the formal definition of an extremal index (with respect to D_n), which is in accordance with the definition given in e.g. [8] for $d = 1$. This result generalizes the claims in [1,16], dealing with the one-dimensional case and the case of index sets being boxes, respectively, to the present framework allowing for a much more involved asymptotic development of the index sets.

The remainder of the paper is devoted to some of the main contributions of the paper, where asymptotic representations of various forms of cluster and cluster counting processes for spatial random fields, satisfying mixing and local conditions as above, are derived. Specifically, we work under the additional assumption that

$$D_n = (\boldsymbol{c_n} C) \cap \mathbb{Z}^d \tag{3}$$

where $\boldsymbol{c_n}$ is a d-dimensional vector with each entrance tending to ∞, and $C \subseteq \mathbb{R}^d$ is a finite union of convex bodies, and standardized together with $(\boldsymbol{c_n})$ such that C has Lebesgue measure 1. Here, $\boldsymbol{c_n}C$ denotes the coordinate-wise multiplication of C by the entries of $\boldsymbol{c_n}$. This in particular ensures that our sufficient geometric assumption, Assumption 2.2, is satisfied. Furthermore we assume that there is a real sequence (x_n) such that the conditions $\mathcal{D}(x_n;K_n;\boldsymbol{c_n})$ and $\mathcal{D}^\ell(x_n;\boldsymbol{k_n};\theta)$ are satisfied with $\theta \in (0,1]$, and such that

$$|D_n|\mathbb{P}(\xi_0 > x_n) \to \tau \in (0,\infty)$$

as $n \to \infty$.

Now, we define two different cluster counting processes of exceedances over the threshold x_n. The first is a spatial version of the classical one-dimensional definition, e.g. seen in [4] and [7]: $\mathbb{Z}^d$ is divided into disjoint (increasing) boxes each potentially being counted as a cluster if there is an x_n-exceedance within the box. For $d = 1$ such a counting process is usually defined on $(0,1]$ via a rescaling of the indices, whereas our geometric construction in (3) allows us to define it on the general set C. The second cluster counting process is to the best of our knowledge previously unseen in the literature in relation to extreme value theory. This counting process is also defined on C but is based on the more intuitive definition that a cluster is formed by the indices of x_n-exceedances which are relatively close to other exceedances. Here, being close corresponds to being within the same box of a size similar to a rescaled version of the boxes from the first type of counting process. We show that both counting processes in fact converge in distribution towards a homogeneous Poisson process on C with intensity $\theta\tau$. A one-dimensional counterpart of the result for the first of the two cluster processes only is found in [8]. A related spatial result under different and much stronger conditions is found in [3] in the simpler spatial scenario, where the index sets are increasing boxes: It is shown that the original exceedance point process (before collecting them into clusters) converges to a compound Poisson point process.

Moreover, we show that the expected size of such clusters is asymptotically equal to $1/\theta$. This is in accordance with the typical interpretation of the extremal index as the reciprocal of the mean number of exceedances in a cluster, but it is indeed not a triviality for the second cluster counting process. That the cluster counting processes converge to a Poisson process in particular means independence between cluster positions in the limit. The extremal independence between clusters is further underlined in the additional result, where we demonstrate that the limiting mean cluster size is in fact independent of the total number of clusters in C.

Related to the two cluster counting processes defined on C, we also define two associated cluster counting processes defined on the original scale in $\mathbb{Z}^d$. Using the distributional convergence results above, we show that both of these original-scale counting processes satisfy the following:

If $(B_n^1),\dots,(B_n^G)$ are disjoint sequences of subsets of $D_n \subseteq \mathbb{Z}^d$ each satisfying the geometric Assumption 2.2, then the joint distribution of cluster counts of either type converges in distribution to $(L^1,\dots,L^G)$, which are independent random variables with each L^g being Poisson distributed with parameter $\theta\tau \lim_{n\to\infty}|B_n^g|/|D_n|$ (assuming that the limit exists). This result is far from being a trivial consequence of the convergence of the cluster point processes defined above, as the asymptotic behavior of the B_n^g sets can be considerably more complex than what is obtained by rescaling to original scale of subsets of the set C introduced in (3).

In the last part of the paper, we consider the non-clustering case of $\theta = 1$ and the usual point process of exceedances over the threshold x_n, now defined on the general set C. We show that this process and its associated original-scale process converges exactly as the cluster counting processes but with $\theta = 1$; see e.g. [8, Chapter 5] for the classical one-dimensional case.

The paper is organized as follows. In Section 2 we introduce the geometric structures applied in the paper. In Section 3 we introduce the mixing and local conditions, and furthermore show the representation (1) of the distribution function of $\max_{v\in D_n}\xi_v$. This then leads to the equivalence (2) and to the result on the existence of an extremal index in Section 4. Sections 5 and 6 are devoted to the convergence of cluster counting processes and the expected size of such clusters, respectively, and Section 7 contains convergence results for the cluster counting processes on the original scale $\mathbb{Z}^d$. Finally, Section 8 contains similar results to the ones in Sections 5 and 7 for the non-clustering case. Proofs that are either very technical or have similarities to proofs in the existing literature are deferred to the supplementary material [14].

2. Geometric assumption and preliminaries

As mentioned in the introduction, we consider a stationary random field $(\xi_v)_{v\in\mathbb{Z}^d}$ and a sequence of very flexible index sets $(D_n)_{n\in\mathbb{N}}$ with $D_n \subseteq \mathbb{Z}^d$. The main purpose of this section is to present the sufficient assumption on the expansion of the index sets, which is in fact formulated in terms of a continuous counterpart.

Before presenting the assumption, we mention the following notation used throughout the paper. We let $|\cdot|$ be a general size-measure understood as follows: $|v|$ is the Euclidean norm of a single one- or multidimensional point v, $|A|$ is Lebesgue measure of a full-dimensional set $A \subseteq \mathbb{R}^d$, and $|A|$ is the number of points in a discrete set $A \subseteq \mathbb{Z}^d$. For two sets $A, B \subseteq \mathbb{R}^d$, we define their Minkowski sum by $A \oplus B = \{a + b \mid a \in A, b \in B\}$, and we let $B(r) = \{u \in \mathbb{R}^d : |u| \le r\}$ be the closed ball in $\mathbb{R}^d$ centered at the origin $0 \in \mathbb{R}^d$ and with radius $r \ge 0$. Furthermore, we will use the following notation for a coordinate-wise scaling of a set $A \subseteq \mathbb{R}^d$: With $\boldsymbol{r}$ (in bold) denoting a vector of d elements $r_1,\dots,r_d$ we write

$$\boldsymbol{r}A = \{(r_1a_1,\dots,r_da_d) \in \mathbb{R}^d : (a_1,\dots,a_d) \in A\}.$$

That is, $\boldsymbol{r}A$ is a compact notation for the linear transformation of A induced by the diagonal matrix with entries $r_1,\dots,r_d$. In particular we obtain for any full-dimensional set A that $\boldsymbol{r}A$ has Lebesgue measure

$$|\boldsymbol{r}A| = |A|\prod_{\ell=1}^{d} r_\ell.$$

If $\boldsymbol{r}$ is a vector of d identical entries r, that is, if the scaling is the same in all directions, we write

$$\boldsymbol{r}A = rA = \{(ra_1,\dots,ra_d) \in \mathbb{R}^d : (a_1,\dots,a_d) \in A\}.$$

Lastly, for a vector $\boldsymbol{r}$ we write $\boldsymbol{r}^{-1}$ or $1/\boldsymbol{r}$ for the vector of elements $r_1^{-1},\dots,r_d^{-1}$.

First, we define a type of continuous set which will be the cornerstone of the index set assumption. In the following definition, a convex body in $\mathbb{R}^d$ is a compact and convex set with non-empty interior; see [15, Chapter 4].

Definition 2.1. A set $C \subseteq \mathbb{R}^d$ is said to be p-convex, if it has the form

$$C = \bigcup_{i=1}^{p} \overline{C}_i ,$$

where $\overline{C}_1, \dots, \overline{C}_p$ are convex bodies in $\mathbb{R}^d$.

The geometric requirement on the discrete index sets (D_n) is in fact given in terms of requirements on an associated p-convex set, which is assumed to exist. Essentially we require that the so-called intrinsic volumes of a down-scaled version of the convex bodies defining the p-convex set are bounded. If the p-convex set increases similarly in all directions, this simply means that the intrinsic volumes has to be comparable in order to certain powers of the volume of the set. The jth intrinsic volume $V_j(C)$ (for $j = 0, \dots, d$) describes the geometry of the convex body C, and additionally, the collection of intrinsic volumes constitutes an essential part in the famous Steiner formula from convex geometry. As indicated in the introduction, the intrinsic volumes have a direct geometrical meaning, among which we mention a few: $V_0(C) = 1$, $V_1(C)$ is proportional to the mean width, $V_{d-1}(C)$ is half the surface area, and $V_d(C) = |C|$ is the volume of C. Moreover, the intrinsic volumes satisfy some important properties including non-negativity, i.e. $V_j(C) \geq 0$, homogeneity, i.e. $V_j(\gamma C) = \gamma^j V_j(C)$ for all $\gamma > 0$, and monotonicity, i.e. $V_j(C) \leq V_j(D)$ for $C \subseteq D$.

We can now state the sufficient assumption on the index sets. Due to stationarity of the random fields considered in the paper, we can and do without loss of generality assume that $0 \in D_n$, even though this is not stated in the assumption. Note also that the assumption as such does not mention the intrinsic volumes but rather a surrounding box with volume proportional to the size of the index set. However, as described in the subsequent lemma, this is essentially equivalent to the requirement on the intrinsic volumes presented in (5). The assumption below is therefore slightly more general than that of e.g. [17] in which connectivity combined with the intrinsic volume requirements are assumed, and the same order of expansion in all directions is required.

Assumption 2.2. For the sequence $(D_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{Z}^d$ there exists a sequence $(C_n)_{n\in\mathbb{N}}$ of p-convex sets such that $D_n = C_n \cap \mathbb{Z}^d$, where

$$C_n = \bigcup_{i=1}^{p} C_{n,i}$$

and $|C_n| \to \infty$ as $n \to \infty$. Moreover, there is a sequence of d-dimensional vectors $(\boldsymbol{c_n})_{n\in\mathbb{N}}$ each with elements $0 < c_{n,1}, \dots, c_{n,d} < \infty$ such that all of the following is satisfied:

(i) $c_{n,\ell} \to \infty$ as $n \to \infty$ for all $\ell = 1, \dots, d$.
(ii) $c_{n,1} \cdots c_{n,d} \sim |C_n|$ as $n \to \infty$.
(iii) There exists $0 < c < \infty$ such that

$$C_n \subseteq \boldsymbol{c_n}[-c, c]^d \tag{4}$$

for all n.

In the remainder of the paper, when considering a sequence of sets satisfying Assumption 2.2, we will refer to the vectors $\boldsymbol{c_n}$ as scaling vectors.

Lemma 2.3. *Let C_n be a union of convex bodies as defined in Assumption 2.2, and let $(\boldsymbol{c_n})$ be scaling vectors satisfying Assumption 2.2 (i)–(ii). If (4) is satisfied then*

$$\sum_{i=1}^{p} V_j(\boldsymbol{c_n^{-1}} C_{n,i}) \quad \text{is bounded in } n \text{ for each } j = 1, \ldots, d-1. \tag{5}$$

If C_n is also connected then (4) and (5) are equivalent.

Proof. The first claim of the lemma follows simply be the monotonicity and homogeneity of the intrinsic volumes. The second claim follows from Theorem 2.4(iii) below. □

The next example illustrates how Assumption 2.2 looks like when C_n is connected and expands at the same pace in all directions. This assumption was the one used in the papers [13,17,18].

Example 1. Let $D_n = C_n \cap \mathbb{Z}^d$, where $C_n = \cup_{i=1}^{p} C_{n,i}$ is a p-convex set for all n. If C_n is connected and $\boldsymbol{c_n} = (|C_n|^{1/d}, \ldots, |C_n|^{1/d})$ for all n, then Assumption 2.2 is satisfied with this scaling vector $\boldsymbol{c_n}$ if and only if

$$\frac{\sum_{i=1}^{p} V_j(C_{n,i})}{|C_n|^{j/d}} \quad \text{is bounded in } n \text{ for each } j = 1, \ldots, d-1.$$

We approximate the index sets D_n by a union of certain increasing boxes J_z^n, $z \in \mathbb{Z}^d$, which are implicitly given in terms of an underlying d-dimensional vector $\boldsymbol{k_n} = (k_{n,1}, \ldots, k_{n,d})$ of elements tending to ∞ at sufficiently slow rates given later. The full approximation scheme is given after this paragraph. The entries of $\boldsymbol{k_n}$ determine the size of these boxes relative to the size of D_n in a certain direction. In particular, as it will be shown in Theorem 2.4(ii), we can approximate D_n by a union of approximately $k_n^* = \prod_{\ell=1}^{d} k_{n,\ell}$ such boxes, with the approximation improving for increasing n. In the remainder of the paper, we will employ the notation $k_n^* = \prod k_{n,\ell}$. The approximation scheme is very similar to that of [17], although in that paper $\boldsymbol{k_n} = (k, \ldots, k)$ is the same in each direction and independent of n.

Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of sets satisfying Assumption 2.2, and let C_n be the associated p-convex sets, i.e. $D_n = C_n \cap \mathbb{Z}^d$ for all $n \in \mathbb{N}$. Furthermore, let $(\boldsymbol{c_n})$ be the sequence of d-dimensional scaling vectors appearing in the assumption, each with elements $c_{n,1}, \ldots, c_{n,d}$. Let $(\boldsymbol{k_n})$ be a sequence of vectors with elements $k_{n,1}, \ldots, k_{n,d} \in \mathbb{N}$ satisfying that $k_{n,\ell} \to \infty$ and $k_{n,\ell} = o(c_{n,\ell})$ for all $\ell = 1, \ldots, d$. Note that this will be implied by growth assumptions specified later. For each $n \in \mathbb{N}$ we define the integers

$$t_{n,\ell} = \lfloor c_{n,\ell}/k_{n,\ell} \rfloor, \qquad \text{for all } \ell = 1, \ldots, d,$$

and we collect these integers in the vector $\boldsymbol{t_n}$. This vector notation will be used throughout the remainder of the paper. For each $z = (z_1, \ldots, z_d) \in \mathbb{Z}^d$ we now define I_z^n to be the box with corner $\boldsymbol{t_n} z$ and side-lengths given by the entries of $\boldsymbol{t_n}$, i.e.

$$I_z^n = \boldsymbol{t_n}\big(z + [0,1)^d\big) = \bigtimes_{\ell=1}^{d} \big[z_\ell t_{n,\ell}, (z_\ell + 1) t_{n,\ell}\big).$$

The idea of the sets I_z^n is that they can be used to approximate the C_n-sets better and better by increasing n. Let P_n be the set of indices z for which I_z^n is contained in C_n, and let Q_n be the set of indices z for which I_z^n is intersected by C_n. That is,

$$P_n = \{z \in \mathbb{Z}^d \,:\, I_z^n \subseteq C_n\}, \quad \text{and} \quad Q_n = \{z \in \mathbb{Z}^d \,:\, I_z^n \cap C_n \neq \varnothing\}.$$

Moreover, we let $p_n = |P_n|$ and $q_n = |Q_n|$ be the size of those sets and note that, by construction,

$$\limsup_{n\to\infty} \frac{p_n}{k_n^*} \leq 1 \leq \liminf_{n\to\infty} \frac{q_n}{k_n^*}.$$

To approximate the index sets D_n we use the lattice points $J_z^n = I_z^n \cap \mathbb{Z}^d$ of I_z^n, and define

$$D_n^- = \bigcup_{z \in P_n} J_z^n \qquad \text{and} \qquad D_n^+ = \bigcup_{z \in Q_n} J_z^n. \tag{6}$$

Since $J_z^n \subseteq D_n$ for all $z \in P_n$, and $z \in Q_n$ for all J_z^n with $J_z^n \cap D_n \neq \varnothing$, we have the approximation

$$D_n^- \subseteq D_n \subseteq D_n^+. \tag{7}$$

As mentioned, the following theorem gives the quality of the approximation scheme, when approximating D_n using the sets J_z^n. The proof can be found in Section A of the supplementary material [14].

Theorem 2.4. *Let $(D_n)_{n\in\mathbb{N}}$ satisfy Assumption 2.2, and let C_n be the p-convex set associated with D_n by $D_n = C_n \cap \mathbb{Z}^d$. Furthermore, let $(\boldsymbol{c_n})$ be the sequence of d-dimensional scaling vectors each with elements $0 < c_{n,1}, \ldots, c_{n,d} < \infty$ introduced in Assumption 2.2. Let $(\boldsymbol{k_n})$ be a sequence of d-dimensional vectors with elements tending to infinity at a rate $k_{n,\ell} = o(c_{n,\ell})$ for each ℓ. Then*

(i) *$|D_n| \sim |C_n|$ as $n \to \infty$,*
(ii) *the sequences p_n and q_n, defined above, satisfy that*

$$\lim_{n\to\infty} \frac{p_n}{k_n^*} = \lim_{n\to\infty} \frac{q_n}{k_n^*} = 1,$$

(iii) *there is a set K_n, that can be chosen independently of the sequence of vectors $(\boldsymbol{k_n})$, defined by*

$$\begin{aligned} K_n &= \mathbb{Z}^d \cap \boldsymbol{c_n}[-c, c]^d \\ &= \mathbb{Z}^d \cap \bigtimes_{\ell=1}^{d} \left[-c \cdot c_{n,\ell}\,,\, c \cdot c_{n,\ell}\right] \end{aligned}$$

for some $0 < c < \infty$, such that $D_n^+ \subseteq K_n$ for all $n \in \mathbb{N}$.

If the sets (C_n) are all connected and satisfy (5) instead of (4), then Assumption 2.2 still holds and in particular (i)–(iii) *does so too.*

3. Behavior of the maximum

In this paper we consider a stationary random field $(\xi_v)_{v\in\mathbb{Z}^d}$ and a sequence of index sets $(D_n)_{n\in\mathbb{N}}$ with $D_n \subseteq \mathbb{Z}^d$ satisfying Assumption 2.2 above. Under the assumption of certain mixing and local conditions, we present results relating the distribution of the tail of the individual variables to the distribution of the maximum of the field over D_n. As mentioned in the previous section, the geometric approximation used in this paper is to some extend a generalization of that of [17] in that the approximation is constructed with n-dependent $\boldsymbol{k_n}$. Moreover, the conditions and associated results of this section naturally generalize those of [17] by allowing index sets to increase with a different pace in different directions and, perhaps more importantly, a certain degree of clustering of extremes.

Before presenting conditions and results, we introduce some relevant notation used throughout the paper. For a d-dimensional vector $\boldsymbol{\gamma}$, we say that two subsets A, B of $\mathbb{Z}^d$ are $\boldsymbol{\gamma}$-separated if $B \subseteq (A \oplus \boldsymbol{\gamma} B(1))^c$. If $\boldsymbol{\gamma} = (\gamma, \dots, \gamma)$ for some $0 < \gamma < \infty$, we say that the sets are γ-separated. Moreover, for two vectors, as for instance $\boldsymbol{\gamma_n} = (\gamma_{n,1}, \dots, \gamma_{n,d})$ and $\boldsymbol{c_n} = (c_{n,1}, \dots, c_{n,d})$ used below, we write $\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ if $\gamma_{n,\ell}/c_{n,\ell} \to 0$ for each $\ell = 1, \dots, d$. Lastly, for any $A \subseteq \mathbb{Z}^d$ we shorten $\max_{v \in A} \xi_v$ by $M_\xi(A)$.

In the remainder of the paper we consider index sets $(D_n)_{n \in \mathbb{N}}$ satisfying Assumption 2.2. In particular, Theorem 2.4 holds. In fact, the mixing condition $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$ given shortly is described in terms of the surrounding box $K_n \supseteq D_n^+$ that exists due to Theorem 2.4(iii).

Condition $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$. The condition $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$ is satisfied for the stationary field $(\xi_v)_{v \in \mathbb{Z}^d}$ if there exists an increasing sequence $(\boldsymbol{\gamma_n})$ of d-dimensional vectors such that: 1) $\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ as $n \to \infty$ and 2) for all $n \in \mathbb{N}$ and all $\boldsymbol{\gamma_n}$-separated sets $A, B \subseteq K_n$ where at least one is a box, it holds that

$$\left|\mathbb{P}(M_\xi(A \cup B) \le x_n) - \mathbb{P}(M_\xi(A) \le x_n)\mathbb{P}(M_\xi(B) \le x_n)\right| \le \alpha_n, \tag{8}$$

where $\alpha_n \to 0$ as $n \to \infty$.

Example 2. Suppose $(\xi_v)_{v \in \mathbb{Z}^d}$ is an m-dependent random field for some $m \in \mathbb{N}$: For all m-separated sets $A, B \subseteq \mathbb{Z}^d$ it holds that $(\xi_v)_{v \in A}$ and $(\xi_v)_{v \in B}$ are independent. If furthermore $(D_n)_{n \in \mathbb{N}}$ is a sequence of sets satisfying Assumption 2.2, then for any sequence (x_n) the condition $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$ is satisfied. The sequence $(\boldsymbol{\gamma_n})$ should satisfy $\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ as $n \to \infty$ and $\gamma_{n,\ell} \ge m$ eventually for each $\ell = 1, \dots, d$. Note that (α_n) can be chosen to be 0 eventually.

If $(\xi_v)_{v \in \mathbb{Z}^d}$ is an i.i.d. random field, it is well known that

$$|D_n|\mathbb{P}(\xi_0 > x_n) \to \tau \quad \text{if and only if} \quad \mathbb{P}(M_\xi(D_n) \le x_n) \to \exp(-\tau) \tag{9}$$

as $n \to \infty$; see e.g. [8, Theorem 1.5.1.]. However, as the following example illustrates, the approximate independence implied by the mixing condition $\mathcal{D}$ is not enough to guarantee the equivalence (9) to hold true for dependent fields. The example is a spatial generalization of the classical example of a one-dimensional process with local maximum occurring in clusters of a given size; see e.g. [2, Example 4.4.2].

Example 3. Let $(D_n)_{n \in \mathbb{N}}$ be a sequence of sets satisfying Assumption 2.2, and let F be a distribution function chosen such that

$$|D_n|\overline{F}(x_n) = |D_n|(1 - F(x_n)) \to \tau$$

as $n \to \infty$ for some sequence (x_n) and some $\tau \in (0, \infty)$. Let $B \subseteq \mathbb{Z}^d$ be a finite set with $|B| \ge 1$, and let $(Y_z)_{z \in \mathbb{Z}^d}$ be a field of i.i.d. random variables with common distribution function $F^{1/|B|}$. Define the stationary field $(\xi_v)_{v \in \mathbb{Z}^d}$ by

$$\xi_v = \max_{z \in v + B} Y_z,$$

and note that ξ_v has distribution function F and that $(\xi_v)_{v\in\mathbb{Z}^d}$ is m-dependent for m large relative to the span of B. Thus condition $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$ is satisfied, cf. Example 2. From the convergence of the tail of F it is seen that

$$|D_n|\mathbb{P}(Y_0 > x_n) \to \tau/|B|, \tag{10}$$

which will be used shortly. Appealing to Lemma A.1 in the supplementary material [14] we have that the number of points $|D_n \oplus B|$ in the Minkowski sum $D_n \oplus B$ is asymptotically equivalent to $|D_n|$. For $n \to \infty$, we therefore conclude that

$$\begin{aligned}\mathbb{P}\big(\max_{v\in D_n} \xi_v \le x_n\big) &= \mathbb{P}\big(\max_{z\in D_n\oplus B} Y_v \le x_n\big)\\ &= \mathbb{P}\big(Y_0 \le x_n\big)^{|D_n\oplus B|}\\ &\sim \mathbb{P}\big(Y_0 \le x_n\big)^{|D_n|}\\ &\to \exp\big(-\tau/|B|\big),\end{aligned}$$

where the asymptotic equivalence and the convergence follow from (10) by standard arguments.

In fact, under the assumption of the mixing condition above, the distribution function of the maximum $M_\xi(D_n)$ is close to the k_n^*th power of the distribution function of the maximum taken over certain $\boldsymbol{\gamma_n}$-separated boxes. More precisely, the lemma below holds with subsets H_z^n of J_z^n defined by

$$H_z^n = \big\{u \in \mathbb{Z}^d \;:\; z_\ell t_{n,\ell} \le u_\ell \le (z_\ell + 1)t_{n,\ell} - 1 - \gamma_{n,\ell}, \text{ for all } \ell = 1, \dots, d\big\},$$

which are indeed $\boldsymbol{\gamma_n}$-separated for varying $z \in \mathbb{Z}^d$. The proof of the lemma is deferred to Section B of the supplementary material [14].

Lemma 3.1. *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of sets satisfying Assumption 2.2, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$ for some sequence $(x_n)_{n\in\mathbb{N}}$. If $(\boldsymbol{k_n})$ is a sequence of d-dimensional vectors with elements tending to infinity such that $k_n^*\alpha_n \to 0$ and $\boldsymbol{k_n}\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ as $n \to \infty$, then*

$$\mathbb{P}\big(M_\xi(D_n) \le x_n\big) \le \mathbb{P}^{k_n^*}\big(M_\xi(H_0^n) \le x_n\big) + o(1) \tag{11}$$

as $n \to \infty$. If furthermore

$$\limsup_{n\to\infty} |D_n|\mathbb{P}(\xi_0 > x_n) < \infty, \tag{12}$$

then

$$\begin{aligned}\mathbb{P}\big(M_\xi(D_n) \le x_n\big) &= \mathbb{P}^{k_n^*}\big(M_\xi(H_0^n) \le x_n\big) + o(1)\\ &= \mathbb{P}^{k_n^*}\big(M_\xi(J_0^n) \le x_n\big) + o(1)\end{aligned} \tag{13}$$

as $n \to \infty$.

Before proceeding, we give a brief remark on the equality (13). In the one-dimensional case, it is stated in [7] that the equality holds even without the tail assumption (12). This is however not the case unless stronger assumptions on the increase rate of $\boldsymbol{k_n}$ are made.

Now let $\preceq$ be an arbitrary translation invariant total order on $\mathbb{Z}^d$ and define

$$A_v^n = \{z \in v + (\boldsymbol{t_n}\,[-1,1]^d \cap \mathbb{Z}^d) \;:\; v \prec z\},$$

which has varying side-length given by the d-dimensional vector $\boldsymbol{t_n}$. The following lemma, which generalizes [16, Theorem 3.1] to the flexible geometric set-up of the present paper, shows that the distribution of the maximum $M_\xi(D_n)$ is essentially determined by the simultaneous distribution of (ξ_v) for $v \in 0 \cup A_0^n$. As it follows along the lines of [16, Theorem 3.1], the proof can be found in Section B of the supplementary material [14].

Lemma 3.2. *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of sets satisfying Assumption 2.2, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$ for some sequence $(x_n)_{n\in\mathbb{N}}$. If $(\boldsymbol{k_n})$ is a sequence of d-dimensional vectors with elements tending to infinity such that $k_n^*\alpha_n \to 0$ and $\boldsymbol{k_n}\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ as $n \to \infty$, then*

$$\mathbb{P}\big(M_\xi(D_n) \le x_n\big) \le \exp\Big(-|D_n|\mathbb{P}\big(M_\xi(A_0^n) \le x_n < \xi_0\big)\Big) + o(1) \tag{14}$$

as $n \to \infty$. If furthermore (12) is satisfied then

$$\mathbb{P}\big(M_\xi(D_n) \le x_n\big) = \exp\Big(-|D_n|\mathbb{P}\big(M_\xi(A_0^n) \le x_n < \xi_0\big)\Big) + o(1) \tag{15}$$

as $n \to \infty$.

Appealing to (13), the corollary below, which will be used multiple times in later sections of the paper, is a consequence of Lemma 3.2.

Corollary 3.3. *Let the assumptions of Lemma 3.2 including (12) be satisfied. Then*

$$k_n^*\mathbb{P}\big(M_\xi(J_0^n) > x_n\big) = |D_n|\mathbb{P}\big(M_\xi(A_0^n) \le x_n < \xi_0\big) + o(1) \tag{16}$$

With the results above in mind, it is natural to consider the local condition $\mathcal{D}^\ell$ below. The condition, that introduces potential clustering of exceedances of x_n, is defined for $\theta \in [0,1]$.

Condition $\mathcal{D}^\ell(x_n; \boldsymbol{k_n}; \theta)$. The condition $\mathcal{D}^\ell(x_n; \boldsymbol{k_n}; \theta)$ is satisfied for the stationary field $(\xi_v)_{v\in\mathbb{Z}^d}$ if

$$\mathbb{P}\big(M_\xi(A_0^n) \le x_n \mid \xi_0 > x_n\big) \to \theta \tag{17}$$

as $n \to \infty$.

The following result is a generalization of [17, Theorem 5] allowing clustering of extremes in the sense of (17), and allowing the more general expansion of $(D_n)_{n\in\mathbb{N}}$ as assumed in Assumption 2.2.

Theorem 3.4. *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of sets satisfying Assumption 2.2, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$ for some sequence $(x_n)_{n\in\mathbb{N}}$. Moreover, let $(\boldsymbol{k_n})$ be a sequence of d-dimensional vectors with elements tending to infinity satisfying $k_n^*\alpha_n \to 0$ and $\boldsymbol{k_n}\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ as $n \to \infty$, such that $\mathcal{D}^\ell(x_n; \boldsymbol{k_n}; \theta)$ is satisfied for some $\theta \in (0,1]$. Then, for all $0 \le \tau < \infty$,*

$$|D_n|\mathbb{P}(\xi_0 > x_n) \to \tau$$

if and only if

$$\mathbb{P}\big(\max_{v\in D_n} \xi_v \le x_n\big) \to \exp(-\theta\tau)$$

as $n \to \infty$.

Proof. Since $\tau < \infty$ the first implication is an immediate consequence of (15). Assuming conversely that $\mathbb{P}(M_\xi(D_n) \le x_n) \to \exp(-\theta\tau)$, an application of (14) combined with the fact that $\theta > 0$ implies (12). The implication now follows from the equality (15). □

For certain fields, clustering of extremes is not obtained in a slowly increasing neighborhood as in condition $\mathcal{D}^\ell(x_n; \boldsymbol{k_n}; \theta)$ but rather in a fixed neighborhood of a given size. In the literature, a range of anti-clustering conditions are found giving rise to such fixed neighborhood clustering. Below we include a condition from [16] which is a multi-dimensional counterpart of the condition $D^{(m+1)}(x_n)$ from [1] and which is satisfied by e.g. m-dependent fields under the assumption of (12); see [16, Section 5.1]. The condition essentially says, that any potential clustering in the sense of (17) is limited to the fixed subset

$$A_v^{(m)} = \{z \in v + ([-m,m]^d \cap \mathbb{Z}^d) : v \prec z\}$$

of A_0^n, where m is a non-negative integer.

Condition $\mathcal{D}^{(m)}(x_n; \boldsymbol{k_n})$. Let m be a non-negative integer. The condition $\mathcal{D}^{(m)}(x_n; \boldsymbol{k_n})$ is satisfied for the stationary field $(\xi_v)_{v\in\mathbb{Z}^d}$ if

$$|D_n|\,\mathbb{P}\Big(M_\xi(A_0^{(m)}) \le x_n < \xi_0\,,\, M_\xi(A_0^n \setminus A_0^{(m)}) > x_n\Big) \to 0 \tag{18}$$

as $n \to \infty$, with the convention that $M_\xi(A_0^{(0)}) = M_\xi(\varnothing) = -\infty$.

As an obvious consequence of Lemma 3.2 we obtain:

Corollary 3.5. *Let the assumptions of Lemma 3.2 including (12) be satisfied. Let m be a non-negative integer. Then, as $n \to \infty$,*

$$\mathbb{P}\big(M_\xi(D_n) \le x_n\big) = \exp\Big(-|D_n|\mathbb{P}\big(M_\xi(A_0^{(m)}) \le x_n < \xi_0\big)\Big) + o(1)$$

if and only if condition $\mathcal{D}^{(m)}(x_n; \boldsymbol{k_n})$ is satisfied.

Obviously, the condition (18) is satisfied if

$$|D_n| \sum_{v\in A_0^n\setminus A_0^{(m)}} \mathbb{P}\Big(M_\xi(A_0^{(m)}) \le x_n < \xi_0\,,\, \xi_v > x_n\Big) \to 0. \tag{19}$$

In fact, setting $m = 0$ in (19) corresponds to the anti-clustering condition $\mathcal{D}'(x_n)$ of [17] (which however, was formulated slightly incorrect) now with direction- and n-dependent $\boldsymbol{k_n}$.

Clearly, under the assumption of the anti-clustering condition $\mathcal{D}^{(m)}(x_n; \boldsymbol{k_n})$, the local condition $\mathcal{D}^\ell$ is equivalent to the much simpler convergence

$$\lim_{n\to\infty} \mathbb{P}\big(M_\xi(A_0^{(m)}) \le x_n \mid \xi_0 > x_n\big) = \theta, \tag{20}$$

at least if $\liminf_{n\to\infty}|D_n|\mathbb{P}(\xi_0 > x_n) > 0$. The following lemma, which follows simply by rearranging the probabilities, gives this and another relation between the conditions which are used throughout the paper.

Lemma 3.6. *Assume that* $\liminf_{n\to\infty}|D_n|\mathbb{P}(\xi_0 > x_n) > 0$. *Then the relations below hold:*

(i) *If* $\mathcal{D}^{(m)}(x_n;\boldsymbol{k_n})$ *is satisfied for some* $m \in \mathbb{N}_0$, *then* $\mathcal{D}^{\ell}(x_n;\boldsymbol{k_n};\theta)$ *is equivalent to* (20) *for this* m.

(ii) *Assume that* $\limsup_{n\to\infty}|D_n|\mathbb{P}(\xi_0 > x_n) < \infty$. *If* $\mathcal{D}^{\ell}(x_n;\boldsymbol{k_n};1)$ *is satisfied, then* $\mathcal{D}^{(m)}(x_n;\boldsymbol{k_n})$ *and* (20) *hold for* $\theta = 1$ *and all* $m \in \mathbb{N}_0$. *In particular,* $\mathcal{D}^{\ell}(x_n;\boldsymbol{k_n};1)$ *is satisfied if and only if* $\mathcal{D}^{(m)}(x_n;\boldsymbol{k_n})$ *and* (20) *hold for* $\theta = 1$ *and some* $m \in \mathbb{N}_0$, *which is satisfied if and only if* $\mathcal{D}^{(m)}(x_n;\boldsymbol{k_n})$ *and* (20) *hold for* $\theta = 1$ *and all* $m \in \mathbb{N}_0$.

The corollary to Theorem 3.4 given below does in fact not follow directly by an application of Lemma 3.6, however, it follows by almost identical arguments as Theorem 3.4 utilizing Corollary 3.5.

Corollary 3.7. *Let* $(D_n)_{n\in\mathbb{N}}$ *be a sequence of sets satisfying Assumption 2.2, and let* $(\xi_v)_{v\in\mathbb{Z}^d}$ *be a stationary field satisfying* $\mathcal{D}(x_n;K_n;\boldsymbol{c_n})$ *for some sequence* $(x_n)_{n\in\mathbb{N}}$. *Moreover, let* $(\boldsymbol{k_n})$ *be a sequence of* d*-dimensional vectors with elements tending to infinity satisfying* $k_n^*\alpha_n \to 0$ *and* $\boldsymbol{k_n}\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ *as* $n \to \infty$, *such that* $\mathcal{D}^{(m)}(x_n;\boldsymbol{k_n})$ *is satisfied for some* $m \in \mathbb{N}_0$. *Assume furthermore that* (20) *is satisfied for such* m *and for some* $\theta \in (0,1]$. *Then, for all* $0 \le \tau < \infty$,

$$|D_n|\mathbb{P}(\xi_0 > x_n) \to \tau$$

if and only if

$$\mathbb{P}\big(\max_{v\in D_n} \xi_v \le x_n\big) \to \exp(-\theta\tau)$$

as $n \to \infty$.

Remark. Note that a field satisfying the conditions of the Corollary with $m = 0$, but with $\mathcal{D}^{(m)}(x_n;\boldsymbol{k_n})$ replaced by the stronger requirement (19), necessarily satisfies (20) with $\theta = 1$. Thus, in this case, the result simplifies to Theorem 5 of [17].

Example 3 (Continued). Recall the setup of Example 3. We concluded that one implication of the corollary was true with $\theta = 1/|B|$. As we will see now, under the initial assumption that (x_n) is chosen such that $\limsup_{n\to\infty}|D_n|\mathbb{P}(\xi_0 > x_n) < \infty$, the reverse implication is in fact also true since the anti-clustering condition $\mathcal{D}^{(m)}(x_n;\boldsymbol{k_n})$ is satisfied for sufficiently large m, and since (20) is satisfied with $\theta = 1/|B|$: As already mentioned, $(\xi_v)_{v\in\mathbb{Z}^d}$ is m-dependent for m sufficiently large. Therefore, $D^{(m)}(x_n;\boldsymbol{k_n})$ is satisfied for any vector sequence $(\boldsymbol{k_n})$ with the proper growth rates; see the comment prior to condition $\mathcal{D}^{(m)}(x_n;\boldsymbol{k_n})$. Now fix such an m. Then,

$$\begin{aligned}
&\mathbb{P}\big(M_\xi(A_0^{(m)}) \le x_n < \xi_0\big)\\
&\quad = \mathbb{P}\big(M_\xi(A_0^{(m)}) \le x_n\big) - \mathbb{P}\big(M_\xi(A_0^{(m)}) \le x_n, \xi_0 \le x_n\big)\\
&\quad = \mathbb{P}\big(Y_z \le x_n \text{ for all } z \in A_0^{(m)} \oplus B\big) - \mathbb{P}\big(Y_z \le x_n \text{ for all } z \in (A_0^{(m)} \cup \{0\}) \oplus B\big).
\end{aligned}$$

Let the elements of B be ordered as follows (relative to the underlying translation invariant order $\preceq$):

$$b_1 \prec b_2 \prec \cdots \prec b_{|B|}.$$

Then, utilizing the fact that $b_j - b_i \in A_0^{(m)}$ for all $i < j$ due to the choice of m, we see that

$$(A_0^{(m)} \cup \{0\}) \oplus B = (A_0^{(m)} \oplus B) \cup \{b_1\},$$

with $b_1 \notin A_0^{(m)} \oplus B$. Consequently, and using the i.i.d. structure of the random variables $(Y_z)_{z\in\mathbb{Z}^d}$, the probability above simplifies to

$$\mathbb{P}\big(M_\xi(A_0^{(m)}) \le x_n < \xi_0\big) = \mathbb{P}(Y_0 \le x_n)^{|A_0^{(m)}\oplus B|}\,\mathbb{P}(Y_0 > x_n).$$

Since $A_0^{(m)} \oplus B$ is a finite set independent of n, the first factor of the product converges to 1 as $n \to \infty$. Realizing secondly that $\mathbb{P}(Y_0 > x_n)$ is asymptotically equivalent to $\mathbb{P}(\xi_0 > x_n)/|B|$ shows that (20) is satisfied with $\theta = 1/|B|$.

4. The extremal index

In this section we utilize results from the previous section in order to relate local conditions such as $\mathcal{D}^\ell$ to the existence of the so-called extremal index. Essentially, the field has extremal index θ if the two limits from Theorem 3.4 are satisfied for all $\tau > 0$: In accordance with the definition given in e.g. [2,8], we say that the stationary field $(\xi_v)_{v\in\mathbb{Z}^d}$ has extremal index $\theta \in [0,1]$ (with respect to (D_n)) if for each $\tau > 0$ there exists a sequence $x_n(\tau)$ such that

$$|D_n|\mathbb{P}(\xi_0 > x_n(\tau)) \to \tau \tag{21}$$

and

$$\mathbb{P}\big(M_\xi(D_n) \le x_n(\tau)\big) \to \exp(-\theta\tau).$$

as $n \to \infty$. Typically, one chooses the sequence $x_n(\tau)$ such that (21) is satisfied, which is possible if and only if

$$\frac{\mathbb{P}(\xi_0 > x)}{\lim_{y\uparrow x}\mathbb{P}(\xi_0 > y)} \to 1$$

as $x \to \infty$ (cf. [8, Theorem 1.7.13]) and which for instance is the case if the distribution of ξ_0 is in any of the three classical maximum domains of attraction. Moreover, if (21) is satisfied for some $\tau > 0$ it holds for all $\tau > 0$.

Theorem 4.1. *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of sets satisfying Assumption 2.2, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$ where for all $0 < \tau < \infty$ the sequence $x_n = x_n(\tau)$ is chosen such that $|D_n|\mathbb{P}(\xi_0 > x_n(\tau)) \to \tau$. Let $\alpha_n = \alpha_n(\tau)$ and $\gamma_n = \gamma_n(\tau)$ be the mixing constants from the condition. Then the field has extremal index $\theta \in [0,1]$ with respect to (D_n) if and only if there exists a sequence of vectors $(\boldsymbol{k_n}) = (\boldsymbol{k_n}(\tau))$ with elements tending to infinity satisfying $k_n^*\alpha_n \to 0$ and $\boldsymbol{k_n}\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ as $n \to \infty$, such that $\mathcal{D}^\ell(x_n(\tau); \boldsymbol{k_n}(\tau); \theta)$ is satisfied for all $\tau > 0$.*

Proof. The claim follows easily from Lemma 3.2 equation (15). □

Corollary 4.2. *Let the assumptions of Theorem 4.1 be satisfied. Assume furthermore that there exists a sequence of vectors $(\boldsymbol{k_n}) = (\boldsymbol{k_n}(\tau))$ with elements tending to infinity satisfying $k_n^*\alpha_n \to 0$ and $\boldsymbol{k_n}\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ as $n \to \infty$, such that the anti-clustering condition $\mathcal{D}^{(m)}(x_n(\tau); \boldsymbol{k_n}(\tau))$ holds for some $m \in \mathbb{N}_0$ and all $\tau > 0$. Then the field has extremal index $\theta \in [0,1]$ with respect to (D_n) if and only if*

$$\lim_{n\to\infty}\mathbb{P}\big(M_\xi(A_0^{(m)}) \le x_n(\tau) \mid \xi_0 > x_n(\tau)\big) = \theta$$

for all $\tau > 0$.

Proof. Applying Theorem 4.1 and Lemma 3.6(i) gives the result. □

Example 3 (Continued). The field $(\xi_v)_{v\in\mathbb{Z}^d}$ with $\xi_v = \max_{z\in v+B} Y_z$ introduced in Example 3 has extremal index $1/|B|$ with respect to (D_n).

Remark. Theorem 4.1 and Corollary 4.2 both suggest that the extremal index could be estimated by a properly defined runs estimator; see for instance [16, Remark 4.1] for an estimator when the index set is a d-dimensional box. However, as exemplified in Example 3 and shown in Theorem 6.1, the typical interpretation of the extremal index as the reciprocal mean cluster size is valid for either of the cluster-definitions given in the next section, and hence estimating the index as the number of clusters divided by the number of exceedances is also a natural approach.

5. Cluster counting processes

In the remainder of the paper we focus on sets D_n of the specific structure defined below. Moreover, we only consider choices of the corresponding sequence x_n such that $|D_n|\mathbb{P}(\xi_0 > x_n)$ converges to a finite, non-zero limit. In particular, this means that the local condition $\mathcal{D}^\ell(x_n;\boldsymbol{k_n};\theta)$ is implied by the anti-clustering condition $\mathcal{D}^{(m)}(x_n;\boldsymbol{k_n})$ in combination with (20) for some $\theta\in[0,1]$; see Lemma 3.6(i). Though only formulated under the assumption of $\mathcal{D}^\ell$, all results from Section 5 therefore remain true under the combined assumption of $\mathcal{D}^{(m)}$ and (20).

Now assume for some $p\in\mathbb{N}$ that C is a p-convex set. Then define the sequence $(C_n)_{n\in\mathbb{N}}$ as

$$C_n = \boldsymbol{c_n} C,$$

where $(\boldsymbol{c_n})_{n\in\mathbb{N}}$ is a sequence of d-dimensional vectors $\boldsymbol{c_n} = (c_{n,1},\dots,c_{n,d})$ such that $0 < c_{n,\ell}\to\infty$, and where we without loss of generality will assume that $|C| = 1$. We now construct the index sets

$$D_n = C_n \cap \mathbb{Z}^d$$

for all $n\in\mathbb{N}$, and note in particular that $(D_n)_{n\in\mathbb{N}}$ satisfies Assumption 2.2 with $(\boldsymbol{c_n})$ playing the role of the scaling vector from the assumption. Furthermore, (x_n) and $(\boldsymbol{k_n})$ will be sequences, such that (ξ_v), (D_n), (x_n) and $(\boldsymbol{k_n})$ jointly satisfy conditions specified in the relevant theorems.

As mentioned in the introduction we introduce two definitions of cluster counting processes, N_n and $\tilde{N}_n$, respectively, counting the number of clusters (of the relevant type) of indices, where ξ_v is above the level x_n within D_n. First we define the measure N_n, based on the grid formed by the J_z^n-sets, and show Theorem 5.1. Afterwards, we introduce the more intuitive cluster measure $\tilde{N}_n$ and show a similar convergence theorem for that.

In the definition of the measure N_n, we count the set J_z^n as a cluster, if $M_\xi(J_z^n) > x_n$. For each n we define the random measure N_n on C by counting the number of clusters in a scaled version of the index set as follows

$$N_n(A) = \sum_{z\in\mathbb{Z}^d} \mathbf{1}_A\Big(z\frac{\boldsymbol{t_n}}{\boldsymbol{c_n}}\Big)\mathbf{1}_{\{M_\xi(J_z^n)>x_n\}}$$

for all $A\in\mathcal{B}(C)$. Note that the cluster is counted as placed in the set A, if the down-scaled corner-point $z\boldsymbol{t_n}/\boldsymbol{c_n}$ of the set J_z^n is in A. We will later refer to such points, each representing a cluster, as *cluster points*. It should be emphasized that N_n depends crucially on the choice of the $\boldsymbol{k_n}$-sequence.

In Theorem 5.1 below we show that the sequence of random measures $(N_n)_{n\in\mathbb{N}}$ *converges in distribution with respect to the vague topology* towards a homogeneous Poisson measure N, denoted $N_n \xrightarrow{vd} N$. Details on this type of convergence can be found in [6, Chapter 4], and it is defined as

$$\int_C f \,\mathrm{d}N_n \to \int_C f \,\mathrm{d}N \qquad \text{for all } f \in \hat{C}_C,$$

where $\hat{C}_C$ denotes all bounded, continuous functions $f : C \to \mathbb{R}_+$ with compact support. In the present case, where the limiting measure is a homogeneous Poisson measure, the convergence can equivalently be formulated as

$$\big(N_n(A_1),\dots,N_n(A_K)\big) \xrightarrow{\mathcal{D}} \big(N(A_1),\dots,N(A_K)\big)$$

as $n \to \infty$ for all $K \in \mathbb{N}$ and all $A_1,\dots,A_K \in \mathcal{B}(C)$ with $|\partial A_k| = 0$ for $k = 1,\dots,K$. See [6, Theorem 4.11]. Here $\xrightarrow{\mathcal{D}}$ denotes weak convergence.

Theorem 5.1. *Let $(D_n)_{n\in\mathbb{N}}$ be defined as above, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$ for some sequence $(x_n)_{n\in\mathbb{N}}$. Moreover, let $(\boldsymbol{k_n})$ be a sequence of d-dimensional vectors with elements tending to infinity satisfying $k_n^*\alpha_n \to 0$ and $\boldsymbol{k_n}\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ as $n \to \infty$, such that $\mathcal{D}^\ell(x_n; \boldsymbol{k_n}; \theta)$ is satisfied for some $\theta \in (0,1]$. If for some $0 < \tau < \infty$,*

$$|D_n|\mathbb{P}(\xi_0 > x_n) \to \tau$$

then $N_n \xrightarrow{vd} N$, where N is a homogeneous Poisson measure on C with intensity $\theta\tau$.

Proof. According to [6, Theorem 4.18] it suffices to show that $\mathbb{E}N_n(A) \to \mathbb{E}N(A)$, where $A \subseteq C$ is a box on the form $A = \bigtimes_{j=1}^d (a_j, b_j]$, and that $\mathbb{P}(N_n(B) = 0) \to \mathbb{P}(N(B) = 0)$ for all B, where $B \subseteq C$ is a finite union of boxes each on the form $\bigtimes_{j=1}^d (a_j, b_j]$.

First we find that

$$\mathbb{E}N_n(A) = \mathbb{P}\big(M_\xi(J_0^n) > x_n\big) \sum_{z\in\mathbb{Z}^d} \mathbf{1}_A\Big(z\frac{\boldsymbol{t_n}}{\boldsymbol{c_n}}\Big).$$

Recalling that

$$t_{n,\ell} = \lfloor c_{n,\ell}/k_{n,\ell} \rfloor,$$

it is easily seen that

$$\sum_{z\in\mathbb{Z}^d} \mathbf{1}_A\Big(z\frac{\boldsymbol{t_n}}{\boldsymbol{c_n}}\Big) \sim k_n^*|A|$$

as $n \to \infty$. Combining this with Corollary 3.3 and (17) gives

$$\begin{aligned}
\mathbb{E}N_n(A) &\sim |A|k_n^*\mathbb{P}\big(M_\xi(J_0^n) > x_n\big) \\
&\sim |A||D_n|\mathbb{P}(\xi_0 > x_n)\mathbb{P}\big(M_\xi(A_0^n) \le x_n \mid \xi_0 > x_n\big) \\
&\sim |A|\theta\tau \\
&= \mathbb{E}N(A).
\end{aligned}$$

Now let $B \subseteq C$ be a finite union of boxes. Define $B_n = \boldsymbol{c_n} B$ and furthermore

$$B_n^- = \bigcup_{z : I_z \subseteq B_n} I_z^n, \qquad B_n^+ = \bigcup_{z : I_z \cap B_n \neq \varnothing} I_z^n,$$

such that $B_n^- \subseteq B_n \subseteq B_n^+$. By arguments as in Theorem 2.4, $|B_n^-|/|B_n| \to 1$ and $|B_n^+|/|B_n| \to 1$, respectively. Also note that both B_n^- and B_n^+ themselves will be unions of (at most) the same number of boxes as B. Then the sequences $(B_n^- \cap \mathbb{Z}^d)_{n\in\mathbb{N}}$ and $(B_n^+ \cap \mathbb{Z}^d)_{n\in\mathbb{N}}$ together with $(x_n)_{n\in\mathbb{N}}$ and $(\xi_v)_{v\in\mathbb{Z}^d}$ satisfy the conditions of Theorem 3.4 with

$$|B_n^- \cap \mathbb{Z}^d|\,\mathbb{P}(\xi_0 > x_n) \to |B|\tau \quad \text{and} \quad |B_n^+ \cap \mathbb{Z}^d|\,\mathbb{P}(\xi_0 > x_n) \to |B|\tau,$$

since $|B_n \cap \mathbb{Z}^d|/|D_n| \to |B|$. Then, by Theorem 3.4,

$$\mathbb{P}(M_\xi(B_n^- \cap \mathbb{Z}^d) \le x_n) \to \exp(-|B|\theta\tau) = \mathbb{P}(N(B) = 0),$$
$$\mathbb{P}(M_\xi(B_n^+ \cap \mathbb{Z}^d) \le x_n) \to \exp(-|B|\theta\tau) = \mathbb{P}(N(B) = 0).$$

The desired limit follows, since

$$\mathbb{P}(M_\xi(B_n^+ \cap \mathbb{Z}^d) \le x_n) \le \mathbb{P}(N_n(B) = 0) \le \mathbb{P}(M_\xi(B_n^- \cap \mathbb{Z}^d) \le x_n).$$

This concludes the proof. □

Next we define the alternative cluster counting process $\tilde{N}_n$ based on what intuitively is interpreted as clusters. For this we define the set

$$Q_n^- = \left\{ z \in \mathbb{Z}^d \ : \ z\frac{\boldsymbol{t_n}}{\boldsymbol{c_n}} \in C \right\}.$$

That is all indices z where J_z^n can be counted by N_n. Note that $P_n \subseteq Q_n^- \subseteq Q_n$. Furthermore, let $\tilde{D}_n$ be the union of the corresponding J_z^n-sets

$$\tilde{D}_n = \bigcup_{z\in Q_n^-} J_z^n.$$

Now we consider the rescaled set of indices

$$\Phi_n = \{ v/\boldsymbol{c_n} \ : \ v \in \tilde{D}_n\,,\, \xi_v > x_n \},$$

where the field is above x_n. Note that both of the measures N_n and $\tilde{N}_n$ are based on variables ξ_v in the same extended index set $\tilde{D}_n$. To define $\tilde{N}_n$, we divide Φ_n into a number of disjoint clusters of points such that different clusters are separated by a certain distance. More precisely, we say that $u, u' \in \Phi_n$ are in the same cluster, if there is a sequence of distinct elements $u = u_0, u_1 \dots, u_R = u'$ in that cluster such that

$$|u_{i,\ell} - u_{i-1,\ell}| \le \frac{1}{k_{n,\ell}}$$

for all $\ell = 1, \dots, d$ and $i = 1, \dots, R$. The distances $1/k_{n,\ell}$, $\ell = 1, \dots, d$ are asymptotically equivalent to the side lengths of the rescaled boxes $\boldsymbol{c_n^{-1}} J_z^n$, and therefore the clusters produced this way are somewhat comparable to those counted by N_n. The procedure uniquely divides Φ_n into X_n disjoint clusters, where $0 \le X_n \le |\Phi_n|$. If $X_n \ge 1$, let these clusters be denoted C_i^n for $i = 1, \dots, X_n$. The ordering is arbitrary and will not be relevant subsequently. For each cluster, we define the cluster point x_i^n, meaning the point in $\mathbb{R}^d$ that represents the cluster, as the point in C_i^n closest to the mean of the points in the cluster, i.e.

$$x_i^n = \arg\min_{x \in C_i^n} \sum_{x' \in C_i^n} |x - x'|^2.$$

In principle, we could have used any systematically chosen point from C_i^n as the cluster point representing the cluster — the above is just an intuitively natural choice. Alternatively, we could have used the actual mean of the points in C_i^n, but it will be helpful in the subsequent arguments that the cluster point itself corresponds to (the rescaling of) an extremal observation.

Based on the cluster points $x_1^n, \dots, x_{X_n}^n$ we define a random point measure on C as

$$\tilde{N}_n(A) = \sum_{i=1}^{X_n} \mathbf{1}_A(x_i^n)$$

for all $A \in \mathcal{B}(C)$ with the convention that $\tilde{N}_n(A) = 0$ if $X_n = 0$. Note that similar to N_n, the measure $\tilde{N}_n$ will also depend on the sequence $(\boldsymbol{k_n})$.

In Theorem 5.2 below we see that $(\tilde{N}_n)_{n\in\mathbb{N}}$ converges in exactly the same way as $(N_n)_{n\in\mathbb{N}}$. The proof relies on finding a set with sufficiently large probability, where the two measures N_n and $\tilde{N}_n$ are identical on the sets A and B under study in the proof of Theorem 5.1.

Theorem 5.2. *Let $(D_n)_{n\in\mathbb{N}}$ be defined as above, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$ for some sequence $(x_n)_{n\in\mathbb{N}}$. Moreover, let $(\boldsymbol{k_n})$ be a sequence of d-dimensional vectors with elements tending to infinity satisfying $k_n^*\alpha_n \to 0$ and $\boldsymbol{k_n}\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ as $n \to \infty$, such that $\mathcal{D}^\ell(x_n; \boldsymbol{k_n}; \theta)$ is satisfied for some $\theta \in (0,1]$. If for some $0 < \tau < \infty$,*

$$|D_n|\mathbb{P}(\xi_0 > x_n) \to \tau$$

then $\tilde{N}_n \stackrel{vd}{\to} N$, where N is a homogeneous Poisson measure on C with intensity $\theta\tau$.

Proof. Similarly to the proof of Theorem 5.1 it suffices to show that $\mathbb{E}\tilde{N}_n(A) \to \mathbb{E}N(A)$, where $A \subseteq C$ is a box on the form $A = \bigtimes_{j=1}^d (a_j, b_j]$ and that $\mathbb{P}(\tilde{N}_n(B) = 0) \to \mathbb{P}(N(B) = 0)$ for all B, where $B \subseteq C$ is a finite union of boxes on the form $\bigtimes_{j=1}^d (a_j, b_j]$.

Let $m \in \mathbb{N}$ and define the events

$$A_m = \{N(\partial A \oplus [-1/m, 1/m]^d) = 0\},$$

$$A_{m,n} = \{N_n(\partial A \oplus [-1/m, 1/m]^d) = 0\},$$

$$B_m = \{N(\partial B \oplus [-1/m, 1/m]^d) = 0\},$$

$$B_{m,n} = \{N_n(\partial B \oplus [-1/m, 1/m]^d) = 0\},$$

with the convention that N_n and N are 0 outside of C. These sets represent that there is no activity close (with m large) to the boundaries of the sets A and B, and they will help to ensure that clusters counted inside A and B by N_n will also be counted as inside A and B by $\tilde{N}_n$. Note that $\mathbb{P}(A_m^c) \to 0$ and $\mathbb{P}(B_m^c) \to 0$ as $m \to \infty$.

Furthermore consider the finite collection $\mathcal{E}_m$ of overlapping boxes with side lengths $2/m$, indexed by the same $m \in \mathbb{N}$ as above,

$$\mathcal{E}_m = \left\{ \frac{z}{m} + [-1/m, 1/m]^d \ : \ \mathrm{dist}\left(\frac{z}{m}, C\right) \le \frac{1}{m}, z \in \mathbb{Z}^d \right\},$$

where $\mathrm{dist}(u,u') = \max_\ell |u_\ell - u'_\ell|$ is the maximum norm. We define the events

$$E_m = \{N(E) \le 1 \text{ for all } E \in \mathcal{E}_m\},$$
$$E_{m,n} = \{N_n(E) \le 1 \text{ for all } E \in \mathcal{E}_m\},$$

and notice that the overlapping nature of the sets in $\mathcal{E}_m$ ensures that no sufficiently small neighborhood in C experiences more than one of the clusters counted by N_n.

Using the facts that the number of sets in $\mathcal{E}_m$ is of order m^d and that $\mathbb{P}(N(E) \ge 2) = O(1/m^{2d})$ together with Boole's inequality, it is seen that

$$\mathbb{P}(E_m^c) = O(1/m^d)$$

as $m \to \infty$.

From Theorem 5.1 we have $\mathbb{P}(E_{m,n}) \to \mathbb{P}(E_m)$ and $\mathbb{P}(B_{m,n}) \to \mathbb{P}(B_m)$ as $n \to \infty$, so for given $\delta > 0$ we can choose $m, n_0 \in \mathbb{N}$ such that

$$\mathbb{P}(B_{m,n}^c) < \delta \quad \text{and} \quad \mathbb{P}(E_{m,n}^c) < \delta$$

for all $n \ge n_0$.

Note that the ℓth side length of the set $\boldsymbol{c_n^{-1}} J_z^n$ is smaller than $1/k_{n,\ell}$ for $\ell = 1, \dots, d$. Hence, choosing n large enough ensures that for any set $\boldsymbol{c_n^{-1}} J_z^n$, the set and all its neighbors will be fully contained in at least one set from $\mathcal{E}_m$. For instance, choosing n such that $3 \max_\ell 1/k_{n,\ell} < 1/m$ suffices, and this choice will also be used in the proof of Theorem 6.1.

With this choice of n we have $N_n(A) = \tilde{N}_n(A)$ on $A_{m,n} \cap E_{m,n}$ and that $N_n(B) = \tilde{N}_n(B)$ on $B_{m,n} \cap E_{m,n}$: By definition of the two types of clusters any cluster point for $\tilde{N}_n$ will be counted as at least one cluster point for N_n. The set $E_{m,n}$ ensures that a cluster for $\tilde{N}_n$ is also at most one cluster for N_n. Furthermore, assuming the sets $A_{m,n}$ and $B_{m,n}$ gives that there is the same amount of clusters inside A and B, respectively.

Now we easily see that

$$\mathbb{P}(\tilde{N}_n(B) = 0) \in \big(\mathbb{P}(N_n(B) = 0) - 2\delta, \mathbb{P}(N_n(B) = 0) + 2\delta\big)$$

for n large, which together with Theorem 5.1 gives the desired convergence of $\mathbb{P}(\tilde{N}_n(B) = 0)$ by letting $n \to \infty$ and then $\delta \to 0$.

Next we turn to showing the convergence $\mathbb{E}\tilde{N}_n(A) \to \mathbb{E}N(A)$. First we find, using that $N_n(A)$ and $\tilde{N}_n(A)$ are equal on $A_{m,n} \cap E_{m,n}$ for n large,

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{E}\tilde{N}_n(A) &\ge \liminf_{n\to\infty} \mathbb{E}(\tilde{N}_n(A); A_{m,n} \cap E_{m,n}) \\ &= \liminf_{n\to\infty} \mathbb{E}(N_n(A); A_{m,n} \cap E_{m,n}) = \mathbb{E}(N(A); A_m \cap E_m).\end{aligned}$$

The last equality follows, since $\{N_n(A), N_n(E) : E \in \mathcal{E}_m\}$ converges in distribution to $\{N(A), N(E) : E \in \mathcal{E}_m\}$, such that also $(N_n(A), 1_{A_{m,n} \cap E_{m,n}})$ converges in distribution to $(N(A), 1_{A_m \cap E_m})$, and in addition $N_n(A)1_{A_{m,n}\cap E_{m,n}} \leq N_n(A)$, where $\mathbb{E}N_n(A) \to \mathbb{E}N(A)$. Letting $m \to \infty$ shows that $\liminf_{n\to\infty} \mathbb{E}\tilde{N}_n(A) \geq \mathbb{E}N(A)$.

For the upper bound we utilize the inequality $\tilde{N}_n(A) \leq N_n(A \oplus [-1/m, 1/m]^d)$ for n large. Then, with similar considerations,

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{E}\tilde{N}_n(A) \\
&\leq \limsup_{n\to\infty} \mathbb{E}(N_n(A); A_{m,n} \cap E_{m,n}) \\
&\quad + \limsup_{n\to\infty} \mathbb{E}(N_n(A \oplus (-1/m, 1/m]^d); A_{m,n}^c \cup E_{m,n}^c) \\
&= \mathbb{E}(N(A); A_m \cap E_m) + \mathbb{E}(N(A \oplus (-1/m, 1/m]^d); A_m^c \cup E_m^c).
\end{aligned}$$

Letting $m \to \infty$ together with integrability of $N(A)$ and $N(A \oplus (-1/m, 1/m]^d)$ gives $\limsup_{n\to\infty} \mathbb{E}\tilde{N}_n(A) \leq \mathbb{E}N(A)$ as desired. $\square$

6. Mean number of points in clusters

We continue considering the framework used to introduce the cluster processes N_n and $\tilde{N}_n$ in Section 5. In particular, we define $(D_n)_{n\in\mathbb{N}}$ as

$$D_n = (\boldsymbol{c_n} C) \cap \mathbb{Z}^d$$

for a p-convex set $C \subseteq \mathbb{R}^d$ with $|C| = 1$. In this section we study the mean cluster size with respect to both of the cluster definitions. It should be noted that under the assumptions of Theorem 5.1 and 5.2 the limit of the mean number of clusters within C with respect to either definition is $\theta\tau$. The expected number of extremal points above x_n within D_n is (before rescaling)

$$\sum_{v\in D_n} \mathbb{P}(\xi_0 > x_n),$$

which by assumption converges to τ as $n \to \infty$. Thus one would expect that the mean number of extremal points within each cluster in the limit will be $1/\theta$. In Theorem 6.1 below, we show that this is indeed true. However, for the clusters counted by $\tilde{N}_n$ we need to impose a stronger mixing condition than $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$. Furthermore we demonstrate via conditioning that this limit is still valid independently of how many clusters there are in C in total.

In relation to the cluster counting process N_n, where a set J_z^n is counted as a cluster if $M_\xi(J_z^n) > x_n$, we define the number of points in this potential cluster as

$$Y_z^n = \left|\{v \in J_z^n : \xi_v > x_n\}\right|.$$

Clearly, J_z^n represents a cluster if $Y_z^n > 0$. We will be interested in the mean number of points in such a cluster, i.e.

$$\mathbb{E}(Y_0^n \mid Y_0^n > 0).$$

Similarly, for the cluster point measure $\tilde{N}_n$, there are clusters if $X_n = \tilde{N}_n(C) > 0$. In that case let C be a cluster chosen uniformly among the $\tilde{N}_n(C)$ clusters. More specifically, we assume that $C = C_S^n$, where conditioned on $(\tilde{N}_n(C) = \ell)$ for each $\ell \in \mathbb{N}$, the variable S is uniform on $\{1, \dots, \ell\}$ and independent of everything else. Then we consider the mean cluster size defined as

$$\mathbb{E}(|C| \mid \tilde{N}_n(C) > 0).$$

In particular, we will focus on the limit of these conditional means as $n \to \infty$, and results for this are found in Theorem 6.1 below. We almost immediately have a result for the first of the two, while the second result requires the following stronger version of the mixing condition $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$.

Condition $\overline{\mathcal{D}}(x_n; \boldsymbol{k_n}; K_n)$. The condition $\overline{\mathcal{D}}(x_n; \boldsymbol{k_n}; K_n)$ is satisfied for the stationary field $(\xi_v)_{v\in\mathbb{Z}^d}$ if there exists an increasing sequence $(\boldsymbol{\gamma_n})$ of d-dimensional vectors with $\boldsymbol{k_n}\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ such that the following holds: First, for all $\boldsymbol{\gamma_n}$-separated sets $A, B \subseteq K_n$,

$$\left|\mathbb{P}((M_\xi(A) \le x_n) \cap \mathcal{B}) - \mathbb{P}(M_\xi(A) \le x_n)\mathbb{P}(\mathcal{B})\right| \le \alpha_n, \tag{22}$$

where $\mathcal{B} \in \sigma((\xi_{v'} > x_n) : v' \in B\})$. Secondly, for all $v \in K_n$ and $B \subseteq K_n$, where $\{v\}$ and B are $\boldsymbol{\gamma_n}$-separated,

$$\left|\mathbb{P}((\xi_v > x_n) \cap \mathcal{B}) - \mathbb{P}(\xi_v > x_n)\mathbb{P}(\mathcal{B})\right| \le \tilde{\alpha}_n, \tag{23}$$

where $\mathcal{B} \in \sigma((\xi_{v'} > x_n) : v' \in B\})$. The sequences (α_n) and $(\tilde{\alpha}_n)$ satisfy

$$k_n^*\alpha_n \to 0 \qquad \text{and} \qquad \tilde{\alpha}_n = o(\mathbb{P}(\xi_0 > x_n))$$

as $n \to \infty$.

Clearly (22) in itself constitutes a stronger requirement than (8) in condition $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$, while (23) serves as an additional assumption not directly related to (22). Both however, express approximate extremal independence.

Example 2 (Continued). If $(\xi_v)_{v\in\mathbb{Z}^d}$ is m-dependent, then $\overline{\mathcal{D}}(x_n; \boldsymbol{k_n}; K_n)$ is satisfied, similarly to how $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$ is satisfied.

For understanding the formulation of the following theorem, recall that Q_n^- is defined as the set of all $z \in \mathbb{Z}^d$, where the cluster point z (as counted by N_n) of J_z^n is inside D_n.

Theorem 6.1. *Let $(D_n)_{n\in\mathbb{N}}$ be defined as above, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$ for some sequence $(x_n)_{n\in\mathbb{N}}$. Moreover, let $(\boldsymbol{k_n})$ be a sequence of d-dimensional vectors with elements tending to infinity satisfying $k_n^*\alpha_n \to 0$ and $\boldsymbol{k_n}\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ as $n \to \infty$, such that $\mathcal{D}^\ell(x_n; \boldsymbol{k_n}; \theta)$ is satisfied for some $\theta \in (0,1]$. Assume for some $0 < \tau < \infty$ that*

$$|D_n|\mathbb{P}(\xi_0 > x_n) \to \tau.$$

Then as $n \to \infty$

$$\mathbb{E}(Y_0^n \mid Y_0^n > 0) \to 1/\theta. \tag{24}$$

If furthermore $\overline{\mathcal{D}}(x_n; \boldsymbol{k_n}; K_n)$ is satisfied, then

$$\mathbb{E}(|C| \mid \tilde{N}_n(C) > 0) \to 1/\theta \tag{25}$$

as $n \to \infty$, and additionally

$$\max_{z \in Q_n^-} \left| \mathbb{E}(Y_z^n \mid Y_z^n > 0, N_n(C) = \ell) - 1/\theta \right| \to 0 \tag{26}$$

and

$$\mathbb{E}(|C| \mid \tilde{N}_n(C) = \ell) \to 1/\theta \tag{27}$$

for all $\ell \in \mathbb{N}$.

Recall that $\overline{\mathcal{D}}(x_n; \boldsymbol{k_n}; K_n)$ is a stronger requirement than $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$, so requiring $\overline{\mathcal{D}}(x_n; \boldsymbol{k_n}; K_n)$ for (25)–(27) makes the assumption of $\mathcal{D}(x_n; K_n; \boldsymbol{c_n})$ superfluous. The proof of Theorem 6.1 is rather technical and is deferred to Section C of the supplementary material [14].

Example 3 (Continued). Recall that the field $(\xi_v)_{v \in \mathbb{Z}^d}$ with $\xi_v = \max_{z \in v+B} Y_z$ introduced in Example 3 has extremal index $1/|B|$ with respect to (D_n). Let (x_n) be chosen such that $|D_n|\mathbb{P}(\xi_0 > x_n) \to \tau$ for some $0 < \tau < \infty$. Since the field is m-dependent for a sufficiently large m, the condition $\overline{\mathcal{D}}(x_n; \boldsymbol{k_n}; K_n)$ is satisfied. Then, due to Theorem 6.1, the limiting mean number of extremal points within a cluster will not surprisingly be $|B|$ with respect to both cluster definitions.

7. Cluster counting measure on original scale

The results from Section 5 can be used to assess the limiting behavior of the number of clusters that $(\xi_v)_{v \in D_n}$ has above the level x_n in the rescaled set $\boldsymbol{c_n} A$, where A is some fixed subset of C. Describing the number of clusters in more general sequences of sets, only satisfying Assumption 2.2, is not obtained directly from Section 5. Such a result is instead part of Theorem 7.1 below.

Again, we assume that the sequence $(D_n)_{n \in \mathbb{N}}$ has the specific form

$$D_n = (\boldsymbol{c_n} C) \cap \mathbb{Z}^d$$

with $|C| = 1$, such that Assumption 2.2 is satisfied. This will not in itself be the set sequence of interest, but it will contain a more general sequence satisfying Assumption 2.2. As before we let $(\boldsymbol{k_n})$ and (x_n) be sequences satisfying some specified conditions. We define a cluster counting measure L_n on $\mathbb{Z}^d$ as

$$L_n(A) = \sum_{z \in \mathbb{Z}^d} \mathbf{1}_{A \cap D_n}(\boldsymbol{t_n} z) \mathbf{1}_{\{M_\xi(J_z^n) > x_n\}}$$

for all $A \subseteq \mathbb{Z}^d$. That is simply a transformation of the measure N_n back to the original scale and now regarded as a measure on $\mathbb{Z}^d$.

Recalling the definition of the cluster points $x_1^n, \dots, x_{X_n}^n$ used to construct $\tilde{N}_n$, we define the measure $\tilde{L}_n$ on $\mathbb{Z}^d$ as

$$\tilde{L}_n(A) = \sum_{i=1}^{X_n} \mathbf{1}_{A \cap D_n}(\boldsymbol{c_n} x_i^n)$$

for $A \subseteq \mathbb{Z}^d$, with the convention that $\tilde{L}_n(A) = 0$ if $X_n = 0$.

Note that, as opposed to the measures N_n and $\tilde{N}_n$ defined in Section 5 above, both L_n and $\tilde{L}_n$ are measures on the original index set scale.

Theorem 7.1. *Let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathcal{D}(x_n;K_n;\boldsymbol{c_n})$ for some sequence $(x_n)_{n\in\mathbb{N}}$. Moreover, let $(\boldsymbol{k_n})$ be a sequence of d-dimensional vectors with elements tending to infinity satisfying $k_n^*\alpha_n \to 0$ and $\boldsymbol{k_n}\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ as $n\to\infty$, such that $\mathcal{D}^\ell(x_n;\boldsymbol{k_n};\theta)$ is satisfied for some $\theta\in(0,1]$.*

Let $(B_n^1)_{n\in\mathbb{N}},\dots,(B_n^G)_{n\in\mathbb{N}}$ be sequences of sets in $\mathbb{Z}^d$, each satisfying Assumption 2.2, such that $B_n^g \subseteq D_n$ for all n and g, and $B_n^1,\dots,B_n^G$ are pairwise disjoint. Assume furthermore that

$$\lim_{n\to\infty}\frac{|B_n^g|}{|D_n|} = b_g$$

for each $g=1,\dots,G$, where $0\le b_g<\infty$. If for some $0<\tau<\infty$,

$$|D_n|\mathbb{P}(\xi_0 > x_n)\to\tau$$

then

$$(L_n(B_n^1),\dots,L_n(B_n^G)) \xrightarrow{\mathcal{D}} (L^1,\dots,L^G) \tag{28}$$

and

$$(\tilde{L}_n(B_n^1),\dots,\tilde{L}_n(B_n^G)) \xrightarrow{\mathcal{D}} (L^1,\dots,L^G) \tag{29}$$

as $n\to\infty$, where $L^1,\dots,L^G$ are independent random variables with each L^g being Poisson distributed with parameter $b_g\theta\tau$. Here $b_g=0$ means that $\mathbb{P}(L^g=0)=1$.

Proof. We demonstrate the proof of (28). The proof of (29) follows identically replacing N_n and L_n by $\tilde{N}_n$ and $\tilde{L}_n$, respectively.

In addition to the usual set constructions, the proof applies the set construction from Section A of the supplementary material [14], which is similar to that of Section 2 but with a single fixed number $k\in\mathbb{N}$ instead of $\boldsymbol{k_n}$ and with another tuning parameter. We also refer to Section A of the supplementary material [14] for the definition of $\tilde{t}^\ell_{n,k}$ and $\tilde{I}_z^{n,k}$ below.

We apply the construction to the set D_n with tuning $\lambda\equiv 1$. That is,

$$\tilde{t}^\ell_{n,k} = \frac{c_{n,\ell}}{k^{1/d}}$$

for all ℓ, and we thus divide $\mathbb{R}^d$, and thereby also $\boldsymbol{c_n}C$, into the sets $\tilde{I}_z^{n,k}$ indexed by $z\in\mathbb{Z}^d$. Moreover, we let the lattice points be $\tilde{J}_z^{n,k} = \tilde{I}_z^{n,k}\cap\mathbb{Z}^d$. Now we see that

$$L_n(\tilde{J}_z^{n,k}) = N_n\Big(\frac{1}{\boldsymbol{c_n}}\tilde{I}_z^{n,k}\Big),$$

where in fact

$$\frac{1}{\boldsymbol{c_n}}\tilde{I}_z^{n,k} = z/\sqrt[d]{k} + \big[0,1/\sqrt[d]{k}\big)^d$$

is a fixed set independent of n with volume $1/k$. Thus, we immediately have from Theorem 5.1 that for any finite fixed collection of distinct indices $\{z_1,\dots,z_K\}$, it holds that

$$(L_n(\tilde{J}_{z_1}^{n,k}),\dots,L_n(\tilde{J}_{z_K}^{n,k})) \xrightarrow{\mathcal{D}} (N_{z_1},\dots,N_{z_K}),$$

where $N_{z_1},\dots,N_{z_K}$ are independent and Poisson distributed each with parameter $\theta\tau/k$. In particular, this holds for the finite collection of $\tilde{J}_z^{n,k}$-sets that cover D_n across all values of n; see Lemma A.2(ii) in the supplementary material [14].

Now consider the sequences of discrete sets $(B_n^1)_{n\in\mathbb{N}},\dots,(B_n^G)_{n\in\mathbb{N}}$. If $b_g>0$, the set-construction in this proof is identical to a set-construction based on the set B_n^g using the scaling vectors $\boldsymbol{b_n^g}=\boldsymbol{c_n}\,b_g^{1/d}$ and with a tuning parameter $\lambda=b_g$. In line with previous notation, we let $B_{n,k}^{g,-}$ be the union of $\tilde{J}_z^{n,k}$-sets contained in B_n^g, and we let $p_{n,k}^g$ be the number of such sets. If $b_g=0$ for some g, then $p_{n,k}^g=0$ for n large enough, and hence $B_{n,k}^{g,-}=\varnothing$. Note that $B_{n,k}^{1,-},\dots,B_{n,k}^{G,-}$ are disjoint. Let furthermore $B_{n,k}^+$ be the union of all $\tilde{J}_z^{n,k}$-sets that cover all B_n^g jointly for $g=1,\dots,G$. Let $q_{n,k}$ be the number of such sets in this union and note that $p_{n,k}^1+\cdots+p_{n,k}^G\le q_{n,k}$. Define $p_k^g=\liminf_{n\to\infty}p_{n,k}^g$ and $q_k=\limsup_{n\to\infty}q_{n,k}$. Note, since all $p_{n,k}^g,q_{n,k}\in\mathbb{N}$, that all $p_k^g\le p_{n,k}^g$ and $q_k\ge q_{n,k}$ for n large enough. From Lemma A.2(i) of the supplementary material [14] we now have

$$p_k^g\sim b_g k\qquad\text{and}\qquad r_k=o(k)\tag{30}$$

as $k\to\infty$, where $r_k=q_k-(p_k^1+\cdots+p_k^G)$. Choose for each n, k and g a set $B_{n,k}^{g,--}\subseteq B_{n,k}^{g,-}$ consisting of exactly p_k^g sets of the type $\tilde{J}_z^{n,k}$. Let $R_{n,k}=B_{n,k}^+\setminus\bigcup_{g=1}^G B_{n,k}^{g,--}$ and note that $R_{n,k}$ is a union of at most r_k sets of the type $\tilde{J}_z^{n,k}$ for n large enough.

For any fixed selection $z_1,\dots,z_{r_k}$ we have that

$$\mathbb{P}\Big(L_n\Big(\bigcup_{i=1}^{r_k}\tilde{J}_{z_i}^{n,k}\Big)=0\Big)\to\exp\Big(-r_k\frac{\theta\tau}{k}\Big),$$

which will tend to 1 as $k\to\infty$. Also note that for a similar union of at most r_k of the sets $\tilde{J}_z^{n,k}$, the probability will only be larger. Since, across all values of n, there are only finitely many selections of indices z with $\tilde{J}_z^{n,k}$ intersecting D_n, the set $R_{n,k}$ can only consist of finitely many different selections of $\tilde{J}_z^{n,k}$-sets, each selection being of at most r_k sets, across all values of n. Therefore also

$$\liminf_{n\to\infty}\mathbb{P}\big(L_n(R_{n,k})=0\big)\to 1$$

as $k\to\infty$. Now let $\ell_1,\dots,\ell_G\in\mathbb{N}_0$ and $\epsilon>0$ be given and choose $k\in\mathbb{N}$ large enough such that

$$\liminf_{n\to\infty}\mathbb{P}\big(L_n(R_{n,k})=0\big)>1-\epsilon.$$

Then

$$\begin{aligned}
&\limsup_{n\to\infty}\mathbb{P}\big(L_n(B_n^1)=\ell_1,\dots,L_n(B_n^G)=\ell_G\big)\\
&\quad\le\limsup_{n\to\infty}\mathbb{P}\big(L_n(B_n^1)=\ell_1,\dots,L_n(B_n^G)=\ell_G,L_n(R_{n,k})=0\big)+\epsilon\\
&\quad=\limsup_{n\to\infty}\mathbb{P}\big(L_n(B_{n,k}^{1,--})=\ell_1,\dots,L_n(B_{n,k}^{G,--})=\ell_G,L_n(R_{n,k})=0\big)+\epsilon\\
&\quad\le\limsup_{n\to\infty}\mathbb{P}\big(L_n(B_{n,k}^{1,--})=\ell_1,\dots,L_n(B_{n,k}^{G,--})=\ell_G\big)+\epsilon\\
&\quad\le\limsup_{n\to\infty}\sup\mathbb{P}\big(L_n(V_{n,k}^1)=\ell_1,\dots,L_n(V_{n,k}^G)=\ell_G\big)+\epsilon\\
&\quad=\mathbb{P}(L_{1,k}^-=\ell_1)\cdots\mathbb{P}(L_{G,k}^-=\ell_G)+\epsilon,
\end{aligned}$$

where sup in the fifth line is over all choices of disjoint sets $V^1_{n,k},\dots,V^G_{n,k}$ such that each $V^g_{n,k}$ consists of p^g_k (with fixed indices across varying n) sets $\tilde{J}^{n,k}_z$. Note that this is a finite supremum. In the sixth line each $L^-_{g,k}$ is Poisson distributed with parameter $p^g_k\theta\tau/k$. By similar considerations it is obtained that

$$\liminf_{n\to\infty}\mathbb{P}\big(L_n(B^1_n)=\ell_1,\dots,L_n(B^G_n)=\ell_G\big)$$
$$\geq \mathbb{P}(L^-_{1,k}=\ell_1)\cdots\mathbb{P}(L^-_{G,k}=\ell_G)-\epsilon.$$

Letting $k\to\infty$ now gives the desired result since each $L^-_{g,k}\xrightarrow{\mathcal{D}} L^g$ due to (30). □

8. The special case of $\theta = 1$

In this section we again consider the sequence (D_n) of discrete index sets obtained as

$$D_n = (\boldsymbol{c_n}C)\cap\mathbb{Z}^d$$

for some p-convex set C with volume 1. Moreover, we restrict attention to the case $\theta=1$ thus allowing no clustering. We formulate all results under the assumption of the condition $\mathcal{D}^\ell(x_n;\boldsymbol{k_n};1)$ but recall from Lemma 3.6(ii) that it is equivalent to the assumption that $\mathcal{D}^{(m)}(x_n;\boldsymbol{k_n})$ and (20) are satisfied with $\theta=1$ for some (equivalently all) $m\in\mathbb{N}_0$.

We consider the classical point process of exceedances given by

$$\overline{N}_n(A)=\sum_{z\in\mathbb{Z}^d}\mathbf{1}_A\Big(\frac{z}{\boldsymbol{c_n}}\Big)\mathbf{1}_{\{\xi_z>x_n\}}$$

for all $A\in\mathcal{B}(C)$. Not surprisingly, this converges similarly as the point process N_n in Section 5 though without clustering in the limiting process. The result follows by similar (in fact simpler) arguments as Theorem 5.1.

Theorem 8.1. *Let $(D_n)_{n\in\mathbb{N}}$ be defined as above, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathcal{D}(x_n;K_n;\boldsymbol{c_n})$ for some sequence $(x_n)_{n\in\mathbb{N}}$. Moreover, let $(\boldsymbol{k_n})$ be a sequence d-dimensional vectors with elements tending to infinity satisfying $k^*_n\alpha_n\to 0$ and $\boldsymbol{k_n}\boldsymbol{\gamma_n}=o(\boldsymbol{c_n})$ as $n\to\infty$, such that $\mathcal{D}^\ell(x_n;\boldsymbol{k_n};1)$ is satisfied. If for some $0<\tau<\infty$,*

$$|D_n|\mathbb{P}(\xi_0>x_n)\to\tau$$

then $\overline{N}_n\xrightarrow{vd}\overline{N}$, where $\overline{N}$ is a homogeneous Poisson process on C with intensity τ.

This result immediately gives the asymptotic behavior of upper order statistics of the field $(\xi_v)_{v\in\mathbb{Z}^d}$ as formulated in the corollary below. In the corollary, we let

$$\xi^n_{(1)}\geq\xi^n_{(2)}\geq\cdots\geq\xi^n_{(|D_n|)},\qquad n\in\mathbb{N},$$

be the ordered sample of $(\xi_v)_{v\in D_n}$ for all $n\in\mathbb{N}$. In particular, $M_\xi(D_n)=\xi^n_{(1)}$.

Corollary 8.2. *Let $(D_n)_{n\in\mathbb{N}}$ be defined as above, and let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathcal{D}(x_n;K_n;\boldsymbol{c_n})$ for some sequence $(x_n)_{n\in\mathbb{N}}$. Moreover, let $(\boldsymbol{k_n})$ be a sequence of d-dimensional vectors with elements tending to infinity satisfying $k_n^*\alpha_n \to 0$ and $\boldsymbol{k_n}\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ as $n \to \infty$, such that $\mathcal{D}^\ell(x_n;\boldsymbol{k_n};1)$ is satisfied. If for some $0 < \tau < \infty$,*

$$|D_n|\mathbb{P}(\xi_0 > x_n) \to \tau,$$

then, for all k,

$$\mathbb{P}\big(\xi^n_{(k)} \le x_n\big) \to \exp(-\tau)\sum_{j=0}^{k-1}\frac{\tau^j}{j!}$$

as $n \to \infty$.

Proof. The claims follows directly from Theorem 8.1 when realizing that

$$\mathbb{P}\big(\xi^n_{(k)} \le x_n\big) = \mathbb{P}\big(\overline{N}_n(C) \le k-1\big)$$

for all $n \in \mathbb{N}$. □

Naturally, a version of Theorem 7.1 for an exceedance point process on the original index set $\mathbb{Z}^d$ also holds in the present case of $\theta = 1$. To this end define the point process

$$\overline{L}_n(A) = \sum_{z\in\mathbb{Z}^d}\mathbf{1}_{A\cap D_n}(z)\mathbf{1}_{\{\xi_z > x_n\}}$$

for all $A \subseteq \mathbb{Z}^d$. The result below now follows exactly as Theorem 7.1 replacing N_n and L_n by $\overline{N}_n$ and $\overline{L}_n$, respectively.

Theorem 8.3. *Let $(\xi_v)_{v\in\mathbb{Z}^d}$ be a stationary field satisfying $\mathcal{D}(x_n;K_n;\boldsymbol{c_n})$ for some sequence $(x_n)_{n\in\mathbb{N}}$. Moreover, let $(\boldsymbol{k_n})$ be a sequence of d-dimensional vectors with elements tending to infinity satisfying $k_n^*\alpha_n \to 0$ and $\boldsymbol{k_n}\boldsymbol{\gamma_n} = o(\boldsymbol{c_n})$ as $n \to \infty$, such that $\mathcal{D}^\ell(x_n;\boldsymbol{k_n};1)$ is satisfied.*

Let $(B_n^1)_{n\in\mathbb{N}},\dots,(B_n^G)_{n\in\mathbb{N}}$ be sequences of sets in $\mathbb{Z}^d$, each satisfying Assumption 2.2, such that $B_n^g \subseteq D_n$ for all n and g, and $B_n^1,\dots,B_n^G$ are pairwise disjoint. Assume furthermore that

$$\lim_{n\to\infty}\frac{|B_n^g|}{|D_n|} = b_g,$$

for each $g = 1,\dots,G$, where $0 \le b_g < \infty$. If for some $0 < \tau < \infty$,

$$|D_n|\mathbb{P}(\xi_0 > x_n) \to \tau$$

then

$$(L_n(B_n^1),\dots,L_n(B_n^G)) \xrightarrow{\mathcal{D}} (\overline{L}^1,\dots,\overline{L}^G)$$

as $n \to \infty$, where $\overline{L}^1,\dots,\overline{L}^G$ are independent random variables with each $\overline{L}^g$ being Poisson distributed with parameter $b_g\tau$. Here $b_g = 0$ means that $\mathbb{P}(\overline{L}^g = 0) = 1$.

Supplementary Material

Supplementary material: Extremal clustering and cluster counting for spatial random fields (DOI: 10.3150/22-BEJ1561SUPP; .pdf). The supplementary material contains certain proofs of results stated in the main paper: Either very technical proofs or proofs that are of a similar nature as proofs found in the existing literature. All references to the supplementary material are clearly marked in the main paper. In the supplementary material, we refer directly to results and equations in the main paper.

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